KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY
COLLEGE OF SCIENCE
DEPARTMENT OF MATHEMATICS

APPLICATION OF SOBOLEV SPACES TO DIFFERENTIAL EQUATIONS

by

Francis Kwabena Gyau, BSc (Math)
(PG8106312)

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Declaration

I hereby declare that this submission is my own work towards the Master of Philosophy (MPhil) in Pure Mathematics and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgement has been made in the text.

Francis Kwabena Gyau	....	....
(PG8106312)	Signature	Date

Certified by:

Dr E. Prempeh	....	....
(Supervisor)	Signature	Date

Prof S. K. Amponsah	....	....
(Head of Department)	Signature	Date

Dedication

I dedicate this work to my family and friends.

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I appreciate the efforts of my supervisor, Dr Edward Prempeh and all the wonderful professors and teachers who counseled me.Their knowledge and support helped me in understanding and coming up with this work. I thank my family and friends for their unending support and encouragement.

Abstract

This thesis defines the Sobolev Space and provides certain properties using concepts from functional analysis and real analysis. These theories and properties are applied to solutions of some partial differential equations. The concepts used here in finding solutions to differential equation involve analytical properties like continuity, infinite differentiability, continuous derivatives, e.t.c. Specific examples of partial differential equations are taken where the Existence and Uniqueness of their *weak* solutions are studied together with their Regularity and Recovery of their *classical or strong* solutions.

Contents

Chapter 1

INTRODUCTION AND INITIAL DEFINITIONS

This thesis presents a study of the properties of certain Banach Spaces of weakly differentiable functions of several real variables that arise in connection with numerous problems in partial differential equations and other areas of pure and applied mathematics.
These spaces have been associated with a Russian mathematician Sergei Lvovich Sobolev, who worked in mathematical analysis and partial differential equations.

In many problems of mathematical physics and variational calculus,it is not sufficient to deal with classical solutions of differential equations.It is also necessary to introduce the idea of weak derivatives to work in the so-called Sobolev space.

What we seek to achieve is to explore the Sobolev Space and apply its properties to the solutions of partial deferential equations. The beginning chapters gives us the functional analytic tools and elements of real analysis needed to achieve our aim. Chapter 3 discusses the Sobolev Space and Chapter 4 gives examples of partial differential equations where the concept of Sobolev space is applied. We summarize in Chapter 5.

Consider the Dirichlet problem for the Laplace equation in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{aligned} \Delta u &= 0 \quad x \in \Omega \\ u(x) &= \phi(x) \quad x \in \delta\Omega \end{aligned}$$

where $\phi(x)$ is a given function on the boundary $\delta\Omega$ and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u. We have

$$l(u) = \int_{\Omega} \sum_{j=1}^{n} |\frac{\partial u}{\partial x_j}|^2 dx$$

To find the minimum of $l(u)$ on the set of functions satisfying the condition $u|_{\delta\Omega} = \phi$, it is much easier to minimize this functional not in $C^1\left(\overline{\Omega}\right)$, but in a larger class - $W_2^1\left(\Omega\right)$ - consisting of all functions $u \in L_2\left(\Omega\right)$ having the weak derivatives $\partial_j u \in L_2\left(\Omega\right)$ $j = 1, \ldots, n$. If the boundary $\delta\Omega$ is smooth, then the trace of $u(x)$ on $\delta\Omega$ is well defined and the relation $u|_{\delta\Omega} = \phi$ makes sense.

The concepts of Sobolev space enables us to look at the weak solution of this problem. The classical (or strong) solution can always be recovered by showing that any weak solution is a classical solution.

1.1 Concepts from Functional Analysis

Functional analysis play a major role in the study of Sobolev Spaces. Sobolev Spaces allow the use of functional analysis saving us the trouble of developing a new branch of analysis to solve pdes. This section provides definitions and important results in functional analysis.

1.1.1 Linear Space

Definition 1.1. Let $X \neq \varnothing$ be a nonempty set and K a scalar field. Then X is said to be a linear/vector space if the functions

$$X \times X \to X \quad and \quad K \times X \to X$$

known as the addition and scalar multiplication are continuous, i.e.

$$\forall \quad x, y \in X \quad and \quad c \in K$$

$$(x, y) \to x + y \in X \quad and \quad (c, x) \to cx \in X$$

are continuous.

The following properties also holds;

i $x + y = y + x$ and $(x + y) + z = x + (y + z) \quad \forall x, y, z \in X$
Also

$$\exists \underline{0} \in X \texttt{ such that } x + \underline{0} = x \quad \forall\, x \in X$$

$$\exists - x \texttt{ such that } x + (-x) = 0 \quad \forall\, x \in X$$

ii $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y \quad \forall x, y \in X, \forall \lambda \in K$

iii $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot y \quad \forall\, x, y \in X$ and $\forall\, \alpha, \beta \in K$

iv $(\alpha\beta)x = \alpha(\beta \cdot x) \quad \forall\, x \in X, \lambda \in K$

We call X a linear space over the field K.

Unless otherwise stated it is assumed throughout this thesis that all vector spaces referred to are taken from the field of Complex numbers.

1.1.2 Normed Linear Space

Definition 1.2. Let X be a linear space over a field K. A *norm* on X is a real-valued function $\| \cdot \|$, where $\| \cdot \| : X \to [0, +\infty)$ such that for any $x, y \in X, \lambda \in K$ the following conditions are satisfied;

N1:

$$\|x\| \geq 0 \texttt{ and } \|x\| = 0 \texttt{ if, and only if, } x = 0$$

N2:

$$\|\lambda x\| = |\lambda|\|x\|$$

N3:

$$\|x + y\| \leq \|x\| + \|y\|$$

Any nonempty linear space X equipped with a norm(i.e. satisfying these conditions) is known as a *Normed Linear Space*

Example 1.1. Let $X = \mathbb{R}^2$. For any $x = (x_1, x_2)$. Define $\|\cdot\|_2 : X = \mathbb{R} \to [0, \infty)$ by $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

N1: $\|x\|_2 = \sqrt{x_1^2 + x_2^2} \geq 0$

Assume that $x = (x_1, x_2) = 0 \Longrightarrow x_1 = 0 \quad x_2 = 0$

$$\|x\|_2 = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$$

Conversely, assume $\|x\|_2 = 0$

$$\begin{aligned} \|x\|_2 &:= (x_1^2 + x_2^2)^{\frac{1}{2}} = 0 \\ &\Longrightarrow x_1 = 0 = x_2 \quad \therefore x = 0 \end{aligned}$$

N2: For any scalar α

$$\begin{aligned} \|\alpha x\|_2 &= \|\alpha\,(x_1, x_2)\,\|_2 = \|\,(\alpha x_1, \alpha x_2)\,\|_2 \\ &= \left[(\alpha x_1)^2 + (\alpha x_2)\right]^{\frac{1}{2}} \\ &= [\alpha^2\,(x_1^2 + x_2^2)]^{\frac{1}{2}} \\ &= |\alpha|\,(x_1^2 + x_2^2)^{\frac{1}{2}} \\ &= |\alpha|\|x\|_2 \end{aligned}$$

N3: Let $x, y \in X \quad x = (x_1, x_2) \quad y = (y_1, y_2)$

$$
\begin{aligned}
\|x+y\|_2 \quad &:= \|\,(x_1+y_1, x_2+y_2)\,\|_2 = \left[\sum_{i=1}^{2}(x_i+y_i)^2\right]^{\frac{1}{2}} \\
\|x+y\|_2^2 \quad &= \sum_{i=1}^{2}(x_i+y_i)^2 \\
&= \sum_{i=1}^{2}(x_i^2 + 2x_iy_i + y_i^2) \\
&\leq \sum_{i=1}^{2}(x_i^2 + 2|x_iy_i| + y_i^2) \\
&= \sum_{i=1}^{2}x_i^2 + 2\sum_{i=1}^{2}|x_iy_i| + \sum_{i=1}^{2}y_i^2 \\
\sum_{i=1}^{2}|x_iy_i| \quad &\leq \left(\sum_{i=1}^{2}x_i^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{2}y_i^2\right)^{\frac{1}{2}} \quad : \quad \texttt{Cauchy-Schwatz inequality} \\
\therefore \|x+y\|_2^2 \quad &\leq \sum_{i=1}^{2}x_i^2 + 2\left(\sum_{i=1}^{2}x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{2}y_i^2\right)^{\frac{1}{2}} + \sum_{i=1}^{2}y_i^2 \\
&= (\|x\|_2 + \|y\|_2)^2
\end{aligned}
$$

since both sides are non-negative, we have

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

Hence $(\mathbb{R}^2, \|\cdot\|)$ is a normed linear space.

1.1.3 Banach Space

Definition 1.3. A sequence $\{x_n\}$ in a normed linear space X is called a *Cauchy Sequence* if $\forall\epsilon > 0$ there exists an integer N such that $\rho(x, y) = \|x_m - x_n\| < \epsilon$ holds whenever $m, n > N$.
X is thus *complete*.
We call X a *Banach space* if every Cauchy sequence in X converges to a limit in X.

Any complete normed linear space is called a Banach space.

1.1.4 Inner Product Space

Definition 1.4. Let E be a linear space. An *inner product* on E is a scalar-valued function $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{C}$ such that the following conditions are satisfied:
For any $x, y, z \in E$ and $\lambda, \mu \in \mathbb{C}$
I1:

$$\langle x, x\rangle \geq 0 \ \text{ and } \ \langle x, x\rangle = 0 \Leftrightarrow x = 0$$

I2:

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

I3:

$$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$$

A linear space E equipped with a function(or metric) satisfying these is called an*Inner Product Space*.

1.1.5 Hilbert Space

Definition 1.5. An inner product space E is complete if every Cauchy sequence in E converges to an element in E.
Thus, a complete inner product space is called a *Hilbert Space*.

