KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

DYNAMICS OF PRICE STABILIZATION WITH BUFFER STOCK: AN APPLICATION OF COBWEB MODEL USING DELAY DIFFERENTIAL EQUATION

(THE CASE OF MAIZE SUPPLY IN GHANA)

By

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Declaration

I hereby declare that this thesis entitled **Dynamics of Price Stabilization with Buffer Stock: An Application of Cobweb Model Using Delay Differential Equation (The Case of Maize Supply in Ghana)** submitted in partial fulfillment of the degree of Doctor of Philosophy is a record of original work carried out by me under the inspiring supervision of **Doctor Francis T. Oduro,** and that, to the best of my knowledge, it neither contain any materials previously published by another person nor material which has been accepted for the award of any other degree of this University or any other Institution or University, except where due acknowledgment have been made wherever the finding of others have been cited in the study.

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Supervisor Signature Date

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Prof. S. K. Amponsah ....

Head of Department Signature Date

Dedication

I dedicate my dissertation work to God almighty for being my source of wisdom, knowledge and good health and my family. A special feeling of gratitude goes to my loving wife, Augustina Owusu whose words of encouragement and push for tenacity ring in my ears.

I also dedicate this dissertation to my many friends and church family who have supported me throughout the process.

Abstract

This study is intended to use mathematical models for controlling fluctuations in the price of maize by employing a nonlinear continuous time delay differential equation derived from linear demand and nonlinear supply functions of price. These models are formulated from parameters estimated from real economic data of maize price demand and production in the Ashanti Region of Ghana through the use of regression methods. The data is obtained from the Ministry of Food and Agriculture, Statistical Directorate Kumasi-Ghana, pertaining to years from 1994 to 2013.

The results of the study are connected to the assertion that commodity price is dependent on planting time, storage time, relaxation time and total production time. It is proven that if all these individual time segments are combined as one for supply delay to make up total storage time, which is the delay for the buffer, then price oscillations would be drastically reduced.

Also the study is an improvement on the work done by earlier researchers, whose discrete time models are limiting cases of the delay buffer stock model used in this study. The efficiency of a buffer system is proven to be dependent on delay variation suitable enough to be used by buffer stock operators.

It is noted that, the more the buffer stock delay and supply delay are reduced in connection with the type of price scheme operated in the buffer stock scheme, the more stable the price becomes and the effects of buffer stock are felt more by stakeholders. The results of the analysis provide an average stable price of maize as GHC 30.49 compared to the actual average price of GHC 30.27. The equilibrium price in turn provides the average equilibrium weight of 2931.6 and 8217.6 metric tons for demand and supply respectively. The average excess supply that constitutes the stocks in the buffer is also given as 5286 metric tons and they are kept in stock for the next market period (i.e planting period). When at another period (i.e during harvesting period) demand exceeds supply then the

appropriate difference is released from the buffer to the market in order to keep price in equilibrium.

The standard deviation is also reduced to 0.1602 compared to the original 29.48 before the application of buffer stock scheme. Thus, before the application of buffer stock scheme, price oscillated between two price points and could not converge. This affirms the fact that buffer stock acts as a reserve against shortterm shortages and dampens excessive fluctuations.

Inferences are drawn from this study that researchers should rather use continuous time nonlinear delay models as they reflect the realities prevailing in most reallife economic problems. While continuous time delay differential buffer stocks models or equations could be applied in managing unstable market price of maize irrespective of the type of the supply function it is integrated with, being it linear or nonlinear, the discrete time buffer stock models instead work well with linear supply functions.

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Chapter 1

Introduction

1.1 Background of the Study

The price of maize, in recent times tends to fluctuate more than prices of manufactured goods and services. This is largely due to the distortion in production and distribution of agricultural products coupled with the fact that demand and supply are price inelastic (Geoff, 2012). It is believed that in agricultural industry, price instability and scarcity are threat to the distribution of seasonal staple foods such as maize produced by small-scale farmers.

There are a lot of disparities in terms of supply during the harvest and planting seasons which in effect could create inconveniences to stakeholders such as producers, consumers, and government (Sutopo et al., 2012). Thus supplies of staple food in agro-industry tend to be extremely different between the harvest season and the planting season, and this condition triggers price volatility and shortage of food which in turn leads to food security problems such as scarcity and affordability challenges for poor households especially. In almost all business environments, extensive studies have been done on demand because demand uncertainty is clearly inherent in these issues (Liu, 2007).

However, supply uncertainty, appears to have received less attention in terms of research and yet supply also has a significant impact on achieving price stability. It is in fact, critical for profit oriented businesses to know the right amount of commodities to be sent to a particular location on time to turn potential consumer demand of the commodity into revenue (Liu, 2007).

In detecting the source of price volatility in order to establish the reason behind the recurring cycles in the production and prices of staple foods on regular bases paved way for cobweb model. This model explains price fluctuations pertaining to a single market for food commodity which takes one unit of time to produce by farmers (Ezekiel, 1938; Matsumoto, 1998),.

In Ghana, like in other developing countries, a staple food such as maize normally accounts for quite a significant share of the budget of the poor households (FAO, 1997). The poor have limited resources to insure against unexpected price increases. For this class of people, price shocks are very problematic since the poor are not self-sufficient and such situations are likely to jeopardize their capacity to feed themselves. An implementation of food price stabilization policies is the only important way by which governments and other stakeholders in these countries can express their concern regarding the problem (Wright, 2001), since prices of cereal crops such as maize in developing countries are extremely volatile due to elasticity of supply-demand and unstable weather conditions.

1.1.1 Price Stabilization Systems

Many policy interventions have been used by stakeholders to stabilize price, improve quality of maize and increase maize production in Ghana for both local consumption and export to the world market (Anokye and Oduro, 2015; Angelucci, 2012). Price stability depends upon balancing supply and demand because supply and demand uncertainties are major obstacles to achieving stable price of staple foods.

In oder to smooth out the fluctuations in prices, one has to operate a price support scheme through a buffer stock system by which market price of agricultural products are stabilized by buying up supplies of the product when harvests are plentiful and selling stocks to the market when supplies are low (Geoff, 2012). Thus the basic function of a buffer stock system is to store a certain quantity of a particular commodity in boom periods when the price is decreasing and to release a certain quantity of the stored commodity in bust periods when the price is increasing.

1.1.2 Buffer Stock System

The empirical results from Myers et al. (1989) paint a relatively favourable picture of the buffer stock scheme. Thus, while demand shocks are the dominant source of variability during the absence of the scheme, the effects of these shocks are blunted when stock-holding are introduced. The scheme requires reliable supply process which consists of many intermediate steps; starting from producing the commodities from suppliers to shelving the commodities by retailers. There is a raging scientific debate regarding the storage effects as stabilizer on commodity prices, as storage is considered, the core of the explanation of intriguing characteristics of price dynamics (Mitra and Boussard, 2011). Recently, it is hypothesized that endogenous chaotic behavior of markets is the cause of commodity price fluctuations. A nonlinear cobweb model with parameters such as adaptive price expectation, private storage and risk averse agents was developed and as usual, in the theory of competitive storage, the nonlinear cobweb model worked to reduce price variability (Mitra and Boussard, 2011).

Sutopo et al. (2009) observed in previous researches that, buffer stock models have been developed separately based on nonlinear optimization and econometrics methods. While optimizations methods had been used to determine the level of availability with schemes consisting of time and quantity of buffer stock, the econometrics methods on the other hand had been used to determine the equilibrium price using the selling-price and the amount of buffer stock. Therefore they developed a buffer stock model that integrates both methods using decision variables which consist of quantity, time and price to achieve stability. Athanasiou et al. (2008) presented a nonlinear cobweb model with supply increasing according to certain piecewise linear supply function. Athanasiou et al. (2008, 2010) also proposed model under condition of naive price expectation to test effect of governmental interventions through the provision of reserve from buffer stocks operation to weakening commodity price fluctuations. Their model proved that if capacity of the storage for any food commodity is adequately large then there is simple price stabilization scheme, such that the equilibrium price is a global attractor for the system. In addition, Athanasiou et al. (2008) showed that if the government fixes the supply at the average supply, then price stabilization is guaranteed towards average price.

Anokye and Oduro (2013) developed linear cobweb model to study the phenomenon of commodity price fluctuations with the assumptions that there are no external shocks needed to cause price variabilities. The model showed unstable price oscillations around the equilibrium point. However, after incorporating buffer stock model, price stability was achieved but only in the short run because in the long run, the buffer stock system suffered instability until the supply was reviewed.

Edwards and Hallwood (1980) developed linear econometric models (linear supply and demand) to determine the amount of commodities that Government must store, so that the amount of stocks to be released in order to stabilize prices could also be determined. The decision variables of these models are the parameter values of functions and their performance criteria are evaluated based on the level of government expenses.

Price dynamics of a rice market was examined by Brennan (2003) using dynamic programming techniques that parameterized the case of Bangladesh that is characterized with high price elasticity (due to income effects), high storage and interest costs. In this model various storage inventions (both public and private) were explored, and the results showed that both interventions have positive impact in ensuring food security and fair prices for the poor.

Nguyen (1980) proposed a simple rule for the buffer stock authorities to stabilize both price and earnings in all circumstances, except when market is unstable. The model dampens price fluctuations in all periods and not only in the periods when a freely fluctuating price would fall but also outside the chosen limits.

Jha and Srinivasan (2001) evaluated the impact of food buffer stock system on price stabilization of food commodities when private external trade is allowed to compete in a multi-market equilibrium framework adopted in the model as endogenous factor. The results from the analysis indicated that under liberalized trade, buffer stock scheme is ineffective for stabilizing domestic prices. It is therefore prudent to use trade restrictions when buffer scheme is implemented to ensure price stabilization.

This assertion is buttressed by Sutopo et al. (2012), who developed a buffer stock model under free-trade that parameterized trade restrictions, tariffs and an indirect market intervention in accordance with warehouse receipt and collateral management system. The results from this nonlinear programming model with multi-criteria decision variables showed that indirect market intervention is more efficient than its counterpart, the direct market intervention. Soltes et al. (2012), on the other hand, used inventory (buffer stock) model in which price tends to equilibrium point not only monotonically but also oscillates around equilibrium point, and they attributed the cause of the oscillations generated from the system to the order of the differential equations applied in the process.

1.1.3 Time Varying Effect System

The time varying effects on output responses to policies for reducing and/or halting inflation was explored through the use of dynamic general equilibrium model in which time-varying component was introduced as endogenous parameter for analyzing optimal speed of disinflation. The solution of this nonlinear model revealed that output losses would be much larger when disinflation boom disappears. It was also found from the analysis that gradual disinflation of low inflation is undesirable due to its impact on the economy (Evans and Nicolae, 2010). Eduardo and Gergely (2013) studied the dynamics of price in a commodity market governed by balance between demand and supply, by employing a delay differential model. The researchers also did thorough study

of discrete-time case of the model and used the results to obtain new sufficient conditions for global convergence of solution to equilibrium in the continuous-time case. Thus when the delay is large and price is unstable, bound is provided to limit amplitude of oscillations that are sharp. Matsumoto (2010) used nonlinear delay differential and found out that time delay effects have strong stabilizing effect in minimizing
cyclic oscillations.

Ruediger et al. (2013) assessed whether time-varying volatility affect price setting by firms and transmission of monetary policy into the economy. Data analysis were performed to evaluate the impact of idiosyncratic volatility on price setting behavior of firms and also measured effects of volatility on transmission of monetary policy to economy using calibrated business cycle model. The results in twofolds suggest that, sharp business volatility increases the probability of price change and tripling of volatility also causes average quarterly price change to increase. Secondly, the increase in volatility causes monetary policy output to decline (Ruediger et al., 2013).

Anokye and Oduro (2014) also assessed the effects of delay parameter in nonlinear delay equation on price oscillations and it was observed that whenever the delay parameter is reviewed downward, oscillations (price fluctuations) are suppressed. This result indicates that price fluctuations are reduced, if and only if, delay (time-lag) affected factors are improved. The dynamics of human endocrine level was studied over time using nonlinear time delayed dependent model and it was found that daily variation of endocrine as quantified by time-delayed mutual information system intuitively provided the expected diurnal variation in glucose levels amongst random population of human as opposed to the population with no diurnal variation (Albers et al., 2012). Also time-dependent stimulation through microfluidics technology has proven valuable in eliciting previously unseen cellular responses in physiological and cellular processes within a framework of distributed time delay HIV model, and thereby potentially allowing researchers to observe new mechanisms in the pathway of cellular processes (Karyn, 2014).

Mackey (1989) on the other hand developed a price adjustment model having stated the dependent production and storage delays for studying price dynamics in a single commodity market. Conditions for stable equilibrium price are defined in terms of variety of economic parameters. It was found out that whenever price stability failed, Hopf-bifurcation occurs to give rise to oscillatory condition with period in-between two and four times the equilibrium productionstorage delay.

1.2 Statement of Problem

All the models reviewed so far by the study (under buffer stock systems) have not yet considered the effects of delay (time-lag) on price and supply change in response to market dynamics. They also failed to consider the effects of this delay (time-lag) on buffer stock operation in stabilizing prices of food commodities. Mackey (1989), on the other hand, used time-delay parameter and has proven that, storage delay can rather be a destabilizing factor for price.

This study, however, is intended to use mathematical model for controlling prices of maize at the market and yet require that the structure and parameters of the model replicate circumstances occurring at the market so as to prove it that the time varying parameter (delay) is a price stabilizer. The study will, therefore, introduce a time varying parameter model through the use of delay differential equation constructed from demand and supply functions of price. The model would then be integrated with buffer stock model with delay parameter to study dynamics of price. This delayed buffer stock model would mimic the undelayed differential model used by Soltes et al. (2012) to prove the positive impact of the time varying effects on price.

According to Papachristodoulou et al. (2004), nonlinear delay differential equations are very difficult to analyze and for these reasons most researchers are constrained to investigating the properties of their undelayed versions, and this is exactly what Soltes et al. (2012) did in their work.

The parameters of the model would be estimated from maize price and production records in the Ashanti Region of Ghana using regression analysis. The model would then be integrated with buffer stock model that has a supply delay parameter. It is believed that by the introduction of time varying parameter in the said models, stakeholders will appreciate their application and effect on price dynamics and thus be led to devise mechanisms to ensure food availability and the stabilization of food prices.

1.3 Objectives

The general objective of the study is therefore intended to construct, solve and interpret as an extension of existing cobweb models, the continuous-time delay differential buffer stock equation model . This model would determine the average amount of a commodity which must be stored and the average stocks to be released into market based on previous market information in order to stabilize prices of the commodity.

The study also seeks to explore the causes of price instability in perspective of cobweb model and appreciate buffer stock or storage by identifying its characteristics and its importance in stabilizing prices of market commodities.

The specific objectives are as follows:

1. To construct a delay differential equation model which extends continuoustime linear and nonlinear models.
2. To construct delay differential buffer stock model corresponding to 1.

3. To evaluate the ability of the model to stabilize prices through simulationusing price and supply of maize data from Ghana.

4. To use simulation above to compute average price, average stock and supply.

1.3.1 Assumptions

1. Market price is determined by the available supply in a single market andthere are no equal substitutes for maize and no exogenous shocks exist to cause price fluctuations.

2. Starting the scheme with good harvest; certainly, without stocks in thesystem it is impossible to react to a poor harvest.

3. Government determine scheme target price and guarantee to pay farmersthis price for their produce. If market price rises above this price, the market price will prevail. But if the market price falls below this price, then the target price prevails.

4. Government is the only one who operates the inventory and controls themarket under the assumption of naive price expectation.

1.4 Methodology

Difference equations would be used to develop linear and nonlinear supply functions of price which at market equilibrium with linear demand function of price (also to be built from difference equation) constitute discrete time linear cobweb and discrete time nonlinear cobweb models respectively. These cobweb models will also be interpreted and compared.

Linear as well as nonlinear continuous time (delay differential) models which take their basic concepts from functional differential equations and differential (or ordinary differential) equations with initial boundary conditions will also be developed, interpreted and compared. Comparative analysis of the

continuous time models and discrete time cobweb models would also be done. Buffer stock models will be developed and incorporated into these models to stabilize price and (supply) production. Simulation experiments will be performed using MATLAB Solver dde23 to determine reliability of the buffer stock scheme to determine price, make stock and supply to meet market demands during and after harvest. SPSS and MATLAB would be used for mathematical modelling, data
analysis and simulations.

1.5 Justification of the Study

The study seeks to incite stakeholders to appreciate the use of mathematical models for decision making whenever they intend and try to implement policies and find solutions to some basic problems similar to the one contained in this thesis. It also seeks to help make the National Buffer Stock industry more viable and thus provide employment for the youth in the country. It is believed that buffer stock scheme is the only tool through which food security and stability of food prices could be achieved. Although the research is being conducted in the context of Ghanaian environment, the approach would be applicable to other developing countries.

1.6 Scope of Study

This study concentrates on price dynamics of maize in sixteen major market centres in the Ashanti Region including Kumasi, Bekwai, Mampong, Obuasi,
Ejura, Tepa, Abofour in Offinso, Adujaman, Agogo, Nsuta, Efiduase, Juaben, Agona, Nkawie, Obogu and Ejura-Sekyedumase.

1.7 Organization of Thesis

This thesis is organized as follows: Chapter 1 presents background to the study, identification of research gaps for further study, problem statement, the study objectives, the methodology and justification of the study.

Chapter 2 deals with review of relevant literature and comprises different perspectives of researchers on the problems related to price fluctuations of farm commodities and measures that could be put in place to stabilize prices. Chapter 3 consists of mathematical methodologies employing the techniques of continuous time delay differential equations and difference equations constituting discrete time cobweb models. Here the continuous time delay differential buffer stock models and the discrete time buffer stock models would also be developed for a single commodity in a single market.

The main results from data analysis and mathematical modeling are presented in Chapters 4 and 5, in which the reliability of the buffer stock is measured using the real economic data of maize and their simulations. Finally, the conclusion and recommendations of the study are given in Chapter 6.

Chapter 2

Literature Review

2.1 Production and Consumption of Maize

Maize is the most important cereal crop grown and widely consumed in Ghana. It is one of the staple foods whose production has been increasing since 1965 (Morris et al., 1999). In terms of planting area it is number one crop since it accounts for about fifty to sixty percent (50%-60%) of total cereal production. This makes maize the second largest commodity crop in Ghana after cocoa and one of the most important crops for the country's food security (MiDA, 2010).

Maize contributes significantly to consumer diets and according to a nationwide survey carried out in 1990 by Alderman and Higgins (1992), ninety four percent (94%) of all households had consumed maize during the period of the research. Boateng et al. (1990) also found that maize and maize-based foods account for about eleven percent (11%) of household food expenditures by the poor and ten percent (10%) of food expenditures by all other income groups. The per capita maize consumption in Ghana as at the year 2000 was estimated to be 42.5 kg while national consumption in 2006 was projected to be around 943,000 Mt (MoFA, 2000; SRID and MoFA, 2007).

Research has shown that about fifty percent (50%) of the population in Sub-Saharan Africa consume maize and almost the same estimated figure could be found in their basic calories. This staple food has great nutritional values as it contains carbohydrate, protein, iron, oil, fibre, sugar, ash, vitamin B, and minerals. People in the sub-region consume maize in starch-based form found in porridges, pastes, grits and beer. The green maize (i.e fresh on the cob) on the other hand is eaten parched, baked, roasted or boiled and it plays an important role in filling the hunger gap after dry season (Chaudhry, 1983). Consumption of white maize is projected to increase in proportion to population growth due to increase in per capita income. Recent research about domestic production figures did emphasize that, the shortfall between domestic production and domestic consumption could reach 267 000 Mt by 2015 if nothing is done to improve on production of maize (MoFA, 2011).

Cultivation of maize is essentially done by few small holder farmers who depend on traditional tillage and rain-fed conditions which are increasingly erratic for production. Thus total production of maize is closely related to rainfall conditions during a particular season. In periods of good moisture conditions, inefficient storage systems also result in price pressures arising from glut during harvest time and non-availability towards the end of the season (Amanor-Boadu, 2012). Maize is staple food for a significant proportion of the world's population since it is cultivated worldwide (Council, 1990). As a result of this level of consumption, price instability is of great concern in that increase in staple food prices has dire implications on food security situation for the poor in Ghana. In agriculture, planting decisions in regards to quantity of commodity to supply are taken in respect to price level during harvest. Therefore, for farmers to take informed decisions during planting season, they need to have knowledge of the dynamics of price (White and Dawson, 2005).

2.2 Price Trend of Maize

Prices of agricultural food commodities could be used a measure to assess production possibilities and also allocate scarce resources to our advantage within an economy. Interestingly, producers and consumers in competitive food markets are becoming increasingly relevant to changes in price of food commodities (Gortz and Weber, 1986; Kuwornu et al., 2011). It is found that prices of foodstuffs including maize shoot up astronomically, few weeks after the Christmas festivities, affecting consumers negatively (ModernGhana, 2012). It has also been revealed that foodstuff prices exhibited high volatility with maize showing continual increasing price in recent years by as much as twenty three percent (23%) in research conducted by Kuwornu et al. (2011) and also buttressed by the findings of Wodon et al. (2008) that food prices for maize and other cereals increased by twenty to thirty percent (20%-30%) between the last few months of 2007 and the beginning of 2008.

Undoubtedly soaring price of foodstuffs have different radical effects across countries and population groups. The net food exporting countries benefit from improved terms of trade while net food importing countries struggle to meet domestic food demand. The fact that most countries in Africa are net importers of cereals means the poor in these countries are affected by soaring prices thereby limiting their food consumption options. Some will prefer taking less-balanced diets with consequential harmful effects on their health in the long run due to financial difficulties (ISSER, 2008). Sanyang and Jones (2008) also affirm that people tend to eat cheaper and less nutritious food frequently and in lesser quantities, as a result of higher food prices.

Since mid 2007, prices of basic foodstuffs have been increasing rapidly with dire consequences for the poor. Although effects of rising prices at the international market through the first half of 2008 seem to have subsided, price of maize remained well above that of the previous year and would likely to remain volatile for years to come due to rapid economic growth in China and India. This trends are expected to put upward pressure on price as demand of maize simply out-pace supply (Timmer, 2008).

Ghana was not left out in the hook of this international market price crisis and according to Amanor-Boadu (2012), Ghana could not escape price crisis that hit the global commodity markets between the year 2007 and 2010. The price range in crop year 2007/08, was GHC 35.35 per 100kg compared to GHC 12.51, for the previous crop year and GHC 17.91 two crop years later. Ghana experienced the highest variability in market prices in the same year (2007/08) with standard deviation GHC 11.76 and coefficient of variation around thirty three percent (32.6%), which was the highest estimate in the last five crop years.

2.3 Causes of Price Fluctuation

It is well established fact that price fluctuations have negative effects on peoples' welfare, particularly on children's health (Jensen, 2001), food security and growth (Myers, 2006) in developing countries. Extreme price fluctuations of agricultural food commodity mean food insecurity and a cause of great concerns for the poor. Particularly in Ghana where people depend so much on imports, extreme price fluctuations are likely to put food supplies at risk during times of low supply and high demand. Thus uncertainty in price of staple foods is a major constraint to a sustained increase in production (Demeke et al., 2012). The various factors identified to be responsible for volatility of maize price and any other agricultural foodstuffs are discussed in two main categories which are economically defined as endogenous and exogenous factors.

2.3.1 Endogenous Factors

Peaks and troughs in production are directed related to price variability and this is often made worse by host of endogenous causes (Gilbert and Morgan, 2010). The explanation of endogenous factors is tricky, but more plausible, since they are responsible for changes in fundamental supply and demand factors. Thus commodity price are driven by factors such as increased demand, decreased supply and may be cost of fertilizer to mention just a few (Mueller et al., 2011).

Price expectation

Producers expectations about future prices are connected to observations made in relation to previous prices. If expectations are not projected well, it could lead to supply uncertainty and in turn create an inelastic demand that magnifies imbalances to result in large and detrimental price fluctuations (Ezekiel, 1938). Normally farmers in their quest of forming price expectations, look back at the most recent prices and forecast future prices. This backward looking forecasting could sometime turn out to be vital reasons for models' fluctuations (Nerlove, 1958) .

If high prices are expected to continue, farmers would produce so much and thereby end up selling at low prices, and vice versa. Thus high production level and low prices of food commodities are expected to be followed by shortages and high prices of the commodities and create market instability if production decisions taken during planting period are based on expectation that current prices will remain the same after harvest. It means that if current commodity price is rising, producers may expect it continue and accordingly plan to expand production towards the following season, resulting in high levels of production and price collapse. The reverse of the process takes place the following season and the cycle of production and price instability continues (Boussard, 1996; Boussard et al., 2006; Galtier, 2009), especially if farmers have limited information about the future.

Goeree and Hommes (2000) investigated heterogeneity in beliefs or expectations using dynamic cobweb model and it was observed that heterogeneous expectations always lead to price and supply instability and even chaotic market situation. Speculation on commodity markets is one of the root causes of price volatility and should not be considered as random bet that can be smoothed out easily from the price system. Market and non-market factors that influence behavioral and strategic choices of speculators must be given serious attention since speculation caused by demand and supply in physical markets can also serve as price discovery, liquidity and risk-hedging mechanisms (Joachim and Getaw, 2012).

Lagi et al. (2011) also constructed dynamic model that is quantitatively in agreement with food prices and it was proven from the analysis that, investor speculation is one of the dominant causes of price increase in commodity markets. Frankel (2013) modeled oil prices and other storable commodities using econometrics. The model focused on speculative factors, such as trade-off between interest rates and changes of future price expectations by market participants. The analysis went beyond past research to bring to bear new data sources to measure expectations of future prices changes and options data to measure risk perceptions.

Influence of Fuel Demand

The recent steady upward trend in staple food prices is found to be contributed by the increase in crude oil price, since jump in food prices over the last decade has been explained by large increase in crude oil prices which stands out among numerous factors that causes price change. In recent World Bank study, oil prices were found to be major contributing factor to food price increase than several other long-term price drivers (Baffes, 2013). Oil price increase exerts both direct and indirect upward pressures on aggregate prices of food (ISSER, 2008).

Prices of grains or cereals between March 2007 and March 2008 were more than doubled, and the available data suggests, that record high grain prices in 2008 as actually the result of increased production costs driven by high petroleum prices (Mueller et al., 2011). Prices of agricultural commodities are frequently affected by higher energy prices more than usual adverse weather conditions and diversion of some food commodities for the production of bio-fuels (Baffes, 2013). The recent global food commodity price inflation was also attributed to increased demand for biofuels (produced from food grains including maize and oilseeds) and rising energy prices (Trostle, 2008). Global biofuel production tripled between year 2000 and 2007 and was projected to double by the year 2011 (Molony et al., 2010). The price projection is expected to contribute to food price increase and eventually lead to food insecurity if gains from biofuel production are not used effectively to reduce cost of food production (Ansah, 2014).

Baier et al. (2009) also affirm that, the increase in biofuels production in recent times have had a sizeable impact on the price of corn, and estimated that the increase in worldwide biofuels production pushed up corn price by twenty seven percentage (27%) points. According to Donald (2008), the rapid increase in food prices can be linked to the increased production of biofuels from food grains and oilseeds and also the increase in food production costs due to higher energy prices. The contribution of biofuels to rise in food prices raises important policy issues which must be considered in light of their effects on food prices. Kristoufek et al. (2011) examined relationship between wide array of food and fuel commodity prices collected from the United States and European Union and the results indicated significant correlation between food and fuel prices with biofuels linking the two markets. It is also hypothesized that biofuels have been exacerbating price of agricultural commodities. However, magnitude of the impact of biofuel on food prices is unsettled. High food prices have been indirectly accompanied by record high oil prices. (Sexton et al., 2008).

