THE MATHEMATICS OF THE VIBRATING MEMBRANE AS APPLIED TO AFRICAN DRUMS WITH VARYING TENSION COMPARED WITH THOSE OF CONSTANT TENSION:

THE CASE OF THE "DONNO" AND THE "ATUMPAN".

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BY

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DECLARATION

This thesis is the true account of the candidate's own work except for references to other people's work which have been fully acknowledged.

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DEDICATION

This piece is dedicated to my mother

Charlotte Sackey,

My sister Gloria Naa Amaley Sackey and

Priscilla Aopare

ABSTRACT

In this thesis the physical modeling of two Ghanaian percussive drums the "*donno*" of the Akans or "*Lunna*" of the Dagbambas and the "*Atumpan*" of the Akans was approached using the two dimensional wave equations and by imposing boundary and initial conditions on the drumhead. A remark is made about the overtones of these local drums after using matrix laboratory (matlab) to generate their Normal modes using three different types of initial velocity functions.

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CHAPTER ONE

INTRODUCTION

1.1 The History of drums

A Drum is a musical instrument in the percussion family, technically classified as a Membranophone, which literally means "skin sound". It is a Latin and Greek word combined into one. It describes the instrument made by stretching a skin of animal, vegetable, or man-made material so that when it vibrates it produces sound. The history of drums goes back to the seventh century B.C.

In various forms, they have existed before 6000 B.C., and had been found historically, in nearly every culture of the world. Drums consist of a body or a hollowed-shell and a membrane or a piece of animal skin or synthetic material stretched over one end or both ends of the hollowed body. The membrane is called the drumhead or drum skin and it is played by beating on the stretched membrane, either directly with parts of the player's body, or with some sort of implement such as a drumstick, to produce sound. Drums are among the world's oldest and most ubiquitous musical instruments, and the basic design has been virtually unchanged for hundreds of years. The shell almost invariably, has a circular opening over which the drumhead is stretched, but the shape of the remainder of the shell varies widely. In the western musical tradition, the most usual shape is cylindrical, although the Timpani drum for example use bowl-shaped shells. Other shapes include truncated cones (bongo drums) and joined truncated cones (talking drums). Drums with cylindrical shells can be open at one end (as in our local drum-fontomfrom) or more commonly in the Western tradition, they can have another drum head. Sometimes they have a solid shell with no holes in at all though this is rare. It is usual for a drum to have some sort of hole, to let air move through the drum when it is struck. This gives a louder and longer ring to the notes of the drum, thus drums with two drum skin covering both ends of a tubular shell often have a small hole halfway between the two drumheads. The membrane is struck, either with the hand or with a drumstick, and the shell forms a resonating chamber for the resulting sound. The sound of a drum depends on several variables including shell shape, size, thickness of shell, materials of the shell, type of drumhead, tension of the drumhead, position of the drum, location, and how it is struck.

Drums weren't used for entertainment way back then, but had ceremonial, sacred, and symbolic associations. Many civilizations adopted the use of drums, or similar instruments, to warn their people against dangers or to initiate their armies. The drum was a perfect choice because it was easy to make, made a lot of noise, and could be heard loud and clear.

1.2 Importance of Drums to Africa

Africa is a land of many countries, climates, and cultures. It is a place of modern cities, and traditional villages.

One cannot talk of drums and drumming in Africa without making mention of Music, since music is an integral part of every African individual from birth. In traditional

African societies, the absence of music in daily life is unthinkable. Music is used to heal the sick, praise a leader, ensure successful delivery of a child, cure bed wetting, and even to stop a woman from flirting with another woman's husband. Music is also involved with birth, naming of a child, teething, marriage, new moon, death, puberty, agriculture, re-enacting of historical events, hunting, preparation for war, victory celebrations and religious rites. In some African societies music is a dynamic and driving force that animates the life of the entire community. In most cases music goes along with drumming amidst dancing.

Ensemble drumming

Other uses of drums are seen in Artistic performances such as Ensemble drumming. It is practiced throughout West Africa. Drum ensembles play for social occasions, ritual, ceremonies, weddings, funerals, parties, and religious meetings.

Other instruments often join the drums to accompany singing and dancing. Drumming, singing and dancing are often performed in a circular formation going counterclockwise.

Drum ensembles are often led by a master drummer who plays solos against the overlapping patterns. The master drummer also leads the ensemble by playing signals that tell the other players to switch to a different section, change drum patterns, change the tempo, signal the dancer, or end the piece.

Drums are among the most important art forms in Africa, used both as musical instruments and as work of sculpture significant in many ceremonial functions, including communication of messages. (Susan Simandle-The Music and Instruments of West Africa)

1.3 The history and importance of the Talking Drum in West Africa

The talking drum originated from the Mali Empire (10^{th}-16^{th} C.AD) where master drummers who were known as griots would travel throughout the villages playing the news of the day. Today the talking drum can be found throughout West Africa, with many variations of the drum and of its use.

For nearly three and a half centuries, the Walo Walo of Senegal who are the descendants of the ancient Wolof Kingdom of Walo, have played the tama (Senegalese talking drum) by using rhythms extensively to represent words and by tradition the audience know what these rhythms and sounds of the talking drum represent.

The Yoruba of Nigeria have played the talking drum for generations. It is believed to have been originated from Oyo. It was first assembled for the Alaafin (the royal father and custodian of Oyo and the Yoruba heritage), as his musical outfit whenever he goes to war. He used it to motivate his army in the past. The talking drum serves as an important instrument of history. It is use to wake up the Alaafin as early as 5.00 am by reciting Past Alaafin , telling the incumbent Alaafin the challenges they faced ,how they did overcome the challenges as well as the methods they used. The drum goes on to talk about particular songs, dance steps or mannerisms of past Alaafin to enable the incumbent Alaafin know the history of his predecessors.

Something similar could also be said about the Dagbamba of northern Ghana who have developed a high standard of ensemble talking drumming.

It is a common tradition that some chiefs of the smaller neighboring villages together with their entourage, dressed in their traditional regalia's, visit the royal palace early morning on Friday. The drummer troupe stands outside the palace to drum announce their arrival on the talking drum. Even before their arrival the drummers are already in place outside the doors of the royal palace and the master drummer sings the history of all the chiefs who have come, to awake the paramount chief in the process. The leads drummer will actually play the names of the visiting chiefs on the talking drum to announce them, as well as where they are from. They could also include things like who they have come with, and where they are likely to go after their visit, how long it has been since their last visit, and if time is permitted the family's history of their heritage.

The language spoken by the Dabgambas is Dagbanli. It is a tonal language and so all the words you say has a natural rise and fall to them just like in English, the words have an emphasis or stress on a part of the word. By virtue of the tonal nature of the Dabganli language of the Dabgambas, the master drummers can play proverbs on the Luna (talking drum) to describe their roots.

1.4 Cultural background of the "Donno" (hand-held talking drum)

The Dagbamba have a rich musical and cultural history, they are noted for their use of the Luna, or hand-held talking drum. They are one of the cultural groups in the Northern part of Ghana with a very sophisticated oral culture woven around drums and other musical instruments. Thus most of its history, until quite recently, has been based on oral tradition with drummers as professional historians. The drummers belong to a hereditary clan called Lunsi. They serve as verbal artists, genealogists, counselors to royalty, cultural experts, and entertainers. They hold the history of the people through storytelling. According to oral tradition, the political history of Dagbon has its genesis in the life story of a legend called Tohazie (translated as Red Hunter.). Culturally the Dagbon is heavily influenced by Islam. Inheritance is patrilineal. Some of the important festivals of the Dagbon traditional Area include the two Islamic Eid, Damba, and Bugum (fire festival).

Dagomba membranophone ensembles have two distinct types of drums, the Gungon or Brekete which is a low pitched drum with snare and the Luna or Donno which is a pressure sensitive squeeze drum. The Gungon is used in dance performances such as Bamaya and Damba/Takai. The Luna is smaller than the Gungon. This drum is placed underneath the arm and is played with a curved stick. The playing technique of the Luna can imitate the nuances of spoken language (Dagbani) through pitch variation. Bamaya is usually performed on the way to the shrine. Bamaya literally means that the entire place is wet. When the Bamaya dance is being performed the men become subservient, dress in women clothes, wear headbands and lipstick and parade around the village during the harvest season. The story goes that a village that was very dry did the dance to bring rain. It began to rain and rained so much that the villagers were left to perform a mud dance. The lesson that could be drawn from performing this dance is that one must be careful what one desires, because he/she may ends up getting it. The Bamaya is always danced by wearing smock called batakari with a belt having tassels and ornaments that accentuates the movement of the waist. Ankle bells are also worn.

Fig1.The Gungon or Brekete drum

Fig 2. The Luna or Hour-glass drum

Damba/Takai

Damba is an annual New Year's festival celebrated by the dabgamba people of Ghana,-west Africa. The Damba dance used to be a religious festival in times past but in modern day of the Dagbon traditional community it is regarded as a non religious ceremony that is celebrated by the whole community and has been incorporated into the harvest festival. The festival provides a platform for the people to pay homage to the sub chiefs and the paramount chief who sits in state at his palace. The festival is headed by the paramount chief of the area. The Damba involves series of dances performed to commemorate the birth of the prophet Mohammed and are combined in a medley. The first two rhythms are Damba and the rest is Takai. The Damba dance can be segregated into two main parts. The naming of the prophet Mohammed is So Damba and the birthday of the prophet Mohammed is Na Damba. Damba is the signature dance of the Damba Festival which celebrates the Birth of the Holy Prophet Mohammed. Damba is almost an obligatory dance for the Dagomba chiefs, and is associated with festivals celebrating a large harvest on the farm.

Takai is a Dagomba area war dance. This war dance is performed for two reasons: To train warriors and to show what transpired on the field during the war. The dance is traditionally performed by men. The sticks they carry used to be swords in the past. In modern time the theater has put together Damba with Takai which is why you see women dancing it as well. The first rhythm is the traditional rhythm, and the other rhythms were incorporated by the Arts council of Ghana in the 1960's.

1.5 Description of the talking drum

The talking drum is a very important instrument in the culture of the Dagomba people. They refer to it as the Luna. In addition to being one of the oldest indigenous instruments of the Dagbamba, (Dagomba-singular, Dabgamba-plural) it has remained faithful to its original construction and is still made of ordinary materials found in the Dagomba land.

The shell of the Luna is carved from cedar wood into its characteristic hourglass shape that is two gently curving cones connected by a hollow cylinder. Each end of the shell is covered with a goat skin head sewn onto a circular rim made of reed and grass. The two heads are connected by antelope skin tension cords from one end of the shell to the other end. (See fig.2). The drum is played with a constructed curved wooden stick held in the right hand while the Luna is suspended from the left shoulder by a scarf tied to the central cylinder shell and fits snugly into the player's armpit.

The series of strings holding the heads suspended from the circumference and running the length of the drum can be squeezed under the arm. This builds up pressure within the drum which regulates the tension on the drumheads to raise or lower the resultant pitch. In other words when the drum is squeezed, the drumhead tightens and the pitch goes up. Also the pitch comes down when the pressure of the strings are released.

The hand held drum or the "*Donno*" has a number of possible pitch inflections. Based on this characteristic and the fact that tonal languages are use in many African cultures, it is possible to send linguistic messages via the drum. The primary use of the "*Donno*" was to send messages in the past and later found its use in religious chants or poetry, local festivities and dancing.

What makes the talking drum unique is its ability to adapt to the tone of any musical instrument. The uses of the talking drum are many and it is wide spread in West Africa.

In the chapters that follow, the researcher shall review the literatures of earlier writers on the mathematics of some musical instruments and also put into classifications, some musical instruments use in Africa for the chapter two. The chapter three would see us make certain assumptions on a vibrating membrane in order to derive the two dimensional Cartesian wave equation. The Cartesian wave equation will be converted into Plane Polar equation in order to model the circular drum with constant tension in the drum head. The resulting model would further be transformed into Bessel functions of order zero and of order half. For the chapter four, the researcher tries to modify the assumptions made on the derivation of the wave equation so as to model a vibrating membrane with varying tension. This model will also be transformed into Bessel functions of orders zero and one-third using a suitable substitution. The remaining two chapters i.e. chapters five and six will also see us impose some boundary conditions on the models derived in previous chapters and apply matrix laboratory (mat lab) to solve the normal modes of these models. The results chinned out are then summarized, analyzed and interpreted in the light of the thesis topic. Other findings are also discussed and finally conclusions and recommendations made.

CHAPTER TWO

LITERATURE REVIEW

The researcher in this chapter tries to review the literatures of what others have done on how the mathematics of music began, laying emphasis on some musical instruments such as the willow flute, the vibrating string and its application on the guitar, the kettle drum, the Indian local drum and how they are modeled mathematically. The review shall also touch on some African musical instruments, their origins and classifications.

2.1 The History of the Mathematics of Musical Instruments

The study of the mathematics of musical instruments date back at least to the followers of Pythagoras, who on dividing the length of a chord into ratios between the numbers one, two, three, and four of a monochord, that is a single stringed musical instrument produced vibrations of related pitches, which could be replicated by dividing the string into proportions of its length. They discovered harmonic series of notes they considered to be pleasing and corresponded to simple ratios of lengths. Contrary to this in 1634 Marin Mersenne published the first systematic study of harmonics, as “Harmonie Universelle” where he established that the pitch of a bowed note is determined by the frequency at which the string vibrates and that the frequency depended on the type of string , the tension applied, the length and the diameter of the string.

In the mid-17th century, harmonics offered a true mathematico-experimental framework that might not only be applied to strings and pendulums but also to light and gravity. Mersenne understood the production of pure tone and notes that sounded less clearly by a wave from the length of the string to be very complex and attributed it to some partials or harmonics .It was later confirmed that any string will allow for multiple waves to occur, chiefly those that fit neatly along its length. Their wavelengths are determined by dividing the string by whole numbers, as well as the octaves that result from repeated division by two; other important partials result from an odd number of waves fitting along the length of the string (notably the dominant, or fifth note of the scale, produced by division into thirds).The quality and quantity of partials contribute to the overall sound produced.

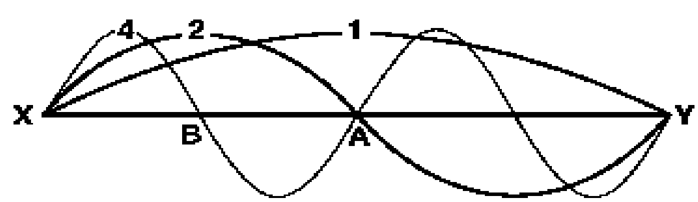


Fig. 2.1 The Harmonics of Pythagoras

Newton was the first to conduct a detailed analysis of the behavior of sound waves under various circumstances and among the first, after Mersenne, to calculate the speed of propagation of sound waves, which he called “successive pulses” of pressure arising from vibrating parts of a tremulous body. Newton was aware that the amplitude of these “pulses” varied both in time and space. Therefore, the obvious mathematical analysis of sound is through differential equations because sound propagates both in time and space. However, a differential equation governing wave behavior was not discovered until

1747, when d’Alembert derived the one-dimensional wave equation for a vibrating string.

The significance of the mathematical behavior of waves in music is far-reaching. Without a proper understanding of the mathematics involved, it would be impossible to build a proper musical instrument or concert hall.

For two millennia since Pythagoras, a number of musical systems have evolved but they still contain potential discords. It is in this light that the researcher makes an attempt to review the works of some earlier writers on wind, string and membrane instruments and their mathematical model.

2.2 The mathematics of the willow flute

Rachel W. Hall and Kresimir Josić (2000-) considered the physical properties of a primitive wind musical instrument called the willow flute.

The operation of this instrument does not depend on finger holes to produce different pitches, but rather by varying the strength of air blown into the instrument , the player selects from a series of pitches called Harmonics whose frequencies are integer multiples of the least tone called the fundamental frequency. The flute is made from a Hollow willow branch or alternatively in modern times PVC pipes in such a way that one end is left open and slot are constructed at the other end into which the player blows, forcing air across a notch in the body. Since the instrument has no finger holes the resulting vibration of the air creates standing waves inside the instrument whose frequency determines the pitch.

These researchers stated that quite a number of different tones can be produced on the willow flute and the possibility of this assertion lies in the mathematics of sound waves. They defined the pressure in a tube by a function U which is dependent on the position x along the length of the tube at time t.

Since the pressure across the tube is close to a constant, the direction is neglected and pressure outside the tube is chosen to be zero. The one-dimensional wave equation

$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, provides a good model of the behavior of air molecules in the tube, where 'α' is a positive constant. Since both ends of the tube are open, the pressure at the ends is the same as the outside pressure; that is, if L is the length of the tube, $U(0,t) = 0 \; and \; U(L,t) = 0$.The solution to the wave equation is a linear combination of solutions of the form

$U_n(x,t) = \sin \frac{n\pi x}{L} \left(a \sin \frac{an\pi t}{L} + b \cos \frac{an\pi t}{L} \right)$, where $n = 1,2,3 \ldots$.., and "a" and "b" are constants. To predict the possible frequencies of tones produced by the willow flute, Rachel W. Hall and Kresimir Josić considered solutions that contain one value of n, and realized that the pressure varies periodically with period ${}^{2L}/_{\alpha n}$ when n and x are fixed as t varies. This implies that the frequency of a willow flute or wind instruments in general will always be defined as $f = {}^{\alpha n}/_{2L}\,; n = 1,2,3 \ldots\ldots$. Based on their findings the researchers concluded that there are two ways to play a wind instrument: either by changing the length of the tube, L or by changing n.

However for the case of a string instrument, which is also governed by the one-dimensional wave equation, there is a third way of playing it, since besides these variations, the value of “α” could be changed by changing the tension through string bending or by using a string of different density.

2.3 A mathematical model of a guitar string

The Guitar is a musical instrument that produces sound when stretched strings are made to vibrate. The strings are made of gut, metal, and nylon or plastic. Types of stringed instruments include: bowed, such as the violin family and viol family; plucked, such as the guitar, ukulele, lute, sitar harp, banjo, and lyre; plucked mechanically, such as the harpsichord; struck mechanically, such as the piano and clavichord; and hammered, such as the dulcimer.

Rasmus Storjohann (2000-) used a mathematical model of a vibrating string to investigate the effect of plucked point and string properties on the sound of a string, and on how the sound of the string evolves over time from the plucked point to the time when the sound fully dies out. Like all physical systems the guitar is constructed so that when it is plucked away from the equilibrium position, it would return to the equilibrium position by having the side effect of producing a pleasant sound.

In the stationary position the string possesses potential energy which is converted to kinetic energy as the string vibrates due to the non-straight and non-equilibrium state. By the time the string is straight, it has so much speed that it overshoots and becomes non-straight in the opposite direction. It will then turn around and move towards a straight shape, the process is repeated again and again with picking up speed and overshoots until the string finally reaches its equilibrium point. The tighter the string, the faster it will move, and the higher the pitch. The heavier the string, the slower it will move, and the lower the pitch. The frequency of a vibrating guitar string is determined by the tension and the weight of the string, and so by tuning the string to a particular frequency say 440Hz, any multiple of this frequency which is not an integer would instantly die out; however the guitar string will naturally vibrate at frequencies that are integral multiples of the frequency tuned to before plucking. This gives rise to the overtone series. The pluck forces the string into some shape that it wants to get away from to gain its equilibrium position. The initial shape of the string is determined to a large degree by the position of pluck and how hard the player uses the finger or nail on the string. These choices have large effect on the resulting sound, since the initial shape of the string at the time of the pluck determines the strengths of the various overtones.

When the string is released it vibrates and after a while the sound dies or decay's. Thus the change in the sound quality from the time of pluck to the time of decay gives the guitar two distinct phases known as the attack and decay. The reason associated to this is that, the harmonics decay at different rates. The decay is caused by a number of different mechanisms namely

1. Resistances to the bending of the string

2. Air resistances breaking the movement of the string

3. Transfer of energy from the string to the body of the guitar;

(This on the whole drains the vibrating string of its energy, making it return to its equilibrium position.)

Rasmus Storjohann on the basis of these decay assumptions cited a paper by V.E Howle and Lloyd N Trefethen called “Eigen values and Musical instrument” to give the details of the relative importance of these different processes. By giving a table to show numbers that quantifies the decay rates of the various Harmonics of the type of string on the guitar as well as a mathematical model of a vibrating string for investigating these processes. Below is a table showing the first four partials and the mathematical model he employed in computing them.

Decay rate of the four first Partial for a guitar equipped with steel or nylon strings.

Partial	Decay rate (steel)	Decay rate (nylon)
1 (fundamental)	0.1	0.4
2 (octave)	0.13	0.55
3 (etc)	0.16	0.7
4	0.18	0.9

Table 2.3.1

$$U(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left(\frac{1}{k(1-k)}\right)\sin(n\pi k)\exp(-\alpha_n t)\sin\left(\frac{n\pi}{L}x\right)\cos\frac{cn\pi}{L}t\ ,$$

where $n = 1,2,3$ .... and k is the plucked point, which lies in the interval $0 < k < 1$.

This expression is the solution to the one dimensional wave equation with some modification by the introduction of the term: $\frac{2}{n^2\pi^2}\left(\frac{1}{k(1-k)}\right)\sin(n\pi k)\exp(-\alpha_n t)$. This model is use to compute the value of U which gives the amplitude of the string from its equilibrium position at any point x from one end of the string at any time t in seconds. L is the length of string in centimeters, k is the point on the string where it is plucked and it is usually expressed as a fraction of the entire string; if the string is plucked at the middle k is assigned the value 0.5 if it is plucked near the ends k is assigned the values 0.1 or 0.9. n gives us the Harmonics or the partials, so that for n=1 we have the fundamental, for n=2, we have the first overtone (the octave),

for n=3 we have the next overtone (octave + fifth) and so on. The ratio of the tension in the string to the mass of the string is represented as c in the mathematical model;$-\alpha_n$ is the decay rate which can be read off from the table 2.3.1.

The remaining terms in the equation are as follows:

$\frac{2}{n^2\pi^2}\left(\frac{1}{k(1-k)}\right)\sin(n\pi k)$ - This expression determines how much there is of each partial at the outset, i.e. when the string has been plucked. It is the only place in the expression where k appears and the only part of the system that depends on where the string is plucked.

$\sin\left(\frac{n\pi}{L}x\right)$ - This describes the shape of the string. Note that this term is always zero at the end points of the string (x=0 and x=L). So that for n=1, a half-wave exist between the ends of the string, for n=2 a whole wave, and for n=3 one and a half wave, etc. These are known as the overtone series.

$\cos\frac{cn\pi}{L}t$ - Gives the vibration of the string as a function of time.

$\exp(-\alpha_n t)$ -This term describes the vibration of the nth partial. Each partial decays with its own decay rate $-\alpha_n$, which explains the phenomenon of attack. And the term

$\exp(-\alpha_n t)\cos\left(\frac{cn\pi}{L}t\right)$ - Describes the complete time dependence of the position of the string, including both the vibration and the exponential decay.

2.4 The mathematics of the Kettle drum (Timpani)

In his lecture note "Playing the Timpani-Vibrations of a circular membrane" Darryl Yong described the Kettle drum or Timpani as a large orchestral percussion drum, made of a metal shell which is hemispherical in shape, with the top covered in stretched vellum. The pitch of the drum can be altered by adjusting screws at the side which changes the tension of the vellum before it is played.

Traditionally, two kettle drums are used in an orchestra, although the use of three is not uncommon in modern piece. The drum is usually played with soft-headed wooden drumsticks or mallet. Besides the appearing of the kettle drum in the percussion group in classical music the sound of it is present in many forms of music from many different parts of the world.

What makes it easier to identify and distinguish the sounds of different types of drums, even without seeing the instruments is that each drum vibrates at characteristic frequencies, depending mainly on the size, shape, tension, and composition of its sound-generating drumhead. In looking at how the kettle drum works mathematically, the researcher looked at the vibration of the circular drumhead and the air in the drum enclosure.

As a first approximation, the vibrations of the timpani's drumhead can be modeled by the wave equation, $u_{tt} = c^2\nabla^2 u$, where c is the speed of waves travelling on the drumhead. The constant c is directly related to the tension of the drumhead. The corresponding pitch that is generated by hitting the drumhead with a mallet can also be adjusted using a foot pedal.

The sound characterized by the timpani is determined by its vibrational modes and their corresponding frequencies. Timpani drums have diameters between 23-29 inches and most timpani players usually prefer playing about 4 to 5 inches from the edge than to strike the drumhead in the center which produces a sound that is somewhat hollow. In his lecture note, however, he showed why this is so by simply giving a mathematical model to explain more about the vibrational modes (Eigen functions) of the circular membrane via solving the wave equation by separation of variables. Since the membrane is circular he said it was convenient to use polar coordinate to explicitly write the displacement of the membrane in the wave equation $u = u(r, \theta, t) = R(r)H(\theta)T(t)$ as $u_{tt} = c^2\left[u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u\right]$. This partial differential equation readily reduces to three different ordinary differential equations in r, θ and t namely

$H'' + n^2H = 0,$

$T'' + c^2\lambda^2 T = 0,$

$r^2R'' + rR' + (\lambda^2r^2 - n^2)R = 0,$

where the separation constants are n^2 and λ^2.

The first O.D.E. in theta has 2π-periodic solutions only if n is an integer in the form

$H_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)\ \ and\ n = 0,1,2 \ldots \ldots$

The second O.D.E. has a general solution of the form

$T(t) = \alpha \cos(c\lambda t) + \beta \sin(c\lambda t)$,

provided $\lambda \geq 0$ so that the frequency of any particular vibration mode is determine by $\frac{2\pi}{(c\lambda)}$.

The solution to the third O.D.E. is the Bessel functions of first and second kind of order n and has the form

$R(r) = CJ_n(\lambda r) + DY_n(\lambda r),$

where $J_n(z)$, $Y_n(z)$ are Bessel functions of the first and second kind respectively. In order to have a finite solution for the membrane, the solution must lie in the interval $0 \leq r \leq a,$ where 'a' is the radius of the drum, so that D assume a zero value. Since the Bessel function of second kind becomes unbounded at r = 0. Also for r = a, we require that

$R(a) = CJ_n(\lambda a) = 0$.

