NEW CONE METRICS ON THE SPHERE

by

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I hereby declare that this submission is my own work towards the PhD and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgment has been made in the text.

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We give an explicit construction of lattices in $PU(1,2)$. A family of these lattices was originally constructed by Livné [15]. Parker [19] constructed these lattices of Livné as the modular group of certain Euclidean cone metrics on the sphere. In this work we give a construction of these lattices which includes that of Parker's as the modular group of certain Euclidean cone metrics on the sphere. Our cone metrics on the sphere had five cone points with cone angles $(\pi-\theta+2\phi,\ \pi+\theta,\ \pi+\theta,\ \pi+\theta,\ 2\pi-2\theta-2\phi)$ Where $\theta>0$, $\phi>0$ and $\theta+\phi<\pi$. These corresponds to a group of five tuples lattices generated by Thurston [27] in his paper *Shapes of Polyhedra and Triangulations of the Sphere* . Hence our choice of θ and ϕ in order to obtain discreteness are as follows:

θ	$2\pi/3$	$2\pi/3$	$2\pi/3$	$2\pi/4$	$2\pi/4$	$(2\pi/5)$	$2\pi/5$	$2\pi/6$
ϕ	$\pi/4$	$\pi/5$	$\pi/6$	$\pi/3$	$\pi/4$	$(2\pi/5)$	$\pi/3$	$\pi/3$

Certain automorphisms which we considered on our cone metrics yielded unitary matrices R_1, R_2 and I_1. Using these matrices, we obtained our fundamental polyhedron D by constructing our vertices, edges and faces to define the polyhedron. Our vertices were obtained by the degeneration of certain cone metrics. The polyhedron D is contained in bisectors whose intersection give us the edges of the polyheron. The faces are also contained in the bisectors. Then finally we proved using Poincaré's

polyhedron theorem that the group Γ generated by the side pairings of D is a discrete subgroup of $PU(1,2)$ with fundamental domain D and presentation:

$$\Gamma = \left\langle J, P, R_1, R_2 : \begin{array}{c} J^3 = R_1^p = R_2^p = (P^{-1}J)^k = I, \\ R_2 = PR_1P^{-1} = JR_1J^{-1}, \qquad P = R_1R_2 \end{array} \right\rangle$$

DEDICATION

I dedicate this work to the LORD God Almighty and to His daughter Joyce Boadi whom He gave to me as a wife. We did this work together, Joyce, and without you there would not have been Dr. Richard Kena Boadi.

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INTRODUCTION

The study of Lattices in Complex Hyperbolic Space has been undertaken by various mathematicians in the field of Geometry for a long time. A key area of interest is how does one give explicit construction of these lattices in the complex hyperbolic space. Among few geometers who undertook these constructions were William P. Thurston in his paper *Shapes of Polyhedra and Triangulations of the Sphere* [27] and generated a list of lattices in complex hyperbolic space. Deligne and Mostow [4] also generated these same list of lattices. Thurston indicated that the spaces of shapes of a polyhedron with given total angles less than 2π at each of its n vertices has a metric, locally isometric to complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^{n-3}$. He indicates that collisions between vertices takes place a finite distance from a nonsingular point, this is the same as saying the metric is not complete. When one completes the metric, a complex hyperbolic cone-manifold is obtained. In some special cases,an orbifold is obtained as the metric completion. In simple terms any collection of n-dimensional Euclidean polyhedra whose $(n-1)$-dimensional faces are glued together isometrically in pairs yields an example of a $cone-manifold$. Thurston indicates that there are only three completely symmetric triangulations of the sphere, which are the tetrahedron, the octahedron and the icosahedron. However, finer triangulations with good geometric properties are often encountered or desired for mathematical, scientific or technological reasons. Example, the kinds of triangulations popularised in modern times by Buckminster Fuller and used for geodesic domes and chemical 'Buckyballs' [27].

In general, there are four major constructions, namely: arithmetic constructions, use of moduli of different objects, algebraic geometry and construction of fundamental domains. In his survey paper on Complex Hyperbolic Lattices, Parker [22] discusses all the major constructions. In as much as one may construct these lattices in several different ways, it is often difficult to do so and so very few explicit constructions are known.

Problem

Parker in attempting to give explicit constructions on some of them, looked at a group in which Ron Livné considered for his PhD thesis [15] in his paper *Cone Metrics on the Sphere and Livné's Lattices* [19]. This group was among the group of lattices generated by Thurston [27]. Livné used the approach of algebraic geometry and constructed these lattices. Parker [19] used Thurston's [27] approach by considering cone metrics on the sphere. He gave explicit construction of these lattices, by giving the fundamental domain and looked at interesting properties on this group. He also gave a presentation of this group proving it using Poincare Polyhedron Theorem.

The unanswered question was how do we extend these constructions to the other groups on the list generated by Thurston and Mostow which were not considered by Parker in the Livné group.

Solution

This is where we came in considering some of the members which had not been looked at and obtained explicit construction of them. We gave fundamental domain of a group of these lattices by using the approach Parker used in constructing the Livné lattices. The results were profound. Our construction led to the generalisation of both the group Parker considered and that which we begun with recalling that generalisation was an earlier agenda. The thesis is therefore arranged as follows: Chapter 1 looks at the construction giving the fundamental domain for the cone structure. Chapter 2 looks at the construction of bisectors and vertices; Chapter 3 looks at the construction of the complex hyperbolic polyhedron D which sets the stage for the final chapter 4 which is the proof that D is a fundamental polyhedron for the group.

CHAPTER 1

CONSTRUCTION OF THE POLYGONS AND AUTOMORPHISMS

1.1 INTRODUCTION

We begin with some key notations and definitions that will be useful in this work and also a section on Complex Hyperbolic Space which is the foundation for this work. Other definitions are given in the other chapters. Further definitions find themselves in the appendix. Let $PU(2,1)$ denote the projective unitary group of signature 2,1. Let $\mathbf{H}^2_{\mathbb{C}}$ denote complex hyperbolic space of dimension 2, and B a bisector of complex hyperbolic space.

Definition (manifold)**.** Let an *n-dimensional chart* be defined as an injection $x : M \to R^n$ whose range is an open subset of R^n. By the projection function $p_i : R^n \to R$, such a chart defines on its domain U a set of coordinate functions $x^i = p_i o x (i = 1, ..., n)$ so that $x = (x^1, ..., x^n)$. A collection of n-dimensional charts called a C^∞ atlas of M into R^n if, when x and y are any two charts whose domains intersect, the change of coordinates $y o x^{-1} : R^n \to R^n$ is a diffeomorphism. A complete C^∞ atlas is such that it is not contained in any other C^∞ atlas of M which determines a C^∞ structure of dimension n on M.

Therefore, a manifold M of dimension n is a set with a given C^∞ structure.

Definition ($(X, G) - manifold$)**.** Let X be a space, and G a group of isometries of X. An $(X, G) - manifold$ is a space equiped with a covering by open sets with homeomorphisms into X, such that the transition maps on the overlap of any two sets is in G.

In our case, the space is modelled after a complete Riemannian n-manifold X. Simply because in general, a cone-manifold is a kind of singular Riemannian metric.

The concept of an (X, G)-cone-manifold is defined inductively by dimension in the following:

Definition. If X is 1-dimensional, an (X, G)-cone-manifold is just an (X, G)-manifold.

Definition. Let X be k-dimenisional, where $k > 1$.For any point $p \in X$, let G_p be the stabilizer of p, and X_p be the set of tangent rays through p. (X, G). Then (X_p, G_p) is a model space of one lower dimension. If Y is any (X_p, G_p)-cone-manifold, there is associated to it a fairly intuitive construction, the *radius r cone* of Y, $C_r(Y)$ for any $r > 0$ such that the exponential map at p is an embedding on the ball of radius r in $T_p(X)$, constructed from the geodesic rays from p in X assembled in the same way that Y is. That is, for each subset of X_p, there is associated a cone in the tangent space at p, and to this is associated (via the exponential map) its radius r cone in X. These are glued together, using local coordinates in Y to form $C_r(Y)$. An (X, G)-cone-manifold is a space such that each point has a neighborhood modelled on the cone of a compact, connected (X_p, G_p)-manifold.

Definition (Lattice)**.** A lattice is a discrete subgroup Γ of a locally compact topological group G with Haar measure, so that the quotient G/Γ has finite volume.

Definition (fundamental domain)**.** Let (X, d) be a metric space and let Γ be a non-trivial group of isometries of X. A subset P of X is said to be a fundamental region for the group Γ if

1. the set P is open in X,
2. the members of $\{\gamma P : \gamma \in \Gamma\}$ are mutually disjoint, and
3. $X = \cup\{\gamma \overline{P} : \gamma \in \Gamma\}$.

A fundamental domain is a connected fundamental region.

Definition (Geodesic)**.** The shortest path between two points in a space.

Definition (totally geodesic space)**.** A space is said to be totally geodesic if for any two elements in the space, there exist a geodesic path between the two elements.

Definition (Tesselation)**.** A tesselation of X is a collection $\mathcal{P}$ of convex polyhedra in X such that

1. the interiors of the polyhedra in $\mathcal{P}$ are mutually disjoint, and
2. the union of the polyhedra in $\mathcal{P}$ is equal to X.

1.2 COMPLEX HYPERBOLIC SPACES

The unit ball in $\mathbb{C}^2$ has a natural metric of constant negative holomorphic sectional curvature (which we normalise to be -1), called the Bergman metric. As such it forms a model for complex hyperbolic 2-space $\mathbb{H}^2_C$ analogous to the ball model of (real) hyperbolic space H^n_R.

1.2.1 Hermitian forms on $\mathbb{C}^{2,1}$

Let $A = (a_{ij})$ be a $k \times l$ complex matrix. The **Hermitian transpose** of A is the $l \times k$ complex matrix $A^* = (\bar{a}_{ji})$ formed by complex conjugating each entry of A and then taking the transpose. Like ordinary transpose of a matrix, the Hermitian tranpose of a product is the product of the Hermitain transposes in the reverse order; i.e. $(AB)^* = B^*A^*$. Also $((A^*)^*) = A$.

Let A be a $k \times k$ complex matrix. A is *Hermitian* if it equals its own Hermitian transpose i.e. $A = A^*$.

Let A be a Hermitian matrix and μ an eigenvalue of A with eigenvectors x. We claim that μ is real.

$$\mu\mathbf{x}^*\mathbf{x} = \mathbf{x}^*(\mu\mathbf{x}) = \mathbf{x}^*A\mathbf{x} = \mathbf{x}^*A^*\mathbf{x} = (A\mathbf{x})^*\mathbf{x} = (\mu\mathbf{x})^*\mathbf{x} = \bar{\mu}\mathbf{x}^*\mathbf{x}$$

Since $\mathbf{x}^*\mathbf{x}$ is real and non-zero we see that μ is real for μ to be equal to $\bar{\mu}$

For any $k \times k$ Hermitian matrix A we can naturally associate a Hermitian form

$$\langle .,. \rangle : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C} \text{ given by } \langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* A \mathbf{z}$$

(note the change in the order)where $\mathbf{w}$ and $\mathbf{z}$ are column vectors in $\mathbb{C}^k$.

Lets consider a complex vector space of (complex) dimension 3 denoted $\mathbb{C}^{2,1}$ which is equipped with a Hermitian form $\langle .,. \rangle$ of signature $(2,1)$ given by a non-singular 3×3 Hermitian matrix J with 2 positive eigenvalues and 1 negative eigenvalue.

There are two standard matrices J which give different Hermitian forms on $\mathbb{C}^{2,1}$. Let $z = (z_1, z_2, z_3)^t$ and $w = (w_1, w_2, w_3)^t$ The first Hermitian form is defined to be: $\langle z, w \rangle_1 = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3}$ from $\langle z, w \rangle_1 = w^* J_1 z$ where

$$J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

the Hermitian matrix

The second Hermitian form is also $\langle z, w \rangle_2 = z_1\overline{w_3} + z_2\overline{w_2} - z_3\overline{w_1}$ from $\langle z, w \rangle_2 = w^* J_2 z$ where

$$J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There are other Hermitian forms which are widely used in the literature.

1.2.2 Three models of complex hyperbolic space

There are three standard models of complex hyperbolic space namely the unit ball model, siegel domain model and the projective model.

If $z \in \mathbb{C}^{2,1}$ then $\langle z, z \rangle$ is real. Thus we may define subsets V_-, V_o, V_+ of $\mathbb{C}^{2,1}$ by

$$V_- = \left\{ z \in \mathbb{C}^{2,1} \,|\langle z, z \rangle < 0 \right\} V_o = \left\{ z \in \mathbb{C}^{2,1} \,|\langle z, z \rangle = 0 \right\} V_+ = \left\{ z \in \mathbb{C}^{2,1} \,|\langle z, z \rangle > 0 \right\}$$

We say that $z \in \mathbb{C}^{2,1}$ is negative, null or positive if z is in V_-, V_o, V_+ respectively. For any non-zero complex scalar λ , since $\langle \lambda z, \lambda z \rangle = |\lambda|^2 \langle z, z \rangle$ the point λz is negative, null or positive if and only if z is negative, null or positive.

We therefore define a projection map P on those points of $\mathbb{C}^{2,1}$ with $z_3 \neq 0$. The projection map is defined by

$$P = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} \in \mathbb{C}^2$$

1.2.3 The Projective Model of Complex Hyperbolic space

$(\mathbb{H}_c^2)$ is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$ which is PV_- and its boundary $\partial \mathbb{H}_c^2$ defined to be the collection of null lines (PV_o)

We can get the other two models from the projection model by taking the section defined by $z_3 = 1$ for the first and second Hermitian forms.

Taking the first Hermitian form with $\langle z, z \rangle_1 < 0$ for $z = (z_1, z_2, 1)^t \in \mathbb{C}^{2,1}$

$$\langle z, w \rangle_1 = z_1 \overline{z_1} + z_2 \overline{z_2} - 1 < 0 \Rightarrow |z_1|^2 + |z_2|^2 < 1$$

Thus $z = (z_1, z_2)$ is in the unit ball in $\mathbb{C}^2$ forming the unit ball model. The boundary of the unit ball model is the sphere S^3 given by

$$|z_1|^2 + |z_2|^2 = 1$$

We say that the standard lift of a point $z = (z_1, z_2)$ in the unit ball (or its boundary) to $\mathbb{C}^{2,1}$ is the column vector $z = (z_1, z_2, 1)^t \in \mathbb{C}^{2,1}$

Taking the second Hermitian form we obtain $z \in \mathbb{H}_c^2$ provided:

$$\langle z, w\rangle_2 = z_1 + z_2\overline{z_2} + \overline{z_1} < 0 i.e. 2Re(z_1) + |z_2|^2 < 0$$

Thus $z = (z_1, z_2)$ is in a domain in $\mathbb{C}^2$ whose boundary is the paraboloid defined by

$$2Re(z_1) + |z_2|^2 = 0$$

This domain is called the Siegel domain and forms the Siegel domain model of H_c^2. The standard lift of a point $z = (z_1, z_2)$ in this domain is the same as in the unit ball model. For the projective model, the metric on H_c^2 known as the *Bergman metric* is given by the distance function defined by the formula

$$\cosh^2\left(\frac{\rho(z, w)}{2}\right) = \frac{\langle z, w\rangle \langle w, z\rangle}{\langle z, z\rangle \langle w, w\rangle}$$

For the ball model and Siegel domain model, the distance between points z and w are obtained by plugging in their standard lifts $\mathbf{z}$ and $\mathbf{w}$ into the above formula.

1.2.4 Cayley Transform

Given two Hermitian forms of signature (2, 1) you could pass between them using a Cayley transform. The following Cayley transform interchanges first and second Hermitian form

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

1.2.5 Isometries

We now consider isometry (metric preserving maps) groups of complex hyperbolic spaces.

Let A be a matrix which preserves the first or second Hermitian form. Then A is a unitary matrix. That is for all v and w in $\mathbb{C}^{2,1}$,

$$\langle Av, Aw\rangle_1 = \langle v, w\rangle_1 \Rightarrow w^* A^* J_1 Av = w^* J_1 v Or \, \langle Av, Aw\rangle_2 = \langle v, w\rangle_2 \Rightarrow w^* A^* J_2 Av = w^* J_2 v$$

We then say that unitary matrices say $A \in U(2,1)$ acts on complex hyperbolic spaces. Since A is unitary with respect to the Hermitian form $\langle ., .\rangle$, we can define A as an element in the projective unitary group $PU(2,1) = U(2,1)/U(1)$ and can show that $PU(2,1)$ acts trivially on H_c^2

Now since the Bergman metric is given in terms of the Hermitian form $\langle ., .\rangle$ it implies that A acts isometrically on the projective model of complex hyperbolic space. Making $PU(2,1)$ a subgroup of the complex hyperbolic isometry group. The following theorem gives the whole isometry group of complex hyperbolic space.

Theorem 1.2.1. *Every isometry of H_c^2 is either holomorphic or else anti-holomorphic. Moreover, each holomorphic isometry of H_c^2 is given by a matrix in $PU(2,1)$ and each anti-holomorphic isometry is given by complex conjugation followed by a matrix in $PU(2,1)$.*

From the definition of a lattice, if G is $PU(2,1)$ then a lattice is a discrete subgroup Γ so that the quotient $\Gamma \setminus \mathbb{H}_c^2$ has finite volume with respect to the Bergman metric.

We conclude by also defining a cone singularity of a manifold which is an isolated point where the total angle is different from 2π. The angle is what we call the *cone angle*. A Euclidean cone metric on the sphere is a metric that is locally isometric to the standard metric on $\mathbb{R}^2$ but having finitely many cone singularities. Consider a cube as an example: it is a Euclidean cone metric on the sphere with eight cone singularities, each with cone angle $3\pi/2$. A simple family of examples of Euclidean cone metrics on the sphere is obtained by taking two copies of the same plane Euclidean polygon and identifying them along their boundary. This is called the *double* of the polygon. The cone angles are then twice the corresponding internal angles of the polygon. We would like to find out what happens when we fix certain cone angles, but allow the cone singularities to move around the sphere. Example, a pillow case, which is the double of a square, and the regular tetrahedron, both have four cone singularities, each with cone angle π. When you move the cone singularities around, you may be able to transform the double square into the regular tetrahedron.

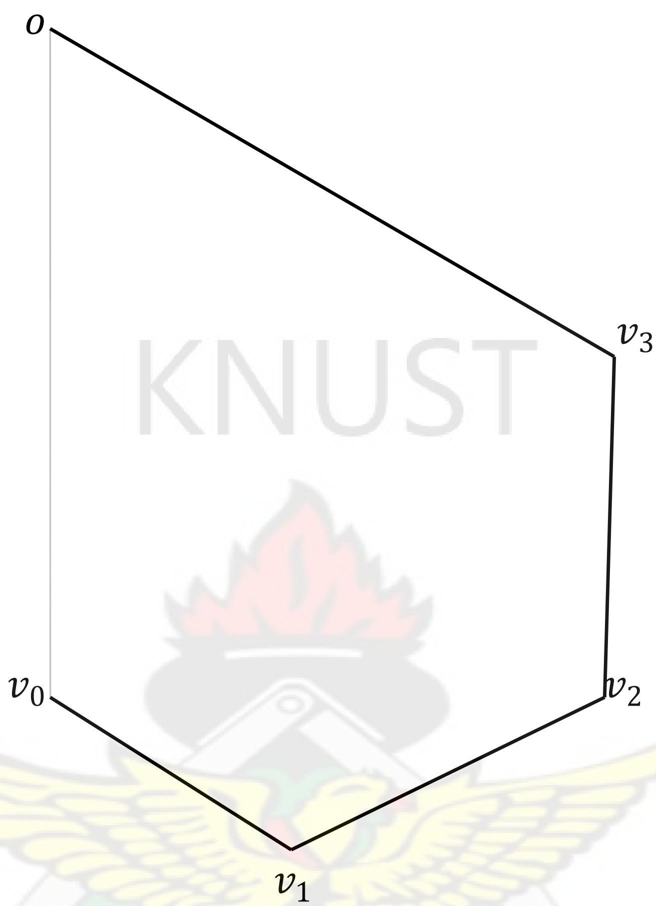

Figure 1.1. Double Pentagon.

1.3 THE FUNDAMENTAL DOMAIN FOR THE CONE STRUCTURES

We consider Euclidean cone metrics on the sphere with five cone points with cone angles

$$(\pi - \theta + 2\phi,\ \pi + \theta,\ \pi + \theta,\ \pi + \theta,\ 2\pi - 2\theta - 2\phi).$$

There are various angles θ and ϕ that one can choose to get a discrete group. Our goal is to try to find a unified construction for all these angles (and to re-prove which angles are allowed for discreteness). Now if you cut the sphere open along a path through the five cone points, we obtain a Euclidean polygon Π. Conversely, if we glue the sides of Π together, we can reconstruct our cone metric on the sphere. By three complex numbers we give an explicit parametrisation of such polygons and show that, in terms of these parameters, the area of the polygon gives a Hermitian form of signature $(1, 2)$. Thurston [27] and Weber [28] describe different ways of doing this. We use the method of Parker used in his paper Cone Metric on the sphere and Livne's lattices which is different from theirs [19]. We look at the case where the cone manifold is the double of a Euclidean pentagon. Cutting the pentagon along four of its sides [See Figure 1.1],with the first cut at the cone point with angle $2(\pi - \theta - \phi)$, then moving along the boundary of the pentagon through the three cone points v_3,v_2 and v_1 with cone angle $\pi + \theta$, ending at the cone point v_o with cone angle $\pi - \theta + 2\phi$. When we cut the double pentagon this way, we get an octagon,which we call Π. This octagon has a reflection symmetry. Using this symmetry to identify the boundary points reconstructs the doubled pentagon with which we began. We now show how to construct Π geometrically. We begin by constructing a fundamental polygon for the cone structure. We start with a big triangle T_3 with angles θ, $\pi - \theta - \phi$ and ϕ. This will only work when $\theta > 0$, $\phi > 0$ and $\theta + \phi < \pi$. We then take off two smaller triangles T_1 with angles ϕ, $\pi/2 + \theta/2 - \phi$ and $\pi/2 - \theta/2$; and T_2 with angles θ, $\pi/2 - \theta/2$ and $\pi/2 - \theta/2$. The corners of the triangles T_1 and T_3 with angles ϕ are the same. The corners of the triangles T_2 and T_3 with angles θ are the same. The base vectors of T_1, T_2 and T_3 are z_1, z_2 and z_3. This is done in order to ensure that the resulting pentagon the three vertices with angles $\pi/2 + \theta/2$ should come together. See Figure 1.2 for the construction.

We have constructed a pentagon whose vertices are the vertex of T_3 and the two vertices of each of T_1 and T_2 not shared by one of the other triangles. This has one edge in common with each of T_1 and T_2. Consider the edge of this pentagon joining the vertices of T_1 with angle $\pi + \theta/2 - \phi$ and the vertex of T_3 with angle $\pi - \theta - \phi$. Reflect the pentagon across this side to form an octagon. This is a fundamental polygon for the cone structure.

