

KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY,
KUMASI

Meshfree Approximation in Nonlinear Black-Scholes Option Pricing Equation with
Transaction Cost

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M.Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

This thesis is dedicated to my parents Rev. George Onwona-Agyeman and Mrs. Doris Onwona-Agyeman who have devoted their lives for me and who have been giving their continuous blessings to me for success and to my dear brothers and sisters with whom I shared big portion of my life.



Abstract

Differential equations play a very important role in the world of finance since most problems in finance are modelled by means of differential equation. These problems are sometimes nonlinear which can be solved by using numerical techniques. We review option pricing models in general, the formulation of the Black-Scholes model, and the Black-Scholes model with transaction costs.

This thesis takes a look at Black-Scholes model with transaction costs with special reference to the model by Guy Barles and Halil Mete Soner. The resulting model is a nonlinear Black-Scholes equation in which the variable volatility is a function of the second derivative of the option price. We solve this nonlinear equation with a special class of numerical technique, called, the meshfree approximation using radial basis function. The method is analysed for stability and thorough comparative numerical results are provided. The numerical results showed that, the value of the option in the case of transaction cost is found to be higher than the analytical value of the standard Black-Scholes model.

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Chapter 1

Introduction

1.1 Background of the Study

Since ancient times a lot of different cultures had been concern about numbers and how to use them for different purposes, either if it was to establish patterns for Agriculture, Medicine, Physics, Chemistry or Astrology or solving more complex problems. Mathematical knowledge has been developed through time; and in our time, we still use this knowledge like the number pi (π) discovered by the Egyptians (Gomez, 2006).

Finance on the other hand is one of the fastest developing areas in the modern banking and corporate world. This, together with the sophistication of modern financial products provides a rapidly growing impetus for new mathematical methods. These mathematical methods are used in pricing financial derivatives which minimize losses caused by price fluctuations of the underlying assets. These assets are financial objects whose value is known at present but is liable to change in future. A derivative is a financial instrument whose value depends on the values of other, more basic, underlying variables (Hull, 2009). Derivatives markets have attracted many different types of traders. These types of traders are: hedgers, speculators and arbitrageurs. Hedgers use derivatives to reduce risk that they face from potential movements in a market variable. Speculators use derivatives to bet the future direction of a market variable hoping to win a large amount with a small initial investment while arbitrageurs take offsetting

positions in two or more instruments to lock in a profit (Hull, 2009).

The simplest derivatives are the forward, futures and options. A future or forward contract is an agreement between two parties to buy or sell an asset at a certain time in the future T , known as the expiration date for a certain price K called the strike price. Each of the parties makes a binding engagement; it is impossible to change later this contract. Unlike forwards, futures are traded on exchanges and more formalized. The holder of the contract makes a profit or loss of $S - K$ if at the expiration date the asset is trading in the marketplace for price S . Futures are widely used to take positions in assets that are not themselves easily traded, such as agricultural commodities.

An option is a security which gives its owner the right to trade in a fixed number of shares of a specified common stock at a fixed price K , at any time on or before a given date T . The act of making this transaction is referred to as exercising the option. The fixed price is termed the strike price, and the given date, the expiration or maturity date. There exist two types of options: the call option and the put option. The call option gives the holder the right to buy the underlying stock for the strike price K by the expiration date T , and the put option gives the holder the right to sell the underlying stock for the strike price K by the expiration date T . Options are divided into two styles by the dates on which they may be exercised. They are the European or American options. The main difference between European and American options is in different exercising time. An American option may be exercised at any time before the expiration date whereas an European option may be exercised only at the maturity date.

The value of the option $V = V(S, t)$ depends on the price of the underlying asset S and the time t . The final condition for $V(S, T)$ is called payoff function. The payoff function for a call option is given by $V_c(S, T) = \max\{S(T) - K, 0\} = (S(T) - K)^+$ and the payoff function for a put option is $V_p(S, T) = \max\{K - S(T), 0\} = (K - S(T))^+$. Black and Scholes (1973) deduced a mathematical model in which the value of an European option can be calculated. Their model was based on certain assumptions. From the model, one can deduce the BlackScholes formula, which gives a theoretical

estimate of the price of European-style options.

According to Klaes (2000), the concept of transaction costs came about as a result of a seminal article by Ronald H. Coase, written in the 1930s. Transaction cost is the expense incurred in buying or selling a security. Transaction costs include commissions, fees, and any direct taxes. In the 1970s, after the limits of the Arrow-Debreu epitome had become apparent, several authors, including Oliver E. Williamson, Kenneth J. Arrow, Armen A. Alchian, and Harold Demsetz, took up the belief of transaction costs and turned it into a useful analytical tool.

In the formulation of the Black-Scholes model, Black and Scholes made the assumption that transactions cost do not exist in hedging a portfolio. In reality they do exist. Ever since, there have been several and successful remodelling of Black-Scholes model that take into account transaction costs. The resulting equation becomes a nonlinear Black-Scholes model with a variable volatility which is a function of the second derivative of the option price. In this case, it is not possible to compute the solution by the analytical formula developed by Black and Scholes and hence the need to resort to numerical methods.

Numerical methods such as the finite difference, finite volume method, and finite element were originally defined on meshes of data points. In such a mesh, each point has a fixed number of predefined neighbours, and this connectivity between neighbours can be used to define mathematical operators like the derivative. These operators are then used to construct the equations to be simulated.

In simulations where the material being simulated can move around especially as in computational fluid dynamics and option pricing problems the connectivity of the mesh can be difficult to maintain without introducing error into the simulation. If the mesh becomes tangled or degenerate during simulation, the operators defined on it may no longer give correct values. The mesh may be recreated during simulation, but this can also introduce error, since all the existing data points must be mapped onto a new and different set of data points.

Meshfree approximation is one of the numerical methods which do not require

a mesh connecting the data points of the simulation domain and are intended to remedy the problems caused by the mesh based methods. In option pricing, Belova and Shmidt (2011) used meshfree methods with radial basis function to approximate the option's value in the Black-Scholes model without transaction cost. This thesis is therefore an extension on the previous research work by Belova and Shmidt (2011) in which transaction cost is incorporated into the Black-Scholes equation.

1.2 Statement of the Problem

In a complete financial market, the value of an option can be obtained analytically by solving the linear Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$

where S is the price of the underlying asset, r is the risk-free interest rate, V is the option price, t is current date, σ is the volatility. The assumptions made in the derivation of the the Black-Scholes model prove to be too restrictive in practice. Therefore different models have been proposed to weaken one or more of these assumptions (Durning, 2005). Thus when transaction cost is incorporated in the Black-Scholes model, the resulting model is a nonlinear Black-Scholes equation with a variable volatility which is a function of the second derivative of the option price. Analytical solution to the valuation of the nonlinear Black-Scholes equation cannot be found and hence, the need to resort to numerical methods. Many numerical methods in solving the Black-Scholes equation such as the finite difference method, finite volume method and binomial method amongst others. In this study, Meshfree Approximation using Radial Basis Function will be employed in solving the Black-Scholes model when there is transaction cost.

1.3 Objectives

The objectives of the study are:

1. To implement meshfree methods using radial basis function for fast and accurate numerical valuation of the nonlinear Black-Scholes model.
2. To compare the option prices with transaction cost computed by the radial basis function with those obtained from the standard Black-Scholes model.
3. To compare the option prices with transaction cost obtained by the radial basis function with option prices without transaction cost in Belova and Shmidt (2011).

1.4 Methodology

Analytical solutions to nonlinear Black-Scholes equations are seldom available. We have to rely on numerical approaches such as binomial approximations, Monte-Carlo methods, finite difference methods, finite element methods, finite volume methods and meshfree methods to get accurate option prices. Traditional methods such as the finite difference scheme or the finite elements do not provide satisfactory results when the dimension of the option pricing problem increases drastically and for nonlinear problems as well. To resolve this issue, we use a special class of numerical methods, namely, the meshfree approximation. These methods are often better suited to cope with changes in the geometry of the domain of interest and for nonlinear problems as well. In this thesis, we apply these methods to solve the Black-Scholes equation with transaction cost. A Matlab programming language is used to implement the method in generating the tables and graphs.

1.5 Significance of the Study

This research will help know how nonlinear models such as the Black-Scholes model with transaction cost can be handled easily using meshfree approximation. The results of the research will go a long way to help financial institutions to know when to include transaction cost in their day to day business activities. This will further boost the confidence of stakeholders in the financial industry to do more business with less risk. Other beneficiaries of the research are stock brokers, directors, and regulators as well as researchers in the academia.

1.6 Organization Of The Thesis

The study is organized in five chapters as follows. The first one provides general background issues of the study. It also provides the statement of problem and sets out the objectives of the study, provides the methods to be use as well as the significance of the study. Chapter 2 reviews pertinent literature related to option pricing models and meshfree methods. Chapter 3 describes the formulation of the Black-Scholes model with transaction costs as well as meshfree methods. Chapter 4 presents data analysis and discussions. The final chapter, which is chapter 5, summarizes the findings of the study.

Chapter 2

Literature Review

2.1 Introduction

As explained in chapter 1, the main objective of this study is to price options with transaction costs using meshfree methods. This Chapter describes, compares, contrasts and evaluates the major theories, arguments, themes, methodologies, approaches and controversies in the scholarly literature on the subject of this study. A general overview of meshfree methods and option pricing models are presented and evaluated.

2.2 Option Pricing Models

The first attempt to explain option prices was by French PhD student Louis Bachelier in 1900 in his thesis, Brownian motion as a model for speculative prices. He proposed that the correct value of an option was the expected value of its payoffs, and by introducing a specific probabilistic model for the underlying price motion, he was able to calculate this expectation and compare his results with market prices. A botanist Robert Brown in 1827 first described the unusual motion exhibited by a small particle that is totally immersed in a liquid or gas. Albert Einstein in 1905 expounded more on this motion. Later, Norbert Wiener gave in a mathematically concise definition of the theory on Brownian Motion in a series of papers originating in 1918. The economist Samuelson in the 1960's proposed the exponential of Brownian motion (Ge-

ometric Brownian motion) for modeling prices which are subject to uncertainty. The geometric Brownian motion is the basic mathematical model for price movements and Black, Scholes and Merton used this principle and came out with the Black-Scholes model in 1973 based on certain assumptions. This has laid the foundation for the rapid growth of markets for derivatives and the starting point for the pricing of other kinds of financial derivatives. The modification of the Black-Scholes formula to cater for a large number of derivatives has taken place while new pricing techniques and models such as stochastic volatility, transaction cost and jump diffusion processes have been developed.

Options have been traded for centuries, but they remained relatively obscure financial instruments until the introduction of a listed options exchange in 1973. Since then, options trading has enjoyed an expansion unprecedented in American securities markets (Cox et al., 1979).

Option pricing theory has a long and illustrious history, but it also underwent a revolutionary change in 1973. At that time, Fischer Black and Myron Scholes presented the first completely satisfactory equilibrium option pricing model called the Black-Scholes model. In the same year, Robert Merton extended their model in several important ways. These path-breaking articles have formed the basis for many subsequent academic studies. In the derivation of their model, Black and Scholes assumed ideal 'conditions' in the market for the stock and option. These conditions are:

- 'Frictionless' markets: there are no transactions costs or differential taxes. Trading takes place continuously in time. Borrowing and short-selling are allowed without restriction and with full proceeds available. The borrowing and lending rates are equal.
- The short-term interest rate is known and constant through time.
- The stock pays no dividends or other distributions during the life of the option.
- The option is 'European' in that it can only be exercised at the expiration date.

- The stock price follows a geometric Brownian motion through time which produces a log-normal distribution for stock price between any two points in time.

Merton (1973) later derived an alternative of the Black-Scholes formula and demonstrated that their basic mode of analysis obtains even when the interest rate is stochastic; the stock pays dividends; and the option is exercisable prior to expiration. Moreover, it was shown that as long as the stock price dynamics can be described by a continuous-time diffusion process whose sample path is continuous with probability one, then their arbitrage technique is still valid.

2.2.1 Black Scholes Model with Transaction Costs

According to Barles and Soner (1998) in a complete financial market without transaction costs, the celebrated Black-Scholes no-arbitrage argument provides not only a rational option pricing formula but also a hedging portfolio that replicates the contingent claim. However, the Black-Scholes hedging portfolio requires continuous trading and therefore, in a market where there are transaction costs, it is prohibitively expensive and leads to nonlinear Black Scholes equation.

As stated by Ankudinova (2008) nonlinear BlackScholes equations have been increasingly, attracting interest over the last two decades, since they provide more accurate values by taking into account more realistic assumptions, such as transaction costs, illiquid markets, risks from an unprotected portfolio or large investor's preferences, which may have an impact on the stock price, the volatility, the drift and the option price itself. In the derivation of the Black-Scholes equation some basic assumptions were made: one of such assumption is that there are no transaction costs in hedging a portfolio. Ever since, there have been several and successful remodelling of Black-Scholes that take into account transaction costs (Ankudinova, 2008).

Leland (1985) came out with an approximate option replication model featuring transaction costs in the Black-Scholes model. He considered a model that allowed the transactions only at discrete times. According to Mawah (2007), Leland took a finite number of portfolio rebalancing periods which was not a perfectly hedged position

with sufficient cash to cover all terminal option values. He then derived an option price that was equal to the Black-Scholes price with adjusted volatility by using delta hedging argument. His adjusted volatility was given as

$$\hat{\sigma} = \sigma \left(1 + \sqrt{\frac{2}{\pi}} \frac{\mu}{\sigma \sqrt{\Delta t}} \right)^{\frac{1}{2}}$$

where σ is the original volatility, μ is the proportional transaction cost and Δt is the transaction frequency. He assumed μ and Δt to be small and kept the ratio $\mu/\sqrt{\Delta t}$ order one. For typical market numbers, this is indeed the case. For example: when $\sigma = 0.2$, $\mu = 0.01$ and one transaction a week, the Leland volatility, $\hat{\sigma}$, is equal to σ times 1.13.