The norm on a normed linear space E is induced by an inner product if, and only if the norm satisfies the parallelogram law,

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \ \forall x, y \in E$$

This theorem is known as the *Jordan-Von Neumann Theorem.*

1.1.6 Topological Space

Definition 1.6. If X is a nonempty set, a *topology* on X is a collection T of subsets of X which satisfies

i. $\varnothing \in T$

ii. $X \in T$

iii. $\bigcap\limits_{j=1}^{n} G_j \in T$ for any finite collection $G_1, \ldots, G_n$ of elements of T

iv. $\bigcup\limits_{\gamma \in \Gamma} G_\gamma \in T$ for any collection $\{G_\gamma | \gamma \in \Gamma\}$ of elements of T.

(X, T) is thus called a topological space.

A topological space (X, T) is said to be **Hausdorff** if every pair of distinct points $x, y \in X$ have disjoint neighbourhoods, i.e

$$U_x \cap U_y = \varnothing$$

where U_x and U_y are the neighbourhoods of x and y respectively.

A **topological vector space** (TVS) is a Hausdorff topological space for which the vector space operations of addition and scalar multiplication are continuous.

Definition 1.7 (Functional)**.** A scalar-valued function defined on a vector space X is called a functional.
The functional f is said to be *linear* provided

$$f(ax+by) = af(x) + bf(y) \;\; \forall x, y \in X \;\; \texttt{and} \;\; \forall a, b \in \mathbb{C}$$

Definition 1.8 (Continuous functions)**.** Let (X, T_x) and (Y, T_y) be two topological spaces. A function $f : X \to Y$ is said to be *continuous* if the preimage $f^{-1}(0) = \{x \in X : f(x) \in O\}$ belongs to T_x for every $O \in T_y$.
The stronger the topology on X or the weaker the topology on Y, the more such continuous functions f there will be.

1.1.7 The Dual Space

Definition 1.9. The set of all continuous, linear functionals on a topological vector space X is called the **dual** of X, denoted by X'.
X' is itself a vector space.

Definition 1.10 (Bounded Linear Functionals)**.** Let X be a normed linear space over the field K. Let $f : X \to K$ be a linear functional. Then f is said to *bounded* if there exists some non-negative constant real number M such that $\forall x \in X, \|f(x)\| \leq M\|x\|$.
M is called the bound for f.

A linear map $f : X \to Y$ is continuous if, and only if it is bounded - thus making continuity and boundedness equivalent as far as linear maps/functionals are concerned. The elements of the dual X' are actually bounded linear functionals, defined by

$$\|f\| = sup|f(x)| \;\; \forall f \in X', \forall x \in X.$$

Theorem 1.1 (Riesz Representation Theorem)**.** *Let H be a Hilbert space and f a bounded linear functional on H. Then there exists a unique vector $w \in H$ such that*

$$f(u) = < u, w > \quad \forall u \in H \qquad and \qquad \|f\| = \|w\|$$

This theorem shows that any bounded linear functional on a Hilbert space can be represented as an inner product with a unique vector in H.

Definition 1.11 (Operators)**.** Let X be a normed space. An *operator* f defined on X into a topological space Y is continuous if and only if

$$f(x_n) \to f(x) \quad \texttt{in } Y \quad \texttt{whenever} \quad x_n \to x \texttt{ in } X.$$

Definition 1.12 (Imbeddings)**.** We say the normed space X is imbedded in the normed space Y denoted by $X \to Y$ if

i. X is a vector subspace of Y

ii. the identity operator I defined on X into Y by $I_x = x \ \forall x \in X$ is continuous.

Definition 1.13 (Support)**.** Let $G \subset \mathbb{R}^n$ be a nonempty set. If u is a function defined on G, we define the *support* of u to be the set

$$supp(u) = \overline{\{x \in G : u(x) \neq 0\}}$$

We say that u has *compact support* in Ω if

i. $\overline{supp(u)} \subset \Omega$

ii. $\overline{supp(u)}$ is compact (closed and bounded)

where the bar refers to the closure. We shall write $G \Subset \Omega$ if $\overline{G} \subset \Omega$ and $\overline{G}$ is compact. Thus u has a compact support in Ω if $supp(u) \Subset \Omega$

1.2 Distributions and Weak Derivatives

Since the concept of weak derivatives is very essential in the study of Sobolev Spaces, a notation for this derivative is introduced here in this section. Other necessary notations are included.

Definition 1.14. A multiindex α is an n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ used to denote the partial derivative operator

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ is interpreted as the degree of α

Useful Notations Let Ω be a domain in $\mathbb{R}$. For any nonnegative integer m let $C^m(\Omega)$ denote the vector space consisting of functions u which, together with all their partial derivatives $D^\alpha u$ of orders $|\alpha| \leq m$, are continuous on Ω.
$C^0(\Omega) \equiv C(\Omega)$ and $C^\infty(\Omega)$ denote functions that are infinitely differentiable (functions that are differentiable are also continuous.
We represent $C_0(\Omega)$ and $C_0^\infty(\Omega)$ as functions in $C(\Omega)$ and C^∞ that have compact support in Ω.

Let $\phi \in C_0^\infty(\Omega)$, an infinitely differentaible function with compact support and $u \in C^m(\Omega)$. Then the weak derivative of $u \in C^m(\Omega)$ is defined by integrating against $\phi \in C_0^\infty(\Omega)$ by parts. Since u has a continuous derivative, we obtain

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = u\phi|_{\partial\Omega} - \int_\Omega \phi \frac{\partial u}{\partial x_i}\, dx = -\int_\Omega \phi \frac{\partial u}{\partial x_i}\, dx$$

We do this several times,and obtain finally;

$$\int_\Omega u D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega \phi D^\alpha u\, dx, \quad |\alpha| \leq m \tag{1.1}$$

where

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

This equation is valid if $u \in C^m(\Omega)$ for every $\phi \in C_0^\infty(\Omega)$.

Definition 1.15 (Test Functions)**.** Let Ω be a domain in $\mathbb{R}$. A sequence $\{\phi_j\}$ of functions belonging to $C_0^\infty(\Omega)$ is said to converge (in the sense of the space $\mathcal{D}(\Omega)$) to the function $\phi \in C_0^\infty(\Omega)$ provided the following are satisfied:

i. there exists $K \Subset \Omega$ such that *supp* $(\phi_j - \phi) \subset K$ for every j, and

ii. $\lim_{j\to\infty} D^\alpha \phi_j(x) = D^\alpha \phi(x)$ uniformly on K for each α.

Equipped with a locally convex topology, $C_0^\infty(\Omega)$ becomes a topological vector space called $\mathcal{D}(\Omega)$. The elements of $\mathcal{D}(\Omega)$ are called **test functions**.

Definition 1.16 (Schwartz Distributions)**.** The dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is called the space of *Schwartz distributions* on Ω.

Definition 1.17 (Locally Integrable Functions)**.** A function u defined almost everywhere on Ω is said to be *locally integrable* on Ω if $u \in L^1(U)$ for every open set $U \Subset \Omega$. We write $u \in L^1_{loc}(\Omega)$.
To every $u \in L^1_{loc}(\Omega)$ there corresponds a distribution $T_u \in \mathcal{D}'(\Omega)$ defined

$$T_u(\phi) = \int_\Omega u(x)\phi(x)\, dx, \quad \phi \in \mathcal{D}(\Omega)$$

1.2.1 Weak Derivative

Finally,let $u \in L^1_{loc}(\Omega)$. If there exists a locally integrable function v such that $T_v = D^\alpha T_u$ in $\mathcal{D}'(\Omega)$, then it is uniquely defined almost everywhere (i.e unique up to sets of measure zero) and equation (1.1) becomes

$$\int_\Omega u D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega \phi v\, dx$$

for all $\phi \in C_0^\infty(\Omega)$. We call $v = D^\alpha u$ the weak-αth partial derivative of u.

1.3 Lebesgue Measure and Integration

Since most of our results are restricted to the Euclidean space $\mathbb{R}$, the Lebesgue measure and other useful properties in *Measure and Integration* are introduced in this section.

1.3.1 Sigma Algebra

Definition 1.18. Let X be a nonempty set. A collection $\mathcal{M}$ of subsets of X is said to be σ-algebra if

i. $\varnothing \in \mathcal{M}$

ii. if $A \in \mathcal{M}$, then its complement $A^c \in \mathcal{M}$

iii. If $\{A_j\}_{j=1}^\infty \in \mathcal{M}$ then $\bigcup\limits_{j=1}^{\infty} \in \mathcal{M}$.

It follows that

i. If $\{A_j\}_{j=1}^\infty \in \mathcal{M}$ then $\bigcap\limits_{j=1}^{\infty} \in \mathcal{M}$

ii. If $A, B \in \mathcal{M}$, then $A - B \in \mathcal{M}$. (recall $A - B = A \cap B^c$)

$(X, \mathcal{M})$ is called a measurable space.

Definition 1.19 (Borel Set)**.** A Borel set,denoted by $B(\mathbb{R})$, is the smallest σ-algebra generated from the set of open sets in $\mathbb{R}$.

1.3.2 Measure

Let $\mathcal{M} \subset P(X)$ be a σ- algebra on a nonempty set X. A measure is a function

$$m : \mathcal{M} \to [0, \infty]$$

such that

i. $m(\varnothing) = 0$

ii. If $\{A_j\}_{j\geq 1} \subset \mathcal{M}$ and A_j are pairwise disjoint then

$$m\left(\bigcup_{j\geq 1} A_j\right) = \sum_{j\geq 1} m(A_j)$$

We call $(X, \mathcal{M}, m)$ a measure space. Any set in $\mathcal{M}$ satisfying these conditions is said to be measurable. If A is measurable, we write $A \in \mathcal{M}$.