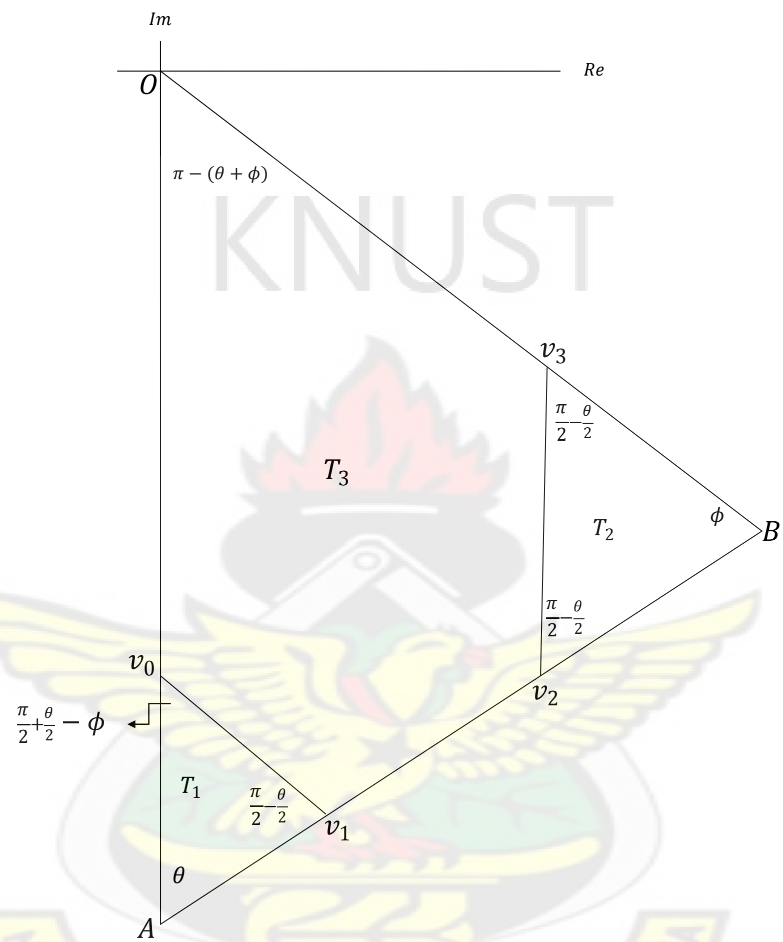

Figure 1.2. Construction of the Pentagon.

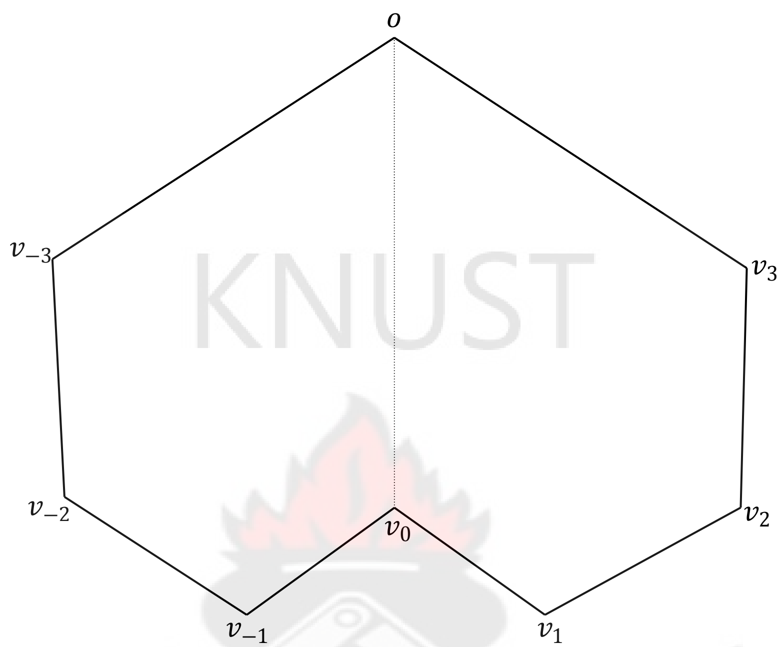

Figure 1.3. Octagon II: Fundamental Domain.

Vertices of triangle T_1 are as follows:

$$\begin{aligned} v_0 &= \frac{-i\sin\theta}{\sin(\theta+\phi)}z_3 + \frac{i\sin\theta}{\sin\phi + \sin(\theta-\phi)}z_1, \\ A &= -iz_3\frac{\sin\theta}{\sin(\theta+\phi)} \\ v_1 &= ie^{-i\phi}z_1 - \frac{i\sin\theta}{\sin(\theta+\phi)}z_3 \end{aligned}$$

Vertices of triangle T_2 are as follows:

$$
\begin{aligned}
v_2 &= -ie^{-i\phi}z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin\theta+\phi)}z_3, \\
B &= -z_3\frac{\sin\phi\, e^{i(\theta+\phi)}}{\sin(\theta+\phi)} \\
v_3 &= -ie^{-i\theta-i\phi}z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}z_3
\end{aligned}
$$

Vertices of triangle T_3 are also as follows:

$$
\begin{aligned}
&0 \\
B &= -z_3\frac{\sin\phi\, e^{i(\theta+\phi)}}{\sin(\theta+\phi)} \\
A &= -iz_3\frac{\sin\theta}{\sin(\theta+\phi)}
\end{aligned}
$$

Our resulting octagon is preserved by reflection in the imaginary axis and we label its vertices so that this reflection interchanges v_j and v_{-j}. Moreover, gluing points of the boundary Π to their image under this reflection reconstructs the doubled pentagon we begun with. Below are the vertices of the octagon.

$$
\begin{aligned}
v_0 &= \frac{-i\sin\theta}{\sin(\theta+\phi)}z_3 + \frac{i\sin\theta}{\sin\phi+\sin(\theta-\phi)}z_1, \\
v_1 &= ie^{-i\phi}z_1 - \frac{i\sin\theta}{\sin(\theta+\phi)}z_3, \\
v_2 &= -ie^{-i\phi}z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin\theta+\phi)}z_3, \\
v_3 &= -ie^{-i\theta-i\phi}z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}z_3, \\
v_{-1} &= ie^{i\phi}z_1 - \frac{i\sin\theta}{\sin(\theta+\phi)}z_3, \\
v_{-2} &= -ie^{i\phi}z_2 + \frac{i\sin\phi\, e^{i\theta+i\phi}}{(\sin\theta+\phi)}z_3, \\
v_{-3} &= -ie^{i\theta+i\phi}z_2 + \frac{i\sin\phi\, e^{i\theta+i\phi}}{\sin(\theta+\phi)}z_3.
\end{aligned}
$$

From the figure;the areas of the triangles are as follows:

$$Area[T_1] = \frac{\sin\theta\sin\phi}{2(\sin\phi + \sin(\theta - \phi))}\,|z_1|^2$$

$$Area[T_2] = \frac{1}{2}\sin\theta\,|z_2|^2$$

$$Area[T_3] = \frac{\sin\theta\sin\phi}{2\sin(\theta + \phi)}\,|z_3|^2$$

The area of the pentagon is therefore: $Area(Pentagon) = Area[T_3 - (T_1 \cup T_2)]$

Hence the area of Octagon(Π)=2*Area(Pentagon) is:

$$Area(\Pi) = \sin\theta(-\frac{\sin\phi}{(\sin\phi + \sin(\theta - \phi))}\,|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta + \phi)}\,|z_3|^2). \qquad (1.1)$$

$$\begin{aligned} Area(\Pi) &= \sin\theta\begin{bmatrix}\overline{z_1} & \overline{z_2} & \overline{z_3}\end{bmatrix}\begin{bmatrix} -\sin\phi/\big(\sin\phi + \sin(\theta - \phi)\big) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sin\phi/\sin(\theta + \phi)\end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= \mathbf{z}^* H \mathbf{z}. \end{aligned}$$

That is H, which is a Hermitian form is given as:

$$H = \sin\theta\begin{bmatrix} -\sin\phi/\big(\sin\phi + \sin(\theta - \phi)\big) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sin\phi/\sin(\theta + \phi)\end{bmatrix}$$

We observe that the area gives a Hermitian form H of signature (1,2) on $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{C}^3$. This leads to a complex hyperbolic structure on the moduli space of such polygons. This is a special case of Proposition 3.3 of [27]. There is a natural way to construct a particular Euclidean cone manifold from Π. The following, σ_j, are edge pairing maps of Π, which are Euclidean isometries which preserve orientation and so are completely determined on each edge by their value on the vertices v_j,v_{j+1}. The maps are

$$\sigma_1(0) = 0, \quad \sigma_1(v_3) = v_{-3}; \quad \sigma_2(v_3) = v_{-2}, \quad \sigma_1(v_2) = v_{-2};$$
$$\sigma_3(v_2) = v_{-2}, \quad \sigma_3(v_1) = v_{-1}; \quad \sigma_4(v_1) = v_{-1}, \quad \sigma_4(v_0) = v_0.$$

Let M be the Euclidean cone manifold given by identifying the edges of Π using the maps σ_j. It is clear that M is homeomorphic to a sphere and has five cone points corresponding to 0,v_0,$v_{\pm1}$,$v_{\pm2}$,$v_{\pm3}$ with cone angles $\pi - \theta + 2\phi$, $\pi + \theta$, $\pi + \theta$, $\pi + \theta$, $2\pi - 2\theta - 2\phi$ respectively. These are examples of the cone manifolds studied by Thurston in [27] in which Parker gave the geometrical construction in [19] and the cone angles correspond to the ball 5-tuples studied by Mostow [18].

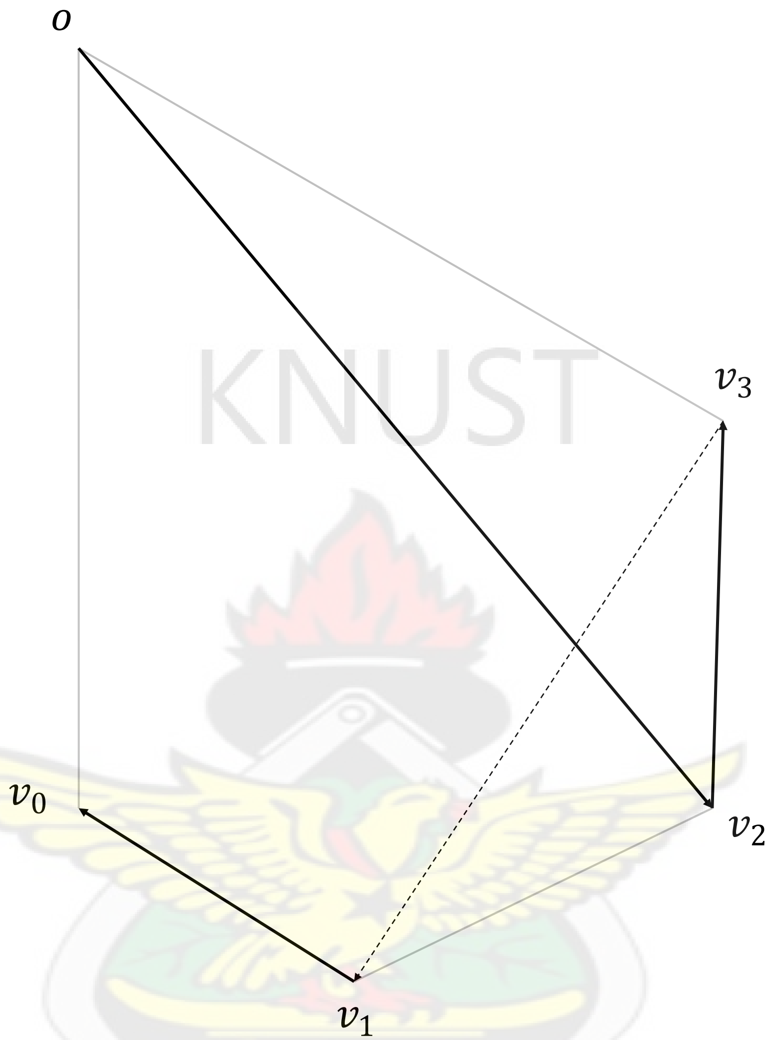

Figure 1.4. The cut for Move R_1.

1.4 MOVES ON THE CONE STRUCTURE

We define automorphisms which we call "'moves"' on such polygons. We consider the "'moves"' looked at by Parker [19] which was in the spirit of Thurston [27]. We define them as follows: Our cone manifold has five cone points with cone angles $2\pi - 2(\theta + \phi)$ and $\pi + \phi - \theta$, corresponding to the vertex 0 and v_0 respectively. The other three vertex have the same cone angles which is $\pi + \theta$. In cutting our cone manifold to get back our octagon, there is no canonical ordering of these three vertices, hence we can change the order of the cut. This results in the moves we will consider namely R_1 and R_2. They are the moves that Parker consider's in his Livné Paper. The third move is called the butterfly move, also considered in Livné paper, which we will consider

The Move R_1

The move R_1 fixes the vertex 0,v_0 and $v_{\pm 1}$ and then interchanges $v_{\pm 2}$ and $v_{\pm 3}$. Hence cutting open the cone manifold, one must begin cutting from 0 and then to $v_{\pm 2}$, then to $v_{\pm 3}$, and then to $v_{\pm 1}$ and v_0; see Figure 1.4. When we cut open the double pentagon, we obtain an octagon, shown in Figure 1.5

Using cut and paste, one can obtain the new octagon from the old. The cut goes from 0 directly to v_2. Then the triangle 0,v_2,v_3 must be glued back on along the edge 0,v_{-3} according to the side identification σ_1. In the same way, the triangle

v_{-1},v_{-2},v_{-3} must be glued by σ_3^{-1} to the side v_1,v_2. See same Figure 1.5 for how it is done.

We now find the new parameters w_1,w_2,w_3 for the new polygon by analysing the vertices.

We write the new vertices as v'_j. Then: $v'_0 = v_0$,$v'_1 = v_1$,$v'_3 = v_2$

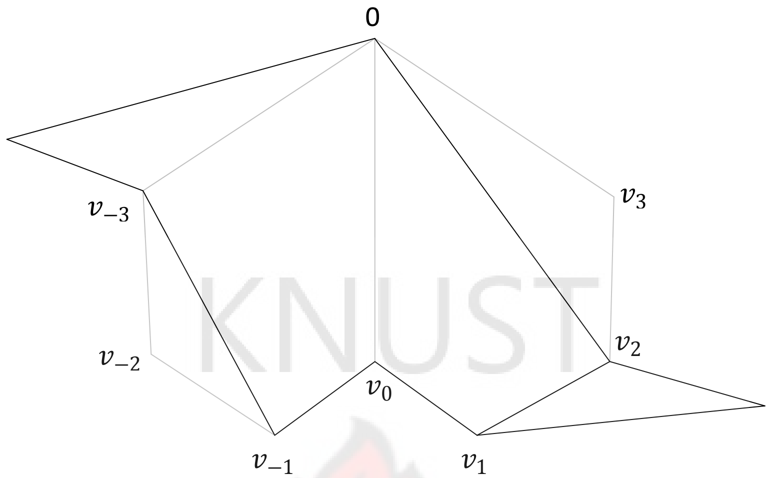

Figure 1.5. The obtained octagon when the Move corresponding to R_1 is applied to the pentagon

Which are:

$$\begin{aligned}
\frac{-i\sin\theta}{\sin(\theta+\phi)}w_3+\frac{i\sin\theta}{\sin\phi+\sin(\theta-\phi)}w_1 &= \frac{-i\sin\theta}{\sin(\theta+\phi)}z_3+\frac{i\sin\theta}{\sin\phi+\sin(\theta-\phi)}z_1,\\
ie^{-i\phi}w_1-\frac{i\sin\theta}{\sin(\theta+\phi)}w_3 &= ie^{-i\phi}z_1-\frac{i\sin\theta}{\sin(\theta+\phi)}z_3,\\
-ie^{-i\theta-i\phi}w_2+\frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}w_3 &= -ie^{-i\phi}z_2+\frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin\theta+\phi)}z_3
\end{aligned}$$

Solving these simultaneous equations give you the following:

$$w_1=z_1,\quad w_2=e^{i\theta}z_2,\quad w_3=z_3$$

In a matrix form, R_1 is as below:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.2}$$

We now verify whether R_1 is a Unitary matrix with respect to the hermitian matrix H obtained. That is we verify $R_1^* H R_1 = H$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \sin\theta \begin{bmatrix} -\sin\phi/\big(\sin\phi + \sin(\theta-\phi)\big) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sin\phi/\sin(\theta+\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \quad \sin\theta \begin{bmatrix} -\sin\phi/\big(\sin\phi + \sin(\theta-\phi)\big) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sin\phi/\sin(\theta+\phi) \end{bmatrix}$$

Hence R_1 is unitary.

The Move R_2

The move R_2 which is more complicated fixes 0, v_0,and $v_{\pm 3}$ but interchanges $v_{\pm 1}$ and $v_{\pm 2}$. This corresponds to a Dehn Twist along a simple closed curve through $v_{\pm 1}$ and $v_{\pm 2}$ that does not separate the other cone points. We obtain the octagon by cutting from 0 to $v_{\pm 3}$, then to $v_{\pm 1}$,$v_{\pm 2}$ and finally to v_0; see Figure 1.7

Using cut and paste to obtain the new octagon from the old, we proceed as follows: The slit goes from 0 to v_3 and then directly to v_1. Hence the triangle v_1,v_2,v_3 should be glued by σ_2 to v_{-2}, v_{-3}. We also analyse the vertices to find the new coordinates;

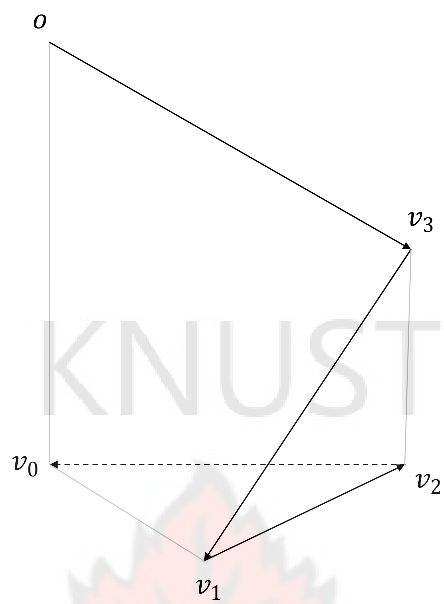

Figure 1.6. The cut for Move R_2

$v_0' = v_0$,$v_2' = v_1$,$v_3' = v_3$ Which are:

$$
\begin{aligned}
\frac{-i\sin\theta}{\sin(\theta+\phi)}w_3 + \frac{i\sin\theta}{\sin\phi+\sin(\theta-\phi)}w_1 &= \frac{-i\sin\theta}{\sin(\theta+\phi)}z_3 + \frac{i\sin\theta}{\sin\phi+\sin(\theta-\phi)}z_1, \\
-ie^{-i\phi}w_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin\theta+\phi)}w_3 &= ie^{-i\phi}z_1 - \frac{i\sin\theta}{\sin(\theta+\phi)}z_3, \\
-ie^{-i\theta-i\phi}w_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}w_3 &= -ie^{-i\theta-i\phi}z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}z_3
\end{aligned}
$$

Solving these simultaneously results in the following:

$$
\begin{aligned}
w_1 &= \frac{1}{(1-e^{-i\theta})\sin(\phi)}(-\sin(\theta)e^{-i\phi}z_1 - (\sin(\phi)-\sin(\theta-\phi))z_2 + (\sin(\phi)+\sin(\theta-\phi))z_3) \\
w_2 &= \frac{1}{(1-e^{-i\theta})\sin(\phi)}(-\sin(\phi)z_1 - \sin(\phi)e^{-i\theta}z_2\sin(\phi)z_3) \\
w_3 &= \frac{1}{(1-e^{-i\theta})\sin(\phi)}(-\sin(\theta+\phi)z_1 - \sin(\theta+\phi)z_2(\sin(\phi)+\sin(\theta)e^{i\phi})z_3)
\end{aligned}
$$

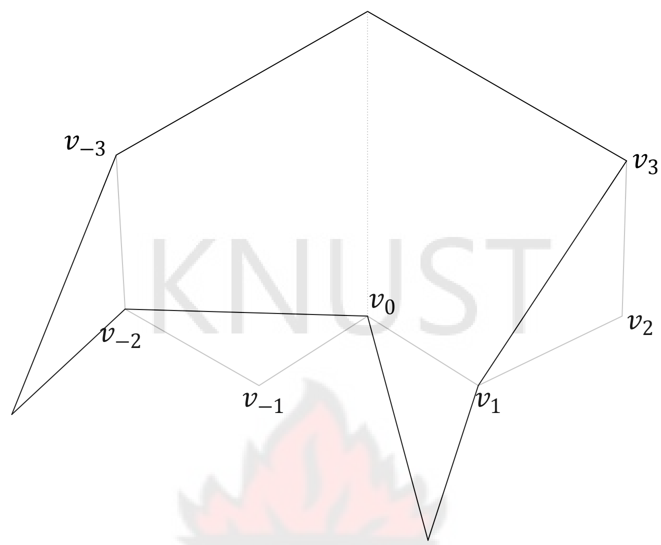

Figure 1.7. The obtained octagon when the Move corresponding to R_2 is applied to the pentagon

In a matrix form R_2 is as below.

$$R_2 = \frac{1}{(1-e^{-i\theta})\sin(\phi)} \begin{bmatrix} -\sin(\theta)e^{-i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi) & -\sin(\phi)e^{-i\theta} & \sin(\phi) \\ -\sin(\theta+\phi) & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix} \quad (1.3)$$

It can be easily verified that R_2 is also unitary (that is $R_2^* H R_2 = H$).

The Move I_1

The third move is a generalisation of the 'butterfly' move discussed by Thurston [27]. A butterfly operation moves one edge of the pentagon across a butterfly-shaped quadrilateral of zero area, yielding a new hexagon of the same area. This should fix z_2 and z_3 and send z_1 to $e^{2i\phi}z_1$. That is, it is given by the matrix:

$$I_1 = \begin{bmatrix} e^{2i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.4}$$

Now all these moves preserve the (signed) area of Π. Our goal will be to consider the group of unitary matrices Γ generated by these moves, namely

$$\Gamma = \langle R_1, R_2, I_1 \rangle$$

and show that Γ is discrete for the various values of θ and ϕ that will be considered. Our interest is in the following cases:

θ	$2\pi/3$	$2\pi/3$	$2\pi/3$	$2\pi/4$	$2\pi/4$	$(2\pi/5)$	$2\pi/5$	$2\pi/6$
ϕ	$\pi/4$	$\pi/5$	$\pi/6$	$\pi/3$	$\pi/4$	$(2\pi/5)$	$\pi/3$	$\pi/3$

CHAPTER 2

CONSTRUCTION OF BISECTORS AND VERTICES

2.1 INTRODUCTION

This chapter shows how that the collection of polygons Π (cone metrics on the sphere) can be parametrised by a subset of complex hyperbolic space. This follows that the moves (automorphisms) act as complex hyperbolic isometries. We will actually be looking at the geometry of the action of the isometries. We construct a polyhedron D whose sides are contained in bisectors and whose vertices correspond to certain cone metrics which have degenerate. Where this degeneration is obtained either from the collision of three cone points or from the collision of two pairs of cone points. So we will be looking at these bisectors and the vertices.

