KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

COMPACTNESS AND ITS APPLICATIONS

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

This work is dedicated to my wife, Augustina Adjei Donkor and my lovely children Godfried Adjei Donkor, Elizabeth Adjei Donkor and Eliana Adjei Donkor.

Abstract

In this study, we explore the concept of compactness as a core of general topology. Compactness together with Continuity and Connectedness form the so called 3 Cs of general topology. We delve into some of the useful theorems in general topology before singling out the less intuitive topic of compactness touching on almost every aspect of the topic. This work does not shy away from the usefulness of compactness in mathematical analysis and hence explains its application therein. The study finally looks at an example of the applications of compactness in a real life situation

Contents

List of Figures

Chapter 1

INTRODUCTION

1.1 Introduction

Considered one of the three main core disciplines of pure mathematics together with algebra and analysis topology is one of the most active areas of modern mathematics.

Topology grew out of geometry, expanding on some of the ideas and loosening some of the structures appearing therein. The word topology, literally, means the study of position or location. Topology is the study of shapes, including their properties, deformations applied to them, mappings between them, and configurations composed of them.

Topology is often described as rubber-sheet geometry. In traditional geometry, objects such as circles, triangles, planes, and polyhedra are considered rigid, with well-defined distances between points and well-defined angles between edges or faces. But in topology, distances and angles are irrelevant. We treat objects as if they are made of rubber, capable of being deformed. We allow objects to be bent, twisted, stretched, shrunk, or otherwise deformed from one to another, but we do not allow the objects to be ripped apart. In the figure below we see four shapes that are very different from a geometric perspective, but are considered equivalent in topology. Any one of the four, if made of rubber, can be deformed to each of the others.

In the next figure we see an example of two objects (a torus and a sphere), each of which cannot be deformed into the other. Under such circumstances, we say the objects are **topologically distinct**

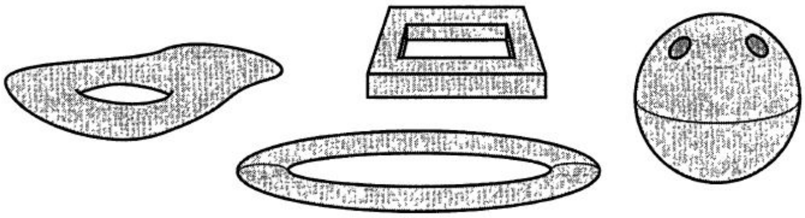


Figure 1.1: These four objects are topologically equivalent

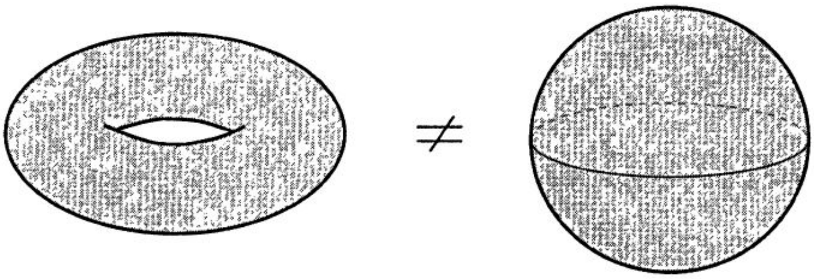


Figure 1.2: The torus and sphere are topologically distinct

1.2 Brief History

A brief history of topology takes us to Koenigsberg in Prussia where the river Pregel overflowed its banks and into the city, dividing it into four parts. Seven bridges were built over the river in its divided course to connect the regions.

The general problem arising was whether one could stroll through the town crossing each of the seven bridges just once. In his paper on the "Seven Bridges of Koenigsberg" which is regarded one of the first academic treatises in modern topology, Leonhard Euler (1707-1783) realised and explained that the problem involved a new mathematical approach that he called the "geometry of position". He wrote

> Recently, there was announced a problem which while it certainly seemed to belong to geometry, was nevertheless so designed that it did not call for the determination of magnitude, nor could it be solved by quantitative calculation; consequently I did not hesitate to assign it to the geometry of position, especially since the solution required only the consideration of position, calculation being of no use. (Newman,

1983)

Euler proved that it was impossible to walk through the whole of the town crossing each bridge just once, given the configuration of the bridges. Following the work of Euler, a number of prominent mathematicians including Carl Friedrich Gauss (1777-1785), August Ferdinand Möbius (1790-1868), Johann Listing (1808-1882), Bernhard Riemann (1826-1866), Felix Klein (1849-1925) and Henri Poincaré (1854-1912) made valuable contributions to the geometry of position over the next century and a half.(Adams & Franzosa, 2008)

The term "Topologie" was introduced in German by Listing in 1847 in his paper *Vorstudien zur Topologie*. It was not after decades before it first appeared in print. The English form topology was used in Listing's obituary in the journal Nature. "Analysis situs" was the expression primarily used for this area of geometry, and in the introduction to his 1895 paper "Analysis situs," Poincaré wrote of the geometry-of-position philosophy:

> The proportions of the figures might be grossly altered, but their elements must not be interchanged and must conserve their relative situation. In other terms, one does not worry about quantitative properties, but one must respect the qualitative properties, that is to say precisely those which are the concerns of Analysis Situs. (Sarkaria, 1999)

In the late nineteenth and early twentieth century there were numerous contributions to the growing discipline that would soon become the field of topology. L. E. J. Brouwer (1881-1966), Georg Cantor (1845-1918), Maurice Fréchet (1878-1973), Felix Hausdorff (1868-1962), Poincaré, Frigyes Riesz (1880-1956), and Hermann Weyl (1885-1955) were some of the mathematicians involved. Hausdorff's 1914 book ;*Grundzüge der Meningeal* (Fundamentals of Set Theory) introduced an axiomatic foundation for topological spaces and thereby initiated the study of topology as an abstract discipline.(Adams & Franzosa, 2008)

Throughout most of the twentieth century, topology developed primarily as a branch of pure mathematics. However, in recent years there has been development in the area growth has been realised in the applications of topology to other fields of mathematics and the sciences as a whole.

1.3 Basics from Topology

1.3.1 Topological Spaces

Definition 1.3.1. *Let X be a set. A **topology** defined on X is a collection of subsets of X each called an **open set**, such that*

O_1: *$\emptyset$ and X are open sets;*

O_2: *$\bigcap_{j=1}^{n} G_j \in \tau$ for every collection $G_1, \ldots G_n$ of elements in τ*

O_3: *$\bigcup_{\gamma \in \Gamma} G_\gamma \in \tau$ for every collection $\{G_\gamma | \gamma \in \Gamma\}$*

*The set X together with a topology τ on X is called a **topological space.** When there is no possibility of confusion we would refer to X as a topological space.*

Example 1.1. *Let X be the three-point set $\{a, b, c\}$. We consider four different collections of subsets of X and will investigate which ones are topologies. In each case assume that the collection contains the empty set and each of the circled sets*

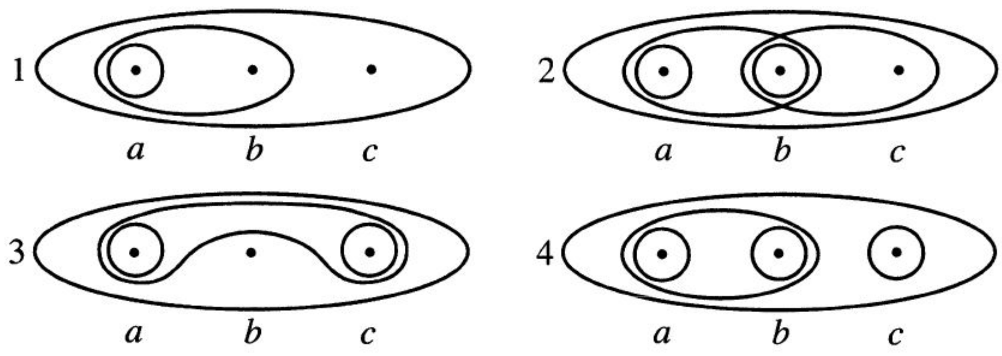


Figure 1.3: Which collections are topologies

Both Φ and X are in each of the four collections. We can see that for collections 1, 2, and 3, the intersection of sets in the collection is also in the collection, and the union of sets in the collection is also in the collection. Hence, collections 1, 2, and 3 depict topologies on X. However, in the case of collection 4, the sets $\{a\}$ and $\{c\}$ are each in the collection, but their union $\{a, c\}$ is not. So collection 4 does not depict a topology on X.

Example 1.2. *Let X be a non-empty set. Define* $T = \{\Phi, X\}$. *Notice that T satisfies all three of the conditions for being a topology. However, if we remove either set, we no longer have a topology. Thus* $\{\Phi, X\}$ *is the minimal topology we can define on X. For obvious reasons, it is called the trivial topology on X.*

Example 1.3. *Let X be a non-empty set and let T be the collection of all subsets of X. Clearly this is a topology, since unions and intersections of subsets of X are themselves subsets of X and therefore are in the collection T. We call this the* ***discrete topology*** *on X. This is the largest topology that we can define on X.*

Theorem 1.3.1 (Kelley, 1955)**.** *A set is open if and only if it contains a neighbourhood of each of its points.*

Proof The union U of all open subsets of a set A is surely an open subset of A. If A contains a neighbourhood of each of its points, then each member x of A belongs to some open subset of A and hence $x \in U$. In this case $A = U$ and therefore A is open. On the other hand, if A is open it contains a neighbourhood (namely, A) of each of its points.

Definition 1.3.2. *Let X be a topological space and* $x \in X$. *An open set U containing x is said to be a neighbourhood of x.*

A subset A of a topological space X is closed if and only if its relative complement $X - A$ is open. This then directly implies that a set is open if and only if it has a closed complement. (The complement of the complement of a set is the set itself.) If *tau* is the trivial topology the complement of X and the complement of the empty set are the only closed sets; that is, only the empty set and X are closed. It is always true that the space and the empty set are closed as well as open, and it may happen, as we have just seen, that these are the only closed sets.

Example 1.4. *The subset* $[a, b]$ *is closed because its complement*

$$\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty)$$

is open. Also since $(-\infty, c) \cup (c, \infty)$ *is open,* $\{c\}$ *is closed. Finally, every set is open implies that every set is closed as well in the discrete topology on* X

Theorem 1.3.2. *Let* X *be a topological space. The following conditions hold.*

(C_1*)* $\emptyset$ *and* X *are closed sets*

(C_2*) Arbitrary intersections of closed sets are closed.*

(C_3*) Finite unions of closed sets are closed.*

Proof.

(i) $\emptyset$ and X are closed because $\emptyset$ is the complement of the open set X whilst the open complement of X is $\emptyset$.

(ii) Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we apply De Morgan's law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

The sets $X - A_\alpha$ are open (by definition) and therefore the right side of the equation above represents the arbitrary union of open sets and thus open. This implies that $\bigcap A_\alpha$ is closed.

(iii) In like manner, if A_i is closed for $i = 1, \ldots, n$ consider the equation

$$X - \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (X - A_i)$$

The set on the right side of this equation is a finite intersection of open sets and thus open. Hence $\bigcup A_i$ is closed

From the above, we realise that instead of using open sets, we could as well specify a topology on a space by giving a collection/family of sets, which in this case will be called closed sets, satisfying the three properties of the theorem above. We could then define open sets as complements of closed sets and proceed just as we did before.

1.3.2 Sequences

A **sequence** is simply defined as a function on the set of non-negative integers. A sequence of real numbers is one whose range is a subset of the set of real numbers. The value of a sequence S at n is denoted, interchangeably, by S_n or $S(n)$. A sequence S is in a set A if and only if $S_n \in A$ for each non-negative integer n. S is eventually in A if and only if there is an integer m such that $S_n \in A$ whenever $n \geq m$. A sequence of real numbers converges to a number s relative to the usual topology if it is eventually in each neighbourhood of s. Using these definitions it turns out that, if A is a set of real numbers, then a point s belongs to the closure of A if and only if there is a sequence in A which converges to s, and s is an accumulation point of A if and only if there is a sequence in $A - [s]$ which converges to s.