Sudden Jumps in Agricultural Input Prices

Agricultural input price increase arises from several sources, including general inflation, weather-related supply problems and/or energy-related price spirals (Chinkook, 2002). Oil prices are blamed for the rise in prices of fertilizers, other farm chemicals and machinery costs and therefore, any further rise in oil prices is expected to quickly translate into increasing production costs. Input like water is also becoming increasingly constraint in availability to agriculture and more costly to procure needed supplies. However, higher energy and oil prices and rising costs of inputs are factored into commodity price projections through higher supply costs (OECD and FAO, 2013). Higher energy prices according to

Joachim (2008), could make agricultural production more expensive by raising the cost of mechanical cultivation, fertilizers and pesticides and transportation of inputs and outputs.

The impact of intermediate input price increases on food prices was analyzed on assumption that the producers can pass the increased production costs through to the final consumers. Findings from the research indicated that intermediate inputs prices increase have a greater impact on food price (Chinkook, 2002). While transportation cost affect cost-to-market for agricultural producers, the variable input cost affect cost of energy intensive farming input such as seed, chemical fertilizer, herbicides and pesticides. These two effects increase costs and reduce agricultural supply, since demand for food is relatively price inelastic. Therefore, any decrease in food supply induced by rising energy prices also leads to significant food price increase (Zilberman et al., 2008).

Lack of Adequate Storage Facilities or/and Low Stock Levels

In some developing countries about twenty five percent (25%) of food produced is never consumed by humans and instead left spoilt or eaten by insects, rats and other pests due to lack of storage facilities. Adequate on-farm storage is crucial for storing surplus food items to enable farmers to supply food beyond harvest period and ensure year round availability of needed food (FAO, 1997), which in effect stabilizes prices of food commodities. Lack of stocks render vulnerable markets to unpredictable disturbances (Wright, 2012), since any minor shock can trigger major price increase if stocks are low (Gilbert and Morgan, 2010). According to Deaton and Laroque (1992, 1996), when stocks are high (or low), then with perturbations stationary, the probability of a low (or high) price during the next period is increased. The recent price volatility of major grains is not anomalous because maize is highly substitutable for calories in the global market. Therefore, when aggregate stocks decline to minimal feasible levels, prices become highly sensitive to small shocks and these are consistent with economics of storage behavior (Wright, 2012).

As a result of poor storage management in Ghana, tones of maize are rotting away in silos making farmers so frustrating. For instance a maxi bag of maize was selling for GHC 32 compared to GHC 55 the previous year. Thus Poor management of agricultural storage facilities could potentially affect the supply of agricultural commodities and their prices (Ghana, 2010). Marketing and storage facilities for agricultural produce in some districts in Ghana are generally poor. Due to poor conditions of existing traditional storage facilities most farming communities experience heavy post harvest losses. These farmers dry and store their maize through traditional method of hanging in their kitchen. However, these processing methods and storage facilities are inefficient and inadequate, and therefore farmers have to either sell their produce at low prices during the harvest or suffer high post harvest losses (NEMA, 2006).

The National Food Reserve Agency also in Malawi 2012 lost 32,222 MT of maize stock due to heavy rains and poor storage facilities. This loss of maize stock resulted in breaks for humanitarian food distributions for food insecured households in the Southern Region. Then in response to the loss of maize stock and rapidly rising maize prices, the Government of Malawi had to resort to importing 35,000 MTs of maize from Zambia (FEWSNET, 2013).

2.3.2 Exogenous Factors

Exogenous factors are that having important economic consequences but not being controlled by economic factors (Lamberton, 1984). Thus factors such as weather (e.g rainfall, temperature pattern), trade liberalization or restrictions or any other non-economic factors that could disturb supply are exogenous.

Climatic Effects

Research has shown that climatic conditions can play a role in reducing agricultural commodity supply and increase consumer prices (Davenport et al., 2013). It is found that recent increased of basic food prices which severely impacted on vulnerable populations worldwide was caused by adverse weather, which causes shortages of agricultural foods (Lagi et al., 2011) as confirmed by ISSER (2008) that poor weather conditions played a role in rising food prices when northern part of Ghana experienced severe flood in 2007 resulting in loss of most farm produce.

Since 2007-08 food crisis, many thoughtful analyses have addressed causes and impacts of high and volatile international food prices so as to be able to propose solutions to the crisis. However, both empirical and theoretical evaluations suggest extreme weather conditions as one of the root causes (Joachim and Getaw, 2012). Agriculture is highly dependent on specific climate conditions for the growth of certain crops. Thus temperature changes, amount of carbon dioxide (CO_2) and weather intensity could have significant effects on crop yields. For instance warmer temperatures make certain crops grow quickly and yet reduce their yields. In other words, crops tend to grow faster in warmer conditions but for some crops such as grains, faster growth reduces the amount of time needed for the seeds to grow and mature (USGCRP, 2009) which in effect brings about food shortage that triggers price volatility and pose humanitarian crisis.

Changes in climatic conditions also affect water availability needed for crop growth and for that farmers forced to continually adapt irrigation for their agricultural production. Climate change has socioeconomic and environmental implications for agriculture since changes in availability and quality of land, soil and water resources are reflected in crop yield which leads to food shortage and thereby causes prices to rise (Institute, 2014). According to Ringler et al. (2010) shifts in crop yield and area for its growth and food price increase are caused by climate change. Changes in these parameters also affect food affordability which leads to less calorie intake to cause malnutrition problems in children. It is projected that cereal production growth in Sub-Saharan Africa would be declined by about three percent (3.2%) as a result of climate change. Extreme weather conditions caused significant jumps in food prices by an average of one hundred and twenty five percent (125%) for corn (or grains) between 2006 and 2008 in both developing and developed countries. For instance, severe droughts in grain-producing regions were the cause of world food price fluctuation (Mazhirov, 2011). As projected in Australia, global frequency and severity of drought is likely to increase as a result of climate change. According to the regional projections South-Eastern Australia will be adversely affected by changes in rainfall patterns as well as rising temperatures to increase severity of droughts and thereby contributing to broad and sustained increases in prices of food (Quiggin, 2010).

Trade Liberalization

Studies have shown that trade liberalization leads to price increase of farm inputs, causing huge problems for small holder farmers. In Ghana food imports is demoralizing small holder farmers since having produced the commodity, farmers have to compete with cheaper imports to obtain economically viable prices for their produce even in village markets. This is simply subjecting domestic agricultural food production to risk (MAGAZINE, 2001).

Trade liberalization affects the poor by changing the prices at which they buy as consumers and sell as producers (Matusz and Tarr, 1999). One major direct effect of international higher food prices on developing countries is that it puts up pressure on local food prices to make food less affordable for consumers.

The real income and welfare of population including the poor are then affected (Plan, 2008). The volatility in international food prices affects households and businesses to the extent that the impact is transmitted to domestic markets.

Since Africa is vulnerable to such international trade impacts, food prices in Africa have become more volatile in recent years (Gerard et al., 2011; G20, 2011). However, these price trends provide incentives for increase in production of foodstuffs to local farmers (Plan, 2008).

Even though high food prices affect the poor a lot, the extent of the impact is dependent on structure of the economy and whether they are net food buyers or sellers (Aksoy and Hoekman, 2010). Most cross-country research gives indication that poor are affected by high food prices because they are net food buyers (Ivanic and Martin, 2008; Wodon et al., 2010).

2.4 Stabilization of Maize Price

Stabilization of food prices is another important area in agricultural economics. While producers expect prices to shoot up in order to earn enough income, consumers always expect prices of foodstuffs to be stable and affordable because foodstuffs are basic necessities of life. Supply variations cause price fluctuations and in turn affects income of producers therefore to suppress supply variations so as to regulate prices to guarantee producers' incomes, governments in many economies often intervene the markets with price stabilization policy (Matsumoto, 1998).

High levels of production instability and large disparity of agricultural commodity prices between import and export commodities in Southern Africa countries have led to the call for market intervention in the form of price stabilization and stock holding of food surpluses (Pinckney, 1991). Gerard et al. (2011) argue that the high and volatile prices of food strengthen the case for governments in developing countries to implement market intervention policy to stabilize food prices, in spite of the practical difficulties associated with the policy. Research interest was triggered towards price fluctuations and how they could be halted when global market in 2007/2008, witnessed extreme increase in prices of agricultural commodities. It is fundamental to understand how price fluctuations arise in the first place before any policies aimed at curtailing price fluctuations are rolled up.

2.4.1 Buffer Stock Scheme

The buffer stock scheme (also known as intervention storage) is a policy used purposely for stabilizing prices in an agricultural commodity market by regulating supply of the commodity through storage system. Under the scheme, commodities are bought and kept in store when there is surplus in market, then when there is economic shortage stocks are released to the market (Morrow, 1980). If buffer stocks are used strategically, prices volatility could be reduced drastically. However, the system requires infrastructure and skills to manage and procure stocks to be able to provide timely response to any unexpected shortfalls in the market. Studies have shown that impacts of factors that affect food commodity price would be overestimated, if inventory effects are not taken into account (Gal et al., 2014).

buffer stock scheme assumes that governments carry out all planning in regards to procurement, inventory management and market operation policy (Bahagia, 2006) and also control the market without having any competition from any storage firms. Local market intervention by governments though considered purely political, it is also justified on economic grounds due to improved macroeconomic and dynamic efficiency derived from stable food prices (Timmer, 1989).

In microeconomic theory, inventories (or buffer stocks) are considered as stabilization factor or something a cost-minimizing firm can use to smooth out production in the face of commodity fluctuations (Blinder and Maccini, 1991). Newbwey (1984) found that with linear demand, dominant producers of a particular commodity choose more stable prices than under perfect competition and price stability increases with their market share. Therefore, with constant elastic demand, competitive degree of price stabilization is achieved.

In buffer stocks schemes, the government either imports or procures commodities from domestic production at set prices, then holds stocks and distributes them at fixed prices through wholesales that are government regulated (Knudsen and Nash, 1990).

Two-Price Scheme

Any buffer stock scheme that is run on two-price policy scheme has two prices namely, floor and ceiling prices. When price drops close to floor price, governments (scheme operator) in ensuring that the price does not fall further start buying up stocks. In same vein, when the price rises close to the ceiling, governments in order to depress prices, sell off its holding (Edwards and Hallwood, 1980). Athanasiou et al. (2008) developed nonlinear cobweb model with buffer which stabilized prices at two equilibrium points based on the capacity of the storage. it was observed that when storage is enlarged then the system has equilibrium price called global attractor. Also, if supply is approximated at average supply then average equilibrium price is achieved and is also referred to as average
price.

Westerhoff and He (2005) developed commodity market behavioral model that encompassed parameters such as consumers, producers and heterogeneous speculators to characterize commodity price fluctuations and evaluate price stabilization schemes. They examined how nonlinear interactions with stakeholders could trigger price fluctuations between bull and bear markets through homoclinic bifurcation. When floor and the ceiling prices were imposed on the model, homoclinic bifurcations were eliminated and price volatility reduced (Westerhoff and He, 2005).

Another buffer stock model was developed by Sutopo et al. (2008) on assumption of limited supply time and continuous consumption and price fluctuation was stabilized through market scheme where stocks are bought from both domestic and import supply points. The model worked with price band policy that bounds domestic price variation between sets of upper and lower bounds.

Market stabilization schemes that run under price band rules have the advantage of transparency whether it is international or domestic system. The effects of the scheme on behavior of prices and aggregate costs of operation are much less straightforward. The price tends to hover around the upper or lower bound of the band (the ceiling or the floor price). The overall effect of the scheme on volatility relative to competitive storage is ambiguous and the reason is that if stocks are released towards ceiling price, price peaks are contained as long as stocks are available. However, in anticipation of this developments private storage is discouraged as price rises to the ceiling to abandon idea of stabilizing production in response to anticipated shortages (Sutopo et al., 2008).

Single Price Pcheme

In this buffer stock scheme the lower and upper bounds are equal, in that this market intervention approach ensures that there is only one fixed price. This scheme ensures that average supply is adjusted regularly to keep up with any broad trends toward increased yield. That is, it must truly be an average of probable yield outcomes at any given point in time (Edwards and Hallwood, 1980).

It is also confirmed in linear buffer stock model developed by Anokye and Oduro (2013) that if price is fixed at a particular level, the average supply of commodity to the market must be reviewed regularly else prices are bound to oscillate in the long run. Anokye and Oduro (2015) employed nonlinear continuous-time delay differential buffer stock model to control prices of maize. This buffer stock model was proven to be dependent on delay variation such that, farther the delay for buffer stock and supply are reviewed downwards, more stable the prices become towards fixed average price point.

Corron et al. (2005) used commodity market model to examine dynamics of correlations between price limiters and growth rate of buffer stocks. It was discovered that optimal price limiters provide commodity market system that is inexpensive to regulate. However, such price schemes have unexpected consequences in irregular commodity price fluctuations between bull and bear markets and even complicate impacts on the size of buffer stocks. On the other hand, by imposing lower price boundary buffer stock become very huge and make storage costs impossible to finance over time (Corron et al., 2005).

Moir and Piggott (1991) presented composite model comprising bufferfund and buffer-stock as an alternative to pure buffer-fund or pure buffer-stock for stabilizing prices. From the results of the analysis each model provided the same stable prices to producers as did by the floor price scheme (or reserve price scheme-which has no ceiling price). However, the combined model proved very effective in terms of cost and degree of stability of the price (Moir and Piggott, 1991).

The effectiveness of buffer stock on price is dependent on the trigger price defined either on fixed-band rule or on adaptive-band rule around the estimated mean price. While the adaptive band incorporates the effects of supply response, fixed band on the other hand disregards behavioural response of producers to reduction in price fluctuations. However, if governments attempt to stabilize price around mean price without taking stabled supply response into account, then governments face steady accumulation of stocks and thus suffer increasing losses (Nurul and Thomas, 1996).

2.4.2 Market Structure

Market structure is another important consideration associated with price stabilization scheme. The extent changes in prices are transmitted into markets from other markets are affected by the overall economic environment and the market structure (i.e oligopolistic, or competitive or the market is spatially integrated or not) where prices are determined by the forces of supply and demand (Lovendal et al., 2007). If markets are segmented then it implies that segments of the market with excess demand do not get feedback from other markets that have excess supply and so transmission of prices across markets is absent. Generally markets are neither totally segmented nor totally integrated (Nurul and Thomas, 1996) and according to Ahmed and Bernard (1989) markets in developing countries are not fully integrated.

Although issues of segmented market is relevant to developing economies where production and consumption are geographically dispersed over wide areas and transportation is a problem, tt is also plausible to some regions in developed countries (Bobenrieth et al., 2006) where market is segmented into local market where farmers sell commodities to storage firms and central market where storage firms sell commodities. Awudu (2000) utilized threshold cointegration tests that allow for asymmetric adjustment towards long-run equilibrium and examined price linkages between principal markets in Ghana. The results of the research indicated that major markets are well integrated and wholesale prices in most of the local markets respond more swiftly to upward price change than they to downward price changes.

2.4.3 Buffer Stock Combined with Trade Liberalization

Dorosh and Shahabuddin (2002) constructed mathematical model to measure the variability of domestic and international grain prices, and also examined the interaction of government intervention and private sector participation in commodity markets. It was found that relative high degree of price stability is achieved due to private sector imports that stabilized markets following major production shortfalls. Domestic grain procurement on the other hand contributed relatively little to rising domestic producer prices during harvest time as only a small percentage of farmers engaged in production of food commodities.

According to Gersfelt and Jensen (2006) minimum market intervention prices are guaranteed to producers for certain agricultural produce in the EU, through combination of sales at floor prices to buffer stock agency and measures taken at the border. It was observed that if intervention price is lowered, floor price at which stock agency starts purchasing also reduced and direct support payment are given to farmers as compensation for reductions in intervention prices. More open and liberalized grain market under the same constraints, incentives and commercial standards facilitate more rapid trade responses to grain surpluses and shortages and make grain market more stabilizing. Thus an effective buffer stock system as part of domestic marketing strategy could be run in conjunction with imports to buffer domestic prices (Nyberg and Rozelle, 1999).

Jie et al. (2013) studied the fluctuation characteristics of grain price in China during the past two decades by using Structural Break Regime Switching Model, and found that grain price growth has become more stable since 2004 with slight growth change regimes. The implementation of minimum grain purchase price policy and the improved market structure among others, found to be the most important contributing factors. When the procurement and distribution of grains are done by only the public agency, it becomes disincentives for the private sector to invest in the market. However it is not advisable to leave the grain economy fully at the behest of the markets owing to the importance of grain food in household consumption and production.

It is therefore, suggested that the government slowly steps out of the food market and let it operates freely, while ensuring effective monitoring and maintenance of an optimal size of buffer stock for grains to avoid extreme food price fluctuations and shortages in the country. The stock purchased and released by the government should also base on prevailing market prices (Munir et al., 2006).

2.5 Model Types in Price Dyanmics/Stabilization

In modeling economic dynamics of commodity markets, many examples of cobweb models with different form of functions (linear or nonlinear) have been demonstrated. Finkenstadt (1995) applied linear supply with nonlinear demand functions. Hommes (1991) used linear demand with nonlinear supply equations. Junhai and Lingling (2007) established a nonlinear model derived from quadratic demand and supply functions where findings indicated that nonlinear cobweb models explain irregular variations observe in real economic data (Jensen and Urban, 1984).

In fact, a great deal of recent economic models entirely discount the role of delays in generating price fluctuations and a good mathematical setting in which to consider this gap is provided by delay-differential equations (Howroyd and Russel, 1984; Eduardo and Gergely, 2013). Loretti et al. (2012) employed delay differential similar to that of (Dibeh, 2007) to study price dynamics of two markets that are coupled via diffusive coupling terms. Two different time delay cases were studied namely, when the two markets experience the same delay time , and when the delay time is different across markets to determine their equilibrium and stability state. Numerical simulations were used to confirm the theoretical findings.

2.5.1 Effects of Nonlinear Models

Although, a lot of attention has been paid to linear difference and differential equations, nonlinear equations have received less attention. However, there has been a significant change of late, as many recent researches conducted in economics are considering nonlinearities (Shone, 2002). Anokye et al. (2014) studied price dynamics of maize in Ghana using linear and nonlinear cobweb models which are constructed from real economic price and production data. It was deduced from their study that researchers rather use nonlinear models to avoid under or overestimation and make better predictions in real economic situations.

The use of nonlinear dynamic models in economics and finance has expanded rapidly in the last two decades (Diks et al., 2008) and in studying the differences between linear and nonlinear models in engineering, SolidWorks (2013) discovered that neglecting nonlinear effects can lead to serious errors as a nonlinear model can help one to avoid overdesign and build better products. SolidNotes (2009) in other words, states that whereas it takes longer time to run a nonlinear solution, yet nonlinear models replicates the actual physical system under study and provide accurate results.

Price dynamics of maize was also studied using on two different types of continuous time delay differential equation models by applying numerical solutions. The results from the study recommended the use nonlinear models to researchers instead of linear models to avoid misleading conclusions when solving real-life economic problems . The recommendation was deduced from the fact that, the results from the nonlinear delay differential model seemed more realistic than that of the linear delay differential model (Anokye and Oduro, 2014).

2.5.2 Effects of Continuous-time Models

Cobweb model of market stability is a discrete-time model that leads to a recurrence equation which describes the sequence of prices when the initial price is not the equilibrium value. One of the implicit assumptions in the demand and supply analysis is that suppliers decide the quantity of goods to be sent to market after they have known the price of that good. In reality, however, most suppliers commit themselves to the supply decision before they know the price of the good in question. Thus, in reality, time is continuous although it is assumed that it evolves discretely so that under this framework the market is cleared once per fixed period of time (Asano, 2012). It is therefore plausible to use continuous-time models, and these models lead to differential equations rather than recurrence equations with price being given as a function of the continuously varying time parameter (Anthony, 1996).

Nowman (1997) applied continuous-time model to study the short-term interest rate in finance and it was noted that, the volatility of short-term rates is highly sensitive. Lindsey and Ryan (1994) also found that the continuous-time model is preferable to the discrete and mixed-time models for the reason that it gives reasonable estimates with relatively few intervals while still making full use of the available information. Continuous-time linear models are also found to be fitting well to irregularly spaced time series data (Jones, 1981; Jones and Ackerson, 1990).

Several other recent papers Brockwell et al. (1991), Tong and Yeung (1991), Brockwell and Hyndman (1992) and Brockwell (1994) have also applied continuous-time models, in modelling conveniently, irregularly spaced data when discrete-time linear models had been found inadequate. However, these differential equations models for lack of consideration on time lag in the modelling makes them less representative of the real situation. Thus the lack of time lags (delay) in these models make them quite different from the realistic problems under consideration.

2.5.3 Effects of Delay Differential Equations

For most farm produce there is a finite time which elapses before a change in production occurs. Consumers on the other hand take buying decisions based on the current market price. This time lag in production could be informed by several factors (Mackey, 1989) and therefore delay should be considered in mathematical modelling to evaluate its effects on production and price.

Delay Differential Equations arise in many applied economic models when traditional simplifications are abandoned for realistic hypotheses (Engelborghs et al., 2002). Delay differential equations have been widely used for many years in control theory and have recently become popular in biological and economical models. Walther (1981) used nonlinear delay differential equations and the results showed that there broad spectrum of dynamic behaviours that exist in DDEs. Meng-xing and Cui (2008) also developed a delayed mathematical model to assess price cooperation certain economic parameters so that when some conditions are satisfied, the existence and stability of periodic price are investigated.

In considering the dynamics of price, production and, consumption of commodities, Belar and Mackey (1989) and Mackey (1989) proposed the price fluctuation models and under relatively mild conditions, determined the stability of equilibrium price. Farahani and Grove (1992) and Cheng (2005) considered naive consumer models and studied the oscillation of equilibrium price.

Chapter 3

Methodology 1

3.1 Introduction

This chapter gives a review of mathematical models that are used in the study. These models will be compared with existing mathematical models in which knowledge gaps have been identified. The models may include the continuoustime linear and nonlinear cobweb models constructed from delay differential equations of price and also consider discrete-time cobweb models obtained from difference equations of price comprising linear and nonlinear.

3.2 Ordinary Differential Equations

Ordinary differential equations (ODEs) are mathematical relations containing one or more functions of one independent variable and its derivatives. They are continuous time equation which can be used to describe dynamically changing phenomena or systems and therefore when incorporating demand and supply functions, the price is given as function of time (Anthony, 1996). Most suppliers in reality commit themselves to the supply decision before they know the price of the good in question as opposed to the standard assumptions under the cobweb theory concerning demand and supply.

Thus ordinary differential equations are equations that relate functions with their derivatives. The functions in most applications, replicate physical quantities while their derivatives replicate rates of change of the quantity. The equation then relate the physical quantities to their rates of change. The role of differential equations in other disciplines cannot be overemphasize, they be could applied in engineering, physics, economics and biology because such phenomenas are common (Asano, 2012; Shapiro, 2011).

Most economic models with temporal dimensions involve relationship between values of variables at specific time and changes of the values over time. An economic growth model may typically contain a relationship between change in capital stock and its output value. When time is considered as continuous variable in function, then their relationship may be modeled as differential equation (Osborne, 2003).

3.2.1 General Definition of Ordinary Differential Equation

Given F, as function of y and t and the derivatives of y, then general form of ordinary differential equation may be written as:

$$F(y,y^{0},y^{00},\ldots y^{n},t) = 0 \tag{3.1}$$

which is also called implicit ordinary differential equation with order n. This implies that equation of the form:

$$F(y,y^{0},y^{00},\ldots y^{n-1},t) = y^{n}$$

may be called explicit ordinary differential equation with order n.

3.2.2 Linear Differential Equation

Differential equation can be said to be linear if F as function of the system is linear combination of derivatives of say x as:

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + ... + a_2(t)x^{00} + a_1(t)x^{0} + a_0(t) = g(t) \tag{3.2}$$

where each $a_i(t)$ is either zero, constant or dependent only on t, and not on x. It is important to note that linear differential equations have no products of the function $x(t)$ and its derivatives and the function or its derivatives occur to power first order. The coefficients $a_i(t)$ and $g(t)$ are continuous functions in t. The linearity of differential equations are determined only by the function $x(t)$ and its derivatives.

In particular, consider the following differential equation whose derivative depends on itself only linearly and having a single variable (say $x(t)$) given as:

$$\frac{dx}{dt} + p(t)x = q(t) \tag{3.3}$$

This can also be referred to as the standard form of the linear ODE or DE. For convenience, sometimes the equation could be written without the t (but the dependence will be implied) on the p and q and write the derivative with a prime as:

$$x^{0} + px = q \tag{3.4}$$

The Equation (3.4), could be solved by integrating factor technique (see for example Shapiro (2011)). Both sides of equation (3.4) is multiplied by integrating factor $\mu(t)$ such that the following equation is obtained:

$$\frac{d}{dt}(\mu x) = \mu x' + \mu p x = \mu q \tag{3.5}$$

Multiplying both sides by dt and integrating gives

$$\int \frac{d}{dt}(\mu(t)x)dt = \int \mu(t)q(t)dt \tag{3.6}$$

Therefore

$$\mu(t)x = \int \mu(t)q(t)dt + c \tag{3.7}$$

by dividing through (3.7) by $\mu(t)$, it could be seen that (3.5) is satisfied by the integrating factor $\mu(t)$ and therefore provide the general solution of equation (3.4) as:

$$x = \frac{1}{\mu(t)}[\int \mu(t)q(t)dt + c], \ where \quad \mu(t) = e^{(\int (p(t)dt))} \tag{3.8}$$

Systems of Differentiation Equations and Stability Analysis:

This section of the study discusses phase portrait of linear differential systems. The stability of equilibrium solutions for any given linear systems be it homogeneous or non-homogeneous will be classified according to shape and behaviour of their phase portraits.

Equilibrium Solution (Critical Point)

Critical point of the system of equations $y' = Ay$ may occur at (y_1, y_2) where $y' = 0, \ that \ is, \ y_1' = 0 = y_2'$. Thus the solution of the homogeneous system $Ay = 0$, gives only one equilibrium solution which occurs at the origin provided $det(A) \neq 0$, or infinitely many solutions ff $det(A) = 0$ (Tseng, 2008).

Example: Given $y' = Ay$, with (0,0) as the only critical point, then $y' = Ax$ is written as:

$$y_1' = a_{11}y_1 + a_{12}y_2, \quad y_2' = a_{21}y_1 + a_{22}y_2$$

For matrix $A = (a_{ij})$, with eigenvalues λ_1 and λ_2, and its corresponding linear eigenvectors $v^1 = (v_1^1, v_2^1)^t$ and $v^2 = (v_1^2, v_2^2)^t$, the general solution is given by

$$y(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}, \tag{3.8i}$$

with c_1 and c_2 as constants. In this case, the characteristic polynomial is also given by:

$$p(\lambda) = \lambda^2 - (a11 + a22)\lambda + \Delta = 0; \Delta = a_{11}a_{22} - a_{12}a_{21}:$$

If matrix A is real, then the characteristic equation has real coefficients. Its roots may be both real or complex conjugates: $\lambda_{1,2} = \rho \pm \sigma$, where ρ and σ are real. For complex conjugates, the eigenvectors v^1 and v^2 are supposed to be complex conjugates as well as constants c_1 and c_2, if the solution (3.8i) have to be real.

