KWAME NKRUMAH UNIVERSITY OF SCIENCE AND

TECHNOLOGY, KUMASI

TRACES IN COMPLEX HYPERBOLIC GEOMETRY

By

DICKSON YAW BADU ANNOR

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY IN PARTIAL FUFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF M.PHIL PURE MATHEMATICS

October, 2015

Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no

material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

Dickson Yaw Badu Annor ....

Student Signature Date

Certi ed by:

Dr. Richard Kena Boadi ....

Supervisor Signature Date

Certi ed by:

Prof. S. K. Amponsah ....

Head of Department Signature Date

Dedication

I dedicate this work to God and to my parents Mr. David K. Annor and Mad. Adwoa Fordjour.

Abstract

This thesis concerns the study of traces in complex hyperbolic geometry. In this thesis we review a paper by Parker. We begin by looking at basic notions of complex hyperbolic geometry, speci cally for the complex hyperbolic space. The main results of the thesis fall into three broad chapters. In the third chapter we reconstruct the proof of proposition 2. Suppose that $A \in SU(2,1)$ has distinct eigenvalues $e^{i\theta}$,$e^{i\varphi}$ and $e^{i\psi}$. We prove that A has a unique xed point in $\mathbf{H}^2_{\mathbb{C}}$ corresponding to one of the eigenspaces. We also amplify calculations given by Parker. In chapter four we prove corollary 3 and 4, and we also prove that $\mathrm{tr}[A,B]\mathrm{tr}[B,A]$ may be expressed as a polynomial function of traces of $A,B,AB,A^{-1}B$ and their inverses. Furthermore, we use equation 18 of Lawton to prove the identity for $|\mathrm{tr}[A,B]|^2$. Finally we discuss the merits on the two ways to parametrise pair of pants groups. As an application, we compute traces of matrices generated by complex re ections in the sides of complex hyperbolic triangle groups in the fth chapter.

Acknowledgements

I express my deepest gratitude to Dr. Richard Kena Boadi for being my supervisor and o ering me so much guidance and help throughout this research and my MPhil. studies. Dr. Boadi's invaluable help, guidance and insights greatly in uenced me. His optimistic attitudes towards both research and life impressed me so much and will always encourage me in my future life.

My special thanks go to Prof. John R. Parker at Durham University, UK for providing me most of the reference materials used for this research. I appreciate your time spent in answering all the e-mails I sent to you and the words of encouragement I had from you.

I am indebted to Dr. Eward Prempeh, Dr. Kwasi Baah Gyam , Mr. F. K. Darkwah and Rev. Dr. Obeng Denteh all of the mathematics department, KNUST for receiving the bene ts of their knowledge and experiences. I would also like to thank the other lecturers at the mathematics department.

I can never forget the immense contributions made by Ansah and Prince, for the great exchange of knowledge during our weekly presentations with our supervisors. I learnt a lot from you guys. And to all my MPhil. colleagues who have nished and are yet to nish I say ayeekoo. You have gone through a lot and you deserve to be honoured for your hard work.

Finally, the most importantly, I always thank my family for their endless love and support. My parents dedicated all their love and e ort to my twenty years' long education. They always support my decisions to pursue my dreams even I need to be 10, 000km away. They keep my heart warm and my belief rm.
I would want to say "ayeekoo" to all thousand and one friends who contributed in adverse ways, nancially and spiritually.

Contents

List of Abbreviation

$\mathbb{R}$ .... The set of real numbers

$\mathbb{Z}$ .... The set of integers

$\mathbb{C}$..The set of complex numbers

A^* .... Hermitian transpose of matrix A tr(A)

.... Trace of matrix A

$\langle z,w\rangle$..Inner product of z and w

det(H)...Determinant of matrix H $\mathbb{C}^k$

....Complex space of dimension k

$\mathbb{C}^{p,q}$ .... Complex matrix of signature (p, q)

$\Re(a)$ .... The real part of $a \in \mathbb{C}$

=(a) .... The imaginary part of a ∈ C ||

.... Absolute value z

_

.... Complex conjugate of z ∈ C w^t

....Tranpose of vector w arg(z)

.... Argument of z I

....Identity matrix

$\partial\mathbf{H}^2_{\mathbb{C}}$ .... Boundary of complex hyperbolic space

List of Figures

Chapter 1

INTRODUCTION

1.1 Background

This thesis consists of the study of complex hyperbolic geometry by means of traces on the complex hyperbolic space. A lot of studies have been done on complex hyperbolic geometry by various mathematicians in the eld of geometry over the years. Among them is Goldman (1999) in his book titled
Complex Hyperbolic Geometry.

Hyperbolic (also called non-Euclidean) geometry is the study of geometry on spaces of constant negative curvature. In dimension 2, surfaces of constant curvature are distinguished by whether their curvature K is positive, zero or negative. Hyperbolic geometry is closely connected to many other parts of mathematics like di erential geometry, complex analysis, topology, dynamical systems including complex dynamics and ergodic theory, relativity, number
theory, Riemann surfaces etc.

In particular, our area of interest in the hyperbolic geometry is the complex case. Questions that can be asked in the real case could also be asked in the complex case. Complex hyperbolic geometry is a particularly rich area of study, enhanced by the con uence of several areas of research including Riemannian geometry, complex analysis, symplectic and contact geometry, lie group theory and harmonic analysis (Goldman, 1999). It has several applications
both in the eld of mathematics and real life.

Pratoussevitch (2005) in her paper *traces in complex hyperbolic triangle groups* presented several formulas for the traces of elements in complex hyperbolic triangle groups generated by complex re ections and applied these formulas to prove some discreteness and non-discreteness for complex hyperbolic triangle groups. In his survey paper published as Parker (2012), Parker studied the connection between the geometry of M and traces of Γ, where M is a complex hyperbolic orbifold written as $\mathbf{H}_{\mathbb{C}}^2/\Gamma$ and Γ is a discrete, faithful representation of $\pi_1(M)$ to Isom($\mathbb{H}^2_{\mathbb{C}}$). He did that by rst considering the case where Γ is a free group on two generators and secondly, he discussed formulae of Pratoussevitch (2005) in the case where Γ is a triangle group generated by complex re ections in three complex lines. Several geometrical information connecting traces and complex hyperbolic space could be seen in Parker (2012).

1.2 Problem statement

Parker in attempting to discuss traces in complex hyperbolic geometry, gave theorems, propositions and corollaries which all talked about trace identities. After reading his paper carefully, we realised some of the propositions and corollaries were left unproven. So the unanswered question was how do we get explicit constructions of the proof of these established propositions and corollaries. However, we followed road maps suggested by Parker.

1.3 Objective

The main objective of the study was to review a paper by Parker (2012) on traces in complex hyperbolic geometry. In order to do this, we made valuable use of equation 18 of Lawton (2007), trace formula which is due to Pratoussevitch (2005) and trace identities by Will (2009).

1.4 Speci c objectives

In order to achieve the main objective of the study, the following speci c objectives were set:

1. To prove that the $\mathrm{tr}[A,B]\mathrm{tr}[B,A]$ may be expressed as a polynomial function of the traces of $A,B,AB,A^{-1}B$ and their inverses.

2. To give two di erent representations for equation 18 of Lawton (2007).

3. To prove the identity for $|\mathrm{tr}[A,B]|^2$.

4. To discuss the application of a trace formula which is due to Pratoussevitch (2005).

5. To state the merits on the two ways to parametrise pair of pants groups.

1.5 Plan of thesis

This thesis is therefore organised as follows. The main results will fall into three broad chapters (3, 4, and 5), each of which is conceived to be self-contained, with its own introduction.

Chapter one looks at the general introduction of the thesis. In Chapter 2 we recall the basic notions of complex hyperbolic geometry, speci cally for complex hyperbolic space. We study the geometry of complex hyperbolic space through complex linear algebra.

In the third chapter we discuss the geometry of isometries; speci cally, classi cation of elements of SU(2, 1) by their trace, traces and eigenvalues for loxodromic maps and eigenvalues and complex displacement for loxodromic maps. Our contributions in this chapter are: ampli cation of the calculations in Parker (2012), proving and reconstructing proofs of propositions.

Chapter 4 looks at two generator groups and Fenchel-Nielsen coordinate. In this chapter we amplify calculations and proofs of Parker (2012). One other result of the chapter is the explicit polynomial for $\mathrm{tr}[A,B]\mathrm{tr}[B,A]$. Also, in an attempt to proof the imaginary part of $\mathrm{tr}[A,B]$ (which was not considered by Parker, 2012) we express equation 18 of Lawton (2007) in terms of $\mathrm{tr}(A),\mathrm{tr}(B),\mathrm{tr}(AB)$ etc. Base on this, we give a proposition and remark on the two di erent representations. Finally we remark on how to parametrise pair of

pants via traces and cross-ratio.

Chapter 5 explains traces for triangle groups. In the last section of this chapter, we give application of a trace formula which is due to Pratoussevitch (2005). We devote chapter 6 for conclusion and recommendation.

Chapter 2

COMPLEX HYPERBOLIC SPACE

2.1 Introduction

In this chapter we review basic features of complex hyperbolic geometry which may be needed later on, speci cally for complex hyperbolic space. We begin with some key de nitions that will be useful in this work and also basic results that will appear through out this thesis in the chapter. Further de nitions nd themselves in the appendix. Let $\mathrm{PU}(2,1)$ denote the projective unitary group of signature (2, 1). Let $\mathbf{H}^2_{\mathbb{C}}$ denotes complex hyperbolic space of dimension 2.

De nition 1 (Matrix) A matrix is a rectangular array of numbers. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{pmatrix}.$$

This matrix is 3 × 4 matrix because there are three rows and 4 columns. The rst row is $(1\ 2\ 3\ 4)$, the second row is $(5\ 2\ 8\ 7)$. The rst column is $\begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$. See for example (Kuttler, 2008).

Operations on matrices

1. $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$(addition)

2. $tA = t[a_{ij}] = [ta_{ij}]$ (scalar multiplication)

3. $-A = -[a_{ij}] = [-a_{ij}]$ (additive inverse)

4. $A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$ (subtraction)

The matrix operations of addition, scalar multiplication, additive inverse and subtraction satisfy the following properties:

1. $A + B = B + A$

2. $(A + B) + C = A + (B + C)$

3. $A + 0 = A$

4. $A + (-A) = 0$

5. $(s + t)A = sA + tA, (s - t)A = sA - tA$

6. $t(A + B) = tA + tB, t(A - B) = tA - tB$

7. $s(tA) = (st)A$

8. $1A = A, 0A = 0, -1(A) = -A$

9. $tA = 0 \Rightarrow t = 0 or A = 0$

where A,B and C are $m \times n$ matrices and s and t are scalars.(Matthews, 1998)

De nition 2 (Invertible matrix) An $n \times n$ (square) matrix A is called invertible/non-singular, if there exists an $n \times n$ matrix B such that $AB = BA = I_n$ where I_n denotes the $n \times n$ identity matrix and the multiplication used is the ordinary matrix multiplication. If this is the case, then matrix B is uniquely determined by A and is called the inverse of A, denoted by A^{-1}. (Gyam , 2012)

De nition 3 (Row/column vector) Matrices which are $n \times 1$ or $1 \times n$ are specially called vectors and are often denoted by a bold letter. Thus

x_1

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a $n \times 1$ matrix also called a column vector while a $1 \times n$ matrix of the form $(x_1 \cdots x_n)$ is referred to as a row vector. (Kuttler, 2008)

Definition 4 (Trace of a matrix) The trace of an $n \times n$ square matrix A is defined to be the sum of the elements of the main diagonal (the diagonal from the upper left to lower right) of $A = [a_{ij}]$ ie.

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

Definition 5 (Determinant of a matrix) The determinant of a square matrix $A = [a_{ij}]$ is a number denoted by $|A|$ or $\det(A)$. This number is defined as the following function of the matrix elements:

$$|A| = \det(A) = \pm \sum a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where the column indices $j_1, j_2, \cdots, j_n$ are taken from the set $\{1, 2, \cdots, n\}$ with no repetition allowed. The plus (minus) sign is taken if the permutation $(j_1, j_2, \cdots, j_n)$ is even (odd).

Theorem 2.1.1 Let A be an $n \times n$ matrix. Then $\operatorname{tr}(A)$ equals the sum of the eigenvalues of A and $\det(A)$ equals to the product of the eigenvalues of A (Kuttler, 2008).

Definition 6 (Eigenvalue and eigenvector) If there exists (possibly complex) scalar λ and vector x such that

$$Ax = \lambda x \text{ or equivalently, } (\lambda I - A)x = 0, x \neq 0$$

then x is the eigenvector corresponding to the eigenvalue λ. Recall that $n \times n$ matrix has n eigenvalues (the roots of the polynomial $\det(\lambda I - A)$).

2.2 Hermitian matrices

Let $A = (a_{ij})$ be a $k \times l$ complex matrix. The Hermitian transpose of A is the $l \times k$ complex matrix $A^* = (\overline{a_{ji}})$ formed by complex conjugating each entry of A and then taking the transpose. The Hermitian transpose of a matrix AB is $(AB)^* = B^*A^*$. Thus, the Hermitian transpose of a product is the product of the Hermitian transposes in the reverse order. Clearly $(A^*)^* = A$.

A $k \times k$ complex matrix H is said to be Hermitian if it equals its own Hermitian transpose i.e. $H = H^*$. A typical example is

$$H = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & -2 \end{bmatrix} = H^*$$

Notice that the diagonal entries must be real, they have to be unchanged by the process of conjugation. Each o diagonal entry is matched with its mirror image across the main diagonal, and $2+i$,$3i$ and $1+i$ are the conjugates of $2-i$,$-3i$,$1-i$ respectively (Strang, 1988).

Let H be a Hermitian matrix and λ an eigenvalue of H with eigenvector $z \neq 0$. We claim that λ is real.

$$\lambda z^* z = z^*(\lambda z) = z^* H z = z^* H^* z = (Hz)^* z = (\lambda z)^* z = \overline{\lambda} z^* z.$$

Since z^*z is length squared, real and positive, we see that λ is real for λ to be equal to $\bar{\lambda}$. Suppose that H is a non-singular Hermitian matrix (that is, all its eigenvalues are non-zero) with p positive eigenvalues and q negative ones. Then we say that H has signature (p,q).

2.3 Hermitian forms on $C^{p,q}$

For each $k \times k$ Hermitian matrix H we can associate a Hermitian form

$$h\cdot,\cdot i : C^k \times C^k \to C \textit{ given by } hz,wi = w^*Hz$$

(notice the change in the order) where w and z are vectors in C^k. Note that the $h\cdot,\cdot i$ is the Hermitian form and is always with respect to a particular Hermitian matrix H. Hermitian forms are sesquilinear, that is they are linear in the rst factor and conjugate linear in the second factor. In other words, Hermitian forms with the following properties are called sesquilinear:

$$hz_1 + z_2,wi = w^*H(z_1 + z_2) = w^*Hz_1 + w^*Hz_2 = hz_1,wi + hz_2,wi;$$

$$h\lambda z,wi = w^*H(\lambda z) = \lambda w^*Hz = \lambda hz,wi;$$

$$hz,\lambda wi = (\lambda w)^*Hz = \bar{\lambda} w^*Hz = \bar{\lambda} hz,wi;$$

$$hw,zi = z^*Hw = z^*H^*w = (w^*Hz)^* = \overline{hz,wi}.$$

where z,z_1,z_2,w are column vectors in C^k and λ a complex scalar (Parker, 2010). Let $h\cdot,\cdot i$ be a Hermitian form associated with the Hermitian matrix H.

We know that the eigenvalues of H are real. We say that

1. $h\cdot,\cdot i$ is non-degenerate if all the eigenvalues of H are non-zero;

2. $h\cdot,\cdot i$ is positive de nite if all the eigenvalues of H are positive;

3. h·,·i is negative de nite if all the eigenvalues of H are negative;

4. h·,·i is de nite if some eigenvalues of H are positive and some are negative.

Suppose that h·,·i is a non-degenerate Hermitian form associated to the $k \times k$ Hermitian matrix H. We say that h·,·i has signature (p,q) where $p + q = k$ if H has p positive eigenvalues and q negative eigenvalues. Thus positive de nite Hermitian forms have signature $(k, 0)$ and negative de nite forms have signature $(0, k)$. We often write $\mathbb{C}^{p,q}$ for $\mathbb{C}^{p+q}$ equipped with a non-degenerate Hermitian form of signature (p,q). This generalises the idea of $\mathbb{C}^p$ with the implied Hermitian form of signature $(p, 0)$.

For real matrices the Hermitian transpose coincides with the ordinary transpose. A real matrix that equals its own transpose is called symmetric. Symmetric matrices de ne bilinear forms on real vector spaces, usually called quadratic forms.

Example 2.3.1 Consider

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H_0' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

It can be seen that H_0 and H^0_0 are both Hermitian. Moreover, since H_0 is a diagonalised matrix, it is easier to see that it has signature (1, 1). H^0_0 also has signature of (1, 1).

Let h·,·i be a Hermitian form associated to the $k \times k$ Hermitian matrix H. A $k \times k$ matrix A is called unitary with respect to H if for all z and w in $\mathbb{C}^k$ we have

$$w^*A^*HAz = \text{hAz,Awi} = \text{hz,wi} = w^*Hz.$$

If the Hermitian form is non-degenerate then unitary matrices form a group. The group of matrices preserving this Hermitian form will be denoted U(H). Sometimes it is only necessary to determine the signature. If h·,·i has signature

(p,q) then we write $U(p,q)$. Since A preserves that form we have

$$w^*A^*HAz = (Aw)^*H(Az) = \langle z,w\rangle = w^*Hz.$$

Therefore letting z and w run through a basis for C^k we have $A^*HA = H$. If H is non-degenerate then it is invertible and this translates to an easy formula for the inverse A:

$$A^{-1} = H^{-1}A^*H.$$

Most of the Hermitian forms we consider will have eigenvalues ±1 and so will be their own inverse. One consequence of this formula is that

$$\det(H) = \det(A^*HA) = \det(A^*)\det(H)\det(A).$$

If $\det(H) \neq 0$ (so the form is non-degenerate) then

$$1 = \det(A^*)\det(A) = \overline{\det(A)}\det(A) = |\det(A)|^2.$$

Thus unitary matrices have unit modulus determinant. The group of those unitary matrices whose determinant is +1 is denoted by $SU(H)$.

Example 2.3.2 Consider the Hermitian forms H_0 and H'_0 in (example 2.3.1).

Suppose that $A \in SU(H_0)$. Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1} = H_0^{-1}A^*H_0 = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}.$$

Therefore $b = \bar{c}$ and $d = \bar{a}$. Hence $1 = ad - bc = |a|^2 - |c|^2$. Hence

$$SU(H_0) = \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} : a,c \in \mathbb{C},\ |a|^2 - |c|^2 = 1 \right\}.$$

Similarly, suppose $A' \in SU(H'_0)$. Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A'^{-1} = H_0'^{-1} A'^{*} H_0' = \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix}.$$

Therefore *a,b,c,d* are all real. Hence

$$H_0') = \left\{ \begin{pmatrix} & \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

a
b

SU(.

That is, $\mathrm{SU}(H_0') = \mathrm{SL}(2, \mathbb{R})$.

2.4 Cayley transform

Given two Hermitian forms H and H^0 of the same signature we can pass between them using a Cayley transform C. That is, we can write

$$H^0 = C^{*}HC.$$

The Cayley transform C is not unique for we may pre-compose and post-compose by any unitary matrix preserving the relevant Hermitian form. The following Cayley transform interchanges the rst and second Hermitian forms

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

In order to see that C is a Cayley transform, we calculate

$$C^*H_1C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = H_2.$$

Also, $C^{-1} = C$ and so $C^*H_2C = H_1$. It is clear that if A is unitary with respect to H then $A^0 = C^{-1}AC$ is unitary with respect to H^0. In order to see this, observe that, using $(C^{-1}AC)^* = C^*A^*C^{*-1}$, we have

$$A^{0*}H^0A^0 = (C^{-1}AC)^*(CHC)(C^{-1}AC) = C^*A^*HAC = C^*HC = H^0.$$

Example 2.4.1 Consider H_0 and H'_0 given by (example 2.3.1) and

$$C_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

One can verify that $H'_0 = C_0^* H_0 C_0$. Furthermore suppose

$$A = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \in \mathrm{SU}(H_0).$$

Then

$$A' = C_0^{-1}AC_0 = \begin{pmatrix} \Re(a) - \Im(c) & \Im(a) + \Re(c) \\ -\Im(a) + \Re(c) & \Re(a) + \Im(c) \end{pmatrix} \in \mathrm{SU}(H'_0) = \mathrm{SL}(2, \mathbb{R}).$$

2.5 Three models of complex hyperbolic space

There are three standard models of complex hyperbolic space, namely:

1. the projective model in $\mathbb{P}^n_{\mathbb{C}}$;

2. the unit ball model in $\mathbb{C}^n$ and

3. the Siegel domain model in $\mathbb{C}^n$.

Let H be a 3×3, non-singular Hermitian form of signature $(2, 1)$. For $z \in \mathbb{C}^{2,1}$ we have $\langle z,z\rangle \in \mathbb{R}$. Let V_-, V_0, V_+ be the subsets of $\mathbb{C}^{2,1}$ defined by

$$V_- = \{z \in \mathbb{C}^{2,1} \mid \langle z,z\rangle < 0\} \tag{2.1}$$

$$V_0 = \{z \in \mathbb{C}^{2,1} - \{0\} \mid \langle z,z\rangle = 0\}, \tag{2.2}$$

$$V_+ = \{z \in \mathbb{C}^{2,1} \mid \langle z,z\rangle > 0\}, \tag{2.3}$$

Vectors in V_-, V_0, V_+ are called negative, null or isotropic, positive respectively.

Example 2.5.1 Consider $\mathbb{C}^{1,1}$ with the Hermitian form given by H_0 in (example 2.3.1). Then

$$V_- = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} : |z_1|^2 < |z_2|^2 \right\},$$

$$V_0 = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} : |z_1|^2 = |z_2|^2 \right\},$$

$$V_+ = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} : |z_1|^2 > |z_2|^2 \right\}.$$

Likewise $\mathbb{C}^{1,1}$ with the Hermitian form given by H'_0,

$$V_- = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : \Im(z_1\overline{z}_2) > 0 \right\},$$

$$V_0 = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \Im(z_1\overline{z}_2) = 0 \right\}$$

$$V_+ = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : \Im(z_1\overline{z}_2) < 0 \right\}.$$

,

De ne an equivalence relation on $C^{2,1}-\{0\}$ by z $\sim$ w if and only if there is a non-zero complex scalar λ so that w = λz. We de ne the standard projection map

$$P : C^{2,1} - \{0\} \longmapsto CP^2 \; by \; P(z) = [z]$$

where [z] is the equivalence class of z. We therefore de ne a projection map P on these points of $C^{2,1}$ with $z_3 \neq 0$ as

$$\mathbb{P} : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} \in \mathbb{C}^2$$

.

Because $\langle\lambda z, \lambda z\rangle = |\lambda|^2\langle z, z\rangle$ we see that for any non-zero complex scalar λ the point λz is negative, null or positive if and only if z is negative, null or positive. The projective model of complex hyperbolic space is de ned to be the collection of negative lines in $C^{2,1}$, that is, $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_-$. The boundary is de ned as the collection of null lines, that is, $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_0$.

In what follows, we de ne the other two standard models of complex hyperbolic space by considering two standard Hermitian forms on $C^{2,1}$. We call these the rst and second Hermitian forms. If the vectors $z = (z_1, z_2, z_3)^t$ and $w = (w_1, w_2, w_3)^t$ are in $C^{2,1}$. The rst Hermitian form is de ned to be :

$$\langle z,w\rangle_1 = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3} \text{ from } \langle z,w\rangle_1 = w^*H_1z \text{ where}$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

the Hermitian matrix. The second Hermitian form is defined to be:

$$\langle z,w\rangle_2 = z_1\overline{w_3} + z_2\overline{w_2} + z_3\overline{w_1} \text{ from } \langle z,w\rangle_2 = w^*H_2z \text{ where}$$

$$H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

the Hermitian matrix.

Both of these forms have the property that each vector in V_- has nonzero third entry. Therefore, we can take the section defined by $z_3 = 1$. This gives a unique point on each complex line in V_-. In other words, given $z = (z_1,z_2) \in \mathbb{C}^2$, we define its standard lift to $\mathbb{C}^{2,1}$ to be column vector

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}$$

in $\mathbb{C}^{2,1}$. Clearly $P(\mathbf{z}) = z$. We consider what it means for $\langle \mathbf{z},\mathbf{z}\rangle$ to be negative for the first and second Hermitian forms respectively. For the first Hermitian form we obtain $z \in \mathbf{H}^2_{\mathbb{C}}$ provided:

$$\langle \mathbf{z},\mathbf{z}\rangle_1 = \overline{z_1}z_1 + \overline{z_2}z_2 - 1 < 0 \Rightarrow |z_1|^2 + |z_2|^2 < 1.$$

Thus $z = (z_1, z_2)$ is in the unit ball in $\mathbb{C}^2$ forming the unit ball model of complex hyperbolic space. The boundary of the unit ball model is the sphere S^3 given by

$$|z_1|^2 + |z_2|^2 = 1.$$

For the second Hermitian form we obtain $z \in \mathbf{H}^2_{\mathbb{C}}$ provided:

$$\langle \mathbf{z}, \mathbf{z} \rangle_2 = z_1 + \bar{z}_2 z_2 + \bar{z}_1 < 0 \Rightarrow 2\Re(z_1) + |z_2|^2 < 0.$$

Thus $z = (z_1, z_2)$ is in a domain in $\mathbb{C}^2$ whose boundary is the paraboloid defined by

$$2\Re(z_1) + |z_2|^2 = 0.$$

This domain is called the Siegel domain and forms the Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$. However, not all the points in $\mathbb{P}(V_0)$ lie in $\mathbb{C}^2 \subset \mathbb{CP}^2$. We have to add an extra point, denoted ∞, on the boundary of the Siegel domain. The standard lift of ∞ is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Of course ∞ is not the only point in $\mathbb{CP}^2 - \mathbb{C}$, that is it not the only point " at infinity". In this respect it is different from the point ∞ on the boundary of the upper half plane model of the hyperbolic plane, which is the only point of $\mathbb{CP}^1$ that is not in $\mathbb{C}$.

Example 2.5.2 For $z \in \mathbb{C}$ the standard lift of z to $\mathbb{C}^{1,1}$ is

$$\mathbf{z} = \begin{pmatrix} z \\ 1 \end{pmatrix} \in \mathbb{C}^{1,1}.$$

If $\mathbb{C}^{1,1}$ has the Hermitian form given by H_0 from (2.1), then we see that $z \in \mathbb{P}V_0$ if and only if $|z| = 1$. Thus $\mathbb{P}V_-$ is the unit disc and $\mathbb{P}V_0$ is the unit circle in $\mathbb{C}$. Similarly for H_0', the point $z \in \mathbb{P}V_-$ if and only if $\Im(z) > 0$ and $z \in \mathbb{P}V_0 \cap \mathbb{C}$ if and only if z is real. We must add an extra point ∞ whose standard lift is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Remark 1: According to Parker (2012) there are other Hermitian forms which are widely used in the literature. In particular, Chen and Greenberg give a close relative of the second Hermitian form. We will refer to this as the third Hermitian form. It is given by

$$\langle z,w\rangle_3 = -z_1\overline{w_2} - z_2\overline{w_1} + z_3\overline{w_3}.$$

It is given by the Hermitian matrix H_3:

$$H_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So far we have de ned complex hyperbolic space as set points. In order to understand its geometry we must give it a metric.

For the projective model, the metric on $\mathbf{H}^2_{\mathbb{C}}$ known as the *Bergman metric* is given by

$$ds^2 = \frac{-4}{\langle \mathbf{z},\mathbf{z}\rangle^2}\det\begin{pmatrix} \langle \mathbf{z},\mathbf{z}\rangle & \langle d\mathbf{z},\mathbf{z}\rangle \\ \langle \mathbf{z},d\mathbf{z}\rangle & \langle d\mathbf{z},d\mathbf{z}\rangle \end{pmatrix}. \quad (2.4)$$

The choice of the constant 4 in the above formula means that the holomorphic sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ is −1. The distance between points $z, w \in \mathbf{H}^2_{\mathbb{C}}$ is given by the formula

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z},\mathbf{w}\rangle\langle \mathbf{w},\mathbf{z}\rangle}{\langle \mathbf{z},\mathbf{z}\rangle\langle \mathbf{w},\mathbf{w}\rangle} = \frac{|\langle \mathbf{z},\mathbf{w}\rangle|^2}{|\mathbf{z}|^2|\mathbf{w}|^2}. \tag{2.5}$$

For the ball model and Siegel domain model one can nd the distance between points z and w and by plugging their standard lifts z and w into the above formula.