By denoting the zeros of Bessel functions as z_{mn} and noting that

$\alpha \cos\phi + \beta \sin\phi = \sqrt{\alpha^2 + \beta^2} \cos(\phi + \phi_0)$ Where $\phi_0$is a phase shift, Darryl Yong gave the solution for computing the vibrational mode of the timpani in a more compact form for the nonaxisymmetric case as

$$u_{mn}(r,\theta,t) = \sum J_n\left(\frac{z_{mn}r}{a}\right)\cos(n\theta)\cos\left(\frac{cz_{mn}t}{a}\right), \quad \text{for} \quad m = 1,2,3 \ldots\ldots and\ n = 0,1,2 \ldots.$$

provided one is after the qualitative behavior of each vibrational mode, the angular and temporal phase shift could be ignored. Finally Darryl Yong said in practice the actual sound that we hear from the timpani is different from the above mathematical model for computing the vibrational mode of the kettle drum due to the following important factors that have been neglected:

The motion of the timpani is damped, which changes the vibrational modes and their frequencies. Sound waves travelling from the timpani to your ear also experience damping. Secondly there are small nonlinear effects such as surface tension which exert their own preference for certain modes by transferring energy from one vibration mode to another. Thirdly, the tension of the drumhead is not uniform across the entire drumhead (c is not really constant). Furthermore, there is a coupling between the vibrations of the membrane, the vibrations of the membrane bowl, and the vibrations of the air particles that eventually reach your ear.

2.5 The Indian Local drum

Siddharthan R et.al presented two theoretical models (or distributions) for the loaded drum head of the Indian musical drums, using the tabla as the prototypical example.

(See Fig. 2.5.1 for picture of the Tabla.)

Fig. 2.5.1 Photograph of the Tabla (Indian local drum)

The Tabla is the most well-known of all Indian drums. The tabla is a paired drum; the first drum is usually placed on the right has a black patch in the centre. This patch is made of a mixture of iron, iron oxides, resin, gum etc. and is stuck firmly onto the membrane. The thickness of this patch decreases radially outwards.

The other drum has a wider membrane, and has also black patch similar to the first one, except that it is not symmetrically placed at the centre but on one side of the membrane. This drum does not produce harmonics but provides lower frequencies in the overall sound while the Tabla is being played. The researchers investigated on how the drum may be made harmonic by considering two different theoretical models. (i.e. their study was on a radial density distribution and how the density variation of the membrane affects the frequencies of the overtones.)

In making the Indian drum produce harmonic sounds the researchers tried several solutions to a membrane with a density variation. The simplest possibility was by considering a loading which varies only with r. This produced a change in the radial part of the wave equation from

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2}\right)R = 0 \ ,$$

where $c^2 = \frac{\tau}{\rho}$ for the uniform membrane to

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{\omega^2}{c^2(r)} - \frac{m^2}{r^2}\right)R = 0,$$

since $\rho = \rho(r)$ so that $c^2(r) = \frac{\tau}{\rho(r)}$,

or equivalently,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{\rho(r)\omega^2}{\tau} - \frac{m^2}{r^2}\right)R = 0 \ ,$$

for the loaded membrane.

The Radial equation was solved numerically for various distributions using the second-order Runge-Kutta method. However the two kinds of loading which was successful and gave interesting patterns were

1. The Step function, i.e. concentric rings with varying density were considered.

2. The Continuous loading, i.e. although the loading is stuck in parts, it becomes more or less continuous after it has been played for some time.

In writing down the wave equation for the loaded membrane, they made these assumptions

1. Normal Modes exist even in the loaded membrane

2. Tension per unit length is the same throughout the membrane, even in the loaded part.

The initial conditions for starting the solution of the wave equation by the Runge-Kutta method were assumed to be identical to those of the Bessel functions.

The allowed frequencies were found in the following way: Firstly, the Radial equation was written as

$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\rho(r)k'^2 - \frac{m^2}{r^2}\right)R = 0$, where $k'^2 = \frac{\omega^2}{\tau}$

Which look exactly like the Bessel equation of order m for the loaded case. Now keeping m = 0, the value of k' was varied and the solution plotted as a graph on the screen until its value became zero at the boundary. The corresponding value of k' was noted. k' was then increased, until the solution again became zero, but this time the solution passed through zero once, meaning that there was a nodal circle. This process was made for a particular value of m and then, m was increased by one, and the same process was continued, and all the allowed frequency values were noted.

By this process they observed m = 0 and m =1 gave the expected results, since the initial values of the Bessel functions and their derivatives are non-zero for the zero and first orders. However, for m = 2 and higher orders, values of both the Bessel function and its derivative become zero at r = 0.

This leads to the solution becoming zero at all points for m = 2 and higher orders. This happens because the Runge-Kutta method depends on the initial value of the solution and its derivative (i.e. r = 0) due to the fact that the iteration begins with the derivative and the initial value as zero, it continues to be so for further values of r. To remedy this problem, the researchers chose to begin the iteration from an infinitesimally small value with r =0.0001 and they were able to compute the values of the Bessel function and its derivatives using the first four terms of the series expansion of the Bessel function.

Out of their investigation they observed that for continuous loading, all the overtones were nearly harmonic with the exception of the fundamental frequency which was found to be higher than it should have been. The results of their finding is summarized in the table below

Summary of the findings

Normal mode	Frequency Ratio			
(Nodal diameter,Circles)	Unloaded	Continuous Loading	Multiple Rings	Tabla
1 (0,0)	1.00	1.07	1.00	1.00
2 (1,0)	1.59	2.00	1.96	2.00
3 (2,0)	3.14	2.98	2.98	3.00
4 (0,1)	2.30	2.99	3.03	3.00
5 (3,0)	3.65	4.00	4.02	4.00
6 (1,1)	2.92	4.00	3.95	4.00
7 (4,0)	3.16	5.01	5.00	5.00
8 (2,1)	3.50	5.01	5.00	5.00
9 (0,2)	3.60	5.02	4.80	5.00
10 (1,2)	4.24	6.02	5.20	
11 (1,3)	5.55	7.80	7.03	
12 (2,2)	4.85	7.00	5.90	
13 (3,1)	4.06	6.04	6.02	
14 (4,1)	4.60	7.09	7.05	

Table 2.2

In conclusion we take a look at the uses and differences of the Tabla with western drum. The key difference between Indian and foreign drums is the absence of the central loading on the membrane and as a result recitals in Indian Classical music is usually accompanied by the Tabla which must be harmonic, otherwise, the aharmonicity would completely disrupt the recital.

On the other hand in most cultures drums play different role in contemporary music. They provide a rhythm to the music produced by the rest of the instruments, and do not produce harmonics.

2.6 Some African Instruments Their Classification and Their Uses

African instruments are believed to be of local origin, or instruments which have become integrated into the musical life of their communities. African musical instruments could be categorized into four main groups, namely Idiophones, Aerophones, Chordophones and membranophones.

(A) Idiophone

An Idiophone (literally, "self sounding") may be broadly defined as any instrument upon which a sound may be produced without the addition of a stretched membrane or a vibrating string or reed. Idiophones may be used musically as rhythm instruments or played independently as melodic instruments. Apart from their musical functions Idiophones are used depending on the African communities from which they evolved as signals for attracting attention, assembling people, or creating an atmosphere (especially during religious rites and ceremonies). They may be use for transmitting messages and at times applied for scaring birds away from newly ploughed fields.

Idiophones found in some African communities includes Shaking Idiophones such as the gourd rattle (maracas) which may be used principally as rhythm instruments held and played as rattles in the hand or as jingling metals worn on the wrist of dancers or even tied to other instruments which serves as modifiers.

Tuned Idiophones are usually tuned and can be used for playing melodies. They include mbira or sansa (hand piano) which is made of a series of wooden or metal strips arranged on a flat sounding board and mounted on a resonator such as a box a gourd or even a tin, and the xylophone.

(B). Membranophones

Membranophones are instruments that make sound from tightly stretched membranes over some sort of frame or shell. The shell of drums are carved out of solid logs of wood, or made from earthenware vessels. They could also be made out of strips of wood bound together by iron hoops. Another material used for making the drum shell is the large gourd or calabash.

Drums appear in a wide variety of shapes and sizes. They may be conical, cylindrical with a bulge in the middle, or a bowl-shaped top, cup-shaped, bottle-shaped or in the shape of an hour glass. Some drums are of light weight and could be played when held and supported under the players armpit, others are heavy and are normally placed on the ground when played. Drums in general could be open at one end – known as single headed drum. When both ends of the shell is covered with animal skin, they are known as double headed drum. The drum heads are fixed by means of glue or nails over the shell, or the drum heads are nailed down by thorns, or suspended by pegs that could be pushed in or out to regulate the tension in the drum head.

The drumhead may also be laced down by strings to another skin at the other end of the shell. The lacing may be Y-shaped, W- shaped or X- shaped. Some drums may have jingles (akasaa) suspended across the drum head as in the case of the male atumpan drum used by the Akans of Ghana. Rattling metals or small bells may be attached to the rim of a drum, as in the Yoruba iya ilu drum found in Nigeria. Apart from their musical uses, some special drums for nonverbal communication may have symbolic significance in some African communities. For communication purposes the sounds of membranophones may function as speech surrogates or as signals (call signals or warning signals).

(C). Aerophones

Aerophones are wind instruments. These instruments achieve the desired sound by forcing air through them which creates vibration.

There are limited wind instruments seen in most African societies. Aerophones are generally classified into three broad groups namely the Flute family, the Horns and Trumpets family and the Reed Pipe family. Flutes are made from materials with natural bore such as bamboo, the husk of cane, the stalks of millet; they could be carved out of wood or from substitutes such as metal or PVC pipes. Flutes may be open-ended or stopped, and are usually played in a vertical or transverse position. Flutes may be used as solo instruments for playing fixed tunes or improvised pieces for conveying signals. They are found in the melody section of an ensemble. The Reed Pipes group are not as wide spread or as significant as the Flute class. Reed instruments are usually made from millet stalk.

The reed is sub grouped as single-reed type and double–reed instruments. Single-reed instruments are found in the Burkina Faso, Northern Ghana, Benin, and Chad, whiles the double reed instruments are mostly found in Nigeria, Cameroon, Kenya and Tanzania. The third class of Aerophones is the Horns and trumpets which are made from animal horns and elephant tusks. These are found in some Akan communities of Ghana.

They are generally designed to be side-blown. There are also trumpets made of bamboo, gourd or metal such as the Ethiopian malakat, which may also be covered with leather or skin. Horns and trumpets may be used for conveying signals and verbal messages as well as for playing music.

(D). Chordophones.

Chordophones are instruments that make sound when stretched strings vibrates. Normally, these strings are stretched over a box or gourd to maximize reverberation. Chordophones are played by plucking, strumming or through friction from a bow. There are five basic types found in Africa namely, bows, lyres, harps, lutes, and zithers. Of these, the oldest and simplest is the musical bow which is still common or wide spread in Africa.

It exist in a variety of forms, such as the earth bow, which is made of a flexible stick stuck in the ground, to whose upper end a piece of string is attached. The string is stretched and buried down in the earth with a piece of stone placed on top of the earth to keep the string in position. They are found in northern Ghana and Uganda. The other forms are the mouth bows and calabash resonator bows. For the mouth bow, the bow resonates in the mouth; when hit at a convenient spot, the shape and size of the mouth cavity are altered so as to amplify selected partials or harmonics produced by the string. And for the bows with calabash resonator, the calabash is placed in the middle of the bow or towards the tip. Usually when it is being played the calabash may be placed on the chest or some part of the body so as to vary its pitch.

Zithers are another type of chordophone found in Africa. One peculiar characteristic of Zithers is that the strings of the instrument are positioned horizontally. Examples of Zithers include the Tube Zither which has its strings run across the shell of a tube such as a hollow bamboo stem. The Trough Zither, it has eight strings which run across the entire length of the resonating wooden trough. The Bow Zither is found in the savannah belt of west Africa. Its design is a U- shaped bow with a calabash resonator attached to its base with five or six strings tied from one side of the bow to the other.

Fig. 2. 6.1 The Trough Zithar

Other African stringed musical instruments or chordophones include the Lyres, Lutes and Harps. The Lute is an instrument whose strings run parallel to its neck. It has a resonating belly such that the strings run from near the base of the belly along the full length of the neck.

Some common Lutes include the harp lutes, and the bow lutes which are found in the savannah belt of West Africa (Guinea, Niger, Gabon and Cameroon).

Fig. 2.6.2 The Harp-lute

The arched or bow harps are closely related to the lute. However, the neck of the harp is arched and the strings run from the neck to the sound box at an angle. Unlike the Lute and Harp the Lyre is an instrument whose strings run from a yoke to a resonator and they are mostly found in the eastern part of Africa. Examples include “begana” lyre found in Ethiopia and the “obukano” lyre found in Kenya. Chordophones are generally played to accompany a solo singer or in poetry recitals, praise singing and narrative songs.

CHAPTER THREE

THE VIBRATING MEMBRANE

3.1 Two-Dimensional Wave Equation

A Vibrating membrane, such as a rectangular drumhead has displacements that satisfy the two dimensional wave equation. We shall be concerned with the two dimensional geometry of the membrane in a plane. Before preparing a model for this problem, we describe a few assumptions concerning the material and behavior of the membrane.

1. The membrane is homogeneous. The density ▪ is constant.

2. The membrane is composed of a perfectly flexible material which offers no resistance to deformation perpendicular to the xy- plane. Motion of each element is perpendicular to the xy- plane.

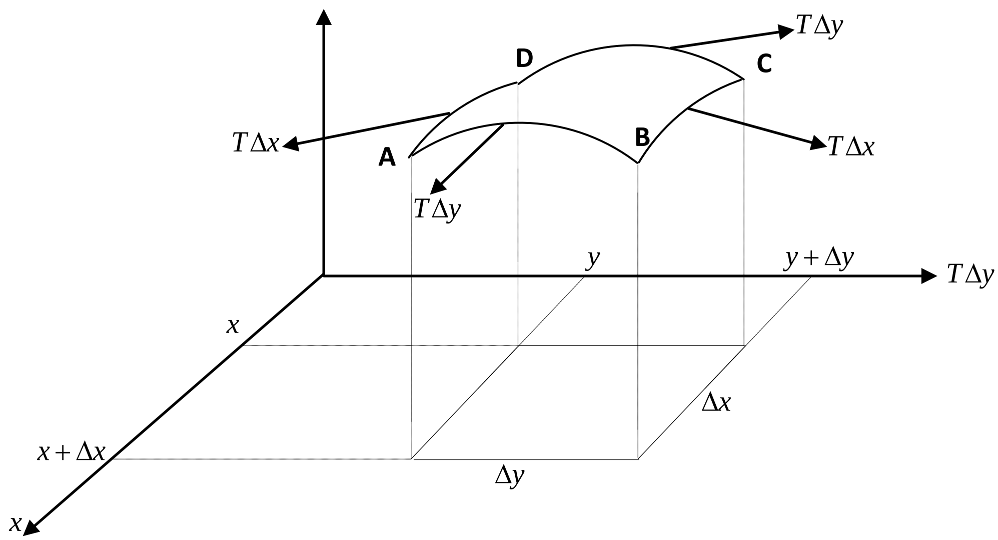
3. The membrane is stretched and fixed along a boundary in the xy- plane.

4. The Tension per unit length T due to stretching is the same in every direction and is constant during the motion. Weight of the membrane is negligible.

5. The Deflection u(x,y,t) of the membrane whiles in motion is relatively small in comparison to the size of the membrane and the angles of inclination are small.

To derive the differential equation which governs the motion of the membrane, we consider the forces acting on a small portion of the membrane as shown in figure 3.1.1 below

An element of stretched vibrating membrane



Fiq.3.1.1. An element and projection of a stretched membrane.

An element of the membrane ABCD in figure 3.1.1 is projected into a small rectangle with edges Δx and Δy parallel to the x and y axes. Deflections and angles of inclination are small enough so that the sides of the element are approximated by Δx and Δy. According to the assumption (4), the forces acting on the edges are approximately TΔx and TΔy , and acts tangential to the membrane.

Horizontal components involve cosines of very small angles of inclination. Since these forces are directed in opposite direction they add to zero approximately. The sum of the Horizontal forces in the x direction (See fig. 3.1.2) is

$$T\Delta y(cos\beta - cos\alpha) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.1.1)$$

And in the y direction the sum is

$$T\Delta x(cos\delta - cos\gamma) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \qquad (3.1.2)$$

The Cross Sections in x and y Plane of a Membrane

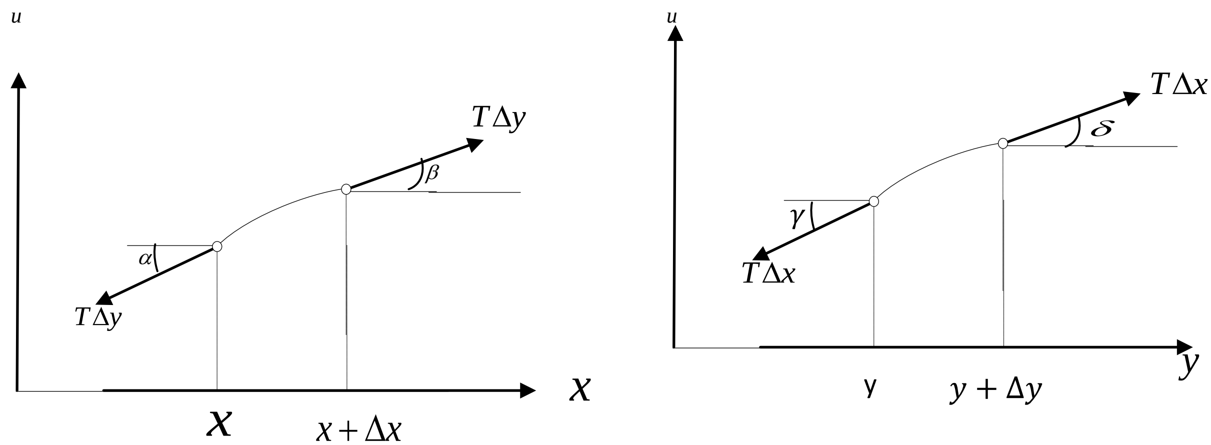


Fig 3.1.2. Cross sections of membrane showing angles of inclination.

From the cross section in the x and y planes, if the horizontal component of $T\Delta y$ is T_{hx}, then from equation (3.1.1)

$$T_{hx} = T\Delta y cos\beta = T\Delta y cos\alpha \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.3)$$

and that of $T\Delta x$ is T_{hy} then from equation (3.1.2) we have

$$T_{hy} = T\Delta x cos\,\delta = T\Delta x cos\gamma \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.4)$$

Equations (3.1.3) and (3.1.4) becomes

$$T\Delta y = \frac{T_{hx}}{cos\beta} = \frac{T_{hx}}{cos\alpha} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.5)$$

and

$$T\Delta x = \frac{T_{hy}}{cos\delta} = \frac{T_{hy}}{cos\gamma} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.6)$$

Adding the forces in the vertical direction and using Newton's second law of motion, we obtain

$$T\Delta y(sin\beta - sin\alpha) + T\Delta x(sin\delta - sin\gamma) = \rho\Delta x\Delta y U_{tt} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.7)$$

If $T\Delta x$ and $T\Delta y$ in equation (3.1.7) are replaced by equations (3.1.5) and (3.1.6) then

$$T_{hx}[tan\beta - tan\alpha] + T_{hy}[tan\delta - tan\gamma] = \rho\Delta x\Delta y U_{tt} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.8)$$

Recognizing that

$tan\beta = U_x(x + \Delta x, y, t)$ and $tan\alpha = U_x(x, y, t)$

$tan\delta = U_y(x, y + \Delta y, t)$ and $tan\gamma = U_y(x, y, t)$ therefore equation (3.1.8) becomes

$$T_{hx}[U_x(x + \Delta x, y, t) - U_x(x, y, t)] +$$

$$T_{hy}\left[U_y(x, y + \Delta y, t) - U_y(x, y, t)\right] = \rho\Delta x\Delta y U_{tt} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.9)$$

If the cosine of the inclinations is all approximately 1, then equation (3.1.9) yields

$$T\Delta y[U_x(x + \Delta x, y, t) - U_x(x, y, t)] +$$

$$T\Delta x\left[U_y(x, y + \Delta y, t) - U_y(x, y, t)\right] = \rho\Delta x\Delta y U_{tt} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.10)$$

Division of equation (3.1.10) by $\rho\Delta x\Delta y$ permits the form

$$\frac{T}{\rho}\left[\frac{U_x(x+\Delta x, y, t) - U_x(x, y, t)}{\Delta x}\right] + \frac{T}{\rho}\left[\frac{U_y(x, y+\Delta y, t) - U_y(x, y, t)}{\Delta y}\right] = U_{tt}(x, y, t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.11)$$

as $\Delta x \to 0$ and $\Delta y \to 0$ in equation (11), then

$$U_{tt}(x, y, t) = c^2\nabla^2\mathrm{U}(\mathrm{x}, \mathrm{y}, \mathrm{t}) \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.12)$$

Where $\nabla^2 U = U_{xx} + U_{yy}$ and $c^2 = {}^{T}\!/_{\rho}$

Equation (3.1.12) is the wave equation in two dimensions.

3.2 The Circular Membrane

This section considers the solution of the two dimensional wave equation of the circular membrane such as the local drums “Kete”, “Fontomfrom”, “Atumpan” and so on which is struck at the centre and an attempt made to describe its transverse movements. We shall transform the two dimensional Cartesian wave equation into its polar form in terms of r and θ using the parametric equations

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.2.1)$$

because the boundary of the membrane can be modeled simply by the polar equation

r= constant.

The researcher shall consider the solution $u(r,t)$ of the vibrating membrane which is radially symmetric, a case which is independent of θ with a single boundary condition

(i). $u(R,t) = 0$ for all $t \geq 0$

which imply that the membrane is fixed at the ends. And with initial conditions

(ii). $u(r,0) = 0$ and $\frac{\partial U}{\partial t}\Big|_{t=0} = f(r)$

The condition in (ii) implies that the membrane is flat initially and at the same time an arbitrary velocity is applied in order for the membrane to start vibrating.

The method of separation of variables is employed to establish two ordinary differential equations using the plane polar form of the two dimensional wave equation. The researcher identified the spatial O.D.E. as the Bessel's differential equation of order zero by applying a suitable substitution its solution is obtained. The time O.D.E. was also identified as the standard simple harmonic differential equation which belongs to a class of second order differential equations with constant coefficient and its solution was also adopted accordingly.

The superposition principle and the Bessel functions orthogonality principle were finally applied to obtain the general solution of the circular vibrating membrane.

3.3 Transformation of the two dimensional Cartesian wave equation into Plane Polar.

By using the parametric equations namely $x = rcos\theta$ and $y = rsin\theta$, we transform the Laplacian,$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in the wave equation, from the rectangular coordinates $u(x, y, t)$ into $u(r, \theta, t)$. Then partially differentiating u with respect to x we get

$$U_x = U_r r_x + U_\theta \theta_x,$$

where the subscripts are the partial derivatives with respect to the indicated variable.

Differentiating again with respect to x, we first have

$U_{xx} = (U_r r_x)_x + (U_\theta \theta_x)_x$

$$= (U_r)_x r_x + U_r r_{xx} + (U_\theta)_x \theta_x + U_\theta \theta_{xx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.3.4)$$

By applying the chain rule again, we find

$(U_r)_x = U_{rr} r_x + U_{r\theta} \theta_x$ and

$(U_\theta)_x = U_{\theta r} r_x + U_{\theta\theta} \theta_x$.

To determine the partial derivatives r_x and θ_x we have to differentiate

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$

Then

$r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$ and

$\theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \left(\frac{-y}{x^2}\right) = \frac{-y}{r^2}$

Differentiating these two functions again, we obtain

$r_{xx} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{-y}{r^3}$ and

$\theta_{xx} = -y \left(\frac{-2}{r^3}\right) r_x = \frac{2xy}{r^4}$

We substitute all these expressions into equation (3.3.4)

Assuming continuity of the first and second partial derivatives, we have $U_{r\theta} = U_{\theta r}$ and we obtain,

$$U_{xx} = U_{rr}(r_x)^2 + U_{r\theta}\theta_x r_x + U_r r_{xx} + U_{\theta\theta}(\theta_x)^2 + U_{r\theta}\theta_x r_x + U_\theta \theta_{xx}$$

$$= U_{rr}(r_x)^2 + 2U_{r\theta}\theta_x r_x + U_{\theta\theta}(\theta_x)^2 + U_\theta \theta_{xx}$$

$$\therefore\ U_{xx} = \frac{x^2}{r^2}U_{rr} - 2\frac{xy}{r^3}U_{r\theta} + \frac{y^2}{r^4}U_{\theta\theta} + \frac{y^2}{r^3}U_r + 2\frac{xy}{r^4}U_\theta \ \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.3.5)$$

In a similar fashion, it follows that

$$U_{yy} = \frac{y^2}{r^2}U_{rr} + 2\frac{xy}{r^3}U_{r\theta} + \frac{x^2}{r^4}U_{\theta\theta} + \frac{x^2}{r^3}U_r - 2\frac{xy}{r^4}U_\theta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.3.6)$$

By adding equations (3.3.2) and (3.3.3), we see that the Laplacian in polar coordinates is

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

The wave equation takes the form

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.3.7)$$

3.4. Vibrating membrane with constant tension

We shall now consider the solution $U(r,t)$ of the wave equation which does not depend on θ,

so that the wave equation (3.3.7) reduces to the simpler form

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.4.1)$$

Subject to the boundary and initial conditions of equations (i) and (ii) under (3.2.1) namely

$$U(R,t) = 0, for\ all\ t \geq 0 \qquad \text{……………………………..} \qquad (3.4.2)$$

since the membrane is fixed along the boundary r = R, and

$$\begin{cases} U(r,0) = 0 & [Initial\ deflection\ ] \\ \frac{\partial U}{\partial t}\ \Big|_{t=0} = f(r) & [Initial\ velocity\ f(r)] \end{cases} \qquad \text{…..…………………….} \qquad (3.4.3)$$

Using the method of separation of variables, we first determine the solution that satisfies the boundary conditions (3.4.2). Let

$$U(r,t) = W(r)G(t) \qquad \text{……………………………} \qquad (3.4.4)$$

By differentiating and inserting (3.4.4) into (3.4.1) and dividing the resulting equation by C^2WG, where $W = W(r)\ and\ G = G(t)$, we obtain

$$\frac{\ddot{G}}{C^2G} = \frac{1}{W}\left(W" + \frac{1}{r}W'\right),$$

where dots denote derivatives with respect to t and primes denote derivatives with respect to r.

