**KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI**

# Valuation of Surrender Option Using Crank-Nicolson and Hopscotch methods

BY

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# DECLARATION

I hereby declare that this submission is my own work towards the award of the M.phil Actuarial Science degree and that, to the best of my knowledge, it contains no material previously published by another person nor that which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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# DEDICATION

This study is dedicated to my senior brother, Mr. Muumin Mac Musah, for his tireless effort which brought me to the level which served as a springboard for me to enrol in this program. May God continue to bless and protect you.

# ABSTRACT

Most of the insurance contracts in Ghana contains the right to early termination and are also path-depend, due to the presence of path-dependence derivatives and the right to early termination of the contract, can make valuation of Life insurance contract in Ghana come with complexities. These complexities are aggravated with introduction of the new parameter (S). Termination of life insurance contract in Ghana among other factors may come as a result of many factors that policyholders face. This study seeks to modify the Black-Scholes partial differential equation by incorporating risk of being multimorbid, and investigate the suitability of using some existing numerical methods (Crank-Nicolson and Hopscotch) to value life insurance contract. Further comparison between the two methods were done to select an efficient method for the modified model. In line with these objectives, simulations for time of an individual to be multimorbid were performed and the survival for risk of multimorbidity computed. This study revealed that, the modified model is stable, consistent and hence suitable to solve. In the numerical analysis of the option valuation using the original Black-Scholes model, Crank-Nicolson method converges faster than Hopscotch method. On the other hand, numerical analysis of the option valuation using the Black-Scholes model with the incorporated multi-morbid survival rate, Hopscotch method converges faster than Crank-Nicolson method. Further, it is observed that, the Hopscotch method converges much faster and give higher values as the step sizes are increased for Black-Scholes partial differential equation of the life insurance contract in Ghana embedded with surrender option. Hence, making the Hopscotch method favour policyholders who might want to surrender in order to receive the surrender value (payoffs).

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# CONTENTS

# LIST OF ABBREVIATION

**SQL** .......... Structured Query Language

**AFT** .......... After Failure Time

**cdf** .......... Cummulative distributive functio

**FCAC** .......... Financial Consumer Agency of Canada

**FRV** .......... Family of Random Variables

**MTM** .......... Market-to-market

**LIC** .......... Life Insurance Company

**GDP** .......... Gross Domestic Product

**PDF** .......... Probability Density function

**PLI** .......... Liability Insurance

**GNP** .......... Gross National Product

**BOPM** .......... Binomial Option Pricing Model

**CS** .......... Capital Stock per Stock

**DBO** .......... Death Bebefit Option

**VAP** .......... Variable Annuity Product

**WHO** .......... World Health Organisation

**CDA** .......... Central Difference Approximation

**CIRS** .......... Cumulative Illness Rating Scale

**CM** .......... Capital Market

**CMPD** .......... Cardio-metabolic and Pain Disorders

**CPD** .......... Cardio-pulmonary Disorders

**EPDE** .......... Eleptic Partial Equation

**GLAD** .......... Gastrointestinal Low Back and Anxiety Disorders

**AIC** .......... Akaike Information Criterion

**ILSA** .......... Italian Longitudinal Study on Aging

**ICED** .......... Index of Co-existent Disease

**PIM** .......... Property Insurance Market

**MEE** .......... Middle East European

**GMM** .......... Generalised Method of Moments

**OECD** .......... Organisation for Economic Co-operation Development

**OW** .......... Output per Worker

**OLS** .......... Ordinary Least Square

**ODE** .......... Ordinary Differencial Equation

**PDE** .......... Partial Differential Equation

**PFA** .......... Pension Fund Asset

**PCI** .......... Property Causality Insurance

**STS** .......... Super-Time-Stepping

**SDE** .......... Stochastic Differencial Equation

**SOR** .......... Successive Over Relaxzation

**SUR** .......... Successive Under Relaxzation

**C-N** ........ Crank-Nicolson

**VAR** ........ Value at Risk

**GARCH** ........ Generalised Auto Regressive Conditional Heteroskedasticity

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# CHAPTER 1

# INTRODUCTION

## 1.1 How Insurance Started

Way back, insurance started in order to provide some sort of help for traders especially, during the periods 5000 BC and 4500 in China and Babylon respectively. Life Insurance dates back to ancient Rome; where 'burial clubs' absorb funeral expenses and helped survivors of members with cash Henncock (2007).