Example 2.5.3 Consider $z \in$ C with $|z| < 1$. We have seen that for the Hermitian form H_0 this point is in $z \in$ PV_-. Moreover, the standard lift of z and its derivative are

$$\mathbf{z} = \begin{pmatrix} z \\ 1 \end{pmatrix}, \qquad d\mathbf{z} = \begin{pmatrix} dz \\ 0 \end{pmatrix}.$$

Plugging this vector and the Hermitian form given by H_0 into (2.4) gives

$$ds^2 = \frac{-4}{(|z|^2-1)^2}\det\begin{pmatrix} |z|^2-1 & \bar{z}dz \\ zd\bar{z} & dzd\bar{z} \end{pmatrix} = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

This is just the Poincare´ metric on the unit disc. Similarly, consider $z \in$ C with $=(z) > 0$. This is in PV_- for the Hermitian form H_0'. Plugging its standard lift into (2.4) gives

$$ds^2 = \frac{-4}{(-2\Im(z))^2}\det\begin{pmatrix} -2\Im(z) & idz \\ -id\bar{z} & 0 \end{pmatrix} = \frac{|dz|^2}{(\Im(z))^2}.$$

This is the Poincare´ metric on the upper half plane. Note that in both of these examples we have the constant 4 and constant curvature −1.

Unitary matrices in $U(2,1)$ acts on $C^{2,1}$ preserving V_+, V_0 and V_-. They also preserve the Bergman metric since it is given solely in terms of the Hermitian form. Therefore unitary matrices act as isometries on complex hyperbolic space. Let us see this action explicitly. Let $z = (z_1, z_2)$ be a point in C^2 and let z be its standard lift to $C^{2,1}$. Then $A \in U(2,1)$ acts as follows:

$$A(z) = P(Az).$$

In other words, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

then

$$\begin{aligned} A(z_1,z_2) &= PA(z_1,z_2) \\ &= P\left(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}\right) \\ &= P\begin{pmatrix} az_1 + bz_2 + c \\ dz_1 + ez_2 + f \\ gz_1 + hz_2 + j \end{pmatrix} \\ &= \begin{pmatrix} (az_1 + bz_2 + c)/(gz_1 + hz_2 + j) \\ (dz_1 + ez_2 + f)/(gz_1 + hz_2 + j) \end{pmatrix}. \end{aligned}$$

This is just a linear fractional transformation in two variables.

Example 2.5.4 Consider $A \in SU(H_0)$:

$$A = \begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix}.$$

Similarly, we see that A acts on the unit disc as the Mo¨bius transformation in $\mathrm{PSU}(H_0)$

$$\begin{aligned} A\mathbf{z} &= \mathbb{P}\begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} \\ &= \mathbb{P}\begin{pmatrix} az+\overline{c} \\ cz+\overline{a} \end{pmatrix} \\ &= \frac{az+\overline{c}}{cz+\overline{a}}. \end{aligned}$$

Also, for H_0', the matrix $A \in \mathrm{SU}(H_0') = \mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane as a Mo¨bius transformation in PSL(2,R).

2.6 PU(2, 1) and its action on complex hyperbolic space

Let U(2,1) be a group of unitary matrices for the Hermitian form h·,·i. Each such matrix A satis es the relation $A^{-1} = H^{-1}A^{*}H$ where A^{*} is the Hermitian transpose of A.

The full group of holomorphic isometries of complex hyperbolic space is the projective unitary group PU(2,1) = U(2,1)/U(1), where U(1) = $\{e^{i\theta}I, \theta \in [0,2\pi)\}$ and I is the 3 × 3 identity matrix.

Sometimes it will be useful to consider SU(2,1), the group of matrices with determinant 1 which are unitary with respect to h·,·i. The group SU(2,1) is a 3-fold covering of PU(2,1). Therefore PU(2,1) = SU(2,1)/$\{I,\omega I,\omega^2 I\}$ where

$\omega = (-1 + i\sqrt{3})/2$ is a cube root of unity. This is direct analogous to the fact that SL(2,C) is a double cover of PSL(2,C) = SL(2,C)/$\{I,-I\}$ (Platis, 2006).

Chapter 3

THE GEOMETRY OF ISOMETRIES

3.1 Introduction

The dynamical behaviour of hyperbolic isometries in $\mathrm{PSL}(2,\mathbb{C})$ may be classi ed as elliptic, parabolic or loxodromic (hyperbolic) and the trace of the corresponding matrix in $SL(2,\mathbb{C})$ distinguishes between these classes. Moreover, for nonparabolic isometries, the geometry of the action in terms of rotation angle or hyperbolic translation length may be read o directly from the trace. Likewise, one may use the trace of an element of $\mathrm{SU}(2,1)$ to decide whether the corresponding complex hyperbolic isometry in $\mathrm{PU}(2,1)$ is elliptic, parabolic or loxodromic (Parker, 2012).

Furthermore, one may deduce information about the geometry of the action of the isometry from this trace. We will discuss elliptic and parabolic isometries which have some subtlety involved, the case of loxodromic isometries is trivial. (Parker, 2012)

3.2 Classi cation of elements of SU(2, 1) by their trace

This section looks at classi cation of elements of $\mathrm{SU}(2,1)$ by their trace. Elements of $\mathrm{SU}(2,1)$ are holomorphic complex hyperbolic isometries and the familiar trichotomy from real hyperbolic geometry applies in the complex hyperbolic setting as well. A holomorphic complex hyperbolic isometry A is said to be:

1. loxodromic if it xes exactly two points of $\partial\mathbf{H}_{\mathbb{C}}^2$;
2. parabolic if it xes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$;

3. elliptic if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

We now show that we can use the trace of $A \in \mathrm{SU}(2,1)$ to decide the class of A. First observe that, if λ_1, λ_2 and λ_3 are the eigenvalues of A, then $\overline{\lambda_1}^{-1}, \overline{\lambda_2}^{-1}$ and $\overline{\lambda_3}^{-1}$ form some permutation of λ_1, λ_2 and λ_3 (Parker, 2010). Let $ch_A(x)$ be the characteristic polynomial of A. Suppose that

$$ch_A(x) = x^3 - a_2x^2 + a_1x - a_0.$$

Then $a_2 = \lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A)$ and $a_0 = \lambda_1\lambda_2\lambda_3 = \det(A) = 1$. The other coefficient is

$$\begin{aligned} a_1 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ &= \lambda_3^{-1} + \lambda_1^{-1} + \lambda_2^{-1} \\ &= \overline{\lambda_1} + \overline{\lambda_2} + \overline{\lambda_3} \end{aligned}$$

Hence, if we denote the trace of A by tr(A) $= \tau$, then $\mathrm{tr}(A^{-1}) = \bar{\tau}$. Putting this in the characteristic polynomial of $A \in \mathrm{SU}(2,1)$ gives

$$ch_A(x) = x^3 - \tau x^2 + \bar{\tau}x - 1.$$

We want to find out when $A \in \mathrm{SU}(2,1)$ has repeated eigenvalues. In other words, we want to find conditions on τ for which $ch_A(x) = 0$ has repeated solutions. This is true if and only if $ch_A(x)$ and its derivative $ch'_A(x)$ have a common root. Clearly

$$ch'_A(x) = 3x^2 - 2\tau x + \bar{\tau}.$$

According to (Kirwan, 1992) cited by Parker (2012), two polynomials have a common root if and only if their resultant vanishes.

The resultant $R(p(x), q(x))$ of two polynomials $p(x)$ and $q(x)$ of degree m and n, respectively, is the determinant of $(m + n) \times (m + n)$ matrix defined as follows. Write the coe cient of $p(x)$ in the rst row followed by $n-1$ zeros. In the next row the coe cients are displaced one row to the right, with one zero to the left and $n-2$ to the right. continue in this fashion until the nth row is $n-1$ zeros followed by the coe cients of $p(x)$. For the last m rows we do the same thing with $p(x)$ and $q(x)$ interchanged (Parker, 2012).

Since $ch_A(x)$ and $ch^{0}_{A}(x)$ have degrees 3 and 2 respectively, the resultant is a 5×5 determinant. Applying the above procedure, we see that

$$R(ch_A(x), ch'_A(x)) = \det \begin{pmatrix} 1 & -\tau & \overline{\tau} & -1 & 0 \\ 0 & 1 & -\tau & \overline{\tau} & -1 \\ 3 & -2\tau & \overline{\tau} & 0 & 0 \\ 0 & 3 & -2\tau & \overline{\tau} & 0 \\ 0 & 0 & 3 & -2\tau & \overline{\tau} \end{pmatrix}$$

$$= 1 \begin{vmatrix} 1 & -\tau & \overline{\tau} & -1 \\ -2\tau & \overline{\tau} & 0 & 0 \\ 3 & -2r & \overline{\tau} & 0 \\ 0 & 3 & -2\tau & \overline{\tau} \end{vmatrix} + 3 \begin{vmatrix} -\tau & \overline{\tau} & -1 & 0 \\ 1 & -\tau & \overline{\tau} & -1 \\ 3 & -2\tau & \overline{\tau} & 0 \\ 0 & 3 & -2\tau & \overline{\tau} \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2\tau & \overline{\tau} & 0 \\ 3 & -2\tau & \overline{\tau} \\ 0 & 3 & -2\tau \end{vmatrix} + \overline{\tau} \begin{vmatrix} 1 & -\tau & \overline{\tau} \\ -2\tau & \overline{\tau} & 0 \\ 3 & -2\tau & \overline{\tau} \end{vmatrix}$$

$$- 1 \begin{vmatrix} -\tau & \overline{\tau} & -1 \\ 3 & -2\tau & \overline{\tau} \\ 0 & 3 & -2\tau \end{vmatrix} + \overline{\tau} \begin{vmatrix} -\tau & \overline{\tau} & -1 \\ 1 & -\tau & \overline{\tau} \\ 3 & -2\tau & \overline{\tau} \end{vmatrix}$$

$$= -8\Re(\tau^3) + 12|\tau|^2 + 2|\tau|^4 - 2\Re(\tau^3)$$

$$+ 3(4\Re(\tau^3) - 9|\tau|^2 + 9 - |\tau|^4 + 2\Re(\tau^3) - |\tau|^2).$$

Therefore

$$R(ch_A(x), ch'_A(x)) = -|\tau|^4 + 8\Re(\tau^3) - 18|\tau|^2 + 27.$$

One has the following well-known theorem:

Theorem 3.2.1 Let $f(\tau) = |\tau|^4 - 8<(\tau^3) + 18|\tau|^2 - 27$. Let $A \in \mathrm{SU}(2,1)$ then:

1. A has an eigenvalue λ with $|\lambda|$ 6= 1 if and only if $f(\mathrm{tr}(A)) > 0$,

2. A has a repeated eigenvalue if and only if $f(\mathrm{tr}(A)) = 0$,

3. A has distinct eigenvalue if and only if $f(\mathrm{tr}(A)) < 0$.

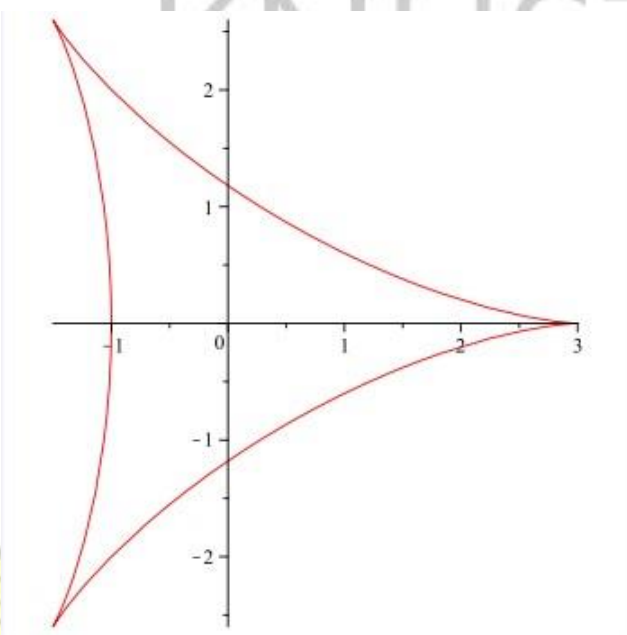

Figure 3.1: The deltoid given by f(τ) = 0. The region with f(τ) < 0 is inside and that with f(τ) > 0 is outside

The curve $f(\tau) = 0$ is a classical curve called a deltoid. The points outside correspond to case (i) in the theorem. This may be seen by considering A with eigenvalue r, r^{-1} and 1 where $r > 1$, which implies tr(A) lies in the interval (3, ∞) and considering A with eigenvalue $e^{i\theta}, e^{-i\theta}$ and 1 whose trace lies in (-1, 3). The rest follows by continuity (Parker, 2012).

From the construction of the deltoid $f(\tau) = 0$ it is clear that part (2) of theorem 3.2.1 follows directly. We now discuss the other cases separately.

Lemma 3.2.2 Let $A \in \mathrm{SU}(2,1)$ and let λ be an eigenvalue of A. Then λ^{-1} is an eigenvalue of A.

Proof. We know that A preserves the Hermitian form de ned by H. Hence, $A^{*}HA = H$ and so $A = H^{-1}(A^{*})^{-1}H$. Thus A has the same set of eigenvalues as $(A^{*})^{-1}$ (they are conjugate). Since the characteristic polynomial of A^{*} is the complex conjugate of the characteristic polynomial of A, so if λ is an eigenvalue of A then $\bar{\lambda}$ is an eigenvalue of A^{*}. Therefore λ^{-1} is an eigenvalue of $(A^{*})^{-1}$ and hence A.

Corollary 1 Suppose $A \in \mathrm{SU}(2,1)$ has an eigenvalue λ with $|\lambda|$ 6= 1. Then $\bar{\lambda}^{-1}$ is a distinct eigenvalue and the third eigenvalue is $\bar{\lambda}\lambda^{-1}$ of absolute value of 1. Moreover, A is loxodromic.

Proof. Using lemma 3.2.2, we see that $\bar{\lambda}^{-1}$ is an eigenvalue. Since $|\lambda|$ 6= 1 it is not equal to λ. As the product of the eigenvalues is 1 we obtain the third eigenvalue.

If v 6= 0 is an eigenvector corresponding to λ then

$$\mathrm{h}v,v\mathrm{i} = \mathrm{h}Av,Av\mathrm{i} = \mathrm{h}\lambda v,\lambda v\mathrm{i} = |\lambda|^{2}\mathrm{h}v,v\mathrm{i}.$$

As $|\lambda|$ 6= 1 we see that v $\in V_0$ and Pv is a xed point of A on $\partial\mathbf{H}^2_{\mathbb{C}}$. Similarly, a non-zero $\bar{\lambda}^{-1}$ eigenvector corresponds to a second xed point on $\partial\mathbf{H}^2_{\mathbb{C}}$. Finally, a non-zero $\bar{\lambda}\lambda^{-1}$ eigenvector lies in V_+ and is a normal vector for the complex line through the two xed points on $\partial\mathbf{H}^2_{\mathbb{C}}$. Hence A has precisely two xed points on $\partial\mathbf{H}^2_{\mathbb{C}}$ and is loxodromic.

Lemma 3.2.3 Suppose that $A \in \mathrm{SU}(2,1)$ has an eigenvalue λ with $|\lambda|$ 6= 1. Then $f(\mathrm{tr}(A)) > 0$.

Proof. Suppose that $re^{i\theta}$ is an eigenvalue of A where r is positive and r 6= 1.

By corollary 1 the other eigenvalues are $(\overline{re^{i\theta}})^{-1} = r^{-1}e^{i\theta}$ and $e^{-2i\theta}$. Therefore $\tau = re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta}$ and $\bar{\tau} = r^{-1}e^{-i\theta} + re^{-i\theta} + e^{2i\theta}$. Hence

$$
\begin{aligned}
|\tau|^2 = \tau\bar{\tau} &= (re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta})(r^{-1}e^{-i\theta} + re^{-i\theta} + e^{2i\theta}) \\
&= 2 + r^2 + r^{-2} + (r + r^{-1})e^{3i\theta} + (r + r^{-1})e^{-3i\theta} + 1 \\
&= (r + r^{-1})^2 + (r + r^{-1})\cos(3\theta) + (r + r^{-1})i\sin(3\theta) \\
&\quad + (r + r^{-1})\cos(3\theta) - (r + r^{-1})i\sin(3\theta) + 1 \\
&= (r + r^{-1})^2 + 2(r + r^{-1})\cos(3\theta) + 1
\end{aligned}
$$

$$
\begin{aligned}
<(\tau^3) &= (re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta})^3 \\
&= (r^2e^{2i\theta} + 2e^{2i\theta} + 2re^{-i\theta} + 2r^{-1}e^{-i\theta} + r^{-2}e^{2i\theta} + e^{-4i\theta}) \\
&\cdot (re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta}) \\
&= (r^3 + 3r + 3r^{-1} + r^{-3})e^{3i\theta} + 3(r + r^{-1})e^{-3i\theta} \\
&+ 3r^2 + 3r^{-2} + e^{-6i\theta} + 6 \\
&= (r + r^{-1})^3\cos(3\theta) + 3(r + r^{-1})^2 + 3(r + r^{-1})\cos(3\theta) + \cos(6\theta).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta}) &= [(r + r^{-1})^2 + 2(r + r^{-1})\cos(3\theta) + 1]^2 \\
&- 8[(r + r^{-1})^3\cos(3\theta) \\
&+ 3(r + r^{-1})^2 + 3(r + r^{-1})\cos(3\theta) + \cos(6\theta)] \\
&+ 18[(r + r^{-1})^2 + 2(r + r^{-1})\cos(3\theta) + 1] - 27 \\
&= (r + r^{-1})^4 + 4(r + r^{-1})^3\cos(3\theta) + 4(r + r^{-1})^2\cos^2(3\theta) \\
&- 8(r + r^{-1})^2\cos(3\theta) + 16(r + r^{-1})\cos(3\theta)
\end{aligned}
$$

$$- 4(r + r^{-1})^2 - 8\cos(6\theta) - 9$$

$$= (r - r^{-1})^2(r + r^{-1} - 2\cos(3\theta))^2 > 0.$$

Lemma 3.2.4 Suppose that $A \in \mathrm{SU}(2,1)$ has three distinct eigenvalues, all of unit modulus. Then $f(\mathrm{tr}(A)) < 0$.

Proof. We write the eigenvalues as $e^{i\theta}, e^{i\varphi}, e^{i\psi}$ where θ, φ and ψ are distinct and $e^{i\theta+i\varphi+i\psi} = 1$. Then $\tau = e^{i\theta} + e^{i\varphi} + e^{i\psi}$ and

$$|\tau|^2 = (e^{i\theta} + e^{i\varphi} + e^{i\psi})(e^{-i\theta} + e^{-i\varphi} + e^{-i\psi})$$

$$= e^{i\theta-i\varphi} + e^{i\theta-i\psi} + e^{i\varphi-i\theta} + e^{i\varphi-i\psi} + e^{i\psi-i\theta} + e^{i\psi-i\varphi} + 3$$

$$= \cos(\theta - \varphi) + i\sin(\theta - \varphi) + \cos(\theta - \psi) + i\sin(\theta - \psi)$$

$$+ \cos(\varphi - \theta) + i\sin(\varphi - \theta) + \cos(\varphi - \psi) + i\sin(\varphi - \psi)$$

$$+ \cos(\psi - \theta) + i\sin(\psi - \theta) + \cos(\psi - \varphi) + i\sin(\psi - \varphi) + 3$$

$$= 3 + 2\cos(\theta - \varphi) + 2\cos(\varphi - \psi) + 2\cos(\psi - \theta)$$

$$\tau^3 = (e^{i\theta} + e^{i\psi} + e^{i\varphi})^3$$

$$= e^{3i\theta} + e^{3i\varphi} + e^{3i\psi} + 3e^{2i\theta+i\varphi} + 3e^{i\theta+2i\varphi} + 3e^{2i\theta+i\psi}$$

$$+ 3e^{i\theta+2i\psi} + 3e^{2i\psi+i\varphi} + 3e^{2i\varphi+i\psi} + 6e^{i\theta+i\varphi+i\psi}$$

$$= \cos(3\theta) + i\sin(3\theta) + \cos(3\varphi) + i\sin(3\varphi) + \cos(3\psi) + i\sin(3\psi)$$

$$= \cos(3\theta) + 3i\sin(3\theta) + \cos(3\varphi) + i\sin(3\varphi) + \cos(3\psi) + i\sin(3\psi)$$

$$+ 3\cos(2\theta + \varphi) + 3i\sin(2\theta + \varphi) + 3\cos(\theta + 2\varphi) + 3i\sin(\theta + 2\varphi)$$

$$+ 3\cos(2\theta + \psi) + 3i\sin(2\theta + \psi) + 3\cos(\theta + 2\psi) + 3i\sin(\theta + 2\psi)$$

$$+ 3\cos(2\psi + \theta) + 3i\sin(2\psi + \theta) + 3\cos(2\varphi + \psi) + 3i\sin(2\varphi + \psi) + 6$$

$$\therefore <(\tau^3) = \cos(3\theta)+\cos(3\varphi)+\cos(3\psi)+6\cos(\theta-\varphi)+6\cos(\varphi-\psi)+6\cos(\psi-\theta)+6.$$

But,

$$\cos(3\theta) = \cos(2\theta + \theta) = \cos(2\theta - \varphi - \psi)$$

$$= \cos[(\theta - \psi) - (\varphi - \theta)]$$

$$= \cos(\theta - \psi)\cos(\varphi - \theta) + \sin(\theta - \psi)\sin(\varphi - \theta).$$

Hence

$$<(\tau^3) = \cos(\theta - \psi)\cos(\varphi - \theta) + \sin(\theta - \psi)\sin(\varphi - \theta)$$

$$+ \cos(\varphi - \psi)\cos(\theta - \varphi) + \sin(\varphi - \psi)\sin(\theta - \varphi) +$$

$$\cos(\psi - \theta)\cos(\varphi - \psi) + \sin(\psi - \theta)\sin(\varphi - \psi)$$

$$+ 6\cos(\theta - \varphi) + 6\cos(\varphi - \psi) + 6\cos(\psi - \theta) + 6.$$

Using this we now calculate

$$f(e^{i\theta} + e^{i\varphi} + e^{i\psi}) = [3 + 2\cos(\theta - \varphi) + 2\cos(\varphi - \psi) + 2\cos(\psi - \theta)]^2$$

$$- 8[\cos(\theta - \psi)\cos(\varphi - \theta) + \sin(\theta - \psi)\sin(\varphi - \theta)$$

$$+ \cos(\varphi - \psi)\cos(\theta - \varphi) + \sin(\varphi - \psi)\sin(\theta - \varphi)$$

$$+ \cos(\psi - \theta)\cos(\varphi - \psi) + \sin(\psi - \theta)\sin(\varphi - \psi)$$

$$+ 6\cos(\theta - \varphi) + 6\cos(\varphi - \psi) + 6\cos(\psi - \theta) + 6]$$

$$+ 18[3 + 2\cos(\theta - \varphi) + 2\cos(\varphi - \psi) + 2\cos(\psi - \theta)] - 27$$

$$= -12 + 4\cos^2(\theta - \varphi) + 4\cos^2(\varphi - \psi) + 4\cos^2(\psi - \theta)$$

$$- 8\sin(\theta - \psi)\sin(\varphi - \theta) - 8\sin(\varphi - \psi)\sin(\theta - \varphi)$$

$$- 8\sin(\psi - \theta)\sin(\varphi - \psi)$$

$$= -4\sin^2(\theta - \varphi) - 4\sin^2(\varphi - \psi) - 4\sin^2(\psi - \theta)$$

$$- 8\sin(\theta - \psi)\sin(\varphi - \theta) - 8\sin(\varphi - \psi)\sin(\theta - \varphi)$$

$$- 8\sin(\psi - \theta)\sin(\varphi - \psi)$$

$= -4(\sin(\theta - \varphi) + \sin(\varphi - \psi) + \sin(\psi - \theta))^2 < 0.$

As these two lemmas exhaust all the possibilities when A has distinct eigenvalues, we have proved theorem 3.2.1.

We now brie y discuss elliptic maps. Suppose rst that A has three distinct eigenvalues of unit modulus.

Proposition 1 Suppose that $A \in \mathrm{SU}(2,1)$ has distinct eigenvalues $e^{i\theta}$,$e^{i\varphi}$ and $e^{i\psi}$. Then A has a unique xed point in $\mathbf{H}^2_{\mathbb{C}}$ corresponding to one of the eigenspaces. There are then three distinct conjugacy classes of elliptic maps with this trace.

If the xed point corresponds to the $e^{i\theta}$ eigenspace then A acts on the tangent space at this point by a unitary matrix with eigenvalues $e^{i\varphi - i\theta}$ and $e^{i\psi - i\theta}$.

Proof. Since A has distinct eigenvalues, A is diagonalisable. Then there exists a basis of eigenvectors for $\mathrm{SU}(2,1)$. Since eigenvectors with distinct eigenvalues are Hermitian orthogonal, an eigenvector v of A in V_- corresponds to a xed point v = Pv ∈ $\mathbf{H}^2_{\mathbb{C}}$. As A has three distinct eigenvalues there are three conjugacy classes depending on which eigenvector lies in V_-. Finally, A is elliptic.

For the second part, we consider the action of $e^{-i\theta}A$ on $\mathbb{C}^{2,1}$ and restrict this to the tangent space. The result follows.

We now consider what happens if A has a repeated eigenvalue and so tr(A) lies on the deltoid. When all three eigenvalues are the same they must be a cube root of unity. These traces are the three vertices of the deltoid. Such maps are parabolic or act as the identity on $\mathbf{H}^2_{\mathbb{C}}$. We now consider the case where A has exactly two distinct eigenvalues(Parker, 2012).

Proposition 2 Suppose that $A \in \mathrm{SU}(2,1)$ has two distinct eigenvalues, one of them repeated. Then the eigenvalues of A are $e^{i\psi}, e^{i\psi}, e^{-2\psi}$ for some ψ with 3ψ 6= 0 $mod\ 2\pi$. Moreover, one of the following three possibilities arises:

1. A xes a complex line L in $\mathbf{H}^2_{\mathbb{C}}$ and rotates a normal vector to L by -3ψ;

2. A xes a point in $\mathbf{H}^2_{\mathbb{C}}$ and acts as $e^{3i\psi}I$ on the tangent space at this point;

3. A xes a point on $\partial\mathbf{H}^2_{\mathbb{C}}$ and there is a complex line L with point on its boundary so that A acts as a parabolic map on L rotates a normal vector to L by -3ψ.

Proof. Suppose that A has repeated eigenvalue λ. Since det(A) = 1, it is clear that the third eigenvalue is λ^{-2}. Using lemma 3.2.2 we see that $\{\lambda,\lambda,\lambda^{-2}\}$ = $\{\overline{\lambda}^{-1}, \overline{\lambda}^{-1}, \overline{\lambda}^{2}\}$. This is a contradiction if $|\lambda|$ 6= 1. Thus $|\lambda| = 1$ and the eigenvalues are $e^{i\psi}, e^{i\psi}, e^{-2i\psi}$ as claimed.

Now we discuss the possible conjugacy classes of A. Let u and w be the eigenvectors corresponding to eigenvalues $e^{i\psi}$ and $e^{-2i\psi}$ respectively. We know that hu,wi = 0 by lemma 6.4 (ii) of Parker (2010) as $e^{3i\psi}$ 6= 1 and $e^{-3i\psi}$ 6= 1 since 3ψ 6= $0(mod 2\pi)$. Therefore the $e^{-2i\psi}$-eigenspace of A is Hermitian orthogonal to $e^{i\psi}$-eigenspace. Since $A \in \mathrm{SU}(2,1)$ it means that two of hu,ui,hu,ui,hw,wi are positive while the other is negative, but $e^{i\psi}$ is repeated eigenvalue and so hu,ui = 0. Hence $e^{-2i\psi}$ cannot be contained in V_0.