2.2 NEW COORDINATES

Complex hyperbolic space can be defined to be the projectivisation of those points in the space for which the Hermitian form is positive. This definition is from the ball model of complex hyperbolic space. We proceed as follows: we know from chapter 1 that the area of Π is given in terms of the Hermitian and it is equivalent to the following:

$$\begin{aligned} H &= \sin\theta \begin{bmatrix} \overline{z_1} & \overline{z_2} & \overline{z_3} \end{bmatrix} \begin{bmatrix} -\sin\phi/\big(\sin\phi+\sin(\theta-\phi)\big) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sin\phi/\sin(\theta+\phi) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= \mathbf{z}^* H \mathbf{z} = \langle \mathbf{z}, \mathbf{z} \rangle > 0. \end{aligned}$$

We achieve the projectivisation by setting $z_3 = 1$. Thus the definition of complex hyperbolic space is as follows:

$$\mathbf{H}^2_{\mathbb{C}} = \left\{ \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} : \langle \mathbf{z}, \mathbf{z} \rangle = \mathbf{z}^* H \mathbf{z} = \frac{-|z_1|^2 \sin\theta \sin\phi}{\left(\sin\phi + \sin(\theta - \phi)\right)} - |z_2|^2 \sin\theta + \frac{\sin\theta\sin\phi}{\sin(\theta+\phi)} > 0 \right\} \quad (2.1)$$

We have already seen that the moves obtained from chapter I_1, R_1, R_2 and their products correspond to unitary matrices with respect to the Hermitian form H. These act projectively on $\mathbf{H}^2_{\mathbb{C}}$ and so lie in $\mathbf{PU}(2,1)$, the holomorphic isometry group of $\mathbf{H}^2_{\mathbb{C}}$. In the same way complex conjugation is an antiholomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$. For convenience, we introduce new coordinates on $\mathbf{H}^2_{\mathbb{C}}$. Consider P, an element of the group of automorphisms defined as $P = R_1 R_2$ as in Parker's [19]. In particular P is a side paring of our fundamental domain D and images of D under powers of P form a cylinder or a torus with a repeating pattern of faces. We write P as a matrix. First lets recall R_1,R_2, and I_1

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$R_2 = \frac{1}{(1-e^{-i\theta})\sin(\phi)} \begin{bmatrix} -\sin(\theta)e^{-i\phi} & -\sin(\phi) - \sin(\theta - \phi) & \sin(\phi) + \sin(\theta - \phi) \\ -\sin(\phi) & -\sin(\phi)e^{-i\theta} & \sin(\phi) \\ -\sin(\theta+\phi) & -\sin(\theta+\phi) & \sin(\phi) + \sin(\theta)e^{i\phi} \end{bmatrix}.$$

$$I_1 = \begin{bmatrix} e^{2i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we find the map $P = R_1R_2$.

$$P = R_1R_2 = \frac{1}{(1-e^{-i\theta})\sin(\phi)}\begin{bmatrix} -\sin(\theta)e^{-i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi)e^{i\theta} & -\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta+\phi) & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix} \tag{2.2}$$

We also find the map $J = R_1R_2I_1$ which we will be interested and check if it has order 3. To do this, we show that the trace is zero.

$$J = R_1R_2I_1 = \frac{1}{(1-e^{-i\theta})\sin(\phi)}\begin{bmatrix} -\sin(\theta)e^{i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi)e^{i(2\phi+\theta)} & -\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta+\phi)e^{2i\phi} & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix}$$

$$trJ = -\sin(\theta)e^{i\phi} - \sin(\phi) + \sin(\phi) + \sin(\theta)e^{i\phi} = 0$$

Hence J has order 3. We now define our second coordinates denoted by $\mathbf{w}$ which is the preimage of the first coordinate. This is given by

$$\begin{aligned} \mathbf{w} &= \begin{bmatrix} w_1 \\ w_2 \\ 1 \end{bmatrix} = \left[P^{-1}(\mathbf{z})\right] \\ &= \frac{1}{(1-e^{i\theta})\sin(\phi)}\begin{bmatrix} -\sin(\theta)e^{i\phi} & -\big(\sin(\phi)+\sin(\theta-\phi)\big)e^{-i\theta} & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi) & -\sin(\phi) & \sin(\phi) \\ -\sin(\theta+\phi) & -\sin(\theta+\phi)e^{-i\theta} & \sin(\phi)+\sin(\theta)e^{-i\phi} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence finding w_1 and w_2 as rational functions of z_1 and z_2, we obtain

$$\begin{aligned} w_1 &= \frac{1}{(1-e^{i\theta})\sin(\phi)}\left[-\sin(\theta)e^{i\phi}z_1 - \big(\sin(\phi)+\sin(\theta-\phi)\big)e^{-i\theta}z_2 + \sin(\phi)+\sin(\theta-\phi)\right] \\ w_2 &= \frac{1}{(1-e^{i\theta})\sin(\phi)}\left[-\sin(\phi)z_1 - \sin(\phi)z_2 + \sin(\phi)\right] \\ 1 &= \frac{1}{(1-e^{i\theta})\sin(\phi)}\left[-\sin(\theta+\phi)z_1 - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin(\phi)+\sin(\theta)e^{-i\phi}\right] \end{aligned}$$

From the third equation

$$
\begin{aligned}
& (1-e^{i\theta})\sin(\phi) \\
= \quad & \left[-\sin(\theta+\phi)z_1 - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin(\phi) + \sin(\theta)e^{-i\phi}\right] \qquad (2.3)
\end{aligned}
$$

Substituting back into the first two equations results in the following

$$
\begin{aligned}
w_1 &= \frac{-\sin(\theta)e^{i\phi}z_1 - \big(\sin(\phi)+\sin(\theta-\phi)\big)e^{-i\theta}z_2 + \sin(\phi) + \sin(\theta-\phi)}{-\sin(\theta+\phi)z_1 - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin(\phi) + \sin(\theta)e^{-i\phi}}, \quad (2.4) \\
w_2 &= \frac{-\sin(\phi)z_1 - \sin(\phi)z_2 + \sin(\phi)}{-\sin(\theta+\phi)z_1 - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin(\phi) + \sin(\theta)e^{-i\phi}}. \quad (2.5)
\end{aligned}
$$

By similar proceedure, we obtain likewise for z_1 and z_2 in terms of w_1 and w_2

$$
\begin{aligned}
\mathbf{z} &= \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} = \Big[P(\mathbf{w})\Big] \\
&= \frac{1}{(1-e^{-i\theta})\sin(\phi)} \begin{bmatrix} -\sin(\theta)e^{-i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi)e^{i\theta} & -\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta+\phi) & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ 1 \end{bmatrix}.
\end{aligned}
$$

and hence

$$
\begin{aligned}
z_1 &= \frac{-\sin(\theta)e^{-i\phi}w_1 - \big(\sin(\phi)+\sin(\theta-\phi)\big)w_2 + \sin(\phi) + \sin(\theta-\phi)}{-\sin(\theta+\phi)w_1 - \sin(\theta+\phi)w_2 + \sin(\phi) + \sin(\theta)e^{i\phi}}, \quad (2.6) \\
z_2 &= \frac{-\sin(\phi)e^{i\theta}w_1 - \sin(\phi)w_2 + \sin(\phi)e^{i\theta}}{-\sin(\theta+\phi)w_1 - \sin(\theta+\phi)w_2 + \sin(\phi) + \sin(\theta)e^{i\phi}}. \quad (2.7)
\end{aligned}
$$

Our reason of keeping track of two coordinates is that it gives a simple description of the polyhedron D in terms of the arguments of z_1,z_2,w_1 and w_2.

2.3 VERTICES

In this section, we obtain some distinguished points of $\mathbf{H}^2_{\mathbb{C}}$ which will be the vertices of our polyhedron. From the section before this, it will be useful to also have our points in the two coordinates: that is $\mathbf{w}$ and $\mathbf{z}$. The distinguished points (cone structures) are obtained by letting some of the cone points approach each other until in the limit they coalesce, and then result in a new point. The complementary angles of this new cone point (that is 2π minus the cone angle) is the sum of the complementary angles of the cone points that have coalesced. Considering this from the view point of the octagon Π considered in Chapter 1, obtaining the new cone points is the same as either expanding or contracting the triangles T_1 and T_2 till some of the vertices become the same point. Suppose such vertices are adjacent to each other then the edge between them has degenerated to a point. We define the following vertices by where various cone points coalesce.

Point	Cone Points	Angle	Cone Points	Angle
p_1	$v_0,\ v_{\pm1}$	2ϕ	$v_{\pm2},\ v_{\pm3}$	2θ
p_2	$v_0,\ v_{\pm3}$	2ϕ	$v_{\pm1},\ v_{\pm2}$	2θ
p_{231}	$v_{\pm1},\ v_{\pm2},\ v_{\pm3}$	$3\theta-\pi$		
p_{23}	$v_0,\ v_{\pm2},\ v_{\pm3}$	$\theta+2\phi-\pi$		
p_{31}	$v_0,\ v_{\pm1},\ v_{\pm2}$	$\theta+2\phi-\pi$		
p_{12}	$v_0,\ v_{\pm1},\ v_{\pm3}$	$\theta+2\phi-\pi$		

One can notice from the above table that $3\theta \geq \pi$ and $\theta + 2\phi \geq \pi$. This will be the case for all the angles we are interested in. Let's look at how we obtain the vertices:

(1) If v_0 and $v_{\pm1}$ coalesce, the triangle T_1 shrinks to a point and so $z_1 = 0$. If $v_{\pm2}$ and $v_{\pm3}$ coalesce then T_2 also shrinks to a point and so $z_2 = 0$. Thus p_1 is

given by $z_1 = z_2 = 0$.

$$p_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Putting $z_1 = z_2 = 0$ into (2.4) and (2.5) gives the following in terms of w. Thus for $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ 1 \end{bmatrix}$

$$w_1 = \frac{\sin\phi + \sin(\theta - \phi)}{\sin\phi + \sin\theta e^{-i\phi}}, \qquad w_2 = \frac{\sin\phi}{\sin\phi + \sin\theta e^{-i\phi}}.$$

(2) v_0 and $v_{\pm 3}$ coalesce, and, $v_{\pm 1}$ and $v_{\pm 2}$ also coalesce, implies $z_1 + z_2 = z_3 = 1$. This also means T_1 and T_2 enlarges to fill up T_3. That is $T_1 + T_2 = T_3$. Therefore $w_1 = w_2 = 0$. So from eqn (2.6) and (2.7).

$$z_1 = \frac{\sin\phi + \sin(\theta - \phi)}{\sin\phi + \sin\theta e^{i\phi}}, \qquad z_2 = \frac{\sin\phi e^{i\phi}}{\sin\phi + \sin\theta e^{i\phi}}.$$

$$p_2 = \begin{bmatrix} \frac{\sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin\theta e^{i\phi}} \\ \frac{\sin\phi e^{i\phi}}{\sin\phi + \sin\theta e^{i\phi}} \\ 1 \end{bmatrix}$$

(3) $v_{\pm 1}$, $v_{\pm 2}$ and $v_{\pm 3}$ coalesce. $v_{\pm 1}$ and $v_{\pm 2}$ coalescing implies $z_1 + z_2 = z_3 = 1$. Also $v_{\pm 2}$ and $v_{\pm 3}$ coalescing also implies $z_2 = 0$. That is T_2 shrinks to zero. Hence $z_1 = 1$.

$$p_{231} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$w_1 = 1, \qquad w_2 = 0$$

(4) v_0, $v_{\pm 2}$ and $v_{\pm 3}$ coalesce. Now for $v_{\pm 2}$ and $v_{\pm 3}$ coalescing shrinks T_2 to zero which implies $z_2 = 0$. $w_1 = 0$ From eqn (2.7): $-\sin\phi w_2 + \sin\phi e^{i\theta} = 0$. Which implies $w_2 = e^{i\theta}$

$$z_1 = (1 - e^{i\theta})\frac{\sin\phi + \sin(\theta - \phi)}{\sin\phi + \sin\theta e^{i\phi} - \sin(\theta + \phi)e^{i\theta}}$$

Now taking the denominator and simplifying, you obtain

$$\begin{aligned}
& \sin\phi + \sin\theta e^{i\phi} - \sin(\theta+\phi)e^{i\theta} \\
= & \sin\phi + \sin\theta\cos\phi + i\sin\theta\sin\phi - \sin\theta\cos\phi e^{i\theta} - \sin\phi\cos\theta e^{i\theta} \\
= & \sin\phi + \sin\theta\cos\phi(1 - e^{i\theta}) + i\sin\theta\sin\phi - i\sin\phi\sin\theta\cos\theta - \sin\phi\cos^2\theta \\
= & \sin\phi(1 - \cos^2\theta) + i\sin\theta\sin\phi(1 - \cos\theta) + \sin\theta\cos\phi(1 - e^{i\theta}) \\
= & \sin\phi\sin^2\theta + i\sin\theta\sin\phi(1 - \cos\theta) + \sin\theta\cos\phi(1 - e^{i\theta}) \\
= & i\sin\phi\sin\theta(1 - \cos\theta - i\sin\theta) + \sin\theta\cos\phi(1 - e^{i\theta}) \\
= & (i\sin\phi\sin\theta + \sin\theta\cos\phi)(1 - e^{i\theta}) \\
= & \sin\theta e^{\phi}(1 - e^{i\theta})
\end{aligned}$$

and substituting it back and simplifying, you obtain

$$z_1 = \frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta e^{i\phi}}$$

$$p_{23} = \begin{bmatrix} \frac{\sin\phi + \sin(\theta-\phi)}{\sin\theta e^{i\phi}} \\ 0 \\ 1 \end{bmatrix}$$

(5) v_0, $v_{\pm 1}$ and $v_{\pm 2}$ coalesce. T_1 shrinks to zero and the horizontal bases of T_2 and T_3 are equal. Hence $z_1 = 0$ and $z_2 = z_3 = 1$. Also $w_2 = 0$

$$p_{31} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$w_1 = (1 - e^{-i\theta})\frac{\sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin\theta e^{-i\phi}}$$

(6) v_0, $v_{\pm 1}$ and $v_{\pm 3}$ coalesce. $z_1 = 0$ and $w_1 = 0$,$w_2 = 1$

$$z_2 = \frac{-\sin\phi + \sin\phi e^{i\theta}}{-\sin(\theta+\phi) + \sin\phi + \sin\theta e^{i\phi}}$$

and simplifying, you obtain:

$$z_2 = e^{i\theta}.$$

$$p_{12} = \begin{bmatrix} 0 \\ e^{i\theta} \\ 1 \end{bmatrix}$$

In coordinates (normalising so last coordinate of $\mathbf{z}$ and $\mathbf{w}$ is 1) we have

Point	z_1	z_2	w_1	w_2
p_1	0	0	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\phi+\sin\theta e^{-i\phi}}$	$\frac{\sin\phi}{\sin\phi+\sin\theta e^{-i\phi}}$
p_2	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\phi+\sin\theta e^{i\phi}}$	$\frac{\sin\phi e^{i\theta}}{\sin\phi+\sin\theta e^{i\phi}}$	0	0
p_{231}	1	0	1	0
p_{23}	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta}\ e^{-i\phi}$	0	0	$e^{i\theta}$
p_{31}	0	1	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta}\ e^{i\phi}$	0
p_{12}	0	$e^{i\theta}$	0	1

In concluding this chapter, we show that the collection of vertices described above is symmetrical with respect to an involution. The polyhedron D will also demonstrate this symmetry when we get to chapter 4. Let us consider the antiholomorphic isometry ι given by $\iota(\mathbf{z}) = R_1R_2R_1(\bar{\mathbf{z}})$, which is the same as $\iota(\mathbf{z}) = PR_1(\bar{\mathbf{z}})$. Which is

$$
\begin{aligned}
\iota \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} &= \frac{1}{(1-e^{-i\theta})\sin(\phi)} \begin{bmatrix} -\sin(\theta)e^{-i\phi} & -e^{i\theta}(\sin(\phi)+\sin(\theta-\phi)) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi)e^{i\theta} & -e^{i\theta}\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta+\phi) & -e^{i\theta}\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix} \begin{bmatrix} \bar{z_1} \\ \bar{z_2} \\ 1 \end{bmatrix} \\
&\sim \begin{bmatrix} \bar{w_1} \\ \bar{w_2}e^{i\theta} \\ 1 \end{bmatrix}. \qquad (2.8)
\end{aligned}
$$

Notice that $\sim$ refers to projective equality. The following lemma deduced from the above equation can be verified using the vertices obtained in the above table of vertices.

Lemma 2.3.1. *The isometry ι has order 2 and acts on the p_j by*

$\iota(p_1) = p_2$, $\iota(p_{231}) = p_{231}$, $\iota(p_{23}) = p_{31}$, $\iota(p_{12}) = p_{12}$.

Proof. Consider $\iota(p_1) = p_2$. Using equation 2.8

$$
\iota \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{(1-e^{-i\theta})\sin(\phi)} \begin{bmatrix} \sin(\phi)+\sin(\theta-\phi) \\ \sin(\phi)e^{i\theta} \\ \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix}
$$

But from equation 2.3 $(1-e^{-i\theta})\sin(\phi) = \sin(\phi)+\sin(\theta)e^{i\phi}$, where $z_1 = z_2 = 0$ Hence

$$\iota(p_1) = \iota \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.9}$$

$$= \begin{bmatrix} \frac{\sin(\phi)+\sin(\theta-\phi)}{\sin(\phi)+\sin(\theta)e^{i\phi}} \\ \frac{\sin(\phi)e^{i\theta}}{\sin(\phi)+\sin(\theta)e^{i\phi}} \\ 1 \end{bmatrix} = p_2 \tag{2.10}$$

Hence the proof for the first one.

Now consider $\iota^2(p_1) = \iota(p_2)$. From equation 2.6, the results with simplification is

$$\frac{1}{(1-e^{-i\theta})\sin(\phi)} \begin{bmatrix} \frac{\sin(\phi)+\sin(\theta-\phi)}{\sin(\phi)+\sin(\theta)e^{i\phi}}[-\sin(\theta)e^{-i\phi} - \sin(\phi) - \sin(\theta-\phi) + \sin(\phi) + \sin(\theta)e^{i\phi}] \\ \frac{\sin(\phi)e^{i\theta}}{\sin(\phi)+\sin(\theta)e^{i\phi}}[-\sin(\phi) - \sin(\theta-\phi) + \sin(\theta)e^{-i\phi} + \sin(\phi) - \sin(\phi)e^{-i\theta}] \\ -\sin(\theta+\phi)\frac{\sin(\phi)+\sin(\theta-\phi)}{\sin(\phi)+\sin(\theta)e^{i\phi}} - \sin(\theta+\phi)\frac{\sin(\phi)}{\sin(\phi)+\sin(\theta)e^{i\phi}} + \sin(\phi) + \sin(\theta)e^{i\phi} \end{bmatrix}$$

Simplifying further also gives:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = p_1$$

Hence ι has order 2.

We now consider $\iota(p_{231}) = p_{231}$

$$\iota\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{1}{(1-e^{-i\theta})\sin(\phi)}\begin{bmatrix}-\sin(\theta)e^{-i\phi}+\sin(\phi)+\sin(\theta-\phi)\\ -\sin(\phi)e^{i\theta}+\sin(\phi)e^{i\theta}\\ -\sin(\theta+\phi)+\sin(\phi)+\sin(\theta)e^{i\phi}\end{bmatrix}$$

$$= \frac{1}{(1-e^{-i\theta})\sin(\phi)}\begin{bmatrix}(1-e^{-i\theta})\sin(\phi)\\ 0\\ (1-e^{-i\theta})\sin(\phi)\end{bmatrix} = p_{231}.$$

Consider $\iota(p_{23}) = p_{31}$

$$\iota(p_{23}) = \iota\begin{bmatrix}\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta}\,e^{-i\phi}\\ 0\\ 1\end{bmatrix}$$

$$= \frac{1}{(1-e^{-i\theta})\sin(\phi)}\begin{bmatrix}-\sin(\phi)-\sin(\theta-\phi)+\sin(\phi)+\sin(\theta-\phi)\\ -\sin(\phi)e^{i\theta}\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta}\,e^{-i\phi}+\sin(\phi)e^{i\theta}\\ -\sin(\theta+\phi)\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta}\,e^{-i\phi}+\sin(\phi)+\sin(\theta)e^{i\phi}\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} = p_{31}$$

$\square$

CHAPTER 3

CONSTRUCTION OF THE COMPLEX HYPERBOLIC POLYHEDRON D

3.1 INTRODUCTION

In this chapter we construct our polyhedron D. In constructing the polyhedron, we need the vertices (which we already have in the previous chapter), the edges and the faces which we are going to find in this chapter. The faces of the polyhedron will be contained in bisectors. We define a bisector as follows.

Definition (Bisector)**.** A Bisector, denoted B, is the locus of points in complex hyperbolic space equidistant from a given, pair of points in complex hyperbolic space, say z_j and z_k.

Using the standard formula for the distance function, we see that $z \in D$ if and only if

$$\cosh^2\left(\frac{\rho(\boldsymbol{p},\boldsymbol{p}_j)}{2}\right) = \cosh^2\left(\frac{\rho(\boldsymbol{p},\boldsymbol{p}_k)}{2}\right) \tag{3.1}$$

$$\frac{\langle \boldsymbol{p},\boldsymbol{p}_j\rangle \langle \boldsymbol{p}_j,\boldsymbol{p}\rangle}{\langle \boldsymbol{p},\boldsymbol{p}\rangle \langle \boldsymbol{p}_j,\boldsymbol{p}_j\rangle} = \frac{\langle \boldsymbol{p},\boldsymbol{p}_k\rangle \langle \boldsymbol{p}_k,\boldsymbol{p}\rangle}{\langle \boldsymbol{p},\boldsymbol{p}\rangle \langle \boldsymbol{p}_k,\boldsymbol{p}_k\rangle} \tag{3.2}$$

If $\boldsymbol{p}_j$ and $\boldsymbol{p}_k$ have the same norm, that is $\langle \boldsymbol{p}_j,\boldsymbol{p}_j\rangle = \langle \boldsymbol{p}_k,\boldsymbol{p}_k\rangle$, then the above reduces to

$$\langle \boldsymbol{p},\boldsymbol{p}_j\rangle \langle \boldsymbol{p}_j,\boldsymbol{p}\rangle = \langle \boldsymbol{p},\boldsymbol{p}_k\rangle \langle \boldsymbol{p}_k,\boldsymbol{p}\rangle$$

$$\left|\langle \boldsymbol{p},\boldsymbol{p}_j\rangle\right|^2 = \left|\langle \boldsymbol{p},\boldsymbol{p}_k\rangle\right|^2$$

$$\left|\langle \boldsymbol{p},\boldsymbol{p}_j\rangle\right| = \left|\langle \boldsymbol{p},\boldsymbol{p}_k\rangle\right|$$

Hence the definition of the bisector B becomes:

$$B = \left\{ \boldsymbol{p} \in \mathbf{H}^2_{\mathbb{C}} : \left| \langle \boldsymbol{p}, \boldsymbol{p}_j \rangle \right| = \left| \langle \boldsymbol{p}, \boldsymbol{p}_k \rangle \right| \right\} \tag{3.3}$$

We consider some properties of bisectors that we will need. Since there are no totally geodesic real hypersurfaces in Complex Hyperbolic Space, Bisectors being real hypersurfaces in Complex Hyperbolic Space are not totally geodesic(ie shortest paths between any two points in the bisector may not necessarily lie in complex hyperbolic space). Rather they are foliated by totally geodesic subspaces in two different ways [19]. The two subspaces are slices and meridians. The first section looks at the definition of the polyhedron. This is followed by the intersection of the Bisectors and then the faces of the polyhedron.