Life insurance served as a way of providing family security, synchronized with the growth of wealthy families during the industrial revolution in England. As a result of rapid economic growth brought in by industrial revolution, traders and manufacturers in England became wealthy and influential and standard of living was high, which of course their families would have found it difficult to continue with that standard of living at the event of their death, unless they provided special financial security for their families and loved ones. To such people, they saw life assurance as an alternative and a better way of providing for family's financial security Sha (2011).

Among the first Life Insurance companies of London (the society for the Assurance of Widows and Orphans) and that of the United States of America (Corporative for the Relief of the poor and Distressed Presbyterian Minister and for the Poor and Distressed Widows and children of Presbyterian Ministers') were founded in 1699 and 1775 respectively Sha (2011).

## 1.2 How Insurance Operates

Operations of insurance, involve individuals or business establishments who make cash payments periodically known as premium into a common scheme, from where policyholders get their compensations from in an event of loss based on advanced agreements made under which type of loss and of range of coverage.

This explains that, there is an agreement between 'two people', where one (insured) pays some amount of money to the other (insurance company) who promises to help the one making the contributions with some future help in case of any loss that the contributor might face, which could have had higher financial consequences without insurance.

Generally, risk is the main term in insurance - the likelihood that both the premium payer and the insurance company affected by the event occurring and the end results such events come along with are uncertain. In addition, the risks in an event could involve far more factors than simply the possibility of happenings of such events.

Therefore, for a fair arrangement for the parties involved premiums collected should appropriately reflect the risk. That is, all aspects of insured risk must be actuarially evaluated and calculated well, considering the kind of event, extent of benefits, nature ( characteristics) of the insured and size (number) of individuals or entities simultaneously under similar risk Gladwell (2005).

The idea of "shared risk" between policyholders and the insurers is of great significance because it is the basis under which the ideas of solidarity is established. Indeed, sustenance of solidarity in insurance would be impossible if each policyholder (participants) in the insurance pool does not take it as a duty in the prevention and mitigation of risks as possible as they can.

### 1.2.1 Types of Insurance Companies

Insurance companies can be grouped as follows:

**General Insurance Companies**: In this type of insurance company, they provide all kinds of insurance apart from life insurance.

**Life Insurance Companies**: Insurance companies that deals in life insurance, pension products and annuities are referred to as life insurance companies.

### 1.2.2 Types of Insurance

The list of types of insurance are more than what one can list in this work, and it is important therefore to choose from the list of insurance policies that best suit an individual's circumstances.

Some types of insurance are; Health insurance, the Disability insurance, Renter's insurance, Auto-mobile insurance, Liability insurance, Life insurance and so on.

According to Financial Consumer Agency of Canada FCAC (2011), insurance is said to be a way of reducing one's potential financial loss or hardship in case of unexpected event. It helps cover the cost of unforeseen events such as theft, illness or property damage to a policyholder. Furthermore, insurance can provide one's spouse, relatives and children with a financial payment upon their death.

### 1.2.3 Life Insurance

The number one purpose of Life Insurance is to guard oneself against any kind of loss of income, as a result of permanent disability or death. Life insurance is also used for retirement planning and as well as insulating. This is a type of insurance you may pay for, but only spouses, children or generally beneficiaries benefit from it. That is, except in some cases where the reason for taking this type of policy is to provide for loved ones, children and families at the time of one's death.

Life Insurance Companies put into the market different types of insurance policies to meet an individual's needs as one's personal circumstances could not remain the same as time goes on. Below is a brief discussion of the three kinds of life insurance in Ghana;

### Term Life or temporary Insurance:

Provides some sort of protection for people for a defined period; generally ten, twenty, twenty five or thirty years. Should the policyholder happens to die, the company pays cash benefits to beneficiary of the deceased during the policy period (term).