Suppose that A is diagonalisable . First suppose that the $e^{-2i\psi}$ is in V_+. As hw,wi > 0, at least one of hu,ui or hu,ui is negative. Since the Hermitian form is non-degenerate and has signature (2, 1) one of them is positive and the other negative. Then $e^{i\psi}$-eigenspace is inde nite and $e^{-2i\psi}$ contains vectors in V_-, V_0 and V_+. Its image under P is a complex line in $\mathbf{H}^2_{\mathbb{C}}$ xed by A. Secondly, suppose that the $e^{-2i\psi}$-eigenspace is in V_- and corresponds to an isolated xed points of A in $\mathbf{H}^2_{\mathbb{C}}$. Using similar argument, $e^{i\psi}$-eigenspcae is in V_+ and so

corresponds to an isolated xed point of A in $\mathbf{H}^2_{\mathbb{C}}$.

Now suppose that A is not diagonalisable. Since $e^{i\psi}$ is a repeated eigenvalue, there exists a vector v that is not a multiple of u and which satis es
Av = $e^{i\psi}$v + u. (To see this, put A into Jordan normal form). Then

$$\text{hv,ui} = \text{h}A\text{v,}A\text{ui} = \text{h}e^{i\psi}\text{v} + \text{u,}e^{i\psi}\text{ui} = \text{hv,ui} + e^{-i\psi}\text{hu,ui.}$$

This implies hu,ui = 0 and u ∈ V_0. This corresponds to a xed point of A on $\partial\mathbf{H}^2_{\mathbb{C}}$. The hyperplane spanned by u and v corresponds to a complex line L in $\mathbf{H}^2_{\mathbb{C}}$ and acts on this line as a parabolic map with xed point Pu. The $e^{-2i\psi}$eigenspace of A is spanned by a polar vector of A and so A acts on a normal vector to L as multiplication by $e^{-3i\psi}$.

An elliptic map is called regular if all its eigenvalues are distinct. Such maps were described in Proposition 1 (chapter 3). Elliptic maps of the type given in Proposition 2 (i) (chapter 3) are called complex re ections in a line and will be discussed in section 5.2. Elliptic maps of the type given in Proposition 2 (ii) and (iii) (chapter 3) are called complex re ections in a point and screw parabolic
or elliptic-parabolic respectively.

Again, using the discriminant function

$$f(\tau) = |\tau|^4 - 8<(\tau^3) + 18|\tau|^2 - 27$$

we can classify isometries of complex hyperbolic plane by the traces of the corresponding matrices. An isometry $A \in \mathrm{SU}(2,1)$ is regular elliptic i $f(\mathrm{tr}(A)) <$
0 and hyperbolic i $f(\mathrm{tr}(A)) > 0$. If $f(\mathrm{tr}(A)) = 0$ there are three cases. If $(\mathrm{tr}(A))^3 = 27$ then A is unipotent. Otherwise, A is either a complex re ection in a complex geodesic or a complex re ection about a point, or A is elliptoparabolic. Note that for real τ the function f factors into $f(\tau) = (\tau+1)(\tau-3)^3$. This means for $A \in \mathrm{SU}(2,1)$ whose trace is real, that A is regular elliptic if and only if $\mathrm{tr}(A) \in (-1,3)$ and hyperbolic i $\mathrm{tr}(A) \notin [-1,3]$ (Pratoussevitch, 2005).

3.3 Traces and eigenvalues for loxodromic maps

A loxodromic matrix A in $\mathrm{SL}(2,\mathrm{C})$ has eigenvalues λ and λ^{-1} where $|\lambda| > 1$. Furthermore writing $\mathrm{tr}(A) = \tau$ then $\tau = \lambda + \lambda^{-1}$. The map sending λ to τ is a

conformal map from the exterior of the unit disc to the complex plane slit along the real axis from −2 to 2 (inclusive).

In this section we want to generalise this result to loxodromic maps in $\mathrm{SU}(2,1)$. The map from eigenvalues to trace is no longer holomorphic but, we are able to show that it is a di eomorphism from the exterior of the unit disc onto the set of points in C with $f(\tau) > 0$, that is the exterior of the deltoid, compare theorem 3.2.1. If $\tau = \mathrm{tr}(A)$ then recall the discriminant function $f(\tau)$ of theorem

3.2.1

$$f(\tau) = |\tau|^4 - 8<(\tau) + 18|\tau|^2 - 27.$$

The main result is:

Proposition 3 Let A be a loxodromic map in $SU(2,1)$ with eigenvalue λ with $|\lambda| > 1$. Then the function Φ gives the trace in terms of the eigenvalue

$$\Phi : \{\lambda \in \mathrm{C} : |\lambda| > 1\} \; 7\!\to \{\tau \in \mathrm{C} : f(\tau) > 0\}$$

given by

$$\Phi(\lambda) = \tau = \lambda + \bar{\lambda}\lambda^{-1} + \lambda^{-1}$$

is a di eomorphism. Moreover, $\Phi(\omega\lambda) = \omega\Phi(\lambda)$, where ω is a cube root of unity and so this di eomorphism is well de ned for elements of $\mathrm{PSU}(2,1)$.

We prove this result by rst showing that the map Φ is surjective.

Lemma 3.3.1 Suppose that $\tau \in \mathrm{C}$ satis es $f(\tau) > 0$ then there exists $\lambda \in \mathrm{C}$ with $|\lambda| > 1$ so that $\tau = \lambda + \bar{\lambda}\lambda^{-1} + \lambda^{-1}$.

Proof. If we can nd such a $\lambda = re^{i\theta}$ then, as in lemma 3.2.3, we must nd r and θ solving

$$|\tau|^2 = (r + r^{-1})^2 + 2(r + r^{-1})\cos(3\theta) + 1,$$

$$<(\tau^3) = (r + r^{-1})^3 \cos(3\theta) + 3(r + r^{-1})^2 + 3(r + r^{-1})\cos(3\theta) + \cos(6\theta).$$

Eliminating $\cos(3\theta)$ from these equations, we must nd $x = (r+r^{-1})^2 > 4$ solving $g(x) = 0$ where

$$g(x) = x^3 - (3|\tau|^2)x^2 + (3 + 2<(\tau^3) - |\tau|^2)x - (|\tau|^2 - 1)^2.$$

Moreover, since

$$((r + r^{-1}) - 1)^2 \leq |\tau|^2 \leq ((r + r^{-1}) + 1)^2$$

such that a solution must satisfy

$$(|\tau| - 1)^2 \leq x = (r + r^{-1})^2 \leq (|\tau| + 1)^2.$$

Note that since $f(\tau) > 0$ we must have $|\tau| > 1$ and so $(|\tau| + 1)^2 > 4$. We now evaluate $g(x)$ at $x = 4, x = (|\tau| - 1)^2$ and $x = (|\tau| + 1)^2$:

$$g(4) = 64 - (3 + |\tau|^2)16 + (3 + 2<(\tau^3) - |\tau|^2)4 - (|\tau|^2 - 1)^2$$

$$= 27 - 18|\tau|^2 + 8<(\tau^3) - |\tau|^4 = -f(\tau) < 0;$$

$$g((|\tau| - 1)^2) = [(|\tau| - 1)^2]^3 - (3 + |\tau|^2)(|\tau| - 1)^4$$

$$+ [3 + 2<(\tau^3) - |\tau|^2](|\tau| - 1)^2 - (|\tau|^2 - 1)^2$$

$$= (|\tau| - 1)^2[(|\tau| - 1)^4 - (3 + |\tau|^2)(|\tau| - 1)^2$$

$$+ 3 + 2<(\tau^3) - |\tau|^2] - (|\tau|^2 - 1)^2$$

$= (|\tau| - 1)^2[-2|\tau^3| + 2<(\tau^3) + |\tau|^2 + 2|\tau| + 1] - (|\tau|^2 - 1)^2$

$= (|\tau| - 1)^2[-2|\tau^3| + 2<(\tau^3)]$

$+ (|\tau| - 1)^2(|\tau|^2 + 2|\tau| + 1) - (|\tau|^2 - 1)^2$

$= -2(|\tau| - 1)^2(|\tau^3| - <(\tau^3)) \leq 0,$

Following the same procedure we see that

$$g((|\tau| + 1)^2) = 2(|\tau| + 1)^2(|\tau^3| + <(\tau^3)) \geq 0.$$

Thus $g((|\tau|-1)^2) \leq 0 \leq g((|\tau|+1)^2)$; that is 0 is a number between $g((|\tau|-1)^2)$ and $g((|\tau|+1)^2)$. Now g is continuous so by intermediate value theorem, there is a number $x = x_0$ between $(|\tau| - 1)^2$ and $(|\tau| + 1)^2$ with $g(x_0) = 0$ so that

$$x_0 > 4,\ x_0 \geq (|\tau| - 1)^2,\ x_0 \leq (|\tau| + 1)^2.$$

From this we can solve $(r + r^{-1})^2 = x_0$ to obtain

$$r = \frac{\sqrt{x_0} + \sqrt{x_0 - 4}}{2} > 1.$$

Substituting into the rst of our equations, we obtain

$$\cos(3\theta) = \frac{|\tau|^2 - (r + r^{-1})^2 - 1}{2(r + r^{-1})} = \frac{|\tau|^2 - x_0 - 1}{2\sqrt{x_0}}.$$

The right hand side lies in [-1, 1] by construction. So we can solve to nd 3θ. Finally, by considering $\arg(\tau)$ we can solve for θ. Writing $\lambda = re^{i\theta}$ gives the result.

Proof. Proposition 3:

Write

$$X = \{\lambda \in C : |\lambda| > 1\}, \qquad Y = \{\tau \in C : f(\tau) > 0\}.$$

From lemma 3.2.3, we see that the image of X under Φ maps onto Y. We calculate the Jacobian of $\tau(\lambda)$:

$$
\begin{aligned}
|J_\tau(\lambda)| &= \left| \frac{\partial\tau}{\partial\lambda} \right|^2 - \left| \frac{\partial\tau}{\partial\overline{\lambda}} \right|^2 \\
&= |1 - \overline{\lambda}\lambda^{-2}|^2 - |\lambda^{-1} - \overline{\lambda}^{-2}|^2 \\
&= (1 - |\lambda|^{-2})(1 - 2|\lambda|^{-1}\cos(3arg(\lambda)) + |\lambda|^{-2}).
\end{aligned}
$$

This is clearly di erent from 0 whenever $|\lambda| > 1$.

Therefore Φ is a local di eomorphism from X onto Y. It is clear that, when $\lambda \in X$ then λ tends to in nity if and only if τ tends to in nity. Likewise, from the proof of lemma 3.2.3, it is clear that Φ extends continuously to a map from the unit circle $\{\lambda \in \mathrm{C} : |\lambda| = 1\}$ to the set $\{\tau \in \mathrm{C} : f(\tau) = 0\}$. Hence Φ extends continuously to a map from $\overline{X}$ to $\overline{Y}$ and is therefore proper. Thus, by Ehresmann's bration theorem we see that Φ is a locally trivial bration (that is, when thought of as a map from an annulus to itself, it is a covering map). Because Φ is a bounded distance from the identity for large values of $|\lambda|$ we see that it has winding number 1 and so Φ is a global di eomorphism (Parker, 2012).

3.4 Eigenvalues and complex displacement for loxodromic maps

A loxodromic element A of $\mathrm{SL}(2,\mathrm{R})$ or $\mathrm{SU}(1,1)$ with eigenvalues λ and λ^{-1} where $|\lambda| > 1$ corresponds to a hyperbolic isometry, which we also denote by A, in $\mathrm{PSL}(2,\mathrm{R})$ or $\mathrm{PU}(1,1)$ respectively. Since A is loxodromic, it has two xed points on the boundary of the hyperbolic plane and these are the projections of the eigenspaces. The geodesic joining these two xed points is called the axis of A, and is denoted $\tilde{\alpha}$. The $\mathbf{H}^1_{\mathrm{C}}/\mathrm{h}A\mathrm{i}$ is

a hyperbolic cylinder (geometrically a catenoid) and $\alpha = \tilde{\alpha}/\langle A\rangle$ is the hyperbolic geodesic around its waist with
hyperbolic length ℓ where

$$|\lambda| = e^{\ell/2}, \qquad |\mathrm{tr}(A)| = 2\cosh(\ell/2).$$

In other words A translates along its axis by a hyperbolic transform length of ℓ.
The ambiguity in the sign of tr(A) exactly corresponds to the choice of lift from $\mathrm{PSL}(2,\mathrm{R})$ to $\mathrm{SL}(2,\mathrm{R})$ or from $\mathrm{PSU}(1,1)$ to $\mathrm{SU}(1,1)$ respectively.

Similarly, when A is in $\mathrm{SL}(2,\mathrm{C})$ its trace corresponds to a complex length. More precisely, suppose tr(A) = $\lambda+\lambda^{-1}$ where $|\lambda| > 1$. Then once again $|\lambda| = e^{\ell/2}$. To find the argument of λ, for any $z \in \tilde{\alpha}$, consider a tangent vector ξ in $T_z(\mathbf{H}^3)$ orthogonal to $\tilde{\alpha}$, the axis of A. Then A sends ξ in $T_z(\mathbf{H}^3)$ to a tangent vector along $\tilde{\alpha}$ by a hyperbolic distance ℓ and rotates the tangent space by an angle φ.
Then

$$\lambda = e^{\ell/2 i\varphi/2}, \qquad \mathrm{tr}(A) = 2\cosh(\ell/2 + i\varphi/2).$$

Since φ is defined mod 2π we see that the imaginary part of $\ell/2+i\varphi/2$ is defined by mod π. This introduces an ambiguity of ± 1 in the trace and this corresponds exactly to the ambiguity introduced when lifting A from $\mathrm{PSL}(2,\mathrm{C})$ to $\mathrm{SL}(2,\mathrm{C})$.

In this section, we illustrate how the geometric action of $A \in \mathrm{SU}(2,1)$ is recorded by trace tr(A). In principle, the relationship is very similar to the case of $\mathrm{SL}(2,\mathrm{R})$ and $\mathrm{SL}(2,\mathrm{C})$ but the function involved are more complicated. The
main result of this section is:

Proposition 4 Let $A \in \mathrm{SU}(2,1)$ be a loxodromic map with axis $\tilde{\alpha}$. Let $\lambda \in \mathrm{C}$ be the eigenvalue of A with $|\lambda| > 1$. Suppose that A has a Bergman translation length ℓ along $\tilde{\alpha}$ and rotates complex line normal to $\tilde{\alpha}$ by an angle φ. Then

$$\lambda = e^{`/2 - i\varphi/3} \quad (3.1)$$

and

$$\mathrm{tr}(A) = 2\cosh(`/2)e^{-i\varphi/3} + e^{2i\varphi/3} \quad (3.2)$$

Furthermore, since φ is de ned mod 2π, the argument of λ and τ are only given mod $2\ \pi/3$ and so these formulae are only well de ned on $\mathrm{PU}(2,1)$.

Proof. It will be convenient to use the Hermitian form H_2 and to conjugate within $\mathrm{SU}(H_2)$ so that A is diagonal:

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda}\lambda^{-1} & 0 \\ 0 & 0 & \overline{\lambda}^{-1} \end{bmatrix}$$

The action of A on $\mathbf{H}^2_{\mathbb{C}}$ is given by

$$\begin{aligned} A(z_1, z_2) &= \mathbb{P}\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda}\lambda^{-1} & 0 \\ 0 & 0 & \overline{\lambda}^{-1} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \\ &= \mathbb{P}\begin{bmatrix} \lambda z_1 \\ \overline{\lambda}\lambda^{-1} z_2 \\ \overline{\lambda}^{-1} \end{bmatrix} \\ &= \begin{pmatrix} \lambda z_1/\overline{\lambda}^{-1} \\ \overline{\lambda}\lambda^{-1} z_2/\overline{\lambda}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} |\lambda|^2 z_1, \overline{\lambda}^2\lambda^{-1} z_2 \end{pmatrix}. \end{aligned}$$

The axis $\tilde{\alpha}$ of A is given by

$$\tilde{\alpha} = \{(-x, 0) \in \mathbb{C}^2 : x > 0\}.$$

Let x be the standard lift of $(-x,0)$ in $\tilde{\alpha}$. Let ℓ the Bergman translation length of A along its axis then

$$\begin{aligned}\cosh(\ell/2) &= \cosh(\rho(A(-x,0),(-x,0))/2)\\ &= \left| \frac{\langle A\mathbf{x},\mathbf{x}\rangle}{\langle \mathbf{x},\mathbf{x}\rangle} \right|\\ &= \left| \frac{-\lambda x - \overline{\lambda}^{-1}x}{-2x} \right|\\ &= (|\lambda| + |\lambda|^{-1})/2.\end{aligned}$$

Therefore, once again we have $|\lambda| = e^{\ell/2}$.

We now consider the argument of λ. The axis $\tilde{\alpha}$ is contained in a unique complex line, the complex axis $\tilde{\alpha}_{\mathbb{C}}$. With our normalisation,

$$\tilde{\alpha}_{\mathbb{C}} = \{(z,0) \in \mathbb{C}^2 : \Re(z) < 0\}.$$

For any point $(-x,0) \in \alpha$, let ξ be a tangent vector in $T_{(-x,0)}(\mathbf{H}^2_{\mathbb{C}})$ orthogonal to $\tilde{\alpha}_{\mathbb{C}}$. Since

$$A(z_1,z_2) = (|\lambda|^2 z_1, \lambda^2\lambda^{-1} z_2),$$

we see that $\xi \in T_{(-x,0)}(\mathbf{H}^2_{\mathbb{C}})$ is sent to $\xi e^{i\varphi}$ then

$$\varphi = arg(\overline{\lambda}^{-2}\lambda^{-1}) = -3arg(\lambda).$$

Hence $\arg(\lambda) = -\varphi/3$. Thus we obtain (3.1). Finally, since $\mathrm{tr}(A) = \lambda + \overline{\lambda}\lambda^{-1} + \overline{\lambda}^{-1}$, we obtain (3.2).

Corollary 2 Let A be as in Proposition 4. The function (3.2) relating ℓ and φ to $\mathrm{tr}(A)$ is local di eomorphism.

Proof. Since it is clear that $\lambda = e^{\ell/2 - i\varphi/3}$ is a local di eomorphism the result follows by composing this map with the function relating λ and $\mathrm{tr}(A)$, and then using proposition 8.

In fact it is just as simple to calculate to the Jacobian directly. Using (3.2), the real and imaginary parts of $\mathrm{tr}(A)$ are:

$$<(\mathrm{tr}(A)) = 2\cosh(`/2)\cos(\varphi/3) + \cos(2\varphi/3),$$

$$=(\mathrm{tr}(A)) = -2\cosh(`/2)\sin(\varphi/3) + \sin(2\varphi/3).$$

Therefore

$$|J_\tau(\ell,\phi)| =$$

$$\det\begin{pmatrix} \sinh(\ell/2)\cos(\phi/3) & -\frac{2}{3}\cosh(\ell/2)\sin(\phi/3) - \frac{2}{3}\sin(2\phi/3) \\ -\sinh(\ell/2)\sin(\phi/3) & -\frac{2}{3}\cosh(\ell/2)\cos(\phi/3) + \frac{2}{3}\cos(2\phi/3) \end{pmatrix}$$

$$= -(2/3)\sinh(\ell/2)(\cosh(\ell/2) - \cos(\phi))$$

This is clearly non-zero when $` > 0$.

Chapter 4

TWO GENERATOR GROUPS AND FENCHEL-NIELSEN COORDINATES

4.1 Introduction

There is a long tradition of studying subgroups of $SL(2,C)$ by relating the traces of groups elements to their geometry. Vogt and Fricke showed that a non-elementary two generator subgroup of $\mathrm{SL}(2,C)$ is determined up to conjugation by the traces of the generators and their production. One aim of this section is to extend this result to two generator subgroups of $SU(2,1)$. We begin by discussing trace relations in $M(3,C)$, then specialising to $\mathrm{SL}(3,C)$ before nally giving the results for $\mathrm{SU}(2,1)$ (Parker, 2012).

We are also interested in the geometry of two generator subgroups of $\mathrm{SU}(2,1)$. In this section, we pay much attention to the case where the generators and their products are all loxodromic. The fundamental group of a three- holed sphere is a free group on two generators. The generators and their product correspond to the these three boundary components. Since we require that these three elements are loxodromic, we can use the results of section 3.4 to give geometric information about the corresponding three-holed sphere. As an application, we discuss how to generalise Fenchel-Nielsen coordinates to complex hyperbolic representations of surface groups (Parker, 2012).

4.2 Trace identities in $M(3,\mathbb{C})$

In this section we derive some trace identities for 3 × 3 matrices. The rst lemma follows by writing $\mathrm{tr}(A)$, $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A^3)$ as homogeneous polynomials in the eigenvalues of A and then solving for the coe cients of the characteristic polynomial.

Lemma 4.2.1 Let $A \in M(3,\mathbb{C})$. Then the characteristic polynomial of A ie. $\mathrm{ch}_A(x)$ is

$$x^3 - \mathrm{tr}(A)x^2 + \frac{\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)}{2}x - \frac{\mathrm{tr}(A)^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3)}{6}.$$

For any $A \in M(3,\mathbb{C})$ de ne $ch(A)$ to be the following matrix (here I is the 3 × 3 identity matrix):

$$\begin{aligned}\mathrm{ch}(A) = {} & A^3 - \mathrm{tr}(A)A^2 + \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))A \\ & - \frac{1}{6}(\mathrm{tr}(A)^3 - \mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3))I.\end{aligned}$$

Then by the Cayley-Hamilton theorem, $ch(A) = 0$, the 3×3 zero matrix. Parker (2012) used a process known as trilinearisation on this identity to obtain the following:

Proposition 5 Let $A,B,C \in M(3,\mathbb{C})$. Then

$$
\begin{aligned}
O = {} & ABC + ACB + BAC + BCA + CAB + CBA \\
& - \mathrm{tr}(A)(BC + CB) - \mathrm{tr}(B)(AC + CA) - \mathrm{tr}(C)(AB + BA) \\
& + (\mathrm{tr}(B)\mathrm{tr}(C) - \mathrm{tr}(BC))A + (\mathrm{tr}(A)\mathrm{tr}(C) - \mathrm{tr}(AC))B + \\
& (\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))C - (\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C) + \mathrm{tr}(ABC) \\
& + \mathrm{tr}(CBA))I + (\mathrm{tr}(A)\mathrm{tr}(BC) + \mathrm{tr}(B)\mathrm{tr}(AC) + \mathrm{tr}(C)\mathrm{tr}(AB))I.
\end{aligned}
$$

Proof. Using the Cayley-Hamilton theorem, as indicated above, for any $A,B,C \in M(3,\mathbb{C})$ we have

$$O = ch(A+B+C) - ch(A+B) - ch(A+C) - ch(B+C) + ch(A) + ch(B) + ch(C).$$

To obtain the result, we expand this expression and simplify, using the fact that $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Corollary 3 For any $A,B, \in M(3,\mathbb{C})$ we have:

$$
\begin{aligned}
O = {} & ABA^{-1} + B + A^{-1}BA \\
& - \mathrm{tr}(A)(BA^{-1} + A^{-1}B) - \mathrm{tr}(A^{-1})(AB + BA) + \mathrm{tr}(A)\mathrm{tr}(A^{-1})B \\
& + (\mathrm{tr}(B)\mathrm{tr}(A^{-1}) - \mathrm{tr}(BA^{-1}))A + (\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))A^{-1} - \\
& (\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B) - \mathrm{tr}(A)\mathrm{tr}(BA^{-1}) - \mathrm{tr}(A^{-1})\mathrm{tr}(AB))I.
\end{aligned}
$$

$$O = ABA + A^2B + BA^2$$
$$- \mathrm{tr}(A)(AB+BA) - \frac{1}{2}\mathrm{tr}(B)A^2 + (\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))A$$
$$+ \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))B - \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))\mathrm{tr}(B)I$$
$$+ (\mathrm{tr}(A)\mathrm{tr}(AB) - \mathrm{tr}(A^2B))I.$$

Proof. For the rst identity, we put $C = A^{-1}$ in the expression from Proposition 5 (chapter 4). This gives

$$O = ABA^{-1} + B + B + B + B + A^{-1}BA$$
$$- \mathrm{tr}(A)(BA^{-1} + A^{-1}B) - 2\mathrm{tr}(B)(I) - \mathrm{tr}(A^{-1})(AB + BA)$$
$$+ (\mathrm{tr}(B)\mathrm{tr}(A^{-1}) - \mathrm{tr}(BA^{-1}))A + (\mathrm{tr}(A)\mathrm{tr}(A^{-1}) - \mathrm{tr}(I))B + (\mathrm{tr}(A)\mathrm{tr}(B)$$
$$- \mathrm{tr}(AB))A^{-1} - (\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}) + \mathrm{tr}(ABA^{-1})$$
$$+ \mathrm{tr}(A^{-1}BA))I + (\mathrm{tr}(A)\mathrm{tr}(BA^{-1}) + \mathrm{tr}(B)\mathrm{tr}(I) + \mathrm{tr}(A^{-1})\mathrm{tr}(AB))I.$$

By simplifying and using $\mathrm{tr}(I) = 3$, we have

$$O = ABA^{-1} + B + A^{-1}BA$$
$$- \mathrm{tr}(A)(BA^{-1} + A^{-1}B) - \mathrm{tr}(A^{-1})(AB + BA)$$
$$+ [\mathrm{tr}(B)\mathrm{tr}(A^{-1}) - \mathrm{tr}(BA^{-1})]A + \mathrm{tr}(A)\mathrm{tr}(A^{-1})B$$
$$+ [\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)]A^{-1} - [\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1})$$
$$+ \mathrm{tr}(B) - \mathrm{tr}(A)\mathrm{tr}(BA^{-1}) - \mathrm{tr}(A^{-1})\mathrm{tr}(AB)]I,$$

For the second identity put $C = A$ into Proposition 5 (chapter 4), so that

$$O = ABA + A^2B + BA^2 + BA^2 + A^2B + ABA$$
$$- \mathrm{tr}(A)(BA + AB) - \mathrm{tr}(B)(A^2) - \mathrm{tr}(A)(AB + BA)$$

$$+ (\mathrm{tr}(B)\mathrm{tr}(A) - \mathrm{tr}(BA))A + (\mathrm{tr}(A)\mathrm{tr}(A) - \mathrm{tr}(A^2))B +$$

$$(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))A - (\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A) + \mathrm{tr}(ABA)$$

$$+ \mathrm{tr}(ABA))I + (\mathrm{tr}(A)\mathrm{tr}(BA) + \mathrm{tr}(B)\mathrm{tr}(A^2) + \mathrm{tr}(A)\mathrm{tr}(AB))I.$$

Dividing both sides by 2, we have

$$O = ABA + A^2B + BA^2 - \mathrm{tr}(A)(AB + BA)$$
$$- \frac{1}{2}\mathrm{tr}(B)A^2 + (\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))A$$
$$+ \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))B - \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))\mathrm{tr}(B)I$$
$$+ (\mathrm{tr}(A)\mathrm{tr}(AB) - \mathrm{tr}(A^2B))I.$$

Corollary 4 For any $A,B \in M(3,\mathbb{C})$ we have

$$\mathrm{tr}[A,B]+\mathrm{tr}[A^{-1},B] = \mathrm{tr}(A)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B)\mathrm{tr}(B^{-1}) +$$
$$\mathrm{tr}(A)\mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(B^{-1})$$
$$- 3 + \mathrm{tr}(AB)\mathrm{tr}(A^{-1}B^{-1}) - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}B^{-1})$$
$$- \mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B)\mathrm{tr}(AB^{-1})$$
$$- \mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(AB^{-1}) - \mathrm{tr}(A)\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}B).$$

Proof. Multiplying the rst identity from Corollary 3 (chapter 4) on the right by B^{-1} gives

$$O = ABA^{-1}B^{-1} + I + A^{-1}BAB^{-1}$$
$$- \mathrm{tr}(A)(BA^{-1}B^{-1}) - \mathrm{tr}(A)A^{-1} - \mathrm{tr}(A^{-1})A$$
$$- \mathrm{tr}(A)(BAB^{-1}) + \mathrm{tr}(A)\mathrm{tr}(A^{-1})I + \mathrm{tr}(B)\mathrm{tr}(A^{-1})AB^{-1}$$
$$- \mathrm{tr}(BA^{-1})AB^{-1} + \mathrm{tr}(A)\mathrm{tr}(B)(A^{-1}B^{-1}) - \mathrm{tr}(AB)(A^{-1}B^{-1})$$
$$- \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1})B^{-1} - \mathrm{tr}(B)B^{-1} + \mathrm{tr}(A)\mathrm{tr}(BA^{-1})B^{-1} +$$
$$\mathrm{tr}(A^{-1})\mathrm{tr}(AB)B^{-1}.$$

Taking traces gives

$$\begin{aligned}
O = {} & \mathrm{tr}(ABA^{-1}B^{-1}) + 3 + \mathrm{tr}(A^{-1}BAB^{-1}) - \mathrm{tr}(A)\mathrm{tr}(A^{-1}) \\
& - \mathrm{tr}(A)\mathrm{tr}(A^{-1}) - \mathrm{tr}(A^{-1})\mathrm{tr}(A) - \mathrm{tr}(A)\mathrm{tr}(A^{-1}) \\
& + 3\mathrm{tr}(A)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B)\mathrm{tr}(A^{-1})\mathrm{tr}(AB^{-1}) \\
& - \mathrm{tr}(BA^{-1})\mathrm{tr}(AB^{-1}) + \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}B^{-1}) \\
& - \mathrm{tr}(AB)\mathrm{tr}(A^{-1}B^{-1}) - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1}) \\
& - \mathrm{tr}(B)\mathrm{tr}(B^{-1}) + \mathrm{tr}(A)\mathrm{tr}(BA^{-1})\mathrm{tr}(B^{-1}) \\
& + \mathrm{tr}(A^{-1})\mathrm{tr}(AB)\mathrm{tr}(B^{-1}).
\end{aligned}$$

Hence, multiplying through by -1 yields the desired result.