Definition 1.3.3. *In a topological space X, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$ if for every neighbourhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$. The idea behind a sequence converging to a point x is that, given any neighbourhood U of x, the sequence eventually enters and stays in U. Under such circumstances, we say that x is the limit of the sequence x_n and we write*

$$\lim_{n\to\infty} x_n = x$$

Example 1.5. *Consider the sequence given by $x_n = \frac{1}{n}$ in $\mathbb{R}$. The sequence* *converges to 0 since every neighbourhood U of 0 contains an open interval* $(-a, a)$ *into which the sequence eventually enters and stays.*

A sequence x_n may not converge to a point and yet by a proper construction, a sequence may be obtained from it which converges. Such a convergent sequence which is constructed from the main sequence is called a **subsequence**.

Definition 1.3.4. *Let* X *and* Y *be topological spaces. A sequence* (f_n) *of functions* $f_n : X \longrightarrow Y$ *is said to converge (pointwise) to a function* $f : X \longrightarrow Y$ *if for each* $x \in X$*, the sequence* $(f_n(X))$ *converges in* Y *to* $f(x)$*.*

Thus, a sequence of continuous functions does not necessarily converge to a continuous function. However, we are guaranteed that the limit function is continuous if the individual sequences $(f_n(x))$ converge at a uniform rate. This is made explicit for the special case where the range is $\mathbb{R}$, in the following definition:

Definition 1.3.5. *A sequence of functions* $f_n : X \longrightarrow R$ *is said to converge uniformly to* $f : X \longrightarrow R$ *if for every* $\epsilon > 0$ *there exists* $N \in \mathbb{Z}_+$ *such that* $|f_n(x) - f(x)| < \epsilon$ *for every* $x \in X$ *and* $n \geq N$*.*

Theorem 1.3.3 (The Uniform Convergence Theorem)**.** *If* (f_n) *is a sequence of continuous functions* $f_n : X \longrightarrow R$ *that converges uniformly to* $f : X \longrightarrow R$*, then f is continuous.*

Proof. Let $U \subset R$ be an open set. We prove that for each $x \in f^{-1}(U)$ there exists an open set $V_x \subset X$ such that $x \in \subset f^{-1}(U)$. Let $\epsilon > 0$ be such that $(f(x^*)?\epsilon, f(x^*) + \epsilon) \subset U$. By uniform convergence, we can pick $N \in \mathbb{Z}_+$ such that $|f_n(x)?f(x)| < \frac{\epsilon}{3}$ for every $n \geq N$ and $x \in X$. Pick $n' \geq N$. Let $U' = (f_{n'}(x) - \frac{\epsilon}{3}, f_{n'}(x) + \frac{\epsilon}{3})$ and let $V_x = f_{n'}^{-1}(U')$ We claim that is V_x open in X, contains x, and satisfies $f(V_x) \subset U$. Given the claim, it follows that for every $x \in U$, there exists an open set V_x in X such that $x \in V_x \subset f^{-1}(U)$. Therefore U is open in X, and f is continuous. □

Definition 1.3.6. *A topological space X is called Hausdorff if for every pair of distinct points x and y in X, there exist neighbourhoods U and V of x and y respectively that are disjoint.*

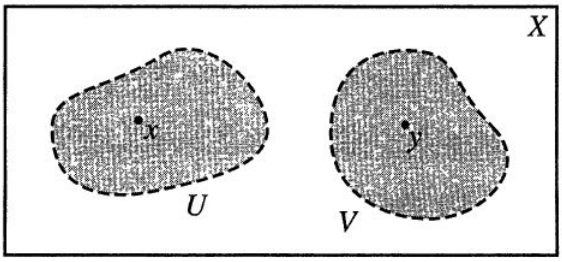


Figure 1.4: Distinct points have distinct neighbourhoods in a Hausdorff space

Example 1.6 (Adams and Franzosa, 2008). *Every set X with the discrete topology is Hausdorff. If x and y are distinct points, then the sets $\{x\}$ and $\{y\}$ are disjoint neighbourhoods of x and y, respectively.*

Theorem 1.3.4. *If X is a Hausdorff space, then every single-point subset of X is closed.*

Proof. Suppose $x \in X$. In order to show that $\{x\}$ is closed, we show that $X - \{x\}$ is open. Let $y \in X - \{x\}$ be arbitrary. Since X is Hausdorff, there are disjoint neighbourhoods U and V containing x and y, respectively. It follows that $x \notin V$ and therefore $y \in V \subset X - \{x\}$. Since every $y \in X - \{x\}$ is in an open set contained in $X - \{x\}$, it implies that $X - \{x\}$ is open.

Theorem 1.3.5. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X*

Proof. Suppose that x_n is a sequence of points of X that converges to x. If $y \neq x$, let U and V be disjoint neighbourhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore, x_n cannot converge to y.

Interior and Closure of Sets

An arbitrary subset A of a topological space X might be neither open nor closed as has been discussed already. However, it is often useful to associate a related open or closed set to A. In particular, we can sandwich each set A between the largest open set contained in A and the smallest closed set containing A. These sets are known as the interior of A and the closure of A, respectively.

Definition 1.3.7. *Let A be a subset of a topological space X. The interior of A, denoted $\mathring{A}$ or $Int(A)$, is the union of all open sets contained in A. The closure of A, denoted $\bar{A}$ or $Cl(A)$, is the intersection of all closed sets containing A.*

Definition 1.3.8. *Define B^n, the* n-**ball**, *to be the set*

$$B^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \ldots x_n^2 \leq 1\}$$

The 2-ball is referred to as the disk. Further, define $\mathring{B}^n$, the open n-ball, *to be the set*

$$\mathring{B}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \ldots x_n^2 \leq 1\}$$

The open 2-ball is referred to as the open disk

Continuity

One of the first experiences one could have with the concept of continuity is in the area of calculus or analysis, where the main focus is on mapping a real line $\mathbb{R}$ to itself.

Definition 1.3.9. *A funtion $f : \mathbb{R} \longrightarrow \mathbb{R}$ is* ***continuous*** *if for every $x_o \in \mathbb{R}$ and every $\epsilon > 0$, $\exists\ \delta > 0$ such that $|f(x) - f(x_o)| < \epsilon$ whenever $|x - x_o| < \delta$.*

The definition above is often known as the $\epsilon - \delta$ definition of continuity. Here, we provide a general definition of continuity for functions that maps from one topological space to another.

Definition 1.3.10. *Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is continuous if $f^{-1}(V)$ is open in X for every open set V in Y.*

This is the **open set definition of continuity**

Example 1.7 (Adams and Franzosa, 2008). *Let $X = a, b, c, d$ and $Y = 1, 2, 3$ have the topologies as depicted in the figure below. Let $f, g, h : X \longrightarrow Y$ be defined by*

$f(a) = 1,\ f(b) = 1,\ f(c) = 2,\ f(d) = 2$

$g(a) = 2,\ g(b) = 2,\ g(c) = 1,\ g(d) = 3,$

$h(a) = 1,\ h(b) = 2,\ h(c) = 2,\ h(d) = 3.$

The function f is continuous, as can be verified by checking that the pre-image of each open set in Y is open in X. Similarly, g is continuous. However, h is not continuous because {2} is open in Y, but $h^{-1}(\{2\}) = \{b, c\}$ is not open in X.

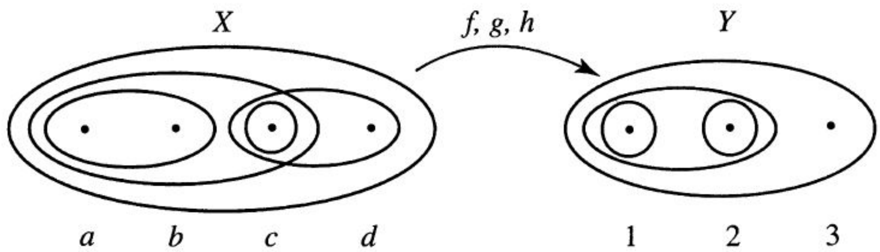


Figure 1.5: f,g,h maps X to Y

Theorem 1.3.6 (Adams and Franzosa, 2008). *A function $f : X \longrightarrow Y$ is continuous in the open set definition of continuity if and only for every $x \in X$ and every open set U containing $f(x)$, there exists a neighbourhood V of x such that $f(V) \subset U$.*

Proof. First, suppose that the open set definition holds for functions $f : X \longrightarrow Y$. Let $x \in X$ and an open set $U \subset Y$ containing $f(x)$ be given. Let

$V = f^{-1}(U)$. It follows that $x \in V$ and that V is open in X since f is continuous by the open set definition. Clearly $f(V) \subset U$, and therefore we have shown the desired result.

Now assume that for every $x \in X$ and every open set U containing $f(x)$, there exists a neighbourhood V of x such that $f(V) \subset U$. We show that $f^{-1}(W)$ is open in X for every open set W in Y. Thus let W be an arbitrary open set in Y. To show that $f^{-1}(W)$ is open in X, choose an arbitrary $x \in f^{-1}(W)$. It follows that $f(x) \in W$, and therefore there exists a neighbourhood V_x of x in X such that $f(V_x) \subset W$, or, equivalently, such that $V_x \subset f^{-1}(W)$. Thus, for an arbitrary $x \in (W)$ there exists an open set such that $x \in V_x \subset f^{-1}(W)$. Hence $f^{-1}(W)$ is open in X.

The next theorem indicates that continuous functions map convergent sequences to convergent sequences demonstrating the idea that continuity preserves proximity.

Theorem 1.3.7. *Assume that $f : X \longrightarrow Y$ is continuous. If a sequence $(x_1, x_2, \ldots)$ in X converges to a point x, then the sequence $(f(x_1), f(x_2), \ldots)$ in Y converges to $f(x)$.*

Proof. Let U be an arbitrary neighbourhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(U)$ is open in X. Furthermore, $f(x) \in U$ implies that $x \in f^{-1}(U)$ The sequence $(x_l, x_2, \ldots)$ converges to x; thus, there exists $N \in \mathbb{Z}_+$ such that $x_n \in f^{-1}(U)$, $\forall n \geq N$. It follows that $f(x_n) \in U \; \forall n \geq N$ and therefore the sequence $f(x_1), f(x_2) \ldots$ converges to f(x).

IMPORTANT NOTE 1. *A continuous function does not necessarily map open sets to open sets. For example, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, given by $f(x) = x^2$ is continuous, but the image of the open set $(-1, 1)$ is $[0, 1)$, which is not open.*

Homeomorphisms

Homeomorphisms provide the most fundamental notion of equivalence between topological spaces. They also preserve all the properties given by a topology, and thereby define a correspondence between points and between open sets in two topological spaces.

Definition 1.3.11. *Let X and Y be topological spaces, and let $f : X \longrightarrow Y$ be a bijection. If the function f and its inverse*

$$f^{-1} : Y \longrightarrow X$$

are continuous, then f is called a ***homeomorphism****. The existence of a homeomorphism between X and Y implies that X and Y are* ***homeomorphic*** *or* ***topologically equivalent*** *which we denote by $X \cong Y$*

The requirement that f^{-1} be continuous is expressive of the fact that for each open set U of X, the inverse image of U under the map $f^{-1} : Y \longrightarrow X$ is open in Y. However, the inverse image of U under the map f^{-1} is the same as the image of U under the map f, (i.e $(f^{-1})^{-1}(U) = f(U)$). This argument provides another definition of homeomorphism as being a bijective correspondence $f : X \longrightarrow Y$ such that $f(U)$ is open if and only if U is open.