The case when eigenvalues are both real

When the two eigenvalues are negative, the solution decays to zero exponentially and the origin will be asymptotically stable. If one eigenvalues is zero and the other is negative, then the origin is stable but not asymptotically stable. On the other hand, if the system has at least one positive eigenvalues, then the origin becomes unstable. Suppose $\lambda_1 > 0$, then in for $c_2 = 0$ in equation (3.8i) and any non-zero value of c_1, the norm of the solution increases without bound which implies instability (Tseng, 2008).

The case when only one eigenvector v^1 exist: The general solution of equation (3.8i) will then be given by:

$$y(t) = [(c_1 + c_2t)v^1 + c_2v^2]e^{\lambda t}$$

If $\lambda > 0$, then the system is unstable and all trajectories diverge from the critical point to infinite-distant away or they converge to the critical point (when $\lambda < 0$). If $\lambda < 0$, then $te^{\lambda t}$ increases until it reaches its finite maximum where $||y(t)||$ is made arbitrarily small by choosing sufficiently small values for c_1 and c_2, to make the system asymptotically stable (Tseng, 2008).

The case of two complex eigenvalues:

In the case when the two eigenvalues are complex equation (3.8i) will have its general solution given by $y(t) = c_1e^{\lambda t}(acos(\mu t) - bsin(\mu t)) + c_2e^{\lambda t}(asin(\mu t) + cos(\mu t))$

Here $\lambda_{1,2} = \rho \pm \sigma$, and when the real part is nonzero, the system either grows or decays exponentially according to the term $e^{\lambda t}$ from the critical point. Therefore, the trajectories will spiral away from the critical point towards infinite-distant away when ($\lambda > 0$) or will spiral toward the critical point and converge when ($\lambda < 0$). This critical point is called spiral point and it is asymptotically stable when $\lambda < 0$, and unstable when $\lambda > 0$.

When λ is zero, the trajectories will neither converge towards critical point nor move to infinite-distant away and this critical point is called center and has a unique stability classification shared by no other: stable (or neutrally stable). (Tseng, 2008).

Nonhomogeneous Linear Systems with Constant Coefficients:

This section discusses stability condition of equilibriums points (critical points) of nonhomogeneous system as given by

$$y' = Ax + b. \qquad 3.8ii$$

where b is a constant vector. The system above is explicitly written as:

$$y_1' = ay_1 + by_2 + g_1$$
$$y_2' = cy_1 + dy_2 + g_2$$

Just as was done earlier with homogeneous system of differential equation, the critical point could be found if we let $y_1' = y_2' = 0$ and solve the resulting nonhomgeneous system of equations. Since the zero vector is not solution of the nonhomogeneous linear system, the origin is no longer the critical point. Instead, the unique critical point (if **A** has nonzero determinant) located at solution of the system of algebraic equations given by:

$$0 = ay_1 + by_2 + g_1$$

$$0 = cy_1 + dy_2 + g_2 \qquad 3.8iii$$

Suppose the critical point, say $(y_1,y_2) = (\alpha,\beta)$, is determined from the system then it could be moved to (0, 0) via translations $\gamma_1 = y_1 - \alpha$ and $\gamma_2 = y_2 - \beta$.

We obtain linear homogeneous system $\gamma' = A\gamma$ after translation. The two systems, before and after the translations should have same coefficient matrix and identical stability classifications for their respective critical points.

Thus if critical points, say $(y_1,y_2) = (\alpha,\beta)$ are plugged into equation 3.8iii, then

$$0 = a\alpha + b\beta + g_1$$

$$0 = c\alpha + d\beta + g_2$$

Now apply the translations $\gamma_1 = y_1 - \alpha$ and $\gamma_2 = y_2 - \beta$. Then $y_1 = \gamma_1 + \alpha$, $y_2 = \gamma_2 + \beta,\ \gamma_1' = yx_1'$, and $\gamma_2' = y_2'$. Substitute them into the system 3.8ii we have

$$y_1' = \gamma_1' = ay_1 + by_2 + g_1 = a(\gamma_1 + \alpha) + b(\gamma_2 + \beta) + g_1,$$

$$y_1 = a\gamma_1 + b\gamma_2 + (a\alpha + b\beta + g_1) = a\gamma_1 + b\gamma_2$$

$$y_2' = \gamma_2 = cy_1 + dy_2 + g_2 = c(\gamma_1 + \alpha) + d(\gamma_2 + \beta) + g_2\, y_2 = c\gamma_1 + d\gamma_2 + (c\alpha + d\beta + g_2) = c\gamma_1 + d\gamma_2$$

for the new variables γ_1 and γ_2, the given system has become

$$\gamma_1' = a\gamma_1 + b\gamma_2$$

$$\gamma_2' = c\gamma_1 + d\gamma_2$$

which is in the form of homogeneous system $\gamma' = A\gamma$, having critical points at the origin and same coefficient matrix **A**, compare to original system (Tseng, 2008).

Example Find the equilibrium point of the following system of differential equation and define its stability

$y_1^0 = y_1 - 2y_2 - 1\ y_2^0$

$= 2y_1 - 3y_2 - 3$

If (3, 1) is the critical point, when $y_1' = y_2' = 0$, then the characteristic polynomial $\lambda^2 + 2\lambda + 1 = 0$, gives repeated eigenvalue of $\lambda = -1$, and one linear eigenvector. This means at (3, 1) the system is asymptotically stable improper node.

Example 2 Define stability of the following system of differential equation, given critical point (1, 1):

$y_1^0 = -2y_1 - 6y_2 + 8\ y_2^0$

$= 8y_1 + 4y_2 - 12$ The solution matrix has characteristic equation given by $\lambda^2 - 2\lambda + 40 = 0$, that results in complex conjugate eigenvalues having positive real part $\lambda = 1$. At critical point (1, 1), the system becomes unstable spiral.

3.2.3 Nonlinear Differential Equation

Nonlinear differential equations are defined as any equations that cannot be written in the form of equation (3.2), and these include all equations that have y, y^0, y^{00}, etc., raised to any power (such as y^2 or $(y^0)^3$) or nonlinear functions of y or any derivative to any order such as $sin(y)$ or e^{ty} or any product of function of these. The order of differential equations are degree of the highest derivative contained in the equation (Shapiro, 2011).

Only few methods are available for solving nonlinear differential equations exactly and yet some of these methods require that the equations have particular symmetries before they could be applied . Nonlinear differential equations sometimes exhibit complicated behavior and even their existence, uniqueness and extendability are hard problems (Boyce and DiPrima, 1967). For qualitative analysis of nonlinear differential equations, the following common methods are applied;

- Linearization by means of Taylor's expansion.
- Changing of variables into something easier to study.
- Bifurcation theory.
- Perturbation theory (Boyce and DiPrima, 1967).

Lets us consider the following nonlinear differential equation (3.9) obtained from the traditional market equations of demand, $D_t = \alpha - \beta(p_t)$ and supply, $S_t = \delta - \gamma(p_{t-1}) - \rho(p_{t-1}^2)$ which is in stable state. This like all nonlinear differential equations has many ways of solving it. By transforming it to linear form, it could be solved using Taylor's expansion.

At equilibrium, it is assumed change in price is proportional to the difference between supply and demand functions. The lag operator is also applied so as to simplify the equation as given by:

$$p^0(t) = \lambda[c - bp + ap^2] \; or \; p^0(t) + bp - ap^2 = c \tag{3.9}$$

where $c = (\alpha - \delta)$, $b = (\beta - \gamma)$ and $a = \rho$. This equation has two fixed points (equilibrium points) obtained from solving

$$\lambda[c - bp + ap^2] = 0 \tag{3.10}$$

Therefore:

- if $\frac{b^2}{4a^2} - \frac{c}{a} > 0$ we have two equilibrium points, $p^*_{1,2} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$
- if $\frac{b^2}{4a^2} - \frac{c}{a} = 0$ only one equilibrium point exists, $p^*_{1,2} = \frac{b}{2a}$
- if $\frac{b^2}{4a^2} - \frac{c}{a} < 0$ then there are no equilibrium points.

Taking the first order Taylor expansion about the point p*, we have:

$$f(p) = f(p^*) + f^0(p^*)(p - p^*) \tag{3.11}$$

where $f^0(p^*) = -b + 2ap^*$ and for p* being a fixed point (equilibrium), $f(p^*) = 0$. On considering that $p* = p^*_1$, then:

$$f'(p^*_1) = -b + 2a\left(\frac{b \pm \sqrt{b^2 - 4ac}}{2a}\right) = \sqrt{b^2 - 4ac} \tag{3.12}$$

Then from equation (3.11):

$$f(p) = (p - p^*_1)\sqrt{b^2 - 4ac} \tag{3.13}$$

Therefore equation (3.13) is a first-order linear approximation about the this equilibrium and the linear approximate solution is given by:

$$p_t = p^*_1 + (p_0 - p^*_1)e^{t(\sqrt{b^2 - 4ac})} \tag{3.14}$$

Equilibrium and Stability Condition

It is found from the above solution that fixed point p_1^* is locally unstable, since $c = (\alpha - \delta)$, $b = (\beta - \gamma)$ and $a = \rho$ are all positive and so as $t \to \infty$ then $p_t \to \infty$ (Shone, 2002).

Similarly, the approximate solution of the differential equation at equilibrium point p_2^* is also given as:

$$p_t = p_2^* + (p_0 - p_2^*)e^{-t(\sqrt{b^2-4ac})} \tag{3.15}$$

where p_2^* is a locally stable equilibrium therefore as $t \to \infty$ then $p_t \to p_2^*$.

Now for the repeated equilibrium point $p_{1,2}^* = \frac{b}{2a}$

$$f'(p_{1,2}^*) = -b + 2a\left(\frac{b}{2a}\right) = 0 \tag{3.16}$$

For which no first-order linear approximations exist for this equilibrium point.

Equilibrium and Stability Condition of Nonlinear Systems:

In building on the eigenvalue analysis for linear systems, we consider the following nonlinear differential system of equations:

$$x' = P(x,y)$$

$$y' = Q(x,y)$$

where P and Q are functions of two variables: $x = x(t)$ and $y = y(t)$, but they are both not linear. Unlike linear systems, nonlinear systems could have more than one critical points. However, like the linear systems, if we let $x' = y' = 0$, then critical points can be found by solving the resulting systems;

$$0 = P(x,y)$$

$$0 = Q(x,y)$$

whose solution is critical points of the system of differential equations. On the phase portrait, are multiple critical points and each trajectory is influenced by more than one critical points which results in chaotic appearance of the phase portrait. Consequently, stability of each critical point is obtained locally on case-by-case basis (Tseng, 2008). Usually, linearizations of P and Q at critical points are used to approximate the behavior of nearby trajectories. In process the nonlinear system is converted to linear system, whose phase portrait approximates local behaviour of the original nonlinear system near the critical point (Tseng, 2008).

Generally, given that the systems P and Q have the critical points $(x,y) = (\alpha,\beta)$, one can determine stability of the system by linearization using first three (3) terms in the Taylor series expansion for each function as follows:

$$x^0 = P(x,y) \approx P(\alpha,\beta) + P_x(\alpha,\beta)(x-\alpha) + P_y(\alpha,\beta)(y-\beta)$$
$$y^0 = Q(x,y) \approx Q(\alpha,\beta) + Q_x(\alpha,\beta)(x-\alpha) + Q_y(\alpha,\beta)(y-\beta)$$

If (α,β) is critical point and $P(\alpha,\beta) = 0 = Q(\alpha,\beta)$, then above linearizations becomes;

$$x^0 \approx P_x(\alpha,\beta)(x-\alpha) + P_y(\alpha,\beta)(y-\beta)$$
$$y^0 \approx Q_x(\alpha,\beta)(x-\alpha) + Q_y(\alpha,\beta)(y-\beta)$$

and like the linear nonhomogeneous system, the critical point could be translated to (0, 0) and still maintains its type and stability using the substitutions $\chi = x-\alpha$ and $\gamma = y-\beta$. After translation, the approximated system becomes;

$$x^0 = P_x(\alpha,\beta)x + P_y(\alpha,\beta)y$$
$$y^0 = Q_x(\alpha,\beta)x + Q_y(\alpha,\beta)y$$

which is now homogeneous linear system with coefficient matrix:

$$\hat{\mathbf{A}} = \begin{bmatrix} P_x(\alpha,\beta) & P_y(\alpha,\beta) \\ Q_x(\alpha,\beta) & Q_y(\alpha,\beta) \end{bmatrix}.$$

Taking the first partial derivatives of the matrix and substituting $x = \alpha$ and $y = \beta$, the following Jacobian matrix is obtained;

$$\hat{\mathbf{J}} = \begin{bmatrix} P_x & P_y \\ Q_x & Q_y \end{bmatrix}.$$

and it has to be calculated once for each nonlinear system. For each critical point of the system, coefficient matrix of the linearized system about a given point $(x,y) = (\alpha,\beta)$, is computed and then use the resulted eigenvalues to approximate its behaviour and stability (Tseng, 2008).

Example: Given the following system of differential equations, determine critical points and its behaviour:

$$x' = x - y \qquad y' = x^2 + y^2 - 2$$

By setting $x' = 0 = y'$ and solving the system, the critical points found as (1, 1) and (-1,-1). The Jacobian matrix is also given by;

$$\hat{\mathbf{J}} = \begin{bmatrix} 1 & -1 \\ 2x & 2y \end{bmatrix}.$$

At point (1,1), the linearized system has coefficient matrix:

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

with eigenvalues $\hat{\lambda} = \frac{3 \pm \sqrt{7}i}{2}$ and the results show an unstable spiral point. Also at (-1,-1), the linearized system gives coefficient matrix:

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}.$$

with eigenvalues $\hat{\lambda} = \frac{-1 \pm \sqrt{17}}{2}$ and an unstable saddle point.

Example 2: Compute the critical points and define the behaviour of the equilibrium point of the following system of differential equations;

$$x' = x - xy$$
$$y' = y + 2xy$$

with critical points computed as (0,0) and (1/2,1), and Jacobian matrix given by;

$$\hat{\mathbf{J}} = \begin{bmatrix} 1 - y & -x \\ 2y & 1 + 2x \end{bmatrix}.$$

then point (0,0), the linearized system has the following coefficient matrix:

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The characteristic equation provides repeated eigenvalues $\lambda = 1$, and for this result, while linear system will have an unstable proper node (star point), nonlinear system will have an unstable node.

At critical point $(\frac{1}{2},1)$, the linearized system has the following coefficient matrix:

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}$$

which has eigenvalues $\lambda = \pm 1$, with results indicating unstable saddle point for the critical point (Tseng, 2008).

3.2.4 Initial Value Problems

If differential equation is given together with specified value (or an initial condition) of the unknown function for a given point in domain of the solution then we have an initial value problem. Thus the problem of finding solution for a function of say y of x when its derivative and initial value y_0 at a particular point say x_0 given, is an initial value problem (Wood, 2015).

In mathematics, any dynamic systems that are modelled usually amount to solving an initial value problem. This means that such systems specify how they will evolve with time given initial conditions.

Theorem 1: If the functions $p(t)$ and $g(t)$ are continuous in an interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists unique function $y = \varphi(t)$ that satisfies the differential equation:

$$\frac{dy}{dt} + p(t)y = q(t), \quad (refer\ to\ equation.\ 3.5)$$

for each $t \varepsilon I$. The unique solution must satisfy the initial condition $y(t_0) = y_0$ where y_0 is an arbitrary initial value. Generally, problem of initial value could be solved using either integrating factor approach (and write the solution as shown in equation 3.10) or method of separation of variables. If general solution of the differential equation can be determined then either method should work. Solution of initial values involves the following steps:

1. Finding general solution of the system.
2. Determining constant value(s) using the initial conditions and obtain the
3. Actual solution of the system.

Example: Given the following differential equation and initial condition;

$$\frac{dy}{dx} = xsin(x)y + \frac{xsin(x)}{y}, \quad y(0) = 1.$$

It implies that

$$\frac{dy}{dx} = xsin(x)(y + \frac{1}{y}),$$

$$\frac{dy}{dx} = xsin(x)(\frac{y^2+1}{y})$$

$$\int \frac{y}{y^2+1})dy = \int xsin(x)dx$$

By substitution LHS becomes

$$\frac{1}{2}\int \frac{1}{u}du = \int xsin(x)dx$$

$$\frac{1}{2}\ln|y^2+1| + c = \int xsin(x)dx$$

By integration by parts, the RHS also becomes

$$\frac{1}{2}\ln|y^2+1| + c = x(-cos(x)) - \int 1*(-cos(x))dx$$

$$\frac{1}{2}\ln|y^2+1| + c_1 = -xcos(x) + \int cos(x)dx$$

$$\frac{1}{2}\ln|y^2+1| + c_1 = -xcos(x) + sin(x) + c_2$$

$$\ln|y^2+1| = 2sin(x) - 2xcos(x) + c$$

$$y^2+1 = e^{2sin(x)-2xcos(x)+c}$$

$$y^2+1 = ke^{2sin(x)-2xcos(x)}$$

$$y = \pm\sqrt{ke^{2sin(x)-2xcos(x)} - 1}$$

To obtain general solution as:

$$y(x) = \pm\sqrt{ke^{2sin(x)-2xcos(x)} - 1}$$

For $y(0) = 1$, i.e. at $x = 0$, actual solution is given as:

$$y(0) = \pm\sqrt{ke^{2sin(0)-2xcos(0)} - 1}$$

$$1 = \pm\sqrt{ke^{0} - 1} = \pm\sqrt{k-1}$$

$$1 = k - 1$$

$$k = 2$$

$$y(x) = \sqrt{2e^{2sin(x)-2xcos(x)} - 1}$$

3.2.5 Existence and Uniqueness of Solutions

Sometimes initial value problems are solved and it is assumed that the solution is valid and is the only solution one may have, most especially if differential equations techniques are used to model physical systems. Therefore one must know that the differential equation has a solution before time and/or energy is spent trying to find it. As much as possible it is expected of the solution to be to be unique if the solution is to be a useful predictive tool (Lindblad, 2014).

This is an amazing theorem as it says not only that a solution exists, but that solution is unique. It also tells one where that solution exists, in the interval over which function is continuous and containing the initial point (Lindblad, 2014).

Linear Differential Equations:

Supposing the following problem is given to determine the interval within which the solution is valid;

$$xy^{0} + 2y = 4x^{2}; \qquad y(1) = 2$$

Then one divides through the function by x to obtain the following

$$y' + \frac{2}{x}y = 4x$$

Computing the integrating factor we have

$$\mu(x) = exp(\int \frac{2}{x}dx) = e^{2\ln x} = x^{2}$$

Multiply through by the integrating factor we obtain

$$x^{2}y^{0} + 2xy = 4x^{3}$$

Therefore

$$x^{2}y = x^{4} + c$$

$$y = x^{2} + \frac{c}{x^{2}}$$

Applying the initial condition, $c = 1$, and this provides the actual solution as follows:

$$y = x^{2} + \frac{1}{x^{2}}, \quad x > 0$$

This equation sought a solution in interval containing $x = 1$. Since the coefficients in the equation are continuous except at $x = 0$, then from theorem 1, the given initial value problem has solution that is valid at least in the interval $0 < x < \infty$.

However, if the initial condition is changed to $y(1) = 1$, then from the general solution, $c = 0$. Hence actual solution is given as:

$$y = x^2,$$

which is bounded and continuous even in the neighbourhood of $x = 0$.

Nonlinear Equations:

Nonlinear equations are noted to be more complicated to solve than linear ones (Lindblad, 2014).

Theorem 2: If the functions $f(x,y)$ and $\partial(f)/\partial(y)$ are continuous in some rectangle $R : |x| \le a$, $|y| \le b$ containing the point (x_0,y_0), then in some interval, $|x| \le h \le a$ contained in $|x| \le a$, there is a unique solution $y = \varphi(x)$ of the initial value problem:

$$y' = f(t, y), \quad y(x_0) = y_0 \tag{3.17}$$

Example Suppose that an initial value problem;

$$\frac{dy}{dx} = y^{\frac{1}{3}}, \quad y(0) = 0$$

is also given to find the interval of solution for for $x \ge 0$. By the separation of variables method

$$y^{-\frac{1}{3}}dy = dx$$

so

$$\frac{3}{2}y^{\frac{2}{3}} = x + c$$

$$y = [\tfrac{2}{3}(x + c)]^{\frac{3}{2}}$$

Then if $c = 0$, then the initial condition is satisfied and so

$$y(x) = \psi_1(x) = [\tfrac{2}{3}(x)]^{\frac{3}{2}}, \quad x \ge 0$$

The function on the other hand can be given as;

$$y(x) = \psi_2(x) = -[\tfrac{2}{3}(x)]^{\frac{3}{2}}, \quad x \ge 0$$

which also satisfies the initial value problem. Moreover, the following function;

$$y(x) = \varphi(x) = 0, \qquad x \geq 0$$

is yet another solution to the same initial value problem. For non-negative arbitrary value x_0, the piecewise functions;

$$y = \gamma(x) = \begin{cases} 0, & if \quad 0 \leq x < x_0 \\ \pm[\frac{2}{3}(x)]^{\frac{3}{2}}, & if \quad x \geq x_0, \end{cases},$$

are continuous and differentiable particularly (at $x = x_0$), and are also solutions to the initial value problem. This means the problem has multiple solutions which illustrates the troublesome nature of initial value problems equations are nonlinear. The singularity of the solution is dependent on the differential equation and its initial conditions.

One important distinction of a linear initial-value problem from a nonlinear one is the fact that the size of interval for existence of solution to linear initial-value problem does not depend on the initial value of the dependent variable. Instead, it depends only on the points of discontinuity of the coefficient function $p(t)$ and the non-homogeneous term $g(t)$. This is often referred to as the statement that linear problems have fixed singular points (Kwa, 2011). On the other hand, the size and endpoints of interval of existence for solution to nonlinear initial-value problem are also dependent on the initial value of the unknown function. This is often referred to as the statement that nonlinear problems have movable singular points (Kwa, 2011).

Method of Successive Approximation:

This method is used to solve and approximate solutions of differential, integral and integro-differential equations and also prove the existence of the solutions (Boyce and DiPrima, 1967). Other uses of method of successive approximation include, obtaining of a qualitative characterization of a solution. It is one of the mathematical methods of solving problems by means of sequence of approximations that converge. The sequence is constructed recursively such that each new approximation is calculated on the basis of the preceding approximation.

From theorem 2, it is assumed that there is function $y = \varphi(x)$ that satisfies the initial value problem, and $f[x,\varphi(x)]$ is continuous function of x only. Hence one can integrate $\frac{dy}{dt} = f(t,y)$), from the initial point $x = 0$ to an arbitrary value of x, to obtain;

$$\phi(x) = \int_0^x f[x,\phi(x)]dt \qquad 1.19a,$$

where the initial condition $\varphi(0)$ has been used.

There is another relation satisfied by the solution of the initial value problem $y' = f(x,y)$, $\qquad y(0) = 0$, which was provided by the integral equation.

Conversely, if integral equation 1.19a is satisfied by the continuous function $y = \varphi(x)$, then the same function also satisfies the initial value problem. One can prove this by substituting $x = 0$ in equation 1.19a, and thus obtain $y(0) = 0$. Furthermore, it is known that the integrand in equation 1.19a is continuous and therefore $\varphi' = f[x,\varphi(x)]$.

In conclusion, one can conveniently say that the initial value problem is equivalent to integral equation in the sense that, any solution of one function is also solution of the other function (Boyce and DiPrima, 1992).

Now by the application of the method of successive approximation it will be shown that integral equation 1.19a has a unique solution starting with the initial function φ_0, either arbitrarily or approximating the initial value problem.

If one chooses that:

$$\varphi_0(x) = 0$$

then presumably φ_0 satisfies the initial condition $y(0) = 0$, and not the differential equation $y' = f(x,y)$.

From equation 1.19a, $\varphi_1(x)$ is obtained from $\varphi_0(x)$ as given by;

$$\phi_1(x) = \int_0^x f[x,\phi_0(x)]dt$$

Similarly,

$$\phi_2(x) = \int_0^x f[x, \phi_1(x)]dt$$

and so in general

$$\phi_{n+1}(x) = \int_0^x f[x, \phi_n(x)]dt$$

Generally, each sequence of functions $\varphi_n(x) = \varphi_0(x), \varphi_1(x), \varphi_2(x), ..., \varphi_n(x)...$, satisfies the initial condition $y(0) = 0$, but none satisfies the differential equation. If it is found in the process that $\varphi_{k+1}(x) = \varphi_k(x)$, for $n = k$, then $\varphi_k(x)$ is the solution of equation 1.19a and the initial value problem, therefore we terminate the sequence at this point. However, it is very difficult for this to occur and so the entire infinite sequence of piecewise functions are considered and checked if every member of the sequence exist, do not break down or interrupt in the process at any stage and/or the sequences converge. Then at this stage, one can find unique function and check whether its limit properties satisfy the integral solution (1.19), as well as the initial value problem (Boyce and DiPrima, 1992).

Example Solve the following problem using method of successive approximation;

$$y' = 2x(1 + y), \qquad y(0) = 0.$$

The following integral equation holds if $y = \varphi(x)$;

$$\phi(x) = \int_0^x 2t[1 + \phi(t)]dt, \qquad 1.19b.$$

Given that $\varphi_0(x) = 0$, it follows that;

$$\phi_1(x) = \int_0^x 2t[1 + \phi_0(t)]dt = \int_0^x 2tdt = x^2.$$

Similarly,

$$\phi_2(x) = \int_0^x 2t[1 + \phi_1(t)]dt = \int_0^x 2t[1 + t^2]dt = x^2 + \frac{x^4}{2}.$$

and

$$\phi_3(x) = \int_0^x 2t[1 + \phi_2(t)]dt = \int_0^x 2t[1 + t^2 + \frac{x^4}{2}]dt = x^2 + \frac{x^4}{2} + \frac{x^6}{2.3}.$$

Equating $\varphi_1(x)$,$\varphi_2(x)$ *and* $\varphi_3(x)$ suggest that

$$\phi_n(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots + \frac{x^{2n}}{n!}.$$

for each $n \geq 1$, using mathematical induction. We must now find out if $\varphi_n(x)$ is true for $n = k$, and $n = k + 1$.

Then it follows that

$$\phi_{k+1}(x) = \int_0^x 2t[1 + \phi_k(t)]dt$$

$$\phi_{k+1}(x) = \int_0^x 2t[1 + t^2 + \frac{x^4}{2!} + \ldots + \frac{t^{2k}}{k!}]dt$$

$$= x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots + \frac{x^{2k+2}}{(k+1)!}.$$

Hence proof of induction is complete and that $\varphi_n(x)$ is the *nth* partial sum of the infinite series;

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k!}, \qquad 1.19c$$

The $\lim_{n\to\infty}\varphi_n(x)$ exists if and only if the series 1.19c converges. By applying the ratio test, it is seen that for each x;

$$\left|\frac{x^{2k+2}}{(k+1)!}\frac{k!}{x^{2k}}\right| = \frac{x^2}{k+1} \to 0, \qquad \text{as } k \to \infty$$

thus 1.19c converges for all x, and this confirms that;

$$\hat{\phi(x)} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k!}, \qquad 1.19c$$

is solution of the equation 1.19b, and the fact that x is defined in interval on the x-axis this function can be differentiated or integrated term by term (Boyce and DiPrima, 1992).