Definition 1.20 (Outer measure)**.** Let $m^* : P(\mathbb{R}) \to [0, +\infty)$ and for any $A \subseteq \mathbb{R}$, we define

$$m^*(A) = inf \sum_{n=1}^{\infty} l(I_n)$$

as the outer measure of A.
where $\{I_n\}_{n=1}^{\infty}$ is a sequence of open intervals such that $A \subseteq \bigcup\limits_{n=1}^{\infty} I_n$.
We then have the following observations

i. $m^*(\varnothing) = 0$

ii. $m^*(A + x) = m^*(A)$: translation invariant

iii. $m^*(x) = 0$

iv. If $A \subset B$ then $m^*(A) \subseteq m^*(B)$

v. If I is any interval then $m^*(I) = l(I)$: length of I.

vi. If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of sets in $P(\mathbb{R})$ then

$$m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} l(A_n): \quad \texttt{countably subadditivity}$$

Definition 1.21. A set $A \in P(\mathbb{R})$ is said to be m^* - measurable if for every $E \in P(\mathbb{R})$

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$$

Definition 1.22 (Lebesgue measure)**.** We denote $m := m^*|\mathcal{M}$ as the outer measure restricted to measurable sets.
Thus, if $A \in \mathcal{M}$ then we write $m(A)$ instead of $m^*(A)$. We call m the *Lebesgue* measure having the following properties:

i. $m(I) = l(I)$, where I is any interval on $\mathbb{R}$.

ii. it is countably additive, i.e $m^*(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} l(A_n)$

iii. it is translation invariant

Also if $\{A_j\}_{j=1}^{\infty} \in \mathcal{M}$ such that $A_1 \subset A_2 \subset \ldots$ then

$$\bigcup_{j=1}^{\infty} A_j = \lim_{j \to \infty} m(A_j)$$

1.3.3 Measurable functions

Definition 1.23. Let A be a measurable set, let $f : A \to \overline{\mathbb{R}}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function, then f is Lebesgue measurable if the set $f > t := \{x \in A : f(x) > t\}$ $\forall t \in \mathbb{R}$ is measurable.

Proposition 1.1. *The following statements are equivalent:*

i. $\{f > t\} \in \mathcal{M}$

ii. $\{f \geq t\} \in \mathcal{M}$

iii. $\{f < t\} \in \mathcal{M}$

iv. $\{f \leq t\} \in \mathcal{M}$

Moreover $\{f = t\} \in \mathcal{M}$

Proposition 1.2. *The following are deduced from the definition of measurable functions*

i. If f is measurable,so is $|f|$

ii. If f and g are measurable and real-valued, so are $f + g$ and fg.

iii. f is measurable if, and only if $f^{-1}(I) := \{x \in D(f) : f(x) \in I\} \in \mathcal{M}$ is measurable for all open intervals I.

Theorem 1.2. *If f is a measurable function and $B \in B(\mathbb{R})$,a Borel set, then $f^{-1}(B)$ is a measurable set.*

Definition 1.24 (Borel function)**.** A function $f : A \to \mathbb{R}$ is called a Borel function if $A \in B(\mathbb{R})$ and $\{f > t\} \in \mathcal{M}$ $\forall t \in \mathbb{R}$.

Definition 1.25 (Characteristic function)**.** Let $A \subset \mathbb{R}^n$ be any given set. The characteristic function of A,denoted by $\mathbb{1}_A$ or χ_A, is defined by

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Definition 1.26 (Simple function)**.** A simple function f is a real-valued function with prescribed scalars $a_1, a_2, \ldots, a_k$ such that

$$s := \sum_{i=1}^{k} a_i \chi_{A_i}(x)$$

where $A_i \in \mathcal{M}$ $a_i \in \mathbb{R}$ and χ_{A_i} is the characteristic function.

s is measurable $\iff$ $A_i s$ are all measurable.

Definition 1.27 (Almost Everywhere)**.** A property is said to hold almost everywhere (a.e) if the set of points it fails is a set of measure zero.

Theorem 1.3 (Egoroff)**.** *Let $A \in \mathcal{M}$,$m(A) < +\infty$, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions and $f_n \to f$ pointwise, then $\exists E \in \mathcal{M}$ with $m(E) < +\infty$ such that $f_n \to f$ uniformly on $A|E$.*
That is, pointwise convergence (almost everywhere) on a set of finite measure implies uniform convergence on a slightly smaller set.

Theorem 1.4 (Lusin)**.** *Let $f : [a, b] \to \mathbb{R}$ be a finite-valued measurable function. Then $\forall \epsilon > 0 \; \exists A \subset [a, b]$,measurable, $m(A) < \epsilon$ such that f is continuous on $[a, b]|A$.*
Moreover $\forall \epsilon > 0 \; \exists$ a continuous function $g \in C^o([a, b])$ such that $g(x) = f(x) \; \forall x \in [a, b]|A$ or $m(\{g(x) \neq f(x)\}) < \epsilon$

Definition 1.28 (Pointwise and Uniform Convergence)**.** Let $D \subset \mathbb{R}$ and $\{f_n\}$ be a sequence defined on D.
We say $\{f_n\}$ converges *pointwise* to f on D if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in D$$

. However, we say $\{f_n\}$ converges *uniformly* to f on D if

$$\lim_{n\to\infty} \sup \{|f_n(x) - f(x)| : x \in D\} = 0$$

1.3.4 The Lebesgue Integral

If s is a simple function on a measurable set E with finite measure and $\{a_1, a_2, \ldots, a_n\}$ is the set of nonzero values of s, then $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ such that $A_i = \{x : s(x) = a_i\} \in \mathcal{M}$. This representation is called **canonical representation** if the $a_i s$ are distinct and not equal to zero and $A_i s$ are disjoint.

Definition 1.29. Let $A_i \subset E$, we define

$$\int_E s(x)\, dx = \sum_{i=1}^{n} a_i m(A_i)$$

Let s, p be simple functions on E. If f is a measurable nonnegative-valued function on E,we define

$$\int_E f(x)\, dx = \sup \int_E s(x)\, dx = \inf \int_E p(x)\, dx \;\; \texttt{s.t} \;\; s(x) \leq f(x) \leq p(x)$$

as the Lebesgue integral of f over E.

We say f is Lebesgue integrable on E if the integral is *finite*. The class of integrable functions on A is denoted by $L^1(A)$

Proposition 1.3. *Let s, p be simple functions on E and $c \in \mathbb{R}$. Then*

i. $\int\limits_E (s+p) = \int\limits_E s + \int\limits_E p$

ii. $\int\limits_E cs = c\int\limits_E s$

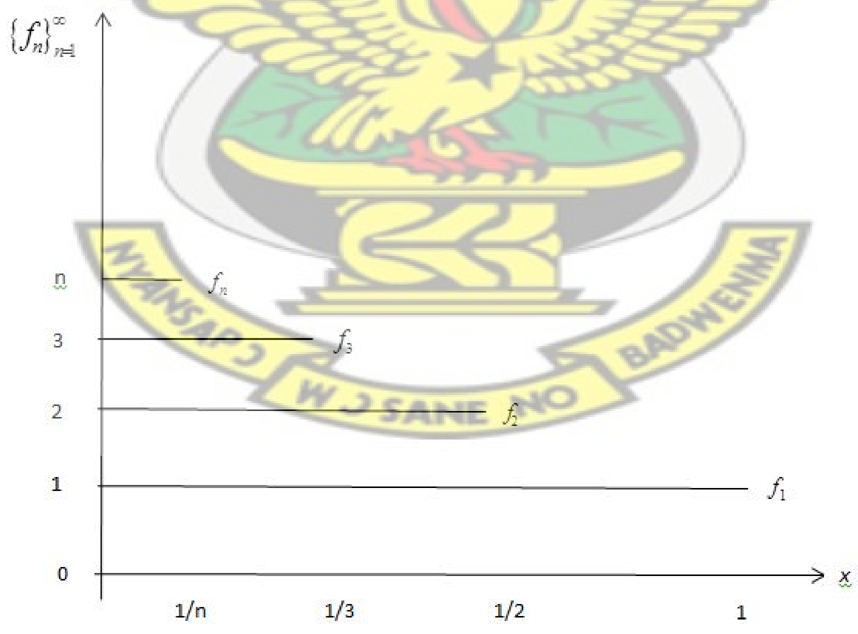
iii. If $s \leq p$ *then* $\int\limits_E s \leq \int\limits_E p$

1.3.5 Bounded Convergence Theorem

Theorem 1.5. *Let E be measurable, $m(E) < +\infty$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $f_n \to f$ pointwise a.e on E. Assume $\exists$ a positive real number M such that $|f| \leq M$ a.e on E. Then*

$$\lim_{n\to\infty} \int\limits_E f_n = \int\limits_E f$$

Example 1.2. Let $E = [0,1]$ and $f_n = n1_{[0,\frac{1}{n}]}$.