Leland, however, made implicit use of the approximation

$$W(t + \Delta t) - W(t) \approx \sqrt{\frac{2}{\pi}} \sqrt{\Delta t},$$

where $W(\cdot)$ is the one-dimensional Brownian motion $\sqrt{\Delta t} \rightarrow 0$, $W(t + \Delta t) - W(t)$ converges to zero like $\sqrt{\Delta t}$. Therefore, it is reasonable to use the approximation $W(t + \Delta t) - W(t) \approx c^* \sqrt{\Delta t}$, where c^* is an arbitrary constant. Then the adjusted volatility of the option price is

$$\hat{\sigma} = \sigma \left(1 + c^* \frac{\mu}{\sigma \sqrt{\Delta t}} \right)^{\frac{1}{2}}.$$

The arbitrary constant c^* is chosen in ways relating to the risk associated in market models with transaction cost (Kusuoka, 1995). Leland derived his model by assuming the convexity of the resulting option price.

Avellaneda and Paras (1994) made an extension of this approach to general option pricing. Boyle and Vorst (1992) discussed option replication in a discrete time framework with transaction costs. Their model was an extension of the Cox-Ross-Rubinstein binomial option pricing model to cover the case of proportional transaction costs. They used the central limit theorem and showed that, as the time step Δt and the transaction cost μ approaches zero, the price of the discrete option converges to a

Black-Scholes price with an adjusted volatility $\hat{\sigma}(1)$ (Barles and Soner, 1998). Here, Boyle and Vorst represented Δt as the mean time length for a change in the value of the stock, and not the transaction frequency.

A different approach to option pricing with transaction costs is to introduce preferences. Hodges and Neuberger (1989) considered the difference between the maximum utility from final wealth when there is no option liability and when there is such a liability. They then, postulated that the price of the option should be equal to the unique cash increment which offsets this difference. It is therefore surprising that in the absence of market frictions, the option price obtained from utility maximization is equal to the Black-Scholes price (Mawah, 2007). Therefore, the Black-Scholes option pricing theory can be extended by use of utility maximization approach. Davis et al. (1993) further developed this theory in the presence of transaction costs.

Barles and Soner (1998) applied the utility function approach of Hodges and Neuberger with the help of an asymptotic analysis of partial differential equations. They then obtained a nonlinear Black-Scholes equation with an adjusted volatility. The volatility they obtained turned out to be a function of the second derivative of the price of the option. The risk factor in the nonlinear model was represented by a free parameter, a . Their derivation of the nonlinear Black-Scholes option pricing formula uses the utility maximization definition and asymptotic analysis. They also derived an upper estimate on the probability of missing the hedge by a given amount. Barles and Soner (1998) used the exponential utility function given as

$$U^\varepsilon(\xi) = 1 - \exp\left(-\frac{\xi}{\varepsilon}\right), \quad \xi \in \mathbb{R}^1$$

with a parameter $\varepsilon > 0$, where $1/\varepsilon$ is equal to the product of the risk-aversion factor and the number of options to be sold. They represented the proportional transaction cost as μ , p as the stock price at time t , and $\Psi^{\varepsilon, \mu}(p, t)$ as the option price with utility function U^ε and, studied the behavior of $\Psi^{\varepsilon, \mu}$ as $\varepsilon \rightarrow 0$, $\mu \rightarrow 0$, $\frac{\mu}{\sqrt{\varepsilon}} = a$, where a is any constant.

According to Barles and Soner (1998), Whalley and Wilmott in their recent

paper studied the limit of $\Psi^{\varepsilon, \mu}$, as $\mu \rightarrow 0$, while keeping ε fixed. They used formal, matched asymptotics and obtained detailed information about the dependence of $\Psi^{\varepsilon, \mu}$, and the optimal hedging strategy on the parameter μ .

2.3 Meshfree Methods

The valuation of financial options leads to mathematical models which are often challenging to solve. Since the Black Scholes equation was discovered in 1973, it has been used as a standard pricing formula for different kinds of options. The assumptions underlying this famous formula do not always hold and the original equation has been generalised to accommodate many new kinds of options. This means that an exact solution to the corresponding equation cannot always be found and we must then resort to approximate or numerical methods. There are three numerical techniques that are commonly employed in finance to price options. The methods are the numerical solution of partial differential equations (PDEs), Monte Carlo Simulation and Binomial methods. The binomial and numerical solution of partial differential equation methods require less computation time for problems of low dimensionality as compared to Monte Carlo simulation (Forsyth et al., 1998).

As stated by Sidahmed (2011), meshfree methods originated about thirty years ago. He stated that, the starting point which seems to have the longest continuous history is the smooth particle hydrodynamics (SPH) method. Lucy (1977) used smooth particle hydrodynamics (SPH) to model the astrophysical phenomena without boundaries such as exploding stars and dust clouds. According to Sidahmed (2011), comparing SPH method to other methods in these times of prolific academicians, the rate of publications was very modest for many years and is mainly reflected in the work of Monaghan. Monaghan (1988) described a turbulence model for the particle method, Smoothed Particle Hydrodynamics (SPH). His model made few assumptions, conserved linear and angular momentum, satisfied a discrete version of Kelvin's circulation theorem, and was computationally efficient. The model was based on a Lagrangian similar to that used for the Lagrangian averaged Navier-Stokes (LANS)

turbulence model, but with a different smoothed velocity. The results from the model were in good agreement with the experimental and computational results of Clercx and Heijst for two-dimensional turbulence inside a box with no-slip walls. The smoothed velocity preserved the shape of the spectrum of the unsmoothed velocity, but reduces the magnitude for short length scales by an amount which depended on a parameter ϵ . He called this the SPH- ϵ model.

The most widely cited pioneering work that used the Radial Basis Function approach was that of Kansa. Kansa (1990) first used it for solving some problems in computational fluid dynamics. He presented multiquadrics (MQ) scheme developed for spatial approximations for these problems. The MQ is a grid free scheme suited for scattered data interpolation. Kansa later used Multiquadric RBFs for parabolic, hyperbolic and elliptic PDEs. He showed that multiquadrics is more efficient than finite difference schemes which require many more operations to achieve the same degree of accuracy.

Swegle et al. (1995) showed the origin of the so-called tensile instability through a dispersion analysis of the linearized equations and proposed a viscosity term to stabilize it. They achieved this by carrying out a von Neumann stability analysis of the SPH algorithm which identifies the criterion for stability or instability in terms of the stress state and the second derivative of the kernel function. Their analysis explained the observation that the method is unstable in tension while apparently stable in compression but showed that it was possible to construct kernel functions which were stable in tension and unstable in compression. The analysis and the stability criterion provided them the insight into possible methods for removing the instability.

Belytschko et al. (1996) examined meshless approximations based on moving least-squares, kernels, and partitions of unity. They showed that the three methods were in most cases identical except for the important fact that partitions of unity enable p -adaptivity to be achieved. They then described methods for constructing discontinuous approximations and approximations with discontinuous derivatives. Next, they reviewed: discretization (collocation and Galerkin), quadrature in Galerkin and fast

ways of constructing consistent moving least-square approximations.

Johnson and Beissel (1996) proposed a normalized smoothing function algorithm that can improve the accuracy of smooth particle hydrodynamics (SPH) impact computations. Their approach consists of adjusting the standard smoothing functions for every node (and every cycle) such that the normal strain rates are computed exactly for conditions of constant strain rates (linear velocity distributions). This, in turn, generally improves accuracy for non-uniform strain rates and therefore significantly improves the accuracy for free boundaries, for non-uniform arrangements of SPH nodes, and for small smoothing distances.

Sharan et al. (1997) used the multiquadric (MQ) approximation scheme for the solution of elliptic partial differential equations with Dirichlet and Neumann boundary conditions. They took two-dimensional Laplace, Poisson, and biharmonic equations describing the various physical processes as the test examples. They concluded that the agreement found between the computed and exact solutions was very good.

Fedoseyev et al. (2002) formulated an improved Kansa-MQ method with PDE collocation on the boundary (MQ-PDECB). They added an additional set of nodes adjacent to the boundary and, correspondingly, added an additional set of collocation equations obtained via collocation of the PDE on the boundary. They applied the MQ-PDECB method to several model 1D and 2D linear and nonlinear elliptic PDEs and have presented results of their numerical experiments.

Giinther and Liu (1998) described a computational algorithm based on d'Alembert's principle that can be used for general constraints both in meshless methods and finite elements. They developed a method of partitioning meshless shape functions suitable for imposing linear boundary conditions and then extended the approach for nonlinear constraints.

Li et al. (2003) developed a meshless method for modelling groundwater contaminant transport using collocation method with radial basis functions. Their numerical results are presented for several cases: pure diffusion; advection and dispersion for continuous source; advection and dispersion for instantaneous source; advection

and dispersion for patch-source. They showed from their results that their method was accurate.

Wen et al. (2006) reproduced a mesh free method based on kernel approximation and point collocation for analysis of metal ring compression. They introduced corrected kernel functions to the stabilization of free-surface boundary conditions. They compared the solution of symmetric ring compression problem with a conventional finite element solution.

Kindelan et al. (2010) introduced a radial basis function collocation method for computing solutions to the time-dependent radiative transfer equation. They used finite differences to discretize the time coordinate, a discrete ordinate method to discretize the directional variable, and an expansion in radial basis functions to approximate the spatial dependence of the solution. They concluded that the radial basis function method does not require any mesh or grid, achieves spectral accuracy in multi-dimensions for arbitrary node layouts, and extremely simple to implement.

Islam et al. (2009) discussed meshfree interpolation method for the numerical solution of the coupled nonlinear partial differential equations. This paper formulated a simple classical radial basis functions (RBFs) collocation (Kansa's) method for the numerical solution of the coupled Korteweg-de Vries (KdV) equations, coupled Burgers equations, and quasi-nonlinear hyperbolic equations. They assessed the accuracy of their method in terms of the error in L_1 and L_2 norms, number of nodes in the domain of influence, time step length; and parameter free and parameter dependent RBFs. They performed numerical experiments to demonstrate the accuracy and robustness of the method for the three classes of partial differential equations (PDEs).

Tatari and Dehghan (2010) proposed a technique for solving partial differential equations using radial basis functions. The radial basis functions are very suitable instruments for solving partial differential equations of various types. However, the matrices which result from the discretization of the equations are usually ill-conditioned especially in higher dimensional problems. They proposed a method for solving the partial differential equations and generalized to solve higher-dimensional problems.

Wang and Liu (2002) proposed a point interpolation meshless method based on combining radial and polynomial basis functions. The interpolation function obtained passes through all scattered points in an influence domain and thus shape functions are of delta function property. They then suggested that the implementation of essential boundary conditions was much easier than the meshless methods based on the moving least-squares approximation. Wang and Liu further stated that, the partial derivatives of shape functions are easily obtained.

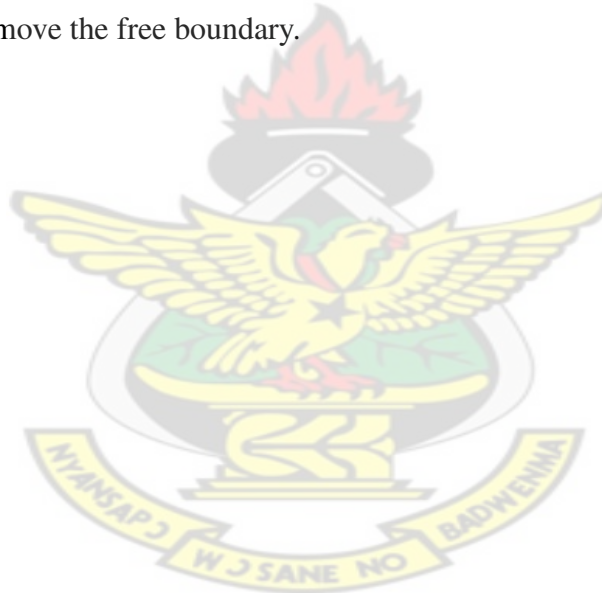
In another paper by Wang et al. (2002), they studied the numerical analysis of Biots consolidation process by radial point interpolation method. They proposed an algorithm to solve Biots consolidation problem using meshless method called a radial point interpolation method (radial PIM). They stated that, the radial PIM is advantageous over the meshless methods based on moving least-square (MLS) method in implementation of essential boundary condition and over the original PIM with polynomial basis in avoiding singularity when shape functions are constructed. In time domain they proposed fully implicit integration scheme to avoid spurious ripple effect. They studied some examples with structured and unstructured nodes and compared with closed-form solution or finite element method solutions.

Dai et al. (2004) proposed a mesh free model for the static and dynamic analysis of functionally graded material (FGM) plates based on the radial point interpolation method (RPIM). They studied the convergence rate and accuracy and compared with the finite element method (FEM).

More recently, meshfree methods have been applied to options. Goto et al. (2007) studied options valuation by using radial basis function approximation. They described the valuation scheme of European, Barrier, and Asian options of single asset by using radial basis function approximation. The option prices were governed with BlackScholes equation. They discretized the equation with Crank-Nicolson scheme and then, approximated the option with the radial basis functions with unknown parameters. It was concluded in their paper that, in the European and the Barrier options, the prices were governed with Black-Scholes equation. The governing option of the

Asian option, however, was different from the rest them. In that case, one has to adopt the other radial basis functions than that for the original Black-Scholes equation.

Belova et al. (2008) presented meshfree approximation scheme based on the radial basis function methods for the numerical solution of the options pricing model. Their paper dealt with the valuation of the European, Barrier, Asian, American options of a single asset and American options of multi assets. The option prices were modelled by the Black-Scholes equation. They used the theta (θ) method to discretize the equation with respect to time. The option price was then approximated in space with radial basis functions (RBF) with unknown parameters using the multiquadric radial basis functions (MQ-RBF). In case of American options a penalty method was used. This was achieved by adding a small and continuous penalty term to the Black-Scholes equation to remove the free boundary.