Uniqueness of the Solution:

For uniqueness of the function 1.19c, it is assumed that $\varphi(x)$ *and* $\psi(x)$ are solutions of the initial value problem and both satisfy the integral equation 1.19b. Then by subtraction

$$\phi(x) - \psi(x) = \int_0^x 2t[\phi(t) - \psi(x)]dt,$$

Taking absolute values of both sides, then for $x > 0$,

$$|\phi(x) - \psi(x)| \leq \int_0^x 2t|\phi(t) - \psi(x)|dt,$$

If x is made to lie within $0 \leq x \leq \hat{A}/2$, where $\hat{A}$ is arbitrary, then $2t \leq \hat{A}$, and

$$|\phi(x) - \psi(x)| \leq \hat{A}\int_0^x 2t|\phi(t) - \psi(x)|dt, \quad 1.19d,$$

At this point a function $\hat{U}$ is introduced and this function defined by

$$\hat{U}(x) = \int_0^x |\phi(t) - \psi(x)|dt,$$

It follows that

$$\hat{U}(0) = 0, \qquad 1.19e,$$

$$\hat{U}(x) \geq 0, \quad for \quad x \geq 0, \qquad 1.19f$$

Furthermore, $\hat{U}$ is differentiable and $\hat{U}'(x) = |\varphi(x) - \psi(x)|$. Hence by 1.19d

$$\hat{U}(x) - A\hat{U}(x) \leq 0 \qquad 1.19g,$$

By multiplying equation 1.19g by non-negative quantity e^{-Ax}, gives

$$[e^{-Ax}\hat{U}(x)]' \leq 0 \qquad 1.19h,$$

Then, by integrating equation 1.19h with limit $[0,x]$, and using equation 1.19e, we have

$$e^{-Ax}\hat{U}(x) \leq 0 \qquad for\ x \geq 0,$$

Hence $\hat{U}(x) \leq 0$ *for* $x \geq 0$, but from equation 1.19g $\hat{U}(x) = 0$; *for each* $x \geq 0$, which implies $U'(x) \equiv 0$ and therefore $\psi(x) \equiv \varphi(x)$ contradicts the hypothesis, and for that matter the initial value problem cannot have two different solutions (Boyce and DiPrima, 1992).

3.2.6 Difference Equations

Discrete-time systems are described by difference equations whereas continuoustime systems are described by differential equations. Difference equation constructs relations between an input signal and an output signal at discrete-time interval. Usually the output signal is calculated based on the past and current input signals (CTMS, 2015).

According to Neusser (2012) difference equation (also known as dynamical system) describes evolution of some variable denoted by X_t over time index t that takes on discrete values and typically runs over all integers Z. By interpretation, t as time index introduces concept of past, present and future which makes difference equation a function that specifies how to compute the value of variable of interest at period t given past values of the variable and time. In most cases the system may be initialized at some point $t_0 = 0$, so that t runs over all natural numbers, i.e. $t \varepsilon N$.

3.2.7 General Definition of Difference Equation

Given F, as function of x and t, the general form of difference equation can be written as:

$$\hat{F}(X_t, X_{t-1}, X_{t-2}, \ldots, X_{t-p}, t) = 0 \quad (3.18)$$

where the variable $\hat{X}_t$ is endogenous or dependent variable and is n-vector, i.e

$\hat{X}_t \varepsilon R^n, n \geq 1$, with n as dimension of the system. The difference between the largest and the smallest time index of dependent variable explicitly involved is order of the difference equation. Therefore in equation (3.18), p is the order. If time index appears as argument of the function F, then it is non-autonomous equation. On the other hand, if time is not a separate argument and enters only as a time index of the dependent variable, then the equation is said to be autonomous or time-invariant (Neusser, 2012).

3.2.8 Linear Difference Equation

A linear difference equation, in particular first order homogeneous difference equation normally takes the general form;

$$aq_n + bq_{n-1} = 0 \tag{3.19}$$

where q_n, with $n = 0,1,2,3,...$ are unknown and a and b as fixed constants. Equation (3.19) is solved by rewriting it as;

$$q_n = (\frac{-b}{a})q_{n-1} = \alpha q_{n-1}$$

with $\alpha = (\frac{-b}{a})$. The equation is now solved recursively as follows

$$q_n = \alpha q_{n-1} = \alpha(\alpha q_{n-2}) = \alpha_2 q_{n-1} = ... = \alpha_n q_0$$

If the initial condition at time 0 is given as q_0, then state of the system at time n is given by $q_n = \alpha^n q_0$. Therefore, general solution of equation (3.19) is given by;

$$q_n = C(\frac{-b}{a})^n$$

First order inhomogeneous equation:

In considering another linear equation of the form;

$$aq_n + bq_{n-1} = c_n \tag{3.20}$$

where c_n is a given sequence and q_n is unknown. For this equation, one for instance may take $c_n = c$; $c_n = cn$; $c_n = c\alpha^n$;, and is called inhomogeneous equation because of the term c_n. In solving such equations the following simple fact may be useful.

Linearity principle: Supposing $\widehat{q_n}$ is a solution of the inhomogeneous $aq_n + bq_{n-1} = c_n$, and q_n a e solution of the homogeneous equation $aq_n + bq_{n-1} = 0$, then $\widehat{q}_n + \widetilde{q}_n$ is a solution of the inhomogeneous equation $aq_n + bq_{n-1} = c_n$. Indeed we have

$$a\widehat{q}_n + b\widehat{q}_{n-1} = c_n$$

$$aq_n + q_{n-1} = 0 \text{ e}$$

e

and thus adding the two equations give

$$a(\widehat{q}_n + \widetilde{q}_n) + b(\widehat{q}_{n-1} + \widetilde{q}_{n-1}) = c_n$$

The general solution of a first order inhomogeneous equation could obtained using the following three steps;

1. finding solution of the homogeneous equation
2. finding particular solution of the inhomogeneous term and
3. finding general solution of the inhomogeneous equation by adding up solutions of the steps 1 and 2 above.

Example : Solve the following inhomogeneous equation where the term $c_n = c$,

$$aq_n + bq_{n-1} = c$$

First the particular solution is computed, and since c is a constant, one can let $q_n = d$, so we will find

$$ad + bd = c; \quad \text{or} \quad d = \frac{c}{a+b}$$

Then from equation (3.19), general solution of the inhomogeneous difference equation is obtained by;

$$q_n = C(\frac{-b}{a})^n + \frac{c}{a+b}$$

Equilibrium and Stability Analysis:

The stability of inhomogeneous difference equation is dependent on the absolute value of $\alpha = (\frac{-b}{a})$. If $\alpha <$ 1, then system is stable and converges towards equilibrium point $\frac{c}{a+b}$. Else it will diverge form the equilibrium point $\frac{c}{a+b}$ if the absolute value of α > 1. However, if the absolute value of α = 1, in system will oscillate in a period-2 cycle in between two equilibrium points (Neusser, 2012).

3.2.9 Nonlinear Difference Equation

Stability condition of linear dynamic systems are fairly easy to determine but the same cannot be said about nonlinear systems, since they can create complex cycle phenomena. The complexity of nonlinear systems is illustrated using the following quadratic equation;

$$y_{t+1} = ay_t - by_{t2}$$

First the fixed points of the quadratic system has to be established, and these are computed as follows;

$$\hat{y}^* = ay^* - by^{*2} = ay^*(1 - \frac{by^*}{c})$$

and the points given by;

$$\hat{y}^* = 0 \quad \text{and} \quad y^* = \frac{a-1}{b}$$

Equilibrium and Stability Analysis:

From results above, we now determine whether the fixed points are stable or unstable by using the following approach on the assumption that y^* is equilibrium point of the nonlinear difference equation;

$$\hat{y}_{t+1} = f(y_t) \quad \text{or} \quad \hat{f}(y_t) = ay_t - by_t^2$$

where $\hat{f}$ is continuously differentiable at $\hat{y}^*$. By taking derivative of the nonlinear difference equation we observe that;

1. If $|\hat{f}^0(y^*)| < 1$, then $\hat{y}^*$ is asymptotically stable ot attractor.

2. If $|\hat{f}'(y^*)| > 1$, then $\hat{y}^*$ is unstable or repellor

3. If $|\hat{f}'(y^*)| = 1$, then $\hat{y}^*$ is in oscillation between two equilibrium points.

Example: Let us consider the stability condition of the following nonlinear system;

$$\hat{y}_{t+1} = 2y_t - y_t^2$$

The fixed points can be found from

$$\hat{a} = 2\hat{a} - \hat{a}^2$$

$$\hat{a}^2 - \hat{a} = 0$$

$$\hat{a}(\hat{a} - 1) = 0$$

$$\hat{a} = 0 \quad \hat{a} = 1$$

To determine stability of the system, let

$$\hat{y} = f(x) = 2x - x^2$$

then for

$$\hat{f}'(x) = 2 - 2x$$ we have

$$\hat{f}'(0) = 2 \quad \text{and} \quad \hat{f}'(1) = 0$$

since

$$|\hat{f}'(0)| > 1, \quad \hat{a} = 0 \text{ is unstable}$$

and since

$$|\hat{f}'(1)| < 1, \quad \hat{a} = 1 \text{ is stable}$$

Many researchers have conducted research using these existing models, for example researchers like Farahani and Grove (1992) and Cheng (2005) developed naive consumer models and studied the oscillation of equilibrium price base on ordinary differential equations. Most of these economic models have utterly ignored the ability of delays in generating economic fluctuations. The lack of time lags (delays) in these models makes them quite different from the realistic problem they seek to model. Therefore delay-differential equation is introduced in the study to deal with this gap (Eduardo and Gergely, 2013; Howroyd and Russel, 1984).

Meanwhile, above existing differential equation models would only help one to grasp the concept of delay differential equation models which would be applied directly in this study to model demand and supply as they contain delay parameter which reflects supply response to market changes.

3.3 Delay Differential Equations

In mathematics, delay differential equations (DDEs) are defined as type of differential equation in which derivative of the unknown function at certain time is given in terms of values of the function at previous times. Thus in DDEs there are constraints that its time evolution can only depend on specific past values of the state variable at discrete or continuous times. They arise in many applied models when traditional simplifications are abandoned for more realistic assumptions (Engelborghs et al., 2002). Delay differential equations are also important class of dynamical systems and according to Roussel (2005), they are applied in either natural or technological control systems. In these systems, a controller monitors the state and makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action.

DDEs belong to class of systems with functional state, i.e. partial differential equations (PDEs) which are infinite dimensional as opposed to ordinary differential equations (ODEs) having a finite dimensional state vector (Jean-Pierre, 2003).

Generally, delay differential equation could be written in the form:

$$y^0(t) = f(y(t),y(t-\tau_1),y(t-\tau_2),...,y(t-\tau_n)) \tag{3.21}$$

where the quantities τ_i are positive constants which define fixed discrete delays. There are other DDEs, notably equations with state-dependent delays where τ_i^0s depend on y or equations with distributed delays where the right-hand side of the differential equation is weighted integral over past states.

Unlike the finite-dimensional dynamical systems where initial conditions are given and a small set of numbers specified; namely the initial values of the state variables and perhaps the initial time in non-autonomous systems, to solve delay differential equation more is required. One has to provide an initial function which denotes behavior of the system prior to time (say 0, assuming that we start at $t = 0$). The function has to cover a period as long as the longest delay, since we will be looking back to earlier values of say y in time that far (Roussel, 2005).

In other words, dynamics induced by delay equation could be understood if it is considered in same terms as done in differential equations, where the solution consists of sequence of y values at increasing values of t. However, from theoretical perspective, solution of DDE is mapping from functions on interval say $[t-\tau,t]$, into functions on the interval $[t,t+\tau]$ through stepwise fashion using the method of steps (Roussel, 2005).

For instance in considering delay differential equation with single delay given by:

$$y^0(t) = f(y(t),y(t-\tau)) \tag{3.22}$$

we will need initial condition defined by $\varphi : [-\tau,0] \to R^n$, so that solution on interval $[0,\tau]$ will also be given by $\hat{\psi}(0)$ which is solution to the inhomogeneous initial value problem:

$$\hat{\psi}^{0}(t) = f(\hat{\psi}(t), \hat{\varphi}(t-\tau)) \qquad (3.23)$$

where $\hat{\psi}(0) = \hat{\varphi}(0)$. The procedure can be continued for the successive intervals using solution of the previous interval as inhomogeneous term. In practice initial value problem for DDEs are solved numerically (Roussel, 2005; Rodney, 1977; Wim and Silviu-Iulian, 2007).

Generally, supposing $y^{0}(t) = f(y(t), y(t-\tau))$ and $\hat{\varphi}(t) = 1$, then DDE initial value problem is solved by integration as given by:

$$y(t) = y(0) + \int_{s=0}^{t} (\tfrac{d}{dt} y(s)) ds = 1 + a \int_{s=0}^{t} (s - \tau) ds$$

which implies that $y(t) = at+1$, when the initial condition is set at $y(0) = \hat{\varphi}(0) = 1$.

Similarly, on interval $t\varepsilon[\tau, 2\tau]$, one integrates and fits the previous condition such that:

$$y(t) = y(\tau) + \int_{s=\tau}^{t} (\tfrac{d}{dt} y(s)) ds = (at+1) + a \int_{s=\tau}^{t} (a(s-\tau)+1) ds$$

$$y(t) = (at+1) + a \int_{s=0}^{t-\tau} (as+1) ds$$

$$y(t) = (a\tau+1) + a(t-\tau)(\tfrac{a(t-\tau)}{2} + 1)$$

Example:

We can now use the preceding method (method of steps) to solve DDE initial value problem where interval of the history function is specifically defined:

$y^{0}(t) = x(t-1)$, *for* $t > 0$ and history function

$y(t) = 1, \quad for -1 \leq t \leq 0$

The Method of Steps mimics the Method of Successive Approximation and it's applied as follows. The method involves solving the equation on one interval at a time.

On the first interval [0,1],

$$\int_0^t y'(s)ds = y(s)|_0^t = y(t) - y(0)$$

This implies that

$$y(t) = y(0) + \int_0^t y'(s)ds$$

If $y'(t) = y(t-1)$, then so is $y'(s) = y(s-1)$.

$$y(t) = y(0) + \int_0^t y(s-1)ds$$

On the interval $0 \le s \le 1$, also means $-1 \le s-1 \le 0$, so that history function $y(s-1) = 1$, on this interval

$$y(t) = y(0) + \int_0^t 1ds$$

$$y(t) = 1 + s|_0^t$$

$$y(t) = 1 + (t-0)$$

$$y(t) = 1 + t$$

which is the first piecewise function obtained on the interval [0,1]. The solution on this interval is used to find next solution on [1,2],

$$\int_1^t y'(s)ds = y(s)|_1^t = y(t) - y(1)$$

To get

$$y(t) = y(1) + \int_1^t y'(s)ds$$

From the solution [0,1], $y(1) = 1 + 1 = 2$, and $y'(s) = y(s-1)$, and so

$$y(t) = 2 + \int_1^t y(s-1)ds$$

On the interval $1 \le s \le 2$, also means $0 \le s-1 \le 1$, so that history function $y(s-1) = 1 + (s-1)$, on this interval

$$y(t) = 2 + \int_1^t [1 + (s-1)]ds$$

$$y(t) = 2 + \int_1^t sds$$

$$y(t) = 2 + \tfrac{1}{2}s^2|_1^t$$

$$y(t) = 2 + \tfrac{1}{2}t^2 - \tfrac{1}{2}$$

$$y(t) = \tfrac{3}{2} + \tfrac{1}{2}t^2$$

To find next solution on [2,3],

$$\int_2^t y'(s)ds = y(s)|_2^t = y(t) - y(2)$$

To get

$$y(t) = y(2) + \int_2^t y'(s)ds$$

From the solution [1,2], $y(2) = \frac{3}{2} + \frac{1}{2}(2)^2 = \frac{7}{2}$, and $y^0(s) = y(s-1)$, and so

$$y(t) = \frac{7}{2} + \int_2^t y'(s-1)ds$$

On the interval $2 \le s \le 3$, also means $1 \le s - 1 \le 2$, so that history function $y(s-1) = \frac{3}{2} + \frac{1}{2}(s-1)^2$, on this interval

$$y(t) = \frac{7}{2} + \int_2^t [\frac{3}{2} + \frac{1}{2}(s-1)^2]ds$$

$$y(t) = \frac{7}{2} + \int_2^t \frac{3}{2} + \frac{1}{2}(t^2 - 2t + 1)ds$$

$$y(t) = \frac{7}{2} + [\frac{3}{2}t + \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{2}t]|_2^t$$

$$y(t) = \frac{7}{2} + [2t - \frac{1}{2}t^2 + \frac{1}{6}t^3|_2^t$$

$$y(t) = \frac{1}{6} + 2t - \frac{1}{2}t^2 + \frac{1}{6}t^3$$

and this process continues until a unique solution is obtained, then the process is terminated. This function is supposed to be converging one and satisfies the properties of limits in continuity. The difficulty of this process makes it easy to see the value of software that can solve DDEs numerically.

Continuity Analysis:

Continuity analysis is done to ascertain whether piecewise functions obtained by the method of steps are continuous. Limits analysis will be used to verify if they are also defined in the interval within which they are derived. Given that;

$$y(t) = \begin{cases} 1 + t, & 0 \le t \le 1 \\ \frac{3}{2} + \frac{1}{2}t^2 & 1 \le t \le 2, \\ \frac{1}{6} + 2t - \frac{1}{2}t^2 + \frac{1}{6}t^3 & 2 \le t \le 3, \end{cases}$$

$$\lim_{t \to 1^+} ((1 + t)) = 1 + 1 = 2,$$

$$\lim_{t \to 1^-} (\frac{3}{2} + \frac{1}{2}t^2) = \frac{3}{2} + \frac{1}{2}(1) = 2$$

These results prove that the first two functions have passed continuity test because left hand limit is equal to right hand limit in the interval [0, 2].

Also in the interval [2, 3];

$$\lim_{t \to 2^+} (\frac{3}{2} + \frac{1}{2}t^2) = \frac{3}{2} + \frac{1}{2}(2) = \frac{7}{2},$$

$$\lim_{t \to 2^-} (\frac{1}{6} + 2t - \frac{1}{2}t^2 + \frac{1}{6}t^3) = \frac{1}{6} + 4 - 2(\frac{8}{6}) = \frac{7}{2},$$

These results also prove that the second and third piecewise functions are continuous in the given interval and one can conclude that the functions are continuous and defined in each interval.

Specifically, if p denotes the price of a certain article and $S = S(p)$ and $D = D(p)$ denote the supply and demand functions or curves of price for that product, respectively, then as the price increases, so also does the supply, whereas as the price increases the demand decreases. The price $p = p^*$ where the two curves intersect is where $S(p^*) = D(p^*)$ and is called market equilibrium. A class of behavior related to this equation is known as cobweb phenomena (Ezekiel, 1938; Gettrick, 2013; Goldberg, 1996; Aarts, 2014).

3.3.1 Cobweb Theorem

The cobweb theorem purports to explain persistent fluctuations of prices of agricultural produce in some selected agricultural markets. The theorem was developed in the 1930s under the assumption of static price expectations where the predicted price equaled actual price in the last market period (Pashigian, 2008).

Thus cobweb model is an economic model that explains reasons for fluctuations in prices and quantities or supplies that occur in some markets. This model is affected by time expectations of prices are formed, since fluctuations in prices and quantities are influenced by way the price expectations are adapted .

This phenomenon of market behaviour is explained by Kaldor (1934) in agricultural market setting such that, If weather conditions for instance are not favourable during a year, then quantity supplied of a certain crop will be quite small and creates excessive demand or shortage to cause prices to be unusually high. In anticipation that prices would remain high farmers plant more so as to supply more the following year. If supply happens to be so high, prices decrease to meet consumers' demand. When farmers realize that prices are low, they cut down supply the following year, resulting in high prices again.

This cycle, also known as cobweb phenomena continues until equilibrium is reached after few fluctuations. Thus in every unstable market situation, equilibrium is reached when in absolute terms, the marginal demand is higher than the marginal supply (Kaldor, 1934).

3.3.2 Price Expectations

The future expectations or beliefs of economic agents have direct influence on their decisions taken today. Predictions of Producers of farm produce and Investors about future commodity or stock prices for instance may affect commodity or financial market movements today (Hommes, 2013). A classic example is ”dotcom bubble”, the rapid run up of stock prices in financial markets worldwide in late 1990s and their subsequent crash. The rise in stock prices was in anticipation of favourable economic fundamentals such as new communication technology and the internet. An over-optimistic estimate of future growth of information and communication technology (ICT) industries seemed to have contributed and strongly reinforced stock prices to rise excessively in the year 1995 to 2000, leading to extreme stock markets valuation worldwide and their subsequent fall in the year 2000 to 2003 (Galbraith and Hale, 2004; Keith et al., 2010; Gaither, 2006).

Another recent expectations driven crisis is the year 2008 to 2012 financialeconomic crisis. It is hard to believe that the decline of worldwide financial markets in 2008 of more than fifty percent (50%) was completely driven by changes in economic fundamentals. Rather, the large decline was strongly amplified by pessimistic expectations and market psychology. The predictions, expectations or beliefs of consumers and producers about future state of the economy are part of " the law of motion". The beliefs or expectations of consumers and producers are among contributing variables that describe how the economy evolves over time. Therefore, the theory of expectations is crucial component of any dynamic economic model since market economy is a highly nonlinear expectations feedback system (Hommes, 2013).

Static or Naive Expectation

The naive expectation hypothesizes that the most reliable forecast of future price is current price. This price expectation condition disregards any possible producer knowledge of future supply or demand shifts and its effects on price. The naive expected price variance and covariance for period t are formed using the squared difference between lagged price and the mean price of data from sample beginning through period $t-1$. Naive price expectations are widely used in early literature of researches (Ezekiel, 1938; Hommes, 2013).
If producers apply naive expectations, then in context of cobweb model, their prediction equals last observed price and the equation takes the form:

$$p_t^e = p_{t-1} \tag{3.24}$$

When supply is increasing and demand is decreasing, then provided they are bounded, there are only two conditions concerning price dynamics in regards to the ratio of marginal supply over marginal demand at steady state (i.e. when demand and supply intersect):

- If $|S^0(p^*)/D^0(p^*)| < 1$, then the steady state is (locally) stable and price converges to p^*.
- If $|S^0(p^*)/D^0(p^*)| > 1$, then the steady state is unstable and price diverges from p* and converges to a stable 2-cycle.

Thus due to nonlinearity of the supply curve, prices converge to stable 2-cycle price fluctuations (Ezekiel, 1938; Hommes, 2013; Irma et al., 1999).

Adaptive Expectations

Adaptive expectations in economics is a condition under which stakeholders form price expectations about future based on what has happened in the past. This economic theory attach much importance to past events in predicting future (Evans and Honkapohja, 2001). Thus an adaptive expectation states that if price of commodities (or inflation) for instance has been higher in the past, then stakeholders are most likely to use this as reference to revise expectations for the future.

Adaptive price expectation is stated in the following equation based on the theory of cobweb modeling, where p^e is next year's price of goods that is currently expected, p^e_{t-1} this year's price of the good that was expected last year and p this year's actual price of the commodity (or rate of inflation).

$$p^e = p^e_{t-1} + \lambda(p_{t-1} - p^e_{t-1}) \quad (3.25)$$

where λ is between 0 and 1. This condition states future price expectations formed currently in respect to past expectations and an error-adjustment of which current expectations are raised (or lowered) according to the gap between actual and previous expectations. The error-adjustment is known as partial adjustment. In other words, expected price is the weighted average of previous price and previous expected price (Nerlove, 1958; Evans and Honkapohja, 2001).

The theory of adaptive expectations is applicable to all previous periods and so current price expectations can be equally given as:

$$p^e = (1-\lambda)\Sigma_{j=0}^{\infty}(\lambda^j p_j) \qquad (3.26)$$

where p_j is the actual price and j the years in past. The current expected price of commodities is the weighted average of all past prices at the market. Usually, the weights get smaller and smaller as one moves further in the past. Due to stochastic shock, once forecasting error is made, agents will be unable to forecast price correctly again even when prices experience no further shocks since part of their errors is incorporated in the forecast (Nerlove, 1958; Evans and Honkapohja, 2001). Nerlove (1958) also stressed that agents form price expectations using recent past data and these lead to over or under supply due systematic forecast errors made in doing so, and they are considered endogenous fluctuations in the cobweb model.

Rational Expectations Theory

The rational price expectations make readily available better forecast rules when observations are made that adaptive expectations or any other fixed-weight distributed lag formula have provided poor forecasts in certain contexts. Rational expectations hypothesize that future predictions of economically relevant variables are not systematically wrong and so errors that appear in the predictions are random (Muth, 1961; Sargent, 1987; Savin, 1987; Evans and Honkapohja, 2001). Alternatively, the expectations are model-consistent and that future prices as predicted by the model are always considered valid.

It is assumed that on average, expectations made by agents may be correct but individually wrong because forecasts do not differ steadily from the market equilibrium results and that any deviations observed are only random. In a typical economic model, rational expectations are modelled on assumption that expected value of a variable is equal to the expected value predicted by the model (Muth, 1961; Sargent, 1987; Savin, 1987).

But how can agents have rational expectations or perfect foresight in a complex, nonlinear world, when the true law of motion is unknown and prices and quantities move irregularly on a strange attractor exhibiting sensitivity to initial conditions?

Rational farmers would discover the regular cyclic pattern in prices, learn from their systematic mistakes, change expectations accordingly and the hog cycle would disappear, so the argument goes. In rational expectations equilibrium, agents use economic theory, and compute their expectations as the conditional mathematical expectation derived from the market equilibrium equations. Therefore, in cobweb model, by taking conditional mathematical expectations on both sides of the market equilibrium, one derives that the rational expectations forecast is exactly given by the steady state price p^*. Thus in context of cobweb model without uncertainty (i.e., $\varepsilon_t \equiv 0$), the forecast is given by:

$$p^e{}_t = p^* \tag{3.27}$$

This assumed to be always exactly right and the rational expectations coincides with perfect foresight. In a noisy cobweb world with uncertainty, the rational expectations forecast $p^e{}_t = p^*$ will be correct on average so that agents make no systematic mistakes, since forecasting errors are proportional to the exogenous random demand shocks ε_t (Muth, 1961; Hommes, 2013).

3.3.3 Cobweb Modelling from Delay Differential Equation

The linear demand function of price will be given as follows if price of a good is denoted as p, :

$$D(p(t)) = a - \alpha p(t) \text{ where } a, \text{and } \alpha \text{ are constants} \tag{3.28}$$

This curve is generally negatively sloped-decreasing in mathematical term with a and α as positive constants. On the other hand, the linear supply function of price with delay τ is positively sloped-increasing and it can be given by:

$$S(p(t)) = b + \beta p(t - \tau) \text{ where } b, \beta \text{ and } \tau \text{ are constants} \tag{3.29}$$

Continuous Time Linear Model:

The following equation is derived on the assumption that, price change is mathematically relative to difference between supply and demand functions (Soltes et al., 2012; Mackey, 1989; Anokye et al., 2014).

$$p^0(t) = [D(p(t)) - S(P(t))] \tag{3.30}$$

This delay differential equation (DDE) describes how market price changes over time. Whenever demand exceeds supply, price rises and whenever supply exceeds demand, price falls. And only the market price at time $t - \tau$, has effect on the current supply such that:

$$p^0(t) = [(a - b) - \alpha p(t) - \beta p(t - \tau)], \text{ where } \tau > 0, \text{ on } [0,d], d > 0 \tag{3.31}$$

This equation with a single delay like all delay differential equations could be solved in stepwise manner using the principle of method of steps. Equation (3.31) would have initial function (also known as history function) as $p(t) = s(t)$ defined interval $[-\tau,0]$ and then its solution is mapped onto solutions of other functions. Thus the solution of this equation will be mapped on functions on interval $[t - \tau,t]$ onto functions on intervals $[t,t + \tau]$, $[t + \tau,t + 2\tau]$,..., from time points $t = 0,\tau,2\tau,...$

Thus the solutions of system will provide sequence of piecewise functions $p_0(t),p_1(t),p_2(t),...$, defined over contiguous time interval of length τ (Roussel, 2005). This system of initial value problem is normally solved by numerical approach and in this study, MatLab solver dde23, which applies the principles of Runge-Kutta would be used to solve it because MatLab solver dde23 is developed to solve such systems with distinct degree of accuracy.