The expressions on both sides must be equal to a constant, and this constant must be negative, say, $-k^2$, where k is a non-zero real constant in order to obtain solutions that satisfy the boundary conditions. Thus,

$$\frac{\ddot{G}}{C^2G} = \frac{1}{W}\left(W" + \frac{1}{r}W'\right) = -k^2 .$$

This yields the two ordinary linear differential equations.

$\ddot{G} + \lambda^2 G = 0$, …………………………….. (3.4.5)

where $\lambda = ck$ and

$W" + \frac{1}{r}W' + k^2 W = 0$, which on multiplying through by r^2 yields

$r^2 W" + rW' + (kr)^2 W = 0$ …………………………. (3.4.6)

Introducing a new independent variable $s = kr$,we may write equation (3.4.6) as follows:

Let $s = kr$, then

$W' = \frac{dW}{dr} = \frac{dW}{ds}.\frac{ds}{dr} = \frac{dW}{ds}k$ and $W" = \frac{d^2W}{ds^2}k^2$

Using these substitutions and taking out the common factor k^2 we obtain

$s^2 W" + sW' + s^2 W = 0$, ………………………… (3.4.7)

where $W = W(s)$, $W' = \frac{dW}{ds}$ $and\ W'' = \frac{d^2W}{ds^2}$.

This is clearly Bessel's equation with parameter zero.

The general solution is

$W = C_1 J_0(s) + C_2 Y_0(s)$,

where $J_0(s)$ and $Y_0(s)$ are the Bessel functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, while $Y_0(s)$ becomes unbounded as $s \to 0$, we cannot use $Y_0(s)$ and must choose $C_2 = 0$ and $C_1 \neq 0$, if not W will yield a trivial solution. Thus setting $C_1 = 1$, we have

$$W(r) = J_0(s) = J_0(kr) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.8)$$

Now applying the boundary condition (3.4.2), we have

$$W(R) = J_0(kR) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.9)$$

Since $G \equiv 0$ would imply $U \equiv 0$, we require that the boundary condition is satisfied wheneverW$(R) = J_0(kR) = 0$.

The Bessel function J_0 has (infinitely many) real zeros and are slightly irregularly spaced. Let us denote the positive zeros of $J_0(s)$ by $s = \alpha_1, \alpha_2, \alpha_3 \ldots..$ having the exact numerical values to four decimal places as follows;

$$\alpha_1 = 2.4048, \alpha_2 = 5.5201, \alpha_3 = 8.6537, \alpha_4 = 11.7915,\ \alpha_5 = 14.9309, \ldots\ldots\ldots$$

Hence equation (3.4.9) would imply

$$kR = \alpha_n \ \Rightarrow k = k_n = \frac{\alpha_n}{R}\ ,\ n = 1,2,3 \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots \qquad (3.4.10)$$

Thus the functions

$$W_n(r) = J_0(k_n r) = J_0\left(\frac{\alpha_n}{R} r\right), n = 1,2,3 \ldots. \qquad \ldots\ldots\ldots\ldots\ldots \qquad (3.4.11)$$

are solutions of equation (3.4.6) which vanish at $r = R$.

The time O.D.E. of equation (3.4.5) may be solved to give the solution of the form

$$G_n(t) = A_n \cos\left(\frac{C\alpha_n}{R} t\right) + B_n \sin\left(\frac{C\alpha_n}{R} t\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.4.12)$$

with $\lambda = \lambda_n = ck_n$.

However for $0 < t < \infty$ an attempt is made to transform the time O.D.E. into Bessel function of order $\pm\frac{1}{2}$, by proceeding as follows

Multiply the time O.D.E. by t to get

$$t\ddot{G} + \lambda^2 tG = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.4.13)$$

Let

$$t = x^m \quad \Rightarrow x = t^{\frac{1}{m}}, \qquad \frac{dx}{dt} = \frac{1}{m} t^{\frac{1}{m}-1} = \frac{1}{m} x^{1-m}$$

Then

$$\frac{dG}{dt} = \frac{dG}{dx} \cdot \frac{dx}{dt} = \frac{1}{m} x^{1-m} \frac{dG}{dx} \quad \text{and}$$

$$\frac{d^2G}{dt^2} = \frac{d}{dx}\left\{\frac{1}{m} x^{1-m} \frac{dG}{dx}\right\}\left\{\frac{1}{m} x^{1-m}\right\}$$

$$= \frac{1}{m^2} x^{1-m}\left[(1-m)x^{-m}\frac{dG}{dx} + x^{1-m}\frac{d^2G}{dx^2}\right]$$

$\Rightarrow$

$$t\frac{d^2G}{dt^2} = \frac{1}{m^2} x^{1-m}\left[(1-m)\frac{dG}{dx} + x\frac{d^2G}{dx^2}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.4.14)$$

Also

$$t\lambda^2 G = \lambda^2 x^m \cdot \frac{x^{1-m}}{m^2} \cdot m^2 \cdot x^{m-1} G = [(\lambda m)^2 x^{2m-1} G] \frac{x^{1-m}}{m^2} \dots\dots\dots\dots\dots \quad (3.4.15)$$

Adding (3.4.14) and (3.4.15) and dividing through by $\frac{x^{1-m}}{m^2}$ the time O.D.E.

becomes

$$t\frac{d^2G}{dt^2} + t\lambda^2 G = (1-m)\frac{dG}{dx} + x\frac{d^2G}{dx^2} + (\lambda m)^2 x^{2m-1} G = 0 \quad \dots\dots\dots\dots\dots \quad (3.4.16)$$

Let

$2m - 1 = 1 ,\Rightarrow m = 1 \;\; and \; also \; 1 - m = 0$

So that equation (3.4.16) becomes

$$x\frac{d^2G}{dx^2} + \lambda^2 x G = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (3.4.17)$$

Let

$$G = x^p w \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \quad (3.4.18)$$

So that differentiating with respect to t

$$\frac{dG}{dx} = px^{p-1}w + x^p w' = \frac{x^p}{x}[pw + xw']$$

$$\frac{d^2G}{dx^2} = p(p-1)x^{p-2}w + px^{p-1}w' + px^{p-1}w' + x^p w''$$

$$= x^{p-2}[x^2 w'' + 2pxw' + p(p-1)w] \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \quad (3.4.19)$$

$\therefore$ On substituting equations (3.4.18) and (3.4.19) into (3.4.17) we obtain

$$x^{p-1}[x^2w'' + 2pxw' + p(p-1)w + \lambda^2x^2w] = 0$$

$\Rightarrow$

$$x^2w'' + 2pxw' + [p(p-1) + \lambda^2x^2]w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.20)$$

Finally set

$2p = 1 \quad \Rightarrow \quad p = \frac{1}{2}$ and so equation (3.4.20) yields

$$x^2w'' + xw' + \left[(\lambda x)^2 - \left(\frac{1}{2}\right)^2\right]w = 0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (3.4.21)$$

which is Bessel's O.D.E. of order $\pm\frac{1}{2}$.

$$\therefore \quad w = C_1J_{\frac{1}{2}}(\lambda x) + C_2J_{-\frac{1}{2}}(\lambda x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.22)$$

But $\quad G = x^pw\,, p = \frac{1}{2}\,, x = t^{\frac{1}{m}}\,, m = 1\,, \lambda = ck_n = \frac{\alpha_n}{R}$

Therefore substituting we obtain the time solution as

$$G_n(t) = t^{\frac{1}{2}}\left[A_nJ_{\frac{1}{2}}\left(\frac{c\alpha_n}{R}t\right) + B_nJ_{-\frac{1}{2}}\left(\frac{c\alpha_n}{R}t\right)\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (3.4.23)$$

where A_n and B_n are real numbers.

Hence the functions

$$U_n(r,t) = W_n(r)G_n(t)$$

$$= t^{\frac{1}{2}}\left[A_nJ_{\frac{1}{2}}\left(\frac{c\alpha_n}{R}t\right) + B_nJ_{-\frac{1}{2}}\left(\frac{c\alpha_n}{R}t\right)\right]J_0\left(\frac{\alpha_n}{R}r\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (3.4.24)$$

where $U_n, n = 1,2,3 \ldots..$ are solutions of the wave equation (3.4.1), satisfying the boundary condition (3.4.2). These are the Eigen functions of our problem and the corresponding Eigen values are $\frac{C\alpha_n}{R}, n = 1,2,3 \ldots.$

Next, we note that the original partial differential equation is linear and as such applying the superposition principle on the various particular solutions or normal modes would give the general solution in the form

$$U(r,t) = \sum_{n=1}^{\infty} W_n(r)G_n(t)$$

$$= \sum_{n=1}^{\infty} t^{\frac{1}{2}} \left[A_n J_{\frac{1}{2}}\left(\frac{c\alpha_n}{R}t\right) + B_n J_{-\frac{1}{2}}\left(\frac{c\alpha_n}{R}t\right)\right] J_0\left(\frac{\alpha_n}{R}r\right) \quad \ldots\ldots\ldots\ldots.. \quad (3.4.25)$$

With the real constants A_n and B_n to be determined using the initial conditions (3.4.3).

To apply the initial condition we note that Bessel function of order n is

$$J_n(t) = \sum_{m=1}^{\infty} \frac{(-1)^m\left(\frac{t}{2}\right)^{2m+n}}{m!(m+n)!}$$

$$= \frac{t^n}{2^n\Gamma(n+1)}\left\{1 - \frac{t^2}{2.2(n+1)} + \frac{t^4}{2.4.2^2(n+1)(n+2)} - \frac{t^6}{2.4.6.2^3(n+1)(n+2)(n+3)} + \cdots\cdots\cdots\cdots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \ldots \quad (3.4.26)$$

Setting $t = \frac{c\alpha_n}{R} t \text{ } and \text{ } n = \pm\frac{1}{2}$ into (3.4.26) we obtain the expansion for the solution of the time O.D.E . so that

$$G_n(t) = \frac{A_n\left(\frac{c\alpha_n}{2R}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\left\{t - \frac{\left(\frac{c\alpha_n}{R}\right)^2}{3!}t^3 + \frac{\left(\frac{c\alpha_n}{R}\right)^4}{5!}t^7 - \cdots\cdots\cdots\cdots\right\} +$$

$$\frac{B_n\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\left\{1-\frac{\left(\frac{c\alpha_n}{R}\right)^2}{2!}t^2+\frac{\left(\frac{c\alpha_n}{R}\right)^4}{4!}t^4-\cdots\cdots\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.27)$$

Now at $t = 0$, $\quad G_n(0) = \frac{B_n\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}$ $then$

$$U(r,0) = \sum_{n=1}^{\infty}\frac{B_n\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}J_0\left(\frac{r\alpha_n}{R}\right) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.28)$$

Multiplying both sides by $rJ_0\left(\frac{r\alpha_m}{R}\right)$ and integrating with respect to r from 0 to R we obtain

$$\sum_{n=1}^{\infty}\frac{B_n\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\int_0^R r\,J_0\left(\frac{r\alpha_n}{R}\right)\,J_0\left(\frac{r\alpha_m}{R}\right)dr = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.29)$$

By the orthogonality principle of Bessel functions we get

$$\frac{B_n\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\cdot\frac{R^2J_1^2(\alpha_n)}{2} = 0$$

$$\Rightarrow B_n = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.30)$$

Similarly as $t = 0$, $\quad \dot{G}_n(0) = \frac{A_n\left(\frac{c\alpha_n}{2R}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}$ and

$$U_t(r,0) = \sum_{n=1}^{\infty}\frac{A_n\left(\frac{c\alpha_n}{2R}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}J_0\left(\frac{\alpha_n r}{R}\right) = f(r) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.31)$$

Multiplying both sides by $rJ_0\left(\frac{r\alpha_m}{R}\right)$ and integrating with respect to r from 0 to R we obtain

$$\sum_{n=1}^{\infty} \frac{A_n\left(\frac{c\alpha_n}{2R}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \int_0^R r\, J_0\left(\frac{r\alpha_n}{R}\right) J_0\left(\frac{r\alpha_m}{R}\right) dr = \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_m}{R}\right) dr \ldots\ldots\ldots.. \qquad (3.4.32)$$

By the orthogonality principle of Bessel functions we get

$$\frac{A_n\left(\frac{c\alpha_n}{2R}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \cdot \frac{R^2 J_1^2(\alpha_n)}{2} = \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_m}{R}\right) dr$$

$$\Rightarrow A_n = \frac{2\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)}{R^2 J_1^2(\alpha_n)} \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \qquad (3.4.33)$$

∴ the complete solution becomes

$$U(r,t) = \sum_{n=1}^{\infty} \frac{2t^{\frac{1}{2}}}{R^2 J_1^2(\alpha_n)} \left[\left\{\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr\right\} \times J_{\frac{1}{2}}\left(\frac{c\alpha_n}{R} t\right)\right] J_0\left(\frac{\alpha_n r}{R}\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (3.4.34)$$

This equation (3.4.34) for $U(r,t)$ is the mathematical expression of the motion of the circular drum with constant tension in the drum head.

CHAPTER FOUR

THE MATHEMATICS OF THE VIBRATING MEMBRANE WITH VARYING TENSION

4.1 Modified Two-Dimensional Wave Equation

The researcher shall be concerned with the two dimensional geometry of the membrane in a plane with a little but significant modification in the assumptions of a vibrating membrane. Thus in preparing a model for this problem of how the "*donno*" (the single talking drum) sounds the way it does, we shall retain the description of the assumptions concerning the material and behavior of the membrane, as in the case of the vibration of a circular membrane with constant tension except that this time we model the circular membrane with varying tension.

Therefore the underlying assumptions of the single talking drum is as follows

1. The membrane is homogeneous. The density ▪ is constant.

2. The membrane is composed of a perfectly flexible material which offers no resistance to deformation perpendicular to the xy -plane. Motion of each element is perpendicular o the xy -plane.

3. The membrane is stretched and fixed along a boundary in the xy plane.

4. The Tension per unit length T is very small and varies periodically with time due to the press and release of the strings during the motion.

5. The deflection $U(x, y, t)$of the membrane while in motion is relatively small in comparison to the size of the membrane; the angles of inclination of the deflections are small.

The researcher aimed at a solution of the form $U(r, t)$ which is radially symmetric due to the circular physical orientation of the drum and assumed a periodic function for the tension, applied.

The Method of separation of variables is applied on the partial differential equation that describes the dynamics of the drum to obtain two second order ordinary differential equations with variable coefficients. These ordinary differential equations are solved and with the boundary and initial conditions imposed, the mathematical expression describing the motion of the single talking drum is obtained. We proceed as follows:

4.2 Vibrating Membrane with Varying Tension

We shall now consider the solution $U(r, t)$ of the wave equation which does not depend on θ, so that the wave equation in polar coordinates reduces to the new form with the tension modified to vary as a periodic function of time

$$\frac{\partial^2 U}{\partial t^2} = \frac{F(t)}{\rho}\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.1)$$

where $F(t) = \sin t$, $\rho = a^2$. This modification in the wave equation is introduced in the modeling of the single talking-drum, commonly called the "*donno*" of the Akans of Ghana, since the tension is made to vary by pressing and releasing the strings within a small range of time when it is played.

Subject to the boundary and initial conditions namely

$$U(R,t) = 0, for\ all\ t \geq 0\ , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.2)$$

since the membrane is fixed along the boundary r = R .

and for $0 < t < \infty$

$$\begin{cases} U(r,0) = 0 & Initial\ deflection \\ \frac{\partial U}{\partial t}\Big|_{t=0} = f(r) & [Initial\ velocity\ f(r)] \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.3)$$

Using the method of separation of variables, we first determine the solution that satisfies the boundary conditions (4.2.2). Let

$$U(r,t) = W(r)G(t) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.4)$$

By differentiating and inserting (4.2.4) into (4.2.1) and dividing the resulting equation by WG, we obtain

$\frac{a^2\ddot{G}}{\sin tG} = \frac{1}{W}\left(W" + \frac{1}{r}W'\right)$, where dots denote derivatives with respect to t and primes denote derivatives with respect to r.

Let the separation constant be $-k^2$, then

$$\frac{a^2\ddot{G}}{\sin tG} = \frac{1}{W}\left(W" + \frac{1}{r}W'\right) = -k^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.5)$$

This yields the two ordinary linear differential equations, one purely of the spatial co-ordinate r and the other purely of the time co-ordinate t.

$$\ddot{G} + \lambda^2 \sin t\, G = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.6)$$

where $\lambda = \frac{k}{a}$ and

$$W" + \frac{1}{r}W' + k^2W = 0 ,$$

which on multiplying through by r^2 yields

$$r^2W" + rW' + (kr)^2W = 0 \qquad \text{……………………….} \qquad (4.2.7)$$

The equation (4.2.7) is identified as the Bessel's differential equation of order zero and by applying a suitable substitution its solution is obtained as

$$W = C_1J_0(s) + C_2Y_0(s) \qquad \text{………………………….} \qquad (4.2.8)$$

where $J_0(s)$ and $Y_0(s)$ are the Bessel functions of the first and second kind of order zero. Since the deflection of the membrane is always finite, we cannot use $Y_0(s)$ and must choose $C_2 = 0$. Also $C_1 \neq 0$ else we shall get the trivial solution i.e. $W = 0$.

On setting $C_1 = 1$, then

$$W(r) = J_0(s) = J_0(kr) \qquad \text{…………………………….} \qquad (4.2.9)$$

Now applying the boundary condition (3.4.2), we have

$$W(R) = J_0(kR) = 0 \qquad \text{……………………………….} \qquad (4.2.10)$$

Since $G \equiv 0$ would imply $U \equiv 0$, we require that the boundary condition is satisfied whenever $W(R) = J_0(kR) = 0$. We note that $J_0(x) = 0$ has infinitely many zeros and so we let $s = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ .... be the positive zeros of $J_0(s)$.

. Then $kR = Rk_n = \alpha_n \qquad n = 1,2,3$ ....

$\Rightarrow \quad k_n = \frac{\alpha_n}{R}$ becomes the Eigen values and the corresponding spatial solution is

$$W_n(r) = J_0\left(\frac{r\alpha_n}{R}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.11)$$

Next we consider the time O.D.E. equation (4.2.6)

$$\ddot{G} + \lambda^2 \sin t\, G = 0$$

Now the interspersed varying of the tension of the vibrating membrane through the twist and release of the strings attached to the drum is done in small intervals of time. Thus for small time$t$, $\sin t \approx t$ and the time O.D.E. then becomes

$$\ddot{G} + \lambda^2 tG = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (4.2.12)$$

This form of the time O.D.E. is identified as the standard Air's differential equation which is oscillatory. Clearly the O.D.E. is a linear second order differential equation with variable coefficients. It's solution could be converted into Bessel function of order one-third which are known to be oscillatory like the periodic functions of sine and cosine, as follows.

For $0 < t < \infty$

Multiply the time O.D.E. by t to get

$$t\ddot{G} + \lambda^2 t^2 G = 0$$

Let

$$t = x^m \quad \Rightarrow x = t^{\frac{1}{m}}, \qquad \frac{dx}{dt} = \frac{1}{m} t^{\frac{1}{m}-1} = \frac{1}{m} x^{1-m}$$

Then

$$\frac{dG}{dt} = \frac{dG}{dx} \cdot \frac{dx}{dt} = \frac{1}{m} x^{1-m} \frac{dG}{dx} \quad \text{and}$$

$$\frac{d^2G}{dt^2} = \frac{d}{dx}\left\{\frac{1}{m} x^{1-m} \frac{dG}{dx}\right\}\left\{\frac{1}{m} x^{1-m}\right\}$$

$$= \frac{1}{m^2} x^{1-m}\left[(1-m)x^{-m}\frac{dG}{dx} + x^{1-m}\frac{d^2G}{dx^2}\right]$$

$\Rightarrow$

$$t\frac{d^2G}{dt^2} = \frac{1}{m^2} x^{1-m}\left[(1-m)\frac{dG}{dx} + x\frac{d^2G}{dx^2}\right] \qquad \dots\dots\dots\dots\dots\dots \qquad (4.2.13)$$

Also

$$(t\lambda)^2 G = \lambda^2 x^{2m} \cdot \frac{x^{1-m}}{m^2} \cdot m^2 \cdot x^{m-1} G = [(\lambda m)^2 x^{3m-1} G]\frac{x^{1-m}}{m^2} \quad \dots\dots\dots\dots \quad (4.2.14)$$

Adding equations (4.2.13) and (4.2.14) and dividing through by $\frac{x^{1-m}}{m^2}$ the time O.D.E. becomes

$$t\frac{d^2G}{dt^2} + (t\lambda)^2 G = (1-m)\frac{dG}{dx} + x\frac{d^2G}{dx^2} + (\lambda m)^2 x^{3m-1} G = 0 \quad \dots\dots\dots\dots \quad (4.2.15)$$

Let

$$3m - 1 = 1 ,\Rightarrow m = \frac{2}{3} \quad and\ also\ 1 - m = \frac{1}{3}$$

So that equation (4.2.15) becomes

$$x\frac{d^2G}{dx^2} + \frac{1}{3}\frac{dG}{dx} + \left(\frac{2}{3}\lambda\right)^2 xG = 0 \qquad \dots\dots\dots\dots\dots\dots.. \qquad (4.2.16)$$

Let

$$G = x^p w \qquad \text{………………………..} \qquad (4.2.17)$$

So that differentiating with respect to t

$$\frac{dG}{dx} = px^{p-1}w + x^p w' = \frac{x^p}{x}[pw + xw'] \qquad \text{…………………………} \qquad (4.2.18)$$

$$\frac{d^2G}{dx^2} = p(p-1)x^{p-2}w + px^{p-1}w' + px^{p-1}w' + x^p w''$$

$$= \frac{x^p}{x^2}[x^2w'' + 2pxw' + p(p-1)w] \qquad \text{…………………………} \qquad (4.2.19)$$

$\therefore$ On substituting (4.2.17), (4.2.18) and (4.2.19) into (4.2.16) we obtain

$$\frac{x^p}{x}[x^2w'' + 2pxw' + p(p-1)w] + \frac{1}{3}\frac{x^p}{x}[pw + xw'] + \frac{x^p}{x}\left(\frac{2}{3}\lambda\right)^2 x^2 w = 0$$

$\Rightarrow$

$$x^2w'' + \left(2p + \frac{1}{3}\right)xw' + \left[p^2 - p + \frac{p}{3} + \left(\frac{2}{3}\lambda\right)^2 x^2\right]w = 0 \qquad \text{…………} \qquad (4.2.20)$$

Finally set

$2p + \frac{1}{3} = 1 \quad \Rightarrow \quad p = \frac{1}{3}$ and so equation (4.2.20) yields

$$x^2w'' + xw' + \left[\left(\frac{2}{3}\lambda x\right)^2 - \left(\frac{1}{3}\right)^2\right]w = 0 \qquad \text{…………………..} \qquad (4.2.21)$$

Which is Bessel’s O.D.E. of order $\frac{1}{3}$.

$$\therefore\ \ W = C_1 J_{\frac{1}{3}}\left(\frac{2}{3}\lambda x\right) + C_2 J_{-\frac{1}{3}}\left(\frac{2}{3}\lambda x\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (4.2.22)$$

But $\quad G = x^p w\ , p = \frac{1}{3}\ , x = t^{\frac{1}{m}}\ , m = \frac{2}{3}\ , \lambda = \frac{k_n}{a} = \frac{\alpha_n}{aR}$

Therefore substituting we obtain the time solution as

$$G_n(t) = t^{\frac{1}{2}}\left[C_1 J_{\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right) + C_2 J_{-\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right)\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.23)$$

Hence the functions

$$U_n(r,t) = W_n(r)G_n(t) = t^{\frac{1}{2}}\left[A_n J_{\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right) + B_n J_{-\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right)\right] J_0\left(\frac{\alpha_n r}{R}\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (4.2.24)$$

where $U_n's\ \ n = 1,2,3 \ldots \ldots$ Are called the eigen functions and are the solutions of the wave equation

$\frac{\partial^2 U}{\partial t^2} = \frac{\sin t}{a^2}\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right)$, satisfying the boundary condition $U(R,t) = 0$.

Clearly we see that there are infinitely many Eigen values and to each value of α_n there corresponds a particular solution (Eigen function) with the A_n and B_n being arbitrary constants.

Now since the original partial differential equation is linear, any linear combination of the particular solutions would also be a solution. Accordingly we take the linear combination of the U_n as the general solution of the wave equation, that is

$$U(r,t) = \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} t^{\frac{1}{2}}\left[A_n J_{\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right) + B_n J_{-\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right)\right] J_0\left(\frac{\alpha_n r}{R}\right) \ldots\ldots\ldots \ (4.2.25)$$

The arbitrary constants A_n and B_n in the solution must now be chosen so that the boundary conditions at $t = 0$ are satisfied.

We note that near the starting time at $t = 0, J_{-\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right) \cong \left\{\frac{\left(\frac{3aR}{\alpha_n}\right)^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}\right\} t^{\frac{-1}{2}} \to \infty \; as \; t \to 0.$

since the starting time is a branch point of both $j_{\frac{1}{3}}$ and $j_{\frac{-1}{3}}$ and neither functions are odd nor even.

This is always so when the order of Bessel function is fractional. Thus we choose $B_n = 0$.