Chapter 3

Methodology

3.1 Black-Scholes Model

According to Shcherbakov and Szwackiewicz (2010) the Black-Scholes model assumes that the financial market consists of shares and risk-free financial instruments (bonds, bank account). Buying and selling these instruments is a continuous process, meaning that their prices processes can be modelled by stochastic differential equations (SDE) in the case of a risk instrument or by ordinary differential equations (ODE) in the case of a risk-free instrument. Stochastic differential equations are utilized in pricing derivative assets because they give a formal model of how an underlying assets price changes over time (Ntwiga, 2005). Ntwiga further stated that, if S_t is the price of a security, a trader will be interested in knowing dS_t ; the next instant's incremental change in the security price. He stated that the dynamic behaviour of the asset price in a time interval dt can then be represented by the stochastic differential equation given by

$$dS_t = \alpha(S_t, t)dt + \sigma(S_t, t)dW_t \text{ for } t \in [0, \infty)$$

where dW_t is an innovation term representing an unpredicted event that occur during the infinitesimal interval dt , $\alpha(S_t, t)$ is the drift parameter and $\sigma(S_t, t)$ the diffusion parameter which depends on the level of observed asset price S_t and on time t .

3.1.1 Stochastic Process

Following Hull (2009), any variable whose value changes over time in an uncertain way is said to follow a stochastic process. A stochastic process can be grouped into two namely: discrete time and continuous time stochastic processes. A discrete time stochastic process is one where the variable can change only at certain fixed points in time whereas continuous time stochastic process is one where the changes occur at any time.

Wiener Process

A Wiener process is a particular type of Markov process with a mean change of zero and a variance rate of 1.0 per year (Hull, 2009). A Markov process is a type of stochastic process where the past history of the variable and the way the present has emerged from the past are irrelevant but the present value of the variable is relevant for predicting the future. Hull stated that a Wiener process had been used in Physics to describe the motion of a particle that is subject to a large number of small molecular shocks and is sometimes referred to as Brownian motion (Hull, 2009).

Mathematically, a generalized Wiener process for a variable x can be defined in terms of dz as

$$dx = a dt + b dz$$

where a and b are constants known as drift rate and variance rate respectively. Thus a generalized Wiener process has an expected drift rate of a and a variance rate of b^2 .

Itô Process

A further type of stochastic process known as an Itô process can be defined (Hull, 2009). An Itô process is a generalized Wiener process in which the drift rate and the variance rate are functions of the value of the underlying variable and time t given as

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t,$$

where $a(X_t, t)$ is the drift rate, $b(X_t, t)$ is the variance rate and dW_t is the Standard Wiener process.

Itô Lemma

As stated by Hull (2009), the price of the stock option is a function of the underlying stock's price and time. That is, the price of any derivative is a function of the stochastic variable underlying the derivative and time. Ito (1942) discovered a very important result in the area of the behavior of functions of stochastic variables. This result is known as Itô lemma.

Suppose that the value of a variable S_t follows the Itô process

$$dS_t = a(S_t, t)dt + \sigma(S_t, t)dW_t$$

where a and σ are functions of S and t . Itô lemma shows that a function G of x and t follows the process

$$dG_t = \left(\frac{\partial G}{\partial S_t} a_t + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S_t^2} \sigma_t^2 \right) dt + \frac{\partial G}{\partial S_t} \sigma_t dW_t$$

where dW_t is the Wiener process. Thus G_t follows an Itô process with the drift rate

$$\frac{\partial G}{\partial S_t} a_t + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S_t^2} \sigma_t^2$$

and the variance rate

$$\left(\frac{\partial G}{\partial S_t} \right)^2 \sigma_t^2.$$

If a variable $S(t)$ follows a Geometric Brownian motion, then it obeys a stochastic differential equation of the form

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

and Itô's lemma is given for any function $G(S, t)$ as

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dW,$$

where μ and σ are constants.

Black Scholes Equation

In the early 1970s, Fischer Black, Myron Scholes and Robert Merton achieved a major breakthrough in the pricing of stock options (Hull, 2009). This involved the development of what has become known as the Black-Scholes model. The model has had a huge influence on the way that traders price and hedge options.

Let a stock price S , follow

$$dS = \mu S dt + \sigma S dW \quad (3.1)$$

where μ is the drift rate, σ is the volatility and W follows a Wiener process. Now, suppose that V is the price of a call option or other derivative contingent on S . The variable V must be some function of S and t . Hence, by Itô lemma

$$dV = \left(\frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S} \sigma S dW. \quad (3.2)$$

The discrete forms of equations 3.1 and 3.2 are

$$\Delta S = \mu S \Delta t + \sigma S \Delta W \quad (3.3)$$

and

$$\Delta V = \left(\frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial V}{\partial S} \sigma S \Delta W \quad (3.4)$$

The Wiener process underlying V and S are the same and can be eliminated by choosing an appropriate portfolio of the stock and the derivative. We, thus, choose a

portfolio of

$$-1 : \text{derivative} \\ + \frac{\partial V}{\partial S} : \text{Shares}$$

The holder is short one derivative and long an amount $\frac{\partial V}{\partial S}$ of shares. We define Π as the value of the portfolio and we have

$$\Pi = -V + \frac{\partial V}{\partial S} S. \quad (3.5)$$

The change $\Delta\Pi$ in the value of the portfolio in time Δt is given by

$$\Delta\Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S \quad (3.6)$$

We substitute equations 3.3 and 3.4 into equation 3.6 to give

$$\Delta\Pi = -\left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\Delta t - \frac{\partial V}{\partial S}\sigma S\Delta W + \frac{\partial V}{\partial S}(\mu S\Delta t + \sigma S\Delta W) \quad (3.7)$$

Rearranging equation 3.7, we obtain

$$\Delta\Pi = -\frac{\partial V}{\partial S}\mu S\Delta t + \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\Delta t - \frac{\partial V}{\partial S}\sigma S\Delta W + \frac{\partial V}{\partial S}\mu S\Delta t + \frac{\partial V}{\partial S}\sigma S\Delta W \quad (3.8)$$

We further simplify to obtain the change of the portfolio as

$$\Delta\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\Delta t. \quad (3.9)$$

Since the portfolio is now risk-less due to the elimination of the ΔW term, it must then earn a return similar to other short term risk-free securities. Therefore

$$\Delta\Pi = r\Pi\Delta t, \quad (3.10)$$

where r is the risk-free interest rate. Substitutions of equations 3.5 and 3.9 into equation 3.10 yields

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \right) \Delta t = r \left(V - \frac{\partial V}{\partial S} S \right) \Delta t \quad (3.11)$$

Further simplification leads to the the Black-Scholes differential equation given by

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV. \quad (3.12)$$

3.2 Nonlinear Black-Scholes type equations

The value of an option can be determined by the classical Black-Scholes theory in an idealized financial market. The assumptions that are made in the derivation of this model prove to be too restrictive in practice. Therefore, in recent years different models have been proposed to weaken one or more of these assumptions (During, 2005). During further stated that, these models would result in strongly or fully nonlinear, possibly degenerate, parabolic diffusion-convection equations for the option value V as a function of the underlying security S and the time t

$$V_t + \frac{1}{2} \hat{\sigma}(S, t, V_s, V_{ss})^2 S^2 V_{ss} + rS V_s - rV = f(S, t, V_s), \quad S > 0, \quad t \in (0, T)$$

where r denotes the riskless interest rate, $\hat{\sigma}$ is the nonlinear volatility, and the nonlinear function f models different effects.

The source of such nonlinearities is so called market incompleteness. A market is said to be complete if and only if there exists a unique equivalent martingale measure (During, 2005). On the other hand, a market is called incomplete if not every contingent claim is reachable. The presence of transaction costs, trading restrictions and additional randomness in the model are some of the typical reasons for market incompleteness.

3.2.1 Volatility Models

In the standard Black-Scholes model, the essential parameter that is not directly observable and is assumed to be constant, is the volatility σ . Many approaches have been proposed to improve the model by treating the volatility in different ways using a modified volatility function $\hat{\sigma}(\cdot)$ to model the effects of transaction costs, illiquid markets and large traders (Ankudinova, 2008).

3.2.2 Transaction Costs Models

One of the assumptions in the Black-Scholes model is that, the model requires a continuous portfolio adjustment in order to hedge the position without any risk. This adjustment easily becomes expensive in the presence of transaction costs since an infinite number of transactions is needed (Kwok, 2008). Ankudinova (2008) stated that the hedger needs to find the balance between the transaction costs that are required to rebalance the portfolio and the implied costs of hedging errors. Because of this “imperfect” hedging the option might be over or underpriced up to the extent where the riskless profit obtained by the arbitrageur is offset by the transaction costs, so that there is no single equilibrium price but a range of feasible prices. In a market with transaction costs, it has been shown that there is no replicating portfolio for the European Call option and the portfolio is required to dominate rather than replicate the value of the option.

3.2.3 The Transaction Cost Model of Leland

The main idea of Leland (1985) was to relax the hedging conditions so as to trade at discrete times, which reduce the expenses of the portfolio adjustment. He made the assumption that the transaction cost $\mu|\alpha|S/2$ where μ denotes the transaction cost and α the number of assets bought ($\alpha > 0$) or sold ($\alpha < 0$) at price S is proportional to the monetary value of the assets bought or sold.

Consider a replicating portfolio with Δ units of the underlying assets and the bond B ,

then we have

$$\Pi = \alpha S + B.$$

The change in the portfolio after a small change in time Δt becomes

$$\Delta \Pi = \alpha \Delta S + rB \Delta t - \frac{\mu}{2} |\Delta \alpha| S, \quad (3.13)$$

where ΔS is the change in price S , $\alpha \Delta S$ represents the change in value, $rB \Delta t$ represents the change the bond growth in Δt time and $\Delta \alpha$ represents the change in the number of assets such that the last term becomes the transaction cost due to portfolio change. Applying Itô lemma to the value of the option $V(S, t)$, this gives us

$$\Delta V = V_s \Delta S + \left(V_t + \frac{\sigma^2}{2} S^2 V_{ss} \right) \Delta t. \quad (3.14)$$

If the option V is replicated by the portfolio Π , then their values have to match at all times and there can be no risk-free profit. With this no-arbitrage argument we get

$$\Delta \Pi = \Delta V \quad (3.15)$$

Comparing equations (3.13) and (3.14), we get $\alpha = V_s$ and

$$rB \Delta t - \frac{\mu}{2} |\Delta \alpha| S = \left(V_t + \frac{\sigma^2}{2} S^2 V_{ss} \right) \Delta t \quad (3.16)$$

Leland then showed that,

$$\frac{\mu}{2} |\Delta \alpha| S = \frac{\sigma^2}{2} Le S^2 |V_{ss}| \Delta t, \quad (3.17)$$

where Le is the *Leland number*, given by

$$Le = \sqrt{\frac{2}{\pi}} \left(\frac{\mu}{\sigma \sqrt{\Delta t}} \right) \quad (3.18)$$

where Δt is the transaction frequency and μ is the transaction cost. Substituting (3.17) and $B = \Pi - \alpha S = V - SV_s$ into equation (3.16) becomes

$$rV - rSV_s - \frac{\sigma^2}{2} Le S^2 |V_{ss}| = V_t + \frac{\sigma^2}{2} S^2 V_{ss}. \quad (3.19)$$

Therefore the option price is the solution of the nonlinear Black-Scholes equation

$$V_t + \frac{1}{2} \hat{\sigma}^2 S^2 V_{ss} + rSV_s - rV = 0 \quad (3.20)$$

where $\hat{\sigma}$ is the modified volatility given as

$$\hat{\sigma}^2 = \sigma^2 (1 + Le \text{sign}(V_{ss})), \quad (3.21)$$

with σ representing the historical volatility and Le the Leland number.

3.2.4 The transaction cost model of Boyle and Vorst

Boyle and Vorst (1992) used the central limit theorem derived from the binomial model with transaction costs and discrete trading processes that as the time step Δt and the transaction cost μ approach zero, the price of the option converges to a Black-Scholes price with the modified volatility $\hat{\sigma}$ of the form

$$\hat{\sigma}^2 = \sigma^2 \left(1 + Le \sqrt{\frac{\pi}{2}} \text{sign}(V_{ss}) \right). \quad (3.22)$$

Boyle and Vorst assumed convexity of V such that $\hat{\sigma}^2 = \sigma^2 (1 + Le \sqrt{\pi/2})$ and equation (3.20) turns into the linear Black-Scholes equation.

3.2.5 Transaction Cost Model of Hodges and Neuberger

A different approach to model transaction costs was suggested by (Hodges and Neuberger, 1989). They consider a utility function and made the assumption that the behavior of the investor is characterized by this function. The utility function is the measure

of the relative satisfaction of the investor from the input. Hodges and Neuberger show that the BlackScholes price is the difference between the maximum utility from the final wealth with and without option liability. They conclude that the price of the option in a market with transaction costs should be equal to the unique cash increment which offsets this difference.

3.2.6 Transaction Cost Model of Barles and Soner

They apply the utility function approach of Hodges and Neuberger with the help of an asymptotic analysis of partial differential equations.

Let us consider a financial market which consists of one bond and one stock, the price is governed by the dynamics:

$$dS(l) = S(l)[\alpha dl + \sigma dW(l)], \quad l \in [t, T] \quad (3.23)$$

with initial data $S(t) = s$, $W(\cdot)$ a standard one-dimensional Brownian motion, α is the constant mean return rate and σ is the constant volatility.