This diagram shows a sequence of functions (not uniformly bounded).

$$f_n \xrightarrow[pointwise]{a.e} f = \begin{cases} 0 & x \in \left(0, \frac{1}{n}\right] \\ \infty & x = 0 \end{cases}$$

$$\int_E f_n = \int n\mathbb{1}_{[0,1]} = n \cdot \frac{1}{n} = 1$$

$$\int_E f = \int_0^{\frac{1}{n}} f + \int_{\frac{1}{n}}^1 0 = 0 \texttt{ (for n very large)}$$

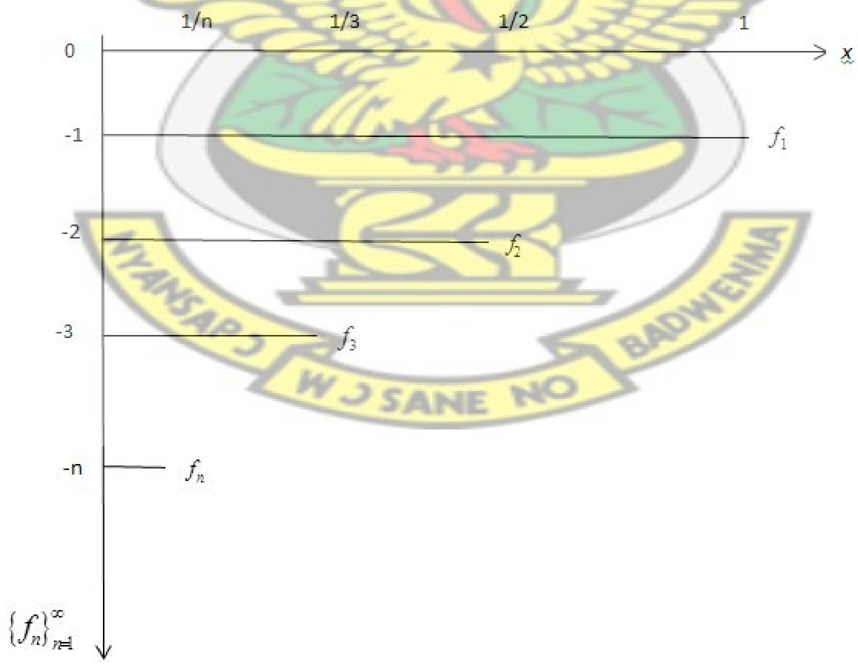
$$0 = \int_E f \neq \int_E f_n = 1$$

Hence the Bounded Convergence Theorem fails.
This is because $\{f_n\}_{n=1}^{\infty}$ is **NOT** uniformly bounded

Lemma 1.1 (Fatou's Lemma). *Let $f_n : E \to [0, +\infty]$, $E \in \mathcal{M}$ be a sequence of non-negative measurable functions and suppose $f_n \xrightarrow[pointwise]{a.e} f$ on E. Then*

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

Example 1.3. Let $E = [0, 1]$ and $f_n = n1_{[0,\frac{1}{n}]}$.



This produces a sequence of (negative) measurable functions (not uniformly bounded).

$$f_n \xrightarrow[pointwise]{a.e} f = \begin{cases} 0 & : x \in \left(0, \frac{1}{n}\right] \\ -\infty & : x = 0 \end{cases}$$

$$\int_E f_n = \int -n\mathbb{1}_{[0,\frac{1}{n}]} = \int_0^{\frac{1}{n}} dn = -n \times \frac{1}{n} = -1$$

$$\int_E f = \int_0^{\frac{1}{n}} f + \int_{\frac{1}{n}}^1 0 = 0 \texttt{ (for n very large)}$$

$$0 = \int_E f \not\leq \liminf_{n\to\infty} \int_E f_n = -1$$

Hence the Fatou's lemma fails since the given sequence is **NOT** non-negative.

Theorem 1.6 (Monotone Convergence Theorem)**.** *Let $f_n : E \to [0, +\infty]$, $E \in \mathcal{M}$ be a sequence of non-negative measurable functions and $f_n \leq f_{n+1}$ a.e and let $f_n \xrightarrow{pointwise} f$ a.e. Then*

$$\lim_{n\to\infty} \int_E f = \int_E \lim_{n\to\infty} f$$

Theorem 1.7 (Dominated Convergence Theorem)**.** *Let $f_n : E \to [0, +\infty]$, $E \in \mathcal{M}$ be a sequence of measurable functions on E such that $f_n \xrightarrow{pointwise} f$ a.e on E. If $\exists$ a Lebesgue integrable function g such that $|f_n(x)| \leq g(x)$ $\forall x \in E$, then*

$$\lim_{n\to\infty} \int_E f = \int_E \lim_{n\to\infty} f$$

Chapter 2

$L^p(\Omega)$ SPACES

2.1 Definition and Properties

2.1.1 The Space $L^p(\Omega)$

Definition 2.1. Let Ω, measurable, be a domain in $\mathbb{R}$ and let $p \in \mathbb{R}, 1 \leq p < \infty$. Then $L^p(\Omega)$ denotes the class of all measurable functions f on Ω for which

$$\int_\Omega |f(x)|^p \, dx < \infty \tag{2.1}$$

This means that for a function $f \in L^p(\Omega)$

i. f must be measurable and

ii. $\displaystyle\int_\Omega |f(x)|^p \, dx < \infty$

The space $L^p(\Omega)$ is a vector space and elements of $L^p(\Omega)$ are equivalence classes of measurable satisfying (2.1)

Lemma 2.1. *If $1 \leq p < \infty$ and $a, b \geq 0$, then*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p) \tag{2.2}$$

Proof If $p = 1$, then (2.2) is trivial. For $p > 1$, the function $\phi : x \to x^p$ is convex, that is $\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$ $\forall \lambda \in [0, 1]$.
For $\lambda = \frac{1}{2}$, we have

$$\begin{aligned}
\phi\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) &\leq \tfrac{1}{2}\phi(x) + \tfrac{1}{2}\phi(y) \\
\left(\tfrac{x}{2} + \tfrac{y}{2}\right)^p &\leq \tfrac{x^p}{2} + \tfrac{y^p}{2} \\
(x+y)^p &= 2^p\left(\tfrac{x^p}{2} + \tfrac{y^p}{2}\right)^p \leq 2^p\left(\tfrac{x^p}{2} + \tfrac{y^p}{2}\right) = 2^{p-1}(x^p + y^p)
\end{aligned}$$

This completes the proof.
If $f, g \in L^p(\Omega)$, then

$$|f(x)+g(x)|^p \leq (|f(x)|+|g(x)|)^p \leq 2^{p-1}\left(|f(x)|^p+|g(x)|^p\right)$$

confirming that $f+g \in L^p(\Omega)$ since $\displaystyle\int_\Omega |f(x)+g(x)|^p\,dx < \infty$.

$f = g$ a.e is an equivalence relation on $L^p(\Omega)$. By $f \in L^p(\Omega)$, we mean $[f] \in L^p(\Omega)$ such that

$$[f] := \{g \in L^p(\Omega) : f = g \quad a.e\}$$

2.1.2 L^p Norm

Definition 2.2. A norm on an $L^p(\Omega)$ space is a function $\|\cdot\|$ defined by

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}}$$

such that $1 \leq p < \infty$ and $f \in L^p(\Omega)$.

Unless otherwise started, we may write $\|f\|_p$ instead of $\|f\|_{L^p(\Omega)}$

Example 2.1. Let $f \in L^p(\Omega)$
N1:$\|f\|_p := \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \geq 0$ since $\displaystyle\int_\Omega |f(x)|^p\,dx < \infty$
Assume $\|f\|_p = 0 \Longrightarrow \displaystyle\int_\Omega |f(x)|^p\,dx = 0 \Rightarrow |f| = 0$ a.e then $f = 0$
Conversely if $f = 0$, then $\|f\|_p = 0$
N2: Let $f \in L^p(\Omega)$ and $c \in \mathbb{C}$

$$\|cf\|_p := \left(\int_\Omega |cf(x)|^p\,dx\right)^{\frac{1}{p}} = |c|\|f\|_p$$

N3: $f, g \in L^p(\Omega)$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p, \qquad \texttt{known as } \textit{Minkowki's inequality}$$

Hence $\|\cdot\|$ is a norm on $L^p(\Omega)$.

2.1.3 Cauchy's Inequality

Theorem 2.1. *If* $1 < p < \infty$, $1 < q < \infty$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

p and q are called the conjugate pair.

Proof For $p = q = 2$. Let $a, b \geq 0$
Then $(a-b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0$

$$\Rightarrow 2ab \leq a^2 + b^2$$

$$\therefore ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

For other values of p and q which are conjugate pair,
Let $\alpha = a^p, \beta = b^q$,we show that $\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} \leq \frac{\alpha}{p} + \frac{\beta}{q}$
Using the logarithm function which is concave, that is,

$$\phi\left(\lambda x + (1-\lambda)y\right) \geq \lambda\phi(x) + (1-\lambda)\phi(y) \quad \forall \lambda \in [0,1]$$

then for the case $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$\begin{aligned} \log\left(\tfrac{1}{p}\alpha + \tfrac{1}{q}\beta\right) &\geq \tfrac{1}{p}\log\alpha + \tfrac{1}{q}\log\beta \\ \log\alpha^{\frac{1}{p}} + \log\beta^{\frac{1}{q}} &\leq \log\left(\tfrac{\alpha}{p} + \tfrac{\beta}{q}\right) \\ \log\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} &\leq \log\left(\tfrac{\alpha}{p} + \tfrac{\beta}{q}\right) \end{aligned}$$

since the log function is increasing,

$$\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} \leq \frac{\alpha}{p} + \frac{\beta}{q} \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

2.1.4 Hölder's Inequality

Theorem 2.2. *If $1 < p < \infty$, $1 < q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \leq \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\,dx\right)^{\frac{1}{q}} \tag{2.3}$$

where $L^1(\Omega)$ is the space of Lebesgue integrable function.