### Whole Life Insurance:

This provides protection to the insured for his or her whole life and beneficiaries are supposed to benefit from such contract when the policyholder dies.

### Universal Life Insurance:

Whole life insurance with more flexibility; This type of contract allows the insured to maintain his or her policy and have the opportunity to make changes regarding the death benefits or amount paid at regular intervals (premium).

## 1.3 Financial Derivatives

Pricing of newly created products in the financial institutions is a challenge in recent years. The use and application of financial mathematics has taken to the exploiting of better and advanced mathematical methodologies like, partial differential equations and stochastic equations among others to enable researchers handle these challenges in the financial institutions. In this work, pricing of surrender option (contract) is considered.

A financial derivative, derives its value from an underlying assets; financial derivatives depend on some characteristics exhibited by the underlying asset or assets. There should be obligations and rights that exist between the writer of the security and policyholder, so that there could be a way to pay or deliver future compensation (cash) depending on the nature and circumstance of an unexpected event in the future. We can describe the value (future value) of derivative as a stochastic process because of its uncertainty. Stocks, interest rates and foreign currencies are among the groups of underlying assets and Swaps, options, forwards and futures constitute the main types of derivatives.

## 1.4 Options

Let's consider this scenario; A Samsung company called you this morning with an offer that in four months' time you will have an option to purchase the company's shares from them at a price of 35 Ghana cedis per share (based on an agreement between you and the company today).

The main point is you are now having the *option* to buy this company's shares. Few months (four moths) from today, you can find out if the market price would favour you or not when you decide to exercise the right (option). (In an ideal case, you would like to exercise the right if the market price were more than 35 cedis, which you could re-sell for an immediate profit.) This kind of deal has no downside for you because, four months from now you either make some profit or walk away with no loss. The company on the other side, have no potential of making gains and has an unlimited potential of losing. To compensate, you should be made to pay a certain amount in advance to enter into such an option contract.

This is an *European call* described above. Shares from Samsung are an example of an **asset**; a financial quantity with a defined current value but an unknown future value. With this scenario created and introducing some notations can help one with the discussions on the two basic kinds of options Davis (2005).

Basically, holder of an option automatically possess a right to buy or sell at or before validity period of the option is elapsed. It is a right and not an obligation for the holder to exercise this right. An option could be described exercised when the holder of the asset decides to purchase/hand over the stocks that are related to the option at a certain price. In all cases, the writer of the option cannot be ignored as a party to such a contract Hull (2003).

## 1.5 The Two Basic Kinds of Options

There are two option types, these are the American-styled and the European-styled options.

### American-Styled Option

An American call or put options allows holder (who has the right) to purchase (sell) the actual asset that is related to the option up at any time t, before $T$, for a certain $V$ price (strike price). We denote this type of option by $l(p)$. The pay-off of the American styled (call) at T is given:

$$l = maximum(S_t - V, 0). \tag{1.1}$$

Pay-off of the American styled put is:

$$R = maximum(V - S_T, 0). \tag{1.2}$$

### European styled Option

An European option provides the holder a right to sell or buy for example, stock which has an initial stock price $S$, at a defined future time $T$ and for a certain price $V$. We denote this option price by l(p). The pay-off of an European call at maturity is:

$$L = maximum(S_T - V, 0). \tag{1.3}$$

to calculate the value of the European put:

$$R = maximum(V - S_T, 0). \tag{1.4}$$

The link between European call and put options (put-call parity), denoted as:

$$L + Ve^{-rt} = R + S \tag{1.5}$$

Note r represents risk-less rate in equation (1.5) and S denotes initial price of the stock.