Consider the process of bonds owned $X(s)$ and the process of shares owned $Y(s)$. We let the trading strategy $(L(s), M(s))$ be a pair of non-decreasing processes with $L(t) = M(t) = 0$, which are interpreted as the cumulative transfers measured in shares of stock. $M(s)$ is measured in shares from stock to bond and $L(s)$ is measured in shares from bond to stock. Let $\mu \in (0, 1)$ be the proportional transaction cost and initial values x and y , $s \in [t, T]$, The processes $X(s)$ and $Y(s)$ evolve according to

$$X(s) = x - \int_t^s S(\tau)(1 + \mu)dL(\tau) + \int_t^s S(\tau)(1 - \mu)dM(\tau), \quad s \in [t, T] \quad (3.24)$$

and

$$Y(s) = y + L(s) - M(s), \quad s \in [t, T] \quad (3.25)$$

where the first integral represents buying shares of stock at a price increased by the proportional transaction cost μ and the second integral represents selling stock at a

reduced price of the transaction cost. We add the amount of the stocks bought and subtract the amount for the stocks sold to the initial amount of stocks owned in equation (3.25). According to the utility maximization approach of Hodges and Neuberger, the price of a European Call option can be found by the difference between the maximum utility of the terminal wealth when there is no option liability and when there is such a liability. Let the exponential utility function be

$$U(\xi) = 1 - e^{-\gamma\xi}, \quad \xi \in \mathbb{R}, \quad (3.26)$$

where $\gamma > 0$ is the risk aversion factor. Boyle and Vorst (1992) considered two optimization problems. The first value function is the expected utility from the final wealth when there are no option liabilities taken over the transfer processes

$$V_a(x, y, S(t), t) := \sup_{L(\cdot), M(\cdot)} E[U(X(T) + Y(T)S(T))], \quad (3.27)$$

and the second value function is the expected utility from the final wealth assuming that we have sold N European call options taken over the transfer processes

$$V_b(x, y, S(t), t) := \sup_{L(\cdot), M(\cdot)} E[U(X(T) + Y(T)S(T) - N(S(T) - K)^+)]. \quad (3.28)$$

According to Hodges and Neuberger (1989) the price of each option is equal to the maximal solution Λ of the algebraic equation

$$V_a(x + N\Lambda, y, S(t), t) = V_b(x, y, S(t), t).$$

This means that the option price Λ equals the increment of the initial capital at time t that is needed to cope with the option liabilities arising at T (Ankudinova, 2008). By a linearity argument selling N options with risk aversion factor of γ yields the same price as selling one option with risk aversion factor γN .

This leads to performing an asymptotic analysis as $\gamma N \rightarrow \inf$. We then consider

$$U(\xi) = 1 - e^{-\gamma N \xi}.$$

We set

$$\gamma N = 1/\varepsilon.$$

Then, we have

$$U_\varepsilon(\xi) = 1 - e^{-\xi/\varepsilon}, \quad \xi \in \mathbb{R}.$$

Our optimization problem then becomes

$$V_a(x, y, S(t), t) = 1 - \inf_{L(\cdot), M(\cdot)} E[e^{-1/\varepsilon(X(T)+Y(T)S(T))}]$$

and

$$V_b(x, y, S(t), t) = 1 - \inf_{L(\cdot), M(\cdot)} E[e^{-1/\varepsilon(X(T)+Y(T)S(T)-(S(T)-K)^+)}].$$

Now, define z_a and $z_b : \mathbb{R} \times (0, \infty) \times (0, T) \rightarrow \mathbb{R}$ by

$$V_a(x, y, S(t), t) = 1 - e^{-1/\varepsilon(x+yS(t)-z_a(y, S(t), t))}$$

$$V_b(x, y, S(t), t) = 1 - e^{-1/\varepsilon(x+yS(t)-z_b(y, S(t), t))}.$$

Then,

$$z_a(y, S(t), T) = 0 \text{ and } z_b(y, S(t), T) = (S(T) - K)^+$$

and the option price is given by

$$\Lambda(x, y, S(t), t; 1/\varepsilon, 1) = z_b(y, S(t), t) - z_a(y, S(t), t).$$

Barles and Soner (1998) state that the value functions V_a and V_b are the unique solutions of the dynamic programming equation

$$\min\{-V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} - rSV_s, -V_y + S(1+\mu)V_x, V_y - S(1-\mu)V_x\} = 0 \quad (3.29)$$

by the theory of stochastic optimal control. This leads to a dynamic programming equation for z_a and z_b , which are independent of the variable x .

Supposing that the proportional transaction cost $\mu = a\sqrt{\varepsilon}$ for some constant $a > 0$, Barles and Soner prove that as $\varepsilon \rightarrow 0$ and $\mu \rightarrow 0$,

$$z_a \rightarrow 0 \text{ and } z_b \rightarrow V,$$

where V is the unique (viscosity) solution of the nonlinear BlackScholes equation

$$V_t + \frac{1}{2}\hat{\sigma}^2 S^2 V_{ss}^2 + rSV_s - rV = 0 \quad (3.30)$$

where

$$\hat{\sigma}^2 = \sigma^2(1 + \Psi(e^{r(T-t)}a^2S^2V_{ss})) \quad (3.31)$$

is the nonlinear volatility. σ denotes the constant volatility of the underlying, $a = \mu/\sqrt{\varepsilon}$ and $\Psi(x)$ is the solution to the following nonlinear ordinary differential equation

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x) - x}}, \quad x \neq 0, \quad (3.32)$$

with the initial condition

$$\Psi(0) = 0. \quad (3.33)$$

Barles and Soner analysed the ordinary differential equation in equations (3.32) and (3.33) and stated that

$$\lim_{x \rightarrow \infty} \frac{\Psi(x)}{x} = 1 \text{ and } \lim_{x \rightarrow -\infty} \frac{\Psi(x)}{x} = -1 \quad (3.34)$$

Because of equation (3.34), the function $\Psi(\cdot)$ is treated as identity for large arguments. The volatility then becomes

$$\hat{\sigma}^2 = \sigma^2(1 + e^{r(T-t)}a^2S^2V_{ss}) \quad (3.35)$$

3.3 Meshfree Approximation

In this section we give a short description of a modern method that is a competitor of the finite difference method (FDM) and the finite element method (FEM) (Duffy, 2006). Thus, we discuss the meshless method that attempts to resolve some of the shortcomings of Finite Difference Method and Finite Element Method. A mesh is defined as any of the open spaces or interstices between the strands of a net that is formed by connecting nodes in a predefined manner. In the traditional difference schemes such as the finite element, finite volume and finite difference methods, the spatial domain is discretised into meshes. The meshfree method is used to establish a system of algebraic equations for the whole problem domain without the use of a predefined mesh. Meshfree methods essentially use a set of nodes scattered within the problem domain as well as on the boundaries to represent the problem domain and its boundaries.

Meshless methods are being used nowadays in many different areas of sciences and engineering, for example, scattered data modeling, problems involving moving discontinuities such as cracks and shocks, multi-scale resolution, non-uniform sampling, computer graphics, neural networks, etc (Sidahmed, 2011). The most important features of mesh free methods which make them very powerful are as follows:

- The shape functions of meshfree method can be constructed to have any desired order of continuity.
- Meshfree method does not require mesh alignment sensitivity.
- In meshless methods, the connectivity of the nodes is determined at run-time, hence no a-priori mesh is required.
- Meshfree method is easy to program and understand. (Sidahmed, 2011)

The key idea of the meshfree methods is to provide a stable and accurate numerical solutions for partial differential equations or integral equations with all kinds of possible

boundary conditions with a set of arbitrarily distributed nodes (or particles) without using any mesh that provides the connectivity of these nodes or particles.

3.4 Constructing the meshfree shape functions

There are different approaches of constructing the meshfree shape function. Smooth Particle Hydrodynamics Approach (SPH) and Moving Least-Squares Approach were suggested by Belytschko et al. (1996) while Liu (2002) proposed Point Interpolation Method.

3.4.1 Smooth Particle Hydrodynamics Approach

Smooth Particle Hydrodynamics Approach (SPH) is the oldest of the meshfree methods according to (Liu and Liu, 2003). Sidahmed stated that the rationale for this method was provided by invoking the notion of a kernel approximation for solution $v(x)$ on a domain Ω generated by

$$v_h(x) = \int_{\Omega} Z(x - \xi, h) v(\xi) d\xi. \quad (3.36)$$

$Z(x - \xi, h)$ is a weight function or kernel, $v_h(x)$ is the approximation, and h is a measure of the size of the support.

The weight function Z is a monotonically decreasing function and satisfies the properties below.

$$Z(x - \xi, h) > 0 \quad \text{over } \Omega \quad (3.37)$$

$$Z(x - \xi, h) = 0 \quad \text{outside } \Omega \quad (3.38)$$

$$\int_{\Omega} Z(x - \xi, h) d\xi = 1 \quad (3.39)$$

$$Z(s, h) \rightarrow \delta(s) \quad \text{as } h \rightarrow 0 \quad (3.40)$$

Exponential, cubic spline and quadratic spline functions are the three commonly used

weight functions. These are:

The exponential weight function

$$Z(\bar{c}) = \begin{cases} e^{-(\frac{\bar{c}}{\alpha})^2} & \text{for } \bar{c} \leq 1 \\ 0 & \text{for } \bar{c} > 1, \end{cases}$$

The cubic spline weight function

$$Z(\bar{c}) = \begin{cases} \frac{2}{3} - 4\bar{c}^2 + 4\bar{c}^3 & \text{for } \bar{c} < \frac{1}{2} \\ \frac{4}{3} - 4\bar{c} + 4\bar{c}^2 - \frac{4}{3}\bar{c}^3 & \text{for } \frac{1}{2} < \bar{c} \leq 1 \\ 0 & \text{for } \bar{c} > 1, \end{cases}$$

The quadratic spline weight function

$$Z(\bar{c}) = \begin{cases} 1 - 6\bar{c}^2 + 8\bar{c}^3 - 3\bar{c}^4 & \text{for } \frac{1}{2} < \bar{c} \leq 1 \\ 0 & \text{for } \bar{c} > 1, \end{cases}$$

The following weight function is normally used (for 1-D problems) in Smooth Particle Hydrodynamics methods.

$$Z(\bar{c}) = \frac{2}{3h} \begin{cases} 1 - \frac{2}{3}\bar{c}^2 + \frac{3}{4}\bar{c}^3 & \text{for } \bar{c} \leq 1 \\ \frac{1}{4}(2 - \bar{c})^3 & \text{for } 1 < \bar{c} \leq 2 \\ 0 & \text{for } \bar{c} \geq 2, \end{cases}$$

where α is constant, $\bar{c} = c/h$ and h is the smoothing length.

In Smooth Particle Hydrodynamics, the main idea is to obtain a simple formula for $v_h(x)$ in terms of nodal values $v_I \equiv v(x_I)$, $I = 1 : n_N$. Quadrature approaches are usually used. For example, the quadrature can be performed by the trapezoidal rule in one dimension, which gives

$$v_h(x) = \sum_I Z(x - x_I) v_I \Delta x_I \quad (3.41)$$

for a sequentially numbered set of nodes x_I . For interior nodes Δx , is

$$\Delta x = (x_{I+1} - x_{I-1})/2.$$

On the left end,

$$\Delta x_{nN-1} = (x_{I+1} - x_b)/2$$

where x_b is coordinate of the left boundary, with a similar expression on the right. The sum needs to be taken only over the point x_I where $z(x - x_I) > 0$ (Sidahmed, 2011).

The quadrature is more difficult to come to the grips with in multi dimensions. We use formulas of the type

$$v_h(x) = \sum_I Z(x - x_I) v_I \Delta U_I \quad (3.42)$$

are used, where ΔU_I represents the volume of node I . We can write equation (3.42) in the following form

$$v_h = \sum_I \phi(x) v_I, \quad (3.43)$$

where $\phi_I(x)$ are the Smooth Particle Hydrodynamics shape functions given by

$$\phi_I(x) = Z(x - x_I) \Delta U_I.$$

3.4.2 Moving Least-Squares Approach

In this method, we let $v(x)$ be the function of a field variable defined in the domain Ω . $v_h(x)$ is the approximation of $v(x)$ at point x . The Moving Least-Squares (MLS) approximates the field function in the form of series representation

$$v_h(x) = \sum_{j=1}^m q_j(x) a_j(x) \equiv Q^T(\mathbf{x}) a(\mathbf{x}) \quad (3.44)$$

where m is the number of terms of monomials (polynomial basis), $q_i(x)$ are monomial basis functions, and $a(\mathbf{x})$ is a vector of coefficients given by

$$a(\mathbf{x}) = [a_0(x), a_1(x), \dots, a_m(x)]^T, \quad (3.45)$$

which are functions of x .

A complete polynomial basis of order m in 1 dimensional space is given by

$$Q(\mathbf{x}) = [Q_0(x), Q_1(x), \dots, Q_m(x)]^T = [1, x, x^2, \dots, x^m] \quad (3.46)$$

whereas in 2D space, it is given by,

$$Q(\mathbf{x}) = Q(x, y) = [1, x, y, xy, x^2, y^2, \dots, x^m, y^m]^T, \quad (3.47)$$

If we assume that the support domain of x contains a set of n local nodes $x_1, x_2, \dots, x_n$, then equation (3.44) is used to calculate the approximated values of the field function at the nodes

$$v_h(x, x_I) = Q^T(x_I) a(x), \quad I = 1, 2, \dots, n. \quad (3.48)$$

We then construct a functional of weighted residual using the approximated values of the field function and the nodal parameters $v_I = v(x_I)$

$$J = \sum_I^n Z(x - x_I) [v^h(x, x_I) - v(x_I)]^2 \quad (3.49)$$

$$= \sum_I^n Z(x - x_I) [Q^T(x_I) a(x) - v(x_I)]^2 \quad (3.50)$$

where weight function is $Z(x - x_I)$, and v_I is the nodal parameter of the field variable at node I with compact support the same weight functions as in SPH are used.