Proof Recall Cauchy's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with equality occurring if and only if $a^p = b^q$.
If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $f(x)g(x) = 0$ a.e in Ω, and (2.3) is satisfied.
Otherwise substitute

$$a = \frac{|f(x)|}{\|f\|_p} \quad \texttt{and} \quad b = \frac{|g(x)|}{\|g\|_q}$$

in the above inequality and integrating over Ω, we have;

$$\begin{aligned} \int_\Omega \frac{|f(x)||g(x)|}{\|f\|_p\|g\|_q}\,dx &\leq \frac{1}{p}\int_\Omega \left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\int_\Omega \left(\frac{|g(x)|}{\|g\|_q}\right)^q \\ &= \frac{1}{p} + \frac{1}{q} = 1 \\ \int_\Omega |f(x)g(x)|\,dx &\leq \|f\|_p\|g\|_q \end{aligned}$$

Hence

$$\int_\Omega |f(x)g(x)|\,dx \leq \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\,dx\right)^{\frac{1}{q}}$$

Corollary 2.1. *If $p > 0, q > 0$ and $r > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then*

$$fg \in L^r(\Omega) \quad \textbf{and} \quad \|fg\|_r \leq \|f\|_p\|g\|_q$$

2.1.5 Minkowski's Inequality

Theorem 2.3. *If $1 \leq p < \infty$, then*

$$\|f+g\|_p \leq \|f\|_p + \|f\|_p \tag{2.4}$$

Proof For $p = 1$, the inequality holds since

$$\int_\Omega |f(x) + g(x)|\,dx \leq \int_\Omega |f(x)|\,dx + \int_\Omega |g(x)|\,dx$$

For $1 < p < \infty$, let $w \geq 0$ then $\|w\|_q \leq 1$. By Hölder's inequality;

$$\begin{aligned} \int_\Omega (|f(x)| + |g(x)|)\,w(x)\,dx &\leq \int_\Omega |f(x)|w(x)\,dx + \int_\Omega |g(x)|w(x)\,dx \\ &\leq \|f\|_p\|w\|_q + \|g\|_p\|w\|_q = (\|f\|_p + \|g\|_p)\,\|w\|_q \\ &\leq \|f\|_p + \|g\|_p \end{aligned}$$

It follows that

$$\|f+g\|_p \leq \|f\|_p + \|f\|_p$$

2.2 $L^\infty(\Omega)$ Space

Here, we look at the case $p = \infty$. The space $L^\infty(\Omega)$ are such that $\Omega \in \mathfrak{M}(\mathbb{R})$,which is, the space of all bounded measurable functions on Ω or the space of all measurable functions which are bounded except possibly on a subset of measure zero.

Definition 2.3. A function f that is measurable is said to be *essentially bounded* on Ω if there is a constant t such that $|f(x)| \leq t$ a.e on Ω.
The greatest lower bound of all such t is called the essential supremum of $|f|$ on Ω, and

is denoted by $ess\sup_\Omega |f(x)|$.
$L^\infty(\Omega)$ is the space of all measurable functions f that are essentially bounded on Ω; and functions equal a.e on Ω.

$$ess\sup_\Omega |f| = \inf\{t : f \leq t \texttt{ a.e in } \Omega\}$$

Proposition 2.1. *$L^\infty(\Omega)$ is a vector space*

Proof (a) Let $f \in L^\infty(\Omega)$, $c \in \mathbb{C}$,$c \neq 0$. Thus f is measurable and $ess\sup_\Omega |f(x)| < \infty$
Since af is measurable and

$$\begin{aligned} ess\sup_\Omega |af| &= \inf\{t : |cf| \leq t \texttt{ a.e in } \Omega\} \\ &= |c| \inf\{t : |f| \leq t' \texttt{ a.e in } \Omega\} \quad t' = t/c \\ &= |c| ess\sup_\Omega |f| \end{aligned}$$

$$\therefore cf \in L^\infty(\Omega)$$

(b) Let $f, g \in L^\infty(\Omega)$. Then $f + g$ is measurable and $ess\sup_\Omega |f| < \infty$, $ess\sup_\Omega |g| < \infty$
From the definition of essential supremum;

$$|f| \leq ess\sup_\Omega |f| \quad and \quad |g| \leq ess\sup_\Omega |g|$$

2.2.1 $L^\infty(\Omega)$ Norm

Definition 2.4. The functional $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = ess\sup_\Omega |f(x)|$$

is the norm on $L^\infty(\Omega)$.

Example 2.2. Let $f \in L^\infty(\Omega)$
N1:$\|f\|_\infty = ess\sup_\Omega |f(x)| \geq 0$ since $|f| \geq 0$
Assume $\|f\|_p = 0$, then $|f(x)| \leq 0$ a.e on $\Omega \Rightarrow |f(x)| = 0$ a.e $\Leftrightarrow f = 0$ a.e
Conversely if $f = 0$ a.e, then $\|f\|_\infty = ess\sup_\Omega |f(x)| = 0$
N2: Let $f \in L^\infty(\Omega)$ and $c \in \mathbb{C}$

$$\|cf\|_\infty = |c|\|f\|_\infty$$

N3: $f, g \in L^\infty(\Omega)$, then $f + g$ is measurable and $|f| \leq \|f\|_\infty$ a.e and $|g| \leq \|g\|_\infty$ a.e

$$|f + g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty < \infty \quad a.e$$

$$\|f + g\|_\infty - \epsilon < |f + g| \leq \|f\|_\infty + \|g\|_\infty < \infty \quad a.e \quad \forall\epsilon > 0$$

As $\epsilon \to 0$,

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

Hence $L^\infty(\Omega)$ is a normed linear space.

Proposition 2.2. *If $f \in L^\infty(\Omega)$ then $f \in L^p(\Omega)$ and*

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$$

2.3 The Completeness of $L^p(\Omega)$

Theorem 2.4. *$L^p(\Omega)$ is a Banach space if $1 \leq p \leq \infty$*

Proof First assume $1 \leq p < \infty$ and let $\{f_n\}$ be a Cauchy sequence in $L^p(\Omega)$. There is a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that

$$\|f_{n_{j+1}} - f_{n_j}\| \leq \frac{1}{2^j}, \quad j = 1, 2, \ldots$$

Let $g_m(x) = \sum_{j=1}^{m} |f_{n_{j+1}}(x) - f_{n_j}(x)|$. Then

$$\|g_m\| \leq \sum_{j=1}^{m} \frac{1}{2^j} < 1, \quad m = 1, 2, \ldots$$

Putting $g(x) = \lim_{m\to\infty} g_m(x)$, we obtain by Monotone Convergence Theorem;

$$\int_\Omega |g(x)|^p \, dx = \lim_{m\to\infty} \int_\Omega |g_m(x)|^p \, dx \leq 1.$$

Hence $g(x) < \infty$ a.e on Ω and the series

$$f_{n_1}(x) + \sum_{j=1}^{\infty} \left(f_{n_{j+1}}(x) - f_{n_j}(x)\right) \longrightarrow f(x) \tag{2.5}$$

a.e on Ω by Dominated Convergence Theorem.
Since (2.5) telescopes, we have

$$\lim_{m\to\infty} f_{n_m}(x) = f(x) \quad \texttt{a.e in } \Omega.$$

For any $\epsilon > 0$ there exists N such that if $n, m \geq N$, then $\|f_m - f_n\|_p < \epsilon$. Hence by Fatou's Lemma

$$\begin{aligned} \int_\Omega |f(x) - f_n(x)|^p \, dx &= \int_\Omega \lim_{j\to\infty} |f_{n_j}(x) - f_n(x)|^p \, dx \\ &\leq \liminf_{j\to\infty} \int_\Omega |f_{n_j}(x) - f_n(x)|^p \, dx \leq \epsilon^p \end{aligned}$$

if $n \geq N$. Hence $f = (f - f_n) + f_n \in L^p(\Omega)$ and $\|f - f_n\|_p \to 0$ as $n \to \infty$.
Therefore $L^p(\Omega)$ is complete and so, Banach.

For the case $p = \infty$. Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\Omega)$, then $\exists$ a set $A \subset \Omega$, $m(A) = 0$ such that if $x \notin A$, then for every $n, m = 1, 2, \ldots$

$$|f_n(x)| \leq \|f_n\|_\infty, \qquad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty.$$

Therefore,$\{f_n\}$ converges uniformly on $\Omega|A$ to a bounded function f. Thus $f \in L^\infty(\Omega)$ and $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.
Hence $L^\infty(\Omega)$ is also complete and a Banach space.

Corollary 2.2. *$L^2(\Omega)$ is the Hilbert space with respect to the inner product*

$$< f, g >= \int_\Omega f(x)\overline{g(x)} \, dx.$$

Hölder's inequality for $L^2(\Omega)$ is simply the Cauchy-Schwartz inequality

$$| < f, g > | \leq \|f\|_2 \|g\|_2.$$

Definition 2.5 (Mollifiers)**.** Let J be a nonnegative, real-valued function such that $J \in C_0^\infty(\mathbb{R}^n)$ and

i. $J(x) = 0$ if $|x| \geq 1$, and

ii. $\int_{\mathbb{R}^n} J(x) \, dx = 1$

We shall take for any $k > 0$,

$$J(x) = \begin{cases} k \exp\left(\frac{-1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

If $\epsilon > 0$, the function

$$J_\epsilon(x) = \epsilon^{-n} J(x/\epsilon)$$

satisfying the properties above, is called a *mollifier*. The convolution

$$J_\epsilon * u(x) = \int_{\mathbb{R}^n} J_\epsilon(x-y)u(y)\,dy$$

is called a *mollification* of u.

Theorem 2.5 (Properties of Mollification)**.** *Let u be a function which is defined on $\mathbb{R}^n$ and vanishes identically outside Ω.*

(a) If $u \in L^1_{loc}(\mathbb{R}^n)$, then $J_\epsilon * u \in C^\infty(\mathbb{R}^n)$.

(b) If $u \in L^1_{loc}(\Omega)$ and $supp(u) \Subset \Omega$, then $J_\epsilon * u \in C_0^\infty(\Omega)$ provided

$$\epsilon < dist\,(supp(u), bdry(\Omega))$$

(c) If $u \in L^p(\Omega)$ where $1 \leq p < \infty$, then $J_\epsilon * u \in L^p(\Omega)$. Also

$$\|J_\epsilon * u\|_p \leq \|u\|_p \quad \textit{and} \quad \lim_{\epsilon \to 0+} \|J_\epsilon * u - u\|_p = 0$$

(d) If $u \in C(\Omega)$ and if $G \Subset \Omega$, then $\lim_{\epsilon \to 0+} J_\epsilon * u(x) = u(x)$ uniformly on G.