4.3 Traces identities in SL(3,C)

Suppose A is in SL(3,C), then the characteristic polynomial of A may be de ned by lemma 4.3.1.

Lemma 4.3.1 Let $A \in \mathrm{SL}(3,\mathbb{C})$. The characteristic polynomial of A is

$$\mathrm{ch}_A(x) = x^3 - \mathrm{tr}(A)x^2 + \mathrm{tr}(A^{-1})x - 1.$$

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A. Then $\lambda_1\lambda_2\lambda_3 = \det(A) = 1$. This gives the constant term in $ch_A(x)$. We see that $\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}$ are eigenvalues of A^{-1}. Thus, using both of these facts, we see that the linear term in $ch_A(x)$ is

$$\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_3 = \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} = \mathrm{tr}(A^{-1}).$$

By Cayley-Hamilton theorem, we see that for $A \in \mathrm{SL}(3,\mathbb{C})$ we have

$$O = A^3 - \mathrm{tr}(A)A^2 + \mathrm{tr}(A^{-1})A - I. \tag{4.1}$$

Lemma 4.3.2 Let $A \in \mathrm{SU}(2,1)$. Then

1. $\mathrm{tr}(A^2) = (\mathrm{tr}(A))^2 - 2\mathrm{tr}(A^{-1})$;

2. $\mathrm{tr}(A^3) = (\mathrm{tr}(A))^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^{-1}) + 3$.

Proof.

i) Multiplying equation (4.1) by A^{-1} gives;

$$A^2 = \mathrm{tr}(A)A - \mathrm{tr}(A^{-1})I + A^{-1}.$$

Taking traces we see that

$$\mathrm{tr}(A^2) = \mathrm{tr}(A)\mathrm{tr}(A) - \mathrm{tr}(A^{-1})\mathrm{tr}(I) + \mathrm{tr}(A^{-1})$$

$$= \mathrm{tr}(A)^2 - 3\mathrm{tr}(A^{-1}) + \mathrm{tr}(A^{-1})$$

$$= (\mathrm{tr}(A))^2 - 2\mathrm{tr}(A^{-1});$$

ii) Taking traces in equation (4.1) and then substituting for $\mathrm{tr}(A^2)$ gives

$$\mathrm{tr}(A^3) = \mathrm{tr}(A)\mathrm{tr}(A^2) - \mathrm{tr}(A^{-1})\mathrm{tr}(A) + 3$$

$$= \mathrm{tr}(A)((\mathrm{tr}(A))^2 - 2\mathrm{tr}(A^{-1})) - \mathrm{tr}(A^{-1})\mathrm{tr}(A) + 3$$

$$= (\mathrm{tr}(A))^3 - 2\mathrm{tr}(A)\mathrm{tr}(A^{-1}) - \mathrm{tr}(A)\mathrm{tr}(A^{-1}) + 3 =$$

$$(\mathrm{tr}(A))^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^{-1}) + 3.$$

Proposition 6 Let $A,B \in \mathrm{SL}(3,\mathrm{C})$. Then $\mathrm{tr}[A,B]\mathrm{tr}[B,A]$ may be expressed as a polynomial function of the traces of $A,B,AB,A^{-1}B$ and their inverses.

Proof. Write $A = MN$ and $B = NM$ in the expression for corollary 4 (chapter 4). This gives

$$\mathrm{tr}[MN, NM] + \mathrm{tr}[N^{-1}M^{-1}, NM]$$

$$= \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(NM)\mathrm{tr}(N^{-1}M^{-1})$$

$$+ \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(NM)\mathrm{tr}(N^{-1}M^{-1})$$

$$- 3 + \mathrm{tr}(MNNM)\mathrm{tr}(M^{-1}N^{-1}N^{-1}M^{-1})$$

$$- \mathrm{tr}(MN)\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}N^{-1}M^{-1})$$

$$- \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(N^{-1}M^{-1})\mathrm{tr}(MNNM)$$

$$+ \mathrm{tr}(M^{-1}N^{-1}NM)\mathrm{tr}(MNN^{-1}M^{-1})$$

$$- \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(NM)\mathrm{tr}(MNN^{-1}M^{-1})$$

$$- \mathrm{tr}(MN)\mathrm{tr}(N^{-1}M^{-1})\mathrm{tr}(M^{-1}N^{-1}NM)$$

$$= 2\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(MN)^2\mathrm{tr}(M^{-1}N^{-1})^2$$

$$- 3 + \mathrm{tr}(M^2N^2)\mathrm{tr}(M^{-2}N^{-2}) - \mathrm{tr}(MN)^2\mathrm{tr}(M^{-2}N^{-2})$$

$$- \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M^2N^2) + \mathrm{tr}[M,N]\mathrm{tr}[N,M]$$

$$- \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})(\mathrm{tr}[M,N] + \mathrm{tr}[M^{-1},N]) - - - - - (1)$$

Using corollary 4 (chapter 4), $\mathrm{tr}[M,N] + \mathrm{tr}[M^{-1},N]$ can be expressed in terms of the traces of $M,N,MN,M^{-1}N$ and their inverses. That is

$$\mathrm{tr}[M,N] + \mathrm{tr}[M^{-1},N] = \mathrm{tr}(M)\mathrm{tr}(M^{-1}) + \mathrm{tr}(N)\mathrm{tr}(N^{-1}) +$$

$$\mathrm{tr}(M)\mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(N^{-1})$$

$$- 3 + \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(M^{-1}N^{-1})$$

$$- \mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(MN) + \mathrm{tr}(M^{-1}N)\mathrm{tr}(MN^{-1})$$

$$- \mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(MN^{-1}) - \mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N). - - - - - (2)$$

If M and N are in SL(3,C) we can use their characteristic polynomials to write

$$M^2 = \mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1}, \qquad N^2 = \mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1}$$

$$M^{-2} = M - \mathrm{tr}(M)I + \mathrm{tr}(M^{-1})M^{-1}, N^{-2} = N - \mathrm{tr}(N)I + \mathrm{tr}(N^{-1})N^{-1}$$

Hence

$$M^2N^2 = (\mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1})(\mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1})$$

$$= \mathrm{tr}(M)\mathrm{tr}(N)MN - \mathrm{tr}(M)\mathrm{tr}(N^{-1})M + \mathrm{tr}(M)MN^{-1}$$

$$- \mathrm{tr}(M^{-1})\mathrm{tr}(N)N + \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})I - \mathrm{tr}(M^{-1})N^{-1}$$

$$- \mathrm{tr}(N_{-1})M_{-1} + M_{-1}N_{-1}$$

Taking traces gives

$$\mathrm{tr}(M^2N^2) = \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN) - \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})$$

$$+ \mathrm{tr}(M)\mathrm{tr}(MN^{-1}) - \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2$$

$$+ \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1}N^{-1}) - - - - - - - - (3)$$

Using similar argument gives the following:

$$\mathrm{tr}(M^2N^{-2}) = \mathrm{tr}(M)\mathrm{tr}(MN) - \mathrm{tr}(M)^2\mathrm{tr}(N)$$

$$+ \mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})$$

$$+ \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1}) - \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})^2$$

$$+ \mathrm{tr}(M^{-1}N) + \mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) - - - - - - (4)$$

$$\mathrm{tr}(M^{-2}N^2) = \mathrm{tr}(N)\mathrm{tr}(MN) + \mathrm{tr}(MN^{-1}) - \mathrm{tr}(M)\mathrm{tr}(N)^2$$

$$+ \mathrm{tr}(M)\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(M^{-1}N)$$

$$- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N^{-1}) - - - - (5)$$

$$\mathrm{tr}(M^{-2}N^{-2}) = \mathrm{tr}(MN) + \mathrm{tr}(M)\mathrm{tr}(N) + \mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})$$

$$- \mathrm{tr}(M)\mathrm{tr}(N^{-1})^2 + \mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N)$$

$$- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N) + \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) - - - - - (6)$$

Thus it su ces to express the trace of $[MN,NM]$ and $[N^{-1}M^{-1},NM]$ in terms of these other traces. To do this, rst write

$$[MN,NM] = MN^2MN^{-1}M^{-2}N^{-1}$$

$$[NM,MN] = NM^2NM^{-1}N^{-2}M^{-1}$$

and substitute for N^2, N^{-2}, M^2 and M^{-2} as above to have

$$[MN,NM]$$
$$= M\left(\mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1}\right)MN^{-1}\left(M - \mathrm{tr}(M)I + \mathrm{tr}(M^{-1})M^{-1}\right)N^{-1}$$

$$= \mathrm{tr}(N)MNMN^{-1}MN^{-1} - \mathrm{tr}(N)\mathrm{tr}(M)MNMN^{-1}N^{-1}$$

$$+ \mathrm{tr}(N)\mathrm{tr}(M^{-1})MNMN^{-1}M^{-1}N^{-1} - \mathrm{tr}(N^{-1})MMN^{-1}MN^{-1}$$

$$+ \mathrm{tr}(N_{-1})\mathrm{tr}(M)MMN_{-1}N_{-1} - \mathrm{tr}(N_{-1})\mathrm{tr}(M_{-1})MMN_{-1}M_{-1}N_{-1}$$

$$+ MN_{-1}MN_{-1}MN_{-1} - \mathrm{tr}(M)MN_{-1}MN_{-1}N_{-1}$$

$$+ \mathrm{tr}(M^{-1})MN^{-1}MN^{-1}M^{-1}N^{-1} - - - - - - - - - - - - - (7)$$

and

$$[NM,MN]$$

$$= N(\mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1})NM^{-1}(N - \mathrm{tr}(N)I + \mathrm{tr}(N^{-1})N^{-1})M^{-1}$$

$$= \mathrm{tr}(M)NMNMNM^{-1} - \mathrm{tr}(M)\mathrm{tr}(N)NMNM^{-2}$$

$$+ \mathrm{tr}(M)\mathrm{tr}(N_{-1})NMNM_{-1}N_{-1}M_{-1} - \mathrm{tr}(M_{-1})N_2M_{-1}NM_{-1}$$

$$+ \mathrm{tr}(M_{-1})\mathrm{tr}(N)N_2M_{-2} - \mathrm{tr}(M_{-1})\mathrm{tr}(N_{-1})N_2M_{-1}N_{-1}M_{-1}$$

$$+ NM_{-1}NM_{-1}NM_{-1} - \mathrm{tr}(N)NM_{-1}NM_{-2}$$

$$+ \mathrm{tr}(N^{-1})NM^{-1}NM^{-1}N^{-1}M^{-1}. ------------(8)$$

Then using corollary 3 (chapter 4) to substitute for expressions as MNM, MNM^{-1}. Putting equations 2, 3, 4, 5, 6, 7 and 8 into equation 1, eventually yields the polynomial:

$$|\mathrm{tr}[M,N]|^2 = -5\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) + 3 - \mathrm{tr}(M)^2\mathrm{tr}(N)^2\mathrm{tr}(MN)$$

$$- \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1}) + \mathrm{tr}(M)^2\mathrm{tr}(N)\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN)$$

$$- \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N) + \mathrm{tr}(M)^2\mathrm{tr}(N)^2\mathrm{tr}(MN)\mathrm{tr}(M^{-1})^2$$

$$- \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)\mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M)^3\mathrm{tr}(N)^3$$

$$+ \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})\mathrm{tr}(MN) + \mathrm{tr}(M)^3\mathrm{tr}(N^{-1})\mathrm{tr}(N) + \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN^{-1})$$

$$+ \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N) - \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1})^2\mathrm{tr}(N)$$

$$+ \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M)\mathrm{tr}(MN^{-1})\mathrm{tr}(MN)$$

$$- \mathrm{tr}(M)^3\mathrm{tr}(MN^{-1})\mathrm{tr}(N) + \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(N)\mathrm{tr}(N^{-1})$$

$$- \mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})^2 + \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN^{-1})$$

$$- \mathrm{tr}(M)\mathrm{tr}(M^{-1})\mathrm{tr}(MN^{-1})\mathrm{tr}(M^{-1}N) + \mathrm{tr}(M)\mathrm{tr}(M^{-1})^2\mathrm{tr}(N)\mathrm{tr}(MN^{-1})$$

$$- \mathrm{tr}(M)\mathrm{tr}(MN^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2\mathrm{tr}(MN)$$

$$+ \mathrm{tr}(M^{-1})\mathrm{tr}(M)\mathrm{tr}(N)^3 + \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})$$

$$- \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2\mathrm{tr}(M)\mathrm{tr}(N^{-1})^2 + \mathrm{tr}(M^{-1})^2\mathrm{tr}(N)^2\mathrm{tr}(M^{-1}N)$$

$$- \mathrm{tr}(M^{-1})^3\mathrm{tr}(N)^3 + \mathrm{tr}(M^{-1})^2\mathrm{tr}(N)^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1})$$

$$- \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(MN) - \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M)\mathrm{tr}(N)$$

$$- \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(M)\mathrm{tr}(N^{-1})^3$$

$$- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N) + \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1})\mathrm{tr}(N)$$

$- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(MN)$

$- \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(N) - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})$

$+ \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(N^{-1})^2 - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N)$

$+ \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1})^2\mathrm{tr}(N) - \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1})^2$

$+ \mathrm{tr}(MN)^3 + \mathrm{tr}(MN)^2\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1}) - \mathrm{tr}(MN)^2\mathrm{tr}(M)\mathrm{tr}(N^{-1})^2$

$+ \mathrm{tr}(MN)^2\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N) - \mathrm{tr}(MN)^2\mathrm{tr}(M^{-1})^2\mathrm{tr}(N)$

$+ \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(MN)^2\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)$

$- \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M)^2\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M)\mathrm{tr}(MN^{-1})$

$- \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M^{-1})\mathrm{tr}(N)^2 + \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})$

$+ \mathrm{tr}(M^{-1}N^{-1})^3 + \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(M^{-1})$

$+ \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(N^{-1})$

$- \mathrm{tr}(MN)\mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(M^{-1}N^{-1})^2 - \mathrm{tr}(M^{-1})^2\mathrm{tr}(MN)^2\mathrm{tr}(M^{-1}N^{-1})$

$+ \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1}N)\mathrm{tr}(MN^{-1})$

$- \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(MN^{-1})$

$- \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N)$

$+ \mathrm{tr}[M,N]\mathrm{tr}[N^{-1},M^{-1}] + \mathrm{tr}[N,M]\mathrm{tr}[M^{-1},N^{-1}].$

Note that the last two terms could be expanded by applying corollary 3 on equations 7 and 8.

4.4 Trace parameters for two generator groups of SU(2,1)

Let Y be a three holed sphere (sometimes called pair of pants). If the boundary curves are denoted by α,β,γ, then the fundamental group of Y is

$$\pi_1(Y) = \mathrm{h}[\alpha],[\beta],[\gamma] : [\alpha\beta\gamma] = id\mathrm{i}.$$

In fact, π_1 is a free group generated by any two of $[\alpha],[\beta],[\gamma]$ where $[\alpha],[\beta]$ and $[\gamma]$ are the homotopy classes in π_1 representing the boundary curves.

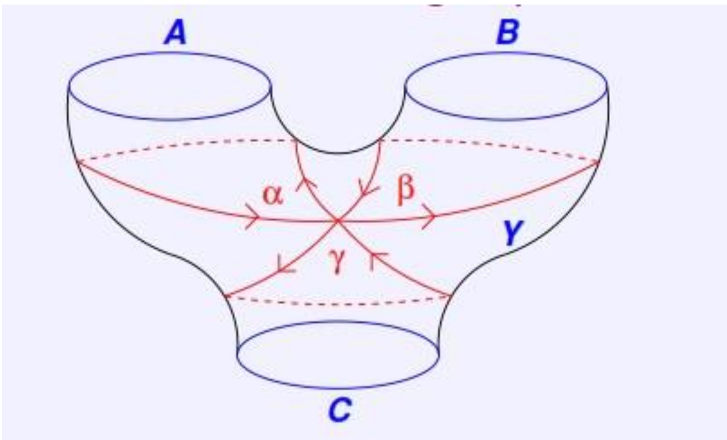

Figure 4.1: Pair of pants

We want to study representations (conjugacy class of homomorphism) $\rho : \pi_1(Y) \longrightarrow \Gamma_Y < \mathrm{SU}(2,1)$. Let $A = \rho([\alpha]), B = \rho([\beta]), C = \rho([\gamma])$, then $\rho(\pi_1(Y)) = \Gamma_Y$ is a subgroup of $\mathrm{SU}(2,1)$ generated by A,B,C with $ABC = I$. In other words, $C = (AB)^{-1} = B^{-1}A^{-1}$.

According to Parker (2012), it is well known that for $\mathrm{SL}(2,\mathrm{R})$ or $\mathrm{SU}(1,1)$ (the holomorphic hyperbolic isometry groups of the upper plane and Poincar$\acute{e}$disc respectively) then the group generated by A and B is completely determined up to conjugation by $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(AB)$. Geometrically, under mild hypotheses, $\mathrm{h}A,B\mathrm{i}$ corresponds to a representation ρ_0 of π which gives Y a hyperbolic metric. The mild hypotheses are that $\mathrm{h}A,B\mathrm{i}$ should be discrete, faithful (or free), totally loxodromic and that the axes of A,B and AB should bound a common region in the hyperbolic plane. We suppose that $A = \rho([\alpha]), B = \rho([\beta]), C = \rho([\gamma])$ are all loxodromic. Let α,β,γ be the axes of A,B,C (geodesic joining xed points).

Then lengths of $A = \rho([\alpha]), B = \rho([\beta]), C = \rho([\gamma])$ are given by

$$|\mathrm{tr}(A)| = 2\cosh(`(\alpha)/2),$$

$$|\mathrm{tr}(B)| = 2\cosh(`(\beta)/2), |\mathrm{tr}(C)|$$

$$= 2\cosh(`(\gamma)/2).$$

In fact, our mild hypotheses about the axes of A,B and C imply that

$$\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C) < 0$$

and so we may choose a lift from $\mathrm{PSL}(2,\mathrm{R})$ to $\mathrm{SL}(2,\mathrm{R})$ where all three traces are negative. Conversely, given $`(\alpha),`(\beta),`(\gamma)$ in $\mathrm{R}_+$ we can construct a hyperbolic metric on $Y$ whose boundary geodesics have these lengths. This in turn gives rise to a group $\mathrm{h}A,B\mathrm{i}$ satisfying $|\mathrm{tr}(A)| = 2\cosh(`(\alpha)/2)$ etc (Parker, 2012).

Similarly, if $\mathrm{h}A,B\mathrm{i}$ is a discrete, free, geometrically nite and totally loxodromic subgroup of $\mathrm{SL}(2,\mathrm{C})$ then we have a similar picture, but the lengths of the boundary curves are now complex, as discussed in the introductory part of section 3.4. The main di erence is that, not all triples of complex lengths give rise to discrete, free, totally loxodromic, geometrically nite group (Parker, 2012).

We now want to play a similar game using complex hyperbolic representations of $\pi_1(Y)$. Again the representations we will be interested in will be discrete, free, totally loxodromic and geometrically nite. We will also add the hypothesis that $\mathrm{h}A,B\mathrm{i}$ is Zariski dense. A subgroup of $\mathrm{PSU}(2,1)$ is Zariski dense if and only if its action on CP^2 does not have a global xed point. $_2$ ^{2}. Equivalently, it does not x a point on $\mathbf{H}_\mathrm{C}$ or preserve a complex line in $\mathbf{H}_\mathrm{C}$ Consider ρ : $\pi_1(Y) \to \mathrm{SU}(2,1)$. Then ρ is irreducible if and only if its image is Zariski dense (Parker, 2012).

The main question is what are the data we need to completely determine hA,Bi up to conjugation. Our rst observation is that $\mathrm{SU}(2,1)$ has complex dimension four and so we do not expect to be able to determine hA,Bi using only three complex numbers (Parker, 2012).

Theorem 4.4.1 (Wen's theorem): Suppose that $A,B \in \mathrm{SU}(2,1)$ and that hA,Bi is Zariski dense. Then hA,Bi is determined up to conjugation within $SU(2,1)$ by

$$\mathrm{tr}(A),\ \mathrm{tr}(B),\ \mathrm{tr}(AB),\ \mathrm{tr}(A^{-1}B),\ \mathrm{tr}[A,B].$$

Remark 2: According to Parker (2012) Wen's theorem refers to A and B in $\mathrm{SL}(3,\mathbb{C})$ and also requires the traces of A^{-1},B^{-1},$A^{-1}B^{-1}$ and AB^{-1}. Namely, one would expect to only need to use four traces to describe hA,Bi. In fact, one needs an extra one, tr$[A,B]$ and this satis es relations with the other traces.

In what follows, we want to consider $A,B,C \in \mathrm{SU}(2,1)$ with $ABC = I$. It is clear that $\mathrm{tr}(AB) = \mathrm{tr}(C^{-1}) = \mathrm{tr}(C)$. We want to express the other parameters in a way that is symmetrical with respect to cyclic permutations of A,B and C. First we consider the trace of $A^{-1}B$.

Lemma 4.4.2 let A,B,C be element of $\mathrm{SU}(2,1)$ so that $ABC = I$. Then

$$\begin{aligned}\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B) &= \mathrm{tr}(B^{-1}C) - \mathrm{tr}(B^{-1})\mathrm{tr}(C)\\ &= \mathrm{tr}(C^{-1}A) - \mathrm{tr}(C^{-1})\mathrm{tr}(A).\end{aligned}$$

Proof. We already know that

$$A^3 - \mathrm{tr}(A)A^2 + \mathrm{tr}(A^{-1})A - I = 0.$$

Multiplying on the right by $A^{-1}B$ gives

$$A^2B - \mathrm{tr}(A)AB + \mathrm{tr}(A^{-1})B - A^{-1}B = 0$$

$$A^2B - \mathrm{tr}(A)AB = A^{-1}B - \mathrm{tr}(A^{-1}B).$$

Taking traces and using $AB = C^{-1}$ gives

$$\mathrm{tr}(C^{-1}A) - \mathrm{tr}(C^{-1})\mathrm{tr}(A) = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B).$$

This shows equality between the rst and third expressions. Cyclically permuting A,B and C gives the second as well.

Therefore by using $\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$ instead of $\mathrm{tr}(A^{-1}B)$ we give symmetric parameters. Furthermore, trivially we have

$$\mathrm{tr}[A,B] = \mathrm{tr}[B,C] = \mathrm{tr}[C,A] = \overline{\mathrm{tr}[B,A]} = \overline{\mathrm{tr}[C,B]} = \overline{\mathrm{tr}[A,C]}.$$

We saw in corollary 4 and proposition 6 (chapter 4) that, the real and absolute value of $\mathrm{tr}[A,B]$ were determined by the other parameters. We now illustrate this explicitly.

We now express equation 18 of Lawton (2007) in terms of $\mathrm{tr}(A),\mathrm{tr}(B),\mathrm{tr}(AB)$ etc.

Lemma 4.4.3 There exists a polynomial $Q \in$ R so $Q - t_{(5)}t_{(-5)} \in ker(\Pi)$, where $t_{(5)}$ and $t_{(-5)}$ are generators of R, $t_{(5)} = \mathrm{tr}[A,B], t_{(-5)} = \mathrm{tr}[B,A], \Pi$ is a surjective algebra morphism and in particular

$$\begin{aligned}
Q = {} & 9 - 6\mathrm{tr}(A)\mathrm{tr}(A^{-1}) - 6\mathrm{tr}(B)\mathrm{tr}(B^{-1}) - 6\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(AB) \\
& - 6\mathrm{tr}(A^{-1}B)\mathrm{tr}(AB^{-1}) + \mathrm{tr}(A)^3 + \mathrm{tr}(B)^3 + \mathrm{tr}(AB)^3 + \mathrm{tr}(A^{-1}B)^3 \\
& + \mathrm{tr}(A^{-1})^3 + \mathrm{tr}(B^{-1})^3 + \mathrm{tr}(B^{-1}A^{-1})^3 + \mathrm{tr}(AB^{-1})^3 \\
& - 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(A^{-1}) - 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(AB)\mathrm{tr}(A)
\end{aligned}$$

$- 3\mathrm{tr}(AB^{-1})\mathrm{tr}(B)\mathrm{tr}(AB) - 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1})\mathrm{tr}(B^{-1}A^{-1})$

$+ 3\mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(A) + 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(B)\mathrm{tr}(A^{-1})$

$+ 3\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) + 3\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(B^{-1}A^{-1})$

$+ \mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(A) + \mathrm{tr}(AB)\mathrm{tr}(B^{-1})\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)$

$+ \mathrm{tr}(AB^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(A^{-1}B)\mathrm{tr}(A) + \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}B)\mathrm{tr}(B)$

$+ \mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(AB)\mathrm{tr}(A) + \mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(AB^{-1})$

$+ \mathrm{tr}(AB)\mathrm{tr}(A^{-1}B) + \mathrm{tr}(AB^{-1})^2\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B^{-1})$

$+ \mathrm{tr}(A^{-1}B)^2\mathrm{tr}(AB)\mathrm{tr}(B) + \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1})\mathrm{tr}(AB^{-1})$

$+ \mathrm{tr}(A)^2\mathrm{tr}(B)\mathrm{tr}(A^{-1}B) + \mathrm{tr}(A)\mathrm{tr}(B^{-1})^2\mathrm{tr}(B^{-1}A^{-1})$

$+ \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B) + \mathrm{tr}(A)^2\mathrm{tr}(AB)\mathrm{tr}(B^{-1})$

$+ \mathrm{tr}(AB^{-1})\mathrm{tr}(A)\mathrm{tr}(B)^2 + \mathrm{tr}(AB^{-1})\mathrm{tr}(B)\mathrm{tr}(B^{-1}A^{-1})^2$

$+ \mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1})\mathrm{tr}(AB^{-1})^2 + \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)$

$+ \mathrm{tr}(A)^2\mathrm{tr}(AB)\mathrm{tr}(B^{-1}) + \mathrm{tr}(AB^{-1})\mathrm{tr}(A)\mathrm{tr}(B)^2$

$+ \mathrm{tr}(A^{-1}B)\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})^2 + \mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)^2$

$+ \mathrm{tr}(A)\mathrm{tr}(AB)\mathrm{tr}(AB^{-1})^2 + \mathrm{tr}(A^{-1})\mathrm{tr}(AB)\mathrm{tr}(A^{-1}B)^2$

$+ \mathrm{tr}(A^{-1})\mathrm{tr}(AB^{-1})\mathrm{tr}(AB)^2 + \mathrm{tr}(A)\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1})^2$

$- 2\mathrm{tr}(B^{-1}A^{-1})^2\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}) - 2\mathrm{tr}(AB)^2\mathrm{tr}(B)\mathrm{tr}(A)$

$- 2\mathrm{tr}(AB^{-1})^2\mathrm{tr}(A^{-1})\mathrm{tr}(B) - 2\mathrm{tr}(A^{-1}B)^2\mathrm{tr}(A)\mathrm{tr}(B^{-1})$

$+ \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1})^2\mathrm{tr}(B^{-1}A^{-1}) + \mathrm{tr}(A)^2\mathrm{tr}(B)^2\mathrm{tr}(AB)$

$+ \mathrm{tr}(AB^{-1})\mathrm{tr}(A^{-1})^2\mathrm{tr}(B)^2 + \mathrm{tr}(A^{-1}B)\mathrm{tr}(A)^2\mathrm{tr}(B^{-1})^2$

$- \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})^2\mathrm{tr}(B)\mathrm{tr}(A) - \mathrm{tr}(A^{-1}B)\mathrm{tr}(B)^2\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})$

$- \mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(A)^2\mathrm{tr}(A^{-1})\mathrm{tr}(B) - \mathrm{tr}(AB)\mathrm{tr}(A^{-1})^2\mathrm{tr}(A)\mathrm{tr}(B^{-1})$

$- \mathrm{tr}(B^{-1}A^{-1}\mathrm{tr}(B)^2\mathrm{tr}(B^{-1})\mathrm{tr}(A) - \mathrm{tr}(AB)\mathrm{tr}(B^{-1})^2\mathrm{tr}(B)\mathrm{tr}(A^{-1})$

$- \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(A)^2 - \mathrm{tr}(A^{-1}B)\mathrm{tr}(B)\mathrm{tr}(A)\mathrm{tr}(A^{-1})^2$

$- \mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})^3\mathrm{tr}(A) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)^3\mathrm{tr}(A) - \mathrm{tr}(A^{-1})^3\mathrm{tr}(B^{-1})\mathrm{tr}(B)$

$- \mathrm{tr}(A)^3\mathrm{tr}(B^{-1})\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)\mathrm{tr}(A^{-1})\mathrm{tr}(B)$

$- \mathrm{tr}(A^{-1}B)\mathrm{tr}(AB)\mathrm{tr}(B)\mathrm{tr}(A)\mathrm{tr}(B^{-1}) - \mathrm{tr}(A^{-1})\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB^{-1})\mathrm{tr}(AB)$

$- \mathrm{tr}(A^{-1})\mathrm{tr}(A)\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1}) + \mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})^2\mathrm{tr}(A)^2\mathrm{tr}(B) +$

$\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})^2\mathrm{tr}(B)^2\mathrm{tr}(A)$.