Now to apply the initial conditions we note that Bessel functions of order n is

$$J_n(t) = \sum_{m=1}^{\infty} \frac{(-1)^m\left(\frac{t}{2}\right)^{2m+n}}{m!(m+n)!}$$

$$= \frac{t^n}{2^n\Gamma(n+1)}\left\{1 - \frac{t^2}{2.2(n+1)} + \frac{t^4}{2.4.2^2(n+1)(n+2)} - \frac{t^6}{2.4.6.2^3(n+1)(n+2)(n+3)} + \cdots\cdots\cdots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.26)$$

Setting $t = \frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}} \; and \; n = \frac{1}{3}$ into (4.2.26) we obtain the expression for the solution of the time O.D.E. so that

$$G_n(t) = t^{\frac{1}{2}}\left[A_n J_{\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR}t^{\frac{3}{2}}\right)\right]$$

$$= \frac{A_n\left(\frac{\alpha_n}{3aR}\right)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)}\left\{t - \frac{\left(\frac{\alpha_n}{aR}\right)^2}{12}t^4 + \frac{\left(\frac{\alpha_n}{aR}\right)^4}{504}t^7 - \cdots\cdots\cdots\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.2.27)$$

With $t = 0$, $\dot{G_n}(0) = \frac{A_n\left(\frac{\alpha_n}{3aR}\right)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)}$ *then*

$$U_t(r,0) = \sum_{n=1}^{\infty} \frac{A_n\left(\frac{\alpha_n}{3aR}\right)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} J_0\left(\frac{\alpha_n r}{R}\right) = f(r) \qquad \text{...………………} \qquad (4.2.28)$$

Multiplying both sides by $rJ_0\left(\frac{r\alpha_m}{R}\right)$ and integrating with respect to r from 0 to R we obtain

$$\sum_{n=1}^{\infty} \frac{A_n\left(\frac{\alpha_n}{3aR}\right)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} \int_0^R r\, J_0\left(\frac{r\alpha_n}{R}\right) J_0\left(\frac{r\alpha_m}{R}\right) dr = \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_m}{R}\right) dr \text{……….} \quad (4.2.29)$$

By the orthogonality principle of Bessel functions we get

$$\frac{A_n\left(\frac{\alpha_n}{3aR}\right)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} \cdot \frac{R^2 J_1^2(\alpha_n)}{2} = \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr$$

$$\Rightarrow A_n = \frac{2\left(\frac{3aR}{\alpha_n}\right)^{\frac{1}{3}}\Gamma\left(\frac{4}{3}\right)}{R^2 J_1^2(\alpha_n)} \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr \qquad \text{…………………} \qquad (4.2.30)$$

$\therefore$ the complete solution becomes

$$U(r,t) = \sum_{n=1}^{\infty} \frac{2t^{\frac{1}{2}}}{R^2 J_1^2(\alpha_n)} \left[\left\{\left(\frac{3aR}{\alpha_n}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right) \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr\right\} J_{\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR} t^{\frac{3}{2}}\right)\right] J_0\left(\frac{\alpha_n r}{R}\right)$$

$$\text{…………………..} \qquad (4.2.31)$$

This equation (4.2.31) for $U(r,t)$ is the mathematical expression of the motion of the hour-glass like drum or the single talking African drum.

CHAPTER FIVE

APPLICATION OF TWO- DIMENSIONAL WAVE EQUATION TO THE LOCAL DRUMS IN GHANA

5.1. Computation of Vibrational Modes Using the Two Models

In this chapter an attempt is made to calculate the vibrational modes for each of the two models of drums, namely, a drum with constant tension discussed in chapter three and a drum with varying tension as a function of time, discussed in chapter four using the matrix laboratory (matlab). The chapter will be composed under the following subtopics (a). The vibrational modes of a drum head with constant tension using $f(r) = 1$, The normal modes of a vibrating drum head with constant tension using $f(r) = r\left(2 - \frac{r}{2}\right)^2$, The normal modes of a vibrating drum head with constant tension using $f(r) = \left(2 - \frac{r}{2}\right)^2$, (b). The vibrational modes of a drum head with varying tension as a function of time using $f(r) = 1$, The normal modes of a vibrating drum head with varying tension using $f(r) = r\left(2 - \frac{r}{2}\right)^2$, The normal modes of a vibrating drum head with varying tension using $f(r) = \left(2 - \frac{r}{2}\right)^2$ and (c). The Description of the Normal modes of each of the two models (a) and (b).

5.2. The vibrational modes of a drum head with constant tension using $f(r) = 1$.

To evaluate the vibrational mode of a circular drum using the model of chapter three, the researcher considered the boundary value problem having the conditions

$U(R,0) = 0$ as, the condition at the boundary and initial velocity $U(r,0) = 0$ *and* $U_t(r,t) = f(r)$; the radius used in the computations was taken to be 3.50 inches.

Three different initial velocity functions were used, namely a velocity which is invariant for instance $f(r) = 1$; the second type of velocity function used is one dependent on the radius and the best candidate selected was a polynomial function of r which has at least one trough and one crest. In this case a cubic function i.e. of the form

$f(r) = r\left(2 - \frac{r}{2}\right)^2$, was chosen.

The third initial velocity used is also dependent on the radius and had only one trough with no crest was a quadratic function of r i.e. $f(r) = \left(2 - \frac{r}{2}\right)^2$.

The choice of an initial velocity function with repeated roots of the form $f(r) = \left(a - \frac{r}{b}\right)^2$, where the constants a and b are chosen so that the centre would have the largest amplitude and least amplitude near the highest value of r, is adapted to ensure that at least a nodal point exist on the drum head because the entire drumhead do not vibrate at the same time.

The model for a vibrating circular membrane with fixed tension is a function of r and t

$$U_n(r,t) = \sum_{n=1}^{\infty} \frac{2t^{\frac{1}{2}}}{R^2 J_1^2(\alpha_n)} \left[\left\{\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr\right\} \times J_{\frac{1}{2}}\left(\frac{c\alpha_n}{R} t\right)\right] J_0\left(\frac{\alpha_n r}{R}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.2.1)$$

And so by allowing t to take integral values from 0 to 9, while's r takes on values

r = 0 ,1 ,1.5 ,2 ,2.25 , 2.5 ,3 ,3.25 ,3.5 and 4 with

$A_n = \frac{2t^{\frac{1}{2}}}{R^2 J_1^2(\alpha_n)} \left[\left\{\left(\frac{2R}{c\alpha_n}\right)^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr\right\}\right]$, c=1, and α_n being equal to the first ten zeros of the Bessel function of order zero. $J_x = J_0\left(\frac{\alpha_n r}{R}\right)$ and $J_y = J_{\frac{1}{2}}\left(\frac{c\alpha_n}{R} t\right)$, Matrix Laboratory codes were applied to compute these expression in the following tables

Bessel function of order zero for drumhead with fixed tension

J_x=									
1	1	1	1	1	1	1	1	1	1
0.8854	0.4684	-0.0346	-0.3585	-0.3667	-0.1229	0.167	0.2996	0.204	-0.0267
0.7516	0.0204	-0.3997	-0.1599	0.2431	0.2262	-0.1121	-0.2412	-0.002	0.2142
0.5808	-0.3078	-0.1954	0.2885	0.0332	-0.2468	0.0742	0.1817	-0.1415	-0.1022
0.4861	-0.3863	0.0146	0.2547	-0.2088	-0.0406	0.2079	-0.1336	-0.0733	0.1795
0.3877	-0.4003	0.1973	0.0631	-0.2206	0.1996	-0.0448	-0.1198	0.18	-0.106
0.1887	-0.2604	0.2766	-0.2492	0.1885	-0.1075	0.0206	0.0579	-0.1164	0.147
0.0921	-0.1356	0.1632	-0.18	0.1876	-0.1872	0.1795	-0.1654	0.1458	-0.1219
0	0	0	0	0	0	0	0	0	0
-0.1634	0.2256	-0.2396	0.2158	-0.1633	0.0931	-0.0179	-0.0501	0.1008	-0.1273

Table 5.2.1 gives values of Bessel function of order zero varying r

Bessel function of order half for drumhead with fixed tension

j_y=									
0	0	0	0	0	0	0	0	0	0
1.8316	1.906	0.9442	-0.294	-1.0453	-0.9482	-0.2148	0.5668	0.854	0.5037
2.0026	-0.0172	-1.0474	0.4051	0.6383	-0.5842	-0.2962	0.626	-0.0016	-0.5575
1.4707	-1.1002	0.7965	-0.4747	0.1535	0.1318	-0.3478	0.471	-0.4931	0.4216
0.5533	0.0243	-0.3414	0.5146	-0.5661	0.5126	-0.3781	0.1945	0.0022	-0.1772
-0.374	0.852	-0.1379	-0.5292	0.3183	0.2974	-0.3902	-0.0932	0.3819	-0.0785
-0.9797	-0.0298	0.4762	0.5211	0.2114	-0.182	-0.3861	-0.2917	-0.0027	0.2568
-1.0863	-0.7197	-0.5751	-0.4928	-0.438	-0.3981	-0.3675	-0.343	-0.3228	-0.3058
-0.7227	0.0344	0.4315	0.4469	0.1707	-0.1669	-0.3361	-0.2485	0.0032	0.2253
-0.0955	0.6343	-0.131	-0.3864	0.2473	0.214	-0.2941	-0.0634	0.2847	-0.0628

Table 5.2.2 gives values of Bessel function of order ½ varying

The first associated coefficients for drumhead with fixed tension

A_n_1=									
0.6528	-0.123	0.05	-0.0269	0.0168	-0.0115	0.0083	-0.0063	0.005	-0.004

Table 5.2.3 gives the coefficient of the normal mode with f(r) = 1

Now by multiplying corresponding columns of tables 5.1, 5.2, 5.3 we obtain matrices of the first ten partials of the vibrating membrane with fixed tension and with these partials the first ten Normal modes were obtained and tabulated below

The 1st Normal mode

U_1_1=									
0	1.1958	1.3074	0.9601	0.3612	-0.2442	-0.6396	-0.7092	-0.4718	-0.0624
0	1.0587	1.1575	0.8501	0.3198	-0.2162	-0.5663	-0.6279	-0.4178	-0.0552
0	0.8987	0.9826	0.7216	0.2715	-0.1835	-0.4807	-0.533	-0.3546	-0.0469
0	0.6945	0.7593	0.5576	0.2098	-0.1418	-0.3715	-0.4119	-0.274	-0.0362
0	0.5812	0.6354	0.4667	0.1756	-0.3109	-0.3447	-0.2293	-0.0303	-0.0303
0	0.4637	0.5069	0.3723	0.1401	-0.0947	-0.248	-0.275	-0.1829	-0.0242
0	0.2256	0.2467	0.1812	0.0682	-0.0461	-0.1207	-0.1338	-0.089	-0.0118
0	0.1101	0.1204	0.0884	0.0333	-0.0225	-0.0589	-0.0653	-0.0434	-0.0057
0	0	0	0	0	0	0	0	0	0
0	-0.1954	-0.2137	-0.1569	-0.059	0.0399	0.1045	0.1159	0.0771	0.0102

Table 5.2.4: _1 in U_1_1 is used to identify the velocity function used

The 2nd Normal mode

U_2_1=									
0	0.9613	1.3095	1.0955	0.3582	-0.349	-0.6359	-0.6206	-0.476	-0.1404
0	0.9489	1.1585	0.9135	0.3184	-0.2653	-0.5646	-0.5864	-0.4197	-0.0918
0	0.8939	0.9826	0.7244	0.2714	-0.1857	-0.4806	-0.5312	-0.3547	-0.0485
0	0.7667	0.7586	0.516	0.2107	-0.1096	-0.3726	-0.4391	-0.2727	-0.0122
0	0.6718	0.6346	0.4144	0.1767	-0.0782	-0.3123	-0.3789	-0.2277	-0.0002
0	0.5575	0.5061	0.3181	0.1412	-0.0527	-0.2495	-0.3104	-0.1813	0.0071
0	0.2867	0.2461	0.1459	0.0689	-0.0188	-0.1216	-0.1569	-0.0879	0.0086
0	0.1419	0.1201	0.07	0.0337	-0.0083	-0.0594	-0.0773	-0.0429	0.0048
0	0	0	0	0	0	0	0	0	0
0	-0.2483	-0.2132	-0.1264	-0.0597	0.0163	0.1054	0.1359	0.0762	-0.0074

Table 5.2.5: _1 in U_2_1 identify the first velocity function used

The 3rd Normal mode

U_3_1=									
0	1.0085	1.2571	1.1353	0.3411	-0.3559	-0.6121	-0.6494	-0.4545	-0.147
0	0.9473	1.1604	0.9121	0.319	-0.2651	-0.5654	-0.5854	-0.4205	-0.0915
0	0.875	1.0035	0.7084	0.2782	-0.1829	-0.4901	-0.5197	-0.3633	-0.0459
0	0.7574	0.7689	0.5082	0.214	-0.1082	-0.3772	-0.4335	-0.2769	-0.0109
0	0.6725	0.6339	0.415	0.1765	-0.0783	-0.3119	-0.3793	-0.2274	-0.0003
0	0.5668	0.4957	0.3259	0.1379	-0.0541	-0.2448	-0.3161	-0.177	0.0058
0	0.2998	0.2317	0.1569	0.0642	-0.0207	-0.1151	-0.1648	-0.082	0.0067
0	0.1496	0.1115	0.0766	0.0309	-0.0094	-0.0555	-0.082	-0.0393	0.0038
0	0	0	0	0	0	0	0	0	0
0	-0.2596	-0.2006	-0.1359	-0.0556	0.0179	0.0996	0.1428	0.071	-0.0058

Table 5.2.6 _1 in U_3_1 identify the first velocity function used

The 4th Normal mode

U_4_1=									
0	1.0164	1.2462	1.1481	0.3273	-0.3416	-0.6261	-0.6361	-0.4665	-0.1366
0	0.9444	1.1643	0.9075	0.324	-0.2702	-0.5604	-0.5902	-0.4162	-0.0953
0	0.8738	1.0053	0.7064	0.2804	-0.1852	-0.4879	-0.5218	-0.3614	-0.0475
0	0.7597	0.7657	0.5119	0.21	-0.1041	-0.3813	-0.4297	-0.2804	-0.0079
0	0.6745	0.6311	0.4182	0.1729	-0.0747	-0.3155	-0.3759	-0.2304	0.0024
0	0.5673	0.495	0.3268	0.137	-0.0532	-0.2456	-0.3153	-0.1778	0.0064
0	0.2978	0.2344	0.1538	0.0677	-0.0242	-0.1116	-0.1681	-0.079	0.0041
0	0.1482	0.1135	0.0743	0.0334	-0.012	-0.053	-0.0844	-0.0372	0.0019
0	0	0	0	0	0	0	0	0	0
0	-0.2579	-0.203	-0.1332	-0.0586	0.021	0.0966	0.1456	0.0684	-0.0036

Table 5.2.7: _1 in U_4_1 identify the first velocity function used

The 5th Normal mode

U_5_1									
0	0.9988	1.2569	1.1507	0.3178	-0.3363	-0.6226	-0.6435	-0.4636	-0.1324
0	0.9509	1.1603	0.9066	0.3275	-0.2721	-0.5617	-0.5875	-0.4172	-0.0968
0	0.8695	1.0079	0.707	0.2781	-0.1839	-0.487	-0.5236	-0.3607	-0.0465
0	0.7591	0.7661	0.5119	0.2097	-0.1039	-0.3812	-0.4299	-0.2803	-0.0078
0	0.6782	0.6288	0.4177	0.1749	-0.0758	-0.3162	-0.3744	-0.231	0.0015
0	0.5712	0.4927	0.3262	0.1391	-0.0544	-0.2464	-0.3136	-0.1784	0.0055
0	0.2945	0.2364	0.1542	0.0659	-0.0232	-0.1109	-0.1695	-0.0784	0.0049
0	0.1449	0.1155	0.0747	0.0316	-0.011	-0.0523	-0.0858	-0.0366	0.0027
0	0	0	0	0	0	0	0	0	0
0	-0.2551	-0.2047	-0.1336	-0.0571	0.0201	0.096	0.1468	0.0679	-0.0043

Table 5.2.8: _1 in U_5_1 identify the first velocity function used

The 6th Normal mode

U_6_1=									
0	1.0097	1.2636	1.1492	0.3119	-0.3397	-0.6205	-0.6389	-0.4617	-0.1349
0	0.9495	1.1595	0.9068	0.3282	-0.2717	-0.5619	-0.5881	-0.4175	-0.0965
0	0.872	1.0094	0.7067	0.2768	-0.1847	-0.4865	-0.5226	-0.3603	-0.0471
0	0.7565	0.7644	0.5123	0.2112	-0.1031	-0.3817	-0.4311	-0.2808	-0.0072
0	0.6777	0.6286	0.4177	0.1752	-0.0756	-0.3163	-0.3746	-0.2311	0.0016
0	0.5734	0.494	0.3259	0.1379	-0.055	-0.246	-0.3127	-0.178	0.005
0	0.2933	0.2357	0.1544	0.0665	-0.0229	-0.1111	-0.17	-0.0786	0.0052
0	0.1429	0.1143	0.075	0.0327	-0.0103	-0.0527	-0.0866	-0.037	0.0031
0	0	0	0	0	0	0	0	0	0
0	-0.2541	-0.2041	-0.1337	-0.0576	0.0198	0.0962	0.1473	0.0681	-0.0045

Table 5.2.9: _1 in U_6_1 identify the first velocity function used

The 7th Normal mode

U_7_1=									
0	1.0079	1.2611	1.1463	0.3088	-0.343	-0.6237	-0.642	-0.4645	-0.1373
0	0.9492	1.1591	0.9063	0.3276	-0.2722	-0.5625	-0.5886	-0.4179	-0.0969
0	0.8722	1.0097	0.707	0.2772	-0.1843	-0.4862	-0.5222	-0.3599	-0.0468
0	0.7563	0.7643	0.5121	0.2109	-0.1033	-0.3819	-0.4313	-0.281	-0.0074
0	0.6773	0.6281	0.4171	0.1745	-0.0763	-0.317	-0.3752	-0.2317	0.0011
0	0.5734	0.4941	0.326	0.1381	-0.0549	-0.2459	-0.3126	-0.1779	0.0051
0	0.2933	0.2356	0.1543	0.0664	-0.0229	-0.1112	-0.1701	-0.0787	0.0051
0	0.1425	0.1138	0.0745	0.0321	-0.0109	-0.0533	-0.0872	-0.0375	0.0027
0	0	0	0	0	0	0	0	0	0
0	-0.254	-0.2041	-0.1337	-0.0575	0.0199	0.0963	0.1473	0.0681	-0.0045

Table 5.2.10: _1 in U_7_1 identify the first velocity function used

The 8th Normal mode

U_8_1=									
0	1.0043	1.2572	1.1433	0.3075	-0.3424	-0.6219	-0.6398	-0.4629	-0.1369
0	0.9482	1.1579	0.9054	0.3273	-0.2721	-0.5619	-0.5879	-0.4175	-0.0968
0	0.873	1.0106	0.7077	0.2775	-0.1844	-0.4866	-0.5227	-0.3603	-0.0469
0	0.7557	0.7635	0.5116	0.2107	-0.1032	-0.3816	-0.4309	-0.2807	-0.0073
0	0.6778	0.6286	0.4175	0.1747	-0.0764	-0.3172	-0.3755	-0.2319	0.0011
0	0.5739	0.4946	0.3264	0.1382	-0.055	-0.2461	-0.3128	-0.1781	0.0051
0	0.2931	0.2354	0.1542	0.0664	-0.0229	-0.1111	-0.17	-0.0786	0.0052
0	0.1431	0.1145	0.075	0.0323	-0.011	-0.0536	-0.0875	-0.0378	0.0026
0	0	0	0	0	0	0	0	0	0
0	-0.2539	-0.2039	-0.1335	-0.0575	0.0198	0.0962	0.1472	0.0681	-0.0045

table 5.2.11: _1 in U_8_1 identify the first velocity function used.

The 9th Normal mode

U_9_1=									
0	1.0086	1.2572	1.1408	0.3075	-0.3405	-0.6219	-0.6414	-0.4629	-0.1355
0	0.949	1.1579	0.9049	0.3273	-0.2717	-0.5619	-0.5883	-0.4174	-0.0965
0	0.873	1.0106	0.7077	0.2775	-0.1844	-0.4866	-0.5227	-0.3603	-0.0469
0	0.7551	0.7635	0.5119	0.2107	-0.1035	-0.3816	-0.4307	-0.2807	-0.0075
0	0.6775	0.6286	0.4177	0.1747	-0.0765	-0.3172	-0.3754	-0.2319	0.001
0	0.5746	0.4946	0.3259	0.1382	-0.0546	-0.2461	-0.3131	-0.1781	0.0053
0	0.2926	0.2354	0.1545	0.0664	-0.0231	-0.1111	-0.1698	-0.0786	0.005
0	0.1437	0.1145	0.0746	0.0323	-0.0107	-0.0536	-0.0878	-0.0378	0.0028
0	0	0	0	0	0	0	0	0	0
0	-0.2534	-0.2039	-0.1338	-0.0575	0.02	0.0962	0.147	0.0681	-0.0043

table5.2.12: _1 in U_9_1 identify the first velocity function used

The 10th Normal mode

U_10_1=									
0	1.0066	1.2594	1.1392	0.3082	-0.3402	-0.6229	-0.6402	-0.4638	-0.1352
0	0.9491	1.1579	0.905	0.3273	-0.2717	-0.5619	-0.5883	-0.4174	-0.0965
0	0.8726	1.0111	0.7074	0.2776	-0.1844	-0.4868	-0.5225	-0.3605	-0.0468
0	0.7553	0.7633	0.5121	0.2106	-0.1035	-0.3815	-0.4308	-0.2806	-0.0075
0	0.6772	0.629	0.4174	0.1748	-0.0765	-0.3174	-0.3752	-0.2321	0.001
0	0.5748	0.4944	0.3261	0.1381	-0.0547	-0.246	-0.3133	-0.178	0.0053
0	0.2923	0.2357	0.1542	0.0665	-0.0231	-0.1112	-0.1696	-0.0787	0.005
0	0.144	0.1142	0.0748	0.0322	-0.0108	-0.0535	-0.0879	-0.0377	0.0028
0	0	0	0	0	0	0	0	0	0
0	-0.2532	-0.2042	-0.1336	-0.0576	0.02	0.0963	0.1469	0.0682	-0.0044

table 5.2.13: _1 in U_10_1 identify the first velocity function used.

Tables 5.2.4 –Table 5.2.13 gives the first ten normal mode with the velocity function f(r)=1

5.3. The normal mode of a vibrating drum head with constant tension using $\boldsymbol{f(r) = r\left(2-\frac{r}{2}\right)^2}$.

By using the second initial velocity function considered i.e. $f(r) = r\left(2-\frac{r}{2}\right)^2$ the new coefficient is obtained

The second associated coefficients for drumhead with fixed tension

A_n_2=									
1.1727	-0.1191	-0.0277	-0.0147	-0.0022	-0.0046	0.0001	-0.0022	0.0004	-0.0012

table 5.3.1 gives the coefficient of the normal mode with $f(r) = r\left(2 - \frac{r}{2}\right)^2$

And on multiplying the respective columns of tables 5.1.1,5.1.2 and 5.2.1 the matrices of the first ten Normal modes were also generated and tabulated below.