In an introductory course in abstract algebra one may come across the concept of isomorphism between algebraic objects such as groups or rings. An isomorphism is a bijective correspondence that preserves the algebraic structure involved. The analogous notion in topology, is a homeomorphism, which is a bijective correspondence that preserves the topological structure (given by the collection of open sets in each topological space) involved.

Example 1.8. *Let X and Y be the topologies on the three-point sets shown in Figure 1.11. Define $f : X \longrightarrow Y$ by $f(a) = 1$, $f(b) = 2$, $f(c) = 3$. Then f is a homeomorphism since it is a bijection on points and a bijection between the open sets in X and Y.*

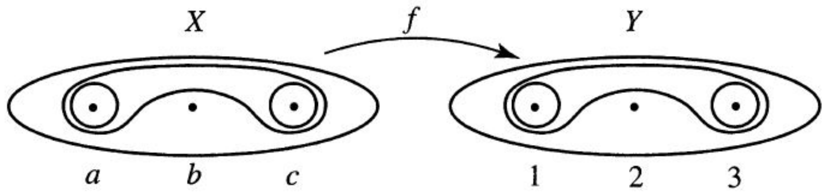


Figure 1.6: A homeomorphism from X to Y

Remark 1. *(1) The function* $id : X \longrightarrow X$*, defined by* $id(x) = x$*, is a homeomorphism. The function is known as the identity map.*

(2) If $f : X \longrightarrow Y$ *is a homeomorphism, then so is* $f^{-1} : Y \longrightarrow X$

(3) If $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *are homeomorphisms, then so is* $g \circ f : X \longrightarrow Z$

1.3.3 Metric Spaces

One of the most frequently used ways of defining a topology on a set is to define the topology in terms of a metric on the set. (i.e the topology on the set is derived from the notion of distance). When this is done, the set together with the metric defined on it becomes a metric space. Metric spaces are topological spaces that result from having a means for measuring distance between points in the underlying set.

Definition 1.3.12. *A* ***metric*** *on a set X is a function*

$$d : X \times X \longrightarrow [0, \infty)$$

with the following properties:

(1) $d(x, y) \geq 0$ *for all* $x, y \in X$*; equality holds if and only if* $x = y$*.*

(2) (Symmetry) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*.*

(3) (Triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x,y,z \in X$

We call $d(x,y)$ the distance between x and y, and the pair (X,d), consisting of the set X and the metric d, a metric space

The above axioms of a metric can be paraphrased to be that the distance between two points is at least 0, and it equals 0 only when the two points are the same. The distance from point x to point y is the same as the distance from point y to point x. Finally, the distance to travel from x to y and then y to z is never shorter than the distance to travel directly from x to z.

Example 1.9. *On $\mathbb{R}$, define $d(x,y) = |x-y|$. This is called the* ***Euclidean metric*** *or* ***standard metric*** *on $\mathbb{R}$. Conditions (i) and (ii) for a metric are immediate. The triangle inequality can be easily verified by considering separately each of the orderings of the three points x, y and z. For instance, in the case that $y \leq x \leq z$, we have*

$$\begin{aligned} d(x,y) + d(y,z) &= (x-y) + (z-y) \\ &\geq z - y \\ &\geq z - x \\ &= d(z,x) \\ &= d(x,z) \qquad \textit{By condition (ii)} \end{aligned}$$

Example 1.10.

$$\delta(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & elsewhere \end{cases}$$

Example 1.11 (Class of examples)**.** *$(X, \|\cdot\|)$, a normed linear space. $d_{\|\cdot\|}(x,y) = \| x - y \|$*

Example 1.12 (Metrics defined on a plane in $\mathbb{R}^2$)**.** *The measure of the distance* *between two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in a plane is defined as*

$$d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$$

and is known as the Euclidean distance formula. There we indicated that d satisfies the three properties that make it a metric on the plane. We call d the ***Euclidean metric*** *or standard metric on $\mathbb{R}^2$. This metric measures the straight-line distance between points in the plane.*

Definition 1.3.13. *Let (X, d) be a metric space. For $x \in X$ and $\epsilon > 0$ define the open ball of radius ϵ centered at x to be the set*

$$B_d(x, \epsilon) = \{y \in X|\ d(x, y) < \epsilon\}$$

and the closed ball of radius ϵ centered at x to be the set

$$B_d(x, \epsilon) = \{y \in X|\ d(x, y) \leq \epsilon\}$$

Theorem 1.3.8. *Let (X, d) be a metric space. The collection of open balls, $\mathcal{B} = B_d(X, \epsilon)|x \in X, \epsilon > 0)$, is a basis for a topology on X.*

To prove this theorem, we employ the following lemma.

Lemma 1.3.1. *Let(X, d) be a metric space. If $x \in X, \epsilon > 0$, and $y \in B_d(X, \epsilon)$, then there exists $\delta > 0$ such that $B_d(y, \delta) \subset B_d(X, \epsilon)$*

Proof. Let δ be equal to $\epsilon - d(x, y)$. Our claim is that $B_d(y, \delta) \subset B_d(X, \epsilon)$. In proving, let $z \in B_d(X, \epsilon)$ be arbitrary, then $d(x, y) < \delta$. This implies

$$\begin{aligned} d(x, y) + d(y, z) &< d(x, y) + \delta \\ &< d(x, y) + (\epsilon - d(x, y)) \\ &= \epsilon \end{aligned}$$

Thus $d(x, z) < \epsilon$. Hence $z \in B_d(x, \epsilon)$ and $B_d(y, \delta) \subset B_d(X, \epsilon)$

Proof of theorem. We investigate whether or not $\mathcal{B}$ is a basis. Obviously every point $x \in X$ is contained in a set in $\mathcal{B}$. In fact, $x \in B_d(X, \epsilon)$ for every $\epsilon > 0$. To satisfy the second condition for a basis is, we must show that if $x \in B_1 \cap B2$, and $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in B$ such that $x \in\subset B_1 \cap B_2$. Let B_1 and B_2 be two sets in $\mathcal{B}$, and suppose that $x \in B_1 \cap B_2$. Then by Lemma above there exist $\delta_i, \delta_2 > 0$ such that $B_d(X, \delta) \subset B_1$ and $B_d(X, \delta) \subset B_2$. Let $\delta = \min\{\delta_i, \delta_2\}$. Then $x \in B_d(X, \delta) \subset B_1 \cap B_2$, as desired. It follows that $\mathcal{B}$ is a basis for a topology on X.

Definition 1.3.14. *Let (X, d) be a metric space. The topology generated by the basis of open balls $\mathcal{B} = \{B_d(X, \epsilon) | x \in X, \epsilon > 0\}$ is the topology induced by the metric d and is called a **metric topology**.*

IMPORTANT NOTE 2. *The discrete topology is one induced by the discrete metric. Open sets don't contain their boundary points*

It is assumed therefore, that when a metric space (X, d) is assumed, it is a topological space with the metric induced by d
The first condition for a basis is trivial, since $x \in B(x, \epsilon)$ for any $\epsilon > 0$. Before checking the second condition for a basis, we show that if y is a point of the basis element $B(x, \epsilon)$, then there exists a basis element $B(y, \delta)$ centered at y that is contained in $B(y, \delta)$. Define δ to be a positive

Example 1.13. *Consider the metric on $\mathbb{R}$ given by $d(x, y) = |x - y|$. The basis elements associated with the metric d are the open intervals*

$$B_d(x, \epsilon) = \{y \in \mathbb{R} | |x - y| < \epsilon\} = (x - \epsilon, x + \epsilon)$$

Every open interval (a, b) in the real line can be expressed in the form $(x - \epsilon, x + \epsilon)$ by setting $x = \frac{a+b}{2}$ and $\epsilon = \frac{b-a}{2}$ it follows that this basis is exactly the basis for the *standard topology on $\mathbb{R}$. Therefore, the topology induced by the standard metric on $\mathbb{R}$ is the standard topology.*

Theorem 1.3.9. *Every metric space is Hausdorff*

Proof Let (X, d) be a metric space. Suppose x and y are distinct points in X with $d(x, y) = \epsilon$. Consider the sets $U = B_d(X, \epsilon/2)$ and $V = B_d(y, \epsilon/2)$. It follows that $x \in U$, $y \in V$, and U and V are open sets, then $U$and V are disjoint. Suppose $U \cap V \neq \emptyset$, and z is in the intersection. Then $d(x, z) < \epsilon/2$ and $d(y, z) < \epsilon/2$. By the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

i.e $d(x, y) < \epsilon$. This contradicts $d(x, y) = \epsilon$. Thus $U \cap V = \emptyset$. Hence, there exist disjoint open sets U and V containing x and y, respectively, implying that X is Hausdorff. The above theorem implies that, a topological space that is not Hausdorff cannot be induced by a metric.

Lemma 1.3.2. *In a metric space a sequence $(X_n)_{n\in\mathbb{N}}$ can only converge to at most one point.*

Proof. Suppose $X_n \xrightarrow{T_\delta} X$ and $X_n \xrightarrow{T_\delta} Y$

$$d(x, y) \leq d(x, X_n) + d(X_n, y)$$
$$d(x, y) \leq \lim_n \big(d(x, X_n) + x(X_n, y)\big)$$

This clearly implies $x = y$ □

Definition 1.3.15. *Let X be a topological space. X is said to be **metrizable** if there exists a metric d on the set X that induces a topology on X*

Definition 1.3.16 (Completeness)**.** *Let (X, d) be a metric space. $(X_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** if and only if, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$d(X_n, X_m) \leq \epsilon$ whenever $n, m > N$. A metric space is ***complete*** *if every Cauchy sequence is converges to a point in the space. A subset $Y \subset X$ is complete if every Cauchy sequence in Y is convergent in Y.*

Example 1.14. *$\mathbb{R}$ is complete with respect to the standard metric by its very definition*

Example 1.15. *The set of rationals in $(0, 1)$ are not complete in $\mathbb{R}$*

Remark 2. *In a complete metric space (X, d) a subset $Y \subset X$ is complete if and only if it is closed.*

1.4 Connectedness

There are several natural approaches that could be taken to rigorously capture the concept of connectedness for a topological space. One approach might be to say that a topological space is connected if it cannot be broken down into two distinct pieces that are separated from each other. Another approach might be to say that a topological space is connected if we can take a continuous walk in the space from any point to any other point. As simple as the concept appears, connectedness has profound implications for topology and its applications.

Definition 1.4.1. *Let X be a topological space.*

(i) We call X connected if there does not exist a pair of disjoint non-empty open sets whose union is X.

(ii) X is disconnected if X is not connected

(iii) If X is disconnected, then a pair of disjoint nonempty open sets whose union is X is called a ***separation of X***

A space can be separated if it can be broken up into two globs - disjoint open sets. Otherwise, one says it is connected. From this idea, we can say

Definition 1.4.2 (Munkres, 2000)**.** *Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X*

Being formulated entirely in terms of the collection of open sets of X, it is obviously a topological property. Put differently, if X is connected, so is any space homeomorphic to X.

Remark 3. *A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and itself.*

If A is a nonempty proper subset of X that is both open and closed in X, then the sets $U = A$ and $V = X - A$ constitute a separation of X, for they are open, disjoint and nonempty, and their union is X. Conversely, if U and V form a separation of X, then U is nonempty and different from X, and it is both open and closed in X. For a subspace Y of a topological space X, there is another useful way of formulating the definition of connectedness:

Lemma 1.4.1. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.*

Proof. Suppose first that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\bar{A}\cap Y$. Since A is closed in Y, $A = \bar{A}\cap Y$; or to say the same thing.$\bar{A}\cap B = \emptyset$. Since $\bar{A}$ is the union of A and its limit points, B contains no limit points of A. A similar argument shows that A contains no limit points of B.