Proposition 7 Suppose that A,B,C are elements of SU(2,1) such that $ABC = I$. Let $a = \mathrm{tr}(A)$,$b = \mathrm{tr}(B)$,$c = \mathrm{tr}(C)$ and $d = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$. Then the equation in lemma 4.4.3 becomes

$$\begin{aligned}
Q = 9 &- 6|a|^2 - 6|b|^2 - 6|c|^2 - 6(\overline{d} + a\overline{b})(d + \overline{a}b) + a^3 + b^3 \\
&+ \overline{c}^3 + (\overline{d} + a\overline{b})^3 + \overline{a}^3 + \overline{b}^3 + c^3 + (d + \overline{a}b)^3 - 3(d + \overline{a}b)\overline{a}c \\
&- 3(\overline{d} + a\overline{b})a\overline{c} - 3(d + \overline{a}b)b\overline{c} - 3(\overline{d} + a\overline{b})\overline{b}c + 3(d + \overline{a}b)a\overline{b} \\
&+ 3(\overline{d} + a\overline{b})\overline{a}b + 3abc + 3\overline{a}\,\overline{b}\,\overline{c} + |a|^2|b|^2 + |b|^2|c|^2 \\
&+ (d + \overline{a}b)(\overline{d} + a\overline{b})|a|^2 + (d + \overline{a}b)(\overline{d} + a\overline{b})|b|^2 + |a|^2|c|^2 \\
&+ (d + \overline{a}b)(\overline{d} + a\overline{b})|c|^2 + (d + \overline{a}b)\overline{b}c + (\overline{d} + a\overline{b})b\overline{c} + \overline{a}^2\overline{b}(d + \overline{a}b)
\end{aligned}$$

$+ a^2b(\overline{d} + a\overline{b}) + ab^2c + a\overline{b}^2\overline{c} + (d + a\overline{b})a^2c + (\overline{d} + a\overline{b})a^2c$

$+ (d + a\overline{b})bc^2 + (\overline{d} + \overline{a}\overline{b})bc^2 + a^2bc + a^2b\,\overline{c} + (d + a\overline{b})\overline{a}b^2$

$+ (\overline{d} + \overline{a}b)b^2\overline{c} + ac(d + a\overline{b})^2 + (\overline{d} + \overline{a}b)\overline{a}\,b^2 + (\overline{d} + \overline{a}b)b^2c$
$+ \overline{a}c(d + ab)^2 + a\,\overline{c}^2(d + a\overline{b}) + ac^2(d + ab) - 2a\,b\overline{c}^2$

$$- 2ab\overline{c}^2 - 2\overline{a}b(d+\overline{a}b)^2 - 2a\overline{b}(\overline{d}+a\overline{b})^2 + \overline{a}^2\overline{b}^2 c + a^2b^2\overline{c}$$

$$+ (d+\overline{a}b)\overline{a}^2b^2 + (\overline{d}+a\overline{b}) - (d+\overline{a}b)a\overline{b}|b|^2 - (\overline{d}+a\overline{b})\overline{a}b|b|^2$$

$$- a|a|^2bc - \overline{a}|a|^2\overline{b}\,\overline{c} - ab|b|^2c - \overline{a}\,\overline{b}|b|^2\overline{c} - (d+\overline{a}b)a|a|^2\overline{b}$$

$$- (\overline{d}+a\overline{b})\overline{a}|a|^2b - |a|^2\overline{b}^3 - |a|^2b^3 - \overline{a}^3|b|^2 - a^3|b|^2$$

$$- (d+\overline{a}b)\overline{a}|b|^2c - (\overline{d}+a\overline{b})a|b|^2\overline{c} - |a|^2b(d+\overline{a}b)\overline{c}$$

$$- |a|^2\overline{b}(\overline{d}+a\overline{b})c + |a|^4|b|^2 + |a|^2|b|^4.$$

Proposition 8 Let $A,B,C \in \mathrm{SU}(2,1)$ with $ABC = I$. Let

$$a = \mathrm{tr}(A),\ b = \mathrm{tr}(B),\ c = \mathrm{tr}(C),\ d = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})tr(B).$$

Then

$$2\Re(\mathrm{tr}[A,B]) = |a|^2 + |b|^2 + |c|^2 + |d|^2 - abc - \overline{a}\overline{b}\overline{c} - 3$$

and

$|\mathrm{tr}[A,B]|^2 =$

$$|a|^2|b|^2|c|^2 + a^2b^2\overline{c}^2 + \overline{a}^2\overline{b}^2c + a^2\overline{b}^2c^2 + \overline{a}^2b\overline{c}^2 + \overline{a}b^2c^2$$

$$+ a\overline{b}^2\overline{c}^2 + |a|^2|b|^2 + |b|^2|c|^2 + |a|^2|c|^2 - ab\overline{c}^2 - 2\overline{a}\overline{b}c^2$$

$$- 2a\overline{b}^2c - 2\overline{a}b^2\overline{c} - 2\overline{a}^2bc - 2a^2\overline{b}\overline{c} + a^3 + \overline{a}^3 + b^3 + \overline{b}^3$$

$$+ c^3 + \overline{c}^3 + 3abc + 3\overline{a}\overline{b}\overline{c} - 6|a|^2 - 6|b|^2 - 6|c|^2$$

$$+ d(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c)$$

$$+ \overline{d}(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c)$$

$$+ (d^2 - 3\overline{d})(\overline{a}b + \overline{b}c + a\overline{c}) + (\overline{d}^2 - 3d)(a\overline{b} + b\overline{c} + \overline{a}c)$$

$$+ |d|^2(|a|^2 + |b|^2 + |c|^2 - 6) + d^3 + \overline{d}^3 + 9.$$

Proof. Using $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)} = \overline{a}$ etc and also $\mathrm{tr}(A^{-1}B) = d + \overline{a}b$ in the expression of corollary 4 (chapter 4) gives

$$2<(\mathrm{tr}[A,B]) = |a|^2 + |b|^2 + |a|^2|b|^2 - 3 + c(\bar d + a\bar b) - ab\overline{(d + ab)}$$
$$-\,\bar a\,\bar b\,\bar c + (d + \bar a b)(\bar d + a\bar b) - \bar a b(\bar d + a\bar b) - a\bar b(d + \bar a b)$$

$$= |a|^2 + |b|^2 + |a|^2|b|^2 - 3 + |c|^2 - abc - \bar a\bar b\bar c$$

$$+ (d + \bar a b)(\bar d + a\bar b) - \bar a b(\bar d + a\bar b) - a\bar b(d + \bar a b)$$

$$= |a|^2 + |b|^2 + |c|^2 + |d|^2 - abc - \bar a\bar b\bar c - 3.$$

For the second part, we simplify the equation given in proposition 7 above to have

$$|\mathrm{tr}[A,B]|^2 = 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 - 6(d + \bar a b)(\bar d + ab) + a^3 + b^3$$

$$+ c^3 + (\bar d + a\bar b)^3 + \bar a^3 + \bar b^3 + \bar c^3 + (d + \bar a b)^3 - 3(d + \bar a b)ac$$
$$- 3(\bar d + a\bar b)\bar a\bar c - 3(d + \bar a b)b\bar c - 3(\bar d + a\bar b)\bar b c + 3(d + \bar a b)a\bar b$$

$$+ 3(\bar d + a\bar b)\bar a b + 3abc + 3\bar a\,\bar b\,\bar c + |a|^2|b|^2 + |b|^2|c|^2$$

$$+ (d + \bar a b)(\bar d + a\bar b)|a|^2 + (d + \bar a b)(\bar d + a\bar b)|b|^2 + |a|^2|c|^2$$

$$+ (d + \bar a b)(\bar d + a\bar b)|c|^2 + (d + \bar a b)\bar b c + (\bar d + a\bar b)b\bar c + \bar a^2\bar b(d + \bar a b)$$

$$+ a^2 b(\bar d + a\bar b) + a\bar b^2 c + \bar a b^2\bar c + (d + \bar a b)a^2 c + (\bar d + a\bar b)\bar a^2\bar c$$

$$+ (d + \bar a b)bc^2 + (\bar d + a\bar b)\bar b\bar c^2 + a^2bc + \bar a^2\bar b\,\bar c + (d + \bar a b)\bar a b^2$$

$$+ (\bar d + a\bar b)b^2 c + ac(\bar d + a\bar b)^2 + (\bar d + a\bar b)a\,\bar b^2 + (d + \bar a b)\bar b^2 c$$

$$+ \bar a\bar c(d + \bar a b)^2 + \bar a\,\bar c^2(\bar d + a\bar b) + ac^2(d + \bar a b) - 2\bar a\,\bar b c^2$$
$$- 2ab\bar c^2 - 2\bar a b(d + \bar a b)^2 - 2a\bar b(\bar d + a\bar b)^2 + \bar a^2\bar b^2 c + a^2b^2\bar c$$

$$+ (d + \bar a b)\bar a^2 b^2 + (\bar d + a\bar b) - (d + \bar a b)a\bar b|b|^2 - (\bar d + a\bar b)\bar a b|b|^2$$

$$- a|a|^2bc - \bar a|a|^2\bar b\,\bar c - ab|b|^2c - \bar a\,\bar b|b|^2\bar c - (d + \bar a b)a|a|^2\bar b$$

$$- (\bar d + a\bar b)\bar a|a|^2 b - |a|^2\bar b^3 - |a|^2 b^3 - \bar a^3|b|^2 - a^3|b|^2$$

$$- (d + \bar a b)\bar a|b|^2 c - (\bar d + a\bar b)a|b|^2\bar c - |a|^2 b(d + \bar a b)\bar c$$
$$- |a|^2\bar b(\bar d + a\bar b)c + |a|^4|b|^2 + |a|^2|b|^4$$

$$= 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 - 6|d|^2 - 6\bar a b\bar d - 6|a|^2|b|^2 - 6a\bar b d$$

$$+ a^3 + b^3 + \bar c^3 + d^3 + \bar a^3 + \bar b^3 + c^3 + \bar d^3 + 3a\bar b\bar d^2 + 3a^2\bar b^2\bar d$$

$$
\begin{aligned}
&+ a^3b^3 + 3a\overline{b}d^2 + 3a^2b^2d + a^3b^3 - 3a\overline{c}d - 3a^2bc - 3ac\,\overline{d} \\
&- 3a^2\overline{b}\,\overline{c} - 3b\overline{c}d - 3\overline{a}b^2\overline{c} - 3\overline{b}c\overline{d} - 3a\overline{b}^2c + 3a\overline{b}d + 3|a|^2|b|^2 \\
&+ 3\overline{a}b\overline{d} + 3|a|^2|b|^2 + 3abc + 3\overline{a}\,\overline{b}\,\overline{c} + |a|^2|b|^2 + |b|^2|c|^2 + |a|^2|d|^2 \\
&+ a|a|^2\overline{b}d + \overline{a}|a|^2b\overline{d} + |a|^4|b|^2 + |b|^2|d|^2 + a\overline{b}|b|^2d + \overline{a}b|b|^2\overline{d} \\
&+ |a|^2|b|^4 + |c|^2|d|^2 + a\overline{b}|c|^2d + \overline{a}b|c|^2\overline{d} + |a|^2|b|^2|c|^2 + \overline{b}cd^2 \\
&+ 2\overline{a}|b|^2\overline{c}d + \overline{a}^2b|b|^2c + b\overline{c}\overline{d}^2 + 2a|b|^2\overline{c}\,\overline{d} + a^2|b|^2\overline{c}\,\overline{c} + \overline{a}^2\overline{b}d \\
&+ \overline{a}^3|b|^2 + a^2b\overline{d} + a^3|b|^2 + a\overline{b}^2c + \overline{a}b^2\overline{c} + a^2cd + a|a|^2bc + \overline{a}^2\overline{c}\,\overline{d} \\
&+ \overline{a}|a|^2\overline{b}\,\overline{c} + bc^2d + \overline{a}b^2c^2 + \overline{b}\,\overline{c}^2\overline{d} + a\overline{b}^2\,\overline{c}^2 + \overline{a}^2bc + a^2\overline{b}\,\overline{c} \\
&+ ab^2d + |a|^2b^3 + \overline{a}\,\overline{b}^2\overline{d} + |a|^2\overline{b}^3 + \overline{b}^2\,\overline{c}d + \overline{a}|b|^2\overline{b}\,\overline{c} + b^2c\overline{d} \\
&+ ab|b|^2c + a\overline{c}d^2 + 2|a|^2b\overline{c}d + \overline{a}|a|^2b^2\overline{c} + \overline{a}c\overline{d}^2 + 2|a|^2\overline{b}c\overline{d} \\
&+ a|a|^2\overline{b}^2c + \overline{a}\,\overline{c}^2d + \overline{a}^2b\overline{c}^2 + ac^2d + a^2\overline{b}c^2 - 2\overline{a}\,\overline{b}c^2 - 2\overline{a}\,\overline{b}\,\overline{c}^2 \\
&- 2\overline{a}bd^2 - 4\overline{a}^2b^2d - 2\overline{a}^3b^3 - 2a\overline{b}\,\overline{d}^2 - 4a^2\overline{b}^2\overline{d} - 2a^3\overline{b}^3 + \overline{a}^2\overline{b}^2c \\
&+ a^2b^2\overline{c} + \overline{a}^2b^2d + \overline{a}^3b^3 + a^2b^2\overline{d} + a^3\overline{b}^3 - a\overline{b}|b|^2d - |a|^2|b|^4 \\
&- \overline{a}\,\overline{b}|b|^2\overline{d} - |a|^2|b|^4 - a|a|^2bc - \overline{a}|a|^2\overline{b}\,\overline{c} - a\overline{b}|b|^2c - \overline{a}\,\overline{b}|b|^2\overline{c} \\
&- a|a|^2\overline{b}d - |a|^4|b|^2 - \overline{a}|a|^2b\overline{d} - |a|^4|b|^2 - |a|^2\overline{b}^3 - |a|^2b^3 \\
&- \overline{a}^3|b|^2 - a^3|b|^2 - \overline{a}|b|^2cd - \overline{a}^2|b|^2bc - a|b|^2\overline{c}d - a^2\overline{b}|b|^2\overline{c} \\
&- |a|^2b\overline{c}d - \overline{a}|a|^2b^2\overline{c} - |a|^2\overline{b}c\overline{d} - a|a|^2\overline{b}^2c + |a|^4|b|^2 + |a|^2|b|^4 \\
&= 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 + a^3 + \overline{a}^3 + b^3 + \overline{b}^3 + c^3 + \overline{c}^3 + d^3 \\
&+ \overline{d}^3 + |a|^2|b|^2 + |a|^2|c|^2 + |b|^2|c|^2 + |a|^2|b|^2|c|^2 + \overline{a}^2b\overline{c}^2 + a^2\overline{b}c^2 \\
&+ |d|^2(|a|^2 + |b|^2 + |c|^2 - 6) + (d^2 - 3\overline{d})(\overline{a}b + \overline{b}c + a\overline{c}) \\
&+ (\overline{d}^2 - 3d)(a\overline{b} + b\overline{c} + \overline{a}c) - 2\overline{a}^2bc - 2a^2\overline{b}\,\overline{c} - 2\overline{a}b^2\overline{c} - 2a\overline{b}^2c \\
&- 2\overline{a}\,\overline{b}c^2 - 2abc^2 + 3abc + 3\overline{a}\,\overline{b}\,\overline{c} + a\overline{b}^2c + ab^2c^2 + ab^2c^2 + a^2b^2c \\
&+ d(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c) \\
&+ \overline{d}(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c).
\end{aligned}
$$

Remark 3: We remark that when we write the formula of the real and modulos of tr$[A,B]$ in terms of traces of A,B,AB and $A^{-1}B$ (see lemma

4.4.3) then there is a set symmetries generated by $(A,B) \to (B,A)$ and $(A,B) \to (A^{-1},B)$ etc. Some of these send [A, B] to itself, others to its inverse. Thus there are two solutions to the quadratic. However, when we write in terms of a,b,c,d (as in proposition 7) there is a three fold cyclic symmetry $a \to b \to c \to a$.

Now when we put the real and modulos of $\mathrm{tr}[A,B]$ together we have

Proposition 9 Let A,B,C elements of $\mathrm{SU}(2,1)$ with $ABC = I$. Then if $\langle A,B,C\rangle$ is Zariski dense, it is determined up to conjugacy by

$$\mathrm{tr}(A),\ \mathrm{tr}(B),\ \mathrm{tr}(C),\ \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B),\ \mathrm{tr}[A,B].$$

Also, the last two of these expressions remain unchanged under cyclic permutations of A,B and C.

Moreover, the group is determined by $\mathrm{tr}(A)$,$\mathrm{tr}(B)$,$\mathrm{tr}(C)$,$\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$ together with the sign of the imaginary part of $\mathrm{tr}[A,B]$.

Remark 4: In proposition 9 Parker (2012) attempts to parametrise pair of pants groups via traces. As seen in the discussion in Parker (2012), since $\mathrm{SU}(2,1)$ has dimension four one cannot determine $\langle A,B\rangle$ up to conjugation. One would expect to only need to use four traces to describe $\langle A,B\rangle$. Ideally, one needs an extra one, $\mathrm{tr}[A,B]$ but the real part and absolute value of $\mathrm{tr}[A,B]$ are determined by other parameters. So Parker (2012) considered a group with three generators $\langle A,B,C\rangle$ whose product is the identity instead of $\langle A,B\rangle$. The reason for doing so is to get a formulae with three fold symmetry.

Example 4.4.1 We now consider an example that shows the traces of A,B,AB and $A^{-1}B$ do not determine the imaginary part of $\mathrm{tr}[A,B]$.

For $\theta \in (-\pi/2,\pi/2)$ let $Q(\theta) \in \mathrm{SU}(2,1)$ be the matrix

$$Q(\theta) = \frac{-e^{-i\theta/6}}{2\cos(\theta/2)} \begin{bmatrix} 1 & \sqrt{2\cos(\theta)} & -e^{i\theta} \\ \sqrt{2\cos(\theta)} & e^{-i\theta} - 1 & \sqrt{2\cos(\theta)} \\ -e^{i\theta} & \sqrt{2\cos(\theta)} & 1 \end{bmatrix}.$$

$$A = \begin{pmatrix} re^{i\varphi} & 0 & 0 \\ 0 & e^{-2i\varphi} & 0 \\ 0 & 0 & r^{-1}e^{i\varphi} \end{pmatrix}, B_\theta = Q(\theta) \begin{pmatrix} se^{i\psi} & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & s^{-1}e^{i\psi} \end{pmatrix} Q(-\theta)$$

Then we have

$$\mathrm{tr}(A) = (r + r^{-1})e^{i\varphi} + e^{-2i\varphi},$$

$$\mathrm{tr}(B_\theta) = (s + s^{-1})e^{i\psi} + e^{-2i\psi}$$

Note that $Q(\theta)^{-1} = Q(-\theta)$. For $r > 1$ and $s > 1$, de ne $A, B \in SU(2,1)$ by

$$\begin{aligned} \mathrm{tr}(AB_\theta) = {} & \frac{1}{2 + 2\cos(\theta)} \left((r + r^{-1})e^{i\phi} + 2\cos(\theta)e^{-2i\phi} \right) \cdot \\ & \left((s + s^{-1})e^{i\psi} + 2\cos(\theta)e^{-2i\psi} \right) \\ & - \frac{1}{2 + 2\cos(\theta)} e^{-2i\phi - 2i\psi} \left((2\cos(\theta) + 2\cos(2\theta) \right), \end{aligned}$$

$$
\begin{aligned}
\mathrm{tr}(A^{-1}B_\theta) = {} & \frac{1}{2+2\cos(\theta)}\left(\; (r+r^{-1})e^{-i\phi} + 2\cos(\theta)e^{2i\phi} \; \right) \cdot \\
& \left(\; (s+s^{-1})e^{i\psi} + 2\cos(\theta)e^{-2i\psi} \; \right) \\
& - \frac{1}{2+2\cos(\theta)} e^{2i\phi-2i\psi} \left(\; (2\cos(\theta) + 2\cos(2\theta) \; \right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}[A,B_\theta] = {} & 3 - \frac{1}{(2+2\cos(\theta))^2}(r-r^{-1})^2(s-s^{-1})^2 \\
& - \frac{8\cos(\theta)}{(2+2\cos(\theta))^2}\left(\; (2-(r+r^{-1})\cos(3\phi)(2-(s+s^{-1})\cos(3\psi) \; \right) \\
& + \frac{2\cos(\theta)}{(2+2\cos(\theta))^2}\left(\; (e^{i\theta}(rs^{-1}+r^{-1}s) + e^{-i\theta}(rs+r^{-1}s^{-1}) \; \right) \cdot \\
& \left(\; r+r^{-1}-2\cos(\theta) \; \right)\left(\; s+s^{-1}-2\cos(3\psi) \; \right)
\end{aligned}
$$

Then it is easy to see that

$$\mathrm{tr}(B_\theta) = \mathrm{tr}(B_{-\theta}),\ \mathrm{tr}(AB) = \mathrm{tr}(AB_{-\theta}),\ \mathrm{tr}(A^{-1}B_\theta) = \mathrm{tr}(A^{-1}B_{-\theta})$$

but $\mathrm{tr}[A,B_\theta] \neq \mathrm{tr}[A,B_{-\theta}]$.

4.5 Cross-ratios

Cross-ratios were generalised to complex hyperbolic space by Korá nyi and Riemann. Following their notation, given that z_1,z_2,z_3,z_4 are quadruple of distinct points on $\partial\mathbf{H}^2_{\mathbb{C}}$. Let $\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3$ and $\mathbf{z}_4$ be corresponding lifts in $V_0 \subset \mathbb{C}^{2,1}$.

Their Korá nyi-Riemann cross-ratio is de ned to be

$$X = [z_1,z_2,z_3,z_4] = \frac{\langle \mathbf{z}_3,\mathbf{z}_1\rangle\langle \mathbf{z}_4,\mathbf{z}_2\rangle}{\langle \mathbf{z}_4,\mathbf{z}_1\rangle\langle \mathbf{z}_3,\mathbf{z}_2\rangle}.$$

Since the z_i are distinct we see that X is nite and non-zero. We note that X is invariant under SU(2,1) and independent of the chosen lifts. We will only use the absolute value $|[z_1,z_2,z_3,z_4]|$ which we call the real cross-ratio. Observe that if two of the entries are the same then the cross-ratio is still de ned and equals one of 0, 1 or ∞. If z_1, z_2, z_3 and z_4 all lie on $\partial\mathbf{H}^2_{\mathbb{C}}$ then we can express

the cross-ratio in terms of the Cygan metric as follows:

$$|[z_1, z_2, z_3, z_4]| = \frac{\rho_0(\mathbf{z}_3, \mathbf{z}_1)^2 \rho_0(\mathbf{z}_4, \mathbf{z}_2)^2}{\rho_0(\mathbf{z}_4, \mathbf{z}_1)^2 \rho_0(\mathbf{z}_3, \mathbf{z}_2)^2}$$

provided none of the four points is ∞. If $z_3 = \infty$ then

$$|[z_1, z_2, z_3, z_4]| = \frac{\rho_0(\mathbf{z}_4, \mathbf{z}_2)^2}{\rho_0(\mathbf{z}_4, \mathbf{z}_1)^2}.$$

(Parker, 2010).

By choosing di erent orderings our four points we may de ne other crossratios. There are symmetries associated with certain permutations. Having said that, it remains that there are only three cross-ratios left. Given distinct points $z_1, z_2, z_3, z_4 \in \partial\mathbf{H}^2_{\mathbb{C}}$, we de ne

$$\mathbb{X}_1 = [z_1, z_2, z_3, z_4],\ \mathbb{X}_2 = [z_1, z_3, z_2, z_4],\ \mathbb{X}_3 = [z_2, z_3, z_1, z_4] \tag{4.2}$$

Then three complex numbers $\mathbb{X}_1, \mathbb{X}_2$ and $\mathbb{X}_3$ satisfy the following identities

$$|\mathbb{X}_2| = |\mathbb{X}_1||\mathbb{X}_3|, \tag{4.3}$$

$$2|\mathbb{X}_1|^2 \Re(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1 + \mathbb{X}_2) \tag{4.4}$$

Note that the norm and real part of $\mathbb{X}_3$ are determined by $\mathbb{X}_1$ and $\mathbb{X}_2$, but that the sign of $\Im(\mathbb{X}_3)$ is not determined Parker (2012).

Let A and B be loxodromic maps in $SU(2,1)$ with attracting xed points A and B be a_A, a_B and repelling xed points be r_A, r_B respectively. Suppose these xed points correspond to attractive eigenvectors $\mathbf{a}_A, \mathbf{a}_B$ and repulsive eigenvectors $\mathbf{r}_A, \mathbf{r}_B$ respectively. Following (4.2), we de ne the rst, second and

third cross-ratios of loxodromic maps A and B to be

$$\mathbb{X}_1(A,B) = [a_B, a_A, r_A, r_B] = \frac{\langle \mathbf{r}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{a}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{r}_A, \mathbf{a}_A \rangle} \tag{4.5}$$

$$\mathbb{X}_2(A,B) = [a_B, r_A, a_A, r_B] = \frac{\langle \mathbf{a}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{a}_A, \mathbf{r}_A \rangle} \tag{4.6}$$

$$\mathbb{X}_3(A,B) = [a_A, r_A, a_B, r_B] = \frac{\langle \mathbf{a}_B, \mathbf{a}_A \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_A \rangle \langle \mathbf{a}_B, \mathbf{r}_A \rangle} \tag{4.7}$$

Since the xed points were assumed to be distinct, none of these cross-ratios is either zero or in nity. These three numbers satisfy the identities of (4.3) and (4.4) Parker (2012).

Theorem 4.5.1 Suppose that A and B are loxodromic elements of $SU(2,1)$ with distinct xed points. Also, suppose that $\langle A,B\rangle$ does not preserve a complex line. Then the group $\langle A,B\rangle$ is determined up to conjugation in $\mathrm{SU}(2,1)$ by: $\mathrm{tr}(A)$,$\mathrm{tr}(B)$,$\mathbb{X}_1(A,B)$,$\mathbb{X}_2(A,B)$ and $\mathbb{X}_3(A,B)$.

It is obvious that, this result is asymmetrical in that it depends on the choice of two of the boundary curves. To get around the di culty, Parker (2012) used the method of Parker and Platis in their paper Complex hyperbolic FenchelNielsen coordinates to show that choosing a di erent pair of boundary coordinates amount to a real change of coordinates (Parker, 2012).

Proposition 10 Let A,B and C be loxodromic elements of $\mathrm{SU}(2,1)$ with $ABC = I$. Then $\mathrm{tr}(C)$,$\mathbb{X}_1(A,C)$,$\mathbb{X}_2(A,C)$ and $\mathbb{X}_3(A,C)$ may be expressed as real analytic functions of $\mathrm{tr}(A)$,$\mathrm{tr}(B)$,$\mathbb{X}_1(A,B)$,$\mathbb{X}_2(A,B)$ and $\mathbb{X}_3(A,B)$.