(e) If $u \in C(\overline{\Omega})$, then $\lim_{\epsilon \to 0+} J_\epsilon * u(x) = u(x)$ uniformly on Ω.

Chapter 3

THE SOBOLEV SPACE $W^{m,p}(\Omega)$

Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p-norms of the function itself as well as its derivatives up to a given order.

Definition 3.1 (The Sobolev Norms). For any function u and m a positive integer, we define a functional $\|\cdot\|_{m,p}$ as

$$\begin{aligned} \|u\|_{m,p} &= \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} \qquad \texttt{if} \quad 1 \le p < \infty \\ \|u\|_{m,\infty} &= \max_{0\le|\alpha|\le m} \|D^\alpha u\|_\infty \end{aligned}$$

where $\|\cdot\|_p$ is the norm in $L^p(\Omega)$. Equivalently $\|u\|_{m,p} = \|u\|_p + \sum_{0\le|\alpha|\le m} \|D^\alpha u\|_p$.

Definition 3.2 (Sobolev Spaces). Let m any positive integer and $1 \le p \le \infty$

(a) $H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$ *wrt* the norm $\|\cdot\|_{m,p}$

(b) $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \texttt{ for } 0 \le |\alpha| \le m\}$

(c) $W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ where $C_0^\infty(\Omega)$ is the vector space of infinitely differentiable functions with compact support.

(d) $u \in W^{m,p}(\Omega)$ if, and only, if

$$\int_\Omega u D^\alpha(\phi)\, dx = (-1)^{|\alpha|} \int_\Omega \phi D^\alpha u\, dx \qquad \forall \phi \in C_0^\infty(\Omega)$$

Specifically, the Sobolev Space consists of all functions $u : \Omega \to \mathbb{R}, u \in L^p(\Omega)$ such that each weak derivative $D^\alpha u$ for $|\alpha| \le m$ exists and belongs to $L^p(\Omega)$.

Remark

i. $W^{0,p}(\Omega) = L^p(\Omega)$

ii. If $1 \leq p < \infty$, $W_0^{0,p}(\Omega) = L^p(\Omega)$ because $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$

iii. For any m, the following chain of imbeddings hold

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega)$$

iv. For each $\lambda, \mu \in \mathbb{R}$ and $u, v \in W^{m,p}(\Omega)$, $\lambda u + \mu v \in W^{m,p}(\Omega)$ and $D^\alpha(u+v) = D^\alpha u + D^\alpha v$. Thus $W^{m,p}(\Omega)$ is a vector space over $\mathbb{R}$

Theorem 3.1. *$W^{m,p}(\Omega)$ is a Banach Space.*

Proof Let u_n be a Cauchy sequence in $W^{m,p}(\Omega)$. Then since the Sobolev norm is defined in terms of the norm on $L^p(\Omega)$, $D^\alpha u_n$ is a Cauchy sequence in $L^p(\Omega)$ for $|\alpha| \leq m$. Since $L^p(\Omega)$ is complete there exist functions u such that $u_n \to u$ and u_α such that $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$ as $n \to \infty$. $L^p(\Omega) \subset L^1_{loc}(\Omega)$ and so to every $u_n \in L^1_{loc}(\Omega)$, there is a distribution $T_{u_n} \in \mathcal{D}'(\Omega)$ defined by

$$\begin{aligned} |T_{u_n}(\phi) - T_u(\phi)| &\leq \int_\Omega |u_n(x) - u(x)||\phi(x)|\, dx \qquad \phi \in \mathcal{D}(\Omega) \\ &\leq \|\phi\|_q \|u_n - n\|_p \quad \texttt{by Holder's inequality} \end{aligned}$$

where (p, q) are the exponent conjugate pair.
As $n \to \infty$ $T_{u_n}(\phi) \to T_u(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$. Similarly $T_{D^\alpha u_n}(\phi) \to T_{u_\alpha}(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$.

$$\Rightarrow T_{u_\alpha}(\phi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\phi) = \lim_{n\to\infty} (-1)^{|\alpha|} T_{u_n}(D^\alpha \phi) = (-1)^{|\alpha|} T_u(D^\alpha \phi)$$

Thus $u_\alpha = D^\alpha u$ in the weak (distribution) sense on Ω for $|\alpha| \leq m$; from which we get $u \in W^{m,p}(\Omega)$. Since $\lim_{n\to\infty} \|u_n - u\|_{m,p} = 0$, the space $W^{m,p}(\Omega)$ is complete.

Corollary 3.1. *The space $H^1(\Omega) = W^{1,2}(\Omega)$ is a Hilbert Space with inner product*

$$\langle u, v \rangle = \int_\Omega uv\, dx$$

Corollary 3.2. *In general $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$*

3.1 Approximation by Smooth Functions on Ω

In this section we wish to establish the fact that any element in $W^{m,P}$ can be approximated by functions smooth on Ω. In other words the set $\{\phi \in C^\infty(\Omega) : \|\phi\|_{m,p} < \infty\}$ is dense in

$W^{m,p}(\Omega)$. We require the notion of infinitely differentiable partitions of unity. We then this theorem *without proof.*

Theorem 3.2 (Partitions of Unity). *Let A be an arbitrary subset of $\mathbb{R}$ and let $\mathcal{O}$ be a collection of open sets in $\mathbb{R}^n$ which cover A, i.e $A \subset \bigcup_{U\in\mathcal{O}} U$. Then there exists a collection Ψ of functions $\Psi \in C_0^\infty(\mathbb{R}^n)$ having the following properties:*

(i) For every $\psi \in \Psi$ and every $x \in \mathbb{R}^n$, $0 \le \psi(x) \le 1$.

(ii) If $K \Subset A$, all but finitely many $\psi \in \Psi$ vanishes identically on K.

(iii) For every $\psi \in \Psi$ there exists $U \in \mathcal{O}$ such that $supp(\psi) \subset U$.

(iv) For every $x \in A$, we have $\sum_{\psi\in\Psi} \psi(x) = 1$.

Such a collection Ψ is called a C^∞ - partition of unity for A subordinate to $\mathcal{O}$.

Lemma 3.1 (Mollification in $W^{m,p}(\Omega)$). *Recall the definition of a mollifier "for any $k > 0$,*

$$J(x) = \begin{cases} k\exp\left(\frac{-1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \ge 1 \end{cases}$$

$$J_\epsilon(x) = \epsilon^{-n} J(x/\epsilon) \quad \epsilon > 0\text{"}$$

Let J_ϵ be so defined and let $1 \le p < \infty$ and $u \in W^{m,p}(\Omega)$. If Ω' is a subdomain with compact closure in Ω, then

$$\lim_{\epsilon\to 0+} J_\epsilon * u = u \qquad \textbf{\textit{in}}\ W^{m,p}(\Omega')$$

Thus,

$$J_\epsilon * u \to u \quad in \quad W^{m,p}(\Omega')$$

Proof Let $\epsilon < d(\Omega', \partial\Omega))$ and $\tilde{u}$ be the zero extension of u outside Ω. That is, $\tilde{u}(x) = u(x)$ if $x \in \Omega$ and 0 otherwise. If $\phi \in \mathcal{D}(\Omega')$,

$$\begin{aligned}\int_{\Omega'} J_\epsilon * u(x) D^\alpha \phi(x)\,dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{u}(x-y) J_\epsilon(y) D^\alpha\phi(x)\,dx\,dy \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n}\int_{\Omega'} D_x^\alpha u(x-y) J_\epsilon(y)\,dx\,dy \\ &= (-1)^{|\alpha|}\int_{\Omega'} J_\epsilon * D^\alpha u(x)\phi(x)\,dx\end{aligned}$$

Thus, the derivative of the mollification is equal to the mollification of the derivative,i.e

$$D^\alpha J_\epsilon * u = J_\epsilon D^\alpha * u$$

in the distributional sense in Ω'. Since $D^\alpha u \in L^p(\Omega)$ for $0 \le |\alpha| \le m$ we have by property (3) of mollifications:

$$\lim_{\epsilon\to 0+} \|D^\alpha J_\epsilon * u - D^\alpha u\|_{p,\Omega'} = \lim_{\epsilon\to 0+} \|J_\epsilon * D^\alpha u - D^\alpha u\|_{p,\Omega'} = 0$$

It follows that $\lim_{\epsilon\to 0+} \|J_\epsilon u - u\|_{m.p,\Omega'} = 0$

Theorem 3.3. *If $1 \le p < \infty$, then*

$$H^{m,p}(\Omega) = W^{m,p}(\Omega)$$

Proof By Corollary 3.2, $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$. We show that $W^{m,p}(\Omega) \subset H^{m,p}(\Omega)$, that is, the set $\{\phi \in C^m(\Omega) : \|\phi\|_{m,p} < \infty\}$ is dense in $W^{m,p}(\Omega)$. It suffices to show that for every $u \in W^{m,p}(\Omega)$ and $\epsilon > 0$, there exists $\phi \in C^\infty(\Omega)$ such that $\|\phi - u\|_{m,p} < \epsilon$. For $k = 1, 2, \ldots$ define

$$\Omega_k = \{x \in \Omega : |x| < k \ \texttt{and} \ dist(x, bdry\Omega) > 1/k\}$$

and let $\Omega_0 = \Omega_{-1} = \emptyset$. Then

$$\mathcal{O} = \left\{U_k : U_k = \Omega_{k+1} \cap \left(\overline{\Omega_{k-1}}\right)^c, k = 1, 2, \ldots\right\}$$

is a collection of open subsets of Ω that covers Ω. Let Ψ be a C^∞ - partitions of unity for Ω subordinate to $\mathcal{O}$. Let $\psi_k \in C_0^\infty(U_k)$ and $\sum_{k=1}^\infty \psi_k(x) = 1$ on Ω.
If $0 < \epsilon < 1/(k+1)(k+2)$, then $J_\epsilon * (\psi_k u)$ has support in the intersection $V_k = \Omega_{k+2} \cap (\Omega_{k-2})^c \Subset \Omega$. Since $\psi_k u \in W^{m,p}(\Omega)$ choose ϵ_k with $0 < \epsilon_k < \frac{1}{(k+1)(k+2)}$, such that

$$\|J_{\epsilon_k} * (\psi_k u) - \psi_k u\|_{m,p,\Omega} = \|J_{\epsilon_k} * (\psi_k u) - \psi_k u\|_{m,p,V_k} < \epsilon/2^k$$

Let $\phi = \displaystyle\sum_{k=1}^\infty J_{\epsilon_k} * (\psi_k u)$. On any $\Omega' \Subset \Omega$ only finitely many terms in the sum can be nonzero. Thus $\phi \in C^\infty(\Omega)$. For $x \in \Omega_k$, we have

$$u(x) = \sum_{j=1}^{k+2} \psi_j(x) u(x), \qquad \texttt{and} \qquad \phi(x) = \sum_{j=1}^{k+2} J_{\epsilon_j} * (\psi_j u)(x).$$

Thus

$$\|u - \phi\|_{m,p,\Omega_k} \le \sum_{j=1}^{k+2} \|J_{\epsilon_j} * (\psi_j u) - \psi_j u\|_{m,p,\Omega} < \epsilon$$

Hence by MCT $\|u - \phi\|_{m,p,\Omega} < \epsilon$.