Conversely, suppose that A and B are disjoint nonempty set whose union is Y, neither of which contains a limit point of the other. Then $\bar{A}\cap B = \emptyset$ and $A\cap\bar{B}=\emptyset$; therefore, we conclude that $\bar{A}\cap Y=A$ and $\bar{B}\cap Y=B$. Thus both A and B are closed in Y, since $A = Y - B$ and $B = Y - A$, they are open in Y as well. □

Example 1.16. *Consider the two topologies on the three-point set X = {a, b, c}. In the first topology, X is connected since there is no pair of disjoint* *nonempty open sets whose union equals X. However, in the second topology, X is disconnected. The pair of open sets,* $U = \{a, b\}$ *and* $V = \{c\}$,*is a separation of X.*

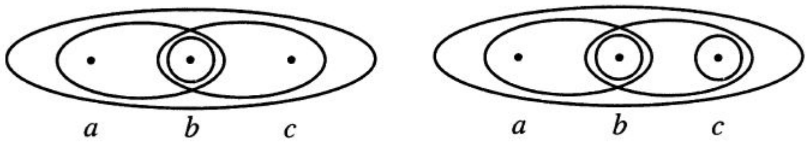


Figure 1.7: Two topologies on $X = \{a, b, c\}$, one connected and the other not.

Example 1.17. *Let* Y *denote the subspace* $[-1, 0) \cup (0, 1]$ *of the real line* $\mathbb{R}$. *Each of the sets* $[-1, 0)$ *and* $(0, 1]$ *is nonempty and open in* Y *(although not in* $\mathbb{R}$*); therefore, they form a separation of Y. Alternatively, note that neither of these sets contains a limit point of the other.*

Path Connectedness

In this section we introduce path connectedness, the second of the two approaches to connectedness that we mentioned at the beginning.

Let X be a topological space and x and y be points in X. We define a path from x to y in X to be a continuous function $f : [0, 1] \longrightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Definition 1.4.3. *A topological space* X *is path connected if for every* $x, y \in X$ *there is a path in* X *from* x *to* y. *A subset* A *of a topological space* X *is path connected in* X *if* A *is path connected in the subspace topology that* A *inherits from* X.

From the above, $\mathbb{R}^n$ is path connected, as is every open ball and every closed ball in $\mathbb{R}^n$

Theorem 1.4.1. *If* X *is a path connected space, then it is connected.*

Proof Let X be a path connected space. We prove that X is connected by showing that it has only one component, or equivalently that every pair of points $x, y \in X$ is contained in some connected subset of X. Thus, let x and y be arbitrary points in X. Since X is path connected, there is a path in X from x to y. The image of such a path is a connected subset of X containing both x and y. Therefore every pair of points in X is contained in a connected subset of X, and it follows that X is connected.

Remark 4. *The two types of connectedness mentioned so far are not the same, but path connectedness does imply connectedness. Again, with reference to the non-equivalence of path connectedness and connectedness, it is important to note that the converse of the above statement is not true, that is, connectedness does not imply path connectedness (as clearly established in the space known as the topologist's whirlpool). It follows that path connectedness is stronger condition on a topological space than connectedness.*

Example 1.18. *Define the unit ball B^n in $\mathbb{R}^n$ by the equation*

$$B^n = \{x| \; ||x|| \leq 1\},$$

where

$$||x|| = ||(x_1, \ldots, x_n)|| = (x_1^n + \ldots + x_n^2)^{1/2}$$

The unit ball is path connected; given any two points x and y of B^n, the straight-line path $f : [0,1] \longrightarrow \mathbb{R}^n$ defined by

$$f(t) = (1-t)x + ty$$

lies in B^n. For if x and y are in B^n and t is in [0,1],

$$||f(t)|| \leq (1-t)||x|| + t||y|| \leq 1$$

A similar argument shows that every open ball $B_d(x, \epsilon)$ and every closed ball

$\bar{B}_d(x, \epsilon)$ *in* $\mathbb{R}^n$ *is path connected.*

Example 1.19. *Define a punctured Euclidean space to be the space* $\mathbb{R}^n - \{0\}$*,where 0 is the origin in* $\mathbb{R}^n$*. If n>1, this space is path connected: Given x and y different from 0, we can join x and y by the straight-line between them if that path does not go through the origin. Otherwise, we can choose a point z not on the line joining x and y, and take the broken-line path from x to z, and then from z to y.*

1.5 Objectives of the Study

1. To explain the concept of compactness as a central topic in general topology.
2. To explore some of the very useful applications of compactness in mathematics and its implications.
3. To ascertain using a specific example the usefulness of compactness in real life.

1.6 Scope and Limitations

1.6.1 Scope of the Study

This research will span a selected collection of works in the field of topology. We will explore pertinent concepts under the broad topic of general topology. Major themes like sequences, connectedness, continuity and especially compactness. in the field, seasoned textbooks on the subjects of general topology and dynamical systems as well as useful internet resource cannot be overstated. This work was produced using LaTeX.

1.6.2 Limitations of the Study

In carrying out this study, the researcher faces a major problem of inadequate finances in securing important computer hardware to help facilitate the execution of the thesis. Another major setback is the unavailability of much needed resources in the form of useful textbooks, inadequate in-depth information on the subject matter. The work is limited to the various works of researchers and academicians in the field.

1.7 Organisation of the Thesis

The content of this study is divided into the following chapters:

- Chapter 1: Introduction
- Chapter 2: Literature Review
- Chapter 3: Compactness
- Chapter 4: Some Applications
- Chapter 5: Conclusion and Recommendation

Chapter 2

LITERATURE REVIEW

2.1 Introduction

This chapter intend to point out the works and opinions of earlier Scientist (Mathematicians) about the compactness and its importance or applications. Compactness is very important in diverse fields in both mathematics and other fields. The applications are realistic in real analysis, topology, functional analysis, differentials and other disciplines as we will see as we go through this paper.

2.2 The Concept Of Compactness

The property of a set being bounded in a metric space is not preserved by homomorphism. Thus for example, the interval (0, 1) and the whole of R are homomorphic under the usual topology. In that to generalize theorems in real analysis like ?a continuous function on closed bounded interval is bounded?. Therefore the need to for a new concept that is compactness.

Compactness may come in various types. As according to (Wong, 1973), the usual concept of compactness, includes the four types: Compactness, countable compactness, sequential Compactness and semi compactness. He also talks about fuzzy topology which he considered less useful. A fuzzy topological space is defined as a pair (X, τ) where $\tau \subseteq L^{\wedge}$ (*all maps from x to L*) and r is closed under arbitrary unions and finite intersections. (Hutton, 1977). (Lowen, 1997) also pointed out that initial and final fuzzy topologies from a categorical point of view are the right concept to generalize the topological ones. With this fuzzy compactness the Tychonoff product theorem is valid but not in the case of weak fuzzy compactness.

(Balder, 1990) presented a general relative sequential compactness criterion for salary integrable functions. With this he extended and included a classical weak compactness criterion for abstract L_1-spaces and the recent relative sequential compactness criterion of Prokhorov type for transition probabilities. In (Rakotoson, 1992) compactness lemma in conjunction with other lemma a regularity result and an integration by parts was applied to parabolic equation.

In the theory of dynamic programming, stochastic process where S_n represents the state of some systems at time n and a_n denote action taking at a time n. By the choice of an action or decision a_n we may control the stochastic development of the subsequent states $(S_n + 1, S_n + 2 \ldots)$. The set of policy of dynamic programming decision model which guarantees the optimal policy is arrived at through reduction of the problem to the compactness of a set of probability measures. (SHAL, 1975).

A dynamic programing problem where transition mechanism depends only on the action taken and does not depend on the current state of the system is called invariant (Assaf, 1980).

According to him, two typical problems which are invariant are the replacement and maintenance problems.

As it is easily seen, the strichartz estimates for the Schrodinger equation are not compact due to the invariance of this equation under some transformation. (Keraani, 2001).

According to (Bribiesca, 2000) in relation between area of the surface enclosing the volume and the contract surface through a measure of discrete is invariant under translation, rotation and scaling. According to him, the term compactness does not refer to point- set topology, but is related to intrinsic properties of objects.

Urban morphology, political districting and accuracy of enumeration units are seen as a result of compactness. Compactness in these areas are measured by methods: perimeter- area- measurement, single perimeters of related circles, direct comparison to standard shape and dispersion of elements of a shape?s area. (Maceachren, 1985).

A set of first other sentences has a model if and only if every finite subset of it has a model. This is the theorem of compactness which is a very important tool in model theory since it provides a method for constantly models of any set of sentences that is finitely consistent. (Wikipedia, 2015).

(Eastaugh, 2012) shows the interrelations between compactness properties of infinitesimal co-object of non-smooth analysis (sub differentials, normal cones, co- derivatives), metric regularity properties of single valued and set valued mappings between Banach spaces and sub differentials calculus rules. According to him, equivalence theorems showing various known compactness properties from co derivatives of set valued mapping form two groups of equivalence.

The compactness theorem which is a model-existence theorem shows that given the consistency of some theory, non-standard models of theory exist. This seen in the case of arithmetic. It is also has very useful applications in proving that certain classes of structures are finitely axiomatishable. (Eastaugh, 2012).

2.3 Compactness in Functional analysis.

According to (Nagy, 2007) Arzela-Ascoli Theorem is also used as a tool in obtaining Functional Analysis results, such as the compactness for duals of compact operators.

(Galaz-Fontes, 1998) established a criterion for a subset of the space of compact linear operators from a reflexive and separable space X into a Banach space Y to be compact.

According to (Robert, 1955) theorem of Arzel & and Ascoli, Characterizing conditionally compact subsets of the Banach space C(X) of continuous functions defined on a compact topological space X, is fundamental for much of functional analysis.

(Akkouchi, 2010) denote the space of all compact linear operators by K(X; Y) which is a closed subspace of L(X: Y).

(Cascales & Orihuela, 2014) show that survey about Topology as a tool in functional analysis would be such a giant enterprise that have been naturally chosen.

From (Galaz-Fontes, 1998) the Banach spaces X; Y, with its usual norm, the space K(X, Y) of all compact linear operators T: X $\rightarrow$ Y is also a Banach space.

(Haslinger, 2011) discuss compactness of the ∂-Neumann operator in the setting of weighted L^2 spaces on C^n leading to the description of relatively compact subsets of L^2 spaces.

As an application of characterization of the ∂-Neumann operator, sufficient condition is derived for compactness of the ∂-Neumann operator on (0,q)-forms in weighted L^2 spaces on C^n. (Haslinger, 2012).

Several characterization of totally bounded sets of precompact operations are given leading to an affirmative solution of the conjecture that a collectively compact set R is totally bound if and only if $R^* = K^* : K \epsilon R$ is collectively compact. (Palmer, 1969).

Given B a Banach space which contains a weakly compact fundamental subset $\exists$ a set (Γ) and a bounded one-to-one nonlinear operator T from B into(Γ).

(Rakotoson & Temam, 2001) gives a new optimal compactness criterion which insures that time dependent bounded sequences in suitable Hilbert spaces contain convergent subsequences.

2.4 Compactness in Differential equation

In solving non-linear partial differential equations, to ensure convergence of a subsequence to a solution is based on the compactness result. (Tartar, 1983).

(DiPerna, 1985) outlines a general program and present some new results dealing with oscillations in weakly convergent solution sequences to systems of conservation laws in which he employs the Young measure and the Tartar-Murat theory of compensated compactness and deals with systems of hyperbolic and elliptic type.

Also (Diperna, 1983) establishing uniform estimates on both the amplitude and derivatives of appropriate solutions in relevant metric and thus appeal to an appropriate compactness.