3.3.4 Continuous Time Non-Linear Model

Considering a simple nonlinear delay differential equation (of quadratic form) for the supply function of price as:

$$S(P(t)) = b + \beta p(t-\tau) - \delta p^2(t-\tau) \tag{3.32}$$

And linear demand function of price also given as:

$$D(P(t)) = a - \alpha p(t) \tag{3.33}$$

where $a, b, \alpha, \beta, \delta$ *and* τ are positive constants. Whenever price increases, supply increases until it exceeds demand where the trend changes. Therefore, price change equation is given as follows;

$$p^0(t) = [(a-b) - \alpha p(t) - \beta p(t-\tau) + \delta p^2(t-\tau] \tag{3.34}$$

This is simple nonlinear delay differential equation (DDE) that mimics the undelayed difference equation used in (Jensen and Urban, 1984) and in practice solution of this equation (3.34) is very tedious using an analytical method. Therefore, a numerical method is applied using MatLab solver dde23 with history function of $p(t) = s(t)$, on $[-\tau,0]$.

3.4 Buffer Stock-Based Pricing Model

This model relaxes the assumption that supply must equal demand in order to maintain price in equilibrium or moderate any possible instability. Thus buffer stock-based pricing model introduces an inventory, buffering the difference between supply and demand and letting prices respond to the level of the inventory or buffer stocks so as to eliminate the instability and cycles that are usually associated in the basic cobweb model (EconModel, 2015).

3.4.1 Delay Differential Model with Buffer Stock

We shall study how the dynamics of price adjustment will be affected when buffer stock is incorporated into equation (3.34). It is assumed that buffer has negative effect on price. Buffer stock model is mathematically formulated as given below:

$$p'(t) = [D(p(t)) - S(p(t))] - \int_{\tau}^{t} [S(p(u)) - D(p(u))]du \quad (3.35)$$

The second term is the integral of past differences which expresses the accumulated stock (say G) in the buffer. If $G > 0$, then G causes downward adjustment of price because government releases stocks to the market. If $G < 0$, the buffer stock operator rather buys from the market to adjust price upward while $G = 0$, denotes no interference from the government (Athanasiou et al., 2008; Soltes et al., 2012; Anokye and Oduro, 2013). Equation (3.35) is a price adjustment model obtained from continuous time delay integro-differential equation that mimics the undelayed integro-differential model used by (Soltes et al., 2012),can be transformed into differential equation by simplifying it into the form given by:

$$p^{00}(t) = [D^0(p(t)) - S^0(p(t))] - [S(p(t)) - D(p(t)] \quad (3.36)$$

The following equation is obtained when the expressions for $S(p(t))$ and $D(p(t))$ in equations (3.32 and 3.33) are fixed into the equation (3.36):

$$p^{00}(t) = p^0(t) - [(b - a) + \alpha p(t) + \beta p(t - \tau) - \delta p(t - \tau)] \quad (3.37)$$

where $p^0(t)$ is the equation (3.34). Equation (3.37) is a continuous-time second order nonlinear delay differential equation which also mimics the second order undelayed differential equation used by Soltes et al. (2012) which can also be solved numerically using MatLab solver dde23 that applies the principles of Runge-Kutta. However, equation (3.37) has to be converted into two first order nonlinear delay differential equations before one can use MatLab dde23 to solve it (Code B.3 at appendix B).

3.4.2 Cobweb Modelling from Difference Equation

Supply and demand models are effective means of modelling how market forces determine the price, the quantity supplied of commodities by producers and the quantity demanded of goods or services by consumers. At the market-clearing equilibrium (MCE), it is assumed that demand must equal supply. Thus cobweb model uses the demand and supply difference equations of price to determine price p_t at t time from price p_{t-1} at time of supply at $t-1$. The market clearing equation that describes the process is a first order non-homogeneous difference equation (Nicholson, 1987).

Linear Cobweb Models:

Given an initial price of p_{t-1}, the market responds at time t a demand quantity D_t determined by the quantity of supply S_t. Therefore current market price P_t decided by demand.

If demand and supply curves are both linear and are given as:

$$D_t = \alpha - \beta p_t \tag{3.38}$$

$$S_t = \delta + \gamma p_{t-1} \tag{3.39}$$

then β represents the slope and α represents intercept for demand equation of price and the parameters γ and δ respectively denote the slope and intercept of supply function of price (Ezekiel, 1938; Goldberg, 1996; Aarts, 2014).

At the Market Clearing Equilibrium (MCE) (3.38) = (3.39) i.e. $D_t = S_t$:

$$p_t = Ap_{t-1} + B \; for \; t = 1,2,3,... \quad (3.40)$$

where $A = -\gamma/\beta$ and $B = (\alpha - \delta)/\beta$. Equation (3.40) constitutes first order cobweb model and could be solved as given by:

$$p_t = A^t p_0 + B(1 - A^t/(1 - A)) \; for \; t = 0,1,2,... \quad (3.41)$$

where $A \neq 1$. The solution of homogenous part from equation (3.41) is also given by: $ph_t = A^t p_0$

Since lag operators do not affect constants when they are applied on them, (Kirchgassner and Wolters, 2007; Anokye and Oduro, 2013; Fulford et al., 1997), the lag operator is applied on equation (3.40) to obtain particular given by:

$$p_t = B/(1 - A) = (\alpha - \delta)/(\beta + \gamma), \text{ where } A = -\gamma/\beta \text{ and } B = (\alpha - \delta)/\beta$$

At equilibrium, it is assumed that price remains constant for all time periods in the system. By letting p_e be the equilibrium price, one can obtain:

$$p_t = p_{t-1} = ... = p_e = (\alpha - \delta)/(\beta + \gamma), \; where \; \alpha \geq \delta \quad (3.42)$$

Therefore the actual solution (i,e in simplified form compared to (3.41)) of the linear equation with equilibrium price p_e of the function p_t will be given as;

$$p_t = A^t(p_0 - p_e) + p_e$$

where p_0 is the initial market price.

Stability condition of Linear Difference Equation

It is observed from above solution that stability of the system is dependent on the value of $|A| = |\gamma/\beta|$, and so if;

1. $|\gamma/\beta| < 1$, then price of the system converges towards equilibrium price p_e,

2. $|\gamma/\beta| > 1$, then the system is unstable and no equilibrium price will be obtained,

3. $|\gamma/\beta| = 1$, then price oscillates in between p_{t-1} and p_t, and
4. $|\gamma/\beta| < 0$, then price oscillation involved whether the system is non-stationary or stationary or damped oscillations obtained

3.4.3 Nonlinear Cobweb Models

If the demand function of price is linear and supply function of price is (of quadratic form) or backward bending nonlinear and they are given as:

$$D_t = \alpha - \beta p_t \tag{3.43}$$

$$S_t = \delta + \gamma p_{t-1} - \rho p_{t-1}^2 \tag{3.44}$$

Then at MCE, (3.43) = (3.44), and the following nonlinear difference equation that constitute nonlinear cobweb model, similar to that of Jensen and Urban (1984) is obtained as:

$$p_t = C - Bp_{t-1} + Ap_{t-1}^2 \quad where\ C = (\alpha - \delta)/\beta,\ B = \gamma/\beta\ and\ A = \rho/\beta \tag{3.45}$$

After applying lagging operator on (3.45), this equation will be similar to quadratic equation having two fixed points given as follows, provided $(1 + B) - 4AC \geq 0$:

$$p_{1,2} = \frac{(1+B) \pm \sqrt{(1+B)^2 - 4AC}}{2A} \tag{3.46}$$

Bifurcation Discussion

This implies that bifurcation occurs when $\frac{(1+B)}{4A^2} - \frac{C}{A} = 0$ or $(1+B) = \pm 2\sqrt{AC}$ and therefore:

$$p_{1,2} = \frac{(1+B)}{2A} \tag{3.47}$$

$\sqrt{\quad}$

- At the left of $(1 + B) = 2\sqrt{AC}$; no equilibrium.
- At $(1 + B) = 2\sqrt{AC}$; we have a 'node' up, i.e. attractive from below and repelling from above.
- At the right of $(1+B) = 2\sqrt{AC}$; we have two equilibria, the smaller one is a 'sink', while the bigger one is a "source" which explains the node behaviour of (3.47).

Note that similar conclusions hold for the other value (Junhai and Lingling, 2007; Fulford et al., 1997).

Stability Condition of Nonlinear Difference Equation

Theorem 3: If an equilibrium point of the following nonlinear autonomous difference equation is given as p^*;

$$p_{t+1} = f(p_t)$$

where f is continuously differentiable at p^*, then

1. If $|f'(p^*)| < 1$, then p^* is an asymptotically stable equilibrium point.
2. If $|f'(p^*)| > 1$, then p^* is unstable.

Proof

According to Elaydi. (2005), if $|f'(p^*)| \leq M < 1$ then, due to continuity of the derivative, there exists an interval $J = (p^* - \gamma, p^* + \gamma)$, $\gamma > 0$, such that $|f'(p)| \leq M < 1$ for all pJ. And also for $p_0 \epsilon J$,

$$|p_1 - p^*| = |f(p_0) - f(p^*)|$$

so from the mean value theorem, there exists $\xi, p_0 < \xi < p^*$, such that

$$|f(p_0) - f(p^*)| = |f'\xi||p_0 - p^*|$$

hence

$$|p_1 - p^*| \leq M|p_0 - p^*|$$

This shows that p_1 is closer to p_* and so also in J because $M < 1$. By mathematical induction;

$$|p_t - p^*| \leq M^t|p_0 - p^*|$$

Therefor for any $\varepsilon > 0$, if $\delta_\varepsilon = min\{\gamma,\varepsilon\}$, then $|p_0 - p^*| < \delta_\varepsilon$ implies $|p_t - p^*| < \varepsilon$ for all $t \geq 0$. So in conclusion p^* is the stable equilibrium point. In addition, since lim−→∞ $|p_t - p^*| = 0$, p_* is asymptotically stable (Elaydi., 2005).

3.5 Buffer Stock Model-At Average Supply

If the buffer stock operator or government in particular decides to set supply of maize at average supply then, it is assumed that stabilization is guaranteed towards the average price and this policy is referred to as keep supply at average. Given the average equilibrium quantity of maize supplied at last time t as shown in the following expression:

$$S_t^A = \frac{\sum_{j=1}^{k} S_{t-j}}{k} + G_{t-1} \quad (3.48)$$

If G_{t-1} denotes control variable which amounts to the quantity of commodity released to the market by government at any time t, then whenever $G < 0$, the government buys and stores $|G_{t-1}|$ units of commodity at time t (Anokye and Oduro, 2013; Athanasiou et al., 2008).

At market clearing condition:

$$D_t = S_t + G_{t-1} \quad (3.49)$$

where

$$G_{t-1} = S_t^A - g(p_{t-1}^e) \tag{3.50}$$

with $g(p_{t-1}^e)$ as the estimated supply at period *t*. The total average quantity of maize available in market at any time *t* is always given as $S_t + G_{t-1}$, which is very close to average equilibrium supply S_t^A. When equations (3.43) and (3.44) are substituted in the equation (3.49), the following nonlinear difference is obtained:

$$\alpha - \beta p_t = \delta + \gamma p_{t-1} - \rho p_{t-1}^2 + (S_t^A - g(p_{t-1}^e)) \tag{3.51}$$

$$p_t = A p_{t-1}^2 - B p_{t-1} + C - D(S_t^A - g(p_{t-1}^e)) \tag{3.52}$$

where $C = \frac{(\alpha-\delta)}{\beta}$, $B = \frac{\gamma}{\beta}$, $A = \frac{\rho}{\beta}$ and $D = \frac{1}{\beta}$

According to Anokye and Oduro (2013) and Athanasiou et al. (2008), market runs short of commodity if the difference between average supply and available supply is positive. Then it means government has to intervene by selling out certain quantity of commodity from stock. Obviously quantity released at period *t* cannot exceed the quantity stored at period *t*−1. Moreover, whenever commodity is in abundance then it means difference between average equilibrium supply and available supply is negative and therefore government quickly buys certain quantity of commodity from the market to keep price stable (Anokye and Oduro, 2013; Athanasiou et al., 2008).

Chapter 4

Main Results 1: Discrete-time Cobweb Models

4.1 Introduction

This session evaluates discrete time cobweb model derived from linear demand and nonlinear supply functions of price and then linear demand and supply functions of price using real economic data of maize price and production from Ashanti Region, Ghana. The cobweb model would be integrated with buffer stock model to study how price fluctuations could be curtailed. The modeling of the various price functions and estimate of their model parameters values in this study are done using SPSS statistical software by applying the principle of regression analysis. The numerical solution of the various discrete time equations are run using Matlab.

4.1.1 Preliminary Analysis of Price and Production Data

The data (from table A.1. at appendix A) are checked to correct any errors and then verified the stationary status for both price and production data sets by applying time series techniques, before formulating the demand and supply functions of price using regression analysis.

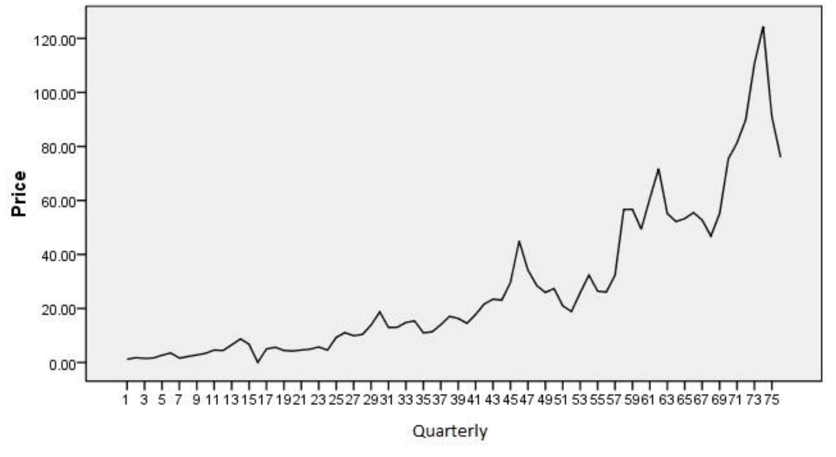

Figure 4.1: Time Series Plot of Price

The above figure shows the time series plot of price in Ghana cedis (Ghc). It indicates that price data is non-stationary and also exhibits nonlinearity characteristics in the form close to quadratic.

4.1.2 Estimates of Demand and Supply Functions

After the preliminary data analysis have been performed to ensure that the price and production data sets (from table A.1. at appendix A) are stationary, regression analysis was employed to formulate price functions for demand and supply. The following table contains estimates of coefficients or the parameters values of the variables used in the various functions of prices.

Table 4.1: Parameter Estimates for Functions of Price

Coefficients of Demand and Supply Functions

Model	Unstandard		Standard			Type
	B	Error	Betta	t	Sign	
1.Price	-96.16	20.26	-0.49	-4.95	0.000	Demand
2.Price	354.28	36.89	1.72	9.60	0.000	
2.$Price^2$	-2.78	0.45	-1.096	-6.14	0.000	Supply
3.Price	167.79	43.23	0.42	3.89	0.000	

The table 4.1 above contains the parameter values of demand and supply functions of price with their p-values showing significant level of the respective parameter values in the functions. The regression was carried out through the origin because in both cases the intercepts were not statistically significant. It contains parameter values of linear demand function of price numbered (1), while the parameter values for quadratic supply function of price are contained in numbered (2) and numbered (3), containing that of linear supply function of price.

Demand Function of Price

The following difference equation that constitutes demand function of price obtained its estimated parameters values from the table 5.1. The parameters values are checked to be statistically significant as shown in the same table mentioned here. The equation was obtained from price data of order two (2) differencing and production data of order one (1) differencing. The confirmation of data stationarity are provided by stationarity test statistics (values) from the Augmented

Dickey-Fuller (ADF) test corresponding to price order of two (2) are given as; ADF test-statistic (-7.252508) is more negative (or greater) than the critical values (-3.540244, -2.909191 and -2.592233) at 1%, 5% and 10% respectively (refer to table A.3, appendix A) while that of production data are also given as; ADF test-statistic (-5.863945) is more negative (or greater) than the critical values (3.531639, -2.905504 and -2.590279) also at 1%, 5% and 10% respectively (refer to table A.4, appendix A). The resulted equation from this analysis is similar to equation (3.38):

$$D_t = -96.16p_t \text{ } where \text{ } \alpha = 0 \tag{4.1}$$

Supply Functions of Price

The following difference equations that constitute supply functions of price also obtained their estimated parameters values from the table 4.1. Equation (4.2) was obtained from price data of order one (1) differencing, with confirmed stationarity values from the Augmented Dickey-Fuller (ADF) test corresponding to differenced order one (1) given as; test-statistic (-6.027941) in magnitude is greater than critical values (-3.531639, -2.905504 and -2.590279) at 1%, 5% and 10% respectively (refer to table A.5, appendix A). The production data of order two (2) differencing is also having the stationarity values given as; ADF teststatistic (-7.637633) is more negative (greater in magnitude) than the critical values (-3.534915, -2.906909 and -2.591024) at 1%, 5% and 10% respectively (refer table A.6, appendix A). The resulted equation is given in (4.2). However, equation (4.3) was obtained with no order of differencing.

These equations whose parameters values also checked to be statistically significant are similar to equation (3.39 and 3.44) respectively:

$$S_t = 167.99p_{t-1} \text{ } where \text{ } \delta = 0 \tag{4.2}$$

$$S_t = 354.28p_{t-1} - 2.78p_{t-1}^2 \quad where \quad \delta = 0 \tag{4.3}$$

4.1.3 Linear Cobweb Model

Analytical Solution: market Clearing Equation for equations (4.1 and 4.2) provide first order linear cobweb model derived from difference equation as follows:

$$p_t = -1.75 p_{t-1} \tag{4.4}$$

Equation (4.4) is similar to equation to (3.40) where $B = 0$ and $A = -1.75$. Since $|\gamma/\beta| > 1$, the system is unstable and there would be no equilibrium price point. Thus prices would never converge towards the equilibrium price point, $p_e = 0$, which is also not realistic because producers are more sensitive to price. If $B = 0$ then in this case, it also mean that every point is an equilibrium point or price, so it only makes sense that it is unstable (Neusser, 2012).

Numerical Solution:

The following graphical representation of equation (4.4) obtained when Matlab code was used to run and solve it numerically. The code is attached at the appendix B (Code B.4).

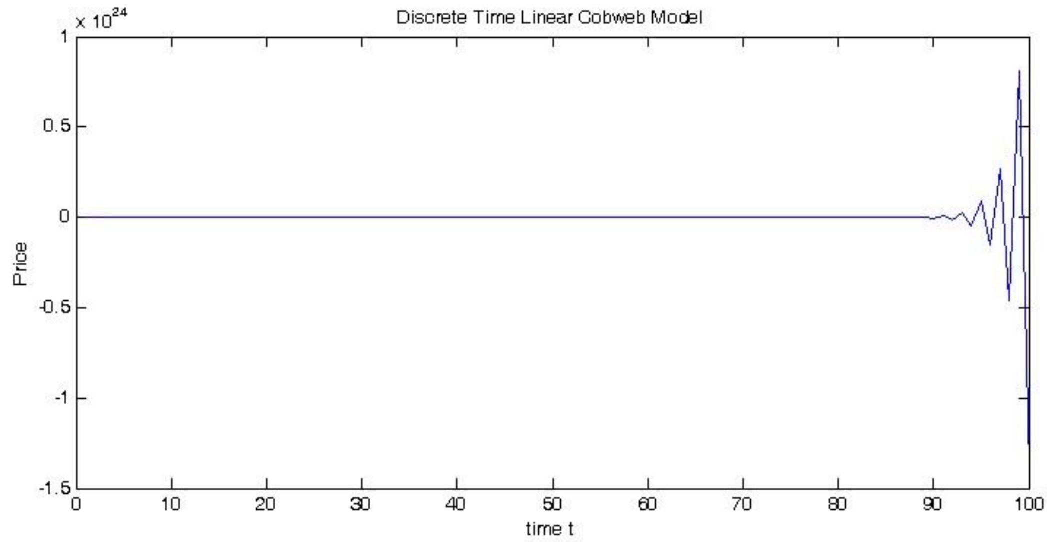

Figure 4.2: Oscillation of Price around Equilibrium-Linear Approach

The figure 4.2, confirms that with time, price of maize would never converge towards equilibrium price point (Ghc=0.00) as shown in the analytical solution. However, price of maize seemed to have stabilized at equilibrium point (Ghc 0.00) as demonstrated in the above graph, long before it started to destabilize around the equilibrium point. This makes the market situation unrealistic as no producers would continue to supply food commodity in that market price condition.

4.1.4 Nonlinear Cobweb Model

Analytical Solution: at the market clearing condition, equations (4.1 and 4.3) provide the following first order nonlinear equation difference which make up nonlinear cobweb model similar to equation (3.45) where the parameter values $C = 0$, $B = -3.684$ and $A = 0.029$:

$$p_t = -3.684p_{t-1} + 0.029p_{t-1}^2 \tag{4.5}$$

This price function is like quadratic equation having two price fixed points,

$$p_{1,2} = \frac{(4.684)\pm\sqrt{(4.684)^2}}{2A},$$

where $P_1 = 161.52$ and $P_2 = 0$. The bifurcation occurs when $P_k = 80.75$ and it's very attractive from below and repelling from above.

This nonlinear cobweb shows that price will be unstable at both $P_1 = 161.52$, and $P_2 = 0$ since $|p^0(161.52)| > 1$ and $|p^0(0)| > 1$, (refer to theorem 3) and therefore fluctuating scope would be bigger and bigger around and in between these price points. In reality price of maize cannot be expected to be $P_2 = 0$ due to price sensitivity of farmers and it thus makes sense that it is unstable too.

Numerical Solution:

Matlab code (Code B.5, appendix B) was written to run the nonlinear cobweb model (equation 4.5) and the following graphical representation was obtained for it.

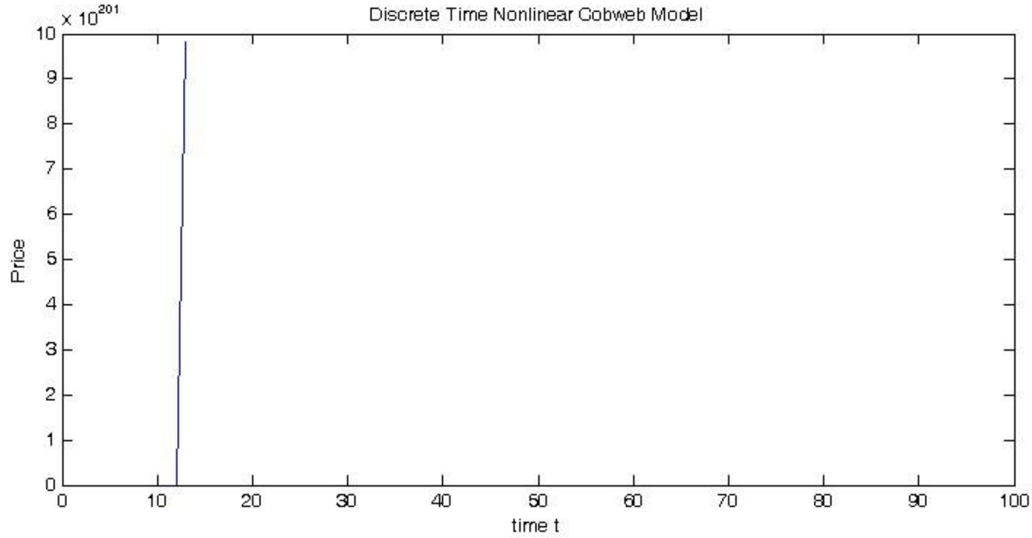

Figure 4.3: Oscillation of Price around Equilibrium-Nonlinear Approach

From figure 4.3, above it is clear that price stability of maize can only be achieved in short term at $P_t = 0$. The reason being that supply curve is steeper than that of demand curve and that, it would cause price to diverge from equilibrium price point. This results support the fact producers are sensitive to price and they would be attracted towards any other price or unstable equilibrium price ($P_t = 161.52$ or GHC 161.52) instead of the zero equilibrium price ($P_t = 0.00$ or GHC 0.00) as demonstrated by the analytical solution.

4.2 Elasticity of Supply Curve and Price Variation

The price stability is dependent on the slope of supply curve or elasticity of supply. The responsiveness of demand and supply to changes in price is quantified using elasticity which is an economic measure designed for such purpose. Mathematically, when the coefficient of price in supply function of price (denoted CPS) is greater than coefficient of price in demand function of price (denoted CPD) then there would be no equilibrium solution. In other words, if ratio of the marginal supply over marginal demand is greater than one, the price system experiences instability (Varian, 1992).

Numerical Presentation:

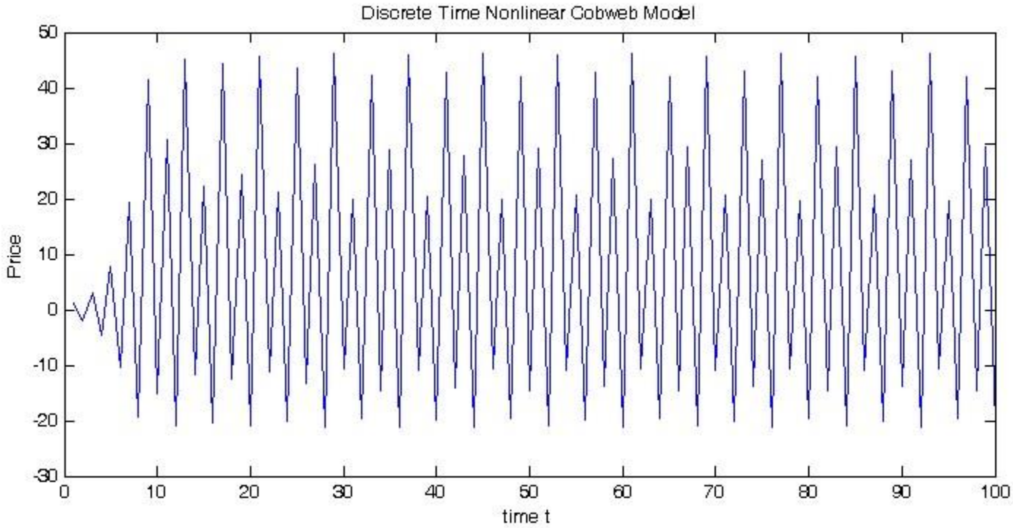

Figure 4.4: Slope of Supply Curve in Nonlinear Model with $|\gamma/\beta| > 1$

The system (equation 4.5) is perturbed on assumption that CPD remains the same, while CPS is varied to observe its effect on price behaviour of maize at the market. The above graph was obtained as a result of price parameter variation using numerical analysis.