3.2 THE POLYHEDRON D

We define the polyhedron D to be those points of $\mathbf{H}^2_{\mathbb{C}}$ for which the arguments of z_1, z_2, w_1 and w_2 lie in the following intervals:

$$D = \left\{ \mathbf{z} = P(\mathbf{w}) \; : \; \begin{array}{ll} \arg(z_1) \in (-\phi, 0), & \arg(z_2) \in (0, \theta), \\ \arg(w_1) \in (0, \phi), & \arg(w_2) \in (0, \theta) \end{array} \right\}. \tag{3.4}$$

This is bounded by eight bisectors:

Bisector	Definition	equivalent definition
$B(J)$	$\arg(z_1) = -\phi$ or $-\phi + \pi$	$\mathrm{Im}\,(z_1 e^{i\phi}) = 0$
$B(J^{-1})$	$\arg(w_1) = \phi$ or $\phi + \pi$	$\mathrm{Im}\,(w_1 e^{-i\phi}) = 0$
$B(P)$	$\arg(z_1) = 0$ or π	$\mathrm{Im}\,(z_1) = 0$
$B(P^{-1})$	$\arg(w_1) = 0$ or π	$\mathrm{Im}\,(w_1) = 0$
$B(R_1)$	$\arg(z_2) = 0$ or π	$\mathrm{Im}\,(z_2) = 0$
$B(R_1^{-1})$	$\arg(z_2) = \theta$ or $\theta + \pi$	$\mathrm{Im}\,(z_2 e^{-i\theta}) = 0$
$B(R_2)$	$\arg(w_2) = 0$ or π	$\mathrm{Im}\,(w_2) = 0$
$B(R_2^{-1})$	$\arg(w_2) = \theta$ or $\theta + \pi$	$\mathrm{Im}\,(w_2 e^{-i\theta}) = 0$

We show that for each of these eight bisectors B :

(i) either the point p_1 or p_2 lies on B;

(ii) three of the four points p_{231}, p_{23}, p_{31} and p_{12} lie on B;

(iii) the fourth of these points lies on the complex spine Σ of B but not on B;

(iv) the spine of B passes through one of p_1 and p_2 and one of p_{231}, p_{23}, p_{31} or p_{12}.

1. Example, we consider $B(J)$. This is given by

$$B(J) = \Big\{(x_1 e^{-i\phi}, z_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ x_1 \in \mathbb{R},\ z_2 \in \mathbb{C}\Big\}.$$

The spine $\sigma(J)$ of $B(J)$ is

$$\sigma(J) = \Big\{(x_1 e^{-i\phi}, 0) \in \mathbf{H}^2_{\mathbb{C}} \ : \ x_1 \in \mathbb{R}\Big\}.$$

The complex spine $\Sigma(J)$ of $B(J)$ is

$$\Sigma(J) = \Big\{(z_1, 0) \in \mathbf{H}^2_{\mathbb{C}} \ : \ z_1 \in \mathbb{C}\Big\}.$$

(i) p_1 is given by $(z_1, z_2) = (0, 0)$. This clearly lies in $B(J)$ and $\sigma(J)$.

(ii) p_{231} is given by $(z_1, z_2) = (1, 0)$. This clearly does not lie on $B(J)$ but does lie on $\Sigma(J)$.

(iii) p_{23} is given by

$$(z_1, z_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta} \, e^{-i\phi}, 0 \right).$$

Since $\big(\sin\phi + \sin(\theta - \phi)\big)/\sin\theta$ is real, this lies on $B(J)$ and $\sigma(J)$.

(iv) p_{31} is given by $(z_1, z_2) = (0, 1)$. This clearly lies on $B(J)$ but does not lie on $\sigma(J)$.

(v) p_{12} is given by $(z_1, z_2) = (0, e^{i\theta})$. This clearly lies on $B(J)$ but does not lie on $\sigma(J)$.

2. Next we consider $B(J^{-1})$. This is given by

$$B(J^{-1}) = \Big\{ (y_1 e^{i\phi}, w_2) \in \mathbf{H}^2_{\mathbb{C}} \;:\; y_1 \in \mathbb{R},\; w_2 \in \mathbb{C} \Big\}.$$

The spine $\sigma(J^{-1})$ of $B(J^{-1})$ is

$$\sigma(J^{-1}) = \Big\{ (y_1 e^{i\phi}, 0) \in \mathbf{H}^2_{\mathbb{C}} \;:\; y_1 \in \mathbb{R} \Big\}.$$

The complex spine $\Sigma(J^{-1})$ of $B(J^{-1})$ is

$$\Sigma(J^{-1}) = \Big\{ (w_1, 0) \in \mathbf{H}^2_{\mathbb{C}} \;:\; w_1 \in \mathbb{C} \Big\}.$$

(i) p_2 is given by $(w_1, w_2) = (0, 0)$. This clearly lies in $B(J^{-1})$ and $\sigma(J^{-1})$.

(ii) p_{231} is given by $(w_1, w_2) = (1, 0)$. This does not lie on $B(J^{-1})$ but does lie on $\Sigma(J^{-1})$.

(iii) p_{23} is given by $(w_1, w_2) = (0, e^{i\theta})$. This lies on $B(J^{-1})$ and $\sigma(J^{-1})$.

(iv) p_{31} is given by $\left(\frac{\sin\phi + \sin(\theta-\phi)}{\sin\theta} \, e^{i\phi}, 0 \right)$. This clearly lies on $B(J^{-1})$ and on $\sigma(J^{-1})$.

(v) p_{12} is given by $(w_1, w_2) = (0, 1)$. This does lie on $B(J^{-1})$ but does not lie on $\sigma(J^{-1})$.

3. Again lets consider $B(P)$. This is given by

$$B(P) = \Big\{(x_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ x_1 \in \mathbb{R}, \ z_2 \in \mathbb{C}\Big\}.$$

The spine $\sigma(P)$ of $B(P)$ is

$$\sigma(P) = \Big\{(x_1, 0) \in \mathbf{H}^2_{\mathbb{C}} \ : \ x_1 \in \mathbb{R}\Big\}.$$

The complex spine $\Sigma(P)$ of $B(P)$ is

$$\Sigma(P) = \Big\{(z_1, 0) \in \mathbf{H}^2_{\mathbb{C}} \ : \ z_1 \in \mathbb{C}\Big\}.$$

(i) p_1 is given by $(z_1, z_2) = (0, 0)$. This lies on $B(P)$ and $\sigma(P)$. NB p_2 is given by

$$(z_1, z_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\phi + \sin\theta e^{i\phi}}, \frac{\sin\phi e^{i\phi}}{\sin\phi + \sin\theta e^{i\phi}}\right)$$

This does not lie on $B(P)$ since the $arg(z_1)$ is not equal to zero.

(ii) p_{231} is given by $(z_1, z_2) = (1, 0)$. This does lie on $B(P)$ and on $\sigma(P)$.

(iii) p_{23} is given by

$$(z_1, z_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta} e^{-i\phi}, 0\right).$$

Clearly does not lie on $B(P)$ and neither on $\sigma(P)$ but clearly does lie on $\Sigma(P)$.

(iv) p_{31} is given by $(z_1, z_2) = (0, 1)$. This clearly lies on $B(P)$ but does not lie on $\sigma(P)$.

(v) p_{12} is given by $(z_1, z_2) = (0, e^{i\theta})$. This clearly lies on $B(P)$ but does not lie on $\sigma(P)$.

4. Now lets consider $B(P^{-1})$. This is given by

$$B(P^{-1}) = \Big\{(y_1, w_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_1 \in \mathbb{R}, \ w_2 \in \mathbb{C}\Big\}.$$

The spine $\sigma(P^{-1})$ of $B(P^{-1})$ is

$$\sigma(P^{-1}) = \Big\{(y_1, 0) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_1 \in \mathbb{R}\Big\}.$$

The complex spine $\Sigma(P^{-1})$ of $B(P^{-1})$ is

$$\Sigma(P^{-1}) = \Big\{(w_1, 0) \in \mathbf{H}^2_{\mathbb{C}} \ : \ w_1 \in \mathbb{C}\Big\}.$$

(i) p_2 is given by $(w_1, w_2) = (0, 0)$. This lies on $B(P^{-1})$ and $\sigma(P^{-1})$.

(ii) p_{231} is given by $(w_1, w_2) = (1, 0)$. This does lie on $B(P^{-1})$ and on $\Sigma(P^{-1})$.

(iii) p_{23} is given by $(w_1, w_2) = (0, e^{i\theta})$. Clearly does lie on $B(P^{-1})$ but not on $\sigma(P^{-1})$.

(iv) p_{31} is given by $(w_1, w_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta} e^{i\phi}, 0\right)$. This does not lie on $B(P^{-1})$ and $\sigma(P^{-1})$ but does lie on $\Sigma(P^{-1})$.

(v) p_{12} is given by $(w_1, w_2) = (0, 1)$. This lies on $B(P^{-1})$ but does not lie on $\sigma(P^{-1})$.

5. Lets consider $B(R_1)$. This is given by

$$B(R_1) = \Big\{(z_1, y_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R}, \ z_1 \in \mathbb{C}\Big\}.$$

The spine $\sigma(R_1)$ of $B(R_1)$ is

$$\sigma(R_1) = \Big\{(0, y_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R}\Big\}.$$

The complex spine $\Sigma(R_1)$ of $B(R_1)$ is

$$\Sigma(R_1) = \Big\{(0, z_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ z_2 \in \mathbb{C}\Big\}.$$

(i) p_1 is given by $(z_1, z_2) = (0, 0)$. This lies on $B(R_1)$ and $\sigma(R_1)$.

(ii) p_{231} is given by $(z_1, z_2) = (1, 0)$. This does lie on $B(R_1)$ and on $\sigma(R_1)$.

(iii) p_{23} is given by

$$(z_1, z_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta} \, e^{-i\phi}, 0 \right).$$

This does lie on $B(R_1)$ and on $\sigma(R_1)$

(iv) p_{31} is given by $(z_1, z_2) = (0, 1)$. This clearly lies on $B(R_1)$ and on $\sigma(R_1)$.

(v) p_{12} is given by $(z_1, z_2) = (0, e^{i\theta})$. This does not lie on $B(R_1)$ and on $\sigma(R_1)$ but lies on $\Sigma(R_1)$.

6. Lets consider $B(R_1^{-1})$. This is given by

$$B(R_1^{-1}) = \Big\{ (z_1, y_2 e^{i\theta}) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R}, \ z_1 \in \mathbb{C} \Big\}.$$

The spine $\sigma(R_1^{-1})$ of $B(R_1^{-1})$ is

$$\sigma(R_1^{-1}) = \Big\{ (0, y_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R} \Big\}.$$

The complex spine $\Sigma(R_1^{-1})$ of $B(R_1^{-1})$ is

$$\Sigma(R_1^{-1}) = \Big\{ (0, z_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ z_2 \in \mathbb{C} \Big\}.$$

(i) p_1 is given by $(z_1, z_2) = (0, 0)$. This lies on $B(R_1^{-1})$ and on $\sigma(R_1^{-1})$.

(ii) p_{231} is given by $(z_1, z_2) = (1, 0)$. This does lie on $B(R_1^{-1})$ but not on $\sigma(R_1^{-1})$.

(iii) p_{23} is given by

$$(z_1, z_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta} \, e^{-i\phi}, 0 \right).$$

This lies on $B(R_1^{-1})$ but not on $\sigma(R_1^{-1})$

(iv) p_{31} is given by $(z_1, z_2) = (0, 1)$. This does not lie on $B(R_1^{-1})$ and also not on $\sigma(R_1^{-1})$ but lies on $\Sigma(R_1)$.

(v) p_{12} is given by $(z_1, z_2) = (0, e^{i\theta})$. This lies on $B(R_1^{-1})$ and on $\sigma(R_1^{-1})$.

7. Lets consider $B(R_2)$. This is given by

$$B(R_2) = \Big\{(w_1, y_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R}, \ w_1 \in \mathbb{C}\Big\}.$$

The spine $\sigma(R_2)$ of $B(R_2)$ is

$$\sigma(R_2) = \Big\{(0, y_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R}\Big\}.$$

The complex spine $\Sigma(R_2)$ of $B(R_2)$ is

$$\Sigma(R_2) = \Big\{(0, w_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ w_2 \in \mathbb{C}\Big\}.$$

(i) p_2 is given by $(w_1, w_2) = (0, 0)$. This lies on $B(R_2)$ and $\sigma(R_2)$.

(ii) p_{231} is given by $(w_1, w_2) = (1, 0)$. This does lie on $B(R_2)$ but not on $\sigma(R_2)$.

(iii) p_{23} is given by $(w_1, w_2) = (0, e^{i\theta})$. Clearly does not lie on $B(R_2)$ neither on $\sigma(R_2)$.

(iv) p_{31} is given by $(w_1, w_2) = \Big(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta}\, e^{i\phi}, 0\Big)$. This lies on $B(R_2)$ but not on $\sigma(R_2)$.

(v) p_{12} is given by $(w_1, w_2) = (0, 1)$. This lies on $B(R_2)$ and on $\sigma(R_2)$.

8. Lets consider $B(R_2^{-1})$. This is given by

$$B(R_2^{-1}) = \Big\{(w_1, y_2 e^{i\theta}) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R}, \ w_1 \in \mathbb{C}\Big\}.$$

The spine $\sigma(R_2^{-1})$ of $B(R_2^{-1})$ is

$$\sigma(R_2^{-1}) = \Big\{(0, y_2 e^{i\theta}) \in \mathbf{H}^2_{\mathbb{C}} \ : \ y_2 \in \mathbb{R}\Big\}.$$

The complex spine $\Sigma(R_2^{-1})$ of $B(R_2^{-1})$ is

$$\Sigma(R_2^{-1}) = \Big\{(0, w_2) \in \mathbf{H}^2_{\mathbb{C}} \ : \ w_2 \in \mathbb{C}\Big\}.$$

(i) p_2 is given by $(w_1, w_2) = (0, 0)$. This lies on $B(R_2^{-1})$ and $\sigma(R_2^{-1})$.

(ii) p_{231} is given by $(w_1, w_2) = (1, 0)$. This does lie on $B(R_2^{-1})$ but not on $\sigma(R_2^{-1})$.

(iii) p_{23} is given by $(w_1, w_2) = (0, e^{i\theta})$. This also lies on $B(R_2^{-1})$ and $\sigma(R_2^{-1})$.

(iv) p_{31} is given by $(w_1, w_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta} e^{i\phi}, 0\right)$. This lies on $B(R_2^{-1})$ but not on $\sigma(R_2^{-1})$.

(v) p_{12} is given by $(w_1, w_2) = (0, 1)$. This does not lie on both $B(R_2^{-1})$ and $\sigma(R_2^1)$ but lies on $\Sigma(R_2^1)$.

Summary of the above is as follows:

Bisector	Definition	Points on spine	Other points	Equidistant from
$B(J)$	$\arg(z_1) = -\phi$	$p_1,\ p_{23}$	$p_{31},\ p_{12}$	$p_{231},\ J^{-1}(p_{231})$
$B(J^{-1})$	$\arg(w_1) = \phi$	$p_2,\ p_{31}$	$p_{12},\ p_{23}$	$p_{231},\ J(p_{231})$
$B(P)$	$\arg(z_1) = 0$	$p_1,\ p_{231}$	$p_{31},\ p_{12}$	$p_{23},\ P^{-1}(p_{31})$
$B(P^{-1})$	$\arg(w_1) = 0$	$p_2,\ p_{231}$	$p_{12},\ p_{23}$	$p_{31},\ P(p_{23})$
$B(R_1)$	$\arg(z_2) = 0$	$p_1,\ p_{31}$	$p_{23},\ p_{231}$	$p_{12},\ R_1^{-1}(p_{31})$
$B(R_1^{-1})$	$\arg(z_2) = \theta$	$p_1,\ p_{12}$	$p_{23},\ p_{231}$	$p_{31},\ R_1(p_{12})$
$B(R_2)$	$\arg(w_2) = 0$	$p_2,\ p_{12}$	$p_{31},\ p_{231}$	$p_{23},\ R_2^{-1}(p_{12})$
$B(R_2^{-1})$	$\arg(w_2) = \theta$	$p_2,\ p_{23}$	$p_{31},\ p_{231}$	$p_{12},\ R_2(p_{23})$

3.3 INTERSECTION OF BISECTORS

In this section we first consider the intersection of bisectors in which both do not pass through p_1 or p_2

1. Suppose $\mathbf{z} \in \mathbf{B}(\mathbf{J}) \cap \mathbf{B}(\mathbf{J^{-1}})$ $z_1 = xe^{-i\phi}$ and $w_1 = ue^{i\phi}$. From the definitions of z_1 and w_1, we have the following

$$
\begin{aligned}
xe^{-i\phi} &= \frac{-u\sin\theta - (\sin\phi + \sin(\theta-\phi))w_2 + \sin\phi + \sin(\theta-\phi)}{-u\sin(\theta+\phi)e^{i\phi} - \sin(\theta+\phi)w_2 + \sin\phi + \sin\theta e^{i\phi}}, \\
ue^{i\phi} &= \frac{-x\sin\theta - e^{-i\theta}(\sin\phi + \sin(\theta-\phi))z_2 + \sin\phi + \sin(\theta-\phi)}{-x\sin(\theta+\phi)e^{-i\phi} - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin\phi + e^{-i\phi}\sin\theta}.
\end{aligned}
$$

Solving for w_2 and z_2

$$
\begin{aligned}
w_2 &= \frac{xu\sin(\theta+\phi) - x(e^{-i\phi}\sin\phi + \sin\theta) - u\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}}, \\
z_2 &= e^{i\theta}\frac{xu\sin(\theta+\phi) - u(e^{i\phi}\sin\phi + \sin\theta) - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - ue^{i\phi}\sin(\theta+\phi)}.
\end{aligned}
$$

2. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J}) \cap \mathbf{B}(\mathbf{P^{-1}})$ then $z_1 = xe^{-i\phi}$ and $w_1 = u$. By definition, w_1 and z_1 are as follows

$$
\begin{aligned}
xe^{-i\phi} &= \frac{-\sin\theta e^{-i\phi}u - (\sin\phi + \sin(\theta-\phi))w_2 + \sin\phi + \sin(\theta-\phi)}{-\sin(\theta+\phi)u - \sin(\theta+\phi)w_2 + \sin\phi + \sin\theta e^{i\phi}}, \\
u &= \frac{-x\sin\theta - e^{-i\theta}(\sin\phi + \sin(\theta-\phi))z_2 + \sin\phi + \sin(\theta-\phi)}{-x\sin(\theta+\phi)e^{-i\phi} - e^{-i\theta}\sin(\theta+\phi)z_2 + \sin\phi + \sin\theta e^{-i\phi}}.
\end{aligned}
$$

Solving for w_2 and z_2

$$
\begin{aligned}
w_2 &= \frac{xue^{-i\phi}\sin(\theta+\phi) - x(e^{-i\phi}\sin\phi + \sin\theta) - ue^{-i\phi}\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}}, \\
z_2 &= e^{i\theta}\frac{xue^{-i\phi}\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - u\sin(\theta+\phi)}.
\end{aligned}
$$

3. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J}) \cap \mathbf{B}(\mathbf{R_2})$ then $z_1 = xe^{-i\phi}$ and $w_2 = u$. By definition, w_2 and z_1 are

as follows

$$\begin{aligned} xe^{-i\phi} &= \frac{-\sin\theta e^{-i\phi}w_1 + (1-u)(\sin\phi + \sin(\theta-\phi))}{-\sin(\theta+\phi)w_1 - u\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ u &= \frac{-x\sin\theta e^{-i\phi} - z_2\sin\phi + \sin\phi}{-x\sin(\theta+\phi)e^{-i\phi} - e^{-i\theta}\sin(\theta+\phi)z_2 + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_2 respectively

$$\begin{aligned} w_1 &= \frac{xu\sin(\theta+\phi) - x(\sin\phi + e^{i\phi}\sin\theta) + (1-u)(\sin\phi + \sin(\theta-\phi))e^{i\phi}}{\sin\theta - x\sin(\theta+\phi)}, \\ z_2 &= \frac{xu\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{e^{-i\phi}(\sin\phi + \sin(\theta-\phi)) - ue^{-i\phi}\sin(\theta+\phi)}. \end{aligned}$$

4. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J}) \cap \mathbf{B}(\mathbf{R_2^{-1}})$ then $z_1 = xe^{-i\phi}$ and $w_2 = ue^{i\theta}$. By definition, z_1 and w_2 are as follows

$$\begin{aligned} xe^{-i\phi} &= \frac{-e^{-i\phi}w_1\sin\theta + (1-ue^{i\theta})(\sin\phi + \sin(\theta-\phi))}{-\sin(\theta+\phi)w_1 - ue^{i\theta}\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ ue^{i\theta} &= \frac{-xe^{-i\phi}\sin\phi - z_2\sin\phi + \sin\phi}{-x\sin(\theta+\phi)e^{-i\phi} - e^{-i\theta}\sin(\theta+\phi)z_2 + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_2 respectively

$$\begin{aligned} w_1 &= \frac{xue^{i(\theta-\phi)}\sin(\theta+\phi) - x(e^{-i\phi}\sin\phi + e^{i(\theta-\phi)}\sin\theta) + (1-ue^{i\theta})(\sin\phi + \sin(\theta-\phi))}{e^{-i\phi}\sin\theta - x\sin(\theta+\phi)e^{-i\phi}}, \\ z_2 &= \frac{xue^{i(\theta-\phi)}\sin(\theta+\phi) - u(e^{i(\theta-\phi)}\sin\theta + e^{i\theta}\sin\phi) - xe^{-i\phi}\sin\phi + \sin\phi}{\sin\phi - u\sin(\theta+\phi)}. \end{aligned}$$

5. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J^{-1}}) \cap \mathbf{B}(\mathbf{P})$ then $w_1 = ue^{i\phi}$ and $z_1 = x$. By definition, w_1 and z_1 are respectively as follows

$$\begin{aligned} ue^{i\phi} &= \frac{-xe^{i\phi}\sin\theta - z_2e^{-i\theta}(\sin\phi + \sin(\theta-\phi)) + \sin\phi + \sin(\theta-\phi)}{-x\sin(\theta+\phi) - z_2e^{-i\theta}\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}, \\ x &= \frac{-u\sin\theta - w_2(\sin\phi + \sin(\theta-\phi)) + \sin\phi + \sin(\theta-\phi)}{-u\sin(\theta+\phi)e^{i\phi} - \sin(\theta+\phi)w_2 + \sin\phi + e^{i\phi}\sin\theta}. \end{aligned}$$

Solving for z_2 and w_2 respectively

$$\begin{aligned} z_2 &= e^{i\theta}\frac{xue^{i\phi}\sin(\theta+\phi)-u(e^{i\phi}\sin\phi+\sin\theta)-xe^{i\phi}\sin\theta+\sin\phi+\sin(\theta-\phi)}{\sin(\theta-\phi)+\sin\phi-ue^{i\phi}\sin(\theta+\phi)}, \\ w_2 &= \frac{xue^{i\phi}\sin(\theta+\phi)+x(e^{i\phi}\sin\theta+\sin\phi)-u\sin\theta+\sin\phi+\sin(\theta-\phi)}{\sin\phi+\sin(\theta-\phi)-x\sin(\theta+\phi)}. \end{aligned}$$

6. Let $\mathbf{z}\in\mathbf{B}(\mathbf{J^{-1}})\cap\mathbf{B}(\mathbf{R_1})$ then $w_1=ue^{i\phi}$ and $z_2=x$. By definition, w_1 and z_2 are respectively as follows

$$\begin{aligned} ue^{i\phi} &= \frac{-z_1e^{i\phi}\sin\theta+(1-xe^{-i\theta})(\sin\phi+\sin(\theta-\phi))}{-z_1\sin(\theta+\phi)-xe^{-i\theta}\sin(\theta+\phi)+\sin\phi+e^{-i\phi}\sin\theta}, \\ x &= \frac{-ue^{i(\theta+\phi)}\sin\phi-w_2\sin\phi+e^{i\theta}\sin\phi}{-u\sin(\theta+\phi)e^{i\phi}-\sin(\theta+\phi)w_2+\sin\phi+e^{i\phi}\sin\theta}. \end{aligned}$$