In (Baldi *et al.*, 2010) a compensated compactness theorem for differential forms of the intrinsic complex of a Carnot group was proved which relies on an L^s-Hodge decomposition for these forms. A kind of compactness plays a very important role in the study of nonlinear partial di5erential equations as well as in the study of nonlinear elliptic equations such as assures the Palais Smale condition. (Ishiwata & Ôtani, 2002).

(Fardoun & Regbaoui, 2015) proves the compactness of solutions of general fourth order elliptic equations which are L^1-perturbations of the Q-curvature equation on compact Riemannian 4-manifolds.

(Wenwen *et al.*, 2013) proposes an effective and efficient approach to computing shape compactness based on the moment of inertia (MI), a well-known concept in physics.

Aubin-type compactness lemma with a nonlinear restriction is established and then apply to solving a nonlinear degenerate parabolic equation. (Meirmanov & Shmarev, 2014).

Controlling compactness can increase sub-grade capability of resisting deformation and improve the pavement structure strength and stiffness. (Wang & GUA, 2009).

(De Pascale et al., 2002) point out some conditions for compactness and how it is applied to boundary value problems.

(Chou, 2014) reports on an experimental investigation of subjective judgments of compactness for electoral districts.

2.5 Compactness in Algebra

(Keimel, 2015) exhibits conditions under which C-C? is not only a space but also an algebra, as in the classical situation leading to the notion of entropicity in the sense of universal algebra.

(Kalton, 1974) derived a simple characterization of weakly compact subsets using criterion of Grothendieck enabling the study of reflexivity and weak sequential convergence.

(Peng, 2010) introduces a notion of tightness for a family of nonlinear expectations and shows that the tightness can be applied to obtain weak compactness in a framework of nonlinear expectation space.

A singular part of bounded sequence in H^1 which is a continuous martingales gives rise to a process of bounded variation. (Dalbaen & Schachermayer, 1999). The compactness of Manifolds is studied using heat kernels. (Gong & Weng, 2001).

The Method of compensated compactness, the Young measure and Lax entropies are used to obtain convergence of $L^p(p < \infty)$ bounded approximating sequences. (Lin, 1992).

(Lions, 1984) presented a new method solving minimization problems in unbounded domains. This was through a principle of equivalence between compactness of all minimizing sequences and some strict sub- addition conditions.

Chapter 3

COMPACTNESS

3.1 Introduction

The notion of a compact topological space is an abstraction of certain important properties of the set of real numbers. Being the least intuitive amongst continuity and connectedness, it is also not nearly so natural. From the beginnings of topology, it was clear that the closed interval $[a, b]$ of the real line had a certain property for proving such theorems as the extreme value theorem and the uniform continuity theorem. This property was the fact that every infinite subset of $[a, b]$ has a limit point - compactness.

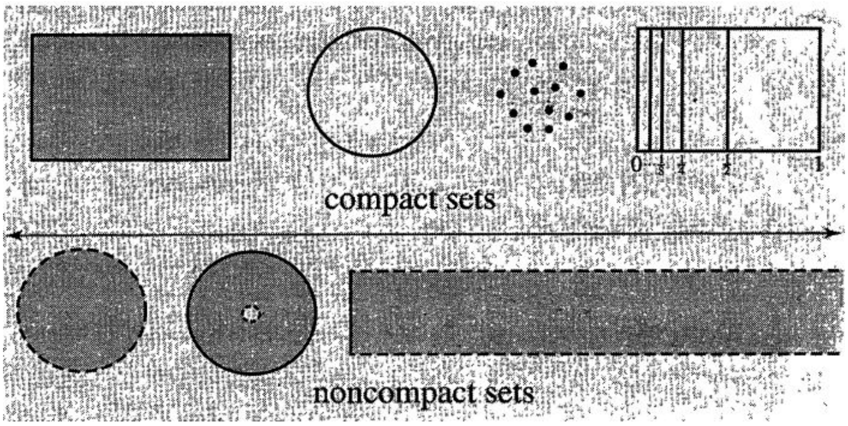


Figure 3.1: Compact and noncompact spaces in $\mathbb{R}^2$

Definition 3.1.1. *Let A be a subset of a topological space X, and let $\mathcal{O}$ be a collection of subsets of X.*

(i) The collection $\mathcal{O}$ is said to be cover of A or cover A if A is contained in the union of the sets in $\mathcal{O}$.

(ii) If $\mathcal{O}$ covers A, and each set in $\mathcal{O}$ is open, then we call $\mathcal{O}$ an open cover of A.

(iii) If $\mathcal{O}$ is a cover of A, and $\mathcal{O}'$ is a sub-collection of $\mathcal{O}$ that also covers A, then $\mathcal{O}'$ is called a subcover of $\mathcal{O}$.

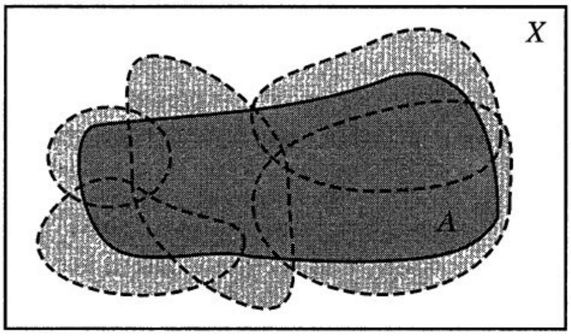


Figure 3.2: An open cover of A

From the definition of basis, it can said that, every basis for a topological space X is an open cover of X.(Adams & Franzosa, 2008)

In defining compact spaces, we are looking at spaces for which every open cover containing infinitely many sets can have a subcover containing finitely many sets.

Remark 5. *It is possible to have an open cover that consists of infinitely many sets as well as ones which consist of finitely many sets.*

Definition 3.1.2. *A topological space X is said to be compact, if every open cover of X has a finite subcover.*

Example 3.1. *The real line $\mathbb{R}$ is not compact, since the covering of $\mathbb{R}$ by open intervals*

$$\mathcal{O} = \{(n, n+2) | n \in \mathbb{Z}\}$$

that is

$$\mathcal{O} = \{\ldots, (-1,1), (0,2), (1,3), (2,4), \ldots\}$$

has no finite subcollection that covers $\mathbb{R}$.

Example 3.2. *All spaces X containing only finitely many points is compact, since every open cover is itself finite. It follows directly that a subcover of this space is finite.*

Remark 6. *(i) All closed intervals in the real line are compact*

(ii) All closed and bounded subsets of $\mathbb{R}^n$ *are compact.*

Lemma 3.1.1 (Adams & Franzosa, 2008)**.** *Let X be a topological space, and assume $A \in X$. Then A is compact in X if and only if every cover of A by sets that are open in X has a finite subcover.*

Proof. Let A be compact in X, and suppose that $\mathcal{O}$ is a cover of A by open sets in X. Then $\mathcal{O}' = \{U \cap A | U \in \mathcal{O}\}$ is a cover of A by open sets in A. Hence,there exists a finite subcover $\{U_1 \cap A, \ldots U_n \cap A\}$ of $\mathcal{O}'$. But then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{O}$. Therefore every cover of A by open sets in X has a finite subcover. Conversely, suppose every cover of A by sets that are open in X has a finite subcover. Let $\mathcal{O}' = \{V_\alpha\}_{\alpha \in \mathcal{B}}$ be a cover of A by open sets in A. Then, by definition of the subspace topology, for each V_α there is an open set U_α in X such that $V_\alpha = U_\alpha \cap A$. It follows that the collection $\mathcal{O}' = \{U_\alpha\}_{\alpha \in \mathcal{B}}$ is a cover of A by open sets in X. Since $\mathcal{O}'$ has a finite subcover $\{U_{\alpha_1}, \ldots U_{\alpha_n}\}$, it follows that $\{V_{\alpha_1}, \ldots V_{\alpha_n}\}$, is a finite subcover of $\mathcal{O}$. Thus every cover of A by open sets in A has a finite subcover, and therefore A is compact. □

Theorem 3.1.1. *Let $f : X \longrightarrow Y$ be continuous, and let A be compact in X. Then $f(A)$ is compact in Y.*

Proof. Let $f : X \longrightarrow Y$ be continuous, and assume that A is compact in X. To show that $f(A)$ is compact in Y, let $\mathcal{O}$ be a cover of $f(A)$ by open sets in Y. Then $f^{-1}(U)$ is open in X for every open set U in $\mathcal{O}$. Hence $\mathcal{O}' = \{f^{-1}(U)|U \in \mathcal{O}\}$ is a cover of A by open sets in X. Since A is compact, the Lemma above implies that there is a finite subcollection of $\mathcal{O}'$, say $\{f^{-1}(U_1), \ldots f^{-1}(U_n)\}$ that covers A. Then the collection of open sets $\{U_1, \ldots U_n\}$, in $\mathcal{O}$ covers $f(A)$. Thus, $\mathcal{O}$ has a finite subcover, implying that $f(A)$ is compact in Y. □

$[0, 1]$ is the prototype of a compact set in $\mathbb{R}$

Remark 7. *(i) For $p \in X, \{x\}$ is compact.*

(ii) The finite union of compact sets is compact.

(iii) Closed subsets of compact sets is compact

Lemma 3.1.2 (Adams & Franzosa, 2008)**.** *Let X be a topological space, and assume $A \in X$. Then A is compact in X if and only if every cover of A by sets that are open in X has a finite subcover.*

Proof. Let A be compact in X, and suppose that $\mathcal{O}$ is a cover of A by open sets in X. Then $\mathcal{O}' = \{U \cap A | U \in \mathcal{O}\}$ is a cover of A by open sets in A. Hence,there exists a finite subcover $\{U_1 \cap A, \ldots U_n \cap A\}$ of $\mathcal{O}'$. But then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{O}$. Therefore every cover of A by open sets in X has a finite subcover. Conversely, suppose every cover of A by sets that are open in X has a finite subcover. Let $\mathcal{O}' = \{V_\alpha\}_{\alpha \in \mathcal{B}}$ be a cover of A by open sets in A. Then, by definition of the subspace topology, for each V_α there is an open set U_α in X such that $V_\alpha = U_\alpha \cap A$. It follows that the collection $\mathcal{O}' = \{U_\alpha\}_{\alpha \in \mathcal{B}}$ is a cover of A by open sets in X. Since $\mathcal{O}'$ has a finite subcover $\{U_{\alpha_1}, \ldots U_{\alpha_n}\}$, it follows that $\{V_{\alpha_1}, \ldots V_{\alpha_n}\}$, is a finite subcover of $\mathcal{O}$. Thus every cover of A by open sets in A has a finite subcover, and therefore A is compact. □

Example 3.3. *Consider $(0, 1]$ as a subspace of $\mathbb{R}$. This space is not compact. The collection $\mathcal{O} = [(\frac{1}{n}, 2) | n \in \mathbb{Z}_+]$ is a cover of $(0, 1]$ by sets that are open in $\mathbb{R}$. There is no finite subcollection of $\mathcal{O}$ that covers $(0, 1]$, and therefore $(0, 1]$ is not compact as a subspace of $\mathbb{R}$.*

Two homeomorphic spaces are either both compact or both noncompact. Every open cover $\mathcal{O}$ of one is sent to an open cover $\mathcal{O}'$ of the other by a homeomorphism. Similarly, every finite subcover of $\mathcal{O}$ is sent to a finite subcover of $\mathcal{O}'$ by the homeomorphism. Since $\mathbb{R}$ and $(0, 1]$ are not compact, it follows that every interval of the form $(a, b), (-\infty, b), (a, \infty), [a, b), [a, \infty), (a, bJ, or(-\infty, b]$ is not compact as well.