We rewrite equation (3.50) in the form

$$J = (Qa - v)^T Z(x) (Qa - v) \quad (3.51)$$

where

$$Q = \begin{bmatrix} q_1(x_1) & q_2(x_1) & \cdots & q_m(x_1) \\ q_1(x_2) & q_2(x_2) & \cdots & q_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ q_1(x_n) & q_2(x_n) & \cdots & q_m(x_n) \end{bmatrix},$$

$$v = [v_1, v_2, \dots, v_n]^T$$

and

$$Z(x) = \begin{bmatrix} Z(x-x_1) & 0 & \cdots & 0 \\ 0 & Z(x-x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z(x-x_1) \end{bmatrix},$$

We obtain the extremum of J in order to find the coefficients $a(x)$

$$\frac{\partial J}{\partial a} = C(x)a(x) - D(x)v = 0, \quad (3.52)$$

where C is the moment matrix and is given by

$$C(x) = Q^T Z(x) Q$$

and D is also given by

$$D(x) = Q^T Z(x).$$

Rearranging equation (3.52), we have

$$a(x) = C^{-1}(x)D(x)v. \quad (3.53)$$

The approximation $v_h(x)$ can then be expressed as

$$v_h(x) = \sum_{I=1}^n \phi_I^k v_I, \quad (3.54)$$

where the shape functions are given by

$$\Phi^k = [\phi_1^k(x), \dots, \phi_n^k(x)] = Q^T(x)C^{-1}(x)D(x) \quad (3.55)$$

where k is the order of the polynomial basis.

3.4.3 Point Interpolation Method

In this method, a function $v(x)$ is defined in the problem domain Ω with a number of scattered field nodes. For a point of interest x_P , the field function $v(x)$ is approximated by using the following series representation:

$$v_h(x, x_P) = \sum_{i=1}^n A_i(x) a_i(x_P)$$

where the number of nodes in support domain of a given point x_P is n , $A_i(x)$ are the basis functions and $a_i(x_P)$ is a coefficient for the basis function $A_i(x)$ corresponding to the given point x_P .

The Point Interpolation Method (PIM) obtains its approximation by letting the interpolation function passing through the function values at each scattered node (Sidahmed, 2011).

The formulation of polynomial PIM begins with the following representation:

$$v_h(x, x_P) = \sum_{i=1}^n q_i(x) a_i(x_P) = Q(x) a(x_P) \quad (3.56)$$

where n is the number of nodes in support domain of a given point x_P , $q_i(x)$ is the basis function of monomials and $a_i(x_P)$ is a coefficient for the monomial $q_i(x)$ corresponding to the given point x_P . The vector a is defined as

$$a(x_P) = [a_1, a_2, \dots, a_n]^T.$$

The coefficients a_i in equation (3.56) can be determined by enforcing that equation (3.56) be satisfied at the n nodes in support domain of point x_P . At node i we can have equation

$$v_i = Q^T(x_i)a_i, \quad i = 1 : n \quad (3.57)$$

where the nodal value of v at $x = x_i$ is v_i .

We can write Equation (3.57) in a matrix form as:

$$V_s = Q_P a, \quad (3.58)$$

where the vector that collects the values of field variables at all the n nodes in the support domain is V_s :

$$V_s = [v_1, v_2, v_3, \dots, v_n]^T.$$

The moment matrix, Q_P , given by

$$Q_P = [Q^T(x_1), Q^T(x_2), Q^T(x_3), \dots, Q^T(x_n)]^T. \quad (3.59)$$

Assuming that the inverse of the moment matrix Q_P exists and using equation (3.58), we have,

$$a = Q_P^{-1} V_s \quad (3.60)$$

Substituting equation (3.60) into equation (3.56), we obtain

$$v_h(x) = \sum_{i=1}^n \phi_i(x) v_i. \quad (3.61)$$

Equation (3.61) can be written in a matrix form as

$$v_h = \Phi(x) V_s \quad (3.62)$$

where $\Phi(x)$ is a matrix of Point Interpolation Method shape functions ϕ_i defined by

$$\Phi(x) = Q^T(x)Q_P^{-1} = [\phi_1, \phi_2, \dots, \phi_n].$$

3.4.4 Radial Basis Functions

A radial Basis function is a real valued function whose value depends only on the distance from some other point, c , called center such that

$$\phi(x, c) = \phi(\|x - c\|).$$

According to Fasshauer (2006), the RBF interpolation deals with univariate basis functions and a specific norm (Euclidean norm) to reduce a multi-dimensional problem into a one-dimensional issue. Duffy (2006) stated that the radial basis function method is independent of the dimension of the problem. Fasshauer explain that, the radial basis function method approximates the value of a function as the weighted sum of radial basis functions. These functions are evaluated on a set of points called centers, which are quasi-randomly scattered over the domain of the problem, (Guarin et al., 2012). The weights are found by matching the approximated and observed values of the function. Once the weights are calculated, they are used to approximate the value of the function at any point over the entire domain.

Following Fasshauer (2006), we consider the set of nodes or centers

$X = [x_1, x_2, \dots, x_N]'$ with $x_j \in \mathbb{R}^s$, $s \geq 1$ and the data values $g_j \in \mathbb{R}$. We assumed that

$$g_j = f(x_j, t) \quad j = 1, 2, \dots, N,$$

where f is an unknown function and t is the time. Let $f(X, t)$ be a linear combination of N certain basis functions such that

$$f(X, t) \simeq \sum_{j=1}^N b_j(t) \phi(\|X - x_j\|) \quad (3.63)$$

where the coefficients $b_j(t)$ are the unknown weights, $\phi(\cdot)$ is the radial basis function and $\| \cdot \|$ is the Euclidean norm. Equation 3.63 can be represented as a system of linear equations given as

$$\begin{bmatrix} f(x_1, t) \\ f(x_2, t) \\ \vdots \\ f(x_N, t) \end{bmatrix} = \begin{bmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_1 - x_2\|) & \cdots & \phi(\|x_1 - x_N\|) \\ \phi(\|x_2 - x_1\|) & \phi(\|x_2 - x_2\|) & \cdots & \phi(\|x_2 - x_N\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_N - x_1\|) & \phi(\|x_N - x_2\|) & \cdots & \phi(\|x_N - x_N\|) \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}. \quad (3.64)$$

The above system of linear equations are solved to obtain the coefficients $b_j(t)$. Once we obtain the weights $b_j(t)$, the value of the function f can be estimated at any set of nodes $\tilde{X} = [\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_M]'$ with $\tilde{x}_m \in \mathbb{R}^s$ for $m = 1, 2, \dots, M$ and time t as

$$f(\tilde{X}, t) \simeq \sum_{j=1}^N b_j(t) \phi(\|\tilde{X} - x_j\|). \quad (3.65)$$

Table 3.1: Some well-known radial basis functions

Name of RBF	$\phi(r), r \geq 0$	Type
Multiquadric	$\sqrt{r^2 + c^2}$	Smooth, global
Inverse multiquadric	$\frac{1}{\sqrt{r^2 + c^2}}$	Smooth, global
Inverse quadratic	$\frac{1}{r^2 + c^2}$	Smooth, global
Gaussian	$e^{-(cr)^2}$	Smooth, global
Cubic	$ r ^3$	Piecewise smooth, global
Thin plate spline	$r^2 \ln r $	Piecewise smooth, global

Source : (Sidahmed, 2011) page 24

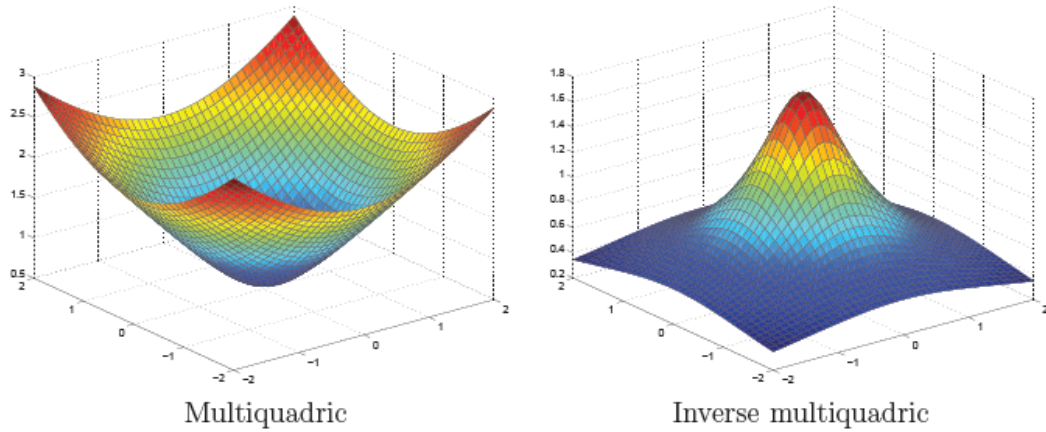


Figure 3.1: Multiquadric and Inverse multiquadric RBFs

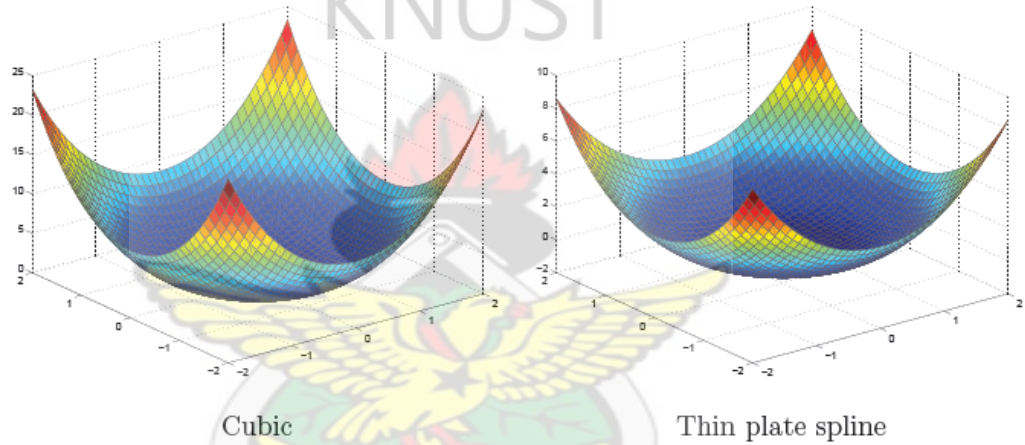


Figure 3.2: Cubic and Thin plate spline RBFs

We now consider the invertibility of collocation matrix since the direct method expressed in equation 3.64 entails inverting the collocation matrix in order to find the coefficients $b_j(t)$.

Definition 3.1. *Fasshauer (2006)*

A real symmetric matrix A is called positive semi-definite if its associated quadratic form is non-negative, i.e,

$$d^T A d = \sum_{j=1}^N \sum_{k=1}^N d_j d_k A_{jk} \geq 0 \dots \dots \dots (3.66)$$

for $d = [d_1, d_2, \dots, d_N]^T \in \mathbb{R}^N$. If the only vector d that turns (3.66) into an equality is

the zero vector, then A is called positive definite.

A very important property of matrices which are positive definite is that, all its eigenvalues are positive. This means that a positive definite matrix is non-singular. We therefore introduce the concept of positive definite function.

Definition 3.2. Fasshauer (2006)

A real-valued continuous function Φ is positive definite on $\mathbb{R}^s$ if and only if it is even and

$$\sum_{j=1}^N \sum_{k=1}^N d_j d_k \Phi(x_j - x_k) \geq 0 \quad (3.67)$$

for any N pairwise different points $x_1, \dots, x_N \in \mathbb{R}^s$, and $d = [d_1, \dots, d_N]^T \in \mathbb{R}^N$. The function Φ is strictly positive definite on $\mathbb{R}^s$ if the only vector d that turns (3.67) into an equality is the zero vector.

It means that the basis functions should be positive definite functions

$$\mathcal{P}f(x) = \sum_{k=1}^N d_k \Phi(x - x_k), \quad x \in \mathbb{R}^s \quad (3.68)$$

According to Belova and Shmidt (2011), the function $\mathcal{P}f(x)$ will yield an interpolant that is translation invariant. Positive definite functions, which are also radial functions, are invariant under all Euclidean transformations: translations, rotations and reflections.

Definition 3.3. Fasshauer (2006)

A function $\Phi : \mathbb{R}^s \rightarrow \mathbb{R}$ is called radial provided there exists a univariate function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(x) = \varphi(r)$, where $r = \|x\|$, and $\|\cdot\|$ is some norm on $\mathbb{R}^s$ - usually the Euclidean norm.

Definition (3.3) implies that for a radial function Φ ,

if $\|x_1\| = \|x_2\|$ then $\Phi(x_1) = \Phi(x_2)$, $x_1, x_2 \in \mathbb{R}^d$. An important application of radial functions is the fact that the interpolation problem becomes insensitive to the dimension s of the space in which the data sites lie. We can work with univariate function φ for all choices of s instead of dealing with a multivariate function Φ (whose

complexity increases with increasing space dimension s). The univariate function φ is called a positive definite radial function on $\mathbb{R}^s$ if and only if the associated multivariate function Φ is positive definite on $\mathbb{R}^s$ and radial.

An important part of the theoretical analysis of the radial basis functions is the integral characteristics.

Definition 3.4. *Fasshauer (2006)*

The Fourier transform of $f \in L_1(\mathbb{R}^s)$ is given by

$$\hat{f}(\omega) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} f(x) \exp^{-i\omega \cdot x} dx, \quad \omega \in \mathbb{R}^s, \quad (3.69)$$

and its inverse Fourier transform is given by

$$\check{f}(x) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} f(\omega) \exp^{ix \cdot \omega} d\omega, \quad x \in \mathbb{R}^s$$

Theorem 3.1. *Fasshauer (2006)* Let $\Phi \in L_1(\mathbb{R}^s)$ be continuous and radial,

$\Phi(x) = \varphi(\|x\|)$. Then its Fourier transform $\hat{\Phi}$ is also radial, i.e., $\hat{\Phi}(\omega) = \varphi(\|\omega\|)$ with

$$F_s \varphi(r) = \frac{1}{\sqrt{r^{s-2}}} \int_0^\infty \varphi(t) t^{\frac{s}{2}} J_{(s-2)/2}(rt) dt,$$

where $J_{s-2/2}$ is the Bessel function of the kind of order $(s-2)/2$.