Solving for z_1 and w_2 respectively

$$\begin{aligned} z_1 &= \frac{xue^{-i(\theta-\phi)}\sin(\theta+\phi)-u(e^{i\phi}\sin\phi+\sin\theta)+(1-xe^{-i\phi})(\sin\phi)+\sin(\theta-\phi)}{e^{i\phi}\sin\theta-ue^{i\phi}\sin(\theta+\phi)}, \\ w_2 &= \frac{xue^{i\phi}\sin(\theta+\phi)+e^{i\theta}\sin\phi-ue^{i(\theta+\phi)}\sin\phi-x(\sin\phi+e^{i\phi}\sin\theta)}{\sin\phi-x\sin(\theta+\phi)}. \end{aligned}$$

7. Let $\mathbf{z}\in\mathbf{B}(\mathbf{J^{-1}})\cap\mathbf{B}(\mathbf{R_1^{-1}})$ then $w_1=ue^{i\phi}$ and $z_2=xe^{i\theta}$. By definition, w_1 and z_2 are respectively as follows

$$\begin{aligned} ue^{i\phi} &= \frac{-z_1e^{i\phi}\sin\theta-(x+1)(\sin\phi+\sin(\theta-\phi))}{-z_1\sin(\theta+\phi)-x\sin(\theta+\phi)+\sin\phi+e^{-i\phi}\sin\theta}, \\ xe^{i\theta} &= \frac{-ue^{i(\theta+\phi)}\sin\phi-w_2\sin\phi+e^{i\theta}\sin\phi}{-u\sin(\theta+\phi)e^{i\phi}-\sin(\theta+\phi)w_2+\sin\phi+e^{i\phi}\sin\theta}. \end{aligned}$$

Solving for z_1 and w_2 respectively

$$\begin{aligned} z_1 &= \frac{xue^{i\phi}\sin(\theta+\phi)-u(e^{i\phi}\sin\phi+\sin\theta)-(x+1)(\sin\phi+\sin(\theta-\phi))}{e^{i\phi}\sin\theta-u\sin(\theta+\phi)e^{i\phi}}, \\ w_2 &= \frac{xue^{i(\theta+\phi)}\sin(\theta+\phi)-x\sin\theta(e^{i\theta}+e^{i(\theta+\phi)})-(ue^{i(\theta+\phi)}+e^{i\theta})\sin\phi}{\sin\phi-xe^{i\theta}\sin(\theta+\phi)}. \end{aligned}$$

8. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P}) \cap \mathbf{B}(\mathbf{P^{-1}})$ then $z_1 = x$ and $w_1 = u$. By definition, z_1 and w_1 are respectively as follows

$$\begin{aligned} x &= \frac{-ue^{-i\phi}\sin\theta - w_2(\sin\phi + \sin(\theta-\phi)) + \sin\phi + \sin(\theta-\phi)}{-u\sin(\theta+\phi) - w_2\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ u &= \frac{-xe^{i\phi}\sin\theta - z_2e^{-i\theta}(\sin\phi + \sin(\theta-\phi)) + \sin\phi + \sin(\theta-\phi)}{-x\sin(\theta+\phi) - z_2e^{-i\theta}\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_2 and z_2 respectively, we have

$$\begin{aligned} w_2 &= \frac{xu\sin(\theta+\phi) - x(\sin\phi + e^{i\phi}\sin\theta) - ue^{-i\phi}\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)}, \\ z_2 &= e^{i\theta}\frac{xu\sin(\theta+\phi) - xe^{i\phi}\sin\theta - u(\sin\phi + e^{i\phi}\sin\theta) + \sin\phi + \sin(\theta+\phi)}{\sin\phi + \sin(\theta-\phi) - u\sin(\theta+\phi)}. \end{aligned}$$

9. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P}) \cap \mathbf{B}(\mathbf{R_2})$ then $z_1 = x$ and $w_2 = u$. By definition, z_1 and w_2 are respectively as follows

$$\begin{aligned} x &= \frac{-w_1e^{-i\phi}\sin\theta + (1-u)(\sin\phi + \sin(\theta-\phi))}{-w_1\sin(\theta+\phi) - u\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ u &= \frac{-x\sin\phi - z_2\sin\phi + \sin\phi}{-x\sin(\theta+\phi) - e^{-i\theta}\sin(\theta+\phi)z_2 + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_2 respectively

$$\begin{aligned} w_1 &= \frac{xu\sin(\theta+\phi) - x(e^{i\phi}\sin\theta + \sin\phi) + (1-u)(\sin(\theta-\phi) + \sin\phi)}{e^{-i\phi}\sin\theta - x\sin(\theta+\phi)}, \\ z_2 &= \frac{xu\sin(\theta+\phi) - u(\sin\phi + e^{-i\phi}\sin\theta) + (1-x)\sin\phi}{\sin\phi - ue^{-i\theta}\sin(\theta+\phi)}. \end{aligned}$$

10. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P}) \cap \mathbf{B}(\mathbf{R_2^{-1}})$ then $z_1 = x$ and $w_2 = ue^{i\theta}$. By definition, z_1 and w_2 are respectively as follows

$$\begin{aligned} x &= \frac{-w_1e^{-i\phi}\sin\theta + (1-ue^{-i\theta})(\sin\phi + \sin(\theta-\phi))}{-w_1\sin(\theta+\phi) - ue^{i\theta}\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ ue^{i\theta} &= \frac{-x\sin\phi - z_2\sin\phi + \sin\phi}{-x\sin(\theta+\phi) - e^{-i\theta}\sin(\theta+\phi)z_2 + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_2 respectively

$$\begin{aligned}
w_1 &= \frac{xue^{i\theta}\sin(\theta+\phi) - x(e^{i\phi}\sin\theta + \sin\phi) + (1-ue^{i\theta})(\sin(\theta-\phi)+\sin\phi)}{e^{-i\phi}\sin\theta - x\sin(\theta+\phi)},\\
z_2 &= \frac{xue^{i\theta}\sin(\theta+\phi) - u(e^{i\theta}\sin\phi + e^{i(\theta-\phi)}\sin\theta) + (1-x)\sin\phi}{\sin\phi - u\sin(\theta+\phi)}.
\end{aligned}$$

11. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P^{-1}}) \cap \mathbf{B}(\mathbf{R_1})$ then $z_2 = x$ and $w_1 = u$. By definition, z_2 and w_1 are respectively as follows

$$\begin{aligned}
x &= \frac{-ue^{i\theta}\sin\phi - w_2\sin\phi + e^{i\theta}\sin\phi}{-u\sin(\theta+\phi) - w_2\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta},\\
u &= \frac{-z_1e^{i\phi}\sin\theta + (1-xe^{-i\theta})(\sin\phi + \sin(\theta-\phi))}{-z_1\sin(\theta+\phi) - xe^{-i\theta}\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}.
\end{aligned}$$

Solving for w_2 and z_1 respectively

$$\begin{aligned}
w_2 &= \frac{xu\sin(\theta+\phi) - x(e^{i\phi}\sin\theta + \sin\phi) + (1-u)e^{i\theta}\sin\phi}{\sin\phi - x\sin(\theta+\phi)},\\
z_1 &= \frac{xue^{-i\theta}\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) + (1-xe^{-i\theta})(\sin\phi + \sin(\theta-\phi))}{e^{i\phi}\sin\theta - u\sin(\theta+\phi)}.
\end{aligned}$$

12. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P^{-1}}) \cap \mathbf{B}(\mathbf{R_1^{-1}})$ then $z_2 = xe^{i\theta}$ and $w_1 = u$. By definition, z_2 and w_1 are respectively as follows

$$\begin{aligned}
xe^{i\theta} &= \frac{-ue^{i\theta}\sin\phi - w_2\sin\phi + e^{i\theta}\sin\phi}{-u\sin(\theta+\phi) - w_2\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta},\\
u &= \frac{-z_1e^{i\phi}\sin\theta + (1-x)(\sin\phi + \sin(\theta-\phi))}{-z_1\sin(\theta+\phi) - x\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}.
\end{aligned}$$

Solving for w_2 and z_1 respectively

$$\begin{aligned}
w_2 &= \frac{xue^{i\theta}\sin(\theta+\phi) - (xe^{i(\phi+\theta)}\sin\theta + e^{i\theta}\sin\phi) + (1-u)e^{i\theta}\sin\phi}{\sin\phi - xe^{i\theta}\sin(\theta+\phi)},\\
z_1 &= \frac{xu\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) + (1-x)(\sin\phi + \sin(\theta-\phi))}{e^{i\phi}\sin\theta - u\sin(\theta+\phi)}.
\end{aligned}$$

13. Let $\mathbf{z} \in \mathbf{B}(\mathbf{R_1}) \cap \mathbf{B}(\mathbf{R_2})$ then $z_2 = x$ and $w_2 = u$. By definition, z_2 and w_2 are respectively as follows

$$\begin{aligned} x &= \frac{-w_1 e^{i\theta}\sin\phi + (e^{i\theta} - u)\sin\phi}{-w_1\sin(\theta+\phi) - u\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ u &= \frac{-z_1\sin\phi + (1-x)\sin\phi}{-z_1\sin(\theta+\phi) - xe^{-i\theta}\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_1 respectively

$$\begin{aligned} w_1 &= \frac{xu\sin(\theta+\phi) - x(e^{i\phi}\sin\theta + \sin\phi) + (e^{i\theta} - u)\sin\phi}{e^{i\theta}\sin\phi - x\sin(\theta+\phi)}, \\ z_1 &= \frac{xue^{-i\theta}\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) + (1-x)\sin\phi}{\sin\phi - u\sin(\theta+\phi)}. \end{aligned}$$

14. Let $\mathbf{z} \in \mathbf{B}(\mathbf{R_1}) \cap \mathbf{B}(\mathbf{R_2^{-1}})$ then $z_2 = x$ and $w_2 = ue^{i\theta}$. By definition, z_2 and w_2 are respectively as follows

$$\begin{aligned} x &= \frac{-w_1 e^{i\theta}\sin\phi + (1-u)e^{i\theta}\sin\phi}{-w_1\sin(\theta+\phi) - ue^{i\theta}\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ ue^{i\theta} &= \frac{-z_1\sin\phi + (1-x)\sin\phi}{-z_1\sin(\theta+\phi) - xe^{-i\theta}\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_1 respectively

$$\begin{aligned} w_1 &= \frac{xue^{i\theta}\sin(\theta+\phi) - x(e^{i\phi}\sin\theta + \sin\phi) + (1-u)e^{i\theta}\sin\phi}{e^{i\theta}\sin\phi - x\sin(\theta+\phi)}, \\ z_1 &= \frac{xu\sin(\theta+\phi) - u(e^{i(\theta-\phi)}\sin\theta + e^{i\theta}\sin\phi) + (1-x)\sin\phi}{\sin\phi - ue^{i\theta}\sin(\theta+\phi)}. \end{aligned}$$

15. Let $\mathbf{z} \in \mathbf{B}(\mathbf{R_1^{-1}}) \cap \mathbf{B}(\mathbf{R_2})$ then $z_2 = xe^{i\theta}$ and $w_2 = u$. By definition, z_2 and w_2 are respectively as follows

$$\begin{aligned} xe^{i\theta} &= \frac{-w_1 e^{i\theta}\sin\phi - u\sin\phi + e^{i\theta}\sin\phi}{-w_1\sin(\theta+\phi) - u\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ u &= \frac{-z_1\sin\phi + (1 - xe^{i\theta})\sin\phi}{-z_1\sin(\theta+\phi) - x\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_1 respectively

$$\begin{aligned} w_1 &= \frac{xue^{i\theta}\sin(\theta+\phi) - x(e^{i(\theta+\phi)}\sin\theta + e^{i\theta}\sin\phi) + (e^{i\theta}-u)\sin\phi}{e^{i\theta}\sin\phi - xe^{i\theta}\sin(\theta+\phi)}, \\ z_1 &= \frac{xu\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) + (1-xe^{i\theta})\sin\phi}{\sin\phi - u\sin(\theta+\phi)}. \end{aligned}$$

16. Let $\mathbf{z} \in \mathbf{B}(\mathbf{R_1^{-1}}) \cap \mathbf{B}(\mathbf{R_2^{-1}})$ then $z_2 = xe^{i\theta}$ and $w_2 = ue^{i\theta}$. By definition, z_2 and w_2 are respectively as follows

$$\begin{aligned} xe^{i\theta} &= \frac{-w_1e^{i\theta}\sin\phi + (1-u)e^{i\theta}\sin\phi}{-w_1\sin(\theta+\phi) - ue^{i\theta}\sin(\theta+\phi) + \sin\phi + e^{i\phi}\sin\theta}, \\ ue^{i\theta} &= \frac{-z_1\sin\phi + (1-xe^{i\theta})\sin\phi}{-z_1\sin(\theta+\phi) - x\sin(\theta+\phi) + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_1 and z_1 respectively

$$\begin{aligned} w_1 &= \frac{xue^{2i\theta}\sin(\theta+\phi) - x(e^{i(\theta+\phi)}\sin\theta + e^{i\theta}\sin\phi) + (1-u)e^{i\theta}\sin\phi}{e^{i\theta}\sin\phi - xe^{i\theta}\sin(\theta+\phi)}, \\ z_1 &= \frac{xue^{i\theta}\sin(\theta+\phi) - u(e^{i(\theta-\phi)}\sin\theta + e^{i\theta}\sin\phi) + (1-xe^{i\theta})\sin\phi}{\sin\phi - ue^{i\theta}\sin(\theta+\phi)}. \end{aligned}$$

The other intersections are the ones in which both bisectors either pass through p_1 or p_2

17.Let $\mathbf{z} \in \mathbf{B}(\mathbf{R_2}) \cap \mathbf{B}(\mathbf{R_2^{-1}})$ then $w_2 = x$ and $w_2 = ue^{i\theta}$

18. Let $\mathbf{z} \in \mathbf{B}(\mathbf{R_1}) \cap \mathbf{B}(\mathbf{R_1^{-1}})$ then $z_2 = x$ and $z_2 = ue^{i\theta}$

19. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P^{-1}}) \cap \mathbf{B}(\mathbf{R_2})$ then $w_1 = u$ and $w_2 = x$

20. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P^{-1}}) \cap \mathbf{B}(\mathbf{R_2^{-1}})$ then $w_1 = u$ and $w_2 = xe^{i\theta}$

21. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P}) \cap \mathbf{B}(\mathbf{R_1})$ then $z_1 = x$ and $z_2 = u$

22. Let $\mathbf{z} \in \mathbf{B}(\mathbf{P}) \cap \mathbf{B}(\mathbf{R_1^{-1}})$ then $z_1 = x$ and $z_2 = ue^{i\theta}$

23. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J^{-1}}) \cap \mathbf{B}(\mathbf{P^{-1}})$ then $w_1 = ue^{i\theta}$ and $w_1 = x$

24. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J^{-1}}) \cap \mathbf{B}(\mathbf{R_2})$ then $w_1 = ue^{i\phi}$ and $w_2 = x$

25. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J^{-1}}) \cap \mathbf{B}(\mathbf{R_2^{-1}})$ then $w_1 = ue^{i\phi}$ and $w_2 = xe^{i\theta}$

26. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J}) \cap \mathbf{B}(\mathbf{P})$ then $z_1 = xe^{-i\phi}$ and $z_1 = u$

27. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J}) \cap \mathbf{B}(\mathbf{R_1})$ then $z_1 = xe^{-i\phi}$ and $z_2 = u$

28. Let $\mathbf{z} \in \mathbf{B}(\mathbf{J}) \cap \mathbf{B}(\mathbf{R_1^{-1}})$ then $z_1 = xe^{-i\phi}$ and $z_2 = ue^{i\theta}$

We now look at the edges obtained when the bisectors intersect as well as the points intersecting. If all points on the spine of the bisectors intersect then we have a Meridian, otherwise a Slice.

Intersections of Bisectors		
Intersections to Consider	Points of Intersection Edges of Intersection	Do all points on Spine intersect Slice(S) or Meridian(M)
$B(J) \cap B(J^{-1})$	p_{12}, p_{23}, p_{31} $p_{23}p_{12}, p_{23}p_{31}, p_{31}p_{12}$	$Notall, hence : Slice(S)$ $\gamma_{p_{23}p_{12}}, \gamma_{p_{23}p_{31}}, \gamma_{p_{31}p_{12}}$
$B(J) \cap B(P)$	p_1, p_{31}, p_{12} $p_1p_{31}, p_1p_{12}, p_{31}p_{12}$	$Notall : Sfor$ $\gamma_{p_1p_{31}}, \gamma_{p_1p_{12}}, \gamma_{p_{31}p_{12}}$
$B(J) \cap B(P^{-1})$	p_{23}, p_{12} $p_{23}p_{12}$	$Notall : Sfor$ $\gamma_{p_{23}p_{12}}$
$B(J) \cap B(R_1)$	p_1, p_{23}, p_{31} $p_1p_{23}, p_1p_{31}, p_{23}p_{31}$	$All : Mfor$ $\gamma_{p_1p_{23}}, \gamma_{p_1p_{31}}$
$B(J) \cap B(R_1^{-1})$	p_1, p_{23}, p_{31} $p_1p_{23}, p_1p_{31}, p_{23}p_{31}$	$All : Mfor$ $\gamma_{p_1p_{23}}, \gamma_{p_1p_{31}}$
$B(J) \cap B(R_2)$	p_{12}, p_{31} $p_{12}p_{31}$	$Notall : Sfor$ $\gamma_{p_{12}p_{31}}$
$B(J) \cap B(R_2^{-1})$	p_{23}, p_{31} $p_{23}p_{31}$	$Notall : Sfor$ $\gamma_{p_{23}p_{31}}$
$B(J^{-1}) \cap B(P)$	p_{12}, p_{31} $p_{12}p_{31}$	$Notall : Sfor$ $\gamma_{p_{12}p_{31}}$

$B(J^{-1}) \cap B(P^{-1})$	p_2, p_{12}, p_{23} $p_2p_{12}, p_2p_{23}, p_{12}p_{23}$	$Notall : Sfor$ $\gamma_{p_2p_{12}} \text{or} \gamma_{p_2p_{23}}$
$B(J^{-1}) \cap B(R_1)$	p_{31}, p_{23} $p_{31}p_{23}$	$Notall : Sfor$ $\gamma_{p_{23}p_{31}}$
$B(J^{-1}) \cap B(R_1^{-1})$	p_{12}, p_{23} $p_{12}p_{23}$	$Notall : Sfor$ $\gamma_{p_{12}p_{23}}$
$B(J^{-1}) \cap B(R_2)$	p_2, p_{31}, p_{12} $p_2p_{31}, p_2p_{12}, p_{31}p_{12}$	$All : Mfor$ $\gamma_{p_2p_{31}}, \gamma_{p_2p_{12}}$
$B(J^{-1}) \cap B(R_2^{-1})$	p_{23}, p_{31} $p_{23}p_{31}$	$Notall : Sfor$ $\gamma_{p_{23}p_{31}}$
$B(P) \cap B(P^{-1})$	p_{231}, p_{12} $p_{231}p_{12}$	$Notall : Sfor$ $\gamma_{p_{231}p_{12}}$
$B(P) \cap B(R_1)$	p_1, p_{231}, p_{31} $p_1p_{231}, p_1p_{31}, p_{231}p_{31}$	$All : Mfor$ $\gamma_{p_1p_{231}}, \gamma_{p_1p_{31}}$
$B(P) \cap B(R_1^{-1})$	p_1, p_{231}, p_{12} $p_1p_{231}, p_1p_{12}, p_{231}p_{12}$	$All : Mfor$ $\gamma_{p_1p_{231}}, \gamma_{p_1p_{12}}$
$B(P) \cap B(R_2)$	p_{231}, p_{12} $p_{231}p_{12}$	$Notall : Sfor$ $\gamma_{p_{231}p_{12}}$
$B(P) \cap B(R_2^{-1})$	p_{231}, p_{12} $p_{231}p_{12}$	$Notall : Sfor$ $\gamma_{p_{231}p_{12}}$

$B(P^{-1}) \cap B(R_1)$	p_{231}, p_{23} $p_{231}p_{23}$	$Notall : Sfor$ $\gamma_{p_{231}p_{23}}$
$B(P^{-1}) \cap B(R_1^{-1})$	p_{12}, p_{23}, p_{231} $p_{12}p_{23}, p_{12}p_{231}, p_{23}p_{231}$	$Notall : Sfor$ $\gamma_{p_{12}p_{23}}, \gamma_{p_{12}p_{231}}, \gamma_{p_{23}p_{231}}$
$B(P^{-1}) \cap B(R_2)$	p_2, p_{12}, p_{231} $p_2p_{12}, p_2p_{231}, p_{12}p_{231}$	$All : Mfor$ $\gamma_{p_2p_{12}}, \gamma_{p_2p_{231}}$
$B(P^{-1}) \cap B(R_2^{-1})$	p_2, p_{23}, p_{231} $p_2p_{23}, p_2p_{231}, p_{23}p_{231}$	$All : Mfor$ $\gamma_{p_2p_{23}}, \gamma_{p_2p_{231}}$
$B(R_1) \cap B(R_1^{-1})$	p_1, p_{23}, p_{231} $p_1p_{23}, p_1p_{231}, p_{23}p_{231}$	$Notall : Sfor$ $\gamma_{p_1p_{23}}, \gamma_{p_1p_{231}}, \gamma_{p_{23}p_{231}}$
$B(R_1) \cap B(R_2)$	p_{31}, p_{231} $p_{31}p_{231}$	$Notall : Sfor$ $\gamma_{p_{31}p_{231}}$
$B(R_1) \cap B(R_2^{-1})$	p_{31}, p_{231}, p_{23} $p_{31}p_{231}, p_{31}p_{23}, p_{231}p_{23}$	$Notall : Sfor$ $\gamma_{p_{31}p_{231}}, \gamma_{p_{31}p_{23}}, \gamma_{p_{231}p_{23}}$
$B(R_1^{-1}) \cap B(R_2)$	p_{12}, p_{231} $p_{12}p_{231}$	$Notall : Sfor$ $\gamma_{p_{12}p_{231}}$
$B(R_1^{-1}) \cap B(R_2^{-1})$	p_{23}, p_{231} $p_{23}p_{231}$	$Notall : Sfor$ $\gamma_{p_{23}p_{231}}$
$B(R_2) \cap B(R_2^{-1})$	p_2, p_{31}, p_{231} $p_2p_{231}, p_2p_{31}, p_{31}p_{231}$	$Notall : Sfor$ $\gamma_{p_2p_{231}}, \gamma_{p_2p_{31}}, \gamma_{p_{31}p_{231}}$

We define the side S of D to be the intersection of $\bar{D}$ with the bisector B.