Theorem 3.1.2. *Let $f : X \longrightarrow Y$ be continuous, and let A be compact in X. Then $f(A)$ is compact in Y.*

Proof. Let $f : X \longrightarrow Y$ be continuous, and assume that A is compact in X. To show that $f(A)$ is compact in Y, let $\mathcal{O}$ be a cover of $f(A)$ by open sets in Y. Then $f^{-1}(U)$ is open in X for every open set U in $\mathcal{O}$. Hence $\mathcal{O}' = \{f^{-1}(U) | U \in \mathcal{O}\}$ is a cover of A by open sets in X. Since A is compact, the Lemma above implies that there is a finite subcollection of $\mathcal{O}'$, say $\{f^{-1}(U_1), \dots f^{-1}(U_n)\}$ that covers A. Then the collection of open sets $\{U_1, \dots U_n\}$, in $\mathcal{O}$ covers $f(A)$. Thus, $\mathcal{O}$ has a finite subcover, implying that $f(A)$ is compact in Y. □

$[0, 1]$ is the prototype of a compact set in $\mathbb{R}$

Remark 8. *(i) For $p \in X, \{x\}$ is compact.*

(ii) The finite union of compact sets is compact.

(iii) Closed subsets of compact sets is compact

In the following lemma, we will find help in proving that the product of finitely many compact spaces is compact - the Tychonoff theorem named after the man who proved it.

Lemma 3.1.3 (Tube Lemma). *Let X and Y be topological spaces, and assume that Y is compact. If $x \in X$, and U is an open set in $X \times Y$ containing $\{x\} \times Y$, then there exists a neighbourhood W of x in X such that $W \times Y \subset U$.*

The set $W \times Y$ is often called a tube about $\{x\} \times Y$

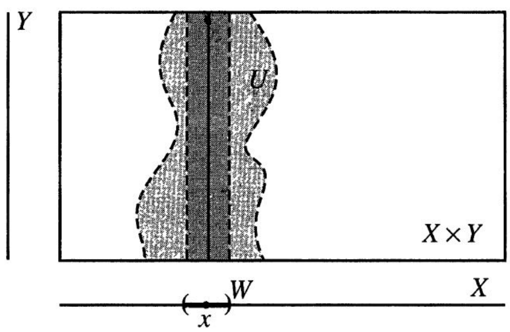


Figure 3.3: The tube $W \times Y$ is contained in the open set U

Proof. For each $y \in Y$ pick open sets W_y in X and V_y in Y such that $(x, y) \in W_y \times V_y \subset U$. The collection of sets $V_{y_{y\in Y}}$ is an open cover of Y. Since Y is compact, finitely many of these sets cover Y, say $V_{y_1}, \ldots, V_{y_n}$. Let $W = \bigcap_{i=1}^{n} W_{y_i}$. Then W is open in X, and W contains x because each W_y contains x. Note that

$$W \times Y \subset \bigcup_{i=1}^{n} (W_{y_i} \times V_{y_i}) \subset U$$

Therefore $W \times Y$ contains $\{x\} \times Y$ and is contained in U. □

Theorem 3.1.3 (Tychonoff)**.** *If X and Y are compact topological spaces, then the product $X \times Y$ is compact.*

Proof. Let $\mathcal{O}$ be an open cover of $X \times Y$. For each $x \in X$, the set $\{x\} \times Y$ is compact in $X \times Y$. Therefore a finite subcollection of $\mathcal{O}_x$ covers $\{x\} \times Y$. Let U_x be the union of the sets in $\mathcal{O}_x$. The set U_x is open in $X \times Y$ and contains $\{x\} \times Y$. By the Tube Lemma, for each $x \in X$ there exists an open set $W_x \subset X$ such that $x \in W_x$ and $W_x \times Y \subset U_x$. Note that $\mathcal{O}_x$ covers $W_x \times Y$.

The collection $\mathcal{W} = \{W_x | x \in X\}$ is an open cover of X. Since X is compact, it is covered by finitely many of the sets in $\mathcal{W}$, say $W_{x_1}, \ldots, W_{x_m}$. It follows that $\mathcal{C} = \mathcal{O}_{x_1} \cup \ldots \cup \mathcal{O}_{x_m}$ covers $X \times Y$. The collection $\mathcal{C}$ is a subcollection of $\mathcal{O}$ and is finite, being a finite union of finite sets. Therefore $\mathcal{O}$ has a finite subcollection that covers $X \times Y$, implying that $X \times Y$ is compact. □

3.2 Compactness in Metric Spaces

In real analysis, the focus is on $\mathbb{R}^n$ with the standard metric and topology - sometimes a set is defined to be compact if it is closed and bounded. The following lemma helps to show that the definition is consistent with the topological definition already presented.

Lemma 3.2.1. *Let $\{[a_n, b_n]\}_{n\in\mathbb{Z}_+}$ be a collection of nonempty closed bounded intervals in $\mathbb{R}$ such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for each $n \in \mathbb{Z}_+$. Then $\bigcup_{n=1}^{\infty} [a_n, b_n]$*

is nonempty.

Proof. Let $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. It implies that

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1.$$

The set $\{a_n\}_{n \in \mathbb{Z}_+}$ is bounded above by each b_n and therefore has a supremum of say, A. Similarly, the set $\{b_n\}_{n \in \mathbb{Z}_+}$ has an infimum say, B. Since $A \leq B$, the interval $[A, B]$ is nonempty.

We claim that $\bigcup_{n=1}^{\infty}[a_n, b_n] = [A, B]$.

We show that $\bigcup_{n=1}^{\infty}[a_n, b_n] \subset [A, B]$. Thus, let $x \in \bigcup_{n=1}^{\infty}[a_n, b_n]$ be arbitrary. Then $x \in [a_n, b_n]$ for all n, implying that $x \geq a_n$ and $x \leq b_n$ for all n. Therefore $x \geq A$ and $x \leq B$; that is, $x \in [A, B]$. Hence $\bigcup_{n=1}^{\infty}[a_n, b_n] \subset [A, B]$.

Now to prove that $[A, B] \subset \bigcup_{n=1}^{\infty}[a_n, b_n]$, let $x \in [A, B]$ be arbitrary. Then $x \geq a_n$ and $x \leq b_n$ for all n. Therefore $x \in \bigcup_{n=1}^{\infty}[a_n, b_n]$. Thus $[A, B] \subset \bigcup_{n=1}^{\infty}[a_n, b_n]$ and it follows that $\bigcup_{n=1}^{\infty}[a_n, b_n] = [A, B]$ □

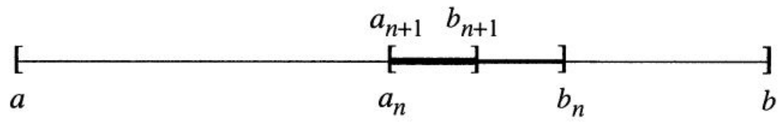
Remark 9. *The lemma does not hold if we replace the closed intervals with open intervals. For example, the collection of nonempty bounded open intervals* $\{(0, \frac{1}{n}) | n \in \mathbb{Z}_+\}$ *satisfies the condition that* $(0, \frac{1}{n+1}) \subset (0, \frac{1}{n})$ *for each* $n \in \mathbb{Z}_+$ *but* $\bigcup_{n=1}^{\infty} = \varnothing$

Theorem 3.2.1. *Every closed and bounded interval is a compact subset of* $\mathbb{R}$ *with the standard topology.*

Proof. Let $\mathcal{O}$ be a cover of $[a, b]$ by open sets in $\mathbb{R}$. We prove by contradiction by assuming that there is no finite subcollection of $\mathcal{O}$ covering $[a, b]$.

Consider the intervals $[a, \frac{a+b}{2}]$ and $\frac{a+b}{2}, b$ obtained by dividing $[a, b]$ in half. The collection $\mathcal{O}$ covers both of these intervals. For at least one of the two, there is no finite subcollection of $\mathcal{O}$ that covers it (otherwise there would be a finite subcollection of $\mathcal{O}$ covering $[a, b]$,contrary to our assumption). Choose such a half of $[a, b]$, and denote it by $[a_1, b_1]$.

In a similar manner, we can choose a half of $[a_l, b_1]$ that is not covered by a finite subcollection of $\mathcal{O}$ and denote it by $[a_2, b_2]$. We repeat this process. In other words, given $[a_n, b_n]$, a subset of $[a, b]$ that is not covered by a finite subcollection of $\mathcal{O}$, choose a half of $[a_n, b_n]$ that is not covered by a finite subcollection of $\mathcal{O}$, and denote it by a_{n+1}, b_{n+1} Consider the collection of intervals $\{[a_n, b_n]\}_{n \in \mathbb{Z}_+}$.



Based on the construction of these intervals, the following statements hold for each $n \in \mathbb{Z}_+$:

1. $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$

2. $b_n - a_n = \frac{b-a}{2^N}$

3. $[a_n, b_n]$ is not covered by a finite subcollection of $\mathcal{O}$

By the preceding lemma, it follows that $\bigcup_{n=1}^{\infty}[a_n, b_n]$ is nonempty. Let x be in this intersection. Then $x \in [a, b]$, and therefore there exists $U \in \mathcal{O}$ such that $x \in U$. Since U is open in $\mathbb{R}$, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Let N be a positive integer large enough so that $\frac{b-a}{2^N} < \epsilon$. Since $x \in \bigcup_{n=1}^{\infty}[a_n, b_n]$, it follows that $[a_N, b_N]$. Also, since $b_N - a_N = \frac{b-a}{2^N} < \epsilon$, it follows that $[a_N, b_N] \subset (x - \epsilon, x + \epsilon) \subset U$. But then $[a_N, b_N]$ is covered by a single set in $\mathcal{O}$, contradicting the fact that $[a_N, b_n]$ is not covered by a finite subcollection of $\mathcal{O}$

Thus there must be a finite subcollection of $\mathcal{O}$ that covers $[a, b]$, and therefore it follows that $[a, b]$ is compact. □

Example 3.4. *The circle S^1 is compact since it is a closed and bounded subset of $\mathbb{R}^2$*

Example 3.5. *The unit sphere S^{n-1} and the closed unit ball B^n in $\mathbb{R}^n$ are* *compact because they are closed and bounded. The set*

$$A = \{x \times (1/x) | 0 < x \leq 1\}$$

is closed in $\mathbb{R}^2$*, but it is not compact because it is not bounded. The set*

$$S = \{x \times (\sin(1/x)) | 0 < x \leq 1\}$$

is bounded in $\mathbb{R}^2$*, but it is not compact because it is not closed.*

Recall that if A and B are subsets of a metric space (X, d), then we define the distance between A and B by

$$d(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$$

Theorem 3.2.2. *Let* (X, d) *be a metric space. If* A *and* B *are disjoint compact subsets of* X*, then* $d(A, B) > 0$*.*

Proof. The distance function $d : X \times X \longrightarrow \mathbb{R}$ is continuous, and the set $A \times B$ is compact in $X \times Y$. The Extreme Value Theorem indicates that d takes on a minimum value on $A \times B$. That is, there exist $a^* \in A$ and $b^* \in B$ such that $d(a^*, b^*) \leq d(a, b)$ for every $a \in A$ and $b \in B$. It follows that $d(A, B) = d(a^*, b^*)$. Since A and B are disjoint, $a^* \neq b^*$ and therefore $d(a^*, b^*) > 0$. Hence, $d(A, B) > 0$ as desired. □

This theorem indicates that there is a positive distance between two disjoint compact sets in a metric space.

Remark 10. *The result of the above theorem does not hold if we replace 'compact' with 'closed.' That is, in a metric space it is possible to have disjoint closed sets* A *and* B *with* $d(A, B) = 0$*.*

3.2.1 Sequential Compactness

Definition 3.2.1 (Willard, 1970). *A space X is sequentially compact if and only if every sequence in X has a convergent subsequence.*

1. Not every compact space is sequentially compact.

2. Every sequentially compact space is countably compact, but not every sequentially compact space is compact. Hence, sequential compactness is neither stronger nor weaker than compactness; just different.