### 1.5.1 Surrender Option

Surrender option (an American-styled put) allows the policyholder to sell back the contract to the original seller (the issuer) and receive a compensation value. In an attempt to fairly value such an option, as well as a best assessment of compensation values are of crucial topics in trying to have the best way to manage a life Insurance contract, both on the solvency and on the competitiveness side.
My major aim of this research is to address the surrender value of a co-morbidity persons who might want to surrender at any time (based on the number of factors which includes; the inability to pay for treatment, severity of chronic diseases that can not be cured) whiles the policyholder is still alive.
In detail, this work considers the single premium, using the Black-Scholes formula and the Crank-Nicolson and Hopscotch methods to determine the surrender value; where i will introduce another parameter in Black-Scholes formula as being the likelyhood of developing multimorbid condition. This will lead to a recursive algorithm that will enable me do easy computing of fair pay-off (surrender value) of a contract of someone with the possibility of being multimorbid.

## Life Insurance Policies Embedded with Options

Over the years, undertakings of insurance have actually developed many ways of attributing more flexibility to life insurance policies as a purpose. These are options that are to enable the insured to make rightful choices related to his credit against the insurer, that could have an impact on both the time to policy maturity and benefits of such a contract (that has potential to change the contract.) For some years undertakings of insurance ignored options that is embedded in their policies, that could give holder the right to sell back the insurance contract to the insurance company, but has given the insured some lack of interest in these options. Twenty years back, situations were worse off as a result of financial markets turmoil, most policyholders had begun to sell back the options embedded in their contracts more often and in an opportunistic way. In fact, insurance companies had to pay benefits which values were more than premiums taken from their policyholders. Hence, the need for important attention to insurance product design so that, to determine the clear-cut value for all kinds of benefits, insurance companies will be aware of what they offered to policyholders. In addition, to have a proper contract pricing, the insurer has to take into account 'dynamic policyholders' behaviour', where 'dynamic' talks of the policyholders' ability to react to external factors (usually economic factors), and thus to exercise embedded options in order to maximise profits. Some kinds of options that are embedded in life insurance contracts are: annuity and lump-sum conversion option, surrender option, resumption option among others Ali (2013).

## Liabilities of Life Insurance

A correct assessment of period and convexity of insurance liabilities and equity measures still remains critical as they continue to be the basic parts of any correct asset-liability management approaches. In order for people to understand and explain the hidden difficulty that are usually encountered by the insurer, then there is a need for all to have a detailed and if possible significant and true concept of risk associated with how insurance operates Ali (2013).

According to Anders and Peter (2002), people or entities who have gone into contract with an insurance company are first to get claims on the assets of the company, where as people holding equity have limited liability; guarantees of rates of interest are the basic components of LICs; based on the principle of contribution (if a risk is insured by multiple companies, and one company has paid out some benefit or claims, that company is entitled to collect proportionate coverage from other companies) are also in position to have right of claim to an equitable share of any excess assets of any investment.

According to Eric (1995), risk-taking initially does occur on the liability side of the balance sheet. Underwriters are those who issue insurance policies which are transformed into liability, due to the time lag between premium inflow and indemnity outflow, mostly the reserves are invested on the financial marketplace and generate (add value) the portfolio of assets of the company.

### 1.5.2 Value of an Option

In dealing with pricing of options, the compensation for surrendering of an option is expressed as: $L_t = f(A_t, t)$; (function time and the underlying asset). Finding surrender value is my primary aim in this study.

The values of call and put options given by $maximum(A_t - V, 0)$, and $maximum(V - A_t, 0)$ for $0 \leq t \leq T$, respectively .This is what (value) the insured receives when he or she happens to surrender before the maturity date. This option is assumed to be an American put option, since it can be exercised at any time of the contract period.

### 1.5.3 Why Investors Will Consider Option Trading

Among some of the reasons an investor will consider option trade include;

a. Options trading help investors with the avoidance of market restrictions and save cost of transactions than trading in stocks

b. There are institutional rules for option and stocks, but stimulating of option trading may depend on the differences of these rules.

c. How an option should be priced is based on a systematic theory and logic.

d. A *speculator* will prefer to gamble in option transactions than stock because the price involvement volatility of an option is more than stocks (Jarrow and Turnbull, 1996), (Chance, 1991), (Kolb, 1999) and (Davis, 2005).