In the figure 4.4 above, it is shown that when CPS is far reduced to 156.12 from 354.28 (refer to equation 4.3), then with CPD still at 96.16 so that $|CPS/CPD| > 1$, the oscillatory behaviour of nonlinear cobweb model has now conformed to the condition of nonlinearity model operated on assumptions of naive price expectation. Thus the system has unstable steady state and so prices will diverge from p* and converge to 2-cycle stable state (Ezekiel, 1938; Hommes, 2013; Irma et al., 1999).

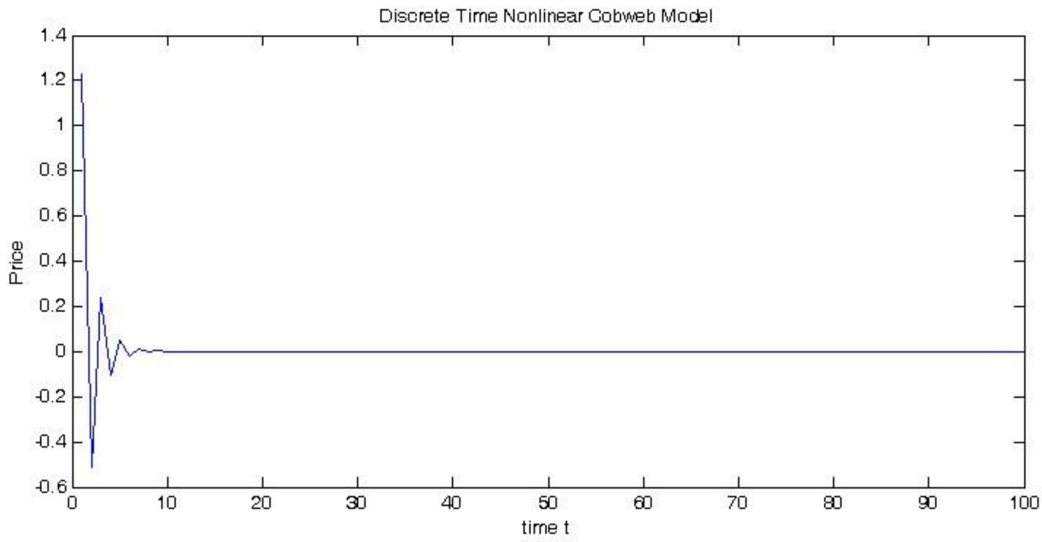

Figure 4.5: Slope of Supply Curve in Nonlinear Model with $|\gamma/\beta| < |0.5|$

CPS is further reduced to 47.08, far from 354.28 (refer to equation 4.3), while CPD remains the same as assumed. From figure 4.5, it is now clear that farther the slope is reduced, more the fluctuations of maize price is stabilized after few periods of instability.

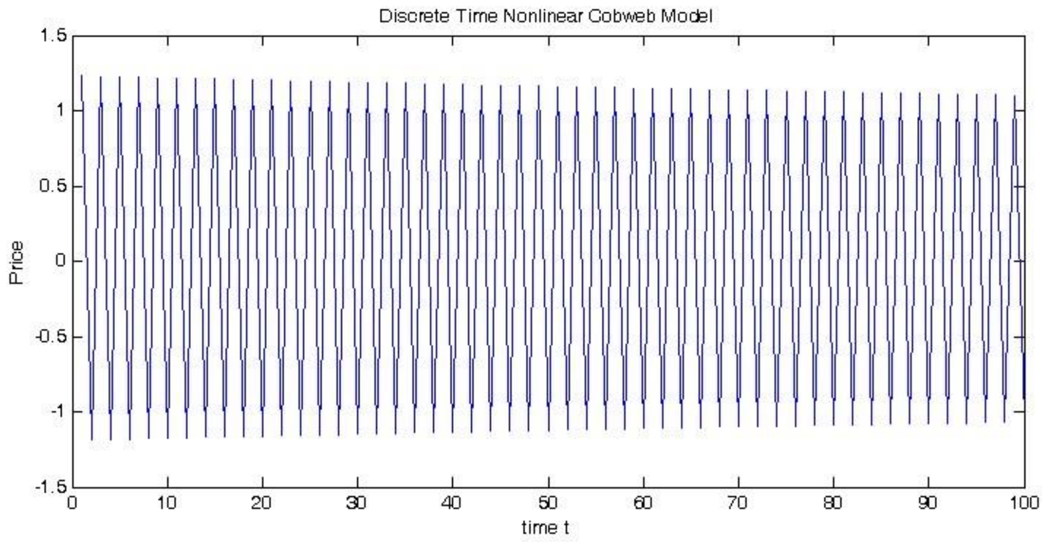

Figure 4.6: Slope of Supply Curve in Nonlinear Model with $|\gamma/\beta| = 1$

From the figure 4.6 above, the price oscillates in 2 cycle between two price points and continues indefinitely without converging. It happened so because both CPS and CPD are 96.16.

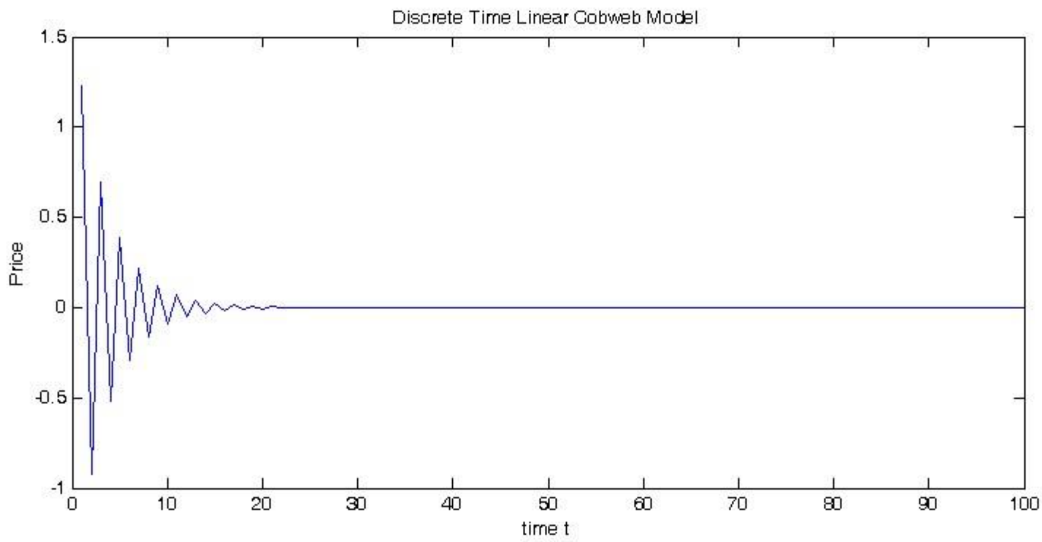

Figure 4.7: Slope of Supply Curve in Linear Model with $|\gamma/\beta| = |0.75|$

When the CPS in linear equation 4.2, is reduced to 72.12 from 167.99, the price was in stable form which is directly opposite to that displayed in figure 4.2. Price of maize would now converge towards an equilibrium price point other than (Ghc=0.00). It will give few oscillations and then converge towards equilibrium.

4.2.1 Nonlinear Cobweb Model with Buffer Stock

When the nonlinear equation (3.52) is applied with supply fixed at the average, where $C = 0$, $B = -3.684$, $D = 0.0104$ and $A = 0.029$, and the average equilibrium supply also given as $S_t^A = 16008.72$, then the following equation is obtained;

$$p_t = 0.029p_{t-1}^2 - 3.684p_{t-1} - 0.0104(S_t^A - g(p_{t-1}^e)) \qquad (4.6)$$

Where the estimated supply $g(p_{t-1}^e)$ at t is the negation of supply function at t with $g(P_0^e) = 0$ at $t = 1$. The equation 4.6 is solved through numerical approach using matlab code (Code B.6, appendix B).

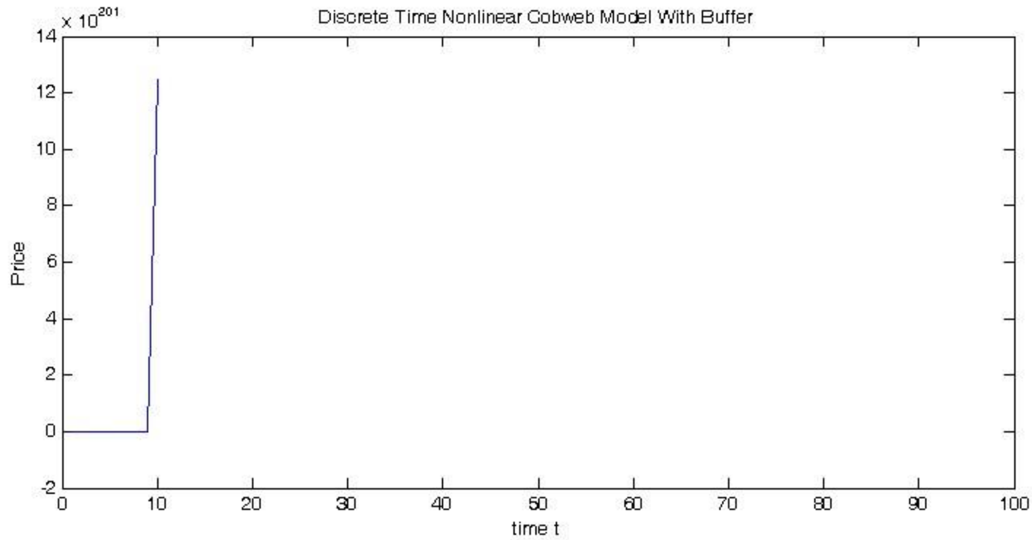

Figure 4.8: Buffer Stock at Average Supply

From figure 4.8, the price behaviour of maize is similar to that of figure 4.3 where the supply slope is far greater than that of demand curve and it would cause price to diverge from equilibrium price point. Producers would be attracted towards any other price or unstable equilibrium price (GhC 161.52) instead of the zero equilibrium price ($P_t = 0.00$). The buffer stock seems to have no effects on the price.

Elasticity of Supply and Buffer Stock

The elasticity of supply (or CPS) is varied to study its effects on price stability in connection with the buffer stock model. As shown in the figure below, the CPS is reduced to 47.05 from 354.28.

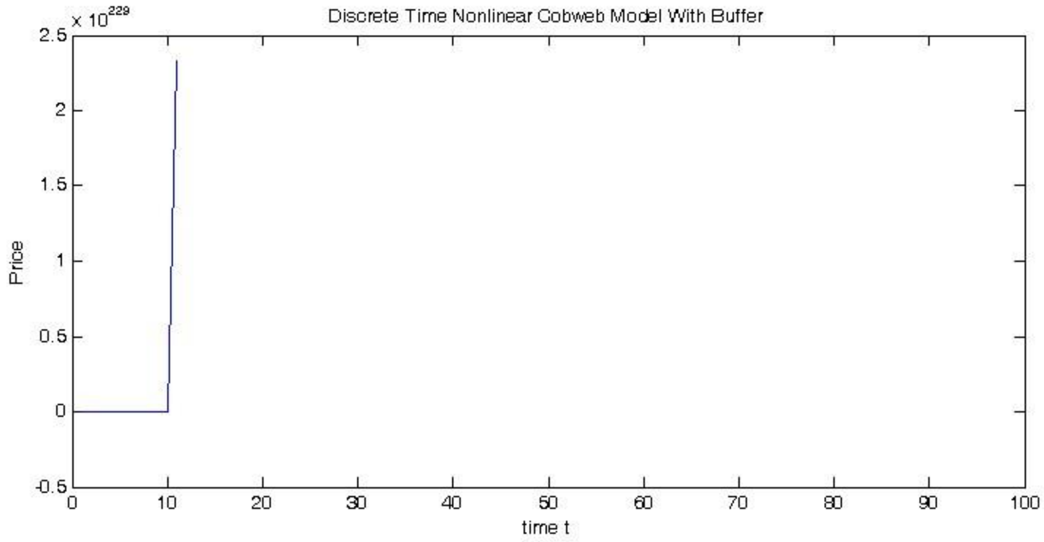

Figure 4.9: Buffer Stock at Average Supply with $|\gamma/\beta| < 0.5$

From the figure above 4.9, it is shown that when discrete time nonlinear cobweb model is incorporated with buffer stock model, prices of maize would still be unstable even though supply elasticity is reviewed downwards.

4.2.2 Structure of the Model and Effects of Buffer Model

It is shown that when the quadratic term in the nonlinear difference equation of supply (equation 4.3) is reviewed downwards then when the function is connected with buffer stock model, more stable the prices of maize would become provided CPS is also reviewed simultaneously downwards. This demonstrates that discrete time buffer stock model (equation 4.6) works as expected in managing price fluctuating systems when the structure of supply function of price is more close to being linear function.

Numerical Presentation:

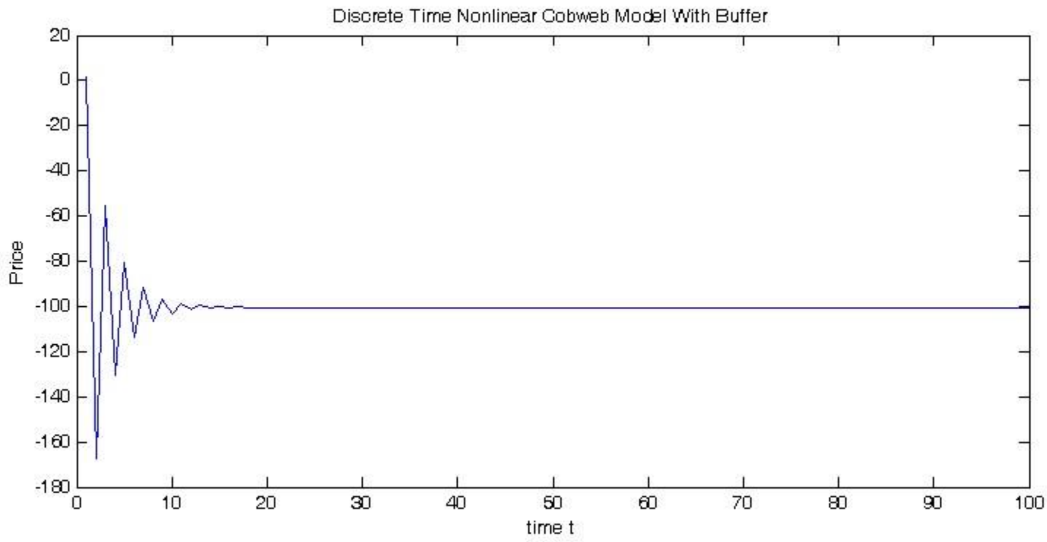

Figure 4.10: Buffer Stock at Average Supply with $|\gamma/\beta| = 0.6$ *and* $\rho < 1$

From the figure above it is shown that price stability of maize is achieved with buffer stock model after few oscillations. Meanwhile CPS should be greater than CPD before one can apply buffer model to achieve price stability.

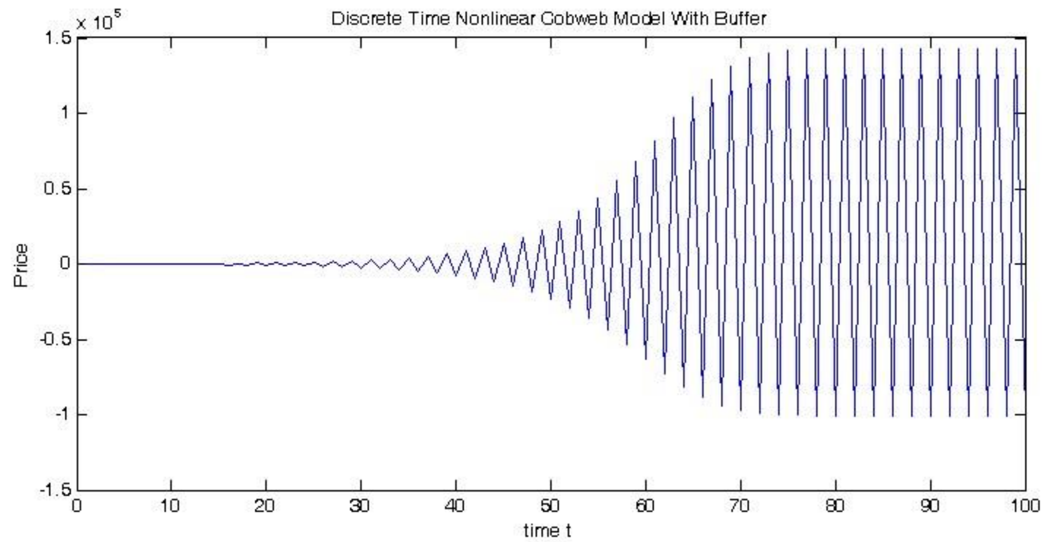

Figure 4.11: Buffer Stock at Average Supply with $|\gamma/\beta| > 1$ *and* $\rho < 1$

It is clear from figure above, that if the coefficient of quadratic term and the CPS are both reduced (i.e *CPS* = 106.92 from 354.28 and ρ = 0.00028 from 2.782 refer to equation 4.3). This suggests the fact that on regular basis, supply of maize has to be reviewed by the buffer operator in order to keep prices at certain level else in the long run buffer will fail. It is found that if CPS and ρ are reviewed, the nonlinear discrete time models begin to work as expected to contain price fluctuations.

It is found that within the intervals [0.1, 1.3], for the ratio of CPS over CPD and [0.0000028, 0.0001] for the ratio of quadratic term over CPD, stability is guaranteed.

These results support the assertion by Jensen and Urban (1984) that discrete time nonlinear models work very well within certain estimated parameter value boundary and beyond the boundary nonlinear models exhibit spectrum of behaviour including chaos.

Any change of coefficients in the models formulated, affects the behaviour of the model which also affect the price, supply and demand as well as quantity of stock that would predicted from the model. Therefore predicted values would not reflect the true market circumstance.

Chapter 5

Main Results 2: Continuous-time Cobweb Model

5.1 Introduction

In this session the notion of time dependence in respect to maize price determination would be discussed thoroughly through the use of mathematical modeling of delay differential equations derived from supply and demand functions of price of maize in Ghana. The session would also demonstrate instability phenomenon involving delay differential equation models having single delay (time-lag). Following this is a process of instability quenching phenomenon which is presented by varying the delay parameter in gradual manner. Proceeded to this would then be buffer stock model incorporated into the delay differential equation model to achieve stability in connections with variation of supply delay parameter and buffer stock delay parameter. The delay variations are done in connection with the type of price scheme that is run by the buffer stock scheme. The study will use real economic data acquired from records of Ministry of Food and Agriculture, Statistical Directorate in Kumasi, Ashanti Region, Ghana.

5.1.1 Parameter Estimates

Modelling of various mathematical functions and their parameter estimates were done by the use of SPSS (refer to table 4.1) and then the numerical solutions of the delay differential equation run using MatLab solver dde23.

5.1.2 Demand Function of Price

The following demand functions of price whose parameter values taken from the table 4.1, is in the form of equation 3.28. The parameter estimates are checked and found to be statistically significant. This function was obtained from price data of order two (2) differencing and production data of order one (1) differencing. The ADF test statistics were used to check the stationary status of the data before deriving the estimates of the models using regression techniques, just as were done for discrete time case in chapter 4.

$$D(p(t)) = -96.16p(t) \ where \ a = 0 \tag{5.1}$$

5.1.3 Supply Function of Price

Similarly, supply functions with time delay τ are given below (in the form of 3.29 and 3.32). Equation (5.2) was obtained from price data of order one (1) differencing and production data of order two (2) differencing. However, equation (5.3) was obtained with no order of differencing in respect to the price and production data sets. Similarly, ADF tests were run to verify the stationary status of the various data sets before estimating the parameters.

$$S(p(t)) = 167.99p(t-\tau) \text{ where } b = 0 \tag{5.2}$$

$$S(p(t)) = 354.28p(t-\tau) - 2.78p^2(t-\tau) \text{ where } b = 0 \tag{5.3}$$

The delay τ expresses time that is needed to realize change of supply in dependence on trend of price. Thus current production depends on the past price.

Analysis of Continuous Time Linear Model

From equations (5.1) and (5.2), an equation for price change is derived and it is in the form of (equation 3.31) where $a = b = 0$ and $\tau = 1$:

$$\frac{1}{96.16}p'(t) = -p(t) - 1.75p(t-1);\ t > 0 \tag{5.4}$$

Analytical Solution: This linear delay differential equation is solved analytically using method of steps.

Given that equation (5.4) is solved on say [0,10], and with history function (initial function) also given as p(t)=1.23 on $t \leq 0$, then for interval [0,1];

$$\int_0^t p'(s)ds = p(s)\mid_0^t = p(t) - p(0) \tag{5.5}$$

$$p(t) = p(0) + \int_0^t p'(s)ds$$

If $p^0(t) = p(t-1)$, then $p^0(s) = p(s-1)$, and so

$$p(t) = p(0) + \int_0^t p(s-1)ds$$

Then on the interval $0 \leq s \leq 1$, implies $-1 \leq s-1 \leq 0$, so that history function p(s-1)=1.23, on this interval;

$$p(t) = 1.23 + \int_0^t [-1.23 - 1.75]ds$$

$$p(t) = 1.23 + (-3.38)s\mid_0^t$$

$$p(t) = 1.23 + (-3.38)t \tag{5.6}$$

This method involves solving the equation (5.4) on one interval at a time. To find the next solution of equation (5.4), implies that one uses the solution from the previous interval [0,1] to obtain the solution on the next interval [1,2], using the same process;

$$p(t) = p(1) + \int_1^t p'(s)ds$$

The solution on [0,1] is $p(t) = 1.23 + (-3.38)t$, and so

$$p(1) = 1.23 + (-3.38)(1) = -2.15.$$

Also on the on the interval $1 \leq s \leq 2$, implies $0 \leq s - 1 \leq 1$, so that previous solution would be $p(s - 1) = 1.23 + (-3.38(t - 1))$, so on this interval;

$$p(t) = -2.15 + \int_1^t p(s-1)ds$$

$$p(t) = -2.15 + \int_1^t [1.23 + (-3.38(s-1))]ds$$

$$p(t) = -2.15 + [4.61s - \tfrac{3.38s^2}{2}] \;|_1^t$$

$$p(t) = \frac{-10.14}{2} + 4.61t - \frac{3.38t^2}{2} \tag{5.7}$$

The solution for equilibrium price points continues in same manner for successive intervals, and it tends to get more complicated as one moves from one interval onto the other.

Like the method of successive approximation applied in existence and uniqueness theorem, method of steps are meant to derive unique solution that satisfies the integral system. If unique solution exists in the said solution interval, then the solution holds for both initial value problem and the integral part. The method of steps therefore generates sequence of functions which only satisfy the initial conditions but do not satisfy the initial value problem given by delay differential equation (Boyce and DiPrima, 1992).

Thus, if in the process, a solution is found to be equal to the immediate solution function before it, then unique solution is obtained for the integral equation (5.5), and the process is terminated. Hence solution is also found for the initial value problem (5.4) (Boyce and DiPrima, 1992).

In general, it is very difficult to obtain a unique solution and so the entire infinite sequence of piecewise functions can be checked to ascertain if every member of the sequence exist, do not break down or interrupt in the process at any stage and/or the sequences converge. Then at this stage, one can find unique function and check whether its limit properties satisfy the integral solution (5.5), as well as the initial value problem (Boyce and DiPrima, 1992).

The difficulty of the process makes it easy to see the value of the application of MatLab solver dde23 for solution. Matlab solver dde23 also follows the principles of method steps using Runge Kutta triple BS(2,3).

Numerical Solution:

The linear delay differential equation is now solved through numerical approach using MatLab solver dde23, with code attached at appendix B (Code B.1). Equation (5.4) is divided by 96.162 so as to make the solution very smooth when it is run. The history function is set at $P(t)$ = 1.23 (initial price from table A.1 at appendix A), when $t \leq 0$, and equation (6.4) on the interval [0,100]. The solution of equation (5.4) is presented in graphical form (see figure 5.1) below;

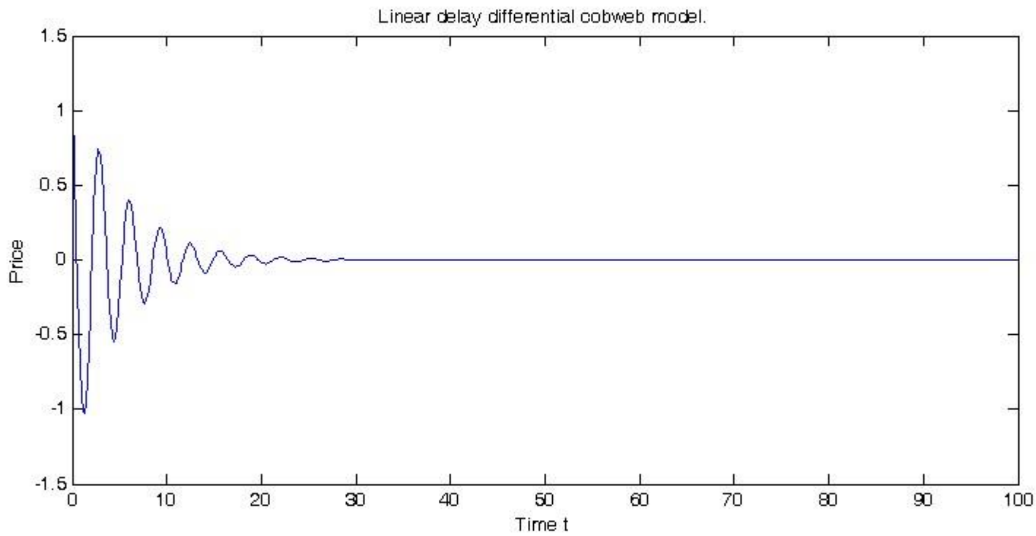

Figure 5.1: Oscillation of Price around Equilibrium

It is clear from the figure 5.1, above that the solution of equation (5.4) oscillates and tends to an equilibrium price of zero with time. However, this equilibrium price is unrealistic, due to the fact that producers are sensitive towards price and, moreover, there is insufficiency of food supply.

Analysis of Continuous Time Nonlinear Model

From equations (5.1) and (5.3), the price equation (5.8) is obtained following equation (3.34) where $\tau = 1$ and $a = b = 0$:

$$\frac{1}{96.16}p'(t) = -p(t) - 3.68p(t-1) + 0.03p^2(t-1); \; t > 0 \tag{5.8}$$

Analytical Solution: Given the history function p(t)=1.23 defined on $t \leq 0$, and solution sequence of equation (5.8) also defined on interval [0,10], then for [0,1];

$$\int_0^t p'(s)ds = p(s) \mid_0^t = p(t) - p(0)$$

$$p(t) = p(0) + \int_0^t p'(s)ds \tag{5.9}$$

If $p^0(t) = p(t-1)$, then $p^0(s) = p(s-1)$, and so

$$p(t) = p(0) + \int_0^t p(s-1)ds$$

Then on the interval $0 \leq s \leq 1$, implies $-1 \leq s - 1 \leq 0$, so that history function p(s-1)=1.23, on this interval;

$$p(t) = 1.23 + \int_0^t [-1.23 - 3.68(1.23) + 0.03(1.23)^2]ds$$

$$p(t) = 1.23 + (-5.72)s \mid_0^t$$

$$p(t) = 1.23 + (-5.72)t \tag{5.10}$$

The method of steps involves solving the equations on one interval at a time. To find the next portion of the solution of equation (5.8), the solution from [0,1] is used to find the solution on the next interval [1,2]. Using similar process;

$$p(t) = p(1) + \int_1^t p'(s)ds$$

The solution function from [0,1] is $p(t) = 1.23 + (-5.72)t$, and so

$$p(1) = 1.23 + (-5.72)(1) = -4.49.$$

Also on the on the interval $1 \leq s \leq 2$, implies $0 \leq s - 1 \leq 1$, so that previous solution would be $p(s-1) = 1.23 + (-5.72(t-1))$, so on this interval;

$$p(t) = -4.49 + \int_1^t p(s-1)ds$$

$$p(t) = -4.49 + \int_1^t [1.23 + (-5.72(s-1))]ds$$

$$p(t) = -4.49 + [6.95s - \tfrac{5.72s^2}{2}] \,|_1^t$$

$$p(t) = \frac{-17.16}{2} + 6.95t - \frac{5.72t^2}{2} \tag{5.11}$$

The solution found for equation (5.8) thus far can be described as piecewise function, and each new step will add a new piece. The solutions tend to get more complicated over each successive interval. It would take ten full iterations of this process simply to solve equation (5.8) on say [0,10], and the solution would be a piecewise function with ten distinct pieces. like equation (5.4), sequence of piecewise functions are generated until unique solution is found to satisfy the initial value problem and the sequence is terminated at that point in the solution interval given.