The transform in the above theorem is also known as a Bessel transform.

Definition 3.5. *Fasshauer (2006)* The Laplace transform of a piecewise continuous function f that satisfies $|f(t)| \leq Me^{-st}$ for some constants a and M is given by

$$\mathcal{L}f(s) = \int_0^\infty f(t) e^{-st} dt, \quad s > a.$$

The theorem below is one of the most important results on positive definite functions.

Theorem 3.2. *Bochner's Theorem*

A (complex-valued) function Φ is positive definite on $\mathbb{R}^s$ if and only if it is the Fourier

transform of finite non-negative Borel measure μ on $\mathbb{R}^s$, i.e.,

$$\Phi = \hat{\mu}(x) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} e^{-ix \cdot y} dt, \quad x \in \mathbb{R}^s.$$

For our well-posed interpolation problem to be guaranteed, we need to extend the Bochner's characterization to strictly positive definite functions. The following theorem gives the extension of Bochner's characterization.

Theorem 3.3. *Fasshauer (2006)*

Let μ be a non-negative finite Borel measure on $\mathbb{R}^s$ whose carrier is not a set of Lebesgue measure zero. Then the Fourier transform of μ is strictly positive definite on $\mathbb{R}^s$.

The carrier of a non-negative Borel measure defined on some topological space Y is given by $Y \setminus \bigcup \{O : O \text{ is an open and } \mu(O) = 0\}$.

The following describes the way to construct strictly positive definite functions.

Collolary 3.1. *Fasshauer (2006)*

Let f be a continuous non-negative function in $L_1(\mathbb{R}^s)$ which is not identically zero. Then the Fourier transform of f is strictly positive definite on $\mathbb{R}^s$.

The following theorem gives the criterion to check whether a given function is strictly positive definite.

Theorem 3.4. *Fasshauer (2006)*

Let Φ be a continuous function in $L_1(\mathbb{R}^s)$. Φ is strictly positive definite if and only if Φ is bounded and its Fourier transform is non-negative and not identically equal to zero.

We therefore shift our attention to positive definite radial functions. The theorems below gives the criterion to check whether a given function is positive definite and radial.

Theorem 3.5. *Fasshauer (2006)*

A continuous function $\varphi : [0, \infty) \rightarrow \mathbb{R}^s$ is positive definite and radial on $\mathbb{R}^s$ if and only

if it is the Bessel transform of a finite non-negative Borel measure μ on $[0, \infty]$, i.e.,

$$\varphi(r) = \int_0^\infty \Omega_s(rt) d\mu(t)$$

where

$$\Omega_s(r) = \begin{cases} \cos r & \text{for } s = 1, \\ \Gamma(\frac{s}{2}) (\frac{2}{r})^{(s-2)/2} J_{(s-2)/2}(r) & \text{for } s \geq 2, \end{cases}$$

and $J_{(s-2)/2}$ is the classical Bessel function of the first kind of order $(s-2)/2$.

Theorem 3.6. Fasshauer (2006)

A continuous function $\varphi : [0, \infty)$ is positive definite and radial on $\mathbb{R}^s$ for all s if and only if it is of the form

$$\varphi(r) = \int_0^\infty e^{-r^2 t^2} d\mu,$$

where μ is a finite non-negative Borel measure on $[0, \infty)$.

3.5 Application of radial basis functions

In this section, we develop efficient mesh free methods based on the radial basis functions (RBFs) to solve nonlinear Black-Scholes model using the transaction costs model of Guy Barles and Halil Mete Soner. The application of RBFs leads to systems of differential equations which are then solved by a time integration scheme. This is done by using a theta (θ) method. The method is analyzed for stability.

3.5.1 Discretization

We approximate the unknown function V (the value of the option in the nonlinear Black-Scholes equation) using the radial basis functions as

$$V(S, t) \simeq \sum_{j=1}^N b_j(t) \phi(S, S_j) \quad (3.70)$$

where b_j are unknown coefficients and $\phi(S, S_j)$ are the radial basis functions. Multi-quadric radial basis function (MQ-RBF) would be used for this problem

$$\phi(S, S_j) = \sqrt{c^2 + \|S - S_j\|^2} \quad (3.71)$$

where S_j is the asset price at the collocation point j for approximating the option price V , $\|S - S_j\|$ denotes the radial distance of each of the N scattered data points S_j . The parameter c is positive and known as shape parameter.

It is well known that the following nonlinear Black-Scholes equation holds for the option price $V(S, t)$ with asset price S at time t

$$\frac{\partial V(S, t)}{\partial t} + \frac{1}{2} \hat{\sigma}^2 S^2 \frac{\partial^2 V(S, t)}{\partial S^2} + rS \frac{\partial V(S, t)}{\partial S} - rV = 0 \quad (3.72)$$

with the nonlinear term given as

$$\hat{\sigma}^2 = \sigma^2 (1 + (e^{r(T-t)} a^2 S^2 V_{ss})). \quad (3.73)$$

Substituting equation (3.73) into equation (3.72) gives

$$\frac{\partial V(S, t)}{\partial t} + \frac{1}{2} S^2 \sigma^2 \left(1 + (e^{r(T-t)} a^2 S^2 \frac{\partial^2 V(S, t)}{\partial S^2}) \right) \frac{\partial^2 V(S, t)}{\partial S^2} + rS \frac{\partial V(S, t)}{\partial S} - rV(S, t) = 0 \quad (3.74)$$

Differentiating equation (3.70), we obtain the following

$$\frac{\partial V(S, t)}{\partial t} = \sum_{j=1}^N \frac{db_j(t)}{dt} \phi(S, S_j), \quad (3.75)$$

$$\frac{\partial V(S, t)}{\partial S} = \sum_{j=1}^N b_j \frac{\partial \phi(S, S_j)}{\partial S}, \quad (3.76)$$

$$\frac{\partial^2 V(S, t)}{\partial S^2} = \sum_{j=1}^N b_j \frac{\partial^2 \phi(S, S_j)}{\partial S^2}, \quad (3.77)$$

Substituting equations (3.75)-(3.77) into equation (3.74) gives

$$\begin{aligned}
& \sum_{j=1}^N \frac{db_j(t)}{dt} \phi(S, S_j) + \frac{1}{2} S^2 \sigma^2 \sum_{j=1}^N b_j(t) \frac{\partial^2 \phi(S, S_j)}{\partial S^2} \left(1 + e^{r(T-t)} a^2 S^2 \sum_{j=1}^N b_j(t) \frac{\partial^2 \phi(S, S_j)}{\partial S^2} \right) \\
& + rS \sum_{j=1}^N b_j(t) \frac{\partial \phi(S, S_j)}{\partial S} - r \sum_{j=1}^N b_j(t) \phi(S, S_j) = 0
\end{aligned} \tag{3.78}$$

Simplifying 3.78 leads to,

$$\begin{aligned}
& \sum_{j=1}^N \frac{db_j(t)}{dt} \phi(S, S_j) + \frac{1}{2} S^2 \sigma^2 \sum_{j=1}^N b_j(t) \frac{\partial^2 \phi(S, S_j)}{\partial S^2} + rS \sum_{j=1}^N b_j(t) \frac{\partial \phi(S, S_j)}{\partial S} \\
& - r \sum_{j=1}^N b_j(t) \phi(S, S_j) + \frac{1}{2} S^2 S^2 \sigma^2 a^2 e^{r(T-t)} \sum_{j=1}^N b_j(t) \frac{\partial^2 \phi(S, S_j)}{\partial S^2} \sum_{j=1}^N b_j(t) \frac{\partial^2 \phi(S, S_j)}{\partial S^2} = 0
\end{aligned} \tag{3.79}$$

By using matrix algebra, we can define the matrix vector products as

$$\begin{aligned}
[\phi]\{b\} &= \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,N} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,1} & \phi_{N,2} & \cdots & \phi_{N,N} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \\
[\phi']\{b\} &= \begin{bmatrix} \phi'_{1,1} & \phi'_{1,2} & \cdots & \phi'_{1,N} \\ \phi'_{2,1} & \phi'_{2,2} & \cdots & \phi'_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi'_{N,1} & \phi'_{N,2} & \cdots & \phi'_{N,N} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \\
[\phi'']\{b\} &= \begin{bmatrix} \phi''_{1,1} & \phi''_{1,2} & \cdots & \phi''_{1,N} \\ \phi''_{2,1} & \phi''_{2,2} & \cdots & \phi''_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi''_{N,1} & \phi''_{N,2} & \cdots & \phi''_{N,N} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}
\end{aligned}$$

Writing equation 3.79 in matrix notation, the fully discretised nonlinear Black-Scholes equation becomes

$$\begin{aligned} [\phi]\{\dot{b}\} + \frac{1}{2}S^2\sigma^2[\phi'']\{b\} + rS[\phi']\{b\} - r[\phi]\{b\} \\ + \frac{1}{2}S^2S^2\sigma^2a^2e^{r(T-t)}[\phi'']\{b\}[\phi'']\{b\} = 0 \end{aligned} \quad (3.80)$$

By rearranging the terms in the above equation, we have

$$\begin{aligned} \{\dot{b}\} + \frac{1}{2}S^2\sigma^2[\phi]^{-1}[\phi'']\{b\} + rS[\phi]^{-1}[\phi']\{b\} - r[\phi]^{-1}[\phi]\{b\} \\ + \frac{1}{2}S^2S^2\sigma^2a^2e^{r(T-t)}[\phi]^{-1}[\phi'']\{b\}[\phi'']\{b\} = 0 \end{aligned} \quad (3.81)$$

From equation (3.81)

$$\begin{aligned} \{\dot{b}\} = rI\{b\} - \frac{1}{2}S^2\sigma^2[\phi]^{-1}[\phi'']\{b\} - rS[\phi]^{-1}[\phi']\{b\} \\ - \frac{1}{2}S^2S^2\sigma^2a^2e^{r(T-t)}[\phi]^{-1}([\phi'']\{b\})^2 \end{aligned} \quad (3.82)$$

where I is the identity matrix. If we define the matrix $[F]$ and the term containing the nonlinearity as $[C'']\{b\}$, then, equation (3.82) becomes

$$\{\dot{b}\} = [F]\{b\} - [C'']\{b\} \quad (3.83)$$

where

$$F = rI - \frac{1}{2}S^2\sigma^2[\phi]^{-1}[\phi''] - rS[\phi]^{-1}[\phi'] \quad (3.84)$$

and

$$[C'']\{b\} = \frac{1}{2}S^2S^2\sigma^2a^2e^{r(T-t)}[\phi]^{-1}([\phi'']\{b\})^2 \quad (3.85)$$

An application of the theta-method for the time discretization of equation (3.83) leads to

$$\begin{aligned} \frac{\{b(t+\Delta t)\} - \{b(t)\}}{\Delta t} = -(1-\theta)[F]\{b(t+\Delta t)\} - \theta[F]\{b(t)\} \\ + (1-\theta)[C'']\{b(t+\Delta t)\} - \theta[C'']\{b(t)\} \end{aligned} \quad (3.86)$$

The nonlinear term gives rise to a nonlinear system of equations whose solution is usually found by Newton's method. By replacing $\{b(t)\}$ in the nonlinear term by $\{b(t + \Delta t)\}$, we obtain linearly implicit scheme given by

$$\frac{\{b(t + \Delta t)\} - \{b(t)\}}{\Delta t} = -(1 - \theta)[F]\{b(t + \Delta t)\} - \theta[F]\{b(t)\} + [C'']\{b(t + \Delta t)\} \quad (3.87)$$

where $0 \leq \theta \leq 1$ denotes the implicitness parameter. After rearranging the terms in (3.87) we obtain

$$[I + (1 - \theta)\Delta t F - \Delta t C'']\{b(t + \Delta t)\} = [I - \theta\Delta t F]\{b(t)\} \quad (3.88)$$

Now, let $H = I + (1 - \theta)\Delta t F - \Delta t C''$ and $G = I - \theta\Delta t F$, then equation (3.88) becomes

$$[H]\{b(t + \Delta t)\} = [G]\{b(t)\}. \quad (3.89)$$

Holding the above equation for the collocation points, we have

$$Ax = b \quad (3.90)$$

where

$$A = [G]$$

$$x = \{b(t)\} \text{ and}$$

$$b = \{b(t + \Delta t)\}[H]$$

The matrices H and G are calculated analytically. The parameter b is determined from the numerical result at the previous time-step $t = T - \Delta t$. We therefore solve equation (3.90) for $\{b(t)\}$. Thus equation (3.90) is solved iteratively from the expiration date $t = T$ to the date of purchase $t = 0$. Once the parameter $\{b(0)\}$ is obtained at the date

$t = 0$, the option price at the date of purchase is estimated from $\{b(0)\}$ by using

$$V(S, t) \simeq \sum_{j=1}^N b_j(t) \phi(\|S - S_j\|).$$

3.6 Stability Analysis of the Numerical Method

To study the stability of the schemes numerically, we define the error at the n^{th} time level by equation

$$e^n = V_{exact}^n - V_{app}^n \quad (3.91)$$

where V_{exact}^n is the exact solution and V_{app}^n is the approximate numerical solution obtained by equation (3.86). For the scheme given by equation (3.86) the error equation at $(n + 1)^{th}$ level can be written as

$$e^n = C e^{n+1} \quad (3.92)$$

where C is the amplification matrix given by

$$C = [I + \theta \Delta t F][I - (1 - \theta) \Delta t F]^{-1}, \quad (3.93)$$

where I is the identity matrix. The numerical scheme is stable if $\rho(C) \leq 1$, where $\rho(C)$ is the spectral radius of C . Substituting equation (3.93) into equation (3.92), we obtain

$$e^n = [I + \theta \Delta t F][I - (1 - \theta) \Delta t F]^{-1} e^{n+1}. \quad (3.94)$$

Simplifying equation (3.94), we obtain

$$[I - (1 - \theta) \Delta t F] e^n = [I + \theta \Delta t F] e^{n+1}. \quad (3.95)$$

It is clear from equation 3.95 that the numerical scheme is stable if all the eigenvalues of the matrix $[I - (1 - \theta)\Delta t F]^{-1}[I + \theta\Delta t F]$ are less than unity, which means that

$$\left| \frac{1 + \theta\Delta t\lambda_F}{1 - (1 - \theta)\Delta t\lambda_F} \right| \leq 1, \quad (3.96)$$

where λ_F is the eigenvalues of the matrix F .