3.2 Approximation by Smooth Functions on $\mathbb{R}^n$

Having shown that any $u \in W^{m,p}$ can be approximated by smooth functions on Ω, we proceed to check if they can be approximated by functions smooth on $\mathbb{R}^n$.

Lemma 3.2. *Given $u \in W^{m,p}(\Omega)$ and $v \in C_0^\infty(\Omega)$, then $uv \in W^{m,p}(\Omega)$*

Proof To show that uv is in the Sobolev Space, we have to compute its weak derivative and have

$$\int_\Omega uvD^\alpha(\phi)\,dx = (-1)^{|\alpha|}\int_\Omega \phi D^\alpha(uv)\,dx$$

for any $\phi \in C_0^\infty(\Omega)$. Note that

$$D^\alpha(uv) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta} D^\beta v D^{\alpha-\beta}u.$$

We prove by induction. Fix $\phi \in C_0^\infty(\Omega)$. Then for $|\alpha| = 1$,

$$\begin{aligned}\int_\Omega uvD^\alpha(\phi)\,dx &= \int_\Omega uD(v\phi) - u\phi D^\alpha v\,dx\\ &= -\int_\Omega (uD^\alpha v + vD^\alpha u)\,\phi\,dx\end{aligned}$$

Since $D^\alpha u$ and $D^\alpha v$ exists for any $u \in W^{m,p}(\Omega)$ and $v \in C_0^\infty(\Omega)$, the above relation is true.

Assume the relations holds for $|\alpha| \leq l$ for any $l < m$ and all $v \in C_0^\infty(\Omega)$. Choose α such that $|\alpha| = l+1$. Then we write $|\alpha| = |\beta| + |\gamma|$ where $|\beta| = l$ and $|\gamma| = 1$. Then for any $v, \phi \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega uvD^\alpha(\phi)\,dx &= \int_\Omega uvD^\beta\left(D^\gamma\phi\right)\,dx\\ &= (-1)^{|\beta|}\int_\Omega\sum_{\sigma\leq\beta}\binom{\beta}{\sigma} D^\sigma v D^{\beta-\sigma}uD^\gamma\phi\,dx\\ &= (-1)^{|\beta|+|\gamma|}\int_\Omega\sum_{\sigma\leq\gamma}\binom{\beta}{\sigma} D^\gamma\left(D^\sigma v D^{\beta-\sigma}u\right)\phi\,dx\\ &= (-1)^{|\alpha|}\int_\Omega\sum_{\sigma\leq\alpha}\binom{\alpha}{\sigma} D^\sigma v D^{\alpha-\sigma}(u)\phi\,dx\\ &= (-1)^{|\alpha|}\int_\Omega \phi D^\alpha(uv)\,dx\end{aligned}$$

Theorem 3.4. *Assume $\Omega \subset \mathbb{R}^n$ is bounded and that $u \in W^{m,p}(\Omega)$ for $1 < p < \infty$.Then there exists functions $u_m \in C_0^\infty(\Omega)$ such that*

$$u_m \to u \quad \textit{in } W^{m,p}(\Omega)$$

Proof Decompose Ω into the following sets;

$$\Omega_i := \{x \in \Omega \mid d(x, \partial\Omega) > 1/i\}, \quad i \in \mathbb{N}\backslash 0$$

and define $V_i = \Omega_{i+3} - \bar{\Omega}_{i+1}$. Choose any $V_0 \subset\subset \Omega$ such that $U = \bigcup_{i=0}^{\infty} V_i$. Let $\phi_{i=0}^{\infty}(V_i)$ can be defined as a partition of unity(Theorem 3.2) for U such that

$$\begin{cases} 0 \leq \phi_i \leq 1 \quad \phi_i \in C_0^{\infty}(\Omega) \\ \sum_{i=0}^{\infty} \phi_i = 1 \quad on\ \Omega \end{cases}$$

By Lemma 3.2, we know that for each i, $\phi_i u \in W^{m,p}(\Omega)$ and $\phi_i u$ has support contained in V_i. We then proceed using the mollification concept to show that the sequence $u^i \in C_0^{\infty}(\Omega)$ is dense in $\phi_i u \in W^{m,p}(\Omega)$. Let J_ϵ be our mollifier function and choose $\delta > 0$. Then given $\epsilon_i > 0$ such that $u^i := J_i * (\phi_i u)$, the following relations are satisfied. For $W_i := \Omega_{i+4} - \bar{\Omega}_i$, $i = 1, 2, \ldots$

$$\begin{cases} \|u^i - \phi u\|_{m,p,\Omega} \leq \frac{\delta}{2^{i+1}} \quad i = 0, 1, 2, \ldots \\ supp(u^i) \subset W_i \quad i = 1, 2, \ldots \end{cases}$$

Define $v(x) := \sum_{i=0}^{\infty} u^i(x) \in C_0^{\infty}(\Omega)$ since $u^i \in C_0^{\infty}(\Omega)$ and $u^i(x)$ can be nonzero for only a finite number of i. Let $u = \sum_{i=0}^{\infty} \phi u$, we have

$$\begin{aligned} \|v - u\| &\leq \sum_{i=0}^{\infty} \|u^i - \phi u\|_{m,p,\Omega} \\ &\leq \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} \\ &= \delta \end{aligned}$$

for each open set $V \subset\subset U$.
Taking the supremum of all such V gives $\|v - u\|_{m,p,\Omega} \leq \delta$. Thus $C_0^{\infty}(\Omega)$ is a dense subset of $W^{m,p}(\Omega)$.

Chapter 4

APPLICATION TO DIFFERENTIAL EQUATION

The aim of this thesis was to explore the properties of Sobolev Spaces and apply them to solving differential equations. In this chapter, certain examples of differential equations (e.g. boundary value problems) are provided.

4.1 Some Examples of Boundary Value Problems

Let $\Omega = [a, b]$ and given any $f \in C(\Omega)$. Consider the boundary value problem

$$\begin{cases} -u'' + u & = f \quad in\ \Omega \\ \quad u & = 0 \quad on\ \partial\Omega \end{cases} \tag{4.1}$$

We wish to use Sobolev spaces to find a weak solution that satisfies this pde. The following steps are adapted in approaching any pde using the concept of Sobolev space.

A. Defining the weak solutions.

B. Establishing the *existence and uniqueness* of a weak solution.

C. Regularity of a weak solution.

D. Recovery of the classical solution.

Note. A *classical solution* is a function $u \in C^2(\Omega)$ that satisfies 4.1 in the usual sense. The concept of weak derivatives is used here to define a weak solution of 4.1 by multiplying by $\phi \in C^1(\Omega)$ and integrating by parts.i.e

$$\phi f = -\phi u'' + \phi u$$

$$\int_\Omega f\phi = -\phi u'|_\Omega + \int_\Omega u'\phi' + \int_\Omega u\phi = \int_\Omega u'\phi' + \int_\Omega u\phi \quad since\ \phi = 0\ on\ \partial\Omega$$

Thus a *weak solution* of 4.1 is a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega f\phi = \int_\Omega u'\phi' + \int_\Omega u\phi \qquad \phi \in H_0^1(\Omega) \tag{4.2}$$

Moreover, given any $f \in L^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ to equation 4.2. We obtain u by

$$\min_{\phi \in H_0^1} \left\{ \frac{1}{2} \int_\Omega (\phi'^2 + \phi^2) - \int_\Omega f\phi \right\}$$

We call this *the Dirichlet's Principle*

We state the Lax - Milgram Theorem, which is key in establishing the existence and uniqueness of a weak solution.

Theorem 4.1 (Lax-Milgram). *Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ be a bilinear mapping satisfying the following inequalities*

i. there exists $\alpha > 0$ such that $|B(u,v)| \leq \alpha \|u\| \|v\|$ for all $u, v \in H$,

ii. there exists $\beta > 0$ such that $\beta \|u\|^2 \leq B(u,u)$ for all $u \in H$.