Continuity Analysis:

Continuity analysis is done on the analytical solution to confirm that the piecewise functions so far obtained are continuous in the interval [0,2], using limits analysis.

Given that;

$$p(t) = \begin{cases} 1.23 + (-5.72)t, & 0 \leq t \leq 1, \\ \frac{-17.16}{2} + (6.95)t - \frac{5.72t^2}{2} & 1 \leq t \leq 2, \end{cases}$$

$\lim_{t\to 1} p(t) = (1.23 + (-5.72)t) = -4.49,$

$\lim_{t\to 2} p(t) = (\frac{-17.16}{2} + (6.95)t - \frac{5.72t^2}{2}) = -6.12,$

These results confirm existence of limits for both functions as t approaches 1 for the first function and t approaches 2 for the second function respectively.

Proof of Continuity:

One (1) as the common interval value between the two solution functions is also put into each of them and the same results obtained; lim $p(t) = (1.23 + (-5.72)t) = -4.49$, $t \to 1^+$

$$\lim_{t\to 1^-} p(t) = \left(\frac{-17.16}{2} + (6.95)t - \frac{5.72t^2}{2}\right) = -4.49,$$

These results prove that the model has passed continuity test since left hand limit is equal to right hand limit and so the model is continuous on the given interval, since the solution functions obtained so far are defined at every point on the interval and they also exhibit no interruption, jump or break.

Numerical Solution:

Based on same assumption for smoothness of the solution using MatLab solver dde23 (code B. 2, appendix B), equation (5.8) is also divided by 96.162 and the history function set at $P(t) = 1.23$, when $t \leq 0$, with (5.8) hold on the interval [0,100]. The solution of equation (5.8) in numerical form is presented graphically as follows (see figure 5.2 below):

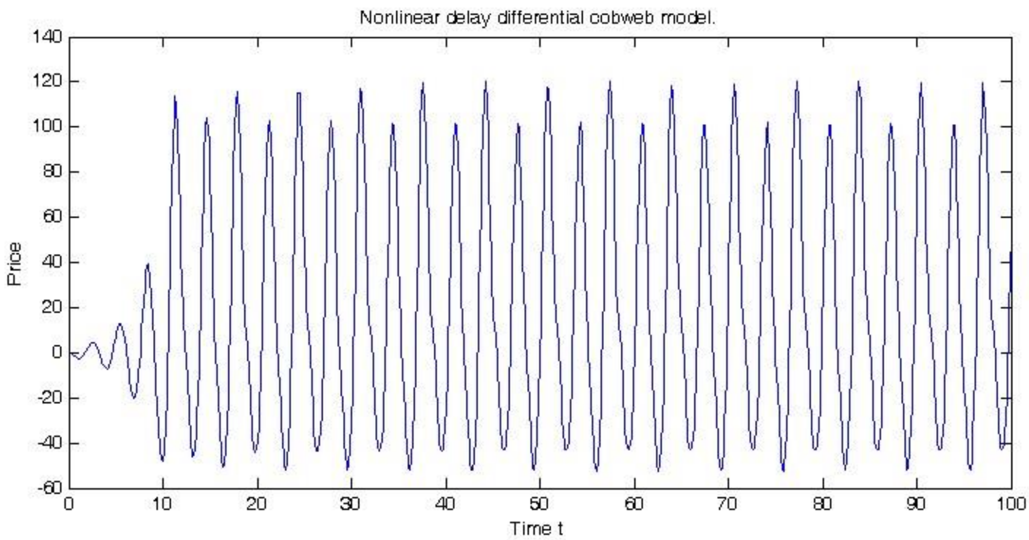

Figure 5.2: Oscillations of Price around 2 Equilibrium Points for $\tau = 1$

It is clear from the Figure 5.2, above that the solution of equation (5.8) would initially show little stability and in a very short time oscillates between two (2) equilibrium (price) points. This conforms to the condition of nonlinear models under naive price expectation (Ezekiel,1938; Hommes,2013; Irma et al., 1999).

5.2 Continuity and Model Truncation Error Analysis

The nonlinear delay differential model (5.8) also meet the spline interpolant fits analysis. This is one of the Matlab numerical techniques used to check model continuity and truncation error. In spline interpolation, the interpolant is a special type of piecewise polynomial function called a spline. For the fact interpolation error are made small even when using low degree polynomials for the spline, spline interpolation is frequently preferred over polynomial interpolation. Spline interpolation avoids the phenomenon of Runge, in which oscillation occurs between points when interpolating using high degree polynomials (Hazewinkel, 2001). Splines usually minimizes the bending and this can only be achieved or proved efficient if polynomials of degree 3 or higher are used (Hazewinkel, 2001).

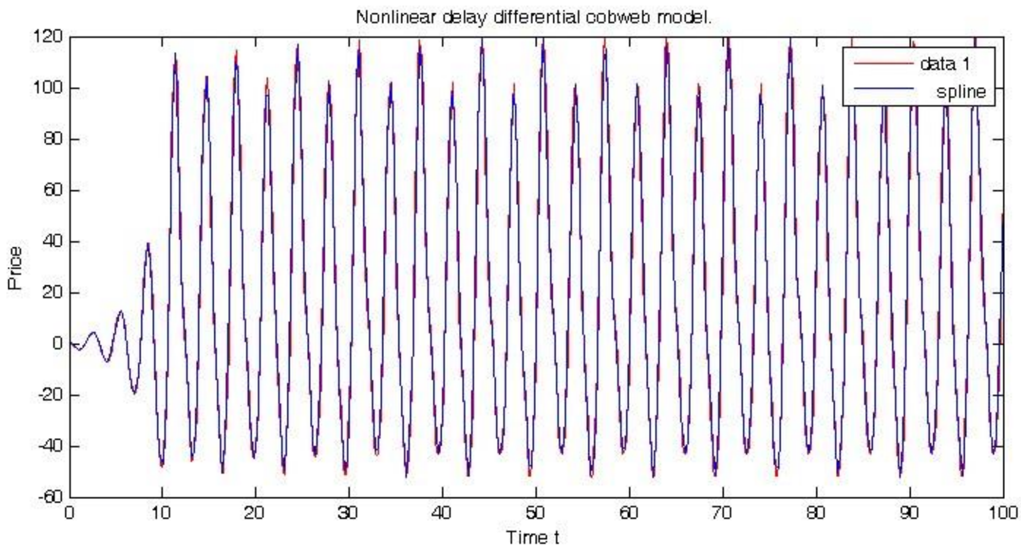

Figure 5.3: Oscillations of Price around 2 Equilibrium Points for $\tau = 1$ (Spline Interpolant)

From figure 5.3, it shows that equation (5.8) could have still performed with order three (3) polynomial and the fact that it fits with spline interpolant means the order (2) polynomial has shown goodness of fit for the study. Therefore the choice of order two (2) polynomial or quadratic for the study is in order. This also shows little stability and in a very short time oscillates between two (2) equilibrium (price) points.

Note that the order two (2) was also chosen because the existing models used order two (2) and so it only makes sense using the same order to prove the point before one tries to go beyond the order.

Residual Analysis Plot:

The Matlab solver dde23 also has the features to track down discontinuities at low order and integrate them using the method of extrapolation.

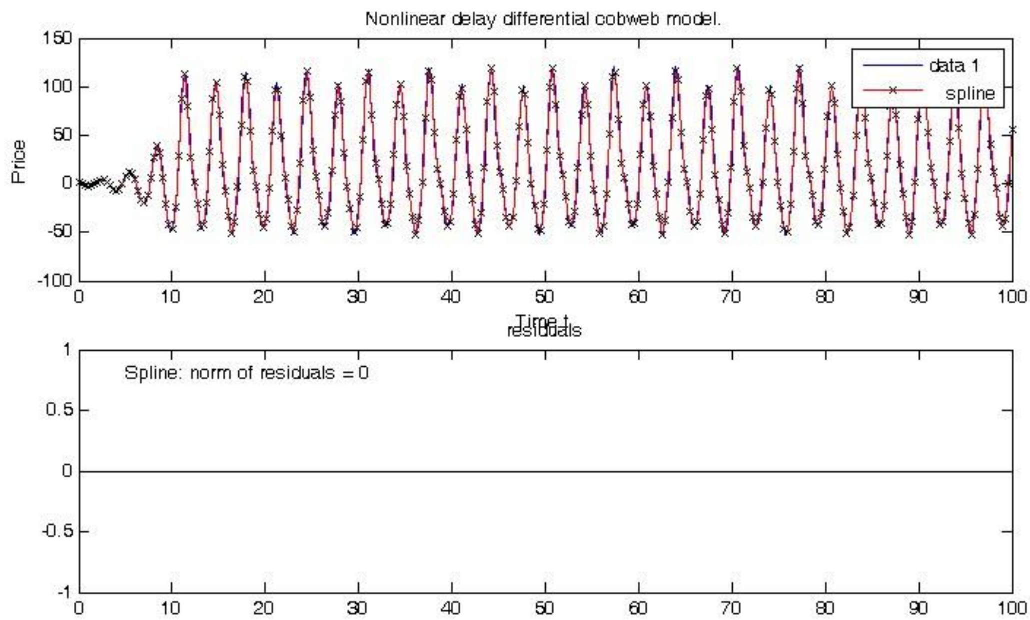

Figure 5.4: Residual Plot of price Oscillation with delay τ = 1

The figure 5.4, confirms the continuity of the model as it fits well with spline interpolation. The residual plot shows residual norm of zero (0), an indication of goodness of fit.

5.3 Time Varying Effects on Price Stability

The effects of system perturbation in regards to delay parameter τ on price oscillations (fluctuations) are discussed. The delay parameter can decrease or increase fluctuations of price (Eduardo and Gergely, 2013). This time varying analysis helps one to have fore knowledge of time needed for supply to respond to price changes and keep the system symmetrical about the equilibrium price.

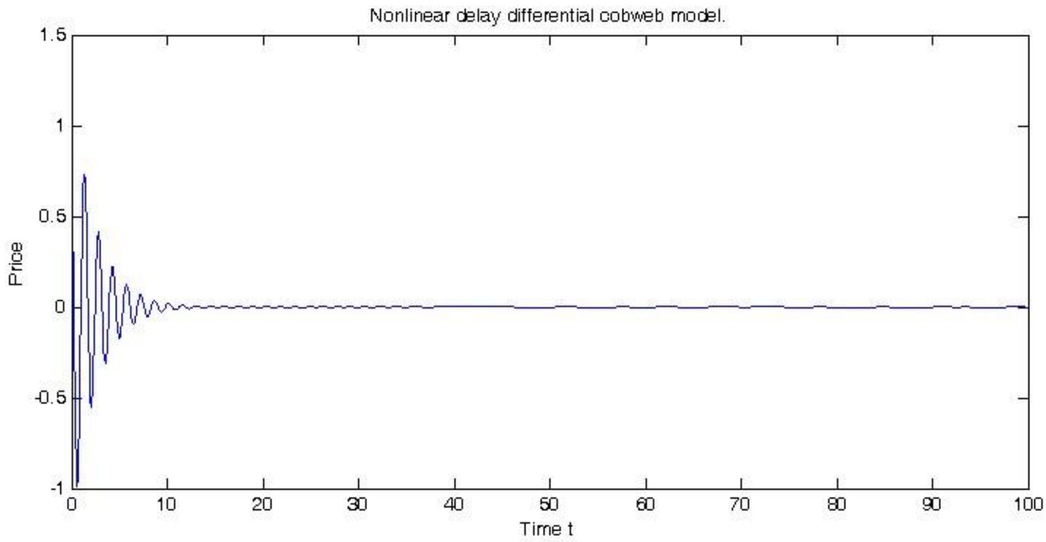

Figure 5.5: Oscillation of price equilibrium with delay $\tau \leq 0.5$

From figure 5.5, it is indicated that price would be in stable equilibrium in the long run when factors affected by time lag in the system are improved. Thus the oscillations with time are suppressed for $\tau \leq 0.5$, as shown above using equation (5. 8).

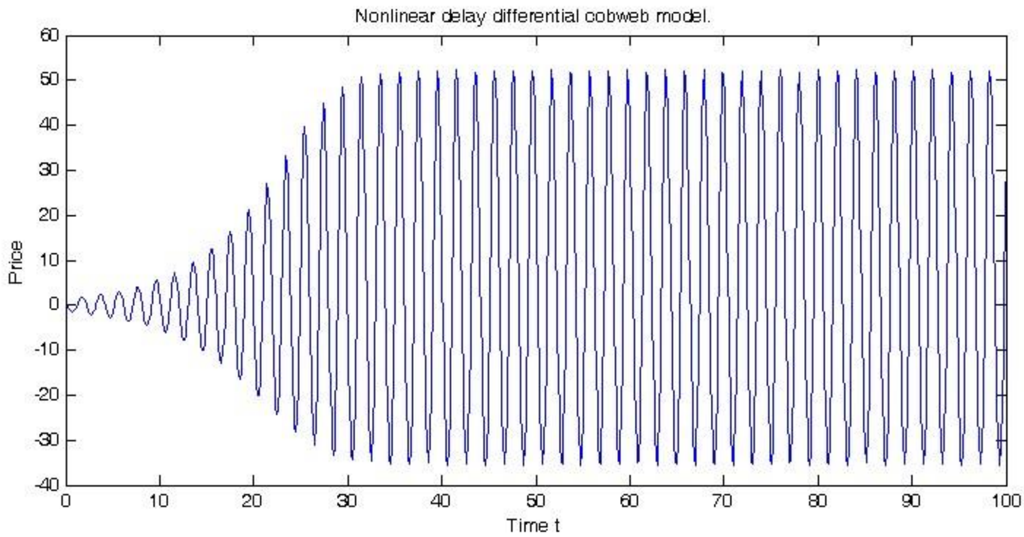

Figure 5.6: Oscillation of price equilibrium with delay $\tau > 0.5$

The price oscillations (fluctuations) start to increase as seen in figure 5.6, above and with time become asymmetric (just like figure 5.2) about the equilibrium for $\tau > 0.5$, using the same equation (5.8).

On the contrary, equation (5.4), for $\tau > 0.5$, would still be in stable equilibrium.

5.4 Continuous Time Nonlinear Model with Buffer Stock

The buffer stock operator is always committed to achieving price stability and sustaining a constant price level. It is clear from figure 5.4, above that if delay in response to supply dynamics is improved then price stability can be achieved.

From equation (3.37), the buffer stock equation is obtained as follows:

$$\frac{1}{96.16}p''(t) = [(-p(t)-3.68p(t-\tau)+0.03p^2(t-\tau))']-[p(t)+3.68p(t-\tau)-0.03p^2(t-\tau)] \tag{5.12}$$

where $b = a = 0$. If delay is fixed at 0.45 for both the inventory (buffer stock) and supply system, the following price oscillation graph is obtained using the Matlab solver dde23 whose written code could be found at appendix B (code B.3):

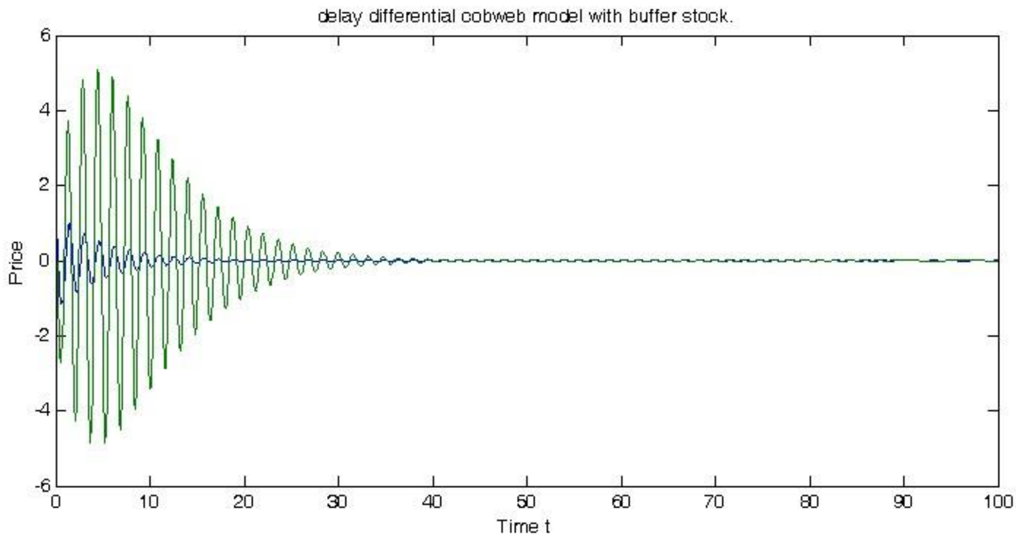

Figure 5.7: Oscillations of Price about equilibrium with Buffer Stock, τ = 0.45

From figure 5.7, above, the buffer stock and the supply are of different oscillation length which signifies no effects on price from the buffer operation. For the buffer stock to have greater impact, the two systems (buffer stock and supply) should synchronize.

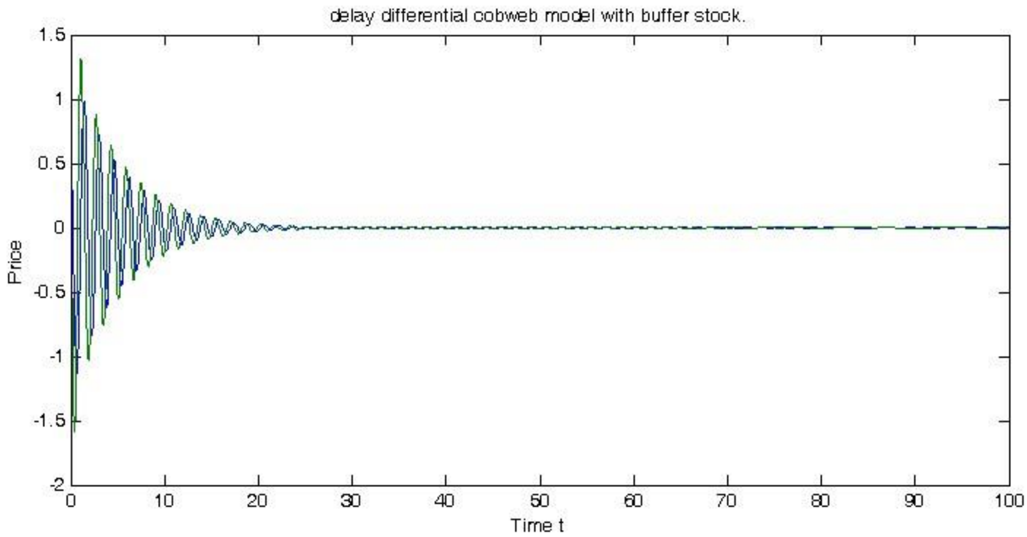

Figure 5.8: Oscillations of Price about Equilibrium with Buffer Stock, τ = 0.22

Now from figure 5.8, when the delay for the buffer stock is reduced to τ = 0.22, and supply delay set at τ = 0.45, the two systems are synchronized to indicate effects of buffer stock operations on price oscillations of maize. The oscillations as well as amplitude (wave length) are also reduced.

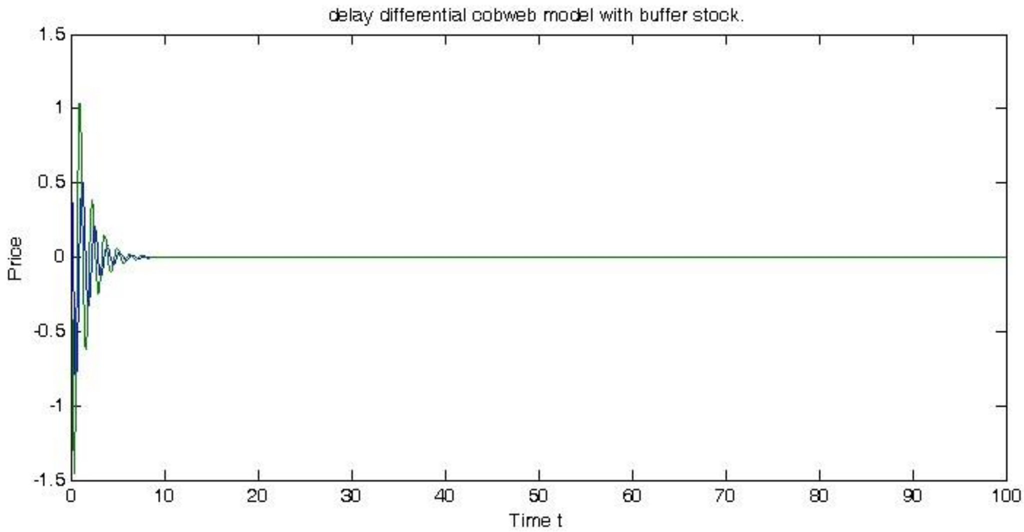

Figure 5.9: Oscillations of Price about Equilibrium with Buffer Stock, τ = 0.20

It is clear from figure 5.9, that if delay parameter (τ = 0.20) for buffer stock model and delay (τ = 0.35) for supply model are further reduced, the more stable price becomes. Now buffer stock scheme have had significant impact on price and they are felt by stakeholders.

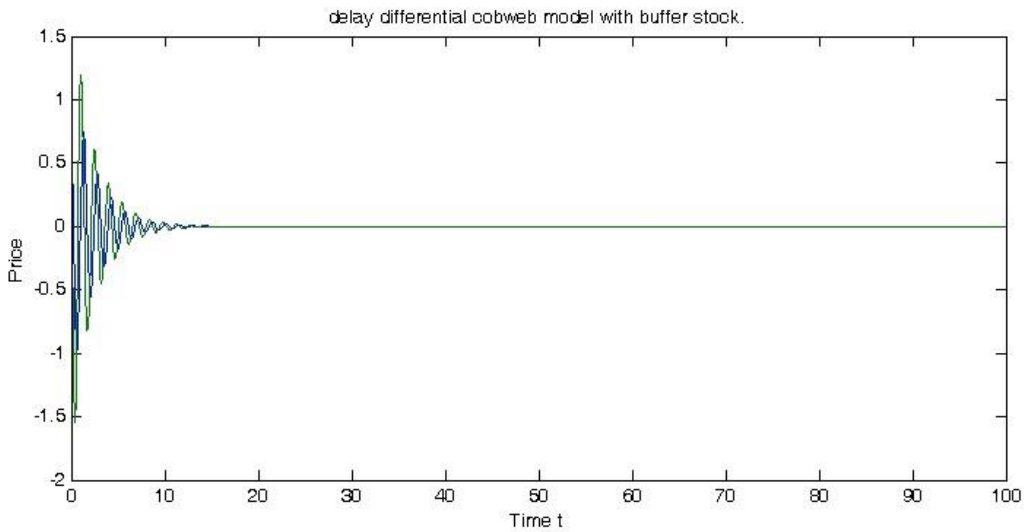

Figure 5.10: Oscillations of Price about Equilibrium with Buffer Stock, τ = 0.21
From figures 5.8 and 5.9, we obtain best average delay times of τ = 0.21 and τ =

0.40 for buffer and supply respectively. These delay times give figure 5.10 which provides an average stable price of maize (from the simulated data in table 6.1 below) as GHC 30.49, that is very close to the mean maize price of GHC 30.27 obtained in the descriptive statistics of the raw data in table A.2 at appendix A. Thus the mean price value (GHC 0.317) from the table 6.1 below is multiplied by 96.16 to obtain the stable equilibrium price of GHC 30.49, which in turn provides the equilibrium average demand and supply respectively as 2931.6 metric tons and 8217.6 metric tons. The average excess supply is given as 5286 metric tons, and they are kept in stock for the next market period (i.e planting period). When at another period (i.e during harvesting period) demand exceeds supply then the appropriate difference is released from the buffer to the market in order to keep price in equilibrium.

Table 5.1: Descriptive Statistics of Maize Price in Simulation

	N	Range	Mean		Std. Dev.	Variance	Skewness		Kurtosis	
	Stat	Stat	Stat	Std.Error	Stat	Stat	Stat	Std.Error	Stat	Std.Error
Price	7	0.490	0.317	0.061	0.160	0.026	-0.398	0.794	0.336	1.587
Valid N	7									

Table 5.1 shows the statistic values of the simulated data (within time points of stability) when the buffer stock was run with Matlab dde23 solver. The values include range, mean, skewness less than 1 (moderate), kurtosis greater than 0 but within the expected value of 3 and their respective standard errors. The standard deviation has been reduced as far as to 0.1602 from 29.48 in table A.2 (appendix A), and the same applies to the variance.

It should be noted that the choice and variation of delay parameter for price stability is informed by the price scheme run by the buffer stock operator. Usually, there are three price schemes that are run in buffer stock schemes which include floor, mean and ceiling price. The choice of any price is mostly influenced by the past information available.

These affirm the fact short-term shortages and excessive price fluctuations in market could be significantly curtailed by buffer stock scheme (Bahagia, 2006). This study disagrees with Mackey (1989), in that, when storage delay is used as a buffer stock delay that is well managed, it could rather be a commodity price stabilizer.

The study also disputes the assertion by Soltes et al. (2012) that the order of the undelayed buffer stock model was the source of price instability. It is proven that if the same model used by Soltes et al. (2012) is fixed with time delay parameter, it makes the model good for managing price fluctuations.

5.5 Elasticity of Supply Curve and Price Variation

At this session, time delay is maintained at one (τ = 1), and then the CPS or elasticity of supply is varied to determine its effects on price stability of maize as modeled in this study.

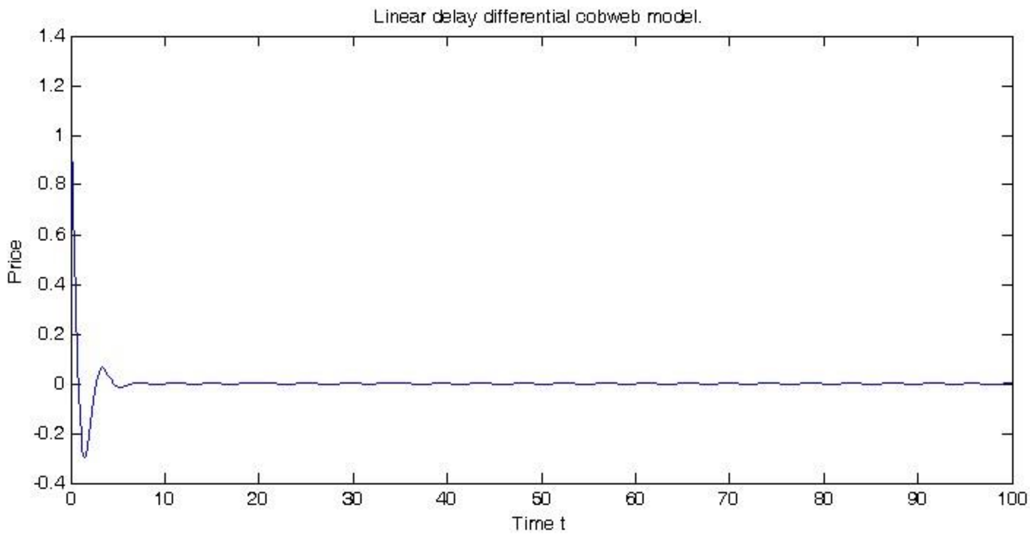

Figure 5.11: Slope of Supply Curve in Linear Model with $|\gamma/\beta|$ = 0.75

It is clear from the figure 5.11, above that oscillations diminished and prices tend to an equilibrium price other than zero as envisaged in the solution of equation

(5.4). This equilibrium price is realistic, and producers would be motivated to respond to price changes at the market as CPS reduced to 72.12 from 167.99 and time delay set at (τ = 1).