Now we consider different cases for θ . Firstly, when $\theta = 1$, we have explicit Euler method. Thus, equation 3.96 reduces to

$$|1 + \theta\Delta t\lambda_F| \leq 1 \quad (3.97)$$

which upon simplification implies that the explicit Euler method will be stable if

$$\Delta t \geq \frac{-2}{\lambda_F} \quad \text{and} \quad \lambda_F \leq 0. \quad (3.98)$$

Secondly, when $\theta = 0$, we have implicit Euler method which is unconditionally stable. This can be seen from equation 3.96 since $\lambda_F \leq 0$. Finally, when $\theta = 0.5$, we have the Crank- Nicholson's method. The Crank-Nicholson's method is unconditionally stable since the inequality 3.96 will hold as long as $\lambda_F \leq 0$ and this does happen.

Chapter 4

Analysis and Results

4.1 Introduction

This chapter presents analysis of the nonlinear Black-Scholes models with transaction cost with special reference to the model by Guy Barles and Halil Mete Soner using multi quadric radial basis approximation. We study the behaviour of the option value at the strike price E for different values of the parameter, a . We present the analysis for the option's delta.

4.2 Numerical Results

Using the multi quadric radial basis function approach, the resulting problems for the Black-Scholes model with transaction cost is solved via Crank-Nicolson's method. We use the same parameters from Belova and Shmidt (2011) with the exception of the parameter a from (Mawah, 2007).

The parameters are presented in Table 4.1.

Table 4.1: Parameter Values

Parameter	Description	Value
S_{max}^*	Maximum asset price	30
N^*	Number of asset data points	121
Δt^*	Number of time steps	0.05
T^*	Expiration date	0.5
E^*	Exercise price	10
r^*	Risk free interest rate	0.05
σ^*	Volatility	0.2
θ^*	Crank-Nicholson method	0.5
c^*	Shape parameter	0.01
a^{**}	Transaction cost parameter	0.005 – 0.01

Source: The parameters with (*) are from Belova and Shmidt (2011) and the parameter with (**) is from Mawah (2007).

We will be comparing prices of call options at the strike price for different values of a . When $a = 0$, we have a Black-Scholes option price which is going to form the basis of the comparisons. Thus, fig 4.1 is a Black-Scholes option price. In figure 4.1, the exact value for the standard Black-Scholes option price for a maximum asset price is 20.2469 whereas the payoff function which is the option price at the expiration date is 20. Thus there is no much difference between the exact value and the payoff function.

Figure 4.2 represents the values of the option with transaction cost and that of the payoff function. The upper curve features transaction costs and the lower one represents the payoff function.

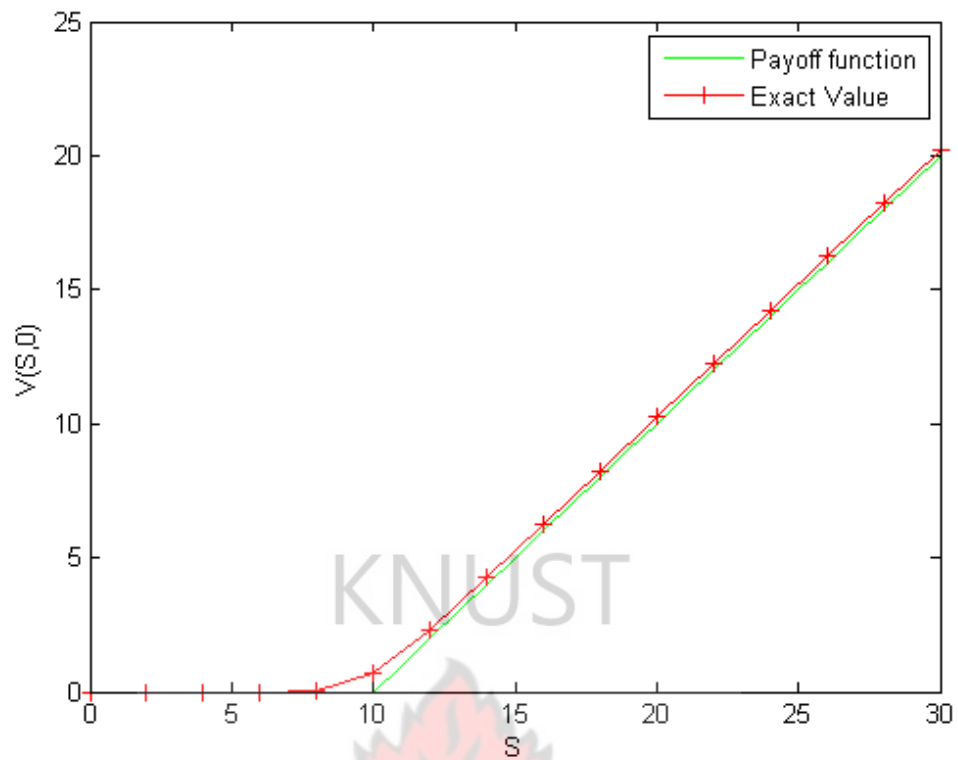


Figure 4.1: Plot of Exact Values versus Asset Price, S

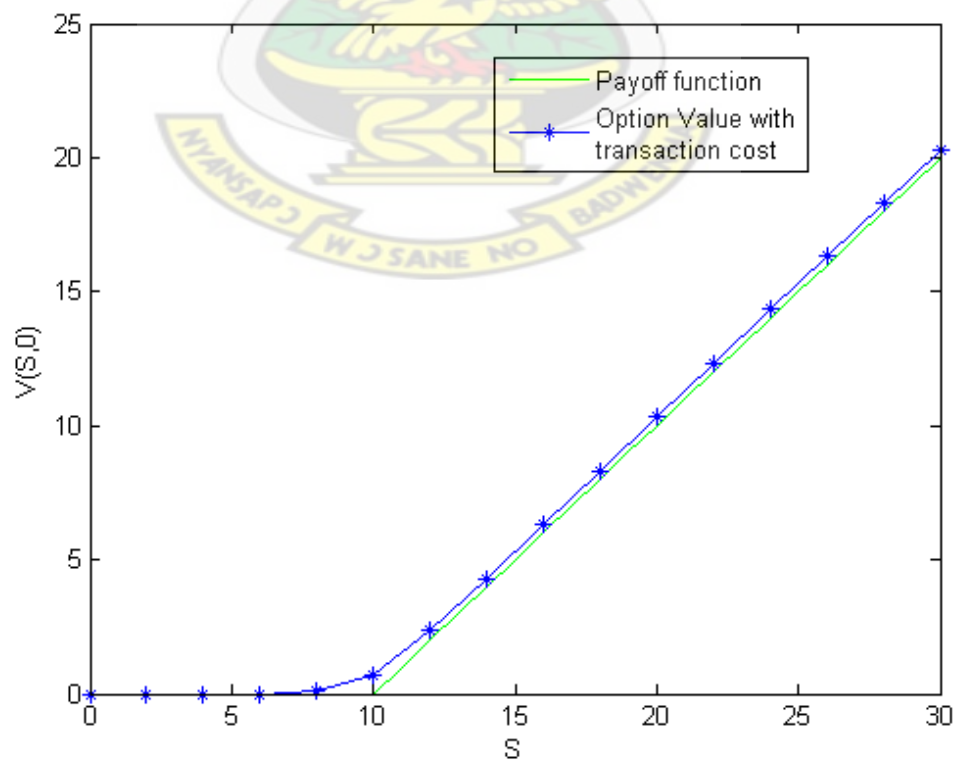


Figure 4.2: Values of the Nonlinear Option Price when $a = 0.005$

Figure 4.3 shows a comparison of results between the options values with transaction cost and the exact values. Even though the results seems identical, there is a substantial difference in the results as shown in table A.2 in the appendix. From table A.2 in appendix A, the exact value of the option and the option value with transaction cost are 20.2469 and 20.2743 respectively for a maximum asset price of 30.0.

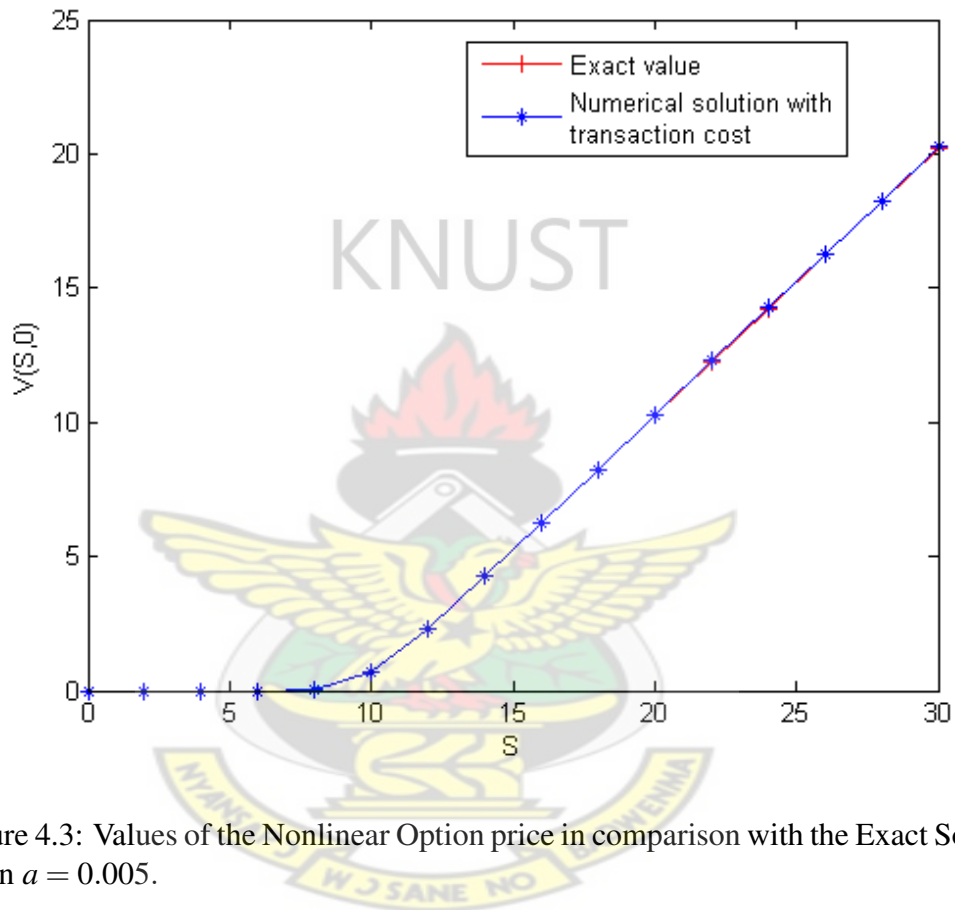


Figure 4.3: Values of the Nonlinear Option price in comparison with the Exact Solution when $a = 0.005$.

Figure 4.4 depicts the superiority of options with transaction cost over options without transaction cost as the graph of the options without transaction cost diverges slightly away for higher values of the asset price, S .

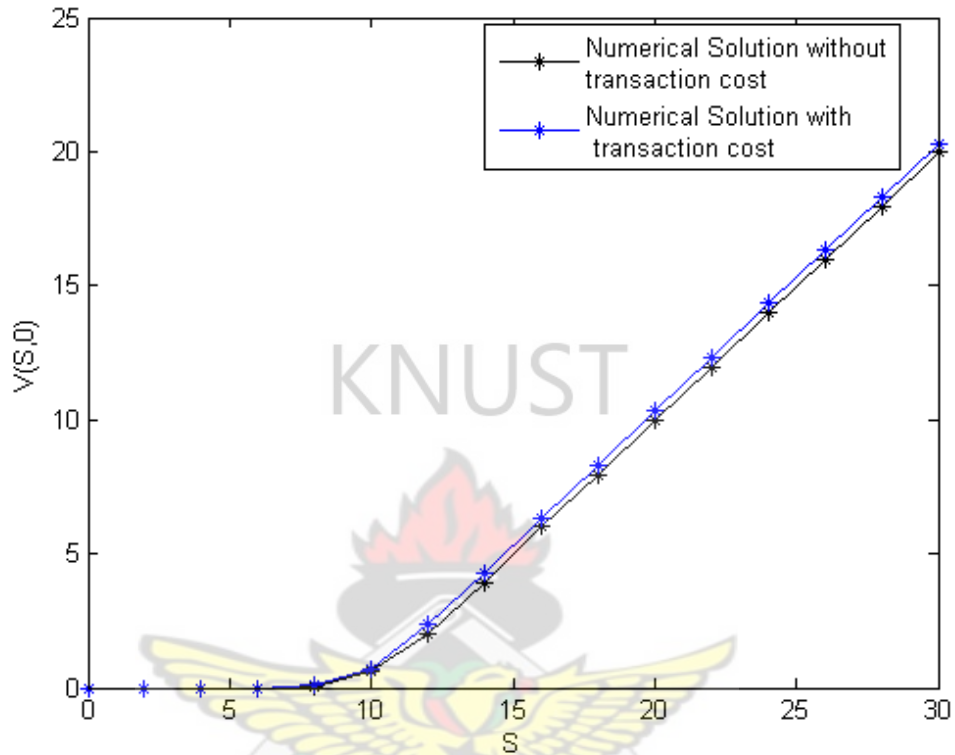


Figure 4.4: The Option Values with and without Transaction Cost when $a = 0.005$.