The summary of the edges and the bisectors which form them as well as whether they are meridians or slices are as follows:

Edges	Bisectors	
$\gamma_{p_1p_{23}}$	$B(J)\cap B(R_1)$	M
$\gamma_{p_1p_{231}}$	$B(P)\cap B(R_1)$	M
$\gamma_{p_1p_{31}}$	$B(J)\cap B(R_1)$	M
$\gamma_{p_1p_{12}}$	$B(J)\cap B(R_1^{-1})$	M
$\gamma_{p_2p_{31}}$	$B(J^{-1})\cap B(R_2)$	M
$\gamma_{p_2p_{231}}$	$B(P^{-1})\cap B(R_2^{-1})$	M
$\gamma_{p_2p_{12}}$	$B(J^{-1})\cap B(R_2)$	M
$\gamma_{p_2p_{23}}$	$B(P^{-1})\cap B(R_2^{-1})$	M
$\gamma_{p_{31}p_{12}}$	$B(J)\cap B(P)$	S
$\gamma_{p_{12}p_{23}}$	$B(J^{-1})\cap B(P^{-1})$	S
$\gamma_{p_{23}p_{231}}$	$B(R_1)\cap B(R_1^{-1})$	S
$\gamma_{p_{31}p_{231}}$	$B(R_2)\cap B(R_2^{-1})$	S
$\gamma_{p_{12}p_{231}}$	$B(P)\cap B(R_2^{-1})$	S
$\gamma_{p_{23}p_{31}}$	$B(J^{-1})\cap B(R_2^{-1})$	S

3.4 THE FACES OF THE POLYHEDRON D

So far we have defined the various dimensional cells in the boundary of D except for one, that is the two-dimensional cells. The zero (which is the vertices) and the one-dimensional (edges) are the ones which have been defined. We are now left with the two-dimensional cells which we call the faces of D.When two pair of sides of D intersect, we obtain a face. It must be noted that a side of the polyhedron D is obtained when a bisector intersects with D. We therefore discuss all the intersections among pairs of sides of D. Our goal in this section is to prove the following:

Proposition 3.4.1. *The interior of each F face of D is homeomorphic to an open ball in $\mathbb{R}^2$ and the boundary of F is made up of edges.*

Recall the various cases of interest below:

θ	$2\pi/3$	$2\pi/3$	$2\pi/3$	$2\pi/4$	$2\pi/4$	$(2\pi/5)$	$2\pi/5$	$2\pi/6$
ϕ	$\pi/4$	$\pi/5$	$\pi/6$	$\pi/3$	$\pi/4$	$(2\pi/5)$	$\pi/3$	$\pi/3$

We begin by proving the following lemma which will be needed

Lemma 3.4.2. *If $\mathbf{z} \in \mathbf{H}^2_{\mathbb{C}}$ then*

$$|z_1| < \frac{\sin\phi + \sin(\theta-\phi)}{\sin(\theta+\phi)}, \qquad |w_1| < \frac{\sin\phi + \sin(\theta-\phi)}{\sin(\theta+\phi)}.$$

Proof. If $|z_1| \geq \big(\sin\phi + \sin(\theta-\phi)\big)/\sin(\theta+\phi)$ then

$$\begin{aligned}
&-\frac{\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \; -\frac{\sin\phi\big(\sin\phi+\sin(\theta-\phi)\big)}{\sin^2(\theta-\phi)} - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \; \frac{\sin^2\phi\big(2\cos\theta-1\big)}{\sin^2(\theta+\phi)} - |z_2|^2 \\
&\leq \; 0. \;\text{ Since } \; 2\cos\theta - 1 \leq 0.
\end{aligned}$$

Similarly for the case when $|w_1| \geq \big(\sin\phi + \sin(\theta - \phi)\big)/\sin(\theta+\phi)$. $\square$

Lets look at this lemma which we will also need.

Lemma 3.4.3. *If* $\mathbf{z} \in \mathbf{H}^2_{\mathbb{C}}$ *then*

(i)

$$|z_1| < \frac{\sin\theta}{\sin(\theta+\phi)},$$

(ii)

$$|z_2| < \frac{\sin\phi}{\sin(\theta+\phi)}.$$

Proof. In order to prove the lemma, first observe from the combination of ϕ and θ of interest that we have $\theta + 2\phi \geq \pi$ and hence

$$\pi - (\theta+\phi) \leq \phi \leq \theta + \phi.$$

Therefore $\sin(\phi) \geq \sin(\theta+\phi)$.

Now we prove (i). If $|z_1| \geq \sin(\theta)/\sin(\theta+\phi)$ then from the area

$$\begin{aligned}
&\frac{-\sin\phi}{\sin\phi + \sin(\theta-\phi)}|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
\leq\; & \frac{\sin\phi}{\sin(\theta+\phi)}\,\frac{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big) - \sin^2\theta}{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big)} - |z_2|^2 \\
=\; & \frac{\sin\phi}{\sin(\theta+\phi)}\,\frac{\sin(\theta+\phi)\sin\phi + \sin^2\theta\cos^2\phi - \cos^2\theta\sin^2\phi - \sin^2\theta}{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big)} - |z_2|^2 \\
=\; & \frac{\sin\phi}{\sin(\theta+\phi)}\,\frac{\sin(\theta+\phi)\sin\phi - \sin^2\phi}{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big)} - |z_2|^2 \\
=\; & -\frac{\sin^2\phi}{\sin^2(\theta+\phi)}\,\frac{\sin\phi - \sin(\theta+\phi)}{\sin\phi+\sin(\theta-\phi)} - |z_2|^2 \\
\leq\; & 0.
\end{aligned}$$

This a contradiction

Similarly, to prove (ii) assume that $|z_2| \geq \sin\phi/\sin(\theta+\phi)$. Then

$$\begin{aligned}
&\frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - \frac{\sin^2\phi}{\sin^2(\theta+\phi)} + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - \frac{\sin\phi}{\sin(\theta+\phi)}\Big(\frac{\sin\phi}{\sin(\theta+\phi)} - 1\Big) \\
&\leq 0
\end{aligned}$$

as $\sin\phi/\sin(\theta+\phi) \geq 1$. □

There are two groups of faces. The first group is obtained by either the two sides of D intersecting both pass through p_1 or both pass through p_2. The second group is obtained by intersecting sides in which one side passes through p_1 and the other through p_2. For the first group, they are all contained in complex lines or Lagrangian planes. They are as follows:

Face	Vertices	Sides	Coordinates
$F(JP)$	$p_1,\ p_{31},\ p_{12}$	$S(J)\cap S(P)$	$Im(z_1e^{i\phi})=Im(z_1)=0$
$F(JR_1)$	$p_1,\ p_{31},\ p_{23}$	$S(J)\cap S(R_1)$	$Im(z_1e^{i\phi})=Im(z_2)=0$
$F(JR_1^{-1})$	$p_1,\ p_{31},\ p_{12}$	$S(J)\cap S(R_1^{-1})$	$Im(z_1e^{i\phi})=Im(z_2e^{-i\theta})=0$
$F(J^{-1}P^{-1})$	$p_2,\ p_{12},\ p_{23}$	$S(J^{-1})\cap S(P^{-1})$	$Im(w_1e^{-i\phi})=Im(w_1)=0$
$F(J^{-1}R_2)$	$p_2,\ p_{31},\ p_{12}$	$S(J^{-1})\cap S(R_2)$	$Im(w_1e^{-i\phi})=Im(w_2)=0$
$F(J^{-1}R_2^{-1})$	$p_2,\ p_{31},\ p_{23}$	$S(J^{-1})\cap S(R_2^{-1})$	$Im(w_1e^{-i\phi})=Im(w_2e^{-i\theta})=0$
$F(PR_1)$	$p_1,\ p_{31},\ p_{231}$	$S(P)\cap S(R_1)$	$Im(z_1)=Im(z_2)=0$
$F(PR_1^{-1})$	$p_1,\ p_{12},\ p_{231}$	$S(P)\cap S(R_1^{-1})$	$Im(z_1)=Im(z_2e^{-i\theta})=0$
$F(P^{-1}R_2)$	$p_2,\ p_{12},\ p_{231}$	$S(P^{-1})\cap S(R_2)$	$Im(w_1)=Im(w_2)=0$
$F(P^{-1}R_2^{-1})$	$p_2,\ p_{23},\ p_{231}$	$S(P^{-1})\cap S(R_2^{-1})$	$Im(w_1)=Im(w_2e^{-i\theta})=0$
$F(R_1R_1^{-1})$	$p_1,\ p_{23},\ p_{231}$	$S(R_1)\cap S(R_1^{-1})$	$Im(z_2)=Im(z_2e^{-i\theta})=0$
$F(R_2R_2^{-1})$	$p_2,\ p_{31},\ p_{231}$	$S(R_2)\cap S(R_2^{-1})$	$Im(w_2)=Im(w_2e^{-i\theta})=0$

These faces are each plane hyperbolic triangles whose boundary comprises of geodesics arcs joining the vertices. Since these geodesics arcs only intersect in their endpoints it implies that each face is homeomorphic to a disc.

We now discuss the other group of faces one by one. These are the faces in which p_1 or p_2 is not a vertice of the face.

Proposition 3.4.4. *The point* $\mathbf{z}$ *lies in* $S(J)\cap S(J^{-1})$ *if and only if* $z_1 = xe^{-i\phi}$ *and* $w_1 = ue^{i\phi}$ *where* $x \geq o$, $u \geq 0$ *and*

$$\big(xu\sin(\theta+\phi)\sin(\theta-\phi)-(x+u)(\sin\phi+\sin(\theta-\phi))\sin\theta+(\sin\phi+\sin(\theta-\phi))^2\big)\geq 0.$$

Proof. w_2 and z_2 are as follows

$$\begin{aligned} w_2 &= \frac{xu\sin(\theta+\phi) - x(e^{-i\phi}\sin\phi + \sin\theta) - u\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}}, \\ z_2 &= e^{i\theta}\frac{xu\sin(\theta+\phi) - u(e^{i\phi}\sin\phi + \sin\theta) - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - ue^{i\phi}\sin(\theta+\phi)}. \end{aligned}$$

We must find conditions on x and u under which $\mathrm{Im}\,(z_2) \geq 0$, $\mathrm{Im}\,(z_2 e^{-i\theta}) \leq 0$, $\mathrm{Im}\,(w_2) \geq 0$ and $\mathrm{Im}\,(w_2^{-i\theta}) \leq 0$.

First we calculate:

$$\mathrm{Im}\,(z_2 e^{-i\theta}) = \frac{-u\sin\phi\, f(x,u,\theta,\phi)}{\left|\sin\phi + \sin(\theta-\phi) - ue^{i\phi}\sin(\theta+\phi)\right|^2}$$

and

$$\mathrm{Im}\,(w_2) = \frac{x\sin\phi\, f(x,u,\theta,\phi)}{\left|\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}\right|^2}.$$

where

$$\begin{aligned} f(x,u,\theta,\phi) &= -xu\sin^2(\theta+\phi) + (x+u)\sin(\theta+\phi)\sin\theta + \sin^2\phi(1-\cos\theta)^2 - \sin^2\theta\cos^2\phi \\ &= \big(\sin\phi - \sin(\theta+\phi)\big)\big(\sin\phi + \sin(\theta-\phi)\big) \\ &\quad + x\sin(\theta+\phi)\sin\theta + u\sin(\theta+\phi)\big(\sin\theta - x\sin(\theta+\phi)\big). \end{aligned}$$

Since $\sin\phi \geq \sin(\theta+\phi)$ and $\sin\theta > x\sin(\theta+\phi)$ we see that $f(x,u,\theta,\phi) > 0$. Therefore the conditions $\mathrm{Im}\,(z_2) \geq 0$ and $\mathrm{Im}\,(w_2 e^{-i\theta}) \leq 0$ imply $x \geq 0$ and $u \geq 0$.

Similarly

$$\mathrm{Im}\,(z_2) = \frac{\big(\sin\theta - u\sin(\theta+\phi)\big)g(x,u,\theta,\phi)}{\left|\sin\phi + \sin(\theta-\phi) - ue^{i\phi}\sin(\theta+\phi)\right|^2}$$

and

$$\mathrm{Im}\,(w_2 e^{-i\theta}) = \frac{-\big(\sin\theta - x\sin(\theta+\phi)\big)g(x,u,\theta,\phi)}{\left|\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}\right|^2}$$

where

$$g(x,u,\theta,\phi) = xu\sin(\theta+\phi)\sin(\theta-\phi) - (x+u)(\sin\phi + \sin(\theta-\phi))\sin\theta + (\sin\phi + \sin(\theta-\phi))^2.$$

Since $\mathrm{Im}\,(z_2) \geq 0$ and $\mathrm{Im}\,(w_2 e^{-i\theta}) \leq 0$ we must have $g(x,u,\theta,\phi) \geq 0$. $\square$

Proposition 3.4.5. *The point z lies in $S(J)\cap S(P^{-1})$ if and only if $z_1 = xe^{-i\phi}$ and $w_1 = 0$ where*

$$0 \le x \le \frac{\sin\phi + \sin(\theta-\phi)}{\sin\theta}.$$

Proof. If $\mathbf{z} \in \mathbf{S}(\mathbf{J}) \cap \mathbf{S}(\mathbf{P}^{-1})$ then $z_1 = xe^{-i\phi}$ and $w_1 = u$. Then we find w_2 and z_2 as follows:

$$\begin{aligned} w_2 &= \frac{xue^{-i\phi}\sin(\theta+\phi) - x(e^{-i\phi}\sin\phi + \sin\theta) - ue^{-i\phi}\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}}, \\ z_2 &= e^{i\theta}\frac{xue^{-i\phi}\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - u\sin(\theta+\phi)}. \end{aligned}$$

$arg(w_1) \in (0,\phi)$,$Im(w_1) \ge 0$. Hence $Im(w_1e^{i\phi}) = u\sin\phi \ge 0$. This implies that $u \ge 0$. Now $\sin\phi + \sin(\theta-\phi) - u\sin(\theta+\phi) > 0$ since $u < (\sin\phi + \sin(\theta-\phi))/\sin(\theta+\phi)$ $arg(z_2) \in (0,\theta)$, $Im(z_2) > 0$. This implies $Im(z_2e^{-i\theta}) < 0$. Therefore

$$\begin{aligned} 0 > \mathrm{Im}\,(z_2e^{-i\theta}) &= \frac{u\sin\phi\sin\theta - xu\sin\phi\sin(\theta+\phi)}{\sin\phi + \sin(\theta-\phi) - u\sin(\theta+\phi)} \\ &= \frac{u\sin\phi(\sin\theta - x\sin(\theta+\phi))}{\sin\phi + \sin(\theta-\phi) - u\sin(\theta+\phi)} \end{aligned}$$

Using part (i) of the lemma, since $x < \sin\theta/\sin(\theta+\phi)$ we have $u \le 0$ and so $u = 0$.

$$\begin{aligned} w_2 &= \frac{-x(e^{-i\phi}\sin\phi + \sin\theta) + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}}, \\ z_2 &= e^{i\theta}\frac{-x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi)}. \end{aligned}$$

Therefore

$$\Re(z_2e^{-i\theta}) = \frac{-x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin\theta - \phi}$$

and using the fact that $\Re(z_2e^{-i\theta}) \ge 0$, the above becomes

$$0 \le -x\sin\theta + \sin\phi + \sin(\theta-\phi)$$

and hence

$$0 \leq x \leq \frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta}.$$

□

Proposition 3.4.6. *The point z lies in $S(J) \cap S(R_2)$ if and only if $z_1 = 0$ and $w_2 = ue^{i\theta}$ where*

$$0 \leq u \leq \frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta}.$$

Proof. By definition $z_1 = xe^{-i\phi}$. Now w_1 and z_2 are as follows

$$\begin{aligned} w_1 &= \frac{xu\sin(\theta+\phi) - x(\sin\phi + e^{i\phi}\sin\theta) + (1-u)(\sin\phi + \sin(\theta-\phi))e^{i\phi}}{\sin\theta - x\sin(\theta+\phi)}, \\ z_2 &= \frac{xu\sin(\theta+\phi) - u(e^{-i\phi}\sin\theta + \sin\phi) - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{e^{-i\phi}(\sin\phi + \sin(\theta-\phi)) - ue^{-i\phi}\sin(\theta+\phi)}. \end{aligned}$$

$arg(w_1) \in (0, \phi)$, $Im(e^{-i\phi}w_1) \leq 0$

$$\operatorname{Im}(e^{-i\phi}w_1) = \frac{x\sin\phi(\sin\phi - u\sin(\theta+\phi))}{\sin\theta - x\sin(\theta+\phi)}$$

$x \geq 0$, and $u < (\sin\phi + \sin(\theta-\phi))/\sin(\theta+\phi)$. Implies $x\sin\phi(\sin\phi - u\sin(\theta+\phi)) > 0$. Which also implies that $\sin\theta - x\sin(\theta+\phi) < 0$. But from Lemma 3.4.3, $\sin\phi - u\sin(\theta+\phi)$ and $\sin\theta - x\sin(\theta+\phi)$ are both positive. This implies that $x = 0$. Now considering z_2, $arg(z_2) \in (0, \theta)$, $\operatorname{Im}(z_2) \geq 0$

$$\operatorname{Im}(z_2) = \sin\phi\frac{xu\sin(\theta+\phi) - u\sin\phi - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - u\sin(\theta+\phi)}.$$

$$0 \leq -u\sin\phi + \sin\phi + \sin(\theta-\phi).$$

□

Proposition 3.4.7. *The point z lies in $S(J) \cap S(R_2^{-1})$ if and only if $z_1 = xe^{-i\phi}$ and $w_2 = ue^{i\theta}$ where $x > 0$ and*

$$x\sin\theta(u\sin(\theta+\phi) - \sin\theta) + (\sin\phi - u\sin(\theta+\phi))(\sin\phi + \sin(\theta-\phi)) > 0$$

Proof. By definition $z_1 = xe^{-i\phi}$ and $w_2 = ue^{i\theta}$. w_1 and z_2 respectively are as follows:

$$\begin{aligned} w_1 &= \frac{xue^{\theta}\sin(\theta+\phi) - x(\sin\phi + e^{i\theta}\sin\theta) + (1-ue^{i\theta})(\sin\phi + \sin(\theta-\phi))}{\sin\theta - x\sin(\theta+\phi)}, \\ z_2 &= \frac{xue^{i(\theta-\phi)}\sin(\theta+\phi) - u(e^{i(\theta-\phi)}\sin\theta + e^{i\theta}\sin\phi) - xe^{-i\phi}\sin\phi + \sin\phi}{\sin\phi - u\sin(\theta+\phi)}. \end{aligned}$$

$arg(w_1) \in (0,\phi)$,$Im(w_1) \geq 0$.

$$\operatorname{Im}(w_1) = \frac{x\sin\theta(u\sin(\theta+\phi) - \sin\theta) + (\sin\phi - u\sin(\theta+\phi))(\sin\phi + \sin(\theta-\phi))}{\sin\theta - x\sin(\theta+\phi)}.$$

But from lemma 3.4.3ii, $\sin\theta - x\sin(\theta+\phi) > 0$. Hence $x\sin\theta(u\sin(\theta+\phi) - \sin\theta) + (\sin\phi - u\sin(\theta+\phi))(\sin\phi + \sin(\theta-\phi)) > 0$ □

Proposition 3.4.8. *The point z lies in $S(P) \cap S(J^{-1})$ if and only if $z_1 = 0$ and $w_1 = u$ where*

$$0 \leq u \leq \frac{\sin\phi + \sin(\theta-\phi)}{\sin\theta}.$$

Proof. By definition $z_1 = x \in \mathbb{R}$ and $w_1 e^{-i\phi} = u \in \mathbb{R}$. Then z_2 and w_2 are as follows:

$$\begin{aligned} z_2 &= e^{i\theta}\frac{\sin(\theta+\phi)e^{i\phi}xu - \big(\sin(\phi)e^{i\phi} + \sin(\theta)\big)u - \sin(\theta)e^{i\phi}x + \big(\sin\phi + \sin(\theta-\phi)\big)}{-\sin(\theta+\phi)e^{i\phi}u + \sin\phi + \sin(\theta-\phi)}, \\ w_2 &= \frac{\sin(\theta+\phi)e^{i\phi}xu - \sin\theta u - \big(\sin\phi + \sin\theta e^{i\phi}\big)x + \sin\phi + \sin(\theta-\phi)}{-\sin(\theta+\phi)x + \sin\phi + \sin(\theta-\phi)}. \end{aligned}$$

We have $0 \leq \operatorname{Im}(z_1 e^{i\phi}) = x\sin\phi$ which imples $x \geq 0$. Also

$$0 \leq \operatorname{Im}(w_2) = \frac{x\sin\phi(u\sin(\theta+\phi) - \sin\theta)}{-\sin(\theta+\phi)x + \sin\phi + \sin(\theta-\phi)}.$$

Since $\sin\phi + \sin(\theta-\phi) > \sin(\theta+\phi)x$ from lemma 3.4.2 the denominator is positive. Using part i of lemma 3.4.3 $u < \sin\theta/(\sin(\theta+\phi)$ making $u\sin(\theta+\phi) - \sin\theta$ negative. This implies $x \leq 0$. Therefore $x = 0$ and

$$\begin{aligned} z_2 &= e^{i\theta}\frac{-(\sin(\phi)e^{i\phi} + \sin(\theta))u + \sin(\phi) + \sin(\theta-\phi)}{-\sin(\theta+\phi)e^{i\phi}u + \sin\phi + \sin(\theta-\phi)}, \\ w_2 &= \frac{-\sin(\theta)u + \sin(\phi) + \sin(\theta-\phi)}{\sin(\phi) + \sin(\theta-\phi)}. \end{aligned}$$

From w_2, $-\sin(\theta)u + \sin(\phi) + \sin(\theta-\phi) >$ □

Proposition 3.4.9. *The point z lies in $S(P) \cap S(P^{-1})$ if and only if $z_1 = w_1$ and that is $x = u$*

Proof. We have $z_1 = x \in \mathbb{R}$ and $w_1 = u \in \mathbb{R}$. Then If $x < \big(\sin\phi + \sin(\theta-\phi)\big)/\sin(\theta+\phi)$ and $u < \big(\sin\phi + \sin(\theta-\phi)\big)/\sin(\theta+\phi)$. We have

$$\begin{aligned} z_2 &= e^{i\theta}\frac{\sin(\theta+\phi)xu - \big(\sin\phi + \sin\theta e^{-i\phi}\big)u - \sin\theta e^{i\phi}x + \sin\phi + \sin(\theta-\phi)}{-\sin(\theta+\phi)u + \sin\phi + \sin(\theta-\phi)}, \\ w_2 &= \frac{\sin(\theta+\phi)xu - \sin\theta e^{-i\phi}u - \big(\sin\phi + \sin\theta e^{i\phi}\big)x + \sin\phi + \sin(\theta-\phi)}{-\sin(\theta+\phi)x + \sin\phi + \sin(\theta-\phi)}. \end{aligned}$$

We as well have

$$0 \geq \mathrm{Im}\left(z_2 e^{-i\theta}\right) = \frac{(u-x)\sin\theta\sin\phi}{\sin\phi + \sin(\theta-\phi) - \sin(\theta+\phi)u}.$$

and

$$0 \leq \mathrm{Im}\left(w_2\right) = \frac{(u-x)\sin\theta\sin\phi}{\sin\phi + \sin(\theta-\phi) - \sin(\theta+\phi)x}.$$

Meaning $x = u$. In this case, $z_2 = e^{i\theta}(1-x)$ and $w_2 = (1-x)$. □

Proposition 3.4.10. *The point z lies in $S(R_1^{-1}) \cap S(R_2^{-1})$ if and only if $z_2 = xe^{i\theta}$ and $w_2 = ue^{i\theta}$ where*

$$\frac{\sin\phi}{\sin(\theta+\phi)} < u < \frac{\sin\phi + \sin(\theta-\phi)}{\sin(\theta+\phi)}$$

and

$$u(x - u^2 + 1)\sin(\theta+\phi) - (u+x)\sin\phi - \sin(\theta-\phi) \leq 0$$

Proof. w_1 and z_1 respectively are as follows:

$$\begin{aligned} w_1 &= \frac{xue^{2i\theta}\sin(\theta+\phi) - x(e^{i(\theta+\phi)}\sin\theta + e^{i\theta}\sin\phi) + (1-u)e^{i\theta}\sin\phi}{e^{i\theta}\sin\phi - xe^{i\theta}\sin(\theta+\phi)}, \\ z_1 &= \frac{xue^{i\theta}\sin(\theta+\phi) - u(e^{i(\theta-\phi)}\sin\theta + e^{i\theta}\sin\phi) + (1-xe^{i\theta})\sin\phi}{\sin\phi - ue^{i\theta}\sin(\theta+\phi)}. \end{aligned}$$