To conclude this section, we show how these mixed trace and cross-ratio coordinates are related to the trace coordinates we found in previous section.

Proposition 11 Let A and B be loxodromic maps in $\mathrm{SU}(2,1)$ with $\mathrm{tr}(A) = \lambda + \bar{\lambda}\lambda^{-1} + \lambda^{-1}$ and $\mathrm{tr}(B) = \mu + \mu\mu^{-1} + \bar{\mu}^{-1}$. Let $\mathbb{X}_1$,$\mathbb{X}_2$,$\mathbb{X}_3$ be the cross-ratios of their xed points given by (4.5), (4.6) and (4.7). Then the traces of AB,$A^{-1}B$ and $[A,B]$ are given by

$$\begin{aligned}\mathrm{tr}(AB) = {} & (\lambda + \overline{\lambda}^{-1})\overline{\mu}\mu^{-1} + \overline{\lambda}\lambda^{-1}(\mu + \mu^{-1}) - \lambda\overline{\lambda}^{-1}\overline{\mu}\mu^{-1} \\ & + \mathbb{X}_1(\overline{\lambda}^{-1} - \overline{\lambda}\lambda)(\overline{\mu}^{-1} - \overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_1(\lambda - \overline{\lambda}\lambda^{-1})(\mu - \overline{\mu}\mu^{-1}) \\ & + \mathbb{X}_2(\lambda - \overline{\lambda}\lambda^{-1})(\overline{\mu}^{-1} - \overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_2(\overline{\lambda}^{-1} - \overline{\lambda}\lambda)(\mu - \overline{\mu}\mu^{-1}),\end{aligned}$$

$$\begin{aligned}\mathrm{tr}(A^{-1}B) = {} & (\lambda^{-1} + \lambda)\mu\overline{\mu}^{-1} + \overline{\lambda}\lambda^{-1}(\mu + \mu^{-1}) - \lambda\lambda^{-1}\overline{\mu}\mu^{-1} \\ & + \mathbb{X}_1(\overline{\lambda}^{-1} - \overline{\lambda}\lambda^{-1})(\overline{\mu}^{-1} - \overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_1(\lambda^{-1} - \lambda\overline{\lambda}^{-1})(\mu - \overline{\mu}\mu^{-1}) \\ & + \mathbb{X}_2(\lambda^{-1} - \lambda\overline{\lambda}^{-1})(\overline{\mu}^{-1} - \overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_2(\overline{\lambda} - \lambda\overline{\lambda}^{-1})(\mu - \overline{\mu}\mu^{-1})\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}[A,B] = {} & 3 - 2\Re(\overline{\mathbb{X}}_1(\lambda - \overline{\lambda}\lambda^{-1})(\lambda^{-1} - \overline{\lambda}\lambda^{-1})(\overline{\mu} - \mu\mu^{-1})(\mu^{-1} - \overline{\mu}\mu^{-1})) \\ & - 2\Re(\mathbb{X}_2(\lambda - \overline{\lambda}\lambda^{-1})(\lambda^{-1} - \lambda\overline{\lambda})(\overline{\mu} - \mu\overline{\mu}^{-1})(\overline{\mu}^{-1} - \overline{\mu}\mu^{-1})) \\ & + (1 - 2\Re(\mathbb{X}_1 + \mathbb{X}_2))(|(\lambda - \overline{\lambda}\lambda^{-1})(\mu - \overline{\mu}\mu^{-1})|^2 + |(\lambda^{-1} - \lambda\overline{\lambda}^{-1}) \\ & \cdot (\mu^{-1} - \mu\overline{\mu}^{-1})|^2) + |\mathbb{X}_1(\overline{\lambda} - \lambda\overline{\lambda}^{-1})(\overline{\mu} - \mu\overline{\mu}^{-1}) + \overline{\mathbb{X}}_1(\lambda^{-1} - \lambda\overline{\lambda}^{-1}) \\ & \cdot (\mu^{-1} - \mu\overline{\mu}^{-1}) + \mathbb{X}_2(\lambda^{-1} - \lambda\overline{\lambda}^{-1})(\overline{\mu} - \mu\overline{\mu}^{-1}) + \overline{\mathbb{X}}_2(\overline{\lambda} - \lambda\overline{\lambda}^{-1}) \\ & \cdot (\mu^{-1} - \mu\overline{\mu}^{-1})|^2 + (|\mathbb{X}_2|^2 - |\mathbb{X}_1|^2\mathbb{X}_3)(|\lambda - \overline{\lambda}\lambda^{-1}|^2 - |\lambda^{-1} - \lambda\overline{\lambda}^{-1}|^2 \\ & - |\lambda^{-1} - \lambda\overline{\lambda}^{-1}|)(|\mu - \overline{\mu}\mu^{-1}|^2 - |\mu^{-1} - \mu\overline{\mu}^{-1}|^2).\end{aligned}$$

Remark 5: In theorem 4.5.1 Parker (2012) again tries to parametrise pair of pants group by using traces of two elements and cross-ratios. Even with this, there is a problem of a sign. This time it is the sign of the imaginary part of $\mathbb{X}_3$. Furthermore, this ambiguity is the same as the ambiguity in the sign of the $\Im(\mathrm{tr}[A,B])$ (found in remark 4). Now from proposition 4.15 in Parker (2012), we can express $\mathbb{X}_1$ and $\mathbb{X}_2$ in terms of $\lambda, \mu, \mathrm{tr}(AB)$ and $\mathrm{tr}(A^{-1}B)$ which give the trace coordinates found in the previous section. So combining trace and cross-ratio we can parametrise pair of pants by considering the group $\langle A,B\rangle$. The merit of this method is that, we can still determine conjugation in $\mathrm{SU}(2,1)$ with only two elements $A,B \in \mathrm{SU}(2,1)$.

4.6 Twist-bend parameters

Let P be a surface of genus $g \geq 2$. We may decompose P into Y-pieces (so called pair of pants) and use this decomposition to de ne complex Fenchel-Nielsen coordinates. The trace coordinates from proposition 8 (chapter 4) give

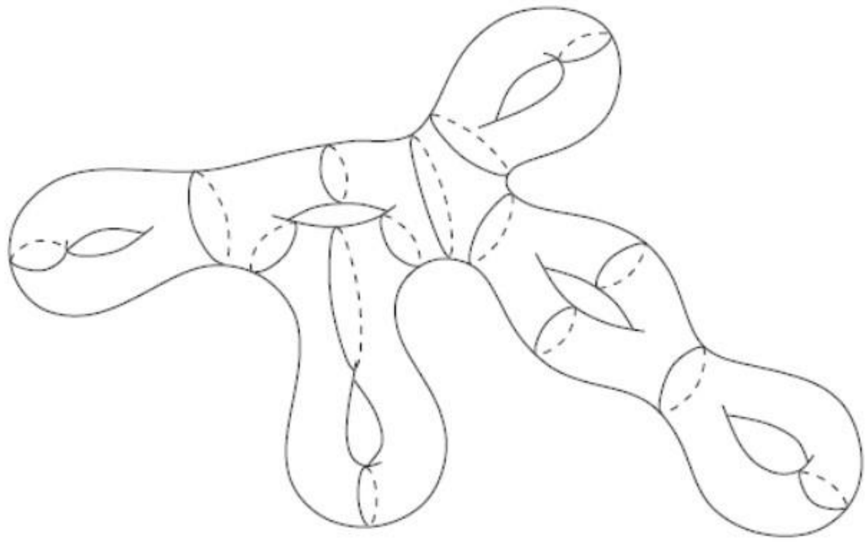

Figure 4.2: Decomposing of a surface into pair of pants

the Fenchel-Nielsen complex lengths via proposition 4 (chapter 3). The remainder of this section gives a brief sketch on how to de ne Fenchel-Nielsen twist-bends.

For hyperbolic surfaces, a Fenchel-Nielsen twist about a simple, closed, oriented curve α involves cutting the surface along α and then re-attracting so that points on one side are moved a hyperbolic distance k relative to the other side. It is often useful to think about doing this with the lift $\tilde{\alpha}$ of α to the hyperbolic plane. If $\tilde{\alpha}$ is the geodesic in the upper half plane with endpoints 0 and ∞, then this process involves applying the dilation $K : z \mapsto e^{k}z$ to the part of the hyperbolic plane on one side of $\tilde{\alpha}$, say the part with $\Re(z) > 0$. We allow k to be negative and this corresponds to moving in the opposite direction relative to the orientation of α. If we twist by a hyperbolic distance k equal to the length of α then the Fenchel-Nielsen twist is the same as a Dehn twist (Parker, 2012).

We can generalise the de nition of Fenchel-Nielsen twist to twist-bends in hyperbolic 3-space $\mathbf{H}^2_R$. The easiest way to describe this is to suppose that the universal cover of the surface is a hyperbolic plane inside $\mathbf{H}^3_R$ and that $\tilde{\alpha}$ is again the geodesic with endpoints 0 and ∞. As well as applying a FenchelNielsen twist, we may also rotate through an angle θ in the plane normal to $\tilde{\alpha}$. This is called a bend and corresponds to applying the rotation $K : z \mapsto e^{i\theta}z$. Doing a twist through distance k and a bend through angle θ gives twist -bend parameter $k + i\theta$. It corresponds to applying the loxodromic surface results in a pleated surface where the bending locus is the geodesic α. There is a relationship

between traces and bending for such surfaces (Parker, 2012).

This description can be extended to the case of complex hyperbolic representations of surface groups even though there is no longer a hyperbolic surface. However, the local picture is the same. Namely, a twist will be a hyperbolic translation of distance k along a geodesic $\alpha(or\tilde{\alpha})$ and a bend through

angle θ in the plane normal to be the complex line containing $\tilde{\alpha}$ (Parker, 2012).

We can describe twist-bends on the level of the fundamental groups. This works for all the cases described above, but we only give details in the case of $\mathrm{SU}(2,1)$. There are two cases, (a) when the parts of the surface on either side of α in di erent three-holed spheres and (b) when they are in the same three-holed sphere. From group theory point of view, (a) corresponds to a free product with amalgamation and (b) to an HNN extension. The de nition of twist-bends in

each case are similar but not quite the same (Parker, 2012).

Let Y be a three holed sphere with oriented boundary geodesics α,β,γ and let $\rho_0 : \pi_1 \mapsto \mathrm{SU}(2,1)$ be a corresponding representation with $\rho_0([\alpha]) = A, \rho_0([\beta]) = B$ and $\rho_0([\gamma]) = C$ where $ABC = I$. Suppose that Y^0 is another

such surface. We have $\alpha', \beta', \gamma', \rho_0', A', B', C'$ all as above. We must decide when we can attach Y and Y^0 along α and α^0. We can do so when α^0 has the same length as α but the opposite orientation. Hence we must have A' being conjugate to A^{-1} (Parker, 2012).

There are two cases to consider. First we must investigate what happens when we attach two distinct three holed spheres along a common boundary. For our initial configuration, we suppose $A' = A^{-1}$. Attaching Y and Y' is then the same as taking the free product of $\langle A,B\rangle$ and $\langle A',B'\rangle$ with amalgamation along the common subgroup $\langle A\rangle = \langle A'\rangle$. The resulting group is then

$$\langle A,B\rangle *_{\langle A\rangle} \langle A',B'\rangle = \langle A,B,A',B' \mid A' = A^{-1}\rangle = \langle A,B,B'\rangle.$$

In this case a twist-bend consists of fixing the surface corresponding to $\langle A,B\rangle$ and moving the surface corresponding to $\langle A',B'\rangle$ by a hyperbolic translation along the axis of A(the twist) and a rotation around the complex axis of A(the bend). In other words, we take a map K that commutes with $A' = A^{-1}$ and we conjugate $\langle A',B'\rangle$ by K. Thus the new group is

$$\langle A,B\rangle *_{\langle A\rangle} K\langle A',B'\rangle K^{-1} = \langle A,B\rangle *_{\langle A\rangle} \langle KA'K^{-1},KB'K^{-1}\rangle = \langle A,B,KA'K^{-1},KB'K^{-1} \mid$$
$$KA'K^{-1} = A^{-1}\rangle = \langle A,B,KB'K^{-1}\rangle.$$

Note that if we swap the roles of Y and Y' then the same process yields a twistbend associated to the matrix K^{-1}. That is, the new group is $\langle A',B',K^{-1}B'K\rangle$ which is conjugate to $\langle A,B,KB'K^{-1}\rangle$ (Parker, 2012).

Secondly, we must consider the case where we close a handle. In this case we consider Y and want to glue two of its boundary components. Suppose one of them is represented by A then the other must be conjugate to A^{-1}, say it is $BA^{-1}B^{-1}$. Note that if A and $BA^{-1}B^{-1}$ correspond to boundary components of the same three holed sphere, this means the third boundary component is $C = (BA^{-1}B^{-1})^{-1}A^{-1} = [B,A]$. Then in order to chose the handle we take HNN extension associated to the isomorphism $\varphi : \mathrm{h}A\mathrm{i} \longrightarrow \mathrm{h}BAB^{-1}\mathrm{i}$ given by $\varphi(A) = BA^{-1}B^{-1}$. It is clear that we may do this by adjoining the stable letter B to obtain

$\mathrm{h}A,BA^{-1}B^{-1}\mathrm{i}*_{\varphi} = \mathrm{h}A,BA^{-1}B^{-1},B \mid BA^{-1}B^{-1} = \varphi(A)\mathrm{i} = \mathrm{h}A,B\mathrm{i}$. Because we want to keep track of the stable letter, we will write extension as

$$\mathrm{h}A,BA^{-1}B^{-1}\mathrm{i} *_{\varphi} (B)$$

If K is a map that commutes with A then we also have $\varphi(A) = (BK)A^{-1}(BK)^{-1}$. Therefore we can take an isomorphic HNN extension by adding the stable letter BK instead of B:

$$\mathrm{h}A,(BK)A^{-1}(BK)^{-1}\mathrm{i} * \varphi(BK) = \mathrm{h}A,BK\mathrm{i}.$$

Thus, in the case of closing a handle performing a complex twist associated to a map K that commutes with A involves changing the stable letter of the HNN extension from B to BK (Parker, 2012).

In both cases, the geometry of the complex twist is record by tr(K) in exactly the same way that tr(A) is related to $`(a) + i\varphi(a)$ as described in proposition 4 (chapter 2). Therefore if K corresponds to a twist through distance $k \in$ R and bend through angle $\theta \in (-\pi,\pi]$ then we have

$$\mathrm{tr}(K) = 2\cosh(k/2)e^{-i\theta/3} + e^{2i\theta/3}.$$

Note that there are subtleties about the direction of twist and the sign of $<(k)$.

Chapter 5

TRACES FOR TRIANGLE GROUPS

5.1 Introduction

This section discusses groups generated by three complex re ections. DeligneMostow came out with groups and their groups have connection with groups generated by three complex re ections (Parker, 2012).

Suppose that Δ is a group generated by three complex re ections in $SU(2,1)$ all with the same angle. In this section Parker (2012) gives combinatorial formulae for the traces of elements of Δ. These formulae are due to Pratoussevitch (2005) generalising earlier work of Sandler (1995). We then apply these formulae to nd the traces of elements of Δ as in proposition 12.

5.2 Re ections

Consider $\mathbb{R}^{n+1}$ with the standard inner product. Let Π be a hyperplane through the origin and let n be a normal vector Π. Thus the orthogonal complement $\Pi^{\perp}$ of Π is spanned by n. Every vector $\mathbf{x} \in \mathbb{R}^{n+1}$ can be written as the sum of two orthogonal vectors, that is

$$\mathbf{x} = \left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right) + \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n},$$

where

$$\left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right) \in \Pi, \quad \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} \in \Pi^{\perp}.$$

Speci cally, re ection R in Π is obtained by multiplying the component in $\Pi^{\perp}$ by -1

$$R(\mathbf{x}) = \left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right) + (-1)\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}.$$

Then

$$R(\mathbf{x}) = \left(\mathbf{x} - \frac{(\mathbf{x}\cdot\mathbf{n})}{(\mathbf{n}\cdot\mathbf{n})}\mathbf{n} \right) + (-1)\frac{(\mathbf{x}\cdot\mathbf{n})}{(\mathbf{n}\cdot\mathbf{n})}\mathbf{n} = \mathbf{x} - 2\frac{(\mathbf{x}\cdot\mathbf{n})}{(\mathbf{n}\cdot\mathbf{n})}\mathbf{n}.$$

This is represented by a matrix in $O(n,1)$ with determinant -1. The hyperplane model of (real) hyperbolic n-space is given by

$$\{x \in R^{n,1} : (x,x) = -1, x_{n+1} > 0\}.$$

Then R maps this hyperboloid to itself.

One can generalise this whole idea from the real world to the complex world, by considering C^{n+1} with a hermitian form H which we assume to be nondegenerate. But, at this point, no restrictions on its signature. The Hermitian form associated to H is given as:

$$hz,wi = w^{*}Hz \textit{ for } z \textit{ and } w \textit{ in } C^{n+1}.$$

Since we are interested in complex hyperbolic space $\mathbf{H}^{n}_{C}$ we mainly think of the case where the form H has signature $(n,1)$ but this is not necessary for the de nition of complex re ections.

Suppose that Π is a complex hyperplane in C^{n+1}. That is $\Pi^{\perp}$ is spanned by $n \in C^{n+1}$ and so we have $\Pi = \{z \in C^{n+1} : hz,ni = 0\}$. Any $z \in C^{n+1}$ may then be decomposed into component in Π and $\Pi^{\perp}$ as

$$= \left(\mathbf{z} - \frac{}{}\mathbf{n} \right) +$$

hz · ni hz · ni z
———n, hn · ni hn · ni

where

$$\left(\mathbf{z} - \frac{\langle \mathbf{z}\cdot\mathbf{n}\rangle}{\langle \mathbf{n}\cdot\mathbf{n}\rangle}\mathbf{n} \right) \in \Pi, \quad \frac{\langle \mathbf{z}\cdot\mathbf{n}\rangle}{\langle \mathbf{n}\cdot\mathbf{n}\rangle}\mathbf{n} \in \Pi^{\perp}.$$

At this point we give a clear di erence between real and complex re ections. A complex re ection will preserve the decomposition of z into components in Π and $\Pi^{\perp}$, will pointwise x the component in Π and will preserve the norm of z. However, since $\Pi^{\perp}$ is a complex line we have greater freedom than we did before: we may multiply the component in $\Pi^{\perp}$ by any complex number with modulus 1 (Parker, 2012). Hence we de ne the re ection in Π with angle ψ to be the map R(z) given by

$$R(\mathbf{z}) = \left(\mathbf{z} - \frac{\langle \mathbf{z} \cdot \mathbf{n} \rangle}{\langle \mathbf{n} \cdot \mathbf{n} \rangle}\mathbf{n} \right) + e^{i\psi}\frac{\langle \mathbf{z} \cdot \mathbf{n} \rangle}{\langle \mathbf{n} \cdot \mathbf{n} \rangle}\mathbf{n} = \mathbf{z} + (e^{i\psi} - 1)\frac{\langle \mathbf{z} \cdot \mathbf{n} \rangle}{\langle \mathbf{n} \cdot \mathbf{n} \rangle}\mathbf{n}. \qquad (5.1)$$

The map R is given by a matrix $\mathrm{U}(H)$ with n eigenvalues +1 and 1 eigenvalue $e^{i\psi}$.

Hence its determinant is $e^{i\psi}$. In order to obtain a map in $\mathrm{SU}(H)$ we must multiply this matrix by $e^{-i\psi/(n+1)}$.

In what follows, we are interested in the case where $n = 2, H$ has signature (2, 1) and n $\in V_+$. This means that, in terms of its action of $\mathbf{H}^2_{\mathbb{C}}$, the re ection R xes a complex line $L = \mathbb{P}\Pi \cap \mathbf{H}^2_{\mathbb{C}}$. However, it will be useful to consider the space of groups generated by three complex re ections (all with the same angle) for a Hermitian form H and then consider the subspace where H has the correct signature.

We now use examples of signature (1, 1) to illustrate that complex re ections in this case are just rotations, that is elliptic matrices.

Example 5.2.1 1. Consider $\mathbb{C}^{1,1}$ where the Hermitian form is given by H_0, as in (2.3.1). Let Π be the complex line in $\mathbb{C}^{1,1}$ with polar vector n where

$$\mathbf{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then using (5.1) the re ection in Π with angle ψ is given by

$$R(z) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (e^{i\psi} - 1)\frac{z_1}{1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} e^{i\psi} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

The matrix in the last line is in U(1,1). Since we are dealing with traces, we want to lift R to a matrix in SU(1,1). Hence we multiply the U(1,1) matrix by $e^{-i\psi/2}$ to get

$$R = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}.$$

2. Consider $\mathbb{C}^{1,1}$ where the Hermitian form is given by H_0'. Let Π be the complex line in $\mathbb{C}^{1,1}$ with polar vector $\mathbf{n}$ where

$$\mathbf{n} = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Then $\langle \mathbf{n}, \mathbf{n} \rangle = 2$. Then using similar argument, the reflection in Π with angle ψ is

$$R(z) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (e^{i\psi} - 1)\frac{iz_1 + z_2}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} (e^{i\psi}+1)z_1 - i(e^{i\psi}-1)z_2 \\ i(e^{i\psi}-1)z_1 + (e^{i\psi}+1)z_2 \end{pmatrix}$$
$$= \begin{pmatrix} e^{i\psi/2}\cos(\psi/2) & e^{i\psi/2}\sin(\psi/2) \\ -e^{i\psi/2}\sin(\psi/2) & e^{i\psi/2}\cos(\psi/2) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

To obtain a matrix of determinant 1 we must multiply by $e^{-i\psi/2}$ to get

$$R = \begin{pmatrix} \cos(\psi/2) & \sin(\psi/2) \\ -\sin(\psi/2) & \cos(\psi/2) \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

5.3 Complex re ections in SU(2,1)

We now give more details for the case we are most interested in, namely where n = 2 and H has signature (2, 1). Let Π be a complex hyperplane in $\mathbb{C}^{2,1}$ with normal vector n $\in \mathbb{C}^{2,1}$ with $\langle n,n\rangle > 0$. The complex re ection with angle ψ xing Π is given by (5.1). Since R is represented by a matrix in SU(2,1), we multiply (5.1) by $e^{-i\psi/3}$ to obtain

$$R(z) = e^{-i\psi/3}\left(\mathbf{z} + (e^{i\psi}-1)\frac{\langle \mathbf{z},\mathbf{n}\rangle}{\langle n,n\rangle}\mathbf{n} \right) = e^{-i\psi/3}\mathbf{z} + (e^{2i\psi/3} - e^{-\psi/3})\frac{\langle \mathbf{z},n\rangle}{\langle n,n\rangle}\mathbf{n} \qquad (5.2)$$

We now relate tr(RA) and tr(A) for any $A \in$ SU(2,1).

Lemma 5.3.1 Let R be complex re ection in the hyperplane orthogonal to n with angle ψ given by (5.2). Let A be any element of SU(2,1). Then

tr(

Proof. We have $RA) = e^{-i\psi/3}\mathrm{tr}(A) + (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle A\boldsymbol{n}, \boldsymbol{n}\rangle}{\langle \quad \rangle}$ n,n

$$RA(\mathbf{z}) = e^{-i\psi/3}A\mathbf{z} + \frac{(e^{2i\psi/3} - e^{-\psi/3})}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n}\langle A\mathbf{z}, \mathbf{n}\rangle$$
$$= e^{-i\psi/3}A\mathbf{z} + \frac{(e^{2i\psi/3} - e^{-\psi/3})}{\langle \mathbf{n}, \mathbf{n}\rangle} \mathrm{nn}^{*}HA\mathrm{z}.$$

Therefore, the matrix of RA is

$$e^{-i\psi/3}A + \frac{(e^{2i\psi/3} - e^{-\psi/3})}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{nn}^{*}HA.$$

Now if a matrix can be written in the form uv* for column vectors u and v, then its trace is just v*u. Thus

$$\mathrm{tr}(\mathrm{nn}^{*}HA) = \mathrm{tr}(\mathrm{n}(A^{*}H\mathrm{n})^{*}) = (A^{*}H\mathrm{n})^{*}\mathrm{n} = \mathrm{n}^{*}HA\mathrm{n} = \mathrm{h}A(\mathrm{n}),\mathrm{ni}.$$

Hence

$$\mathrm{tr}(RA) = e^{-i\psi/3}\mathrm{tr}(A) + \frac{(e^{2i\psi/3} - e^{-i\psi/3})}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathrm{tr}(\mathrm{nn}^{*}HA)$$
$$= e^{-i\psi/3}\mathrm{tr}(A) + (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle A\mathbf{n}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}.$$

Setting A to be the identity matrix, we see the fact (which we already knew from our consideration of eigenvalues, proposition 2 in chapter 3)

$$\mathrm{tr}(R) = 3e^{-i\psi/3} + (e^{2i\psi/3} - e^{-i\psi/3}) = e^{2i\psi/3} + 2e^{-i\psi/3}.$$

5.4 Equilateral triangle groups

We consider the complex triangle group generated by three complex re ections R_1,R_2,R_3 of order p with the property that there is an element J of order 3 so that

$$J^3 = I,\ R_2 = JR_1J^{-1},\ R_3 = JR_2J^{-1} = J^{-1}R_1J. \tag{5.3}$$

We call hR_1,R_2,R_3i an equilateral triangle group if it satis es the condition (5.3) (Zhao, 2011).

Suppose that we are given three complex lines L_1,L_2 and L_3 in $\mathbf{H}^2_{\mathbb{C}}$. These correspond to hyperplanes Π_1,Π_2 and Π_3 in $\mathbb{C}^{2,1}$ with normal vectors n_1,n_2 and n_3 with hn_j,n_ji > 0. For $j = 1,2,3$, consider complex re ections R_j with angle

about complex lines with polar vectors n_j (Parker, 2012). Using (5.2) we have

$$R_j(\mathbf{z}) = e^{-i\psi/3}\mathbf{z} + (e^{2i\psi/3} - e^{-\psi/3})\frac{\langle \mathbf{z}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{n}_j \tag{5.4}$$

Note that this formula is preserved if n_j is sent to λn_j for any $\lambda \in \mathbb{C} - \{0\}$.

Suppose rst that the three complex lines L_1,L_2,L_3 from an equilateral triangle. That is, there is a J map of order 3 cyclically permuting them. In other

words $J \in SU(2,1)$ satis es $\Pi_2 = J\Pi_1, \Pi_3 = J\Pi_2 = J^{-1}\Pi_1$ and $n_2 = Jn_1, n_3 =$

$Jn_2 = J^{-1}n_1$. Thus

hn_1,n_1i = hn_2,n_2i = hn_3,n_3i,hn_2,n_1i = hn_3,n_2i = hn_1,n_3i.

Note that if ω is a cube root of unity, all these formulae remain valid if, for $j = 1,2,3$ we send n_j to $\omega^j n_j$.

The map J will have eigenvalues 1, ω and $\bar{\omega}$ and so tr(J) = 0. Using this fact, the following result is an easy corollary of lemma 5.3.1.

Lemma 5.4.1 Let R be a complex re ection with angle ψ xing a complex line

L with polar vector n. Let $J \in SU(2,1)$ be a regular elliptic map of order 3. Then

$$\operatorname{tr}(RJ) = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle J\boldsymbol{n}, \boldsymbol{n}\rangle}{\langle \boldsymbol{n}, \boldsymbol{n}\rangle}.$$

From lemma 5.4.1, we de ne the variable τ to be this trace (where indices are taken cyclically):

$$\tau = \mathrm{tr}(R_j J) = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle J\mathbf{n}_j, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}. \quad ($$

Sending n_j to $\omega^j \mathrm{n}_j$ means that τ is multiplied by ω. Therefore given R_1, R_2 and R_3 the map τ is only de ned up to multiplication by a cube root of unity.

Furthermore, if L_j and L_{j+1} meet with angle θ (by symmetry this is the same for all three pairs of lines) then

$$\cos(\theta) = \frac{|\langle \mathrm{n}_{j+1}, \mathrm{n}_j\rangle|}{\langle \mathrm{n}_j, \mathrm{n}_j\rangle} = \frac{|\tau|}{2\sin(\psi/2)}. \quad (5.6)$$

This shows that the traces lead to geometrical information about the group.