3. The countable product of sequentially compact spaces is sequentially compact.

3.2.2 Limit Point Compactness

In this section, we introduce one formulation of the notion of compactness that is frequently used. Though weaker in general than compactness, it coincides with compactness for metrizable spaces.

Definition 3.2.2. *A space X is said to be limit point compact if every infinite subset of X has a limit point.*

In many ways this property is more natural and intuitive than that of compactness. In the early days of topology, it was given the name 'compactness', while the open covering formulation was called 'bicompactness'. Later, the word 'compact' was shifted to apply to the open covering definition, leaving this one to search for a new name. It still has not found a name on which everyone agrees. On historical grounds, some call it 'Fréchet compactness'; others call it the 'Bolzano-Weierstrass property'. The term 'limit point compactness' has been invented. It seems as good a term as any; at least it describes what the property is about.

Theorem 3.2.3. *Compactness implies limit point compactness, but not conversely.*

Proof. Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point. We prove the contrapositive - if A has no limit point, then A must be finite.

Suppose A has no limit point. Then A contains all its limit points, so that A is closed. Furthermore, for each $a \in A$ we can choose a neighbourhood U_a of a such that U_a intersects A in the point a alone. The space X is covered by the open set $X - A$ and the open sets U_a; being compact, it can be covered by finitely many of these sets. Since $X - A$ does not intersect A, and each set U_a contains only one point of A, the set A must be finite. □

3.2.3 One-point Compactifications

Definition 3.2.3 (Willard, 1970). *A compactification of a space X is an ordered pair (τ, h) where τ is a compact Hausdorff space and h is an embedding of X as a dense subset of τ. In many cases h will be an inclusion map, so that $X \subset \tau$. In other cases, we can agree to write X when we mean $h(X)$ (homeomorphic spaces are, to a topologist, the same), so that we can again write $X \subset K$. Whenever one of these situations occurs we say simply that τ is a compactification of X, and think of τ as containing X as a dense subspace.*

1. Compact sets are closed and bounded in a metric space
2. Sequences have convergent subsequences in a compact subset of a metric space,
3. Compact metric spaces are complete, and
4. Continuous functions on compact spaces attain minimum and maximum values

Definition 3.2.4 ((Adams & Franzosa, 2008)). *A topological space X is locally compact if every $x \in X$ has a neighbourhood that is contained in a compact subset of X.*

Example 3.6. *Every compact space is automatically locally compact since each $x \in X$ has X both as a neighbourhood and as a compact set containing the neighbourhood.*

Example 3.7. *The real line $\mathbb{R}$ is locally compact since for each $x \in \mathbb{R}$ we have $x \in (x-1, x+1) \subset [x-1, x+1]$, and $[x-1, x+1]$ is compact.*

3.2.4 Paracompactness

Paracompact spaces were first introduced as a natural generalization of compact spaces still retaining enough structure to enjoy many of the properties of compact spaces, yet sufficiently general to include a much wider class of spaces. The notion of paracompactness gained stature with the proof, by A. H. Stone, that every metric space is paracompact and the subsequent use of this result in the solutions of the general metrization problem by Bing, Nagata and Smirnov. The central role played by paracompactness, or paracompact-like properties, in some of the current areas of intensive investigation in topology ensure it a permanent place alongside metrizability and compactness among the most important concepts in general topology. (Willard, 1970)

Theorem 3.2.4 (A. H Stone)**.** *Every metric space is paracompact*

Proof. Let $\mathcal{O}$ be an open cover of the metric space (X, d). For $U \in \mathcal{O}$ $n = 1, 2, 3, \ldots$ let $U_n = \{x \in U | d(x, X - U) \geq \frac{1}{2^n}\}$, then

$$d(U_n, X - U_{n+1}) \geq 1/2^n - 1/2^{n+1} = 1/2^{n+1}$$

Let $\leqq$ be a well ordering of the elements of $\mathcal{O}$. For each $n = 1, 2, 3, \ldots$ and $U \in \mathcal{O}$, let

$$U_n^* = U_n - \bigcup\{V_{n+1} : V \in \mathcal{O}, V \leqq U\}.$$

For each $U, V \in \mathcal{O}$ and each $n = 1, 2, 3, \ldots$, we have

$$U_n^* \subset X - V_{n+1}$$

or

$$V_n^* \subset X - U_{n+1}$$

(depending on which comes first in the well ordering.) In either case

$$d(U_n^*, V_n^*) \geq 1/2^{n+1}$$

Hence, defining an open set U_n^o for each $U \in \mathcal{O}$ and $n \in \mathbb{N}$ by

$$U_n^o = \{x \in X | d(x, U_n^{ast}) < 1/2^{n+3}\}$$

we have $d(U_n^o, V_n^o) \geq 1/2^{n+2}$, so $\tau_n = \{U_n^o | U \in \mathcal{O}\}$ is discrete for each n. Hence, $\tau = \bigcup \tau_n$ is a $\sigma-$discrete and thus $\sigma-$locally finite. Moreover, τ refines $\mathcal{O}$ and covers X. (If $x \in X$, find the first $U \in \mathcal{O}$ to which x belongs, and then $x \in U_n^0$ for some n) $\square$

Chapter 4

APPLICATIONS OF COMPACTNESS

4.1 The Extreme Value Theorem

The Extreme Value theorem is a topologically based theorem that is often introduced in a calculus course. It concerns real-valued functions on a compact domain.

To begin, we need the following lemma, which indicate that every compact subset of the real line contains a maximum value and a minimum value.

Lemma 4.1.1. *Let A be a compact subset of $\mathbb{R}$. Then there exist $m, M \in A$ such that $m \leq a \leq M$ for all $a \in A$.*

Proof. Here we prove the existence of the maximum value M. The proof of the existence of the minimum value m is similar. Since A is compact, it is closed and bounded in $\mathbb{R}$. Therefore A is bounded from above. It follows that the set A has a least upper bound; denote it by M. Of course, a M for all $a \in A$. We claim that $M \in A$. We prove the claim by contradiction; thus suppose that $M \notin A$. Since A is closed, it follows that there exists an $\epsilon > 0$ such that $(M - \epsilon, M + \epsilon) \cap A = \varnothing$. Then $M - \frac{\epsilon}{2}$ is an upper bound for A that is smaller than M, a contradiction. Therefore $M \in A$, and A has a maximum value. □

Theorem 4.1.1 (Extreme Value Theorem)**.** *Let $f : X \longrightarrow Y$ be continuous, where Y is an ordered set in the order topology. It X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$*

Proof. f is continuous and X is compact implies the set $A = f(X)$. Then by the lemma above, A has a minimum element m and a maximum element M. Then since m and M belong to A, we must have $m = f(c)$ and $M = f(d)$ for some points c and d of X.

If A has no largest element, then the collection

$$\{(-\infty, a)|a \in A\}$$

forms an open covering of A. Since A is compact, some finite subcollection

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

covers A. If a_i is largest of the elements $a_1, \ldots, a_n$, then a_i belongs to none of these sets, contrary to fact that they cover A.

A similar argument shows that A has a smallest element. □

Example 4.1. *In this simple application of the Extreme Value Theorem, we view the surface of the Earth as a sphere and the surface temperature as a continuous function on the sphere. Since the sphere is compact, the Extreme Value Theorem ensures that somewhere on the Earth there is a point or there are points where the temperature is hotter than those at every other point on Earth, and there are also points where the temperature is colder than those at every other point on Earth.(Adams & Franzosa,2008)*

Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous. Then f takes on a maximum value and a minimum value on $[a, b]$.

4.2 An Example from Optimization

The publishers at Christ Academy International would like to catch the new wave of interest in Christian Magazines. They are planning to publish a magazine to compete with those currently on the market and use the profit to further promote the true word of God. The financial committee has received the all clear to commit a GHC 700,000 budget to the first run of the magazine. The budget is to be allocated among several aspects of the magazine's production, including an authorship contract, editorial costs, printing, advertising, and distribution. Suppose that there are n such variables, $v_1 \ldots, v_n$. The profit that the company can expect to make on this venture depends on how the resources are allocated. For example, if they opt for printing the magazine in a high-cost Möbius-band format and a minimal advertising budget, they will realize a smaller profit than if they had used a standard magazine format and a larger advertising budget. Thus, we regard the profit, P, as a function of the variables $v_1, \ldots, v_n$. It is natural to assume that the profit, P is continuous. The domain of P is the subset of $\mathbb{R}^n$ given by

$$D = \{(v_1, \ldots, v_n) | v_1, \leq 0, \ldots, v_n \leq 0, v_1 + \ldots + v_n \leq 700,000\}$$

Since D is closed and bounded, it is compact. The Extreme Value Theorem indicates that there is a choice for optimum profit, that is the choice of allocation of resources that will result in a maximum profit for the proposed magazine project.

While the Extreme Value Theorem does not tell us how to find the maximum and minimum values of a function, it guarantees that they exist. This is typical of this topology theorem. The Extreme Value Theorem asserts the existence of a point with specified properties (a maximum and a minimum point), but does not provide the exact location for it.

4.3 Further Applications in Calculus

The Mean Value Theorem makes the geometrically plausible assertion that a differentiable function f on an interval $[a, b]$ will, at some point, attain a slope equal to the slope of the line through the endpoints $(a, f(a))$ and $(b, f(b))$.i.e

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for at least one point $c \in (a, b)$ Although the result itself is geometrically obvious,

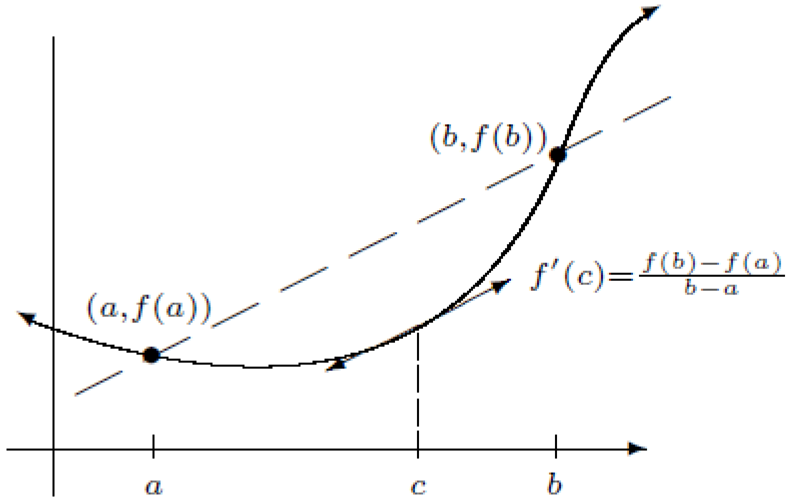


Figure 4.1: The Mean Value Theorem

the Mean Value Theorem is the cornerstone for almost every major theorem pertaining to differentiation. (Abbott, 2010). The Mean Value Theorem can be stated in various degrees of generality, each one important enough to be given its own special designation. One such case is the following theorem

Theorem 4.3.1 (Rolle's Theorem). *Let $f : [a, b] \longrightarrow R$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Proof. Because f is continuous on a **compact** set, f attains a maximum and a minimum. If both the maximum and minimum occur at the endpoints, then f is necessarily a constant function and $f'(x) = 0$ on all of (a, b). In this case, we can choose c to be any point we like. On the other hand, if either the maximum or minimum occurs at some point c in the interior (a, b), then it follows from the Extreme Value Theorem that $f'(c) = 0$. $\square$

4.3.1 Mean Value Theorem

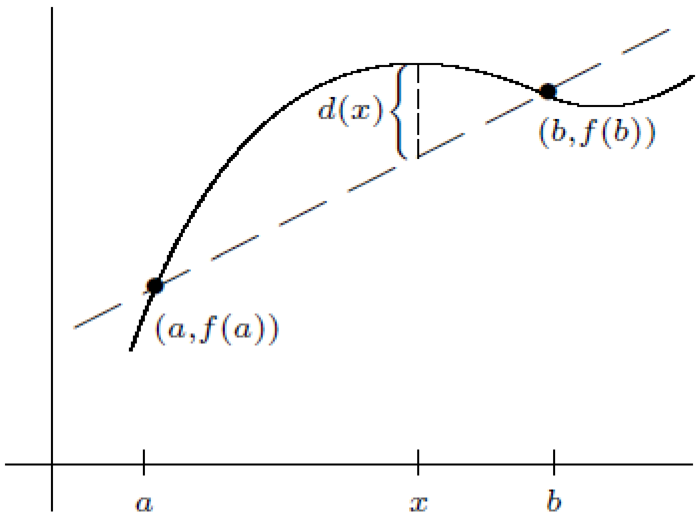
Theorem 4.3.2. *If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. The Mean Value Theorem reduces to Rolle's Theorem in the case where $f(a) = f(b)$. The equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)$$

For the difference between this line and the function $f(x)$, let



$$d(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)\right],$$

and observe that d is continuous on $[a,b]$, differentiable on (a,b), and satisfies $d(a) = 0 = d(b)$. Thus, by Rolle's Theorem, there exists a point $c \in (a,b)$ where $d'(c) = 0$. Because

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we get

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

which completes the proof. □

4.4 Topology meets Real Analysis

4.4.1 Bolzano-Weierstrass Theorem

For the purpose of illustration, let us consider an example on the divergence criterion.