## 1.6 Problem Statement

Life insurance contracts and pension plans are complex financial securities that come in many variations. In literature, contracts which offer a guaranteed return equipped with the right to terminate the contract prior to maturity, do not take into account the risk and controversy component of people with the risk of having multi-morbidity condition at some point in time. The solution to the stochastic differential equation that incorporates individual risk of being multimorbid cannot be solved explicitly. Existing numerical methods may not be suitable for the modified model, hence the need to modify Black-Scholes formula using Hopscotch and Crank-Nicolson numerical methods.

## 1.7 Objective

i. To modify the Black-Scholes model by incorporating the risk of being multimorbid (S) into the model.

ii. To investigate suitability of a numerical method approach of solving the modified model

iii. To compare and select an efficient numerical method (Hopscotch and Crank-Nicolson ).

## 1.8 Methodology

All agents are assumed to operate in continuous time frictionless economy with a perfect financial market, so that tax effects, transaction cost, divisibility, liquidity, and short-sales constraints and other imperfections can be ignored. As regards the specific contract, I also ignored the effects of expense charges, lapses and mortality. Since path-dependence prohibits the derivation of closed-form valuation formulas Ali (2013), the problem can be reduced to allow for the development and implementation of the Crank-Nicolson and Hopscotch methods for fast and accurate numerical valuation of life insurance contracts for persons with the risk of having multimorbidity condition at any point whiles still in the life insurance contract.

## 1.9 Justification

The study is one of the life insurance products-the so-called participation policy embedded with surrender option. This study considers Crank-Nicolson, Hopscotch method and a modified Black-Scholes formula to evaluate surrender value of people with the probability of having multi morbidity condition. Hence, the introduction of the survival rate (S) into the Black-Scholes.

## 1.10 Thesis Organization

This thesis is organised into five main chapters. Chapter 1 presents introduction of the thesis. This consists of background of study, research problem statement, objectives of the study, methodology, thesis justification and organisation. Chapter 2 is literature review, which looks at briefly works done by other researchers on the topic. Chapter 3 is formulation of the mathematical model. Chapter 4 deals with analysis of data collected, formulation of model instances, algorithms, computational procedures, results and discussion. Chapter 5 looks at summary, conclusions and recommendation of the results.

# CHAPTER 2

# LITERATURE REVIEW

## 2.1 Introduction

The chapter looks into the review of related works on insurance, insurance liabilities, surrender options and other related models from other writers who have contributed meaningfully and added knowledge in this area of study.

## 2.2 Definition and meaning of Insurance

Despite the diverse definitions of the term insurance by different writers and researchers, they seem to address the same issue in many different ways. Insurance could mean a promise of compensation for any potential future uncertain event that cause losses. Insurance, helps people with financial protection against uncertain losses by reimbursing losses with insurance company during crisis. Per an individual's preference, one can choose from the wide range of insurance options insurance companies offer their clients.

A lot of insurance companies sell different comprehensive coverage which come with affordable premiums. The regular and fixed insurance payments (premiums) are made based on calculations to the total insurance value (amount). Mainly insurance is used as an effective tool of **risk management** as quantified risk of different volumes can be insured. Mphasis (2009), explain insurance in terms of law and economics. The writer explained insurance as the way of managing risk, where primarily, it is to address the issue of hedging against the risk of a loss (contingent loss). Insurance can therefore be explained as the transfer of equitable risk of losses, from one party to the other or one organisation to the other in exchange for fixed regular payments, or said to be a guaranteed small loss to guard against (prevent) a large but catastrophic loss and that could be devastating.

FCAC (2011), gave the definition of insurance as a way of reducing one's potential financial loss, it help cover any cost of unforeseen and unexpected events such as property damages (example; fire, flood), theft, illness and flood. Also, insurance provides financial payments to one's loved ones upon their death. Pal, K., Bolda, B. and Garg, M. (2007), defined insurance as a co-operate mechanism to spread the loss caused by a particular risk over a number of persons who are exposed to it and who agree to ensure themselves against that risk.