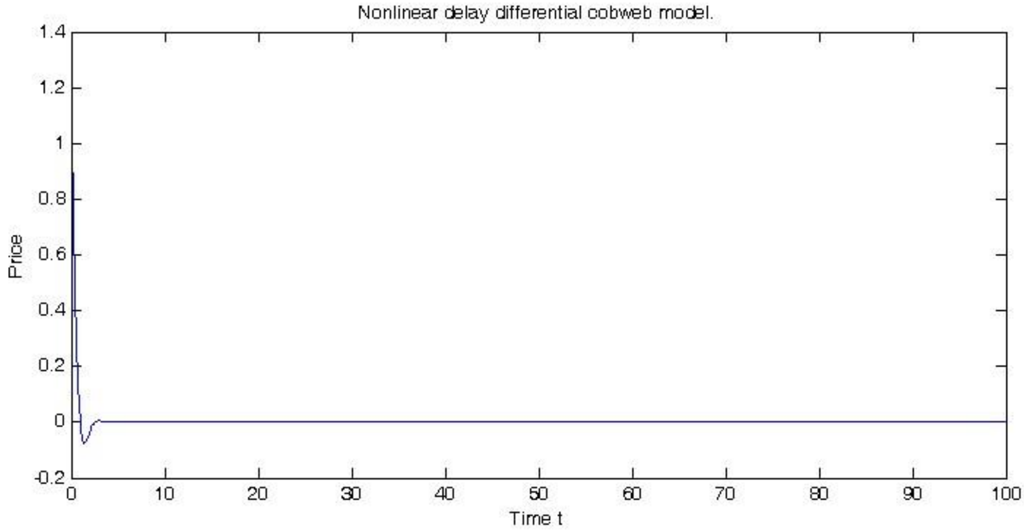

Figure 5.12: Slope of Supply Curve in Nonlinear Model with $|\gamma/\beta|$ = 0.75

From the figure 5.12, price oscillations almost diminished and prices tend to an equilibrium price rather than oscillating between two equilibrium price points as were observed in the solution of equation (5.5), if CPS is far reduced to 72.12 from 354.28. However, delay remained at τ = 1, yet prices converged to equilibrium.

It is clear from the above graph that if time delays set at τ = 1 for both buffer and supply delays and also maintain the quadratic term as ρ = 2.78, when CPS = 57.7, then prices of maize would be stabilized irrespective of the type of supply model linked with the buffer stock model which in essence contradicts that of the case in discrete time cobweb models. The coefficient of the quadratic term in the nonlinear supply function was also reduced to 0.78 from 2.78 and it provided similar graph. Therefore, it deduced that delay differential buffer stock model works as expected in managing price fluctuations with delay differential supply

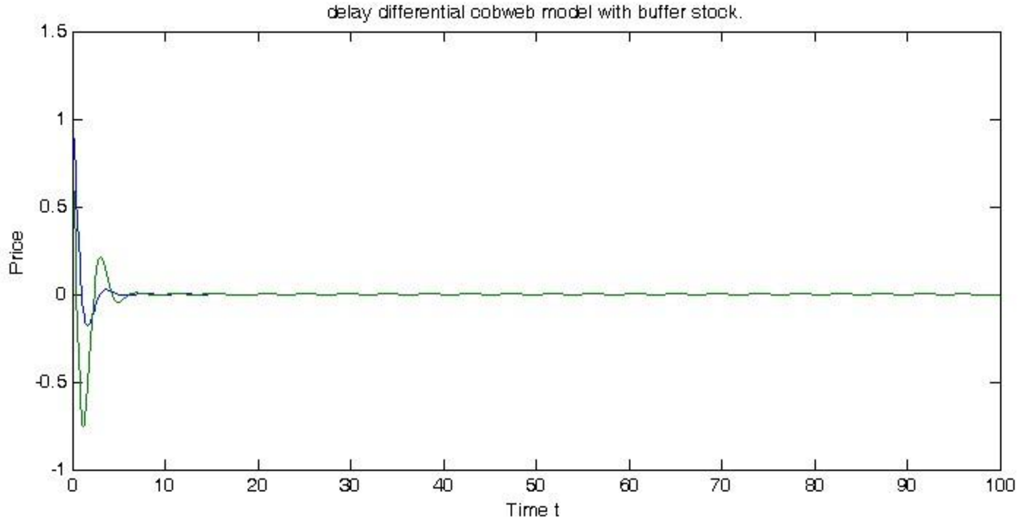

Figure 5.13: Buffer Stock and Supply with $|\gamma/\beta|$ = 0.6 and same ρ = 2.78

model irrespective of the type of supply model being it linear or nonlinear. This makes delay differential model very robust in modeling real life conditions.

5.6 Stock Dynamics and Reliability

Quantity of Maize to Store during Harvesting

It is found that average quantity of maize demanded at the market during harvesting is 2931.6 metric tons, while the average quantity of maize supplied by farmers in harvesting season is also 8217.6 metric tons. Therefore the difference between the average supply and demand constitutes the average quantity of maize to be kept in stock during harvesting which is 5286 metric tons.

Quantity of Maize to Release to Market during Planting

If the market dynamics remain the same such that maize will sell at GHC 30.49, and quantity supplied and demanded of maize would be just as happened in the harvesting period, then in the planting season, the buffer stock system could be run to meet market demand at 2931.6 without any shortage. The reason is that, there would still be a reserve of 2354.4 metric tons in stock to take care of any upward demand surge. Thus a reliable buffer stock operation would be guaranteed using delay differential buffer stock model.

Now it is established that time (delay) affects price maize and it is also continuous because price, supply, demand and stock trend could be changed at any time during harvesting or planting season.

Chapter 6

Conclusions and Recommendations

6.1 Summary

The models in this study performed on the assumptions that price at the market is established based on supply available in a single market.

It is found that suppliers do not undertake decisions regarding production and price of their farm produce in only one time as assumed in the case of discrete time cobweb models. At any time when there is the need, they will review decisions concerning supply and price of their farm produce, since it is established in study that time is continuous in the sense of delay, and generation of price fluctuations is associated with delay. The more the delay of suppliers responding to market changes, the more fluctuating the price becomes.

It is deduced that linear cobweb models though, provide an acceptable estimate of most problems researchers come across but in real economic situation there is the need to use nonlinear models as it is shown in this study to avoid under or overestimation and make better predictions especially when one wants to manage price fluctuations using buffer stock models. The reason is that buffer part of the model is also incorporated with delay. Continuous time delay differential buffer stock model works as expected in managing price fluctuations irrespective of the type of supply function it integrates with, being it linear or nonlinear.

6.2 Discrete Time Cobweb Models

A nonlinear cobweb model was presented with backward bending or quadraticliked supply function of price and linear demand function of price derived from difference equations models. It was compared with linear cobweb model which provided an unstable zero equilibrium price point. This was found unrealistic because of producers' sensitivity towards price of farm produce. However, the nonlinear model provided two equilibria price points of which one is also zero but unstable and the other unstable at non-zero price. which is realistic and a reflection of maize price in Ghana due to inflation (even prices of petrol are reviewed every two weeks) and it has counter effects on food prices.

6.3 Continuous Time Cobweb Models

This section of the study discusses stability conditions of two continuous-time cobweb models developed from linear and nonlinear delay differential functions of price.

Results of model evaluation and analysis have shown that nonlinear delay differential buffer stock model would oscillate between two price points and would not converge with time. Therefore no stable equilibrium price would be achieved just as was observed in the discrete case.

However, after showing little oscillations, the linear model converged to stable zero equilibrium price point which is unrealistic due to same reason of farmers' price sensitivity mentioned.

6.4 Time Varying Effects On Price

The delay parameter is found to be associated with price oscillations and this is also discussed. It was observed that for delay value of $\tau \leq 0.5$, price oscillations of nonlinear delay differential equation models are suppressed. This indicates that if all factors affected by delay (time-lag) are improved, then price fluctuations can be reduced and thereby achieve price stability. In contrast, for delay value $\tau > 0.5$ or $\tau \leq 0.5$, linear delay differential equation remained in stable equilibrium. It is also shown that efficiency of buffer stock system is dependent on delay variation selected in line with price scheme to be used by the buffer stock operator.

6.5 Continuous Time Cobweb Model With Buffer

This section of the study used delay differential buffer stock model which mimic an undelayed integro-differential equation model developed by Soltes et al. (2012) for controlling prices by incorporating delay parameter in both the supply function and buffer stock part of the model.

The results of the study dispute an assertion by Mackey (1989) whose argument is based on the fact that price of commodity is dependent on time associated with planting, storage, relaxation and total production. It is proven that if these respective time parameters are put together as one for supply delay and constitute storage time, delay for the buffer, then variability in price is likely to reduce drastically . The study also improved on researches done by Athanasiou et al. (2008), since their models are discrete-time dependent which is a limiting case of the delay differential buffer stock model.

It is realized that, if delays for buffer stock and supply are reviewed in conformity with price scheme run by the buffer operator, then price would be more stable for the impacts of buffer stock scheme felt by stakeholders.

The model evaluation results also provided average maize stable price of GHC 30.49 that is close to actual average price of GHC 30.27. GHC 30.49 in turn provided equilibrium average demand and supply respectively as 2931.6 metric tons and 8217.6 metric tons. The average excess supply that constitutes the stocks in the buffer is also given as 5286 metric tons and they are kept in stock for the next market period (i.e planting period). When at another period (i.e during harvesting period) demand exceeds supply then the appropriate difference is released from the buffer to the market in order to keep price in equilibrium. The standard deviation also reduced to 0.1602 compared to 29.48 by raw data in the table A.2 (at appendix A), and the same applies to the variance.

6.6 Discrete and Continuous Time Models

Continuous time delay differential buffer stock models could be applied in managing unstable market price of maize irrespective of the type of the supply function it is integrated with, being it linear or nonlinear. The continuous time delay model makes time an important factor and price stability dependent. The larger the time delay, the more price fluctuations are generated.

However, it is shown that when the coefficient of quadratic terms in both discrete and continuous time nonlinear supply equations of prices are reviewed downwards then when they are connected with buffer stock models, price stability could be achieved provided the coefficients of price in supply functions are also reviewed simultaneously downwards. Meanwhile delay differential buffer stock can also achieve price stability without perturbing the coefficient of quadratic term in the supply function. This demonstrates that discrete time buffer stock model works as expected in managing price fluctuating systems when the shape of supply function of price is more close to being linear function

The discrete time cobweb models and continuous time delay differential models could be applied to achieve price stability if the marginal supply is less than marginal demand.

6.7 Recommendations

The findings of the study are interpreted in line with its limitations, of which some could be addressed in further studies to evaluate the impact of buffer stock on price when private storage is allowed to compete with the government.

When farmers reduce their production of one produce because they expect a lower price, they are likely to increase their production in the same or another produce in another market, where price is favourable. Therefore in further studies I will recommend that more than one market is considered to see how price stabilization could be guaranteed in-between two or more markets.

The study is also limited to only naive price expectation and it is proposed that further studies consider the effects of adaptive and rational price expectations and their relationship with time effects on price stabilization and then also consider other shocks like inflation.

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Appendix A

Tables of Data and Statistic Outputs

Table 6.1: Price and Production of Maize in Ashanti Region

	Prices (100 kg) in GHC				Production (Metric Tons)			
Year	Qtr1	Qtr2	Qtr3	Qtr4	Qtr1	Qtr2	Qtr3	Qtr4
1994	1.23	1.78	1.49	1.67	10198.13	14830.01	12403.34	13878.19
1995	2.69	3.53	1.64	2.22	15947.12	20917.51	9730.17	13171.87
1996	2.80	3.41	4.61	4.45	10152.87	12357.34	16710.75	16125.71
1997	6.51	8.73	6.74	-	17268.13	23180.01	17886.86	-
1998	5.06	5.62	4.43	4.26	19354.45	21514.80	16930.53	16287.52
1999	4.64	4.94	5.76	4.60	15016.85	15978.78	18646.42	14875.28
2000	9.32	11.05	9.90	10.39	14663.71	17381.97	15579.11	16342.87
2001	13.87	18.76	12.98	13.04	13403.83	18127.28	12539.71	12595.85
2002	14.83	15.38	10.97	11.36	25347.56	26300.98	18760.61	19417.51
2003	14.03	17.06	16.32	14.49	14649.34	17818.76	17039.90	15132.00
2004	17.80	21.67	23.46	23.06	12629.82	15372.77	16646.07	16362.01
2005	29.67	45.02	34.25	28.56	11638.64	17660.19	13436.17	11203.67
2006	25.91	27.39	20.98	18.87	15226.02	16097.10	12328.32	11090.56
2007	25.85	32.34	26.41	26.06	13190.26	16500.28	13474.43	13296.04
2008	32.34	56.66	56.72	49.39	10103.38	17700.44	17717.10	15428.41
2009	60.79	71.77	55.13	52.21	15780.74	18631.95	14311.43	13552.55
2010	53.29	55.51	52.76	46.82	21597.84	22497.63	21385.72	18976.81
2011	55.10	75.41	81.34	89.95	10572.89	14470.08	15607.96	17260.74
2012	110.69	124.44	90.97	76.00	18169.81	20426.27	14932.32	12475.61
2013	77.09	76.24	74.86	81.67	17036.49	16847.17	16542.22	18047.12

Table 6.2: Descriptive Statistics of Data in Table A.1

	N	Range	Mean		Std. Dev.	Skewness		Kurtosis	
	Stat	Stat	Stat	Std.Error	Stat	Stat	Std.Error	Stat	Std.Error
Price	79	123.21	30.27	3.32	29.48	1.16	0.27	0.58	0.54
Production	79	16570.81	16008.72	383.99	3413.04	0.56	0.27	0.60	0.54
Valid N	7								

Table 6.3: ADF Test Table for Price Data at Difference Order (2)

Null Hypothesis: D(tseries,2) has a unit root
Exogenous: Constant
Lag Length: 10 (Automatic Based on AIC, MAXLAG=10)

	t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic	-7.25251	0.00000
Test critical values: 1% level	-3.54024	
5% level	-2.90919	
10% level	-2.59223	

*MacKinnon (1996) one-sided p-values.
Augmented Dickey-Fuller Test Equation
Dependent Variable: D(tseries,3)
Method: Least Squares
Included observations: 62 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob
D(tseries(-1),2)	-9.63365	1.32832	-7.25251	0.00000
D(tseries(-1),3)	7.95092	1.27705	6.22601	0.00000
D(tseries(-2),3)	7.23339	1.20875	5.98420	0.00000
D(tseries(-3),3)	6.74555	1.11554	6.04689	0.00000
D(tseries(-4),3)	6.15931	1.00768	6.11236	0.00000
D(tseries(-5),3)	5.33029	0.92018	5.79265	0.00000
D(tseries(-6),3)	4.42399	0.81447	5.43175	0.00002
D(tseries(-7),3)	3.44845	0.68492	5.03485	0.00007
D(tseries(-8),3)	2.70656	0.51002	5.30677	0.00003
D(tseries(-9),3)	1.83330	0.32120	5.70766	0.00001
D(tseries(-10),3)	0.94765	0.15899	5.96389	0.00000
C	0.17573	0.74556	0.23571	0.81462
R-squared	0.87845	Mean dependent var	-0.79742	
Adjusted R-squared	0.76703	S.D. dependent var	15.00684	
S.E. of regression	5.77887	Akaike info criterion	6.51828	
Sum squared resid	1669.76841	Schwarz criterion	6.92998	
Log likelihood	-190.06667	F-statistic	32.85092	
Durbin-Watson stat	2.06225	Prob(F-statistic)	0.00000	

Table 6.4: ADF Test Table for Production Data at Difference Order (1)

Null Hypothesis: D(tseries) has a unit root
Exogenous: Constant
Lag Length: 6 (Automatic Based on AIC, MAXLAG=10)

	t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic	-5.86395	0.00005
Test critical values: 1% level	-3.513164	
5% level	-2.90550	

	10% level	-2.59028

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation

Dependent Variable: D(tseries,2)

Method: Least Squares

Included observations: 67 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob
D(tseries(-1))	-4.76136	0.81197	-5.86395	0.00000
D(tseries(-1),2)	3.12295	0.74567	4.18810	0.00010
D(tseries(-2),2)	2.28032	0.64397	3.54101	0.00079
D(tseries(-3),2)	1.62953	0.51694	3.15228	0.00255
D(tseries(-4),2)	1.03350	0.37837	2.73147	0.00830
D(tseries(-5),2)	0.52157	0.23773	2.19395	0.03219
D(tseries(-6),2)	0.24932	0.12371	2.01529	0.04844
C	152.94270	491.71758	0.31104	0.75687

R-squared	0.76646	Mean dependent var	-133.36791
Adjusted R-squared	0.56212	S.D. dependent var	7862.28556
S.E. of regression	4018.58449	Akaike info criterion	19.54690
Sum squared resid	952792255.87848	Schwarz criterion	19.81015
Log likelihood	-646.82108	F-statistic	27.66230
Durbin-Watson stat	1.98978	Prob(F-statistic)	0.00000

Table 6.5: ADF Test Table for Price Data at Difference Order (1)

Null Hypothesis: D(tseries) has a unit root

Exogenous: Constant

Lag Length: 6 (Automatic Based on AIC, MAXLAG=10)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-6.02794	0.00003
Test critical values:	1% level	-3.53164	
	5% level	-2.90550	
	10% level	-2.59028	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation

Dependent Variable: D(tseries,2)

Method: Least Squares

Included observations: 67 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob
D(tseries(-1))	-2.29794	0.38122	-6.02794	0.00000
D(tseries(-1),2)	1.40664	0.35568	3.95476	0.00021
D(tseries(-2),2)	1.11295	0.35369	3.14664	0.00259
D(tseries(-3),2)	1.27543	0.33979	3.75364	0.00040
D(tseries(-4),2)	1.43969	0.29626	4.85962	0.00001

D(tseries(-5),2)	0.91846	0.23579	3.89519	0.00025
D(tseries(-6),2)	0.47402	0.17245	2.74875	0.00793
C	2.43950	0.88654	2.75170	00786
R-squared	0.66504	Mean dependent var	-0.50821	
Adjusted R-squared	0.37195	S.D. dependent var	10.35923	
S.E. of regression	6.34117	Akaike info criterion	6.64365	
Sum squared resid	2372.41386	Schwarz criterion	6.90690	
Log likelihood	-214.56240	F-statistic	16.73444	
Durbin-Watson stat	1.68831	Prob(F-statistic)	0.00000	

Table 6.6: ADF Test Table for Production Data at Difference Order (2)

Null Hypothesis: D(tseries,2) has a unit root
Exogenous: Constant
Lag Length: 7 (Automatic Based on AIC, MAXLAG=10)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-7.63763	0.00000
Test critical values:	1% level	-3.53492	
	5% level	-2.90691	
	10% level	-2.59102	

*MacKinnon (1996) one-sided p-values.
Augmented Dickey-Fuller Test Equation
Dependent Variable: D(tseries,3)
Method: Least Squares
Included observations: 65 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob
D(tseries(-1)2)	-11.96776	1.56695	-7.63763	0.00000
D(tseries(-1),3)	9.59122	1.48736	6.44847	0.00000
D(tseries(-2),3)	7.76914	1.33673	5.81203	0.00000
D(tseries(-3),3)	5.87002	1.10641	5.30547	0.00002
D(tseries(-4),3)	4.04775	0.83817	4.82928	0.00001
D(tseries(-5),3)	2.42349	0.55909	4.33472	0.00006
D(tseries(-6),3)	1.20543	0.30472	3.95585	0.00022
D(tseries(-7),3)	0.38478	0.12116	3.17589	0.00243
C	16.78251	554.21913	0.03028	0.97595
R-squared	0.90380	Mean dependent var	-199.59815	
Adjusted R-squared	0.81828	S.D. dependent var	13444.96218	
S.E. of regression	4458.13994	Akaike info criterion	19.77074	
Sum squared resid	1113000654.43425	Schwarz criterion	20.07181	
Log likelihood	-633.54900	F-statistic	65.76152	
Durbin-Watson stat	2.10787	Prob(F-statistic)	0.00000	

Appendix B

Written Codes Run for the Models

6.8 DDE Code: for linear DDE

```
function ddeprice1
%DDEPRICE program for DDE23.
%This is cobweb model developed from the use delay differential equations.
%ddeprice are solved on [0,100] with history y1(t) = 1.23, for t <= 0. %The
lags are specified as a vector [1], the delay differential
%equations are coded in the subfunction DDEPRICE, and the history is
%evaluated by the function histprice. Because the history is constant it %could
be supplied as a vector:
%sol = dde23(@ddeprice,[1],1.23,[0,100]); sol
= dde23(@ddeprice, 1, @histprice, [0,100]);
figure; plot(sol.x, sol.y) title('delay diff cobweb
model.'); xlabel('Time t'); ylabel('Price');
%——————————————————————————————————————
function s = histprice(t)
%Constant history function for ddeprice.
s = (1.23);
%——————————————————————————————————————
function dydt = ddeprice(t,y,Z)
%Delay Differential equations function for DDEPRICE.
ylag1 = Z(:,1); dydt = -1*y(1)-
1.75*ylag1(1);
```

6.9 DDE Code: for Nonlinear DDE

```
function ddeprice2
%DDEPRICE Solution for Nonlinear Model with DDE23.
%This is nonlinear cobweb model developed from delay differential equations)
% ddeprice are solved on [0,100] with history y1(t) = 1.23, for t <= 0.
% The lags are specified as a vector [1], the delay differential
% equations are coded in the subfunction DDEPRICE, and the history is %
evaluated by the function histprice. Because the history is constant it %
could be supplied as a vector:
% sol = dde23(@ddeprice,[1],1.23,[0,100]);
sol =
dde23(@ddeprice,1.0,@histprice,[0,100]);
figure; plot(sol.x,sol.y) title('Nonlinear delay
diff cobweb model.'); xlabel('Time t');
ylabel('Price'); x = linspace(0,20,100); y =
deval(sol,x,1)
```

% ——————————————————————————————–
function s = histprice(t)
% Constant history function for ddeprice.
s = (1.23);
% ——————————————————————————————–
function dydt = ddeprice(t,y,Z)
% Delay Differential equations function for DDEPRICE.
ylag1 = Z(:,1); $dydt = -1 * y(1) - 3.684 * ylag1(1) + 0.029 * ylag1(1)^2$;

6.10 DDE Code: DDE with Buffer Stock Model

---

function ddebuffer2
%DDEPRICE Solution for DDE23.
%This code for delay differential equations with buffer stock %ddeprice are solved on [0,100] with history functions $y_1(t) = 1.23$ %and $y_1'(0) = 0$,for $t <= 0$.
% The lags are specified as a vector [a,b] (they are varied), the delay differential %equations are coded in the subfunction DDEPRICE, and the history is %evaluated by the function histprice. Since the history is constant, it %could be supplied as a vector:
%sol = dde23(@ddeprice,[a,b],[1.23;0],[0,100]); sol = dde23(@ddebuffer,[0.40,0.21],@histbuffer,[0,100]);
figure; plot(sol.x,sol.y) title('delay diff cobweb model with buffer stock.'); xlabel('time t'); ylabel('solution y');
x = linspace(0,20,100); y = deval(sol,x,1);
% ——————————————————————————–
function s = histbuffer(t)
% Constant history function for ddeprice.
s = [1.23,1.23];
% ——————————————————————————–
function dydt = ddebuffer(t,y,Z)
% Delay Differential equations function for DDEPRICE.
ylag1 = Z(:,1); ylag2 = Z(:,2); $dydt = [(-1 * y(1) - 3.68 * ylag1(1) + 0.029 * ylag1(1)^2);((-1 * y(1) - 3.68 * ylag1(1) + 0.029 * ylag1(1)^2) - (1 * y(2) + 3.68 * ylag2(2) - 0.029 * ylag2(2)^2))]$;

6.11 Code: Linear Discrete Time Cobweb Model

%MatLab Code for Discrete Time Linear Cobweb Model
n = [1 : 100]; y(1) = 1.23; % Initial value of commodity price for m=2:100; y(m)=-1.75*y(m-1); %Linear cobweb model end

plot(n,y) title ('Discrete Time Linear Cobweb Model'); xlabel('Time t'); ylabel('Price');

6.12 Code: Nonlinear Discrete Time Cobweb Model

%MatLab Code for Discrete Time Nonlinear Cobweb Model n=[1:100]; y(1) = 1.23; % Initial value of commodity price for m=2:100; $y(m) = -3.684 * y(m-1) + (0.029 * (y(m-1))^2)$; % Model in the form of quadratic end
plot(n,y) title ('Discrete Time Nonlinear Cobweb Model'); xlabel('time t'); ylabel('solution y');

6.13 Code: Nonlinear Discrete Time Cobweb Model with Buffer stock

%MatLab Code for Discrete Time Nonlinear Cobweb Model with Buffer Stock n=[1:50]; y(1) = 1.23; % Initial value of commodity price for m=2:50; $y(m) = -.64 * y(m-1) + (0.0001 * (y(m-1))^2) - 0.0104 * (16008.72 - (-.64 * y(m-1) + (0.0001 * (y(m-1))^2)))$ %Model in the form of quadratic end
plot(n,y) title ('Discrete Time Nonlinear Cobweb Model With Buffer'); xlabel('time t'); ylabel('solution y');

List of Publications Based on This Study

Martin Anokye, Francis T. Oduro, John Amoah-Mensah, Prince O. Mensah and Emelia O. Aboagye.(2014). ”Dynamics of Maize Price in Ghana: Linear versus Nonlinear Cobweb Models” *Journal of Economics and Sustainable Development*, **Vol 5, No. 7, April, 2014.**

Anokye Martin. and F.T Oduro.(2014). ”Price Dynamics of Maize in Ghana: An Application of Continuous Time Delay Differential Equations” *British Journal of Mathematics and Computer Science*, **Vol. 4 No. 24, September, 2014.**

Anokye Martin. and F.T Oduro.(2015). ”Maize Price Stabilization in Ghana: An Application of A Continuous-Time Delay Differential Equation Model with Buffer Stock” *British Journal of Mathematics and Computer Science*, **Vol. 6 No. 4 January 2015.**

Presentation at an International Conference

Anokye Martin and F.T. Oduro (2016). ”Stabilization of Maize Price in Ghana Using Continuous-Time Delay Differential Equation Model with Buffer Stock”. At the International Conference of Commonwealth Association of Technical Universities and Polytechnics in Africa (CAPA), on Strategic Involvement of TVET Institutions towards the attainment of Post-2015 Sustainable Development Goals in Africa 5th-
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