All of this has been de ned without reference to any particular Hermitian form. We now make a choice of vectors $\mathrm{n}_1, \mathrm{n}_2$ and n_3. This determines a Hermitian form. We choose

$$\mathrm{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathrm{n}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathrm{n}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (5.7)$$

Together with (5.5), this means that with this choice the Hermitian form must be $\langle z, w\rangle = w^* H z$ where

$$H = \begin{bmatrix} 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\tau & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} \\ (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} & 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\tau \\ (e^{-2i\psi/3} - e^{i\psi/3})\tau & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} & 2 - e^{i\psi} - e^{-i\psi} \end{bmatrix} \quad (5.8)$$

This leads to the following matrices in SU(2,1) for J, R_1, R_2 and R_3:

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{5.9}$$

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \tag{5.10}$$

$$R_2 = JR_1J^{-1} = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ -e^{i\psi/3}\overline{\tau} & e^{2i\psi/3} & \tau \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix} \tag{5.11}$$

$$R_3 = J^{-1}R_1J = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ 0 & e^{-i\psi/3} & 0 \\ \tau & -e^{i\psi/3}\overline{\tau} & e^{2i\psi/3} \end{bmatrix}. \tag{5.12}$$

Sending $\mathbf{n}_j$ to $\omega^j\mathbf{n}_j$ means that J is multiplied by ω. Therefore given complex reflections R_1, R_2, R_3 the symmetry map J is only defined up to multiplication by a cube root of unity. From this it is clear that the groups $\langle R_1, J\rangle$ and $\langle R_1, R_2, R_3\rangle$ are completely determined up to conjugacy by the parameter τ. Therefore, in principle, the trace of any element of $\langle R_1, R_2, R_3\rangle$ may be given as function of τ. Moreover, (5.7) determines the Hermitian form H up to a real multiple.

In order to avoid denominator , we choose

$$\langle \mathbf{n}_j, \mathbf{n}_j\rangle = |e^{2i\psi/3} - e^{-i\psi/3}|^2 = 2 - e^{i\psi} - e^{-i\psi}.$$

This means that

$$\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle = \frac{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}{e^{2i\psi/3} - e^{-i\psi/3}}\tau = (e^{-2i\psi/3} - e^{i\psi/3})\tau.$$

As we indicated in section 5.2 the construction of R_1, R_2 and R_3 in terms of n_1, n_2 and n_3 works whatever the signature of H. It is only when H has signature (2, 1) that these re ections act on complex hyperbolic space as complex re ections in complex lines L_1, L_2 and L_3. We now discuss the geometry when H has other signatures.

Since the trace of H is positive, we see that it must have at least positive eigenvalue. If H has signature (3, 0) then our triangle lies on $\mathbb{C}P^2$ and $\mathrm{h}R_1, R_2, R_3\mathrm{i}$ is a subgroup of $\mathrm{SU}(3)$. If H has signature (2, 1), which is the case we are interested in, then $\mathrm{h}R_1, R_2, R_3\mathrm{i}$ is generated by re ections in complex lines in complex hyperbolic space. If H has signature (1, 2) then $\mathrm{h}R_1, R_2, R_3\mathrm{i}$ is generated by re ections in points in complex hyperbolic space. We now give a criterion for determining when H has signature (2, 1) (Parker, 2012).

Lemma 5.4.2 The signature of the matrix H given by (5.7) is (2, 1) if and only if

$$0 < 3(2 - e^{i\psi} - e^{-i\psi})|\tau|^2 - (1 - e^{-i\psi})\tau^3 - (1 - e^{i\psi})\overline{\tau}^3 - (2 - e^{i\psi} - e^{-i\psi})^2.$$

Proof. We must nd when H has two eigenvalues that are positive and one that is negative. Since the sum of the eigenvalues of H is

$$\mathrm{tr}(H) = 3(2 - e^{i\psi} - e^{-i\psi}) > 0,$$

it is easy to see that all three eigenvalues cannot be negative simultaneously. This means we only need to check when H has negative determinant. Hence

$$\begin{aligned} 0 > \det(H) & \\ &= (2 - e^{i\psi} - e^{-i\psi})^3 + (e^{-2i\psi/3} - e^{i\psi/3})^3\tau^3 + (e^{2i\psi/3} - e^{-i\psi/3})^3\overline{\tau}^3 \\ &\quad - 3(2 - e^{i\psi} - e^{-i\psi})(e^{-2i\psi/3} - e^{i\psi/3})(e^{2i\psi/3} - e^{-i\psi/3})|\tau|^2 \end{aligned}$$

$$= (2 - e^{i\psi} - e^{-i\psi})^3 + (2 - e^{i\psi} - e^{-i\psi})(1 - e^{-i\psi})\tau^3$$

$$+ (2 - e^{i\psi} - e^{-i\psi})(1 - e^{i\psi})\overline{\tau}^3 - 3(2 - e^{i\psi} - e^{-i\psi})^2|\tau|^2.$$

The result follows since $2 - e^{i\psi} - e^{-i\psi} > 0$.

Corollary 5 Suppose that the matrix H given by (5.8) has signature (2, 1). For $j = 1,2,3$ let n_j be given by (5.7) and let L_j be complex line with polar vector n_j.
If L_j and L_{j+1} intersect in $\mathbf{H}^2_{\mathbb{C}}$ then they meet at an angle less than $\pi/3$.
Proof. Since H has signature (2, 1) lemma 5.4.2 implies

$$0 < 3(2 - e^{i\psi} - e^{-i\psi})|\tau|^2 - (1 - e^{-i\psi})\tau^3 - (1 - e^{i\psi})\overline{\tau}^3 - (2 - e^{i\psi} - e^{-i\psi})^2$$

$$\leq 4\sin(\psi/2)|\tau|^3 + 12\sin^2(\psi/2)|\tau|^2 - 16\sin^4(\psi/2)$$

$$= 4\sin(\psi/2)(|\tau| - \sin(\psi/2))(|\tau| + 2\sin(\psi/2))^2.$$

This implies that $|\tau| > \sin(\psi/2)$. Note that the converse of this inequality is not necessary true, since in the second line we used $2<(-(1 - e^{-i\psi})\tau^4) \leq 2|1 - e^{-i\psi}||\tau^3| = 4\sin(\psi/2)|\tau|^3$.

If L_j and L_{j+1} intersect in $\mathbf{H}^2_{\mathbb{C}}$ then, from (5.6), the angle θ between L_j and L_{j+1} is given by

$$\cos(\theta) = \frac{|\langle \mathbf{n}_{j+1}, \mathbf{n}_j \rangle|}{\langle \mathbf{n}_j, \mathbf{n}_j \rangle} = \frac{|\tau|}{2\sin(\psi/2)} > \frac{1}{2}.$$

Therefore $\theta < \pi/3$ as claimed.

5.5 General triangle groups

In this section, we consider three complex lines in general position and the group generated by complex re ections of angle ψ in their sides. Let L_1,L_2 and L_3 be three complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with normal vectors n_1,n_2 and n_3. Suppose that $\mathrm{h}n_1,n_2\mathrm{i} = \mathrm{h}n_2,n_2\mathrm{i} = \mathrm{h}n_3,n_3\mathrm{i} > 0$. De ne

$$\rho = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle \mathbf{n}_2, \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle},$$

$$\sigma = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle \mathbf{n}_3, \mathbf{n}_2\rangle}{\langle \mathbf{n}_2, \mathbf{n}_2\rangle},$$

$$\tau = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle \mathbf{n}_1, \mathbf{n}_3\rangle}{\langle \mathbf{n}_3, \mathbf{n}_3\rangle}.$$

These formulae generalise (5.5) but now, since we no longer have the symmetry J, they are not the trace of any group elements. Using proposition 12 in chapter 5 below, we will be able to relate them to other traces.

From (Goldman, 1999), if L_j and L_k meet with angle θ_{jk} then

$$\cos(\theta_{12}) = \frac{|\rho|}{2\sin(\psi/2)},\ \cos(\theta_{23}) = \frac{|\sigma|}{2\sin(\psi/2)},\ \cos(\theta_{31}) = \frac{|\tau|}{2\sin(\psi/2)}$$

We can use ρ,σ,τ to de ne a Hermitian form. Once again we normalise so that $\langle \mathbf{n}_j, \mathbf{n}_j\rangle = 2 - e^{i\psi} - e^{-i\psi} > 0$. Then

$$H = \begin{bmatrix} 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\rho & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} \\ (e^{2i\psi/3} - e^{-i\psi/3})\overline{\rho} & 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\sigma \\ (e^{-2i\psi/3} - e^{i\psi/3})\tau & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\sigma} & 2 - e^{i\psi} - e^{-i\psi} \end{bmatrix} \quad (5.14)$$

We require that H should have signature (2, 1). Since its trace is positive, the same argument we used before shows that this is equivalent to $\det(H) < 0$. Doing this and arguing in a similar way to the proof of lemma 5.4.2, we have:

Lemma 5.5.1 The matrix H given by (5.14) has signature (2, 1) if and only if

$$0 < (2-e^{i\psi}-e^{-i\psi})(|\rho|^2+|\sigma|^2+|\tau|^2)-(1-e^{-i\psi})\rho\sigma\tau-(1-e^{i\psi})\rho\sigma\,\tau-(\overline{2-e^{i\psi}}-e^{-i\psi})^2.$$

This criterion is equivalent to the one given by Pratoussevitch (2005) in proposition 1. A simple geometric consequence of this inequality, generalising corollary 5 (chapter 5) is:

Corollary 6 The angles θ_{jk} from (5.12) satisfy $\theta_{12} + \theta_{23} + \theta_{31} < \pi$.
Proof. Using the inequality from lemma 5.5.1, we nd

$$\begin{aligned}
0 &< (2 - e^{i\psi} - e^{-i\psi})(|\rho|^2 + |\sigma|^2 + |\tau|^2) \\
&- (1 - e^{-i\psi})\rho\sigma\tau - (1 - e^{i\psi})\overline{\rho}\,\overline{\sigma}\,\overline{\tau} - (2 - e^{i\psi} - e^{-i\psi})^2 \\
&\leq 4\sin^2(\psi/2)(|\rho|^2 + |\sigma|^2 + |\tau|^2) + 4\sin(\psi/2)|\rho| + |\sigma| + |\tau| - 16\sin^4(\psi/2) \\
&= 16\sin^4(\psi/2)(\cos^2(\theta_{12}) + \cos^2(\theta_{23}) + \cos^2(\theta_{31})) \\
&+ 32\sin^4(\psi/2)\cos(\theta_{12})\cos(\theta_{23})\cos(\theta_{31}) - 16\sin^4(\psi/2) \\
&= 16\sin^4(\psi/2)(\cos(\theta_{12})\cos(\theta_{23}) + \cos(\theta_{31}))^2 - 16\sin^4(\psi/2)\sin^2(\theta_{12})\sin^2(\theta_{23}) \\
&= 16\sin^4(\psi/2)(\cos(\theta_{12} - \theta_{23}) + \cos(\theta_{31}))(\cos(\theta_{12} + \theta_{23})\cos(\theta_{31})).
\end{aligned}$$

Since $\theta_{jk} \in (0,\pi/2)$ we see that $\cos(\theta_{12}-\theta_{23})$ and $\cos(\theta_{31})$ are both positive. Thus we must have

$$\cos(\theta_{31}) > -\cos(\theta_{12} + \theta_{23}) = \cos(\pi - \theta_{12} - \theta_{23}).$$

Hence $\theta_{31} < \pi - \theta_{12} - \theta_{23}$ as required.

Matrices for the re ections R_1, R_2, R_3 can be obtained by using H in the formula (5.4):

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \rho & -e^{i\psi/3}\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \tag{5.15}$$

$$R_2 = \begin{bmatrix} e^{2i\psi/3} & 0 & 0 \\ -e^{i\psi/3}\overline{\rho} & e^{2i\psi/3} & \sigma \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \tag{5.16}$$

$$R_3 = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ 0 & e^{-i\psi/3} & 0 \\ \tau & -e^{i\psi/3}\overline{\sigma} & e^{2i\psi/3} \end{bmatrix}. \tag{5.17}$$

5.6 Traces in general triangle groups

Let R_1, R_2 and R_3 be as in (5.15), (5.16) and (5.17). We will be interested in nding a formula for the trace of each element of $\Delta = \langle R_1, R_2, R_3 \rangle$, written as a word in R_1, R_2, R_3 and their inverses. Since cyclic permutation does not a ect the trace it will be easier for us to consider words. Consider an element $R_{a_1}^{\epsilon_1} \ldots R_{a_r}^{\epsilon_r}$ of $\langle R_1, R_2, R_3 \rangle$ where $a_j \in \{1,2,3\}$ and $\epsilon_j \in \{1, -1\}$. For ease of rotation, we make the canonical identi cation between this word and the sequences $a = (a_1, a_2 \ldots a_r)$ and $\epsilon = \{\epsilon_1 \ldots \epsilon_r\}$. We shall regard these indices as begin de ned cyclically, that is $a_{r+1} = a_1$ and $\epsilon_{r+1} = \epsilon_1$.

We need to introduce some notation. For the sequence $a = (a_1 \ldots a_r)$ as above and $j = 1,2,3$ taken cyclically (so when $j = 3$ we have $j+1 = 1$) we de ne

$$z_j(a) = \#\{k \in \{1,\ldots,r\} : a_{k+1} = a_k = j\} \tag{5.18}$$

$$p_j(a) = \#\{k \in \{1,\ldots,r\} : a_{k+1} = j+1, a_k = j\} \tag{5.19}$$

$$n_j(a) = \#\{k \in \{1,\ldots,r\} : a_{k+1} = j, a_k = j+1\} \tag{5.20}$$

It is easy to see that

$$\#\{k \in \{1,\ldots,r\} : a_k = j\} = z_j(a) + p_j(a) + n_{j-1}(a),$$

$$\#\{k \in \{1,\ldots,r\} : a_{k+1} = j\} = z_j(a) + p_{j-1}(a) + n_j(a).$$

By relabelling the sequence a, it is clear that these numbers must be the same.

That is $z_j(a) + p_j(a) + n_{j-1}(a) = z_j(a) + p_{j-1}(a) + n_j(a)$. Therefore we have

$$p_1(a) - n_1(a) = p_2(a) - n_2(a) = p_3(a) - n_3(a).$$

We now de ne the winding number $w(a)$ of the sequence $a = (a_1 \ldots a_r)$ to be

$$w(a) = p_j(a) - n_j(a). \tag{5.21}$$

Similarly, for $\epsilon = (\epsilon_1 \ldots \epsilon_r)$ de ne

$$m_+(\epsilon) = \#\{k \in \{1, \ldots, r\} : \epsilon_k = 1\}, \tag{5.22}$$

$$m_-(\epsilon) = \#\{k \in \{1,...,r\} : \epsilon_k = -1\}. \qquad (5.23)$$

We now give the main result for computing traces which is due to Pratoussevitch (2005).

Proposition 12 Let $a = (a_1...a_r)$ be a cyclic word with $a_k \in \{1,2,3\}$. Let $\epsilon = (\epsilon_1...\epsilon_r)$ with $\epsilon_k = \{1,-1\}$. The . $E = \sum_{j=1}^{n} \epsilon_j$ Then

$$\mathrm{tr}(R_{a_1}^{\epsilon_1} \cdots R_{a_r}^{\epsilon_r}) = (e^{i\psi})^{-E/3} \left(3 + \sum_S \frac{(e^{i\psi}-1)^z(-e^{i\psi})^n(e^{i\psi})^w}{(-e^{i\psi})^{m_-}} \rho^{p_1}\overline{\rho}^{n_1}\sigma^{p_2}\overline{\sigma}^{n_2}\tau^{p_3}\overline{\tau}^{n_3} \right)$$

where the sum is taken overall non-empty subsets of $S = \{k_1,...,k_m\}$ of the set $\{1,...,r\}$. Such a subset determines a subset $a_s = (a_{k_1},...,a_{k_m})$ of a and $\epsilon_s = (\epsilon_{k_1},...,\epsilon_{k_m})$ *of* ϵ. The numbers $p_j, n_j, w = p_j - n_j, z = z_1+z_2+z_3, n = n_1+n_2+n_3$ are determined from a_s by (5.18), (5.19), (5.20) and (5.21). Finally, m_- is determined from ϵ_s *by*

Proof. Let $S = \{k_1,...,k_m\}$ be a non-empty subset of $\{1,...,r\}$ and denote the corresponding subsets of a and by $a_s = (a_{k_1},...,a_{k_m})$ and $\epsilon_s = (\epsilon_{k_1},...,\epsilon_{k_m})$. write $a_{k_l} = b_l$ for $l = 1,...,m$.

Using the expression for R_j given in equation (5.4), we have

$$e^{i\psi/3}R_j\mathbf{z} = \mathbf{z} + (e^{-i\psi}-1)\frac{\langle \mathbf{z}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{n}_j = \left(I + (e^{i\psi}-1)\frac{\mathbf{n}_j\mathbf{n}_j^* H}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{z} \right),$$

$$e^{-i\psi/3}R_j^{-1}\mathbf{z} = \mathbf{z} + (e^{-i\psi}-1)\frac{\langle \mathbf{z}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{n}_j \left(I + (e^{-i\psi}-1)\frac{\mathbf{n}_j\mathbf{n}_j^* H}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{z} \right).$$

Therefore

$$(e^{i\psi/3})^{\epsilon_1} R_{a_1}^{\epsilon_1} \cdots (e^{i\psi/3})^{\epsilon_1} R_{a_r}^{\epsilon_r}$$

$$= \left(I + (e^{\epsilon_1 i\psi} - 1)\frac{\mathbf{n}_{a_1}\mathbf{n}_{a_1}^* H}{\langle \mathbf{n}_{a_1}, \mathbf{n}_{a_1}\rangle} \right) \cdots \left(I + (e^{\epsilon_r i\psi} - 1)\frac{\mathbf{n}_{a_r}\mathbf{n}_{a_r}^* H}{\langle \mathbf{n}_{a_r}, \mathbf{n}_{a_r}\rangle} \right)$$

$$= I + \sum_{S\neq\emptyset} (e^{i\psi} - 1)^{m_+}(e^{-i\psi} - 1)^{m_-} \frac{\mathbf{n}_{b_1}\mathbf{n}_{b_1}^* H \mathbf{n}_{b_2} \cdots \mathbf{n}_{b_{m-1}}^* H \mathbf{n}_{b_m}\mathbf{n}_{b_m}^* H}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle \cdots \langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle}$$

$$= I + \sum_{S\neq\emptyset} \frac{(e^{i\psi} - 1)^{|S|}}{(-e^{i\psi})^{m_-}} \frac{\mathbf{n}_{b_1}\langle \mathbf{n}_{b_2}, \mathbf{n}_{b_1}\rangle \cdots \langle \mathbf{n}_{b_m}, \mathbf{n}_{b_{m-1}}\rangle \mathbf{n}_{b_m}^* H}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle \cdots \langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle}.$$

We can put together the powers of $e^{i\psi}$ on the left hand side to be obtain $(e^{i\psi/3})^E = (e^{i\psi})^{E/3}$. Arguing as in the proof of lemma 5.4.1, we have $tr(\mathbf{n}_{b_1}\mathbf{n}_{b_m}^* H) = \langle \mathbf{n}_{b_1}, \mathbf{n}_{b_m}\rangle$. Hence

$$(e^{i\psi})^{E/3} tr(R_{a_1}^{\epsilon_1} \cdots R_{a_r}^{\epsilon_r}) = 3 + \sum_{S\neq\emptyset} \frac{(e^{i\psi} - 1)^{|S|}}{(-e^{i\psi})^{m_-}} \frac{\langle \mathbf{n}_{b_2}, \mathbf{n}_{b_1}\rangle}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle} \cdots \frac{\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_{m-1}}\rangle}{\langle \mathbf{n}_{b_{m-1}}, \mathbf{n}_{b_{m-1}}\rangle} \cdot \frac{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_m}\rangle}{\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle}$$

$$= 3 + \sum_{S\neq\emptyset} (-e^{-i\psi})^{m_-} (e^{i\psi} - 1)\frac{\langle \mathbf{n}_{b_2}, \mathbf{n}_{b_1}\rangle}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle} \cdots \frac{(e^{i\psi} - 1)\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_m}\rangle}{\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle}.$$

From the de nitions of ρ, σ and τ we have Thus for each sum $S \neq \emptyset$ we have:

$$\frac{(e^{i\psi} - 1)\langle \mathbf{n}_{b_{k+1}}, \mathbf{n}_{b_k}\rangle}{\langle \mathbf{n}_{b_k}, \mathbf{n}_{b_k}\rangle} = \begin{cases} e^{i\psi} - 1 & \text{if } b_{k+1} = b_k; \\ e^{i\psi/3}\rho & \text{if } b_{k+1} = 2,\ b_k = 1; \\ (-e^{i\psi})e^{-i\psi/3}\overline{\rho} & \text{if } b_{k+1} = 1,\ b_k = 2; \\ e^{i\psi/3}\sigma & \text{if } b_{k+1} = 3,\ b_k = 2; \\ (-e^{i\psi})e^{-i\psi/3}\overline{\sigma} & \text{if } b_{k+1} = 2,\ b_k = 3; \\ e^{i\psi/3}\tau & \text{if } b_{k+1} = 1,\ b_k = 3; \\ (-e^{i\psi})e^{-i\psi/3}\overline{\tau} & \text{if } b_{k+1} = 3,\ b_k = 1. \end{cases}$$

$$
\begin{aligned}
(-e^{-i\psi})^{m_-}\frac{(e^{i\psi}-1)\langle \mathbf{n}_{b_2},\mathbf{n}_{b_1}\rangle}{\langle \mathbf{n}_{b_1},\mathbf{n}_{b_1}\rangle}\cdots\frac{(e^{i\psi}-1)\langle \mathbf{n}_{b_1},\mathbf{n}_{b_m}\rangle}{\langle \mathbf{n}_{b_m},\mathbf{n}_{b_m}\rangle}
&= (-e^{-i\psi})^{m_-}(e^{i\psi}-1)^{z_1}(e^{i\psi/3\rho})^{\rho_1}((-e^{i\psi})e^{-i\psi/3}\overline{\rho})^{n_1}\\
&\cdot(e^{i\psi}-1)^{z_2}(e^{i\psi/3}\sigma)^{\rho_2}((-e^{i\psi})e^{-i\psi/3}\overline{\sigma})^{n_2}(e^{i\psi}-1)^{z_3}\\
&\cdot((-e^{i\psi})e^{-i\psi/3}\overline{\tau})^{n_3}\\
&= (-e^{-i\psi})^{m_-}(e^{i\psi}-1)^{z_1+z_2+z_3}(-e^{i\psi})^{n_1+n_2+n_3}\\
&\cdot(e^{i\psi/3})^{p_1-n_1}\rho^{p_1}\overline{\rho}^{n_1}(e^{i\psi/3})^{p_2-n_2}\sigma^{p_2}\overline{\sigma}^{n_2}(e^{i\psi/3})^{p_3-n_3}\tau^{p_3}\overline{\tau}^{n_3}\\
&= (-e^{-i\psi})^{m_-}(e^{i\psi}-1)^{z}(-e^{i\psi})^{n}(e^{i\psi})^{w}\rho^{p_1}\overline{\rho}^{n_1}\sigma^{p_2}\overline{\sigma}^{n_2}\tau^{p_3}\overline{\tau}^{n_3}
\end{aligned}
$$

where in the last line, we have used $w = p_1-n_1 = p_2-n_2 = p_3-n_3, z = z_1+z_2+z_3$ and $n = n_1 + n_2 + n_3$. This means that if we consider

$$(R_{a_1}^{\epsilon_1}\cdots R_{a_r}^{\epsilon_r})^{-1} = R_{a_r}^{-\epsilon_r}\cdots R_{a_1}^{-\epsilon_1}$$

then we must send E to $-E$ and swap m_+ and m_-, p_j and n_j. Using the formula for proposition 11 (chapter 5) we can deduce

$$\mathrm{tr}((R_{a_1}^{\epsilon_1}\cdots R_{a_r}^{\epsilon_r})^{-1}) = \overline{\mathrm{tr}(R_{a_1}^{\epsilon_1}\cdots R_{a_r}^{\epsilon_r})}.$$

An immediate consequence of proposition 11 (chapter 5) is the following; which enables us to nd the trace eld of a triangle group.

Corollary 7 The trace of any element of Δ may be written as a power of $e^{i\psi/3}$ times as polynomial in $|\rho|^2, |\sigma|^2, |\tau|^2, \rho\sigma\tau$ and $\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$ with coe cients in $\mathbb{Z}[e^{i\psi}, e^{-i\psi}]$. In particular, when ψ is a rotational multiple of π then the coe cient may be written in $\mathbb{Z}[e^{i\psi}]$.

Proof. We examine the term coming from S 6= $\emptyset$ as in the proof of proposition 11. First we have $p_j - n_j = w$ and so when $w \geq 0$ we have $p_j \geq n_j$. Thus writing $p_j = w + n_j$ we have

$$\rho^{p_1}\overline{\rho}^{n_1} = \rho^{w+n_1}\overline{\rho}^{n_1} = \rho^{w}(|\rho|^2)^{n_1(s)},$$
$$\overline{\sigma}^{p_2}\sigma^{n_1} = \sigma^{w+n_2}\overline{\sigma}^{n_2} = \sigma(|\sigma|^2)^{n_2(s)},$$

$$\tau_{p^3}\tau_{n^3} = \tau_{w+n^3}\tau_{n^3} = \tau_w(|\tau|_2)_{n^3(s)}$$

and so

$$\rho_{p1}\rho_{n1}\overline{\sigma_{p2}}\sigma_{n2}\tau_{p3}\tau_{n3} = (|\rho|_2)_{n1}(|\sigma|_2)_{n2}(|\tau|_2)_{n3}(\rho\sigma\tau)_{|w|}.$$

Likewise, when $w \leq 0$, writing $n_j = p_j - w_j$ we have

$$\rho_{p1}\rho_{n1}\overline{\sigma_{p2}}\sigma_{n2}\tau_{p3}\tau_{n3} = (|\rho|_2)_{p1}(|\sigma|_2)_{p2}(|\tau|_2)_{p3}(\rho\,\bar\sigma\,\bar\tau)_{|w|}.$$

In each case this a monomial in $|\rho|^2, |\sigma|^2, |\tau|^2, \rho\sigma\tau$ and $\rho\,\bar\sigma\,\bar\tau$.

We give an illustrative example of proposition 11, which is section 8 of Pratoussevitch (2005).