According to Dorfman (2008), from a financial point of view defined insurance as a financial arrangement that redistributes the cost of unexpected losses. This is done by taking premiums from the insured; from different policy holders. In case of event of losses), the insurance company pays the insured of a promised amount in exchange for the premium they have been receiving.

Orice (2006), who had come across various definitions of insurance and found out that, while most definitions differed because they were developed for specific purposes or had changed over time, the definitions shared common key elements; risk transfer and risk spreading. The writer explained that definitions of insurance are developed for various purposes such as different fields of study, categories of insurance, and state or federal statutes. While risk transfer and risk spreading are most important elements, these definitions often include other elements, or parameters, found in definitions. These include;

- indemnification, which is the payment for the loss that might be incurred
- ability of a company to make reasonable calculations (estimates) of future losses;
- ability to quantify losses in monetary amounts; and
- the chance of adverse but random occurrence of events outside the control

of the insured.

### 2.2.1 Insurance In Ghana

The coming of insurance to Ghana cannot be discussed without mentioning of the British merchants activities in the 19th century. Ghana was among the countries where merchants sent their goods from, and by law had to insure these goods before sending them to the United Kingdom. This explains why insurance companies had to send their agents to represent them in Ghana where their trading activities were taking place. Thus, transactions of insurance activities were monitored and carried by agents of foreign insurance companies in Ghana, and among the agents were insurance policy sellers and intermediaries. Local insurance companies began to emerge towards independence, Gold Coast insurance Company was among the first insurance companies in Ghana and was founded in 1955. In 1958 and 1957, other companies like Cooperative and General Insurance companies respectively, were established. Later, Government of the Gold Coast bought these two companies and merged them, which was named State Insurance Company in 1962. Between 1962 and 1970 period, significant improvements were seen in the insurance industry in Ghana because rule, regulations and laws were introduced to protect the insured and guard the insurance industry. This development gave way for more policies like aviation insurance, marine insurance and accident insurance among others, introduced into the insurance market apart from life insurance. Most of these regulations and laws favoured and protected the local insurance companies and that created an opportunity for more insurance companies to spring up Afriyie (2006).

### 2.2.2 Economic Importance of Insurance

Roles of insurance institutions, its links to other sectors and contribution to the growth of the Ghanaian economy is worth discussing. Though, there are numerous works done on relationships between growth of economies and capital markets and economic growth and banks lending; these were casual relationships done. Much attention has not been given to the insurance sector in this regard. However, some researchers have attempted to fill this gap as far as the relationship between insurance and the economy are concerned.

Insurance is not different from banks in the capital markets, which are to satisfy and serve the financial needs of private individuals or house holds as far as financial intermediation is of great importance. Insurance service availability is very essential when it comes to the stability of an economy and acceptance of aggravated risk by business participants. Insurance companies after taking claims, form reserve funds with the pooled premiums and invest in other portfolios to make interest. Among the important roles insurance companies play include enhancement of internal cash flows at the assured, placement of large amounts of assets on the capital market and probably, its contribution to economic growth Peter and Haiss (2006). The authors reviewed empirical evidence, theory and identified channels of influence in other to fill this gap. They used use annual insurance data from twenty nine (29) European countries from 1992 to 2004 period by applying a cross-country panel data analysis. Their findings explained why life insurance has weak growth-supporting role with similarities to that of banks and stocks.