The first Normal mode

U_1_2=									
0	2.1481	2.3485	1.7247	0.6489	-0.4386	-1.1489	-1.274	-0.8476	-0.112
0	1.9019	2.0794	1.5271	0.5745	-0.3884	-1.0173	-1.128	-0.7505	-0.0992
0	1.6144	1.7651	1.2963	0.4877	-0.3297	-0.8635	-0.9575	-0.637	-0.0842
0	1.2476	1.364	1.0017	0.3768	-0.2548	-0.6673	-0.7399	-0.4923	-0.0651
0	1.0441	1.1415	0.8383	0.3154	-0.2132	-0.5584	-0.6192	-0.412	-0.0545
0	0.8329	0.9106	0.6688	0.2516	-0.1701	-0.4455	-0.494	-0.3286	-0.0434
0	0.4053	0.4432	0.3255	0.1224	-0.0828	-0.2168	-0.2404	-0.1599	-0.0211
0	0.1978	0.2163	0.1588	0.0597	-0.0404	-0.1058	-0.1173	-0.078	-0.0103
0	0	0	0	0	0	0	0	0	0
0	-0.3511	-0.3838	-0.2819	-0.106	0.0717	0.1878	0.2082	0.1385	0.0183

table 5.3.2 :_2 in U_1_2 identify the velocity function used

The 2nd Normal mode

U_2_2=									
0	1.9211	2.3506	1.8558	0.646	-0.5401	-1.1454	-1.1883	-0.8517	-0.1876
0	1.7956	2.0804	1.5885	0.5732	-0.4359	-1.0156	-1.0878	-0.7524	-0.1346
0	1.6098	1.7651	1.2989	0.4876	-0.3317	-0.8634	-0.9557	-0.6371	-0.0857
0	1.3174	1.3634	0.9614	0.3777	-0.2235	-0.6684	-0.7663	-0.491	-0.0418
0	1.1318	1.1407	0.7877	0.3165	-0.174	-0.5598	-0.6523	-0.4104	-0.0253
0	0.9237	0.9098	0.6163	0.2527	-0.1295	-0.4469	-0.5283	-0.327	-0.0132
0	0.4644	0.4426	0.2913	0.1232	-0.0563	-0.2177	-0.2627	-0.1589	-0.0015
0	0.2286	0.216	0.141	0.0601	-0.0266	-0.1063	-0.1289	-0.0775	-0.0001
0	0	0	0	0	0	0	0	0	0
0	-0.4023	-0.3834	-0.2523	-0.1067	0.0488	0.1886	0.2275	0.1376	0.0013

table 5.3.3 : _2in U_2_2 identify the velocity function used

The 3rd Normal mode

U_3_2=									
0	1.8949	2.3796	1.8337	0.6554	-0.5363	-1.1586	-1.1723	-0.8636	-0.1839
0	1.7965	2.0794	1.5892	0.5728	-0.436	-1.0152	-1.0884	-0.752	-0.1347
0	1.6202	1.7535	1.3078	0.4838	-0.3333	-0.8582	-0.9621	-0.6323	-0.0872
0	1.3226	1.3577	0.9657	0.3759	-0.2243	-0.6658	-0.7694	-0.4887	-0.0425
0	1.1314	1.1411	0.7874	0.3166	-0.174	-0.56	-0.6521	-0.4106	-0.0252
0	0.9186	0.9155	0.612	0.2546	-0.1287	-0.4495	-0.5251	-0.3294	-0.0125
0	0.4572	0.4507	0.2852	0.1258	-0.0553	-0.2214	-0.2583	-0.1622	-0.0005
0	0.2243	0.2207	0.1374	0.0617	-0.026	-0.1084	-0.1263	-0.0794	0.0005
0	0	0	0	0	0	0	0	0	0
0	-0.396	-0.3903	-0.247	-0.109	0.0479	0.1917	0.2237	0.1405	0.0004

table 5.3.4: _2 in U_3_2 identify the velocity function used

The 4th Normal mode

U_4_2=									
0	1.8993	2.3736	1.8407	0.6478	-0.5285	-1.1662	-1.1651	-0.8702	-0.1783
0	1.795	2.0815	1.5867	0.5755	-0.4388	-1.0124	-1.091	-0.7496	-0.1367
0	1.6195	1.7545	1.3066	0.485	-0.3345	-0.8569	-0.9632	-0.6313	-0.0881
0	1.3238	1.356	0.9677	0.3737	-0.222	-0.668	-0.7673	-0.4906	-0.0409
0	1.1325	1.1396	0.7892	0.3147	-0.172	-0.562	-0.6502	-0.4122	-0.0238
0	0.9189	0.9152	0.6124	0.2541	-0.1282	-0.45	-0.5247	-0.3298	-0.0121
0	0.4561	0.4521	0.2835	0.1277	-0.0572	-0.2195	-0.2601	-0.1605	-0.0019
0	0.2235	0.2218	0.1362	0.063	-0.0274	-0.107	-0.1276	-0.0783	-0.0005
0	0	0	0	0	0	0	0	0	0
0	-0.3951	-0.3916	-0.2455	-0.1106	0.0496	0.1901	0.2253	0.139	0.0016

table 5.3.5: _2 in U_4_2 identify the velocity function use

The 5th Normal mode

U_5_2=									
0	1.9015	2.3722	1.8404	0.6491	-0.5292	-1.1667	-1.1641	-0.8706	-0.1788
0	1.7941	2.082	1.5869	0.5751	-0.4386	-1.0122	-1.0913	-0.7495	-0.1365
0	1.6201	1.7541	1.3066	0.4853	-0.3347	-0.857	-0.963	-0.6314	-0.0882
0	1.3239	1.3559	0.9677	0.3737	-0.2221	-0.668	-0.7673	-0.4906	-0.0409
0	1.132	1.1399	0.7892	0.3144	-0.1718	-0.5619	-0.6504	-0.4122	-0.0237
0	0.9184	0.9155	0.6125	0.2539	-0.1281	-0.4499	-0.5249	-0.3297	-0.012
0	0.4566	0.4519	0.2834	0.1279	-0.0574	-0.2195	-0.2599	-0.1606	-0.002
0	0.224	0.2215	0.1361	0.0633	-0.0275	-0.1071	-0.1275	-0.0783	-0.0006
0	0	0	0	0	0	0	0	0	0
0	-0.3954	-0.3914	-0.2455	-0.1108	0.0497	0.1902	0.2251	0.1391	0.0017

table 5.3.6: _2 in U_5_2 identify the velocity function used

The 6th Normal mode

U_6_2=									
0	1.9059	2.3749	1.8397	0.6467	-0.5306	-1.1659	-1.1623	-0.8698	-0.1798
0	1.7936	2.0817	1.5869	0.5754	-0.4384	-1.0123	-1.0916	-0.7496	-0.1364
0	1.6211	1.7548	1.3064	0.4848	-0.335	-0.8568	-0.9626	-0.6312	-0.0885
0	1.3228	1.3553	0.9679	0.3743	-0.2217	-0.6682	-0.7677	-0.4908	-0.0407
0	1.1318	1.1398	0.7893	0.3145	-0.1718	-0.5619	-0.6505	-0.4122	-0.0236
0	0.9192	0.916	0.6123	0.2534	-0.1283	-0.4497	-0.5245	-0.3295	-0.0122
0	0.4561	0.4516	0.2835	0.1282	-0.0572	-0.2196	-0.2601	-0.1607	-0.0019
0	0.2231	0.221	0.1362	0.0637	-0.0273	-0.1073	-0.1278	-0.0785	-0.0004
0	0	0	0	0	0	0	0	0	0
0	-0.395	-0.3911	-0.2455	-0.111	0.0496	0.1902	0.2253	0.1392	0.0016

table 5.3.7: _2 in U_6_2 identify the velocity function used

The 7th Normal mode

U_7_2=									
0	1.9059	2.3749	1.8397	0.6467	-0.5306	-1.1659	-1.1623	-0.8698	-0.1798
0	1.7936	2.0817	1.5869	0.5754	-0.4384	-1.0124	-1.0916	-0.7496	-0.1364
0	1.6211	1.7548	1.3064	0.4848	-0.335	-0.8568	-0.9626	-0.6312	-0.0885
0	1.3228	1.3553	0.9678	0.3743	-0.2217	-0.6682	-0.7677	-0.4908	-0.0407
0	1.1318	1.1398	0.7892	0.3145	-0.1718	-0.5619	-0.6505	-0.4122	-0.0236
0	0.9192	0.916	0.6124	0.2534	-0.1283	-0.4497	-0.5245	-0.3295	-0.0122
0	0.4561	0.4516	0.2835	0.1282	-0.0572	-0.2196	-0.2601	-0.1607	-0.0019
0	0.2231	0.221	0.1362	0.0637	-0.0273	-0.1073	-0.1278	-0.0785	-0.0004
0	0	0	0	0	0	0	0	0	0
0	-0.395	-0.3911	-0.2455	-0.111	0.0496	0.1902	0.2253	0.1392	0.0016

table 5.3.8: _2 in U_7_2 identify the velocity function used

The 8th Normal mode

U_8_2=									
0	1.9047	2.3736	1.8387	0.6462	-0.5304	-1.1653	-1.1616	-0.8693	-0.1797
0	1.7932	2.0813	1.5866	0.5752	-0.4384	-1.0122	-1.0914	-0.7494	-0.1364
0	1.6214	1.7551	1.3067	0.4849	-0.335	-0.857	-0.9628	-0.6313	-0.0885
0	1.3226	1.355	0.9677	0.3743	-0.2217	-0.6681	-0.7676	-0.4907	-0.0406
0	1.132	1.14	0.7894	0.3146	-0.1718	-0.562	-0.6506	-0.4123	-0.0236
0	0.9194	0.9162	0.6125	0.2534	-0.1284	-0.4498	-0.5246	-0.3296	-0.0122
0	0.456	0.4515	0.2834	0.1282	-0.0572	-0.2196	-0.2601	-0.1607	-0.0019
0	0.2233	0.2212	0.1364	0.0638	-0.0273	-0.1074	-0.1279	-0.0786	-0.0004
0	0	0	0	0	0	0	0	0	0
0	-0.395	-0.3911	-0.2455	-0.111	0.0495	0.1902	0.2253	0.1392	0.0016

table 5.3.9: _2 in U_8_2 identify the velocity function used

The 9th Normal mode

U_9_2=									
0	1.905	2.3736	1.8385	0.6462	-0.5303	-1.1653	-1.1617	-0.8693	-0.1796
0	1.7933	2.0813	1.5866	0.5752	-0.4383	-1.0122	-1.0914	-0.7494	-0.1364
0	1.6214	1.7551	1.3067	0.4849	-0.335	-0.857	-0.9628	-0.6313	-0.0885
0	1.3225	1.355	0.9677	0.3743	-0.2217	-0.6681	-0.7676	-0.4907	-0.0407
0	1.132	1.14	0.7894	0.3146	-0.1718	-0.562	-0.6506	-0.4123	-0.0237
0	0.9194	0.9162	0.6124	0.2534	-0.1283	-0.4498	-0.5246	-0.3296	-0.0122
0	0.456	0.4515	0.2835	0.1282	-0.0572	-0.2196	-0.2601	-0.1607	-0.0019
0	0.2234	0.2212	0.1364	0.0638	-0.0273	-0.1074	-0.1279	-0.0786	-0.0004
0	0	0	0	0	0	0	0	0	0
0	-0.3949	-0.3911	-0.2455	-0.111	0.0496	0.1902	0.2253	0.1392	0.0016

table 5.3. 10: _2 in U_9_2 identify the velocity function used

The 10th Normal mode

U_10_2=									
0	1.9044	2.3742	1.838	0.6465	-0.5302	-1.1656	-1.1613	-0.8696	-0.1795
0	1.7933	2.0813	1.5866	0.5752	-0.4383	-1.0122	-1.0914	-0.7494	-0.1364
0	1.6214	1.7551	1.3067	0.4849	-0.335	-0.857	-0.9628	-0.6313	-0.0885
0	1.3226	1.355	0.9677	0.3742	-0.2217	-0.6681	-0.7676	-0.4906	-0.0407
0	1.1318	1.1401	0.7893	0.3146	-0.1718	-0.562	-0.6505	-0.4123	-0.0236
0	0.9195	0.9161	0.6125	0.2534	-0.1284	-0.4498	-0.5247	-0.3296	-0.0122
0	0.4559	0.4516	0.2834	0.1282	-0.0572	-0.2197	-0.26	-0.1607	-0.0019
0	0.2235	0.2211	0.1364	0.0638	-0.0273	-0.1074	-0.127	-0.0785	-0.0004
0	0	0	0	0	0	0	0	0	0
0	-0.3949	-0.3911	-0.2454	-0.111	0.0495	0.1902	0.2252	0.1392	0.0016

table 5.3.11: _2 in U_10_2 identify the velocity function used

5.4. The normal mode of a vibrating drum head with constant tension using

$$\boldsymbol{f(r) = \left(2 - \frac{r}{2}\right)^2}$$

Now by using the third initial velocity function considered i.e. $f(r) = \left(2 - \frac{r}{2}\right)^2$ the new coefficient is obtained in the table below

The third associated coefficients for drumhead with fixed tension

A_n_3=									
0.8959	0.0973	0.0168	0.0052	0.0032	0.0008	0.0012	0.0001	0.0006	0

table 5.4.1 gives the coefficient of the normal mode with $f(r) = \left(2 - \frac{r}{2}\right)^2$

And on multiplying the respective columns of tables 5.1.1, 5.1.2 and 5.3.1 the matrices of the first ten Normal modes were again generated and tabulated below

The 1st Normal mode

U_1_3=									
0	1.641	1.7941	1.3176	0.4957	-0.3351	-0.8777	-0.9732	-0.6475	-0.0856
0	1.4529	1.5885	1.1666	0.4389	-0.2967	-0.7771	-0.8617	-0.5733	-0.0758
0	1.2333	1.3484	0.9902	0.3725	-0.2518	-0.6596	-0.7314	-0.4866	-0.0643
0	0.9531	1.042	0.7652	0.2879	-0.1946	-0.5098	-0.5652	-0.3761	-0.0497
0	0.7976	0.872	0.6404	0.2409	-0.1629	-0.4266	-0.473	-0.3147	-0.0416
0	0.6363	0.6956	0.5109	0.1922	-0.1299	-0.3403	-0.3774	-0.2511	-0.0332
0	0.3096	0.3385	0.2486	0.0935	-0.0632	-0.1656	-0.1836	-0.1222	-0.0162
0	0.1511	0.1652	0.1213	0.0456	-0.0309	-0.0808	-0.0896	-0.0596	-0.0079
0	0	0	0	0	0	0	0	0	0
0	-0.2682	-0.2932	-0.2153	-0.081	0.0548	0.1434	0.1591	0.1058	0.014

table 5.4.2 _3 in U_1_3 identify the velocity function used

The 2nd Normal mode

U_2_3=									
0	1.8265	1.7924	1.2105	0.498	-0.2522	-0.8806	-1.0433	-0.6441	-0.0239
0	1.5398	1.5877	1.1164	0.44	-0.2579	-0.7785	-0.8945	-0.5717	-0.0469
0	1.2371	1.3484	0.9881	0.3726	-0.2501	-0.6597	-0.7329	-0.4866	-0.0631
0	0.8959	1.0425	0.7982	0.2872	-0.2201	-0.5089	-0.5437	-0.3771	-0.0687
0	0.7259	0.8727	0.6818	0.24	-0.1949	-0.4255	-0.446	-0.316	-0.0654
0	0.562	0.6963	0.5537	0.1912	-0.1631	-0.3392	-0.3493	-0.2524	-0.0579
0	0.2613	0.339	0.2765	0.0929	-0.0848	-0.1649	-0.1654	-0.123	-0.0322
0	0.1259	0.1654	0.1358	0.0453	-0.0421	-0.0804	-0.801	-0.0601	-0.0163
0	0	0	0	0	0	0	0	0	0
0	-0.2263	-0.2936	-0.2395	-0.0805	0.0735	0.1428	0.1433	0.1066	0.0279

table 5.4.3 _3 in U_2_3 identify the velocity function used

The 3rd Normal mode

U_3_3=									
0	1.8423	1.7748	1.2239	0.4923	-0.2545	-0.8726	-1.0529	-0.6369	-0.0261
0	1.5393	1.5883	1.116	0.4402	-0.2578	-0.7788	-0.8942	-0.572	-0.0468
0	1.2307	1.3554	0.9827	0.3749	-0.2492	-0.6629	-0.729	-0.4895	-0.0622
0	0.8928	1.0459	0.7956	0.2883	-0.2197	-0.5104	-0.5418	-0.3785	-0.0683
0	0.7262	0.8724	0.682	0.2399	-0.1949	-0.4254	-0.4461	-0.3159	-0.0655
0	0.5651	0.6928	0.5564	0.1901	-0.1636	-0.3376	-0.3512	-0.251	-0.0583
0	0.2657	0.3341	0.2802	0.0913	-0.0855	-0.1627	-0.1681	-0.121	-0.0328
0	0.1285	0.1626	0.138	0.0444	-0.0425	-0.0791	-0.0817	-0.0589	-0.0166
0	0	0	0	0	0	0	0	0	0
0	-0.2301	-0.2894	-0.2427	-0.0791	0.074	0.1409	0.1456	0.1048	0.0284

table 5.4.4 _3 in U_3_3 identify the velocity function used

The 4th Normal mode

U_4_3=									
0	1.8408	1.7769	1.2214	0.495	-0.2572	-0.8699	-1.0555	-0.6346	-0.0281
0	1.5398	1.5876	1.1169	0.4392	-0.2568	-0.7797	-0.8933	-0.5728	-0.0461
0	1.231	1.355	0.9831	0.3744	-0.2488	-0.6633	-0.7286	-0.4898	-0.0619
0	0.8924	1.0466	0.7949	0.289	-0.2205	-0.5096	-0.5425	-0.3778	-0.0689
0	0.7258	0.8729	0.6813	0.2406	-0.1956	-0.4247	-0.4468	-0.3153	-0.066
0	0.565	0.693	0.5562	0.1903	-0.1637	-0.3374	-0.3514	-0.2508	-0.0585
0	0.2661	0.3336	0.2808	0.0907	-0.0848	-0.1633	-0.1674	-0.1216	-0.0323
0	0.1288	0.1622	0.1385	0.0439	-0.042	-0.0796	-0.0812	-0.0593	-0.0163
0	0	0	0	0	0	0	0	0	0
0	-0.2305	-0.2889	-0.2432	-0.0785	0.0734	0.1415	0.145	0.1053	0.028

table 5.4.5 _3 in U_4_3 identify the velocity function used

The 5th Normal mode

U_5_3=									
0	1.8374	1.779	1.2219	0.4932	-0.2562	-0.8692	-1.0569	-0.634	-0.0273
0	1.541	1.5868	1.1167	0.4399	-0.2572	-0.78	-0.8927	-0.573	-0.0464
0	1.2302	1.3555	0.9832	0.374	-0.2485	-0.6632	-0.7289	-0.4897	-0.0617
0	0.8923	1.0466	0.7949	0.289	-0.2204	-0.5096	-0.5426	-0.3778	-0.0688
0	0.7265	0.8725	0.6812	0.241	-0.1958	-0.4248	-0.4465	-0.3154	-0.0662
0	0.5658	0.6925	0.5561	0.1907	-0.164	-0.3376	-0.3511	-0.2509	-0.0586
0	0.2655	0.334	0.2809	0.0903	-0.0846	-0.1632	-0.1677	-0.1215	-0.0322
0	0.1282	0.1626	0.1386	0.0436	-0.0418	-0.0795	-0.0815	-0.0592	-0.0161
0	0	0	0	0	0	0	0	0	0
0	-0.2299	-0.2893	-0.2433	-0.0782	0.0733	0.1413	0.1452	0.1052	0.0279

table 5.4.6 _3 in U_5_3 identify the velocity function used

The 6th Normal mode

U_6_3=									
0	1.8367	1.7785	1.222	0.4936	-0.256	-0.8694	-1.0572	-0.6342	-0.0271
0	1.5411	1.5869	1.1167	0.4398	-0.2572	-0.78	-0.8927	-0.573	-0.0464
0	1.23	1.3554	0.9833	0.3741	-0.2485	-0.6632	-0.729	-0.4897	-0.0616
0	0.8925	1.0467	0.7949	0.2889	-0.2205	-0.5096	-0.5425	-0.3778	-0.0689
0	0.7265	0.8725	0.6812	0.241	-0.1959	-0.4248	-0.4465	-0.3154	-0.0662
0	0.5656	0.6924	0.5561	0.1908	-0.1639	-0.3376	-0.3511	-0.251	-0.0586
0	0.2655	0.334	0.2809	0.0903	-0.0846	-0.1632	-0.1677	-0.1215	-0.0322
0	0.1283	0.1626	0.1385	0.0435	-0.0418	-0.0795	-0.0814	-0.0592	-0.0161
0	0	0	0	0	0	0	0	0	0
0	-0.23	-0.2893	-0.2433	-0.0782	0.0733	0.1413	0.1452	0.1052	0.0279

table 5.4.7 _3 in U_6_3 identify the velocity function used

The 7th Normal mode

U_7_3=									
0	1.8364	1.7782	1.2216	0.4931	-0.2564	-0.8698	-1.0576	-0.6346	-0.0274
0	1.5411	1.5868	1.1166	0.4398	-0.2573	-0.78	-0.8928	-0.5731	-0.0464
0	1.23	1.3555	0.9833	0.3741	-0.2484	-0.6631	-0.729	-0.4897	-0.0616
0	0.8925	1.0467	0.7948	0.2889	-0.2205	-0.5096	-0.5425	-0.3778	-0.0689
0	0.7264	0.8725	0.6812	0.2409	-0.196	-0.4249	-0.4465	-0.3155	-0.0662
0	0.5656	0.6925	0.5562	0.1908	-0.1639	-0.3376	-0.3511	-0.2509	-0.0586
0	0.2655	0.334	0.2809	0.0903	-0.0846	-0.1632	-0.1677	-0.1215	-0.0322
0	0.1283	0.1626	0.1385	0.0434	-0.0419	-0.0795	-0.0815	-0.0593	-0.0162
0	0	0	0	0	0	0	0	0	0
0	-0.23	-0.2893	-0.2433	-0.0782	0.0733	0.1413	0.1452	0.1052	0.0279

table 5.4.8 _3 in U_7_3 identify the velocity function used

The 8th Normal mode

U_8_3=									
0	1.8365	1.7783	1.2216	0.4931	-0.2564	-0.8698	-1.0576	-0.6346	-0.0274
0	1.5411	1.5869	1.1166	0.4398	-0.2573	-0.78	-0.8928	-0.5731	-0.0464
0	1.23	1.3555	0.9833	0.3741	-0.2484	-0.6631	-0.729	-0.4897	-0.0616
0	0.8925	1.0467	0.7949	0.2889	-0.2205	-0.5096	-0.5425	-0.3778	-0.0689
0	0.7264	0.8725	0.6811	0.2409	-0.196	-0.4249	-0.4465	-0.3155	-0.0662
0	0.5656	0.6924	0.5561	0.1908	-0.1639	-0.3376	-0.3511	-0.2509	-0.0586
0	0.2655	0.334	0.2809	0.0903	-0.0846	-0.1632	-0.1677	-0.1215	-0.0322
0	0.1283	0.1626	0.1385	0.0434	-0.0419	-0.0795	-0.0815	-0.0592	-0.0162
0	0	0	0	0	0	0	0	0	0
0	-0.23	-0.2893	-0.2433	-0.0782	0.0733	0.1413	0.1452	0.1052	0.0279

table 5.4.9 _3 in U_8_3 identify the velocity function used

The 9th Normal mode

U_9_3=									
0	1.8365	1.7783	1.2216	0.4931	-0.2562	-0.8699	-1.0578	-0.6346	-0.0273
0	1.5411	1.5869	1.1166	0.4398	-0.2572	-0.78	-0.8928	-0.5731	-0.0464
0	1.23	1.3555	0.9833	0.3741	-0.2484	-0.6631	-0.729	-0.4897	-0.0616
0	0.8925	1.0467	0.7949	0.2889	-0.2206	-0.5096	-0.5425	-0.3778	-0.0689
0	0.7264	0.8725	0.6811	0.2409	-0.196	-0.4249	-0.4465	-0.3155	-0.0662
0	0.5656	0.6924	0.5561	0.1908	-0.1639	-0.3376	-0.3511	-0.2509	-0.0586
0	0.2655	0.334	0.2809	0.0903	-0.0847	-0.1632	-0.1677	-0.1215	-0.0322
0	0.1283	0.1626	0.1384	0.0434	-0.0419	-0.0795	-0.0815	-0.0592	-0.0162
0	0	0	0	0	0	0	0	0	0
0	-0.23	-0.2893	-0.2433	-0.0782	0.0733	0.1413	0.1452	0.1052	0.0279

table 5.4.10 : _3 in U_9_3 identify the velocity function used

The 10th Normal mode

U_10_3=									
0	1.8365	1.7783	1.2216	0.4931	-0.2562	-0.8699	-1.0578	-0.6346	-0.0273
0	1.5411	1.5869	1.1166	0.4398	-0.2572	-0.78	-0.8928	-0.5731	-0.0464
0	1.23	1.3555	0.9833	0.3741	-0.2484	-0.6631	-0.729	-0.4897	-0.0616
0	0.8925	1.0467	0.7949	0.2889	-0.2206	-0.5096	-0.5425	-0.3778	-0.0689
0	0.7264	0.8725	0.6811	0.2409	-0.196	-0.4249	-0.4465	-0.3155	-0.0662
0	0.5656	0.6924	0.5561	0.1908	-0.1639	-0.3376	-0.3511	-0.2509	-0.0586
0	0.2655	0.334	0.2809	0.0903	-0.0847	-0.1632	-0.1677	-0.1215	-0.0322
0	0.1283	0.1626	0.1384	0.0434	-0.0419	-0.0795	-0.0815	-0.0592	-0.0162
0	0	0	0	0	0	0	0	0	0
0	-0.23	-0.2893	-0.2433	-0.0782	0.0733	0.1413	0.1452	0.1052	0.0279

table 5.4.11: _3 in U_10_3 identify the velocity function used

5.5. The vibrational modes of a drum with varying tension using $\boldsymbol{f(r) = 1}$

To evaluate the vibrational mode of a circular drum using the model of chapter four, the researcher considered the boundary value problem of the talking drum having similar conditions as that of the drum with constant tension in the drum head as follows

The initial boundary condition $U(R,0) = 0$, and initial conditions $U(r,0) = 0$ *and* $U_t(r,t) = f(r)$ respectively. The radius taken for the drumhead is 3.50 inches and on taking the three initial velocity functions as $f(r) = 1$, $f(r) = r\left(2 - \frac{r}{2}\right)^2$ and

$f(r) = \left(2 - \frac{r}{2}\right)^2$.

The model of vibrating circular membrane with varying tension is a function of r and t

$$U_n(r,t) = \sum_{n=1}^{\infty} \frac{2t^{\frac{1}{2}}}{R^2 J_1^2(\alpha_n)} \left[\left\{\left(\frac{3aR}{\alpha_n}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right) \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr\right\} J_{\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR} t^{\frac{3}{2}}\right)\right] J_0\left(\frac{\alpha_n r}{R}\right)$$

………………………5.5.1

And so by letting t to take integral values from 0 to 9 whiles r takes on values r = 0 , 1, 1.5, 2, 2.25, 2.5, 3, 3.25, 3.5 and 4.