Now $arg(w_1) \in (0, \phi)$ then $\mathrm{Im}\,(w_1) \geq 0$. w_1 is reduced to

$$w_1 = \frac{xue^{i\theta}\sin(\theta+\phi) - x(e^{i\phi}\sin\theta + \sin\phi) + (1-u)\sin\phi}{\sin\phi - x\sin(\theta+\phi)}$$

$$\mathrm{Im}\,(w_1) = \frac{x\sin\theta(u\sin(\theta+\phi) - \sin\phi)}{\sin\phi - x\sin(\theta+\phi)}$$

From $x < (\sin\phi + \sin(\theta-\phi))/\sin(\theta+\phi)$

$$\sin\phi - x\sin(\theta+\phi) > \sin\phi - (\sin\phi - \sin(\theta-\phi)) = \sin(\theta-\phi) > 0$$

Then for $\mathrm{Im}\,(w_1) \geq 0$, $u\sin(\theta+\phi) - \sin\phi > 0$. Hence

$$u > \frac{\sin\phi}{\sin(\theta+\phi)}$$

Therefore

$$\frac{\sin\phi}{\sin(\theta+\phi)} < u < \frac{\sin\phi + \sin(\theta-\phi)}{\sin(\theta+\phi)}$$

$arg(z_1) \in (-\phi, 0)$, $\Rightarrow \mathrm{Im}\,(z_1) \leq 0$. Therefore

$$z_1 = \frac{\sin\theta\sin\phi(u(x-u^2+1)\sin(\theta+\phi) - (u+x)\sin\phi - \sin(\theta-\phi))}{\sin^2\phi + u\sin(\theta+\phi)(u\sin(\theta+\phi) - 2\cos\theta\sin\phi)}.$$

Using $u > \sin\phi/\sin(\theta+\phi)$, the denominator becomes $\sin^2\phi(1-\cos\theta) > 0$. Implying

$$u(x-u^2+1)\sin(\theta+\phi) - (u+x)\sin\phi - \sin(\theta-\phi) \leq 0$$

in order to make $\mathrm{Im}\,(z_1) \leq 0$. $\square$

CHAPTER 4

PROOF THAT D IS A FUNDAMENTAL POLYHEDRON FOR THE GROUP

4.1 INTRODUCTION

In this chapter we show that the group Γ generated by R_1, R_2 and I_1 is discrete by showing that D is a fundamental polyhedron for Γ. We use Poincaré's polyhedron theorem to show that Γ is discrete. In view of that we therefore give a good account of the Poincaré's polyhedron theorem in Complex Hyperbolic Space. It must be noted that in terms of proving discreteness of Γ, Theorem 0.2 of [27] could prove that, but we use Poincaré's polyhedron theorem which Parker [19] used because in addition it provides the presentation of the group as well as unlimited information about the geometry of the action of P on complex hyperbolic space. Something which is lacking when you use Theorem 0.2 of [27]. Hence our prove is in the same spirit of Parker's [19]. Our goal will be to prove this theorem:

Theorem 4.1.1. *The group Γ generated by the side pairings of D is a discrete subgroup of PU(1,2) with fundamental domain D and presentation:*

$$\Gamma = \left\langle J, P, R_1, R_2 : \begin{array}{c} J^3 = R_1^p = R_2^p = (P^{-1}J)^k = I, \\ R_2 = PR_1P^{-1} = JR_1J^{-1}, \qquad P = R_1R_2 \end{array} \right\rangle \tag{4.1}$$

Where p and k are given below:

p	3	3	3	4	4	5	5	6
k	4	5	6	3	4	2	3	3

In order to prove the above theorem, we use Poincaré's Polyhedron Theorem.

We therefore discuss Poincaré's theorem first, and then follow it with the proof of the theorem.

4.2 POINCARÉ'S POLYHEDRON THEOREM

We begin with some definitions which will help us to formulate the Poincaré's Polyhedron Theorem.Our formulation follows that of Mostow in [16].

Definition. A *combinatorial polyhedron* is a cellular space homeomorphic to a compact polytope, in particular each of its codimension - 2 cells, called a *face*, is contained in exactly two codimension - 1 cells, called *sides*.

Definition. A polyhedron D is the realisation of a combinatorial polyhedron as a cell complex in a manifold X.

As a convention, D is open.

Definition. A polyhedron is smooth if its cells are smooth.

X in this case will be complex hyperbolic space. The sides of the polyhedron D will all be contained in bisectors and D will be smooth.

Definition. A *Poincaré polyhedron* is a smooth polyhedron D in X with sides S_j and *side pairing maps* $T_j \in Isom(X)$ satisfying:

(S.1) For each side S_j of D there is a side S_k of D and a side pairing map T_j so that $T_j(S_j) = S_k$.

(S.2) If $T_j(S_j) = S_k$ then $T_k = T_j^{-1}$. In particular, if $j = k$ then T_j^2 is the identity.

(S.3) $T_j^{-1}(D) \cap D = \phi$.

(S.4) $T_j^{-1}(\bar{D}) \cap \bar{D} = S_j$.

(S.5) The polyhedron D has only finitely many sides and each side has only finitely many faces.

(S.6) There exists a number $\delta > 0$ so that each pair of disjoint sides is a distance at least δ apart.

The relation coming from (S.2) is called a *reflection relation*.

Now there are some face conditions in addition to the side - pairing conditions (S.1) to (S.6). Let S_1 be a side of D and F be a face in the boundary of S_1, where S_1 and F are of codimension - 1 cell and codimension - 2 cell respectively. Let T_1 be the side pairing map associated to S_1 and consider $T_1(F)$. Since by hypothesis each face is contained in the boundary of exactly two sides, $T_1(F)$ is contained in the boundary of $T_1(S_1)$ and another side, which we call S_2. Then let T_2 also be the side pairing map associated to S_2 and consider $T_2 o T_1(F)$. When we continue in this way we obtain a sequence of faces, a sequence of sides S_j and a sequence of side pairing maps T_j. As the polyhedron has finitely many sides and faces, these sequences must therefore be periodic. Let k be the smallest integer so that all three sequences are periodic with period k. Then we have $T_k o ... o T_2 o T_1(F) = F$ and we denote $T_k o ... o T_2 o T_1$ by T. Then T is called the *cycle transformation* at the face F.

Given a cycle transformation $T = T_k o ... T_2 o T_1$ and a positive integer m, define transformations $U_0, ..., U_{mk-1}$ by

$$\begin{array}{ccc} U_0 = 1, & U_1 = T_1, & \cdots U_{k-1} = T_{k-1} o ... T_2 o T_1, \\ U_k = T, & U_{k+1} = T_1 o T, & \cdots U_{2k-1} = T_{k-1} o ... T_2 o T_1 o T, \\ \vdots & \vdots & \vdots \\ U_{mk-k} = T^{m-1}, & U_{mk-k+1} = T_1 o T^{m-1}, & \cdots U_{mk-1} = T_{k-1} o ... T_2 o T_1 o T^{m-1}. \end{array}$$

Then the face conditions are :

(F.1) Every face is a submanifold of X homeomorphic to a codimension - 2 ball.

(F.2) For each face F with cycle transformation T there is an integer l so that the restriction of T^l to F is the identity.

(F.3) For each face F with cycle transformation T there is an integer m so that $T^{lm} = (T^l)^m$ is the identity on the whole space X. Furthermore, the polyhedra

$U_j^{-1}(D)$ for $j = 0, ..., mlk - 1$ are disjoint and their closures $U_j^{-1}(\bar{D})$ cover a neighbourhood of the interior of F, that is D and its images *tessellate* a neighbourhood of F. The relations $T^{lm} = 1$ from (F.3) are called the *cycle relations*.

Then Poincaré's polyhedron theorem states that

Theorem 4.2.1. *Let D be a Poincaré's polyhedron with side - pairing transformations $T_j \in \sum$ satisfying side pairing conditions (S.1) to (S.6) and face conditions (F.1), (F.2) and (F.3). Then the group Γ generated by the side pairing transformations is a discrete subgroup of Isom(X) and D a fundamental domain. A presentation is given by*

$$\Gamma = \left\langle \sum | \textit{reflection relations, cycle relations} \right\rangle$$

4.3 THE SIDE PAIRING MAPS

Let J be the move on the cone structure defined by $J = PI_1 = R_1R_2I_1$. That is

$$J = \frac{1}{(1 - e^{-i\theta})\sin(\phi)} \begin{bmatrix} -\sin(\theta)e^{i\phi} & -\sin(\phi) - \sin(\theta - \phi) & \sin(\phi) + \sin(\theta - \phi) \\ -\sin(\phi)e^{i(2\phi+\theta)} & -\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta + \phi)e^{2i\phi} & -\sin(\theta + \phi) & \sin(\phi) + \sin(\theta)e^{i\phi} \end{bmatrix} \tag{4.2}$$

Let J, P, R_1 and R_2 be given by (4.2), (2.2), (1.2) and (1.3) respectively. In this section we show that the maps J, P, R_1, R_2 pair the sides of D, and they satisfy the conditions of Poincaré's theorem. The maps pair the sides of D as follows; see Figure 4.1

$$J : S(J) \to S(J^{-1}), \quad P : S(P) \to S(P^{-1}), \quad R_1 : S(R_1) \to S(R_1^{-1}), \quad R_2 : S(R_2) \to S(R_2^{-1}),$$

$$J^{-1} : S(J^{-1}) \to S(J), \quad P^{-1} : S(P^{-1}) \to S(P), \quad R_1^{-1} : S(R_1^{-1}) \to S(R_1), \quad R_2^{-1} : S(R_2^{-1}) \to S(R_2)$$

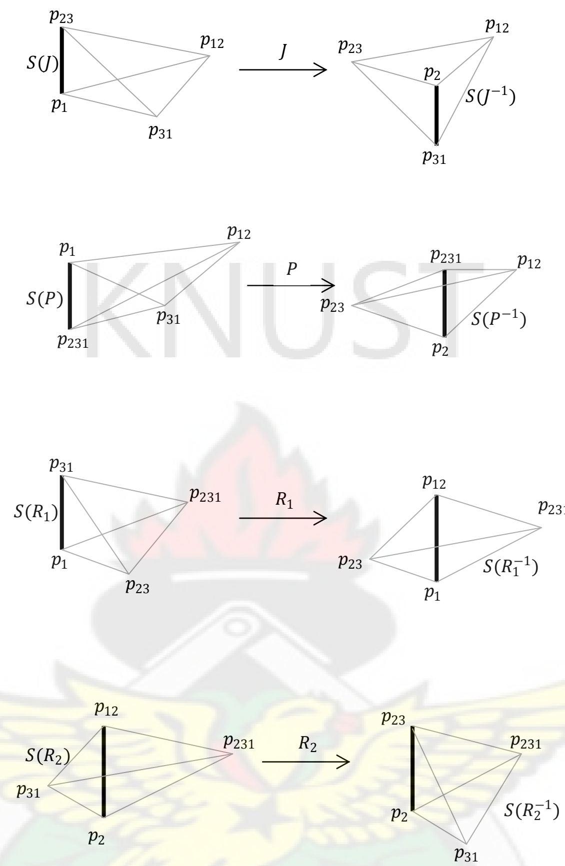

Figure 4.1. Side Pairing map

One can easily verify that the side pairings are consistent with the antitholomorphic involution i which maps D to itself.

Theorem 4.1 will follow after we have shown that Γ satisfies the hypotheses of Poincaré's theorem and that the relations in equation (4.1) are each cycle relations associated to a face cycle of D.

It is clear that the side pairing maps satisfy conditions (S.1) and (S.2). The polyhedron D also satisfies condition (S.5). Because each pair of sides intersect conditions (S.6) becomes unnecessarily. In addition the face condition (F.1) follows from Proposition 3.4.1

We now verify conditions (S.3) and (S.4) for each side.

Lemma 4.3.1. *If T is one of J,P,R_1 or R_2 then $T^{-1}(D) \cap D = T(D) \cap D = \Phi$.*

$$P^{-1}(\overline{D}) \cap \overline{D} = S(P), \quad J^{-1}(\overline{D}) \cap \overline{D} = S(J), \quad R_1^{-1}(\overline{D}) \cap \overline{D} = S(R_1), \quad R_2^{-1}(\overline{D}) \cap \overline{D} = S(R_2),$$

$$P(\overline{D}) \cap \overline{D} = S(P^{-1}), \quad J(\overline{D}) \cap \overline{D} = S(J^{-1}), \quad R_1(\overline{D}) \cap \overline{D} = S(R_1^{-1}), \quad R_2(\overline{D}) \cap \overline{D} = S(R_2^{-1}).$$

Proof. Let us consider the side $S(J)$. If $\mathbf{z} \in D$ then $Im(z_1) \leq 0$ with equality only when $\mathbf{z} \in S(J)$. In the same way, if $\mathbf{z} = \mathbf{P}(\mathbf{w}) \in D$ then $Im(w_1) \geq 0$ and also with equality only when $\mathbf{z} \in S(J^{-1})$. Hence if $P(\mathbf{z}) \in D$, or equivalently $\mathbf{z} \in P^{-1}(D)$, then $Im(z_1) \geq 0$ with equality if and only if $\mathbf{z} \in S(J) = P^{-1}(S(J^{-1})$. Thus (S.3) and (S.4) hold for this side and applying P also for $S(P^{-1})$. The other parts follow similarly. $\square$

In sections 4.4 and 4.5 we obtain the cycle transformation T of each face F. We also find the integers l and m from conditions (F.2) and (F.3). This will conclude our proof of Theorem 4.1. When we consider each face T^l is either going to be the identity, or else F will be contained in a complex line L and the T^l will be a complex reflection of order m that fixes L. This will verify condition (F.2) of Poincar's theorem. We will also verify that the images of D tessellate around the faces formed by intersecting pairs of sides making conditions (F.3) satisfied. Going through this, we will generate the list of cycle relations. This will verify our presentation given in (4.1).

In concluding this section, we describe the method we use in proving the tessellation conditions as used by Parker [19]. We demonstrate that the (open) polyhedron D is disjoint from its image under the relevant side pairings and that these faces are covered by images of D. We recall that D is defined as the intersection of eight halfspaces defined by bisectors. Each face of D is contained in two bisectors and so D is contained in the intersection of the corresponding two halfspaces. Each image of D under appropriate side pairing maps is contained in the intersection of two halfspaces that are the image of one of the original pairs under this map. First, we must show that the intersections are disjoint. Secondly, we then choose a neighbourhood U of the interior of the face that is small enough that it does not meet any of the bisectors defining D except the two we are interested in. We then consider the closures of the halfspace intersections considered above and show that they cover U. Keeping in mind the underlying geometry, we will use linear algebra to codify this picture. This will be easier.

4.4 TESSELLATION AROUND GENERIC FACES

In this section we consider the second group of faces of D as obtained in section 3.4. They are those having not p_1 or p_2 as one of its vertices, i.e. neither contained in complex lines or Lagrangian planes.

Let p_{231} be the fixed point of P. Then p_{231} is given by

$$p_{231} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Lemma 4.4.1. *With p_{231} as above, then*

(i) $\left|\langle \mathbf{z}, p_{231}\rangle\right| < \left|\langle \mathbf{z}, J^{-1}(p_{231})\rangle\right|$ *if and only if* $\mathrm{Im}\,(z_1 e^{i\phi}) > 0$.

(ii) $\left|\langle \mathbf{z}, p_{231}\rangle\right| < \left|\langle \mathbf{z}, J(p_{231})\rangle\right|$ *if and only if* $\mathrm{Im}\,(w_1 e^{-i\phi}) < 0$.

Proof. We prove (i). Then (ii) will follow by applying ι. Since $J^{-1} = I_1^{-1}P^{-1}$ we see that $J^{-1}(p_{231}) = I_1^{-1}P^{-1}(p_{231}) = I_1^{-1}(p_{231})$. We have

$$I_1^{-1}(p_{231}) = \begin{bmatrix} e^{-2i\phi} \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$\begin{aligned}
\langle \mathbf{z}, p_{231}\rangle &= p_{231}^{*}\mathbf{H}\mathbf{z} \\
&= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\theta\sin\phi/\big(\sin\phi + \sin(\theta-\phi)\big) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sin\theta\sin\phi/\sin(\theta+\phi) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \\
&= \frac{-z_1\sin\phi\sin\theta}{\sin\phi + \sin(\theta-\phi)} + \frac{\sin\phi\sin\theta}{\sin(\theta+\phi)}
\end{aligned}$$

and

$$\langle \mathbf{z}, I_1^{-1}(p_{231})\rangle = \frac{-z_1 \sin\phi \sin\theta e^{2i\phi}}{\sin\phi + \sin(\theta-\phi)} + \frac{\sin\phi\sin\theta}{\sin(\theta+\phi)}$$

Hence

$$\begin{aligned}
\left|\langle \mathbf{z}, I_1^{-1}(p_{231})\rangle\right|^2 - \left|\langle \mathbf{z}, p_{231}\rangle\right|^2 &= \left|\frac{-z_1 \sin\phi \sin\theta e^{2i\phi}}{\sin\phi + \sin(\theta-\phi)} + \frac{\sin\phi\sin\theta}{\sin(\theta+\phi)}\right|^2 \\
&\quad - \left|\frac{-z_1 \sin\phi \sin\theta}{\sin\phi + \sin(\theta-\phi)} + \frac{\sin\phi\sin\theta}{\sin(\theta+\phi)}\right|^2 \\
&= \frac{-\sin^2\phi\sin^2\theta}{(\sin\phi+\sin(\theta-\phi))\sin(\theta+\phi)}\Big(z_1e^{2i\phi} - z_1 + \overline{z}_1 e^{-2i\phi} - \overline{z}_1\Big) \\
&= \frac{4\sin^3\phi\sin^2\theta \mathrm{Im}\,(z_1 e^{i\phi})}{(\sin\phi+\sin(\theta-\phi))\sin(\theta+\phi)}.
\end{aligned}$$

Therefore $\left|\langle \mathbf{z}, I_1^{-1}(p_{231})\rangle\right| > \left|\langle \mathbf{z}, p_{231}\rangle\right|$ if and only if $\mathrm{Im}\,(z_1e^{i\phi}) > 0$ as required. □

Proposition 4.4.2. *The polyhedron D and its images under J and J^{-1} tessellate around the face $F(JJ^{-1}) = S(J) \cap S(J^{-1})$. Moreover, the cycle transformation corresponding to this face is J and $l = 3, m = 1$. This gives the cycle relation $J^3 = I$.*

Proof. We know by definition (3.4) that if $\mathbf{z} \in D$ then $\mathrm{Im}\,(z_1e^{i\phi}) > 0$ and $\mathrm{Im}\,(w_1e^{-i\phi}) < 0$. Therefore, using Lemma 4.4.1 we see that

$$D \subset \left\{\mathbf{z} \in \mathbf{H}^2_{\mathbb{C}} : \left|\langle \mathbf{z}, p_{231}\rangle\right| < \left|\langle \mathbf{z}, J^{-1}(p_{231})\rangle\right|, \left|\langle \mathbf{z}, p_{231}\rangle\right| < \left|\langle \mathbf{z}, J(p_{231})\rangle\right|\right\}$$

Now if $\mathbf{z} \in J^{\pm}(D)$ then $J^{\mp}(\mathbf{z}) \in D$. Therefore

$$\left|\langle J^{\mp}(\mathbf{z}), p_{231}\rangle\right| < \left|\langle J^{\mp}(\mathbf{z}), J^{-1}(p_{231})\rangle\right|, \left|\langle J^{\mp}(\mathbf{z}), p_{231}\rangle\right| < \left|\langle J^{\mp}(\mathbf{z}), J(p_{231})\rangle\right|$$

Applying $J^{\pm}$ to each point and using $J^3 = I$, we obtain

$$J^{\pm}(D) \subset \left\{\mathbf{z} \in \mathbf{H}^2_{\mathbb{C}} : \left|\langle \mathbf{z}, J^{\pm}(p_{231})\rangle\right| < \left|\langle \mathbf{z}, J^{\mp}(p_{231})\rangle\right|, \left|\langle \mathbf{z}, J^{\pm}(p_{231})\rangle\right| < \left|\langle \mathbf{z}, p_{231}\rangle\right|\right\}$$

We now show that D, $J(D)$ and $J^{-1}(D)$ tessellate around the face $F(JJ^{-1})$. It is obvious that D, $J(D)$ and $J^{-1}(D)$ are disjoint. We also know that the locus where

$\mathrm{Im}\,(z_1 e^{i\phi}) = \mathrm{Im}\,(w_1 e^{-i\phi}) = 0$ is the face $\mathbf{B}(J) \cap \mathbf{B}(J^{-1})$. This intersects D in the face $\mathbf{S}(J) \cap \mathbf{S}(J^{-1})$. Let U be a neighbourhood of the interior of this face. If we shrink U if necessary, assuming that for all points of U we have

$$arg(z_1) \in (-\phi, 0), arg(z_2) \in (0, \theta), arg(w_1) \in (0, \phi), arg(w_2) \in (0, \theta).$$

then U is in $\overline{D}$ if and only if both $\mathrm{Im}\,(z_1 e^{i\phi}) \geq 0$ and $\mathrm{Im}\,(w_1 e^{-i\phi}) \leq 0$; or equivalently both $\left|\langle \mathbf{z}, p_{231} \rangle\right| \leq \left|\langle \mathbf{z}, J^{-1}(p_{231}) \rangle\right|$ and $\left|\langle \mathbf{z}, p_{231} \rangle\right| \leq \left|\langle \mathbf{z}, J(p_{231}) \rangle\right|$. It is therefore easy to see that $\overline{D}$, $J(\overline{D})$, and $J^{-1}(\overline{D})$ cover U. □

Then tessellation around the face $S(R_1) \cap S(R_2^{-1})$, that is the collection of points satisfying $\mathrm{Im}\,(z_2) = \mathrm{Im}\,(w_2 e^{-i\theta}) = 0$, follows.

Lemma 4.4.3. *Let p_{23}, p_{31}, p_{12} be as in section 2.2. Then*

(i) $\left|\langle \mathbf{z}, p_{23} \rangle\right| < \left|\langle \mathbf{z}, P^{-1}(p_{31}) \rangle\right|$ *if and only if* $\mathrm{Im}\,(z_1) < 0$;

(ii) $\left|\langle \mathbf{z}, p_{12} \rangle\right| < \left|\langle \mathbf{z}, R_1^{-1}(p_{31}) \rangle\right|$ *if and only if* $\mathrm{Im}\,(z_2) > 0$;

(iii) $\left|\langle \mathbf{z}, p_{31} \rangle\right| < \left|\langle \mathbf{z}, R_1(p_{12}) \rangle\right|$ *if and only if* $\mathrm{Im}\,(z_2 e^{-i\theta}) < 0$;

(iv) $\left|\langle \mathbf{z}, p_{31} \rangle\right| < \left|\langle \mathbf{z}, P(p_{23}) \rangle\right|$ *if and only if* $\mathrm{Im}\,(w_1) > 0$;

(v) $\left|\langle \mathbf{z}, p_{12} \rangle\right| < \left|\langle \mathbf{z}, R_2(p_{23}) \rangle\right|$ *if and only if* $\mathrm{Im}\,(w_2 e^{-i\theta}) < 0$.