Proposition 13 Let R_1, R_2 and R_3 be as above. Then for any distinct j,k,l = {1,2,3} we have

$$\mathrm{tr}(R_1R_2) = e^{i\psi/3}(2 - |\rho|^2) + e^{-2i\psi/3},$$

$$\mathrm{tr}(R_1R_2^{-1}) = 1 + 2\cos(\psi) + |\rho|^2,$$

$$\mathrm{tr}(R_1R_2R_3) = 3 - |\rho|^2 - |\sigma|^2 - |\tau|^2 + \rho\sigma\tau,$$

$$\mathrm{tr}(R_3R_2R_1) = 3 - |\rho|^2 - |\sigma|^2 - |\tau|^2 - e^{i\psi}\rho\,\bar\sigma\,\bar\tau,$$

$$\mathrm{tr}(R_1R_2R_3R_3^{-1}) = e^{i\psi/3}(2 - |\rho\sigma - \tau|^2) + e^{-2i\psi/3},$$

$$\mathrm{tr}(R_1^{-1}R_2^{-1}R_3^{-1}) = 3 - (|\rho|^2 + |\tau|^2 + |\sigma|^2) - e^{-i\psi}\rho\tau\sigma,$$

$$\mathrm{tr}[R_1, R_2] = 3 + 2(\cos(\psi) - 1)|\rho|^2 + |\rho|^4,$$

$$\mathrm{tr}(R_1R_2R_3^{-1}R_2^{-1}) = 1 + \cos(\psi)(2 + |\sigma|^2) + |\rho\sigma - \tau|^2.$$

Proof. First consider R_1R_2. We now enumerate all non-empty subsets, their index and winding number, and the contribution they make to the trace. For R_1R_2 the terms are given by the following table:

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{1,2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$

Figure 5.1: tr(R_1R_2)

From this we see that

$$\mathrm{tr}(R_1R_2) = e^{-2i\psi/3}(3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{i\psi}|\rho|^2) = e^{i\psi/3}(2 - |\rho|^2) + e^{-2i\psi/3}.$$

For $R_1R_2^{-1}$ this table becomes: From this we see that

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{1,2\}$	$\{+,-\}$	1	0	1	1	0	0	0	0	0	$\|\rho\|^2$

Figure 5.2: tr$(R_1R_2^{-1})$

$$\mathrm{tr}(R_1R_2^{-1}) = 3 + (e^{i\psi} - 1) - e^{-i\psi}(e^{-i\psi} - 1) + |\rho|^2 = 1 + 2\cos(\psi) + |\rho|^2.$$

Likewise, the table for $R_1R_2R_3$ is fg 5.3:

Therefore

$$\begin{aligned}\mathrm{tr}(R_1R_2R_3) &= e^{-i\psi}(3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{i\psi}|\rho|^2 \\ &\quad - e^{i\psi}|\sigma|^2 - e^{i\psi}|\tau|^2 + e^{i\psi}\rho\sigma\tau) \\ &= 3 - |\rho|^2 - |\sigma|^2 - |\tau|^2 + \rho\sigma\tau.\end{aligned}$$

For $R_3R_2R_1$ we have g 5.4 below:

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{3\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{1,2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
$\{2,3\}$	$\{+,+\}$	0	0	0	0	1	1	0	0	0	$-e^{i\psi}\|\sigma\|^2$
$\{1,3\}$	$\{+,+\}$	0	0	0	0	0	0	1	1	0	$-e^{i\psi}\|\tau\|^2$
$\{1,2,3\}$	$\{+,+,+\}$	0	0	1	0	1	0	1	0	1	$e^{i\psi}\rho\sigma\tau$

Figure 5.3: $\mathrm{tr}(R_1R_2R_3)$

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{3\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2,1\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
$\{3,2\}$	$\{+,+\}$	0	0	0	0	1	1	0	0	0	$-e^{i\psi}\|\sigma\|^2$
$\{3,1\}$	$\{+,+\}$	0	0	0	0	0	0	1	1	0	$-e^{i\psi}\|\tau\|^2$
$\{3,2,1\}$	$\{+,+,+\}$	0	0	0	1	0	1	0	1	-1	$(-e^{-i\psi})^{-3}(e^{i\psi})^{-1}\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$

Figure 5.4: $\mathrm{tr}(R_3R_2R_1)$

Thus

$$\begin{aligned}\mathrm{tr}(R_3R_2R_1) &= e^{-i\psi}(3 + e^{i\psi} - 1 + e^{i\psi} - 1 + e^{i\psi} - 1 \\ &\quad - e^{i\psi}|\rho|^2 - e^{i\psi}|\sigma|^2 - e^{i\psi}|\tau|^2 - e^{i\psi}\overline{\rho}\,\overline{\sigma}\,\overline{\tau}) \\ &= 3 - |\rho|^2 - |\sigma|^2 - |\tau|^2 + \overline{\rho}\,\overline{\sigma}\,\overline{\tau}.\end{aligned}$$

We do the same thing for $R_1R_2R_3R_2^{-1}$.

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{3\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$(-e^{-i\psi})(e^{i\psi}-1)$
$\{1,2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\lvert\rho\rvert^2$
$\{1,3\}$	$\{+,+\}$	0	0	0	0	0	0	1	1	0	$-e^{i\psi}\lvert\tau\rvert^2$
$\{1,2\}$	$\{+,-\}$	1	0	1	1	0	0	0	0	0	$\lvert\rho\rvert^2$
$\{2,3\}$	$\{+,+\}$	0	0	0	0	1	1	0	0	0	$-e^{i\psi}\lvert\sigma\rvert^2$
$\{2,2\}$	$\{+,-\}$	1	2	0	0	0	0	0	0	0	$(-e^{-i\psi})(e^{i\psi}-1)^2$
$\{3,2\}$	$\{+,-\}$	1	0	0	0	1	1	0	0	0	$\lvert\sigma\rvert^2$
$\{1,2,3\}$	$\{+.+,+\}$	0	0	1	0	1	0	1	0	1	$e^{i\psi}\rho\sigma\tau$
$\{1,2,2\}$	$\{+,+,-\}$	1	1	1	1	0	0	0	0	0	$(e^{i\psi}-1)\lvert\rho\rvert^2$
$\{1,3,2\}$	$\{+,+,-\}$	1	0	0	1	0	1	0	1	-1	$(-e^{i\psi})^2(e^{-i\psi})\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$
$\{2,3,2\}$	$\{+,+,-\}$	1	1	0	0	1	1	0	0	0	$(e^{i\psi}-1)\lvert\sigma\rvert^2$
$\{1,2,3,2\}$	$\{+,+,+,-\}$	1	0	1	1	1	1	0	0	0	$(-e^{i\psi})\lvert\rho\rvert^2\lvert\sigma\rvert^2$

Figure 5.5: tr($R_1R_2R_3R_2^{-1})$

Hence

$$
\begin{aligned}
\text{tr}(R_1R_2R_3R_2^{-1}) &= e^{-2i\psi/3}[3+e^{i\psi}-1+e^{i\psi}-1+e^{i\psi}-1-e^{-i\psi}(e^{i\psi}-1)\\
&\quad -e^{i\psi}|\rho|^2-e^{i\psi}|\tau|^2+|\rho|^2-e^{i\psi}|\sigma|^2+e^{-i\psi}(e^{i\psi}-1)^2+|\sigma|^2\\
&\quad +e^{i\psi}\rho\sigma\tau+(e^{i\psi}-1)|\rho|^2+e^{i\psi}\overline{\rho}\,\overline{\sigma}\,\tau+(e^{i\psi}-1)|\sigma|^2\\
&\quad -e^{i\psi}|\rho|^2|\sigma|^2-e^{i\psi}|\rho|^2|\sigma|^2]\\
&= e^{-2i\psi/3}[3e^{i\psi}-e^{i\psi}+1-e^{i\psi}|\tau|^2+e^{i\psi}\rho\sigma\tau\\
&\quad +e^{i\psi}\overline{\rho}\,\overline{\sigma}\,\overline{\tau}-e^{i\psi}|\rho|^2|\sigma|^2]\\
&= e^{-2i\psi/3}[2e^{i\psi}+1-e^{i\psi}(|\tau|^2-\rho\sigma\tau-\overline{\rho}\,\overline{\sigma}\,\overline{\tau}+|\rho|^2|\sigma|^2)]\\
&= e^{i\psi/3}(2-|\rho\sigma-\overline{\tau}|^2)+e^{-2i\psi/3}.
\end{aligned}
$$

Likewise, the table for $R_1^{-1}R_2^{-1}R_3^{-1}$ is:

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
{1}	{−}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
{2}	{−}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
{3}	{−}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
{1, 2}	{−, −}	2	0	1	1	0	0	0	0	0	$-e^{-i\psi}\|\rho\|^2$
{1, 3}	{−, −}	2	0	0	0	0	0	1	1	0	$-e^{-i\psi}\|\tau\|^2$
{2, 3}	{−, −}	2	0	0	0	1	1	0	0	0	$-e^{-i\psi}\|\sigma\|^2$
{1, 2, 3}	{−, −, −}	3	0	1	0	1	0	1	0	1	$-e^{-2i\psi}\rho\sigma\tau$

Figure 5.6: tr($R_1^{-1}R_2^{-1}R_3^{-1}$)

From this we have

$$\text{tr}(R_1^{-1}R_2^{-1}R_3^{-1}) = e^{i\psi}[3 - e^{-i\psi}(e^{i\psi}-1) - e^{-i\psi}(e^{i\psi}-1) - e^{-i\psi}(e^{i\psi}-1)$$

$$- e^{-i\psi}|\rho|^2 - e^{-i\psi}|\tau|^2 - e^{-i\psi}|\sigma|^2 - e^{-2i\psi}\rho\sigma\tau]$$

$$= 3e^{i\psi} - 3(e^{i\psi} - 1) - (|\rho|^2 + |\tau|^2 + |\sigma|^2) - e^{-i\psi}\rho\sigma\tau$$

$$= 3 - (|\rho|^2 + |\tau|^2 + |\sigma|^2) - e^{-i\psi}\rho\tau\sigma$$

Similarly, for $R_1R_2R_1^{-1}R_2^{-1}$ we have the table below:

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
{1}	{+}	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
{2}	{+}	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
{1}	{−}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
{2}	{−}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
{1, 2}	{+, +}	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
{1, 1}	{+, −}	1	2	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)^2$
{1, 2}	{+, −}	1	0	1	1	0	0	0	0	0	$\|\rho\|^2$
{2, 1}	{+, −}	1	0	1	1	0	0	0	0	0	$\|\rho\|^2$
{2, 2}	{+, −}	1	2	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)^2$
{1, 2}	{−, −}	2	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
{1, 2, 1}	{+, +, −}	1	1	1	1	0	0	0	0	0	$(e^{i\psi}-1)\|\rho\|^2$
{1, 2, 2}	{+, +, −}	1	1	1	1	0	0	0	0	0	$(e^{i\psi}-1)\|\rho\|^2$
{1, 1, 2}	{+, −, −}	2	1	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)\|\rho\|^2$
{2, 1, 2}	{+, −, −}	2	1	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)\|\rho\|^2$
{1, 2, 1, 2}	{+, +, −, −}	2	0	2	2	0	0	0	0	0	$\|\rho\|^4$

Figure 5.7: tr[R_1,R_2]

Thus

$$\text{tr}[R_1,R_2] = 3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{-i\psi}(e^{i\psi} - 1) - e^{-\psi}(e^{i\psi} - 1) - e^{i\psi}|\rho|^2$$

$$- e^{-i\psi}(e^{i\psi} - 1)^2 + |\rho|^2 + |\rho|^2 - e^{-i\psi}(e^{i\psi} - 1)^2 - e^{i\psi}|\rho|^2 + (e^{i\psi} - 1)|\rho|^2$$

$$+ (e^{i\psi} - 1)|\rho|^2 - e^{-i\psi}(e^{i\psi} - 1)|\rho|^2 - e^{-i\psi}(e^{i\psi} - 1)|\rho|^2 + |\rho|^4$$

$$= 1 + 2e^{i\psi} - 2e^{-i\psi}(e^{i\psi} - 1)[1 + (e^{i\psi} - 1)]$$

$$+ |\rho|^2[-2e^{i\psi} + 2 + 2(e^{i\psi} - 1) - 2e^{-i\psi}(e^{i\psi} - 1)] + |\rho|^4$$

$$= 1 + 2e^{i\psi} - 2e^{i\psi} + 2 + |\rho|^2(2e^{i\psi} - 2) + |\rho|^4$$

$$= 3 + 2(\cos(\psi) - 1)|\rho|^2 + |\rho|^4$$

Finally, we do the same thing for $R_1 R_2 R_3^{-1} R_2^{-1}$.

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi} - 1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi} - 1$
$\{3\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
$\{1, 2\}$	$\{+, +\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
$\{1, 3\}$	$\{+, -\}$	1	0	0	0	0	0	1	1	0	$\|\tau\|^2$
$\{1, 2\}$	$\{+, -\}$	1	0	1	1	0	0	0	0	0	$\|\rho\|^2$
$\{2, 3\}$	$\{+, -\}$	1	0	0	0	1	1	0	0	0	$\|\sigma\|^2$
$\{2, 2\}$	$\{+, -\}$	1	2	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)^2$
$\{3, 2\}$	$\{-, -\}$	2	0	0	0	1	1	0	0	0	$-e^{i\psi}\|\sigma\|^2$
$\{1, 2, 3\}$	$\{+, +, -\}$	1	1	1	1	0	0	0	0	0	$-\rho\sigma\tau$
$\{1, 2, 2\}$	$\{+, +, -\}$	1	0	1	0	1	0	1	0	1	$(e^{i\psi} - 1)\|\rho\|^2$
$\{1, 3, 2\}$	$\{+, -, -\}$	2	0	0	1	0	1	0	1	-1	$-\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$
$\{2, 3, 2\}$	$\{+, -, -\}$	2	1	0	0	1	1	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)\|\sigma\|^2$
$\{1, 2, 3, 2\}$	$\{+, +, -, -\}$	2	0	1	1	1	1	0	0	0	$\|\rho\|^2\|\sigma\|^2$

Figure 5.8: tr$(R_1 R_2 R_3^{-1} R_2^{-1})$

Thus

$$\begin{aligned}
\mathrm{tr}(R_1 R_2 R_3^{-1} R_2^{-1}) &= 3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{-i\psi}(e^{i\psi} - 1) - e^{-i\psi}(e^{i\psi} - 1) \\
&- e^{i\psi}|\rho|^2 + |\tau|^2 + |\rho|^2 + |\sigma|^2 - e^{-i\psi}(e^{i\psi} - 1)^2 - e^{i\psi}|\sigma|^2 \\
&- \rho\sigma\tau + (e^{i\psi} - 1)|\rho|^2 - \overline{\rho}\,\overline{\sigma}\,\overline{\tau} \\
&= 1 + 2e^{i\psi} - 1 + e^{-i\psi} - e^{i\psi} + 1 - |\sigma|^2 + e^{-i\psi}|\sigma|^2 + |\tau|^2 \\
&+ |\sigma|^2 - \rho\sigma\tau - \overline{\rho}\,\overline{\sigma}\,\overline{\tau} + |\rho|^2|\sigma|^2 \\
&= 1 + e^{i\psi} + e^{-i\psi}(1 + |\sigma|^2) + |\sigma|^2|\rho|^2 - \rho\sigma\tau \\
&- \overline{\rho}\,\overline{\sigma}\,\overline{\tau} + |\rho|^2|\sigma|^2 + |\tau|^2 \\
&= 1 + \cos(\psi)(2 + |\sigma|^2) + |\rho\sigma - \overline{\tau}|^2
\end{aligned}$$

Chapter 6

CONCLUSION AND RECOMMENDATION

6.1 Conclusion

This thesis was divided into six chapters. The main results were presented in three distinct chapters 3, 4 and 5.

In chapter 3 we discussed the geometry of isometries; speci cally, classi cation of elements of SU(2, 1) by their trace, traces and eigenvalues for loxodromic maps and eigenvalues and complex displacement for loxodromic maps. Our contributions in this chapter were: ampli cation of the calculations in Parker (2012) (see for example lemma 3.2.3, lemma 3.2.4, proposition 4 etc), reconstructed existing proof of proposition (see proposition 2) and constructed non-existing proof of proposition (see proposition 1).

Chapter 4 looked at two generator groups and Fenchel-Nielsen coordinate. In this chapter we proved corollaries (see corollary 3 & 4). One other result of the chapter was the explicit polynomial for $\mathrm{tr}[A,B]\mathrm{tr}[B,A]$ (proposition 6). Also, in an attempt to proof the imaginary part of $\mathrm{tr}[A,B]$ see part two of proposition 8 (which was not considered by Parker, 2012) we expressed equation 18 of Lawton (2007) in terms of $\mathrm{tr}(A)$,$\mathrm{tr}(B)$,$\mathrm{tr}(AB)$ etc (see lemma 3.4.3). Based on this, we gave proposition 7 and remarked on the two di erent representations. Finally we discussed the merits on the two ways to parametrise pair of pants groups (see remark 3 and 4).

Chapter 5 explains traces for triangle groups. In the last section of this chapter, we gave application of a trace formula which is due to Pratoussevitch (2005) (see proposition 13).

6.2 Recommendation

We shall consider the following for future work:

1. Try to explain how to eliminate $\mathrm{tr}(AB)$ and $\mathrm{tr}(A^{-1}B)$ using X_1 and X_2 in lemma 4.4.3.

2. In the last section of chapter 4 we use Pratoussevitch's formulae to calculate $\mathrm{tr}[R_1,R_2]$ and you observe this is real. The question is, how does that interact with the ndings of the previous chapter about the ambiguity in the sign of the imaginary part of $\mathrm{tr}[A,B]$?

3. Get simpler formula for $\mathrm{tr}[R_1,R_2]$ in terms of traces of R_1, R_2, R_1R_2 and $R_1^{-1}R_2$.

REFERENCES

1. Baik, H., (2010). *Fenchel − Nielsen coordinates. Trouser decompostion of Riemann surfaces*. Available at www.math.cornell.edu/~hrbaik/Olivetti-Nov2010.pdf

2. Boadi, R. K., (2011). *New Cone Metrics on the Sphere*. PhD thesis Knust-Ghana.

3. Carne, T. K., (2012). *Geometry and groups*. University of Cambridge. Available at https://www.dpmms.cam.ac.uk/⋯/geometry and groups.

4. Cunha, H. and Gusevski, N., (2013). *A note on trace fields of complex hyperbolic groups*, 30123 - 970.

5. Dundas, B. I., (2013). *Differential topology*. Information available at folk.uib.no/nmabd/dt/080627dt.pdf

6. Goldman, W. M., (1999). *Complex hyperbolic geometry*. Oxford Mathematical Monographs. Oxford University Press.

7. Gongopadhyay, K. and Parsad, S., (2014). *On Fenchel − Nielsen coordinates of surface group representations into* $SU(3,1)$. Available at de.arxiv.org

8. Gyam , K. B., (2012). *Solution of inverse eigenvalue problem of certain singular Hermitian matrices*. PhD thesis, Kwame Nkrumah University of Science and Technology.

9. Jin, M., Zeng, W., Ding, N., Gu, X., Yau, S., (2009). *Computing Fenchel − Nielsen coordinates in Teichmuller shape space*. Available at www.ims.cuhk.eduhk/~cis/2009.2/CIS-9-2-04.pdf

10. Kuttler, K., (2008). *An introduction to linear algebra*. Available at

www.smast.umassed.edu/CMMS/Courses/MAR580/Extra.../Linearalgebra.pdf

11. Lawton, S., (2007). *Generators, relations and symmetries in pairs of* 3×3 *unimodular matrices*. Journal of Algebra 313 782 -801

12. Matthews, R. K., (1998). *Elementary linear algebra*. Information available at www.math.fsu.edu/~dli/matthews.pdf

13. Nicodemi, O. E., Sutherland, M. A. and Towsley, G. W., (2007). *An introduction to abstract algebra with notes to the future teacher*. Upper Saddle River, NJ 07458.

14. Parker, J. R., (2009). *Complex hyperbolic lattices*. Discrete groups and geometric structures, ed K. Dekimpe et al. Contemporary mathematics 501, 1 - 42. Available at http://maths.dur.ac.uk 1~dma0jrp/img/Lattices.pdf

15. Parker, J. R. and Platis, I. D., (2008). *Complex hyperbolic Fenchel − Nielsen coordinates*. Topology 47 101-135.

16. Parker, J. R. and Paupert, J., (2009). *Unfaithful complex hyperbolic triangle groups II : higher order reflections*. Paci c J. Maths. 239 357389.

17. Parker, J. R., (2010). *Notes on complex hyperbolic geometry*. Available at https://www.icts.res.in/media/uploads/Program/Files/NCHG.pdf

18. Parker, J. R., (2012). *Traces in complex hyperbolic geometry*, in "Geometry, Topology and Dynamics of Character Varieties". World Scienti c Publishing Co.

Available at maths.dur.ac.uk/~dma0jrp/img/Traces.pdf

19. Platis, I. D., (2006). *Complex cross − ratios and ptolemaean inequality*. Available at www.math.uoc.gr/~jplatis/ptolemaeus.pdf

20. Pratoussevitch, A., (2005). *Traces in complex hyperbolic triangle groups*. Geometriae Dedicate 111, 159 - 185.

21. Sandler, H., (1995). *Traces in* $SU(2,1)$ *and complex hyperbolic ideal triangle groups*. Algebras, Groups and Geometry 12 139-156

22. Series, C., (2013). *Hyperbolic geometry*. Available at https://homepages.warwick.ac.uk/~masbb/Papers?MA448.pdf

23. Strang, G., (1988). *Linear algebra and its applications*. Academic Press, NY, S. Francisco, London.

24. Thompson, J. M., (2010). *Complex hyperbolic triangle groups*. Doctoral thesis, Durham University. Available at Durham E-Thesis Online: http//etheses.dur.ac.uk/478/

25. Will, P., (2009). *Traces, cross − ratios and 2 − generator subgroups of* $SU(2,1)$. Canad. J. Math. 61 1407 - 1436.

26. Zhao, T., (2011). *Geometry of complex hyperbolic lattices*. PhD thesis University of Paris VI.

APPENDIX

De nitions

We brie y de ne concepts from algebra and hyperbolic geometry which are essential for understanding this work.

De nition 7 (Group) Let $*$ be a binary operation on a non empty set G. We say that G is a group under $*$ if the following properties hold:

1. $\forall\ a,b,c \in G, a * (b * c) = (a * b) * c$ (associative).

2. $\exists\ e \in G : \forall\ a \in G, a * e = a = e * a$ (identity).

3. $\forall\, a \in G\ \exists\, b \in G : a * b = e = b * a$ (inverse)

Additionally we say G is abelian if $a \cdot b = b \cdot a$ for all $a,b \in G$. Lets look at some examples.

1. Z,Z_m,Q,R,C are all abelian groups with respect to the addition operation.

2. $GL(2,R)$, the set of invertible 2×2 real matrices with matrix operation is a group, called the general linear group.

3. $SL(2,R) \subset GL(2,R)$ the set of 2×2 matrices with determinant 1 is a group called special linear group.

4. $GL(n,C)$ the set of non-singular $n \times n$ complex matrices. Also called the general linear group of dimension n in complex domain.

5. Orthogonal group is $O(n) = \{T \in GL(n,R) : T^{-1}T = I\}$.

6. Complex special linear group is $SL(n,C) = \{T \in GL(n,C) : det(T) = 1\}$.

7. Unitary group is $U(n) = \{T \in GL(n,C) : T^{*}T = I\}$.
8. Special unitary group is $SU(n) = U(n) \cap SL(n,C)$.

De nition 8 (Group action) Let G be a group and let X be a set. We can de ne an action of G on X. This is a rule for taking $g \in G, x \in X$ and assigning them to an element of X. The map $G \times X$ 7−→ X must satisfy the following:

1. $ex = x$ where e is the identity element in G and $x \in X$.

2. $(g_1g_2) \cdot x = g_1(g_2 \cdot x)$ for all $g_1,g_2 \in G, x \in X$ (associative).

We say that G acts on X or G operates on X. The set X is sometimes referred as a G-set.

For instance, suppose that G = S_4, the group of permutations on the set S ={1,2,3,4}. We illustrate the actions of G on S as in the following examples:

1. $(1\quad 2)(3\quad 4)\cdot 3 = 4$

2. $(1\quad 2)(3\quad 4)\cdot 2 = 1$

3. $(1\quad 2\quad 3\quad 4)\cdot 2 = 3$

4. $(1\quad 3\quad 2)(1\quad 2)\cdot = 3$

5. $(1\quad 3\quad 2)(1\quad 2)\cdot 4 = 4$

De nition 9 (Transitive action) The action of G on X is called transitive if X is nonempty and if for any $x,y \in X \exists g \in G$ such that $g \cdot x = y$.

De nition 10 (Metric space) A metric space (X,d) is a set X together with a distance function(or metric) d on the set X satisfying the following properties:

1. $d(x,y) \geq 0$ for all $x,y \in X$ and $d(x,y)= 0$ if and only if $x = y$;

2. $d(x,y) = d(y,x)$ for all $x,y \in X$;

3. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

In other words, the de nition states that

1. distances are non-negative and the only point at zero distance from x is x itself.

2. the distance is a symmetric function.

3. travelling between two points via an arbitrary third point should not be shorter than the distance between the original two points. That is, distances satisfy the triangle inequality. For points in the Euclidean plane, the triangle inequality states that the lengths of one side of a triangle is less than the sum of the lengths of the other two sides.

The set of real numbers R with the distance function $d(x,y) = |x-y|$ is a metric space. The set of complex numbers C with the distance function $d(z,w) = |z-w|$ is also a metric space.

De nition 11 (Isometry) Let M be a metric space with d as its metric. A map $T : M \longrightarrow M$ is an isometry if it is invertible and preserves distances, so

$$d(T(x),T(y)) = d(x,y) \forall x,y \in M.$$

The set of isometries of M form a group Isom(M) under composition.

Some of the examples of isometries are translations, re ection, glide re ection and rotation. The above de nition is suitable for the space of R or C. In the space of complex hyperbolic, the de nition changes a little. We no more talk of distance-preserving function but rather a metric preserving function.

De nition 12 (Euclidean space) Euclidean n- space denoted $\mathbb{R}^n$ is the metric space with the metric

$$d(\mathrm{x},\mathrm{y}) = |\mathrm{x} - \mathrm{y}|$$

where the right hand side is the Euclidean norm $|\mathrm{x}| = (\mathrm{x}\cdot\mathrm{x})^{1/2}$. The inner product is the usual dot product given by

$$\mathrm{x} \cdot \mathrm{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

where $\mathrm{x},\mathrm{y} \in \mathbb{R}^n$.

De nition 13 (Homomorphism) Given two groups $(G,*)$ and $(H,\bullet)$, a group homomorhpism from $(G,*)$ to $(H,\bullet)$ is a function $h : G \longrightarrow H$ such that for all u and v in G it holds that

$$h(u * v) = h(u) \bullet h(v)$$

where the group operation on the left hand side of the equation is that of G and on the right hand side is that of H.

De nition 14 (Kernel of a homomorphism) The kernel of a homomorphism, h is the set of elements in G which are mapped to the identity in H;

$$\ker(h) = \{u \in G : h(u) = e_H\}.$$

Example, consider the groups $(R,+)$ and $(C^*, \bullet)$. The map $h : R \longrightarrow C^*$ de ned by $h(x) = e^{2\pi ix}\ \forall x \in$ R is a group homomorphism.

De nition 15 (Geodesic) The shortest path between two points in a space.

De nition 16 (Generator of a group) Let Γ be a group. We say that a subset $S = \{\gamma_1, \cdots \gamma_n\} \subset \Gamma$ is a set of generators if every element of Γ can be written as a composition of elements from S and their inverses. We write Γ = hSi.

De nition 17 (Free group) Let F be a group and $X \subseteq F$. Then F is a free on X if for any group G and any map $\theta : X$ 7→ G, there exists a unique homomorphism $\theta^0 : F$ 7→ G with $\theta^0(x) = \theta(x) \forall x \in X$, ie. the diagram commutes.

De nition 18 (A pair of pants) A pair of pants is a complete hyperbolic surface with geodesic boundary, whose interior is homeomorphic to the complement of the three points in the 2-sphere (Baik, 2010). Refer to gure
4.1 (pg 51) for the diagram.

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "i"'] & G \\ F \arrow[ur, "\theta'"'] & \end{tikzcd}$$

De nition 19 (Mo¨bius transformation) A Mo¨bius transformation is a mapping T : C 7→ C of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $a,b,c,d \in$ C and $ad - bc$ 6= 0.

De nition 20 (Di eomorphism) Given two manifolds M and N, a di erentiable map f : M → N is called a di eomorphism if it is a bijection and its inverse $f^{-1} : N \to M$ is di erentiable as well (Wikipedia.org).

De nition 21 (Local di eomorphism) Let X and Y be di erentiable manifolds. A function $f : X \to Y$ is a local di eomorphism, if for each point $x \in X$, there exists an open set U containing x, such that $f(U)$ is open in Y and $f|_U : U \to f(U)$ is a di eomorphism (Wikipedia.org).

De nition 22 (Discrete group) A discrete group is a group equipped with the discrete topology (Wikipedia.org).

De nition 23 (Ehresmann's bration theorem) Let $f : M \to N$ be a submersion. Then f is a locally trivial bration (Dundas, 2013).

De nition 24 (Homotopy) In topology, two continuous functions from one topological space to another are called homotopic (= same, similar and place) if one can be "continuously deformed" into the other, such a deformation being called a homotopy between the two functions (Wikipedia.org).

De nition 25 (HNN extension) Let G be a group with presentation G = h$S|R$i and let $\alpha : H \to K$ be an isomorphism between two subgroups of G. Let t be a new symbol not in S, and de ne

$$G*_{\alpha} = \mathrm{h}S,t|R,tht^{-1} = \alpha(h),\forall h \in H\mathrm{i}.$$

The group $G*_{\alpha}$ is called the HNN extension of G relative to α (Wikipedia.org).

De nition 26 (Fenchel-Nielsen coordinates) Suppose that S is a compact Riemann surface of genus $g > 1$. The Fenchel-Nielsen coordinates depend on the choice of $6g - 6$ curves on S. In order to de ne these coordinates, one can decompose the surface to $2g-2$ pairs of pants, by cutting the surface along $3g-3$ geodesic loops. Two adjacent pairs of pants are glued together along a cutting geodesic loop with an angle, called twisting angle. The lengths of the cutting loops and the twisting angles give the coordinates of the surface, which are the so-called

Fenchel-Nielsen coordinates (Jim et al, 2009).