With $A_n = \frac{2t^{\frac{1}{2}}}{R^2 J_1^2(\alpha_n)} \left[\left\{\left(\frac{3aR}{\alpha_n}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right) \int_0^R r\, f(r)\, J_0\left(\frac{r\alpha_n}{R}\right) dr\right\}\right]$, a=1, and α_n = the first ten zeros of the Bessel functions of order zero. $J_x = J_0\left(\frac{\alpha_n r}{R}\right)$ and $J_y = J_{\frac{1}{3}}\left(\frac{2}{3}\frac{\alpha_n}{aR} t^{\frac{3}{2}}\right)$, Matrix Laboratory codes were applied to compute these expression in the following tables

Bessel function of order zero for drumhead with varying tension

J_x=									
1	1	1	1	1	1	1	1	1	1
0.8854	0.4684	-0.0346	-0.3585	-0.3667	-0.1229	0.167	0.2996	0.204	-0.0267
0.7516	0.0204	-0.3997	-0.1599	0.2431	0.2262	-0.1121	-0.2412	-0.002	0.2142
0.5808	-0.3078	-0.1954	0.2885	0.0332	-0.2468	0.0742	0.1817	-0.1415	-0.1022
0.4861	-0.3863	0.0146	0.2547	-0.2088	-0.0406	0.2079	-0.1336	-0.0733	0.1795
0.3877	-0.4003	0.1973	0.0631	-0.2206	0.1996	-0.0448	-0.1198	0.18	-0.106
0.1887	-0.2604	0.2766	-0.2492	0.1885	-0.1075	0.0206	0.0579	-0.1164	0.147
0.0921	-0.1356	0.1632	-0.18	0.1876	-0.1872	0.1795	-0.1654	0.1458	-0.1219
0	0	0	0	0	0	0	0	0	0
-0.1634	0.2256	-0.2396	0.2158	-0.1633	0.0931	-0.0179	-0.0501	0.1008	-0.1273

Table 5.5.1 values of Bessel function of order zero varying r

Bessel function of order one-third for drumhead with varying tension

j_y=									
0	0	0	0	0	0	0	0	0	
1.9755	2.1818	1.7635	0.9769	0.0834	-0.6647	-1.0818	-1.0931	-0.7477	-0.1947
2.0709	-0.0993	-1.0855	0.2989	0.762	-0.411	-0.5469	0.4785	0.3708	-0.5122
0.7763	-0.5524	0.4657	-0.4183	0.3881	-0.3671	0.3517	-0.3398	0.3303	-0.3225
-0.8651	0.5681	0.5066	-0.3297	-0.4274	0.2099	0.387	-0.1276	-0.3577	0.0635
-0.8363	-0.3677	-0.0903	0.1123	0.2536	0.3337	0.3535	0.319	0.2411	0.1352
0.5942	-0.0015	-0.2937	0.4001	-0.3473	0.1854	0.0154	-0.1826	0.2638	-0.2406
0.5215	0.4186	-0.2549	-0.3154	0.1509	0.2769	-0.0844	-0.2517	0.034	0.2299
-0.6914	-0.4333	-0.0554	0.2446	0.2918	0.1038	-0.1365	-0.2336	-0.1283	0.0712
0.039	-0.1642	0.253	-0.2878	0.2723	-0.2153	0.1306	-0.0343	-0.0566	0.1273

Table 5.5.2 gives values of Bessel function of order 1/3 varying t

The first associated coefficients for drumhead with varying tension

A_n_1=									
0.9556	-0.2068	0.0906	-0.0514	0.0333	-0.0235	0.0175	-0.0136	0.0109	-0.0089

Table 5.5.3 gives the coefficient of the normal mode with f(r) =1

Now on multiplying corresponding columns of tables 5.4.1, 5.4.2 and 5.4.3 we obtain matrices of the first ten partials of the vibrating membrane with varying tension and with these partials the first ten Normal modes were obtained and tabulated below

The 1st Normal mode

U_1_1=									
0	1.8878	1.9789	0.7418	-0.8267	-0.7991	0.5678	0.4983	-0.6607	0.0373
0	1.6715	1.7522	0.6568	-0.732	-0.7076	0.5027	0.4412	-0.585	0.033
0	1.4188	1.4873	0.5575	-0.6213	-0.6006	0.4267	0.3745	-0.4966	0.028
0	1.0964	1.1493	0.4308	-0.4801	-0.4641	0.3298	0.2894	-0.3837	0.0217
0	0.9176	0.9619	0.3606	-0.4018	-0.3884	0.276	0.2422	-0.3211	0.0181
0	0.732	0.7673	0.2876	-0.3205	-0.3099	0.2202	0.1932	-0.2562	0.0145
0	0.3562	0.3734	0.14	-0.156	-0.1508	0.1071	0.094	-0.1247	0.007
0	0.1738	0.1822	0.0683	-0.0761	-0.0736	0.0523	0.0459	-0.0608	0.0034
0	0	0	0	0	0	0	0	0	0
0	-0.3085	-0.3234	-0.1212	0.1351	0.1306	-0.0928	-0.0814	0.108	-0.0061

table 5.5.4 : _1 in U_1_1 identify the velocity function used

The 2nd Normal mode

U_2_1=									
0	1.4365	1.9995	0.8561	-0.9442	-0.7231	0.5681	0.4117	-0.5711	0.0712
0	1.4601	1.7618	0.7103	-0.787	-0.6719	0.5029	0.4006	-0.543	0.0489
0	1.4096	1.4877	0.5599	-0.6237	-0.5991	0.4268	0.3727	-0.4947	0.0287
0	1.2353	1.143	0.3957	-0.444	-0.4875	0.3297	0.3161	-0.4113	0.0112
0	1.0919	0.9539	0.3164	-0.3564	-0.4178	0.2759	0.2757	-0.3558	0.005
0	0.9126	0.7591	0.2419	-0.2735	-0.3403	0.22	0.2279	-0.2921	0.0009
0	0.4737	0.3681	0.1102	-0.1254	-0.1706	0.1071	0.1166	-0.148	-0.0018
0	0.235	0.1794	0.0528	-0.0602	-0.0839	0.0522	0.0576	-0.073	-0.0012
0	0	0	0	0	0	0	0	0	0
0	-0.4103	-0.3188	-0.0955	0.1086	0.1478	-0.0927	-0.101	0.1282	0.0016

table 5.5.4 : _1 in U_2_1 identify the velocity function used

The 3rd Normal mode

U_3_1=									
0	-0.2756	-0.0954	0.1698	-0.0773	0.0656	-0.0183	-0.1193	0.0918	0.0547
0	-0.2175	0.0136	0.0516	-0.0564	0.036	0.0008	-0.0394	0.0419	0.0152
0	-0.0794	0.0468	-0.0199	-0.0185	0.0058	0.0074	0.0113	0.0009	-0.0076
0	0.1046	0.0163	-0.046	0.0283	-0.0214	0.0035	0.0331	-0.028	-0.0145
0	0.1769	-0.0096	-0.0433	0.046	-0.0295	-0.0004	0.033	-0.0346	-0.0128
0	0.2153	-0.0311	-0.0348	0.055	-0.0325	-0.0038	0.0282	-0.0354	-0.0095
0	0.1661	-0.0374	-0.0144	0.0417	-0.0227	-0.0052	0.0135	-0.0227	-0.0031
0	0.0899	-0.0217	-0.0064	0.0225	-0.012	-0.0031	0.0064	-0.0118	-0.0012
0	0	0	0	0	0	0	0	0	0
0	-0.1439	0.0324	0.0125	-0.0361	0.0197	0.0045	-0.0117	0.0197	0.0027

table 5.5.4 : _1 in U_3_1 identify the velocity function used

The 4th Normal mode

U_4_1=									
0	-0.3258	-0.1107	0.1913	-0.0604	0.0598	-0.0389	-0.1031	0.0793	0.0695
0	-0.1995	0.0191	0.0439	-0.0625	0.0381	0.0082	-0.0452	0.0464	0.0099
0	-0.0714	0.0492	-0.0233	-0.0212	0.0067	0.0107	0.0087	0.003	-0.01
0	0.0901	0.0119	-0.0398	0.0332	-0.023	-0.0024	0.0377	-0.0316	-0.0102
0	0.1641	-0.0135	-0.0379	0.0503	-0.031	-0.0056	0.0371	-0.0378	-0.009
0	0.2121	-0.0321	-0.0334	0.056	-0.0329	-0.0051	0.0292	-0.0362	-0.0086
0	0.1786	-0.0336	-0.0197	0.0375	-0.0213	-0.0001	0.0095	-0.0196	-0.0068
0	0.0989	-0.0189	-0.0103	0.0194	-0.011	0.0006	0.0035	-0.0095	-0.0039
0	0	0	0	0	0	0	0	0	0
0	-0.1547	0.0291	0.0171	-0.0325	0.0184	0.0001	-0.0082	0.017	0.0059

table 5.5.5: _1 in U_4_1 identify the velocity function used

The 5th Normal mode

U_5_1=									
0	-0.323	-0.0854	0.2042	-0.0746	0.0682	-0.0504	-0.098	0.089	0.0785
0	-0.2005	0.0098	0.0391	-0.0573	0.035	0.0124	-0.047	0.0428	0.0066
0	-0.0707	0.0554	-0.0202	-0.0246	0.0087	0.0079	0.0099	0.0053	-0.0077
0	0.0902	0.0127	-0.0394	0.0327	-0.0227	-0.0028	0.0379	-0.0313	-0.0099
0	0.1635	-0.0188	-0.0406	0.0533	-0.0328	-0.0032	0.0361	-0.0398	-0.0109
0	0.2115	-0.0377	-0.0363	0.0592	-0.0347	-0.0025	0.0281	-0.0384	-0.0106
0	0.1791	-0.0288	-0.0173	0.0348	-0.0197	-0.0023	0.0104	-0.0178	-0.0051
0	0.0994	-0.0142	-0.0079	0.0168	-0.0094	-0.0016	0.0044	-0.0077	-0.0022
0	0	0	0	0	0	0	0	0	0
0	-0.1552	0.025	0.015	-0.0302	0.0171	0.002	-0.009	0.0154	0.0044

table 5.5.6: _1 in U_5_1 identify the velocity function used

The 6th Normal mode

U_6_1=									
0	-0.3074	-0.0757	0.2129	-0.0796	0.0604	-0.0548	-0.1046	0.0866	0.0836
0	-0.2024	0.0086	0.0381	-0.0567	0.0359	0.0129	-0.0463	0.0431	0.0059
0	-0.0672	0.0576	-0.0182	-0.0257	0.007	0.0069	0.0085	0.0048	-0.0066
0	0.0864	0.0104	-0.0415	0.0339	-0.0208	-0.0017	0.0395	-0.0307	-0.0112
0	0.1629	-0.0192	-0.0409	0.0535	-0.0325	-0.003	0.0363	-0.0397	-0.0111
0	0.2146	-0.0357	-0.0345	0.0582	-0.0363	-0.0034	0.0268	-0.0389	-0.0096
0	0.1775	-0.0298	-0.0182	0.0353	-0.0188	-0.0018	0.0111	-0.0175	-0.0056
0	0.0965	-0.016	-0.0095	0.0177	-0.0079	-0.0007	0.0056	-0.0073	-0.0031
0	0	0	0	0	0	0	0	0	0
0	-0.1537	0.0259	0.0158	-0.0306	0.0163	0.0016	-0.0096	0.0152	0.0049

table 5.5.7 _1 in U_6_1 identify the velocity function used

The 7th Normal mode

U_7_1=									
0	-0.3264	-0.0853	0.219	-0.0728	0.0666	-0.0545	-0.1061	0.0842	0.0859
0	-0.2056	0.007	0.0391	-0.0555	0.037	0.013	-0.0465	0.0427	0.0063
0	-0.0651	0.0586	-0.0189	-0.0265	0.0063	0.0069	0.0086	0.005	-0.0069
0	0.085	0.0097	-0.0411	0.0345	-0.0204	-0.0017	0.0394	-0.0309	-0.011
0	0.159	-0.0212	-0.0396	0.0549	-0.0312	-0.003	0.036	-0.0402	-0.0107
0	0.2155	-0.0353	-0.0348	0.0579	-0.0366	-0.0034	0.0269	-0.0388	-0.0097
0	0.1771	-0.03	-0.0181	0.0355	-0.0187	-0.0018	0.0111	-0.0176	-0.0056
0	0.0931	-0.0177	-0.0084	0.0189	-0.0068	-0.0007	0.0054	-0.0077	-0.0027
0	0	0	0	0	0	0	0	0	0
0	-0.1534	0.026	0.0157	-0.0307	0.0162	0.0016	-0.0096	0.0152	0.0048

table 5.5.8: _1 in U_7_1 identify the velocity function used

The 8th Normal mode

U_8_1=									
0	-0.3115	-0.0918	0.2236	-0.0711	0.0623	-0.052	-0.1027	0.0874	0.0863
0	-0.2011	0.0051	0.0405	-0.055	0.0357	0.0137	-0.0455	0.0437	0.0065
0	-0.0687	0.0602	-0.02	-0.0269	0.0073	0.0063	0.0078	0.0043	-0.007
0	0.0877	0.0085	-0.0402	0.0348	-0.0211	-0.0012	0.04	-0.0303	-0.0109
0	0.157	-0.0203	-0.0402	0.0546	-0.0306	-0.0033	0.0356	-0.0406	-0.0107
0	0.2137	-0.0345	-0.0354	0.0577	-0.0361	-0.0037	0.0265	-0.0391	-0.0097
0	0.1779	-0.0304	-0.0178	0.0356	-0.019	-0.0017	0.0113	-0.0174	-0.0056
0	0.0906	-0.0166	-0.0091	0.0186	-0.0061	-0.0011	0.0048	-0.0082	-0.0028
0	0	0	0	0	0	0	0	0	0
0	-0.1541	0.0264	0.0154	-0.0308	0.0164	0.0014	-0.0098	0.015	0.0048

table 5.5.9 : _1 in U_8_1 identify the velocity function used

The 9th Normal mode

U_9_1=									
0	-0.3196	-0.0878	0.2272	-0.0749	0.0649	-0.0492	-0.1023	0.086	0.0857
0	-0.2028	0.0059	0.0412	-0.0558	0.0362	0.0143	-0.0454	0.0434	0.0063
0	-0.0686	0.0602	-0.02	-0.0269	0.0073	0.0063	0.0078	0.0043	-0.007
0	0.0888	0.0079	-0.0407	0.0353	-0.0215	-0.0016	0.04	-0.0301	-0.0108
0	0.1576	-0.0206	-0.0405	0.0549	-0.0308	-0.0035	0.0355	-0.0405	-0.0107
0	0.2122	-0.0338	-0.0347	0.057	-0.0356	-0.0032	0.0265	-0.0394	-0.0098
0	0.1789	-0.0309	-0.0182	0.036	-0.0193	-0.002	0.0112	-0.0172	-0.0055
0	0.0895	-0.016	-0.0086	0.0181	-0.0057	-0.0007	0.0049	-0.0084	-0.0029
0	0	0	0	0	0	0	0	0	0
0	-0.155	0.0268	0.0158	-0.0312	0.0167	0.0017	-0.0097	0.0149	0.0047

table 5.5.10 : _1 in U_9_1 identify the velocity function used

The 10th Normal mode

U_10_1=									
0	-0.3179	-0.0832	0.2301	-0.0755	0.0637	-0.047	-0.1043	0.0853	0.0846
0	-0.2028	0.0058	0.0412	-0.0558	0.0362	0.0143	-0.0454	0.0434	0.0064
0	-0.0683	0.0612	-0.0194	-0.027	0.007	0.0068	0.0074	0.0041	-0.0072
0	0.0886	0.0074	-0.041	0.0354	-0.0214	-0.0018	0.0402	-0.0301	-0.0107
0	0.1579	-0.0198	-0.04	0.0548	-0.031	-0.0031	0.0352	-0.0407	-0.0109
0	0.2121	-0.0343	-0.035	0.057	-0.0355	-0.0034	0.0268	-0.0393	-0.0097
0	0.1791	-0.0302	-0.0178	0.0359	-0.0194	-0.0017	0.0109	-0.0173	-0.0056
0	0.0892	-0.0166	-0.009	0.0181	-0.0056	-0.0009	0.0051	-0.0083	-0.0027
0	0	0	0	0	0	0	0	0	0
0	-0.1552	0.0262	0.0154	-0.0311	0.0168	0.0015	-0.0095	0.015	0.0049

table 5.5.11: _1 in U_10_1 identify the velocity function used

5.6. The normal mode of a vibrating drum head with varying tension using

$$f(r) = r\left(2 - \frac{r}{2}\right)^2$$

Now on using the second initial velocity function considered i.e $f(r) = r\left(2 - \frac{r}{2}\right)^2$ the new coefficient is then obtained

The second associated coefficients for drumhead with varying tension

A_n_2=									
1.7166	-0.2002	-0.0502	-0.0281	-0.0043	-0.0095	0.0003	-0.0047	0.0009	-0.0027

Table 5.6.1 gives the coefficient of the normal mode with $f(r) = r\left(2 - \frac{r}{2}\right)^2$

And on multiplying the respective columns of tables 5.4.1, 5.4.2 and 5.5.1 the matrices of the first ten Normal modes were also generated and tabulated below.

The 1st Normal mode

U_1_2=									
0	3.3912	3.555	1.3326	-1.4851	-1.4356	1.02	0.8952	-1.1869	0.067
0	3.0026	3.1476	1.1799	-1.3149	-1.2711	0.9031	0.7926	-1.0509	0.0593
0	2.5487	2.6718	1.0015	-1.1161	-1.0789	0.7666	0.6728	-0.892	0.0503
0	1.9696	2.0647	0.774	-0.8625	-0.8338	0.5924	0.5199	-0.6893	0.0389
0	1.6483	1.7279	0.6477	-0.7218	-0.6978	0.4958	0.4351	-0.5769	0.0326
0	1.3149	1.3784	0.5167	-0.5758	-0.5566	0.3955	0.3471	-0.4602	0.026
0	0.6399	0.6708	0.2515	-0.2802	-0.2709	0.1925	0.1689	-0.224	0.0126
0	0.3123	0.3273	0.1227	-0.1367	-0.1322	0.0939	0.0824	-0.1093	0.0062
0	0	0	0	0	0	0	0	0	0
0	-0.5542	-0.581	-0.2178	0.2427	0.2346	-0.1667	-0.1463	0.194	-0.0109

Table 5.6.2: _2 in U_1_2 identify the velocity function used

The 2nd Normal mode

U_2_2=									
0	2.9544	3.5749	1.4432	-1.5988	-1.362	1.0203	0.8113	-1.1001	0.0999
0	2.798	3.1569	1.2317	-1.3682	-1.2366	0.9033	0.7533	-1.0102	0.0747
0	2.5398	2.6722	1.0038	-1.1185	-1.0774	0.7666	0.6711	-0.8902	0.051
0	2.1041	2.0586	0.7399	-0.8275	-0.8564	0.5923	0.5457	-0.716	0.0288
0	1.8171	1.7202	0.605	-0.6779	-0.7262	0.4957	0.4675	-0.6104	0.0199
0	1.4898	1.3704	0.4724	-0.5303	-0.5861	0.3954	0.3806	-0.4949	0.0128
0	0.7537	0.6656	0.2227	-0.2506	-0.2901	0.1924	0.1907	-0.2466	0.0041
0	0.3715	0.3246	0.1077	-0.1213	-0.1422	0.0939	0.0938	-0.1211	0.0017
0	0	0	0	0	0	0	0	0	0
0	-0.6528	-0.5765	-0.1928	0.2171	0.2512	-0.1666	-0.1652	0.2135	-0.0035

table 5.6.3: _2 in U_2_2 identify the velocity function used

The 3rd Normal mode

U_3_2=									
0	2.8659	3.6293	1.4198	-1.6242	-1.3574	1.035	0.8241	-1.0973	0.0872
0	2.8011	3.155	1.2325	-1.3673	-1.2368	0.9028	0.7529	-1.0103	0.0751
0	2.5752	2.6504	1.0131	-1.1083	-1.0792	0.7607	0.6659	-0.8914	0.0561
0	2.1213	2.0479	0.7445	-0.8225	-0.8573	0.5894	0.5432	-0.7166	0.0313
0	1.8158	1.721	0.6047	-0.6783	-0.7261	0.4959	0.4677	-0.6104	0.0197
0	1.4723	1.3812	0.4678	-0.5353	-0.5852	0.3983	0.3832	-0.4944	0.0103
0	0.7292	0.6807	0.2162	-0.2576	-0.2888	0.1965	0.1943	-0.2458	0.0006
0	0.3371	0.3335	0.1039	-0.1255	-0.1414	0.0963	0.0959	-0.1206	-0.0004
0	0	0	0	0	0	0	0	0	0
0	-0.6316	-0.5896	-0.1872	0.2231	0.2501	-0.1702	-0.1683	0.2129	-0.0005

table 5.6.4: _2 in U_3_2 identify the velocity function used

The 4th Normal mode

U_4_2=									
0	2.8385	3.6209	1.4316	-1.615	-1.3606	1.0238	0.833	-1.1042	0.0952
0	2.8109	3.1581	1.2283	-1.3706	-1.2356	0.9068	0.7497	-1.0079	0.0722
0	2.5796	2.6518	1.0113	-1.1098	-1.0787	0.7625	0.6645	-0.8903	0.0548
0	2.1134	2.0455	0.7479	-0.8199	-0.8582	0.5862	0.5458	-0.7186	0.0336
0	1.8088	1.7189	0.6076	-0.6759	-0.7269	0.493	0.4699	-0.6121	0.0217
0	1.4706	1.3807	0.4686	-0.5347	-0.5854	0.3976	0.3837	-0.4948	0.0108
0	0.736	0.6828	0.2133	-0.26	-0.288	0.1993	0.1921	-0.2441	-0.0014
0	0.362	0.335	0.1018	-0.1271	-0.1409	0.0983	0.0943	-0.1194	-0.0018
0	0	0	0	0	0	0	0	0	0
0	-0.6375	-0.5914	-0.1847	0.2251	0.2495	-0.1726	-0.1664	0.2114	0.0013

table 5.6.5: _2 in U_4_2 identify the velocity function used

The 5th Normal mode

U_5_2=									
0	2.8381	3.6176	1.4299	-1.6131	-1.3617	1.0253	0.8323	-1.1055	0.0941
0	2.8111	3.1593	1.2289	-1.3713	-1.2352	0.9062	0.7499	-1.0074	0.0727
0	2.5795	2.651	1.0109	-1.1093	-1.079	0.7629	0.6644	-0.8906	0.0545
0	2.1134	2.0454	0.7478	-0.8198	-0.8583	0.5863	0.5457	-0.7186	0.0336
0	1.8089	1.7196	0.608	-0.6763	-0.7267	0.4927	0.47	-0.6118	0.022
0	1.4706	1.142	0.4689	-0.5351	-0.5852	0.3973	0.3839	-0.4945	0.0111
0	0.736	0.6822	0.213	-0.2596	-0.2882	0.1996	0.192	-0.2443	-0.0017
0	0.3619	0.3334	0.1015	-0.1268	-0.1411	0.0986	0.0942	-0.1196	-0.002
0	0	0	0	0	0	0	0	0	0
0	-0.6374	-0.5908	-0.1844	0.2248	0.2496	-0.1728	-0.1663	0.2116	0.0014

table 5.6.6 _2 in U_5_2 identify the velocity function used

The 6th Normal mode

U_6_2=									
0	2.8445	3.6216	1.4334	-1.6151	-1.3648	1.0236	0.8297	-1.1065	0.0961
0	2.8103	3.1588	1.2285	-1.3711	-1.2348	0.9065	0.7503	-1.0073	0.0724
0	2.5809	2.6519	1.0117	-1.1098	-1.0797	0.7625	0.6638	-0.8908	0.055
0	2.1119	2.0444	0.747	-0.8193	-0.8575	0.5867	0.5464	-0.7184	0.0331
0	1.8086	1.7194	0.6079	-0.6762	-0.7266	0.4928	0.4702	-0.6118	0.0219
0	1.4719	1.3822	0.4696	-0.5355	-0.5858	0.3969	0.3833	-0.4947	0.0115
0	0.7353	0.6818	0.2126	-0.2594	-0.2879	0.1997	0.1922	-0.2442	-0.0019
0	0.3607	0.3337	0.1008	-0.1264	-0.1405	0.0989	0.0947	-0.1194	-0.0024
0	0	0	0	0	0	0	0	0	0
0	-0.6369	-0.5905	-0.1841	0.2247	0.2494	-0.173	-0.1665	0.2115	0.0016

table 5.6.7 : _2 in U_6_2 identify the velocity function used

The 7thNormal mode

U_7_2=									
0	2.8442	3.6214	1.4335	-1.615	-1.3648	1.0236	0.8297	-1.1065	0.0961
0	2.8102	3.1588	1.2285	-1.371	-1.2348	0.9065	0.7503	-1.0073	0.0724
0	2.5809	2.6519	1.0116	-1.1098	-1.0797	0.7625	0.6638	-0.8908	0.055
0	2.1128	2.0444	0.747	-0.8193	-0.8575	0.5867	0.5464	-0.7184	0.0331
0	1.8085	1.7194	0.6079	-0.6762	-0.7266	0.4928	0.4702	-0.6118	0.0219
0	1.4719	1.3822	0.4696	-0.5355	-0.5858	0.3969	0.3833	-0.4947	0.0115
0	0.7353	0.6818	0.2126	-0.2594	-0.2879	0.1997	0.1922	-0.2442	-0.0019
0	0.3607	0.3337	0.1008	-0.1264	-0.1405	0.0989	0.0947	-0.1194	-0.0024
0	0	0	0	0	0	0	0	0	0
0	-0.6368	-0.5905	-0.1841	0.2247	0.2493	-0.173	-0.1665	0.2115	0.0016

table 5.6.8 : _2 in U_7_2 identify the velocity function used

The 8th Normal mode

U_8_2=									
0	2.8493	3.6192	1.435	-1.6144	-1.3662	1.0244	0.8308	-1.1054	0.0963
0	2.8118	3.1581	1.229	-1.3709	-1.2353	0.9067	0.7506	-1.007	0.0725
0	2.5797	2.6524	1.0113	-1.1099	-1.0794	0.7623	0.6635	-0.891	0.0549
0	2.1128	2.044	0.7473	-0.8192	-0.8577	0.5868	0.5466	-0.7182	0.0331
0	1.8079	1.7197	0.6077	-0.6763	-0.7264	0.4927	0.47	-0.612	0.0219
0	1.4713	1.3824	0.4694	-0.5356	-0.5856	0.3968	0.3832	-0.4949	0.0115
0	0.7356	0.6816	0.2127	-0.2594	-0.288	0.1998	0.1923	-0.2441	-0.0019
0	0.3598	0.334	0.1006	-0.1265	-0.1402	0.0988	0.0945	-0.1196	-0.0024
0	0	0	0	0	0	0	0	0	0
0	-0.6371	-0.5904	-0.1842	0.2246	0.2494	-0.173	-0.1666	0.2115	0.0016

table 5.6.9: _2 in U_8_2 identify the velocity function used

The 9thNormal mode

U_9_2=									
0	2.8486	3.6195	1.4353	-1.6148	-1.366	1.0246	0.8309	-1.1055	0.0962
0	2.8116	3.1582	1.229	-1.3709	-1.2352	0.9068	0.7506	-1.007	0.0725
0	2.5797	2.6524	1.0113	-1.1099	-1.0794	0.7623	0.6635	-0.891	0.0549
0	2.1129	2.044	0.7472	-0.8192	-0.8578	0.5868	0.5466	-0.7181	0.0331
0	1.8079	1.7197	0.6076	-0.6762	-0.7264	0.4926	0.47	-0.6119	0.0219
0	1.4712	1.3825	0.4695	-0.5357	-0.5856	0.3968	0.3832	-0.4949	0.0115
0	0.7356	0.6816	0.2126	-0.2593	-0.288	0.1998	0.1923	-0.2441	-0.0019
0	0.3597	0.3341	0.1006	-0.1265	-0.1402	0.0988	0.0945	-0.1196	-0.0024
0	0	0	0	0	0	0	0	0	0
0	-0.6372	-0.5903	-0.1842	0.2246	0.2494	-0.173	-0.1666	0.2115	0.0016

table 5.6.10: _2 in U_9_2 identify the velocity function used

The 10th Normal mode

U_10_2=									
0	2.8491	3.6209	1.4362	-1.6149	-1.3664	1.0253	0.8302	-1.1057	0.0959
0	2.8116	3.1581	1.229	-1.3709	-1.2352	0.9067	0.7506	-1.007	0.0725
0	2.5798	2.6527	1.0114	-1.11	-1.0795	0.7624	0.6634	-0.8911	0.0549
0	2.1128	2.0438	0.7471	-0.8191	-0.8577	0.5867	0.5467	-0.7181	0.0331
0	1.808	1.7199	0.6078	-0.6763	-0.7265	0.4928	0.4699	-0.612	0.0218
0	1.4711	1.3824	0.4694	-0.5357	-0.5855	0.3968	0.3833	-0.4949	0.0115
0	0.7357	0.6818	0.2128	-0.2593	-0.2881	0.1999	0.1922	-0.2442	-0.0019
0	0.3597	0.3339	0.1005	-0.1265	-0.1401	0.0987	0.0945	-0.1196	-0.0024
0	0	0	0	0	0	0	0	0	0
0	-0.6372	-0.5905	-0.1843	0.2246	0.2495	-0.1731	-0.1665	0.2115	0.0017

table 5.6.11 : _2 in U_10_2 identify the velocity function used

5.7. The normal mode of a vibrating drum head with varying tension using

$$\boldsymbol{f(r) = \left(2 - \frac{r}{2}\right)^2}$$

Finally by using the third initial velocity function considered i.e. $f(r) = \left(2 - \frac{r}{2}\right)^2$ the new coefficient for the talking is computed