Then if $f : H \to \mathbb{R}$ is a bounded linear functional on H, there exists a unique element $u \in H$ such that

$$B(u,v) = \langle f, v \rangle \qquad \forall v \in H$$

Proof For any $u \in H$, the mapping $v \mapsto B(u,v)$ is a bounded linear functional on H. Then by Riesz Representation Theorem, there exists a unique element $w \in H$ such that for any $u \in H$

$$B(u,v) = \langle w, v \rangle \quad \forall v \in H$$

Define a mapping $A : H \to H$ by $u \mapsto w$, where w fits the above definition. Thus $B(u,v) = \langle Au, v \rangle \quad v \in H$. We have

(i) A is linear. For any $t, u, v \in H$ and $\lambda \in \mathbb{R}$;

$$\langle A(\lambda t + u), v \rangle = B(\lambda t + u, v) = \lambda B(t,v) + B(u,v) = \langle \lambda At, v \rangle + \langle A, v \rangle = \langle \lambda At + Au, v \rangle$$

(ii) A is bounded. We apply the first inequality;

$$\|Au\|^2 = \langle Au, Au \rangle = B(u, Au) \leq \alpha \|u\| \|Au\|$$

. Hence, $\|Au\| \leq \alpha \|u\|$ for all $u \in H$.

(iii) A is bijective. By the second inequality;

$$\beta\|u\|^2 \leq B(u,u) = \langle Au, u\rangle \leq \|Au\|\|u\|.$$

Hence $\beta\|u\| \leq \|Au\|$ which holds *iff* $Au = 0 \Rightarrow u = 0$. Thus A is injective. By definition, A is onto since to every $w \in H$ there corresponds an element $u \in H$ such that $Au = w$.

Since A is a bounded linear functional, we can apply the Riesz Representation Theorem and obtain

$$B(u,v) = \langle Au, v\rangle = \langle w, v\rangle = \langle f, v\rangle$$

To show that u is unique, suppose both $t, u \in H$ satisfy the above equation. Then

$$\begin{aligned} B(u,v) &= \langle f, v\rangle = B(t,v) \\ \langle Au, v\rangle &= \langle At, v\rangle \end{aligned}$$

So, $Au = At$ and since A is bijective, we have $u = t$.

In STEPS C and D, note that if $f \in L^2$ and $u \in H_0^1$ is the weak solution of (4.1). Then from 4.2;

$$\int_\Omega u'\phi' = \int_\Omega (f-u)\phi \qquad \phi \in C_0^1(\Omega)$$

and from the definition of Sobolev Spaces, $u' \in H^1$ for $f - u \in L^2$. Thus $u \in H^2$ and finally, if $f \in C(\Omega)$, we have the weak solution $u \in C^2(\Omega)$.

Example 4.1. Let $\Omega = (0,1)$. Consider the problem

$$\begin{cases} -(pu')' + qu &= f \quad in\ \Omega \\ \qquad u &= 0 \quad on\ \partial\Omega \end{cases} \tag{4.3}$$

where $p \in C^1(\bar{\Omega})$, $q \in C(\bar{\Omega})$, and $f \in L^2(\Omega)$ with $p, q \geq 0$ on Ω

STEP A: Every classical solution is a weak solution.

If u is the classical solution of (4.3), we have

$$\int_\Omega (pu')\phi' + \int_\Omega (qu)\phi = \int_\Omega f\phi \qquad \phi \in H_0^1(\Omega)$$

This is achieved by multiplying through by $\phi \in H_0^1(\Omega)$ and integrating by parts. Thus any u that satisfies this equation is a weak solution of (4.3).

STEP B: Establish the existence and uniqueness of a weak solution.

Define a symmetric continuous bilinear form on $H_0^1(\Omega)$ such that

$$B(u,\phi) = \int_\Omega (pu')\phi' + \int_\Omega (qu)\phi$$

Then by Lax-Milgram's Theorem, there exists a unique element $u \in H_0^1(\Omega)$ such that

$$B(u, \phi) = \langle f, \phi \rangle = \int_\Omega f\phi \quad \phi \in H_0^1(\Omega)$$

and obtain u by the Dirichlet's principle

$$\min_{\phi \in H_0^1} \left\{ \frac{1}{2} \int_\Omega (p\phi'^2 + q\phi^2) - \int_\Omega f\phi \right\}$$

STEP C: The weak solution is proved to be of class C^2.
We have;

$$\int_\Omega (pu')\phi' = \int_\Omega (f - qu)\phi \quad \phi \in H_0^1(\Omega)$$

Thus $pu' \in H^1$, and so $u' \in H \Rightarrow u \in H^2$. Also, if $f \in C(\bar{\Omega})$ then $pu' \in C^1(\bar{\Omega}) \Rightarrow u' \in C^1(\bar{\Omega})$, and so $u \in C^2(\bar{\Omega})$.
STEP D: Show that any weak solution that is C^2 is a classical solution.

$$\begin{aligned}
\int_\Omega f\phi &= \int_\Omega (pu')\phi' + \int_\Omega (qu)\phi \qquad \phi \in C^1(\bar{\Omega}) \\
\int_\Omega f\phi &= (pu')\phi|_\Omega - \int_\Omega (pu')'\phi + \int_\Omega (qu)\phi \\
\int_\Omega f\phi &= -\int_\Omega (pu')'\phi + \int_\Omega (qu)\phi \qquad \phi = 0 \ on\ \partial\Omega
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= \int_\Omega \left(-(pu')' + qu - f\right)\phi \\
\Rightarrow \quad & -(pu')' + qu - f = 0 \quad a.e \\
& -(pu')' + qu = f \quad a.e\ on\ \Omega \backslash \partial\Omega
\end{aligned}$$

Finally

$$\Rightarrow -(pu')' + qu = f \quad on\ \Omega$$

Hence u is a classical solution of (4.3).

4.2 Elliptic PDE of Second Order

Example 4.2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Consider the Dirichlet problem for the Laplacian

$$\begin{cases} -\Delta u + u &= f \quad in\ \Omega \\ \quad u &= 0 \quad on\ \partial\Omega \end{cases} \tag{4.4}$$

where $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u.

STEP A: Every classical solution is a weak solution.

Note that a weak solution of (4.4) is a function $u \in H_0^1(\Omega)$ which satisfies

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi = \int_\Omega f\phi \quad \phi \in H_0^1(\Omega)$$

where $\nabla u \cdot \nabla \phi = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i}$

If $u \in C^2(\bar{\Omega})$ is a classical solution of (4.4) then $u \in H_0^1(\Omega)$ is its weak solution since $C^2(\bar{\Omega})$ is dense in $H_0^1(\Omega)$ and by the same results in the earlier example.

STEP B: Establish the existence and uniqueness of a weak solution.

Theorem 4.2. *Given any $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H_0^1(\Omega)$ of (4.4)*

Proof Let

$$B(u, \phi) = \int_\Omega (\nabla u \cdot \nabla \phi + u\phi)$$

be a symmetric continuous bilinear form on $H_0^1(\Omega)$ and the mapping $\phi \mapsto \int_\Omega f\phi$ be a bounded linear functional on $H_0^1(\Omega)$. Then by the Lax-Milgram's Theorem, there exist a unique element $u \in H_0^1(\Omega)$ such that

$$B(u, \phi) = \int_\Omega (\nabla u \cdot \nabla \phi + u\phi) = \int_\Omega f\phi$$

which satisfies the definition of a weak solution of (4.4). Similarly u is obtained by the Dirichlet's principle

$$\min_{\phi \in H_0^1} \left\{ \frac{1}{2} \int_\Omega (|\Delta \phi|^2 + |\phi|^2) - \int_\Omega f\phi \right\}$$

STEP C: Regularity of the weak solution. If $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ then $\forall \phi \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi = \int_\Omega f\phi$$
$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega (f - u)\phi$$

Then $u \in H^2$.

STEP D: Show that any weak solution that is C^2 is a classical solution. Let

$u \in H_0^1(\Omega)$ be a weak solution of (4.4).

$$\int_\Omega f\phi = \int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi$$
$$\int_\Omega f\phi = \int_\Omega (-\Delta u + u)\,\phi$$

$$0 = \int_\Omega (-\Delta u + u - f)\,\phi$$
$$\Rightarrow \quad -\Delta u + u - f = 0 \quad a.e$$

Thus $-\Delta u + u = f \quad on\ \Omega$. Hence $u \in C^2(\bar{\Omega})$ is a classical solution of (4.4)

Chapter 5

SUMMARY,CONCLUSIONS AND RECOMMENDATIONS

The main point of this work was to explore the Sobolev Spaces and it's involvement in solving differential equations. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces, rather than in spaces of continuous functions and with the derivatives understood in the classical sense.
Consider this boundary value problem with $\Omega = (0, 1)$

$$\begin{cases} -u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \\ u(0) = u(1) = 0 \end{cases} \tag{5.1}$$

Let b, c and f be given continuous functions. Assume that a classical solution exists, i.e. a twice continuously differentiable function u satisfying (5.1). Then for an arbitrary function v we have

$$\int_\Omega (-u'' + bu' + cu)\, v\, dx = \int_\Omega f v\, dx \tag{5.2}$$

If $v \in C^1(\Omega)$, then by integrating 5.2 by parts, we obtain

$$-uv'|x = 0^1 + \int_\Omega u'v'\, dx + \int_\Omega (bu' + cu)\, v\, dx = \int_\Omega f v\, dx$$

Under the initial conditions of $v(0) = v(1) = 0$, this reduces to

$$\int_\Omega u'v'\, dx + \int_\Omega (bu' + cu)\, v\, dx = \int_\Omega f v\, dx \tag{5.3}$$

Unlike (5.1) or (5.2), equation (5.3) still makes sense if we know only that $u \in C^1(\Omega)$. But we have not yet specified a topological space in which mappings implicitly defined by a weak form of (5.1) such as (5.3) have desirable properties like continuity, boundedness, etc. It turns out that Sobolev spaces,which generalize L^p spaces to spaces of functions whose generalized derivatives also lie in L^p, are the correct setting in which to examine weak formulations of differential equations.

We conclude that the use of Sobolev Spaces in solving differential equations appears to be more reliable than the well-known classical solutions and even the numerical approach since continuity and 'many' derivatives are possible.

The recommendation for this work is simply mathematician should use more of the method of Sobolev Spaces in solving certain types of differential equations - one whose solutions do not have continuity and/or differentiability of higher orders.

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