Example 4.2. *Consider the sequence*

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\right)$$

It is quite clear that the above sequence does not converge to any limit. Notice that

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right)$$

is a subsequence that converges to 1/5. Also,

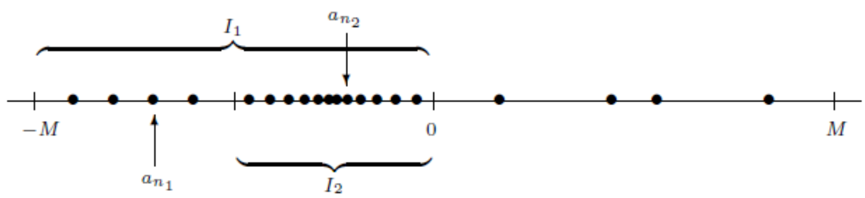
$$(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}\Big)$$

is a different subsequence of the original sequence that converges to -1/5. Here, we have two subsequences of the same sequence converging to two different limits. We can therefore rigorously conclude that the original sequence diverges.

In the example above, it was rather easy to obtain a convergent subsequence, in fact two hiding in the original sequence. For bounded sequences, it turns out that it is possible to find at least one such convergent subsequence.

Theorem 4.4.1 (Abbot, 2010)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $\in N$. Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$. (The midpoint is included in both halves.) Now it, must be that at least one of these closed intervals contains an infinite number of the points in the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some point in the sequence (a_n) satisfying $a_{n_1} \in I_1$. Next, we bisect I_1 into closed intervals of equal length, and let I_2 be



a half that again contains an infinite number of points of the original sequence. Because there are an infinite number of points from (a_n) to choose from, we can select an a_{n_2} from the original sequence with $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of $I_{k?1}$ containing an infinite number of points of (a_n) and then select $n_k > n_{k?1} > \ldots > n_2 > n_1$ so that $a_{n_k} \in I_k$. We want to argue that (a_{n_k}) is a convergent subsequence, but we need a candidate for the limit. The sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$$

form a nested sequence of closed intervals, and by the Nested Interval Property there exists at least one point $x \in \mathbb{R}$ contained in every I_k. This provides us with the candidate we were looking for. It just remains to show that $(a_{n_k}) \to x$.

Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Because x and a_{n_k} are both in I_k. It follows that $|a_{n_k} - x| < \epsilon$ and hence the subsequence is convergent. □

4.4.2 The Fundamental Theorem of Calculus

In Calculus, the derivative is motivated by the problem of finding tangent lines and is given in terms of functional limits of difference quotients. The definition of the integral grows out of the desire to describe areas under non-constant functions and is given in terms of supremums and infimums of finite sums. The Fundamental Theorem of Calculus reveals the remarkable inverse relationship between the two processes. The Riemann Integral The result is stated in two parts. The first is a computational statement that describes how an antiderivative can be used to evaluate an integral over a particular interval. The second statement is more theoretical in nature, expressing the fact that every continuous function is the derivative of its indefinite integral. (Abbot, 2010)

Theorem 4.4.2 (Fundamental Theorem of Calculus). *(i) If $f : [a, b] \to \mathbb{R}$ is integrable, and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f = F(b) - F(a),$$

(ii) Let $g : [a, b] \to \mathbb{R}$ be integrable, and define

$$G(x) = \int_a^x g$$

for all $x \in [a, b]$. Then G is continuous on $[a, b]$. If g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

Proof. (i) Let P be a partition of $[a, b]$ and apply the Mean Value Theorem to F on a typical subinterval $[x_{k-1}, x_k]$ of P. This yields a point $t_k \in (x_{k-1}, x_k)$ where

$$\begin{aligned} F(x_k) - F(x_{k-1}) &= F'(t_k)(x_k - x_{k-1}) \\ &= f(t_k)(x_k - x_{k-1}) \end{aligned}$$

Now consider the upper and lower sums $U(f,P)$ and $L(f,P)$. Because $m_k \leq f(t_k) \leq M_k$(where m_k is the infimum on $[x_{k-1}, x_k]$ and M_k is the supremum), it follows that

$$L(f,P) \leq \sum_{k-1}^{n}[F(x_k) - F(x_{k-1})] \leq U(f,P)$$

But notice that the sum in the middle telescopes so that

$$\sum_{k-1}^{n}[F(x_k) - F(x_{k-1})] = F(b) - F(a).$$

(ii) To prove the second statement, take $x, y \in [a,b]$ and observe that

$$\begin{aligned}|G(x) - G(y)| &= \left|\int_a^x g - \int_a^y g\right| = \left|\int_y^x g\right| \\ &\leq \int_y^x |g| \\ &\leq M|x-y|\end{aligned}$$

where $M > 0$ is a bound on $|g|$. This shows that G is Lipschitz and so is uniformly continuous on $[a,b]$

Now assume that g is continuous at $c \in [a,b]$. In order to show that $G'(c) = g(c)$, we rewrite the limit for $G'(c)$ as

$$\begin{aligned}\lim_{x\to c}\frac{G(x) = G(c)}{x-c} &= \lim_{x\to c}\frac{1}{x-c}\left(\int_a^x g(t)dt - \int_a^c g(t)dt\right) \\ &= \lim_{x\to c}\frac{1}{x-c}\left(\int_c^x g(t)dt\right)\end{aligned}$$

We would like to show that this limit equals $g(c)$. Thus, given an $\epsilon > 0$ we must produce a $\delta > 0$ such that if $|x-c| < \delta$, then

$$\left|\frac{1}{x-c}\left(\int_c^x g(t)dt\right) - g(c)\right| < \epsilon \tag{4.1}$$

The assumption of continuity of g gives us control over the difference $|g(t) - g(c)|$. In particular, we know that there exists a $\delta > 0$ such that

$$|t - c| < \delta \qquad \text{implies} \quad |g(t) - g(c)| < \epsilon$$

To take advantage of this, we cleverly write the constant $g(c)$ as

$$g(c) = \frac{1}{x - c} \int_c^x g(c)dt$$

and combine the two terms in equation 4.1 above into a single integral. Keeping in mind that $|x - c| \geq |t - c|$, we have that for all $|x - c| < \delta$,

$$\begin{aligned}
\left| \frac{1}{x - c} \left(\int_c^x g(t)dt \right) - g(c) \right| &= \left| \frac{1}{x - c} \int_c^x |g(t) - g(c)|dt \right| \\
&\leq \frac{1}{(x - c)} \int_c^x |g(t) - g(c)|dt \\
&< \frac{1}{(x - c)} \int_c^x \epsilon dt = \epsilon
\end{aligned}$$

□

Consider the following theorem

Theorem 4.4.3. *Let $y_o \in \mathcal{O}$ an open subset of $\mathbb{R}^n$, $I \subset \mathbb{R}$ an interval containing t_o. Suppose F is continuous on $I \times \mathcal{O}$ and satisfies the following Lipschitz estimate in y:*

$$\| F(t, y_1) - F(t, y_2) \| \leq L \| y_1 - y_2 \|,$$

for $t \in I, y_j \in \mathcal{O}$. Then the differential equation below has a unique solution on some t-interval containing t_o

$$\frac{dy}{dx} = F(t, y), \qquad y(t_o) = y_o \tag{4.2}$$

Under the hypothesis of the above theorem, if y solves the initial value problem for $t \in [T_o, T_1]$ and $y(t) \in K$ compact in $\mathcal{O}$, for all such $t \in [S_o, S_1]$ with

$S_o < T_o,\ T_1 > T_o$.

Loosely speaking, the Lipschitz condition comes in when you are trying to extend the interval of the solution: there is no point in extending the interval of the solution if you do not even know where there exists one. The only thing available for the proof is to use the compactness of K

Essentially, what the Fundamental theorem tells us is that if F is locally Lipschitz on an open subset U, then for any points $y \in U$, you can find a neighbourhood V of y and a time τ such that there exists a unique solution for every initial condition y_0 in V. So, all these neighbourhoods will form a cover of U (in particular K), and here is where compactness comes in: the finiteness of the subcover will then allow you to extend the interval. Note that you also need to assume your solution stays in the compact set K, otherwise your solution will cease to exist.

Proof. By the Fundamental theorem, for all $y \in K$, there exists an open neighbourhood V_y of y and a time τ_y such that the solution is defined for all initial condition $y_0 \in V_y$ and $|t| \leq \tau_y$. The set $\{V_y \colon y \in K\}$ form an open cover of K, so by compactness of K, there exists a finite subcover $\{V_{y_j} : j = 1, \ldots, n\}$. Define τ to be $\min_j \tau_{y_j}$, then $\forall y_0 \in K$, the solution $y(t)$ exists for $|t| \leq \tau$. But since we assume $y(t) \in K$ for $t \in [T_0, T_1]$, then $y(t)$ is defined for at least $t \in [T_0 - \tau, T_1 + \tau]$ □

Chapter 5

CONCLUSION AND RECOMMENDATION

5.1 Introduction

This chapter summarizes the results of the study, discusses the conclusion arrived at by the researcher and gives recommendations that would be necessary for further research.

5.2 Conclusion

Compactness is very important in diverse fields in mathematics and other fields alike. The applications are realistic in real analysis, topology, functional analysis, differential equations and many other disciplines.

In this study, we broadly discussed general topology, defining some useful terms and exploring some of the essential theorems and their proofs. We continued by reviewing some of the works by earlier scientists and then finally delved into the very less intuitive concept of compactness when compared with continuity and connectedness which makes up the core of general topology, in the subsequent chapter. Some essential applications are discussed in the fourth chapter.

In the third and fourth chapters of this write-up, we delved into the concept of compactness, explained some useful terms, explored some of the pertinent theorems and their proofs. We also succeeded in explaining compactness as a core of topology together with continuity and connectedness, with compactness being the less intuitive. Proceeding to the fourth chapter, we explored some of the very useful theorems in real analysis and calculus which happen to be applications of compactness as well. We also showed by way of an example, the usefulness of compactness in solving a real world problem.

It is important to note that, this research was hindered by a lack of adequate material in terms of text and equipment

5.3 Recommendations

The usefulness of compactness is very vast and therefore could not be fully discussed in this write up. It is therefore recommended that further studies be carried especially with respect to applications in differential equations, the motion of a swinging pendulum and in algebraic number theory.

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