The third associated coefficients for drumhead with varying tension

A_n_3=									
1.3114	0.1636	0.0304	0.0099	0.0064	0.0016	0.0025	0.0003	0.0013	0

table 5.7.1 gives the coefficient of the normal mode with $f(r) = \left(2 - \frac{r}{2}\right)^2$

And on multiplying the respective columns of tables 5.5.1, 5.6.2 and 5.7.1 the matrices of the first ten Normal modes were again generated and tabulated below as follows

The 1st Normal mode

U_1_3=									
0	2.5906	2.7157	1.018	-1.1345	-1.0967	0.7792	0.6838	-0.9067	0.0512
0	2.2938	2.4045	0.9014	-1.0045	-0.971	0.6899	0.6055	-0.8028	0.0453
0	1.947	2.0411	0.7651	-0.8526	-0.8242	0.5856	0.5139	-0.6814	0.0385
0	1.5046	1.5773	0.5913	-0.6589	-0.6369	0.4526	0.3972	-0.5266	0.0297
0	1.2592	1.32	0.4948	-0.5514	-0.533	0.3787	0.3324	-0.4407	0.0249
0	1.0045	1.053	0.3947	-0.4399	-0.4252	0.3021	0.2651	-0.3516	0.0198
0	0.4888	0.5125	0.1921	-0.2141	-0.2069	0.147	0.129	-0.1711	0.0097
0	0.2385	0.2501	0.0937	-0.1045	-0.101	0.0717	0.063	-0.0835	0.0047
0	0	0	0	0	0	0	0	0	0
0	-0.4234	-0.4438	-0.1664	0.1854	0.1792	-0.1274	-0.1118	0.1482	-0.084

table 5.7.2 : _3 in U_1_3 identify the velocity function used

The 2nd Normal mode

U_2_3=									
0	2.9476	2.6995	0.9276	-1.0415	-1.1568	0.779	0.7523	-0.9776	0.0243
0	2.461	2.3969	0.859	-0.961	-0.9992	0.6898	0.6376	-0.836	0.0327
0	1.9543	2.0407	0.7633	-0.8507	-0.8255	0.5856	0.5153	-0.6829	0.0379
0	1.3947	1.5823	0.6191	-0.875	-0.6184	0.4526	0.3761	-0.5048	0.038
0	1.1213	1.3263	0.5297	-0.5873	-0.5098	0.3788	0.3059	-0.4133	0.0352
0	0.8616	1.0595	0.4309	-0.4771	-0.4011	0.3022	0.2377	-0.3232	0.0306
0	0.3959	0.5167	0.2156	-0.2883	-0.1913	0.1471	0.1112	-0.1526	0.0167
0	0.1901	0.2523	0.106	-0.1171	-0.0928	0.0718	0.0537	-0.0739	0.0084
0	0	0	0	0	0	0	0	0	0
0	-0.3429	-0.4475	-0.1868	0.2064	0.1657	-0.1274	-0.0963	0.1322	-0.0144

table 5.7.3: _3 in U_2_3 identify the velocity function used

The 3rd Normal mode

U_3_3=									
0	3.0012	2.6665	0.9418	-1.0261	-1.1596	0.77	0.7446	-0.9793	0.032
0	2.4591	2.3981	0.8585	-0.9615	-0.9991	0.6901	0.6378	-0.8359	0.0325
0	1.9329	2.0539	0.7576	-0.8569	-0.8244	0.5892	0.5184	-0.6822	0.0348
0	1.3842	1.5887	0.6163	-0.6905	-0.6179	0.4544	0.3776	-0.5044	0.0365
0	1.1221	1.3258	0.5299	-0.5871	-0.5098	0.3787	0.3058	-0.4133	0.0354
0	0.8722	1.053	0.4337	-0.4741	-0.4017	0.3005	0.2362	-0.3235	0.0321
0	0.4107	0.5076	0.2196	-0.234	-0.192	0.1446	0.1091	-0.1531	0.0188
0	0.1989	0.2469	0.1083	-0.1146	-0.0933	0.0703	0.0524	-0.0741	0.0096
0	0	0	0	0	0	0	0	0	0
0	-0.3557	-0.4396	-0.1902	0.2027	0.1663	-0.1253	-0.0944	0.1326	-0.0163

table 5.7.4: _3 in U_3_3 identify the velocity function used

The 4th Normal mode

U_4_3=									
0	3.0109	2.6694	0.9377	-1.0294	-1.1585	0.774	0.7415	-0.9769	0.0292
0	2.4557	2.397	0.86	-0.9603	-1	0.6887	0.6389	-0.8368	0.0335
0	1.9314	2.0534	0.7583	-0.564	-0.245	0.5886	0.5189	-0.6826	0.0353
0	1.387	1.5896	0.6151	-0.6915	-0.6176	0.4555	0.3767	-0.5037	0.0357
0	1.1245	1.3265	0.5289	-0.5879	-0.5096	0.3797	0.305	-0.4127	0.0346
0	0.8728	1.0532	0.4334	-0.4743	-0.4016	0.3007	0.236	-0.3234	0.0319
0	0.4083	0.5068	0.2206	-0.2332	-0.1923	0.1436	0.1098	-0.1537	0.0195
0	0.1971	0.2463	0.109	-0.114	-0.0935	0.0696	0.053	-0.0746	0.0101
0	0	0	0	0	0	0	0	0	0
0	-0.3536	-0.439	-0.1911	0.202	0.1666	-0.1244	-0.0951	0.1331	-0.0169

table 5.7.5: _3 in U_4_3 identify the velocity function used

The 5th Normal mode

U_5_3=									
0	3.0114	2.6743	0.9401	-1.0321	-1.1569	0.7718	0.7424	-0.975	0.0309
0	2.4555	2.3952	0.8591	-0.9593	-1.0001	0.6895	0.6386	-0.8375	0.0328
0	1.9315	2.0546	0.7589	-0.857	-0.8241	0.5192	0.5192	-0.6821	0.0357
0	1.387	1.5897	0.6152	-0.6916	-0.6175	0.4554	0.3767	-0.5037	0.0357
0	1.1244	1.3255	0.5284	-0.5874	-0.5099	0.3802	0.3048	-0.4131	0.0343
0	0.8727	1.0521	0.4329	-0.4737	-0.402	0.3012	0.2358	-0.3238	0.0316
0	0.4084	0.5077	0.221	-0.2337	-0.192	0.1432	0.11	-0.1533	0.0198
0	0.1972	0.2473	0.1095	-0.1145	-0.0932	0.0692	0.0532	-0.0742	0.0104
0	0	0	0	0	0	0	0	0	0
0	-0.3537	-0.4398	-0.1915	0.2024	0.1663	-0.1241	-0.0953	0.1328	-0.0172

table 5.7.6: _3 in U_5_3 identify the velocity function used

The 6th Normal mode

U_6_3=									
0	3.0104	2.6736	0.9395	-1.0318	-1.1563	0.7721	0.7429	-0.9748	0.0306
0	2.4556	2.3953	0.8592	-0.9594	-1.0002	0.6895	0.6385	-0.8375	0.0329
0	1.9312	2.0545	0.7587	-0.857	-0.824	0.5881	0.5193	-0.6821	0.0356
0	1.3873	1.5899	0.6153	-0.6916	-0.6176	0.4554	0.3766	-0.5037	0.0358
0	1.1244	1.3255	0.5284	-0.5874	-0.5099	0.3802	0.3084	-0.4131	0.0343
0	0.8725	1.052	0.4328	-0.4736	-0.4019	0.3013	0.2359	-0.3237	0.0315
0	0.4085	0.5078	0.2211	-0.2338	-0.1921	0.1432	0.11	-0.1534	0.0199
0	0.1974	0.2474	0.1096	-0.1145	-0.0933	0.0691	0.0531	-0.0743	0.0105
0	0	0	0	0	0	0	0	0	0
0	-0.3538	-0.4398	-0.1915	0.2025	0.1663	-0.124	-0.0952	0.1328	-0.0172

table 5.7.7: _3 in U_6_3 identify the velocity function used

The 7th Normal mode

U_7_3=									
0	3.0077	2.6723	0.9404	-1.0308	-1.1555	0.7721	0.7427	-0.9752	0.0309
0	2.4552	2.3951	0.8593	-0.9592	-1	0.6895	0.6385	-0.8376	0.0329
0	1.9315	2.0546	0.7586	-0.8571	-0.8241	0.5881	0.5193	-0.6821	0.0356
0	1.3871	1.5898	0.6154	-0.6916	-0.6176	0..4554	0.3766	-0.5037	0.0358
0	1.1239	1.3253	0.5286	-0.5872	-0.5097	0.3802	0.3047	-0.4132	0.0344
0	0.8726	1.052	0.4327	-0.4736	-0.4019	0.3013	0.2359	-0.3237	0.0315
0	0.4085	0.5078	0.2211	-0.2337	-0.192	0.1432	0.111	-0.1534	0.0199
0	0.197	0.2471	0.1098	-0.1144	-0.0931	0.0692	0.053	-0.0743	0.0106
0	0	0	0	0	0	0	0	0	0
0	-0.3538	-0.4398	-0.1915	0.2024	0.1663	-0.124	-0.0952	0.1328	-0.172

table 5.7.8: _3 in U_7_3 identify the velocity function used

The 8th Normal mode

U8_3=									
0	3.0074	2.6724	0.9403	-1.0309	-1.1554	0.7721	0.7426	-0.9752	0.0309
0	2.4551	2.3951	0.8593	-0.9592	-1	0.6895	0.6385	-0.8376	0.0329
0	1.9316	2.0546	0.7586	-0.8571	-0.8241	0.5881	0.5193	-0.6821	0.0356
0	1.387	1.5898	0.6154	-0.6916	-0.6176	0.4554	0.3766	-0.5038	0.0358
0	1.1239	1.3253	0.5286	-0.5872	-0.5097	0.3802	0.048	-0.4123	0.0344
0	0.8726	1.052	0.4327	-0.4736	-0.4019	0.3013	0.2359	-0.3237	0.0315
0	0.4084	0.5078	0.2211	-0.2337	-0.192	0.1432	0.11	-0.1534	0.0199
0	0.197	0.2471	0.1098	-0.1144	-0.0931	0.0692	0.053	-0.0743	0.0106
0	0	0	0	0	0	0	0	0	0
0	-0.3538	-0.4398	-0.1915	0.2025	0.1663	-0.124	-0.0952	0.1328	-0.0172

table 5.7.9: _3 in U_8_3 identify the velocity function used

The 9th Normal mode

U_9_3=									
0	3.0065	2.6729	0.9408	-1.0313	-1.1551	0.7724	0.7426	-0.9754	0.0308
0	2.4549	2.3952	0.8594	-0.9593	-0.9999	0.6895	0.6385	-0.8376	0.0329
0	1.9316	2.0546	0.7586	-0.8571	-0.8241	0.5881	0.5193	-0.6821	0.0356
0	1.3872	1.5898	0.6153	-0.6915	-0.6176	0.4553	0.3766	-0.5037	0.0358
0	1.124	1.3252	0.5285	-0.5871	-0.5098	0.3801	0.3048	-0.4132	0.0344
0	0.8724	1.0521	0.4328	-0.4737	-0.4019	0.3013	0.2359	-0.3237	0.0315
0	0.4086	0.5077	0.2211	-0.2337	-0.1921	0.1432	0.1099	-0.1534	0.0199
0	0.1969	0.2472	0.1099	-0.1144	-0.0931	0.0692	0.0531	-0.0743	0.0106
0	0	0	0	0	0	0	0	0	0
0	-0.0354	-0.4398	-0.1915	0.2024	0.1664	-0.124	-0.0952	0.1328	-0.0172

table 5.7.10: _3 in U_9_3 identify the velocity function used

The 10th Normal mode

U_10_3=									
0	3.0065	2.6729	0.9408	-1.0313	-1.1551	0.7724	0.7426	-0.9754	0.0308
0	2.4549	2.3952	0.8594	-0.9593	-0.9999	0.6895	0.6385	-0.8376	0.0329
0	1.9316	2.0546	0.7586	-0.8571	-0.8241	0.5881	0.5193	-0.6821	0.0356
0	1.3872	1.5898	0.6153	-0.6915	-0.6176	0.4553	0.3766	-0.5037	0.0358
0	1.124	1.3252	0.5285	-0.5871	-0.5098	0.3801	0.3048	-0.4132	0.0344
0	0.8724	1.0521	0.4328	-0.4737	-0.4019	0.3013	0.2359	-0.3237	0.0315
0	0.4086	0.5077	0.2211	-0.2337	-0.1921	0.1432	0.1099	-0.1534	0.0199
0	0.1969	0.2472	0.1099	-0.1144	-0.0931	0.0692	0.0531	-0.0743	0.0106
0	0	0	0	0	0	0	0	0	0
0	-0.0354	-0.4398	-0.1915	0.2024	0.1664	-0.124	-0.0952	0.1328	-0.0172

table 5.7.11: _3 in U_10_3 identify the velocity function used

5.8 Description of the Normal modes of the two models

The first model consists of Bessel functions of order zero and Bessel functions of order half which determines the sound nature of the drum. The second model also comprise Bessel functions of order zero and Bessel functions of order one-third which makes the drum sound the way they do. Clearly the Bessel functions of order zero appear in each of the two models and have frequency $\frac{\lambda_n}{2\pi}$ cycles per unit time; where λ_n are the Eigen values or characteristic values of our problem. Their corresponding Eigen functions are the solutions of the wave equation satisfying the boundary conditions for the two models.

The models are also referred to as the Normal modes. The zeros of the Bessel function of order zero are not evenly spaced and so the forms of the normal modes can easily be obtained from its graph.

Graph of Bessel function of order zero

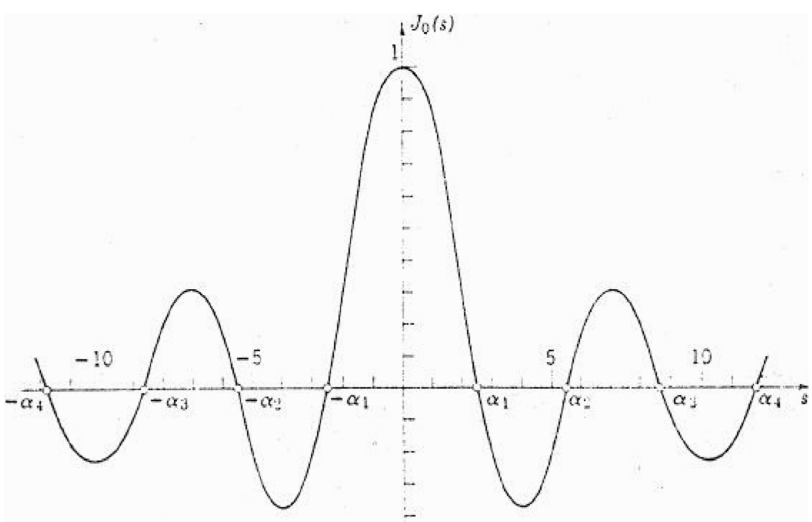


Fig 5.8.1 graph of $J_0(x)$: for showing the normal modes of a drumhead

For both models the form of the normal modes is explained as follows;.

When a drummer plays the drums under consideration, for the first normal mode or for n=1, all the points of the membrane move upward (or downward) at the same time. Here there is no concentric circle or nodal line. When the drummer operates at n=2 or the second normal mode the situation is as follows. The function $W_2(r) = J_0\left(\frac{\alpha_2}{R}r\right)$ is zero for $\frac{\alpha_2}{R}r = \alpha_1$ or $r = \frac{\alpha_{1R}}{\alpha_2}$. The circle $r = \frac{\alpha_{1R}}{\alpha_2}$ is therefore a nodal line and when at some instant the central part of the membrane moves upward, the outer part $r > \frac{\alpha_{1R}}{\alpha_2}$ moves downward and vice versa. Here one nodal line or concentric circle is introduced in the drumhead. In general the nth Normal mode has (n-1) nodal lines which are concentric circles. These are illustrated below in (fig. 5.8.2).

The Normal modes of a circular membrane with vibrations independent of the Angle

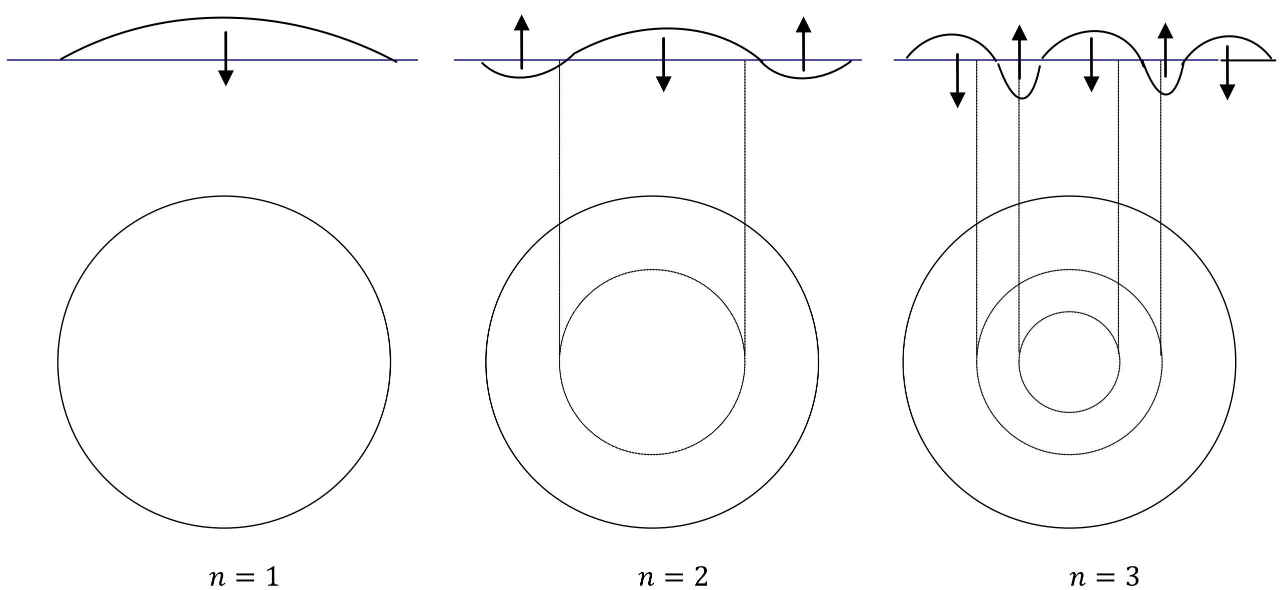


Fig 5.8.2 showing the vibrational modes of a circular drumhead

CHAPTER SIX

ANALYSIS OF RESULTS, CONCLUSION AND RECOMMENDATIONS

The chapter discusses the results of findings from the chapter five under the following subtopics Findings and Discussions, Analysis of Results, Conclusion, Recommendations and area of further studies.

6.1 Findings and Discussions

The first model is made up of Bessel functions of order zero and Bessel function of order half which determines the sound nature of the drum. The second model also involves Bessel functions of order zero and Bessel functions of order one-third which makes the drum sound the way they do. Clearly the Bessel function of order zero appears in both models and by it we are able to determine the frequency of the vibration of the drumhead. Each of the two models possesses frequency of $\frac{\lambda_n}{2\pi}$ cycles per unit time; where λ_n are the Eigen values or characteristic values of our problem.

The graph of fig 6.1.1 below also reveals that Bessel function of order zero $J_0(x)$ has certain common features of the trigonometric functions of damped amplitude. i.e. Jo(x) begin at 1 for $x = 0$ and oscillates as x increases, although they are not periodic in x and that their amplitude decreases slowly from unity as x increases, unlike the graph of $\cos x$ which also has unity amplitude at $x = 0$, $\cos x$ is periodic in nature and oscillate between 1 and -1.

Similarly the graph of Bessel functions of order half for the first model and Bessel function of order one-third replicates the behavior of the periodic sine graph. They are also not periodic and their amplitudes fall off slowly with increasing value of x.

Bessel Functions of orders 1/2, 1/3 and 0: [First Kind]

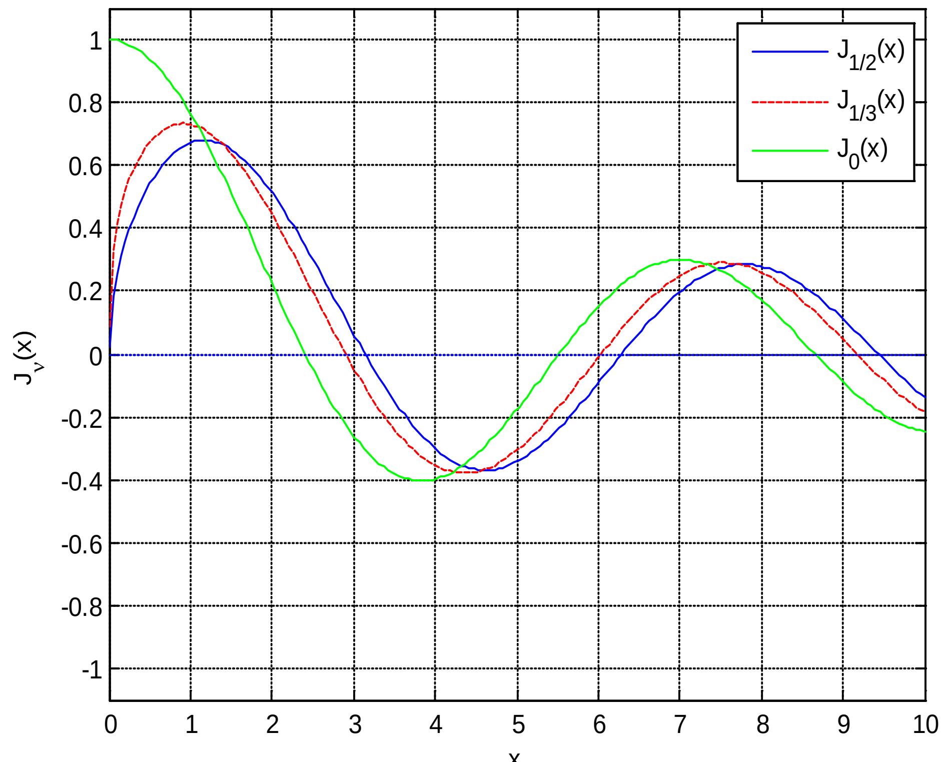


Fig6.1.1 graphs of Bessel functions of orders 1/2; 1/3 and 0

6.2 Analysis of Results

The researcher focus was on the vibration of membrane having varying tension on the drumhead with emphasis on the hour-glass drum or single armpit talking drum ("*donno*"), particularly the way they sound and how they can be interpreted as being a harmonic instrument as well as a rhythm maker. This drum comes in variety of sizes; the larger ones have three sizes of diameters 7.00, 6.50 and 6.00 inches respectively and the smaller one have varying size of 3.5cm, 4.5cm and 5.5cm respectively for their radii.

In order to make a deductive analysis the researcher chose the hour-glass drum having radius 3.5 inches with varying tension in the drumhead and compared it with a hypothetical Circular drum with constant tension in the drumhead having the same radius .Hypothetical circular drum is used in the sense that most circular drum with constant tension have larger diameters than the vibrating membranes or drumheads under consideration. Both drum type was subjected to a boundary value problem with the initial conditions, $U(R,t) = 0, for\ all\ t \geq 0$, initial deflection $U(r,0) = 0$, and initial velocity $U_t(r,0) = f(r)$, using three different types of initial velocity functions $f(r)$ namely $f(r) = 1$, $f(r) = r\left(2 - \frac{r}{2}\right)^2$, and $f(r) = \left(2 - \frac{r}{2}\right)^2$.