(vi) $\left|\langle \mathbf{z}, p_{23} \rangle\right| < \left|\langle \mathbf{z}, R_2^{-1}(p_{12}) \rangle\right|$ *if and only if* $\mathrm{Im}\,(w_2) > 0$;

Proof. This is similar to Lemma 4.4.1. In z coordinates we have

$$p_{23} = \begin{bmatrix} \frac{\sin\phi + \sin(\theta-\phi)}{\sin\theta} e^{-i\phi} \\ 0 \\ 1 \end{bmatrix}, \quad P^{-1}(p_{31}) = \begin{bmatrix} \frac{\sin\phi + \sin(\theta-\phi)}{\sin\theta} e^{i\phi} \\ 0 \\ 1 \end{bmatrix}.$$

$$\langle \mathbf{z}, p_{23} \rangle = -z_1 \sin\phi e^{i\phi} + \frac{\sin\phi \sin\theta}{\sin(\theta+\phi)}$$

and

$$\langle \mathbf{z}, P^{-1}(p_{31})\rangle = -z_1 \sin\phi e^{-i\phi} + \frac{\sin\phi\sin\theta}{\sin(\theta+\phi)}$$

Therefore $\big|\langle \mathbf{z}, p_{23}\rangle\big| < \big|\langle \mathbf{z}, P^{-1}(p_{31})\rangle\big|$ if and only if $\mathrm{Im}\,(z_1) < 0$. This gives (i).

In z coordinates we have

$$p_{12} = \begin{bmatrix} 0 \\ e^{i\theta} \\ 1 \end{bmatrix}, \quad R_1^{-1}(p_{31}) = \begin{bmatrix} 0 \\ e^{-i\theta} \\ 1 \end{bmatrix}.$$

Therefore $\big|\langle \mathbf{z}, p_{12}\rangle\big| < \big|\langle \mathbf{z}, R_1^{-1}(p_{31})\rangle\big|$ if and only if $\mathrm{Im}\,(z_2) > 0$. This gives (ii).

The other parts follow similarly. $\square$

Proposition 4.4.4. *The polyhedron D and its images under R_1^{-1} and R_2 tessellate around the face $F(R_1R_2^{-1}) = S(R_1) \cap S(R_2^{-1})$. Moreover, the corresponding cycle transformation is $R_2P^{-1}R_1$ and $l = m = 1$. This gives the cycle relation $R_2P^{-1}R_1 = I$*

Proof. Notice that $z \in D$ implies z satisfies all six conditions of Lemma 4.4.3. Using Lemma 4.4.3 (ii), (v) we obtain

$$D \subset \left\{ \mathbf{z} \in \mathbf{H}^2_{\mathbb{C}} : \big|\langle \mathbf{z}, p_{12}\rangle\big| < \big|\langle \mathbf{z}, R_1^{-1}(p_{31})\rangle\big|, \big|\langle \mathbf{z}, p_{12}\rangle\big| < \big|\langle \mathbf{z}, R_2(p_{23})\rangle\big| \right\} \tag{4.3}$$

Let us now characterize $R_1^{-1}(D)$. First notice that $z \in R_1^{-1}(D)$ if and only if $R_1(z) \in D$. Thus $R_1(z)$ satisfies the conditions of (3.4). From Lemma 4.4.3 (iii), (iv) we obtain

$$\big|\langle R_1(\mathbf{z}), p_{31}\rangle\big| < \big|\langle R_1(\mathbf{z}), R_1(p_{12})\rangle\big|, \big|\langle R_1(\mathbf{z}), R_1(p_{12})\rangle\big| < \big|\langle R_1(\mathbf{z}), R_1R_2(p_{23})\rangle\big|$$

where we have written $P = R_1R_2$. Thus

$$R_1^{-1}(D) \subset \left\{ \mathbf{z} \in \mathbf{H}^2_{\mathbb{C}} : \big|\langle \mathbf{z}, R_1^{-1}(p_{31})\rangle\big| < \big|\langle \mathbf{z}, p_{12}\rangle\big|, \big|\langle \mathbf{z}, R_1^{-1}(p_{12})\rangle\big| < \big|\langle \mathbf{z}, R_2(p_{23})\rangle\big| \right\} \tag{4.4}$$

In the same way, applying R_2 to Lemma 4.4.3 (vi), (i) we get:

$$R_2(D) \subset \left\{ \mathbf{z} \in \mathbf{H}^2_{\mathbb{C}} : \left|\langle \mathbf{z}, R_2(p_{23})\rangle\right| < \left|\langle \mathbf{z}, p_{12}\rangle\right|, \left|\langle \mathbf{z}, R_2(p_{23})\rangle\right| < \left|\langle \mathbf{z}, R_1^{-1}(p_{31})\rangle\right| \right\} \quad (4.5)$$

Now when we compare equations (4.3), (4.4) and (4.5) the three immediate equations above, we see that D, $R_1^{-1}(D)$ and $R_2(D)$ are all disjoint. Also $\overline{D}$, $R_1^{-1}(\overline{D})$ and $R_2(\overline{D})$ cover a neighbourhood of the interior of the face $F(R_1R_2^{-1})$. This second part of the result is proved in a similar manner to the second part of Proposition 4.4.2. The cycle transformation follows by observing that
S(R$_1$) $\cap\, S(R_2^{-1}) \overset{R_1}{\to} S(P^{-1}) \cap S(R_1^{-1}) \overset{P^{-1}}{\to} S(R_2) \cap S(P) \overset{R_2}{\to} S(R_1) \cap S(R_2^{-1})$.

□

When we apply $R_2^{-1} = P^{-1}R_1$ and R_1 respectively to Proposition 4.4.4 we see that D and its images under R_2^{-1} and P^{-1} tessellate around the face $F(PR_2) = S(P) \cap S(R_2)$ and that D and its images under R_1 and P tessellate around the face $F(R_1^{-1}P^{-1}) = S(R_1^{-1}) \cap S(P^{-1})$. On the otherhand, one could consider a direct argument similar to that given above. In both cases the cycle relation is a cyclic permutation of $R_2P^{-1}R_1 = I$.

4.5 TESSELLATION AROUND FACES IN TOTALLY GEODESIC PLANES

We now show that D and appropriate images tessellate around those faces of D containing either p_1 or p_2. We first look at the faces containing p_1. The result will follow for those faces containing p_2 by applying ι. Our method will be to use the arguments of the bisectors in question. This is because they are defined in terms of their arguments. Hence to show that one intersection is disjoint from the images of another, we have to show that either the argument of z_1 or the argument of z_2 (or both) is different. Conveniently we will describe the argument of z_1 and z_2 by looking at the signs of $Im(z_1)$, $Im(z_1e^{i\phi})$, $Im(z_2)$ and $Im(z_2e^{-i\theta})$.

For example, if $\mathbf{z} = P(\mathbf{w}) \in D$ then $Im(w_1) > 0$, $Im(w_1e^{-i\phi}) < 0$ $Im(w_2) > 0$ and $Im(w_2e^{-i\theta}) < 0$. Hence, if $\mathbf{z} \in P^{-1}(D)$ we have $Im(z_1) > 0$, $Im(z_1e^{i\phi}) < 0$, $Im(z_2) > 0$ and $Im(z_2e^{-i\theta}) < 0$. In the same way, I_1 sends $z_1$to $e^{2i\phi}z_1$ and fixes z_2. Hence if $z \in I_1(D)$ we have $arg(z_1) \in (\phi, 2\phi)$ and $arg(z_2) \in (0, \theta)$. In other words $Im(z_1) > 0$, $Im(z_1e^{i\phi}) > 0$, $Im(z_2) < 0$ and $Im(z_2e^{-i\theta}) < 0$. Likewise, if $z \in I_1P^{-1}(D)$ then $Im(z_1) < 0$, $Im(z_1e^{i\phi}) > 0$, $Im(z_2) > 0$ and $Im(z_2e^{-i\theta}) < 0$. Likewise R_1 maps z_1 to itself and maps z_2 to $e^{i\theta}z_2$. So if $\mathbf{z} \in R_1(D)$ we have $arg(z_1) \in (-\phi, 0)$ and $arg(z_2) \in (\theta, 2\theta)$. In other words, $Im(z_1) < 0$, $Im(z_1e^{i\phi}) > 0$, $Im(z_2) > 0$ and $Im(z_2e^{-i\theta}) > 0$.

Making use of similar arguments, it is easy to show that if $\mathbf{z}$ is in one of the following images of D then $Im(z_1)$, $Im(z_1e^{i\phi})$, $Im(z_2)$ and $Im(z_2e^{-i\theta})$ have the following signs:

	$Im(z_1)$	$Im(z_1e^{i\phi})$	$Im(z_2)$	$Im(z_2e^{-i\theta})$
D	$-$	$+$	$+$	$-$
$P^{-1}(D)$	$+$	$-$	$+$	$-$
$I_1(D)$	$+$	$+$	$+$	$-$
$I_1P^{-1}(D)$	$-$	$-$	$+$	$-$
$R_1(D)$	$-$	$+$	$+$	$+$
$R_1P^{-1}(D)$	$+$	$+$	$+$	$+$
$R_1I_1(D)$	$+$	$-$	$+$	$+$
$R_1I_1P^{-1}(D)$	$-$	$-$	$+$	$+$
$R_1^{-1}(D)$	$-$	$+$	$-$	$-$
$R_1^{-1}P^{-1}(D)$	$+$	$+$	$-$	$-$
$R_1^{-1}I_1(D)$	$+$	$-$	$-$	$-$
$R_1^{-1}I_1P^{-1}(D)$	$-$	$-$	$-$	$-$

Proposition 4.5.1. *The polyhedron D and its images under R_1^{-1}, P^{-1} and $R_1^{-1}P^{-1}$ tessellate around the face $F(PR_1) = S(P) \cap S(R_1)$. Moreover, the corresponding cycle transformation is $P^{-1}R_2^{-1}PR_1$ and $l = m = 1$. This gives the cycle relation $P^{-1}R_2^{-1}PR_1 = 1$.*

Proof. If $z \in F(PR_1)$ then $Im(z_1) = Im(z_2) = 0$. From the table, we read that if $z \in D$ then $Im(z_1) < 0$ and $Im(z_2) > 0$; if $z \in P^{-1}(D)$ then $Im(z_1) > 0$ and $Im(z_2) > 0$; if $z \in R_1^{-1}(D)$ then $Im(z_1) < 0$ and $Im(z_2) < 0$ and if $z \in R_1^{-1}P^{-1}(D)$ then $Im(z_1) < 0$ and $Im(z_2) < 0$. In all four cases $Im(z_1e^{i\phi}) > 0$ and $Im(z_2e^{-i\theta}) < 0$. Thus D, $P^{-1}(D)$, $R_1^{-1}(D)$ and $R_1^{-1}P^{-1}(D)$ are all disjoint. Furthermore, arguing as in Proposition 4.4.2, $\overline{D}$, $P^{-1}(\overline{D})$, $R_1^{-1}(\overline{D})$ and $R_1^{-1}P^{-1}(\overline{D})$

cover a suitably chosen neighbourhood of the interior of $F(PR_1)$.

The cycle transformation follows by observing that

$$S(R_1)\cap S(P) \xrightarrow{R_1} S(P)\cap S(R_1^{-1}) \xrightarrow{P} S(R_2^{-1})\cap S(P^{-1}) \xrightarrow{R_2^{-1}} S(P^{-1})\cap S(R_2) \xrightarrow{P^{-1}} S(R_1)\cap S(P).$$

□

When we apply R_1, PR_1, $R_2^{-1}PR_1 = P$ to Proposition 4.5.1 we see that D and its images tessellate around the faces $F(PR_1^{-1})$, $F(P^{-1}R_2^{-1})$ and $F(P^{-1}R_2)$ respectively. In each case the cycle relation is a cyclic permutation of $P^{-1}R_2^{-1}PR_1 = I$. Making similar arguments we have:

Proposition 4.5.2. *The polyhedron D and its images under R_1^{-1}, $I_1P^{-1} = J^{-1}$ and $R_1^{-1}I_1P^{-1}$ tessellate around the face $F(JR_1) = S(J) \cap S(R_1)$. Moreover, the corresponding cycle transformation is $J^{-1}R_2^{-1}JR_1$ and $l = m = 1$. This gives the cycle relation $J^{-1}R_2^{-1}JR_1 = 1$.*

Proposition 4.5.3. *The polyhedron D and its images under P^{-1}, I_1 and I_1P^{-1} tessellate around the face $F(PJ) = S(P) \cap S(J)$. Moreover, the corresponding cycle transformation is $P^{-1}J$ and $l = 1$, $m = k$. This gives the cycle relation $(P^{-1}J)^k = 1$.*

The above results can be used to show that D and its images tessellate around $F(JR_1^{-1})$, $F(J^{-1}P^{-1})$, $F(J^{-1}R_2)$ and $F(J^{-1}R_2^{-1})$. The cycle transformations of these faces are cyclic permutations of relations which have already been obtained.

Now let us consider $F(R_1R_1^{-1}) = S(R_1) \cap S(R_1^{-1})$. This involves points of ∂D for which $z_2 = 0$. Hence this face is fixed by R_1. The result becomes easy to prove since R_1 is obtained by multiplying z_2 by $e^{i\theta} = e^{2\pi/n}$.

Proposition 4.5.4. *The polyhedron D and its images under powers of R_1 tessellate around the face $F(R_1R_1^{-1}) = S(R_1) \cap S(R_1^{-1})$. Moreover, the corresponding cycle transformation is R_1 and $l = 1$, $m = p$. This gives the cycle relation $R_1^p = 1$.*

When we apply ι we obtain

Proposition 4.5.5. *The polyhedron D and its images under powers of R_2 tessellate around the face $F(R_2^{-1}R_2) = S(R_2^{-1}) \cap S(R_2)$. Moreover, the corresponding cycle transformation is R_2 and $l = 1$, $m = p$. This gives the cycle relation $R_2^p = 1$.*

In conclusion, we have showed that our Theorem 4.1 satisfies Poincaré's polyhedron theorem thereby proving it and generating the associated summary of the faces, cycle element and cycle relation below

Face	Cycle element	Cycle relation
$F(JJ^{-1}) = S(J) \cap S(J^{-1})$	J	$J^3 = I$
$F(R_1R_2^{-1}) = S(R_1) \cap S(R_2^{-1})$	$R_2P^{-1}R_1$	$R_2P^{-1}R_1 = 1$
$F(PR_1) = S(P) \cap S(R_1)$	$P^{-1}R_2^{-1}PR_1$	$P^{-1}R_2^{-1}PR_1 = 1$
$F(JR_1) = S(J) \cap S(R_1)$	$J^{-1}R_2^{-1}JR_1$	$J^{-1}R_2^{-1}JR_1 = 1$
$F(JP) = S(J) \cap S(P)$	$P^{-1}J$	$(P^{-1}J)^k = 1$
$F(R_1R_1^{-1}) = S(R_1) \cap S(R_1^{-1})$	R_1	$R_1^p = 1$
$F(R_2R_2^{-1}) = S(R_2) \cap S(R_2^{-1})$	R_2	$R_2^p = 1$

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APPENDICES

NOTATIONS AND DEFINITIONS

Let $\mathbb{Z}$,$\mathbb{Q}$,$\mathbb{R}$ and $\mathbb{C}$ denote the integers, the rationals, the real and complex numbers respectively.

Definition (Euclidean space)**.** Euclidean n- space denoted $\mathbb{R}^n$ is the metric space with the metric

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$$

where the right hand side is the Euclidean norm $|\mathbf{x}| = (\mathbf{x}.\mathbf{x})^{1/2}$. The inner product is the usual dot product given by

$$\mathbf{x}.\mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

where $\mathbf{x} \in \mathbb{R}^n$

Definition (Isometry)**.** An isometry of $\mathbb{C}$ is a distance-preserving function from $\mathbb{C}$ to $\mathbb{C}$, i.e., a function $f : \mathbb{C} \to \mathbb{C}$ so that for all $z, w \in \mathbb{C}$,

$$|f(z) - f(w)| = |z - w| .$$

Some of the examples of isometries are translations, conjugation and rotation. The above definition is suitable for the space of $\mathbb{R}$ of $\mathbb{C}$. In the space of complex hyperbolic, the definition changes a little. We no more talk of distance-preserving function but rather a metric preserving function.

Definition. A *symmetry* of a set S is a bijection from S to itself.

When the set S is a geometric object, i.e., a subset of Euclidean space, a symmetry of S is a motion(isometry) of the Euclidean space that maps S to itself. If you consider the symmetries of an equilateral triangle in the Euclidean plane, you are likely to get motions(isometries) such as reflections, rotations and translations. These symmetries gives us a basic example of a group.

Definition. A *group* is a set G closed under an operation denoted . with the following properties:

(1) If $a, b, c \in G$, then $(a.b).c = a.(b.c)$ (associative law).

(2) There is an element $e \in G$ satisfying $a.e = e.a = a$ for all $a \in G$. The element e is called the identity element.

(3) If $a \in G$, there is an element $a^{-1} \in G$ satisfying $a.a^{-1} = a^{-1}.a = e$. Where a^{-1} is called the inverse of a.

Additionally we say G is *abelian* if $a.b = b.a$ for all $a, b \in G$. Lets look at some examples.

(1) $\mathbb{Z}$,$\mathbb{Z}_m$,$\mathbb{Q}$,$\mathbb{R}$,$\mathbb{C}$ are all abelian groups with respect to the addition operation.

(2) $GL(2, \mathbb{R})$, the set of invertible 2x2 real matrices with matrix operation is a group, called the *general linear group*.

(3) $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$ the set of 2x2 matrices with determinant 1 is a group called **general linear group.**

(4) $GL(n, \mathbb{C})$ **the set of non-singular nxn complex matrices. Also called the general linear group of dimension** n **in complex domain.**

Definition (group action)**.** A group G is said to act on a set X if there exists a function from $G \times X$ to X, given as $g, x \longmapsto gx$, such that for all $g, h \in G$, and all $x \in X$,

$$1x = x$$

and

$$g(hx) = (gh)x$$

.

Definition (transitive action)**.** A group G is said to act transitively on a set X if for each $x, y \in X$ there is a $g \in G$, such that $gx = y$.

Some quotient groups are as follows: Let $\mathbb{C}^*$ be $\mathbb{C} \setminus \{0\}$. The group $PGL(n, \mathbb{C})$ is the quotient of the group $GL(n, \mathbb{C})$ by the normal subgroup $\{\lambda I : \lambda \in \mathbb{C}^*\}$. The group $PSL(n, \mathbb{C})$ is then the quotient

$$SL(n, \mathbb{C})/(SL(n, \mathbb{C}) \cap \{\lambda I : \lambda \in \mathbb{C}^*\}).$$

Definition (discrete group)**.** Let $GL(n, \mathbb{C})$ be the general linear group of non-singular nxn matrices in the complex plane. $GL(n, \mathbb{C})$ is a topological group. A subgroup G of $GL(2, \mathbb{C})$ is discrete if and only if the subspace topology on G is the discrete topology.

In order to prove that G is discrete, it is only necessary to prove that one point of G is isolated: for example, it is sufficient to prove that:

$$inf\{\|X - I\| : X \in G, X \neq I\} > 0 \tag{4.6}$$

Definition (Cone Metric)**.** Cone metric is metric on the sphere which is locally Euclidean except at a finite number of points. Where these finite number of points have neighbourhoods locally modelled on cones.

Definition (Cone of cone-angle θ)**.** A cone of cone-angle θ is a metric space that can be formed, if $\theta \leq 2\pi$, from a sector of the Euclidean plane between two rays that make an angle θ, by gluing the two rays together.

Generally, a cone of angle θ can be formed by taking the universal cover of the plane minus 0, reinserting 0, and then identifying modulo a transformation that "rotates" by angle θ.

Definition (Apex Curvature)**.** The apex curvature of a cone of cone-angle θ is $2\pi - \theta$.

Definition (Orbifold)**.** An orbifold is a quotient space of a discrete group.

Our orbifolds will be (X, G)-orbifolds, locally modelled on a homogenous space X with a group of isometries G (specifically Lie Group). It can be seen by induction on dimension that an orientable (X, G)-orbifold has an induced metric which makes it into a cone-manifold (using the naturality of the exponential map).

Definition (Holomorphic function)**.** Let Ω be a region (a non-empty connected open subset) of the complex plane. Let f be a complex function defined on Ω. If $z_0 \in \Omega$ and

$$f^{'}(z_0) = lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exist for every $z_0 \in \Omega$, then f is said to be holomorphic (or analytic) in Ω.

Definition (linear fractional transformation)**.** Consider the Reimann sphere $\hat{\mathbb{C}}$. Let $LF(\hat{\mathbb{C}})$ be the set of all linear transformations of $\hat{\mathbb{C}}$. A linear fractional transformation is a continuous map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$\phi(z) = \frac{az + b}{cz + d},$$

where a, b, c, and d are in $\mathbb{C}$ and $ad - bc \neq 0$.

Definition (Möbius Transformation)**.** A Möbius Transformation acting in $\mathbb{R}^n$ is a finite composition of reflections (in spheres or planes).

Each Möbius Transformation is a homeomorphism onto itself. The composition of two Möbius Transformation and so also is the inverse of a Möbius Transformation for if $\Phi = \Phi_1 \ldots \Phi_m$ (where the Φ_j are reflections) the $\Phi^{-1} = \Phi_m \ldots \Phi_1$. Finally, for any reflection Φ say, $\Phi^2(x) = x$ and so the identity map is a Möbius Transformation.

Let $\rho : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$ be complex conjugation, that is, $\rho(z) = \bar{z}$. It can be proved that $M(\hat{\mathbb{C}}) = LF(\hat{\mathbb{C}}) \cup LF(\hat{\mathbb{C}})\rho$

Möbius Transformations of $LF(\hat{\mathbb{C}})$ are called orientation preserving.

Definition (Topological group)**.** A topological group is a group G which is also a topological space, whereby the multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ are continuous functions in G.

Examples of topological groups are $GL(n, \mathbb{C})$ and $SL(n, \mathbb{R})$, and their topology is the metric topology induced by the distance function

$$d(A, B) = |A - B|,$$

which is the matrix norm defined by

$$|A| = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{1/2}$$

Definition. Let Γ be a subgroup of a topological group G. Then Γ is said to be a discrete subgroup of G if for all $\gamma \in \Gamma$ there exists an open neighbourhood Ω of γ in G such that $\Omega \cap \Gamma = \{\gamma\}$.

It can be shown that $SL(n, \mathbb{Z})$ is a discrete subgroup of $SL(n, \mathbb{R})$, and $PSL(n, \mathbb{Z})$ is a discrete subgroup of $PSL(n, \mathbb{R})$

Definition (locally finite fundamental domain)**.** If $\mathcal{S}$ is a collection of subsets of X, then $\mathcal{S}$ is described as locally finite if for each point $x \in X$, there is an open neighbourhood of x which meets only finitely many members of $\mathcal{S}$. We say that a fundamental domain D is a locally finite fundamental domain if the collection $\{\gamma\bar{D} : \gamma \in \Gamma\}$ is a locally finite collection of sets.

Definition (Convex Polyhedron)**.** A convex polyhedron in a metric space X is a non-empty, closed, convex subset of X with finitely many sides and a non-empty interior.