Due to the introduction of transaction cost, there is a rise in the value of option as compared to the option values without transaction cost indicating that transaction cost increases the value of the option as shown in figure 4.4.

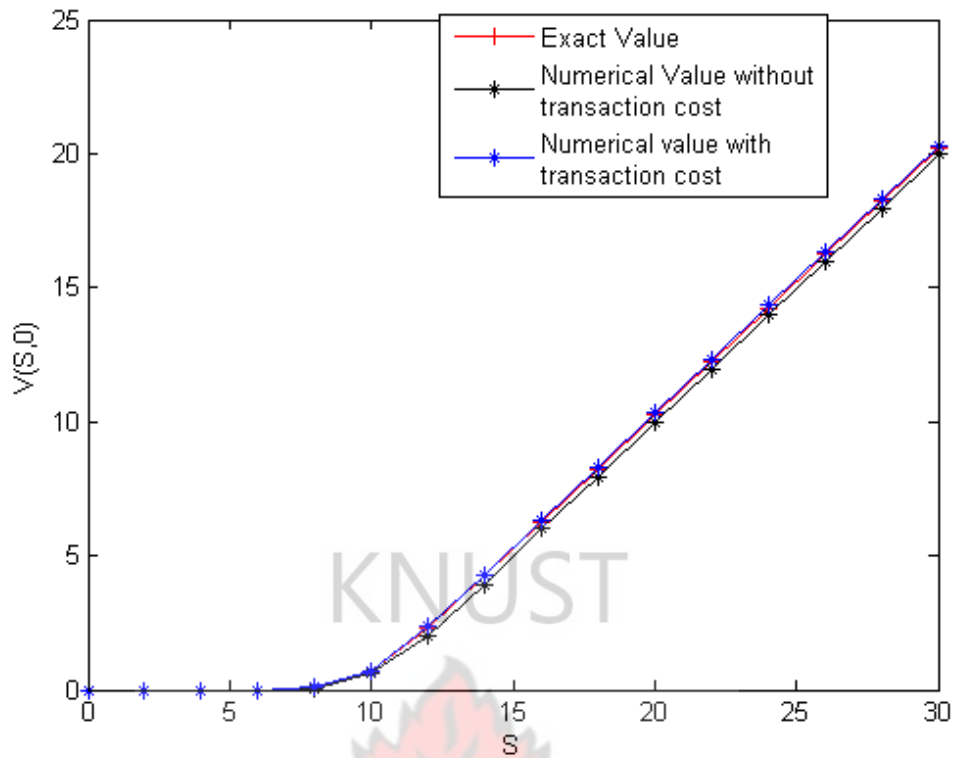


Figure 4.5: Comparison of Exact Values, Option Values with transaction cost and without transaction cost.

The results in figure 4.6 look identical as the difference between the option values with $a = 0.005$ and $a = 0.01$ is not much. For maximum asset price of 30, the option values with transaction costs $a = 0.005$ and $a = 0.01$ are 20.2743 and 20.3056 respectively. This can be confirmed from Table A.3 in Appendix A. It can clearly be seen that when the higher the value of a , the higher the value of the option price.

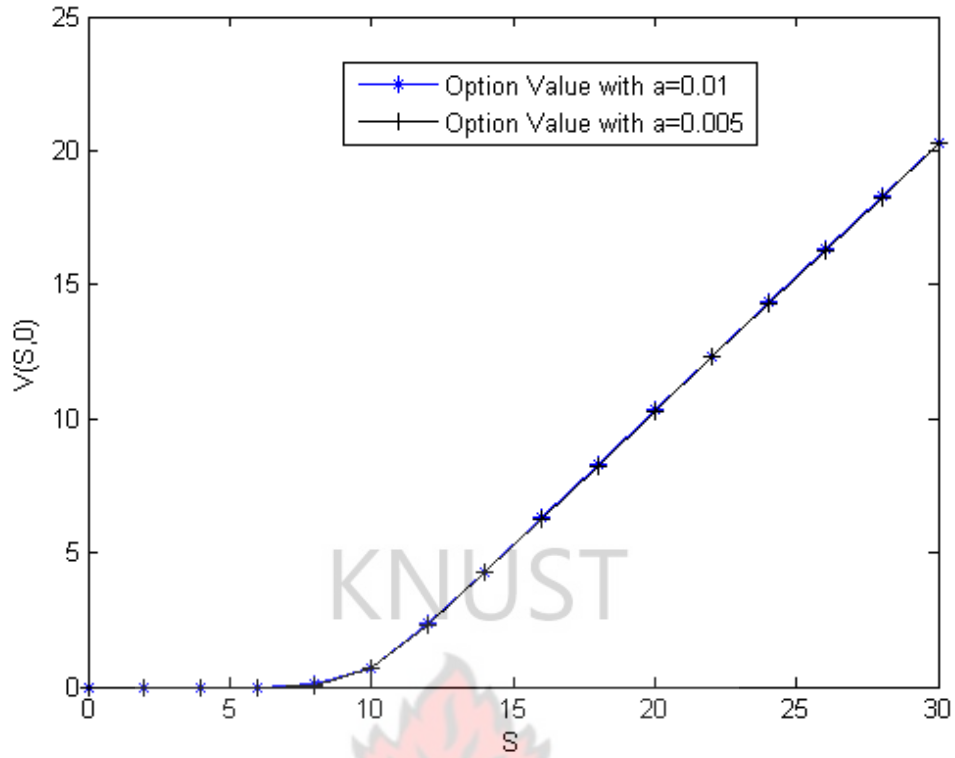


Figure 4.6: Option Values with transaction cost when a is 0.01 and 0.005.

Since the radial basis functions are infinitely differentiable, the computations of the derivatives of the options values are readily available from the derivatives of the basis functions. We can calculate the value of the delta of the option delta from equation 3.76. Table 4.2 gives the values of delta for European options with transaction cost using radial basis function method. From table 4.2, the numerical values of the option's delta lie between 0 and 1 which is in agreement with what is mentioned in (Hull, 2009).

Table 4.2: Values of option's delta when $a = 0.01$.

Asset Price, S	Exact Values of the Option's delta	Numerical Values of the option's delta
8.0	0.1243	0.1241
10.0	0.6643	0.6642
12.0	0.9567	0.9564
14.0	0.9975	0.9972
16.0	0.9999	0.9996
18.0	1.0000	0.9998
20.0	1.0000	0.9999

In order to assess the accuracy of the numerical solution, we calculate the

percentage absolute relative error E_r as

$$E_r(S, t) = \frac{|V_{rbf}(S, t) - V_{exact}(S, t)|}{V_{exact}(S, t)} \times 100.$$

We present the results in figure 4.7.

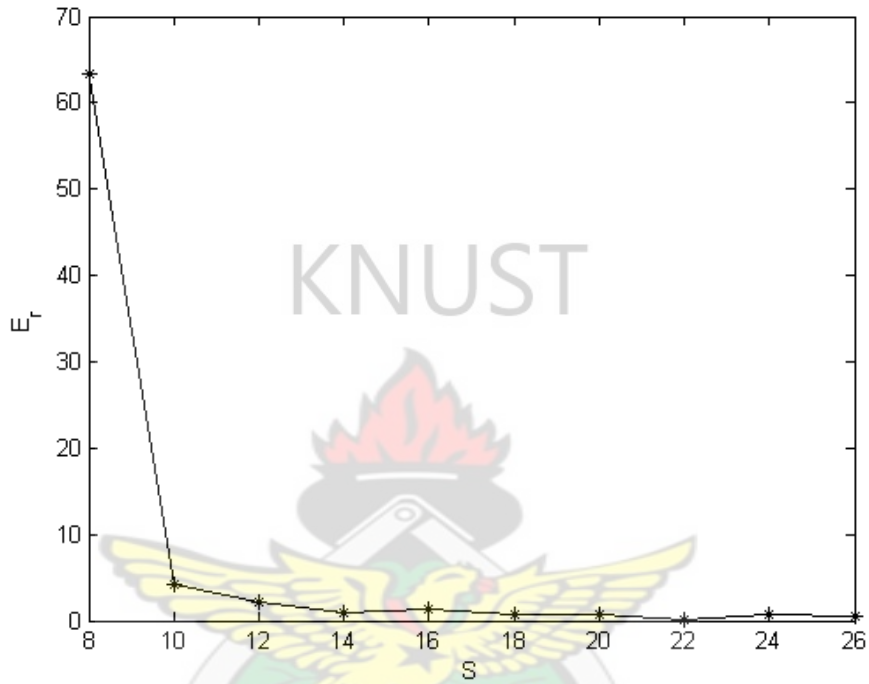


Figure 4.7: Plot of percentage error, E_r , versus asset price, S .

From the figure 4.7, the error decreases for larger values of the asset price S . The modest number of the asset price, $S = 8$, the radial basis function approximation error is almost 63 percent or 63 basis points (bps). For an asset price of 26, the RBF approximation error is only 0.45 basis points.

Chapter 5

Conclusion and Recommendation

5.1 Introduction

In this chapter we conclude based on the discussed results obtained from the analysis in chapter four. Recommendations suggested by the researcher have also been included in this section.

5.2 Conclusion

In this study, we first reviewed option pricing and meshfree methods in general. We used a special class of numerical methods, namely, Meshfree Methods, to study the differential models for pricing options. We applied this method to solve the European options. Our attention was confined to the Black-Scholes model (European option) with transaction cost. The application of radial basis functions led to a system of differential equations which were solved by the Crank- Nicholson Scheme. The method was analysed for stability and we found that it was unconditionally stable. The numerical results describing the payoff function and the values of the option were presented. Some simulations for Greeks, in particular, the option's delta was also performed.

Unlike finite difference method where the derivative of the solution are approximated by finite quotients, the radial basis function method provides a global interpolation formulae for the solution as well the derivatives of the solution. This helps to

compute the Greeks such as the delta without a need to use extra interpolation method.

We compared the influence of transaction costs on the value of the option to that of the standard Black-Scholes option price as well as the value of the option obtained by Belova and Shmidt (2011) without transaction cost. The option values obtained in the presence of transaction costs were slightly higher than the option values in the standard Black-Scholes option and the values of the option in Belova and Shmidt (2011). The levels of transaction can be viewed as deviations from the Black-Scholes option price in reality. These deviations are within 5% of the Black-Scholes price. The option values with transaction cost obtained by using radial basis function deviated from the standard Black-Scholes options were not more than 5%. Therefore the values found in this study are within the acceptable range of option prices with transaction costs.

5.3 Recommendation

We conclude that, the values found in this study when transaction cost are taken into account when valuing option prices using the Black-Scholes model are slightly higher than both the option values without transaction cost and the standard Black-Scholes option pricing model. Even though the difference was not that much, financial institutions such the banks can earn a lot when the transaction under consideration is big enough.

This study looked at the valuation of a fair price of an European option with transaction cost where the asset follows the process

$$\frac{dS}{S} = \mu dt + \hat{\sigma} dz,$$

with μ and $\hat{\sigma}$ being the drift rate and variable volatility respectively. We recommend that future research on this thesis topic should includes a jump-diffusion part in which the asset price motion is given by a process of the form

$$\frac{dS}{S} = \mu dt + \hat{\sigma} dz + (n - 1)dp,$$

where μ is the drift rate, $\hat{\sigma}$ is the volatility of the Brownian part of the motion, dp is the Poisson process and $n - 1$ is an impulse function giving a jump from S to S_n .

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Appendix A

Tables

The numerical results obtained when the shape parameter is $c = 0.01$ from Belova and Shmidt (2011) are shown in Table A.1.

Table A.1: Option Values without Transaction Cost when $c = 0.01$

Asset Price, S	Exact Value	Option Value without transaction cost
0.0	0.0000	0.0000
2.0	0.0000	0.0000
4.0	0.0000	0.0000
6.0	0.0000	0.0000
8.0	0.0456	0.0415
10.0	0.6888	0.5866
12.0	2.2952	2.0421
14.0	4.2496	3.9125
16.0	6.2470	5.9956
18.0	8.2469	7.9654
20.0	10.2469	9.9665
22.0	12.2469	11.9665
24.0	14.2469	13.9662
26.0	16.2469	15.9788
28.0	18.2469	17.9785
30.0	20.2469	19.9611

Table A.2: Option Values with Transaction Cost when $c = 0.01$ and $a = 0.005$

Asset Price, S	Exact Value	Option Value with transaction cost
0.0	0.0000	0.0000
2.0	0.0000	0.0000
4.0	0.0000	0.0000
6.0	0.0000	0.0000
8.0	0.0456	0.0561
10.0	0.6888	0.6983
12.0	2.2952	2.3104
14.0	4.2496	4.2681
16.0	6.2470	6.2781
18.0	8.2469	8.2500
20.0	10.2469	10.2960
22.0	12.2469	12.2877
24.0	14.2469	14.3010
26.0	16.2469	16.2899
28.0	18.2469	18.2654
30.0	20.2469	20.2743

Table A.3: Option Values with transaction cost

Asset Price, S	Option Price with $a = 0.005$	Option Value with $a = 0.01$
0.0	0.0000	0.0000
2.0	0.0000	0.0000
4.0	0.0000	0.0000
6.0	0.0000	0.0000
8.0	0.0561	0.0745
10.0	0.6983	0.7179
12.0	2.3104	2.3461
14.0	4.2681	4.2895
16.0	6.2781	6.3263
18.0	8.2500	8.3041
20.0	10.2960	10.3254
22.0	12.2877	12.3085
24.0	14.3010	14.3432
26.0	16.2899	16.3204
28.0	18.2654	18.2871
30.0	20.2743	20.3056