The Summary of the results in using these initial velocity functions is tabulated below

Summary of Results with a quadratic initial velocity function

Normal mode	velocity function	drum with varying tension	drum with constant tension
	f(r)=(2-r/2)^2		
U_1(1,5)		-1.0045	-0.7076
U_2(0,0)		0	0
U_3(2.5,2)		1.053	-0.0096
U_4(1.5,2)		2.0534	0.0492
U_5(0,2)		3.0114	-0.0854
U_6(1,6)		-1.0002	0.0129
U_7(0,0)		0	0
U_8(2.5,3)		1.052	-0.0354
U_9(1.5,2)		2.0546	0.0602
U_10(0,2)		3.0065	-0.0832

Table 6.2.1

Summary of Results with a Cubic initial velocity function

Normal mode	Velocity function	Drum with varying tension	Drum with constant tension
	f(r)=r(2-r/2)^2		
U_1(1,1)		3.0026	1.9019
U_2(2,2)		2.0586	1.3634
U_2(1.5,3)		1.0038	1.2989
U_3(0,0)		0	0
U_4(1,8)		-1.0079	-0.7496
U_5(2,2)		2.0454	1.3559
U_5(1.5,3)		1.0109	1.3066
U_6(0,0)		0	0
U_7(1,8)		-1.0073	-0.7496
U_8(2,2)		2.044	1.355
U_8(1.5,3)		1.0113	1.3067
U_9(0,0)		0	0
U_10(1,8)		-1.007	-0.7494

Table 6.2.2

From tables 6.2.1 and 6.2.2 the drumhead with varying tension gave different variety of modes which are nearly integral values and are much higher than that of the drum with constant tension. This makes the single armpit talking drum ("*donno*") a nearly harmonic instrument. The key difference between single armpit talking drum ("*donno*") and drums with constant tension like the "*fontonfrom*" is the absence of the strings which holds the drumheads together and can be easily squeezed and released while playing to produce a wide range of tones of different pitches. Based on this characteristic and the fact that tonal languages are used in many African cultures, it is possible to send linguistic messages via the ("*donno*"). Again what makes the single armpit talking drum ("*donno*") unique is its ability to adapt to the tone of any musical instrument and they are also use in religious chants or poetry.

The normal modes for the drum with constant tension using the initial velocity functions dependent on the radius (for both quadratic and cubic functions of r) gave overtones which are purely decimals and not integral. i.e. their higher modes do not occur at frequencies closely related to the fundamental, so the sounds made by the various vibration modes conflict with one another, the result is a collection of unrelated tones that combines into a sound that has no discernible pitch. This fact makes such a drum a rhythmic instrument and not a harmonic instrument. The Drum with constant tension provides the rhythm of any musical piece, rather than add to the harmony due to its inharmonic nature since it overtones are not integral multiples of their fundamental frequency.

Summary of Results with an invariant initial velocity function

Normal modes	velocity function	drum with varying tension	drum with constant tension
	f(r)=1		
U_1(1,1)		1.6715	1.0587
U_2(0,3)		0.8561	1.0955
U_3(0,1)		-0.2756	1.0085
U_3(1.5,2)		0.0468	1.0035
U_4(1.5,2)		0.0492	1.0053
U_5(1.5,2)		0.0554	1.0079
U_6(0,1)		-0.37074	1.0097
U_6(1.5,2)		0.0576	1.0094
U_7(0,1)		-0.3264	1.0079
U_7(1.5,2)		0.0586	1.0097
U_8(0,1)		-0.3115	1.0043
U_9(0,1)		-0.3196	1.0086
U_10(0,1)		-0.3179	1.0066

Table 6.2.3

From table 6.2.3 it can be inferred that the drum with varying tension gave overtones which are much smaller than their fundamental normal mode and their Normal modes are in general not integral as compared to that of the drum with constant tension. This shows that for the talking drum to be made harmonic the choice of the velocity function should not be an invariant. Also it can be seen that all the normal modes for the drum with constant tension and velocity are all near to the integer 1 except that the fundament frequency is much higher than the other normal mode.

An important application of the values of these normal modes from a drum with constant tension in the drumhead is that If one is able to make another drum with constant tension in the drumhead and apply a constant initial velocity such that the normal modes are nearly an integer that is not unity then such pair of drum could be used for communication purposes, a fact employed by the "*Atumpan*" paired drum of the Akans of Ghana which is made to talk when it is played. The "*Atumpan*" paired drum has two distinct pitches one high and the other low. (i.e. the female and the male drum). Thus for a drum with constant tension to be head as talking it should be paired to give at least two contrasting tones so as to imitate the rhythm or flow of the tonal language it tries to imitate.

The "*Atumpan*" drum has enough of a sense of pitch to allow them to be tuned. This makes the sounds usually accompanied by the paired "*Atumpan*" drum have a unique characteristic otherwise, the rhythmic nature of the drum when it is played, would completely disrupt the recitals and appellation being rendered by the drummer who is playing the drums or by a different person.

6.3 Conclusion

This thesis has given us a practical insight into why most drums are rhythm instruments and some could be made to posses both rhythmic and harmonic characteristic. In our quest to investigate the mathematics of vibrating membrane as applied to African drums with varying tension compared with those of constant tension and on using the two dimensional wave equation, we were able to discover that the single armpit talking drum ("*donno*") of the *Akans* of Ghana or the "*Lunna*" of the *Dagbambas* of Ghana) posses rhythmic and harmonic characteristics. Also the "*Atumpan*" drum of the *Akans* of Ghana is a rhythm maker and has a high sense of pitches which enables it to imitate tonal language by imposing a boundary value problem on the two types of drums.

Our local drums can now be considered as having some scientific basis in addition to the notion that it is just an artistic creation of our culture.

6.4 Recommendation

The normal modes obtained for the ("*donno*") drum or the talking drum were nearly integral in nature, instead of exact integers, this the researcher believe was accounted for partly by approximating $\sin t$ by t in the time O.D.E and also by not having the linear combinations of the Bessel functions of orders $J_{\pm\frac{1}{3}}(s)$ and $J_{\pm\frac{1}{2}}(s)$ which are two independent solutions to the spatial O.D.E .in their computations. We recommend that these approximations should be

1. Avoided and rather the exact tension function $f(t) = \sin t$ be used instead of t.
2. the complete Bessel functions of fractional order for the spatial O.D.E should be used,

And their resulting Mathematical Model obtained should be evaluated at an initial time t near zero other than t starting at exactly at zero, since zero is a branch point for Bessel functions of fractional orders. This would give vibrational modes which are not divergent.

Also the researcher believes that integral nature of the Normal modes of the single armpit talking drum (the "*donno*") would also greatly improve the harmony of the drum if an appropriate initial deflection function could be obtained which mimic this physical system. Such a deflection function should

1. Not be an invariant but rather a function dependent on the radius of the drum and
2. This function should be differentiable in the interval $0 \leq r \leq R$ to ensure the existence of the generation of the Fourier- Bessel co-efficients for the mathematical model.

In using the two dimensional wave equation, the researcher replaced the constant c for the tension on the assumption that the tension was a function of time t in order to derive the model for evaluating the vibrational mode of a membrane with varying tension, Thus we recommend that further research be made to include the case where the tension is dependent on time for the derivation of the wave equation.

6.5 Areas of further Studies.

With the application of boundary value Problems, in the study of our local instruments, further scientific studies can be made to find the reason behind the uniqueness of our local drums; this will help in modifying and improve upon their designs thus making them better.

Fig. 6.5 1(a): Picture of the Single Talking drum Fig.6.51(b): Picture of double Talking drum

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Appendix A

Matlab code for evalutating the Normal modes of modes in chapter three and four

% the code is made to evaluate the normal mode of a vibrating membrane

%by varying the tensions in the drumhead

% by keeping the tension in the drumhead constant

%the code is prepared for the talking drum with radius 3.5cm for the two cases.

% in order to make comparisms from the output of the code.

% the following initial and boundary conditions $U = 0, U_t = f(r)\ at\ t = 0$ were used

%with three different velocity functions namely$f(r) = 1, f(r) = r\left(2 - \frac{r}{2}\right)^2$, and

% $f(r) = \left(2 - \frac{r}{2}\right)^2$, these functions were chosen so that nearer the origin the normal

%Mode would be greatest and further from the origin the normal mode would be least.

% Codes for the model in chapter three model ;

% Using $f(r) = 1$

```
syms R w u

disp('CASE1 DRUMHEAD WITH FIXED TENSION')

d=10; a=1; c=1; R=3.5;

    disp([R'])

    disp('u_j are the partials')

    disp('Uj_1,Uj_2,Uj_3 are the Normal modes')

    disp('where _1,_2,_3 are the associated velocity functions used in computing the normal mode')

    disp('j_x are the bessel function of order zero')
```

```
   disp('j_y are the bessel function of order negative one-third and')

   disp('A_n are the co-efficients associated with the donno problem')

% m the first ten zeros of Bessel function of order zero

m=[2.4048,5.5201,8.6537,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346];

u=m/R;

%v the values for the varying r's

v=[0 1 1.5 2  2.25 2.5 3 3.25 3.5 4];

for r=1:d

   v(r);

   x(r,:)=v(r).*u;

end

% j_x computes Bessel fuction of order zero using each v and m in columns

j_x=besselj(0,x)

%intj is the integration in the model

intJ=double(int(w*besselj(0,w.*u),w,0,R));

%A_n, the redefine co-efficients of the model used in computing the normal modese

A_n=double((2./(R.^2.*besselj(1,m)).*besselj(1,m)).*((2./c)./u).^(1/2)*gamma(3/2).*int
J)

% v1 the values for the varying time

v1=[0 1 2 3 4 5 6 7 8 9];

for t=1:d

   v1(t);

   y(t,:)=(v1(t).*u);

end
```

```
% j_y computes the expression involving time in the model using the m’s in columns
j_y= v1(t)^(1/2)*besselj(1/2,y)
% u_j are the partials and U_j are the Normal modes, the _1,_2, and_3 in U_j are the respective velocity function used
u_1=j_x(:,1)*j_y(:,1)'*A_n(1,1);
u_1=j_x(:,1)*j_y(:,1)'*A_n(1,1);
U1_1=u_1
u_2=j_x(:,2)*j_y(:,2)'*A_n(1,2);
U2_1=u_1+u_2
u_3=j_x(:,3)*j_y(:,3)'*A_n(1,3);
U3_1=u_1+u_2+u_3
u_4=j_x(:,4)*j_y(:,4)'*A_n(1,4);
U4_1=u_1+u_2+u_3+u_4
u_5=j_x(:,5)*j_y(:,5)'*A_n(1,5);
U5_1=u_1+u_2+u_3+u_4+u_5
u_6=j_x(:,6)*j_y(:,6)'*A_n(1,6);
U6_1=u_1+u_2+u_3+u_4+u_5+u_6
u_7=j_x(:,7)*j_y(:,7)'*A_n(1,7);
U7_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7
u_8=j_x(:,8)*j_y(:,8)'*A_n(1,8);
U8_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8
u_9=j_x(:,9)*j_y(:,9)'*A_n(1,9);
U9_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9
u_10=j_x(:,10)*j_y(:,10)'*A_n(1,10);
```

```
U10_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_10
```

% Using $f(r) = r\left(2 - \frac{r}{2}\right)^2$,

```
syms R w u
d=10; a=1; c=1; R=3.5;
    disp([R'])
    disp('u_j are the partials')
    disp('Uj_1,Uj_2,Uj_3 are the Normal modes')
    disp('where _1,_2,_3 are the associated velocity functions used in computing the normal mode')
    disp('j_x are the bessel function of order zero')
    disp('j_y are the bessel function of order negative one-third and')
    disp('A_n are the co-efficients associated with the donno problem')
m=[2.4048,5.5201,8.6537,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346];
u=m/R;
%v the values for the varying r's
v=[0 1 1.5 2  2.25 2.5 3 3.25 3.5 4];
for r=1:d
    v(r);
    x(r,:)=v(r).*u;
end
% j_x computes Bessel fuction of order zero using each v and m in columns
j_x=besselj(0,x)
%intj is the integration in the model
```

```
intJ=double(int(w*w*(2-w/2)^(3).*besselj(0,w.*u),w,0,R));

%A_n, the redefine co-efficients of the model used in computing the normal modes

A_n=double((2./(R.^2.*besselj(1,m)).*besselj(1,m)).*((2./c)./u).^(1/2)*gamma(3/2).*int
J)

% v1 the values for the varying time

v1=[0 1 2 3 4 5 6 7 8 9];

for t=1:d

    v1(t);

    y(t,:)=(v1(t).*u);

end

% j_y computes the expression involving time in the model using the m’s in columns

j_y= v1(t)^(1/2)*besselj(1/2,y)

% u_j are the partials and U_j are the Normal modes, the _1,_2, and_3 in U_j are the
respective velocity function used

u_1=j_x(:,1)*j_y(:,1)'*A_n(1,1);

U1_2=u_1

u_2=j_x(:,2)*j_y(:,2)'*A_n(1,2);

U2_2=u_1+u_2

u_3=j_x(:,3)*j_y(:,3)'*A_n(1,3);

U3_2=u_1+u_2+u_3

u_4=j_x(:,4)*j_y(:,4)'*A_n(1,4);

U4_2=u_1+u_2+u_3+u_4

u_5=j_x(:,5)*j_y(:,5)'*A_n(1,5);

U5_2=u_1+u_2+u_3+u_4+u_5
```

```
u_6=j_x(:,6)*j_y(:,6)'*A_n(1,6);

U6_2=u_1+u_2+u_3+u_4+u_5+u_6

u_7=j_x(:,7)*j_y(:,7)'*A_n(1,7);

U7_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7

u_8=j_x(:,8)*j_y(:,8)'*A_n(1,8);

U8_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8

u_9=j_x(:,9)*j_y(:,9)'*A_n(1,9);

U9_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9

u_10=j_x(:,10)*j_y(:,10)'*A_n(1,10);

U10_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_10
```

% Using $f(r) = \left(2 - \frac{r}{2}\right)^2$,

```
syms R w u

d=10; a=1; c=1; R=3.5;

    disp([R'])

    disp('u_j are the partials')

    disp('Uj_1,Uj_2,Uj_3 are the Normal modes')

    disp('where _1,_2,_3 are the associated velocity functions used in computing the normal mode')

    disp('j_x are the bessel function of order zero')

    disp('j_y are the bessel function of order negative one-third and')

    disp('A_n are the co-efficients associated with the donno problem')

m=[2.4048,5.5201,8.6537,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346];

u=m/R;
```

```
%v the values for the varying r's

v=[0 1 1.5 2  2.25 2.5 3 3.25 3.5 4];

for r=1:d

    v(r);

    x(r,:)=v(r).*u;

end

% j_x computes Bessel fuction of order zero using each v and m in columns

j_x=besselj(0,x)

%intj is the integration in the model

intJ=double(int(w*(2-w/2)^(2).*besselj(0,w.*u),w,0,R));

%A_n, the redefine co-efficients of the model used in computing the normal modes

A_n=double((2./(R.^2.*besselj(1,m)).*besselj(1,m)).*((2./c)./u).^(1/2)*gamma(3/2).*int
J)

% v1 the values for the varying time

v1=[0 1 2 3 4 5 6 7 8 9];

for t=1:d

    v1(t);

    y(t,:)=(v1(t).*u);

end

% j_y computes the expression involving time in the model using the m's in columns

j_y= v1(t)^(1/2)*besselj(1/2,y)

% u_j are the partials and U_j are the Normal modes, the _1,_2, and_3 in U_j are the
respective velocity function used

u_1=j_x(:,1)*j_y(:,1)'*A_n(1,1);
```

```
U1_3=u_1

u_2=j_x(:,2)*j_y(:,2)'*A_n(1,2);

U2_3=u_1+u_2

u_3=j_x(:,3)*j_y(:,3)'*A_n(1,3);

U3_3=u_1+u_2+u_3

u_4=j_x(:,4)*j_y(:,4)'*A_n(1,4);

U4_3=u_1+u_2+u_3+u_4

u_5=j_x(:,5)*j_y(:,5)'*A_n(1,5);

U5_3=u_1+u_2+u_3+u_4+u_5

u_6=j_x(:,6)*j_y(:,6)'*A_n(1,6);

U6_3=u_1+u_2+u_3+u_4+u_5+u_6

u_7=j_x(:,7)*j_y(:,7)'*A_n(1,7);

U7_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7

u_8=j_x(:,8)*j_y(:,8)'*A_n(1,8);

U8_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8

u_9=j_x(:,9)*j_y(:,9)'*A_n(1,9);

U9_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9

u_10=j_x(:,10)*j_y(:,10)'*A_n(1,10);

U10_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_10

 disp('CASE 2 DRUMHEAD WITH VARYING TENSION')

% Using  f(r) = 1,

syms R w u

d=10; a=1;R=3.5;

    disp([R'])
```

```
    disp('u_j are the partials')
    disp('Uj_1,Uj_2,Uj_3 are the Normal modes')
    disp('where _1,_2,_3 are the associated velocity functions used in computing the normal mode')
    disp('j_x are the bessel function of order zero')
    disp('j_y are the bessel function of order negative one-third and')
    disp('A_n are the co-efficients associated with the donno problem')
m=[2.4048,5.5201,8.6537,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346];
%c the values for the varying r’s
c=[0 1 1.5 2  2.25 2.5 3 3.25 3.5 4];
for r=1:d
    c(r);
    x(r,:)=c(r).*u;
end
% j_x computes Bessel fuction of order zero using each v and m in columns
j_x=besselj(0,x)
%intj is the integration in the model
intJ=double(int(w*besselj(0,w.*u),w,0,R));
%A_n, the redefine co-efficients of the model used in computing the normal modes
A_n=double((2./(R.^2.*besselj(1,m)).*besselj(1,m)).*((3*a)./u).^(1/3)*gamma(2/3).*intJ)
% c1 the values for the varying time
c1=[0 1 2 3 4 5 6 7 8 9];
for t=1:d
```

```
    c1(t);
     y(t,:)=(2.*c1(t).^(3/2).*u)./(3.*a);
end
% j_y computes the expression involving time in the model using the m’s in columns
j_y= c1(t)^(1/2)*besselj(1/3,y)
% u_j are the partials and U_j are the Normal modes, the _1,_2, and_3 in U_j are the respective velocity function used
u_1=j_x(:,1)*j_y(:,1)'*A_n(1,1);
U1_1=u_1
u_2=j_x(:,2)*j_y(:,2)'*A_n(1,2);
U2_1=u_1+u_2
u_1=j_x(:,3)*j_y(:,3)'*A_n(1,3);
U3_1=u_1+u_2+u_3
u_4=j_x(:,4)*j_y(:,4)'*A_n(1,4);
U4_1=u_1+u_2+u_3+u_4
u_5=j_x(:,5)*j_y(:,5)'*A_n(1,5);
U5_1=u_1+u_2+u_3+u_4+u_5
u_6=j_x(:,6)*j_y(:,6)'*A_n(1,6);
U6_1=u_1+u_2+u_3+u_4+u_5+u_6
u_7=j_x(:,7)*j_y(:,7)'*A_n(1,7);
U7_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7
u_8=j_x(:,8)*j_y(:,8)'*A_n(1,8);
U8_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8
u_9=j_x(:,9)*j_y(:,9)'*A_n(1,9);
```

```
U9_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9

u_10=j_x(:,10)*j_y(:,10)'*A_n(1,10);

U10_1=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_10
```

% Using $f(r) = r\left(2 - \frac{r}{2}\right)^2$,

```
syms R w u

d=10; a=1;R=3.5;

   disp([R'])

   disp('u_j are the partials')

   disp('Uj_1,Uj_2,Uj_3 are the Normal modes')

   disp('where _1,_2,_3 are the associated velocity functions used in computing the normal mode')

   disp('j_x are the bessel function of order zero')

   disp('j_y are the bessel function of order negative one-third and')

   disp('A_n are the co-efficients associated with the donno problem')

m=[2.4048,5.5201,8.6537,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346];

u=m/R;

%c the values for the varying r’s

c=[0 1 1.5 2  2.25 2.5 3 3.25 3.5 4];

for r=1:d

   c(r);

   x(r,:)=c(r).*u;

end

% j_x computes Bessel fuction of order zero using each v and m in columns
```

```
j_x=besselj(0,x)
%intj is the integration in the model
intJ=double(int(w*w*(2-w/2)^3.*besselj(0,w.*u),w,0,R));
%A_n, the redefine co-efficients of the model used in computing the normal modes
A_n=double((2./(R.^2.*besselj(1,m)).*besselj(1,m)).*((3*a)./u).^(1/3)*gamma(2/3).*int
J)
% c1 the values for the varying time
c1=[0 1 2 3 4 5 6 7 8 9];
for t=1:d
    c1(t);
    y(t,:)=(2.*c1(t).^(3/2).*u)./(3.*a);
end
% j_y computes the expression involving time in the model using the m's in columns
j_y= c1(t)^(1/2)*besselj(1/3,y)
% u_j are the partials and U_j are the Normal modes, the _1,_2, and_3 in U_j are the
respective velocity function used
u_1=j_x(:,1)*j_y(:,1)'*A_n(1,1);
U1_2=u_1
u_2=j_x(:,2)*j_y(:,2)'*A_n(1,2);
U2_2=u_1+u_2
u_3=j_x(:,3)*j_y(:,3)'*A_n(1,3);
U3_2=u_1+u_2+u_3
u_4=j_x(:,4)*j_y(:,4)'*A_n(1,4);
U4_2=u_1+u_2+u_3+u_4
```

```
u_5=j_x(:,5)*j_y(:,5)'*A_n(1,5);

U5_2=u_1+u_2+u_3+u_4+u_5

u_6=j_x(:,6)*j_y(:,6)'*A_n(1,6);

U6_2=u_1+u_2+u_3+u_4+u_5+u_6

u_7=j_x(:,7)*j_y(:,7)'*A_n(1,7);

U7_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7

u_8=j_x(:,8)*j_y(:,8)'*A_n(1,8);

U8_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8

u_9=j_x(:,9)*j_y(:,9)'*A_n(1,9);

U9_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9

u_10=j_x(:,10)*j_y(:,10)'*A_n(1,10);

U10_2=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_10
```

% Using $f(r) = \left(2 - \frac{r}{2}\right)^2$,

```
syms R w u

d=10; a=1;R=3.5;

  disp([R'])

  disp('u_j are the partials')

  disp('Uj_1,Uj_2,Uj_3 are the Normal modes')

  disp('where _1,_2,_3 are the associated velocity functions used in computing the normal mode')

  disp('j_x are the bessel function of order zero')

  disp('j_y are the bessel function of order negative one-third and')

  disp('A_n are the co-efficients associated with the donno problem')
```

```
m=[2.4048,5.5201,8.6537,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346];
u=m/R;
%c the values for the varying r's
c=[0 1 1.5 2  2.25 2.5 3 3.25 3.5 4];
for r=1:d
    c(r);
    x(r,:)=c(r).*u;
end
% j_x computes Bessel fuction of order zero using each v and m in columns
j_x=besselj(0,x)
%intj is the integration in the model
intJ=double(int(w*(2-w/2)^(2).*besselj(0,w.*u),w,0,R));
%A_n, the redefine co-efficients of the model used in computing the normal modes
A_n=double((2./(R.^2.*besselj(1,m)).*besselj(1,m)).*((3*a)./u).^(1/3)*gamma(2/3).*int
J)
% c1 the values for the varying time
c1=[0 1 2 3 4 5 6 7 8 9];
for t=1:d
    c1(t);
    y(t,:)=(2.*c1(t).^(3/2).*u)./(3.*a);
end
% j_y computes the expression involving time in the model using the m's in columns
j_y= c1(t)^(1/2)*besselj(1/3,y)
```

```
% u_j are the partials and U_j are the Normal modes, the _1,_2, and_3 in U_j are the respective velocity function used
u_1=j_x(:,1)*j_y(:,1)'*A_n(1,1);
U1_3=u_1
u_2=j_x(:,2)*j_y(:,2)'*A_n(1,2);
U2_3=u_1+u_2
u_3=j_x(:,3)*j_y(:,3)'*A_n(1,3);
U3_3=u_1+u_2+u_3
u_4=j_x(:,4)*j_y(:,4)'*A_n(1,4);
U4_3=u_1+u_2+u_3+u_4
u_5=j_x(:,5)*j_y(:,5)'*A_n(1,5);
U5_3=u_1+u_2+u_3+u_4+u_5
u_6=j_x(:,6)*j_y(:,6)'*A_n(1,6);
U6_3=u_1+u_2+u_3+u_4+u_5+u_6
u_7=j_x(:,7)*j_y(:,7)'*A_n(1,7);
U7_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7
u_8=j_x(:,8)*j_y(:,8)'*A_n(1,8);
U8_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8
u_9=j_x(:,9)*j_y(:,9)'*A_n(1,9);
U9_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9
u_10=j_x(:,10)*j_y(:,10)'*A_n(1,10);
U10_3=u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_10
```

Appendix B

Matlabcode for generating values of Bessel function of order zero, of order, half and of order one-third

```
% This MATLAB file generate values of   Bessel functions of order 0,1/3 and 1/2.

% It generate the first thirty values of the Bessel functions of the above

% order

x=[0:0.1:2.9];

J_0 = besselj(0,x);

J_1x3 = besselj(1/3,x);

J_1x2=besselj(1/2,x);

 fprintf('-------------------------------------------\n')

fprintf('| Order 0 \t | Order 1/3 \t | Order 1/2 \t|\n')

fprintf('-------------------------------------------\n')

fprintf('| %f \t | %f \t  | %f \t |\n',J_0,J_1x3,J_1x2)

fprintf('-------------------------------------------\n')

 %gtext('Bessel functions of the first kind');
```

Appendix C

Matlabcode for graphing Bessel functions of orders half, one-third and order zero

```
xmin = 1e-3;
 xmax = 10;
resolution = 200;
 x = linspace(xmin,xmax,resolution);
 y_zero = 0*x;
 % First, compute the Bessel functions.
 % Bessel functions of the first kind
 J_0=besselj(0,x);
 J_1x2 = besselj(1/2,x);
 %J_1x2 = besselj(0.5,x);
 J_1x3 = besselj(1/3,x);
 %J_1x3 = besselj(1.3,x);
 % set common axis limits
 xplot_min = 0;
 xplot_max = xmax;
 yplot_min = -1.1;
 yplot_max = +1.1;
 % Bessel functions of the first kind
%integer order
%non-integer order
plot(x,J_1x2);
hold on;
```

```
grid on;

title('Bessel Function of orders 1/2, 1/3 and 0:[First Kind]')

 plot(x,J_1x3,'r--');

hold on;

plot(x,J_0,'g-');

 plot(x,y_zero,':');

 legend('J_{1/2}(x)', 'J_{1/3}(x)','J_0(x)');

 xlabel('x');

 ylabel('J_\nu(x)');

 axis([xplot_min xplot_max yplot_min yplot_max]);
```