Beenstock, M., Dickison, G. and Khajuria, S. (1988) used data from 1970 -1981 for 12 countries in a cross-section and pooled time series analysis. Property liability insurance (PLI) premiums were regressed on income, gross national product (GNP) and interest rate development. The author observed a correlation between premium, GNP and interest rate; marginal propensity to insure is high in the long run and it rises with per capita income. Also Beenstock et al. (1988) explained that neither economic cycles nor cyclical income variations can affect insurance consumption. Among the first to conduct studies on casual relationship between insurance industry growth and economic growth were Ward and Zurbruegg (2000). The authors looked into casual relationship between the growth of insurance industry and economic growth, examined short and long dynamic relationships between economic growth measured by annual real GDP and insurance industry, and finally measured by total real premiums for nine countries(OECD countries) for the 1961-1996 period. As additional explanatory variables the authors used changes in government budget surplus, private savings rates, the general population size, general government level of current expenditure, and old age and youth dependency ratios. Which were measured as proportion of the total population under 16 and over 65 years of age, based on bivariate VAR methodology to test for Granger causality. Among the authors findings is that, the causal relationship between economic growth and insurance market development are not the same every where but are different depending on the country. Just as Outreville (1996) and Enz (2000) also concluded that elasticity of the demand for insurance varies itself with the level of income; it becomes less sensitive to income growth in more developed economies. Even though, Ward and Zurbruegg (2000) could not find the exact causes in their attempt to fill this gap, rather suspected that possible causes could be the following: country-specific nature of cultural, regulatory and legal environment, the improvement in financial intermediation and the moral hazard effect of insurance operating in various countries.

Another work carried out in China by Zou and Adams (2006), gives insight into the property insurance market of China from 1997 to 1999 period. Market regulation and Chinese market specialities make this work more appropriate in providing evidence for the socio-political decision model of Hofstede (1995), and the law-and-finance view of La Porta (1998). Their conclusions showed that companies that are highly leveraged with intensive production consume property insurance, and companies that are owned partly by the state or possible tax-loss carry-forward reduces demand. An increase in foreign or managerial ownership as well as improved growth options facilitate demand, while the company size centres inversely.

Contribution by Davies and Hu (2004), added to knowledge is very special with respect to the regressions direction and the variable set up in their contribution to knowledge. The authors conducted a test of causality using data from 1960 to 2003, spanning 43 years and over, for 20 East and Middle East European (EME) countries as well as 18 OECD countries. In this test output per worker (OW) was used as the dependent variable while pension fund asset (PFA) and capital stock per worker (CS) was used on the explanatory side. Results of their work showed that ordinary least square(OLS) regression gives evidence that CS and PFA has positively and significantly impact on OW. In the long run, findings in the dynamic heterogeneity models' support the OLS outputs. In the tests findings the CS and PFA suggested a co-integrated with OW. Another finding showed that the PFA development impacted strongly on the OW in East and Middle East European countries than in the OECD countries and the shock response stays positive but decreases in the long run.

The works of Esho, N., Kirievsky, A., Ward, D. and Zurbruegg, R. focussed on the legal framework in addition to the GDP - Property Causality Insurance Consumption (PCI) link. The authors based causality study on data from different (44 countries) between 1984 to 1998 and includes OLS and fixed-effects estimations and GMM estimation on panel date based. It was also observed that no matter the type of methodology employed, a positive correlation exists between real GDP, the strength of the country's property rights and insurance consumption. Demand of insurance and loss probability are significantly connected, but its connection to risk aversion is quite weak. The authors observed that when GMM estimators were used in the investigation, price was negatively impacted. Although the data employed in the study showed great variations between the developments of countries with different origin in terms of legal PCI price, per capita, GDP amongst others, there was no evidence that legal origin was a significant indicator for PCI consumption. Contrary to other sectors, the property rights importance simply suggested that insurance demand were facilitated by legal environments.

### 2.2.3 Valuation of Life Insurance Contracts Embedding Right to Early Exercise (Surrender Option)

Variable annuities provide guarantees for unsatisfied customers to have their monies refunded and market guarantees on invested principal Blessing and Jun (2002). The authors pointed out that these guarantees had embedded in them unpredictable maturity put options with strike prices. These annuities; where people pay lump-sum can be exercised any time before maturity (surrendered). When the option (lapse option) is rationally surrendered before maturity, it is said to be an American-styled put option; exercised when the strike price is lower than the market value of an underlying asset. Options that are embedded with surrender options have stochastic maturity and that the holder of the option can at any time exercise this lapse. Embedded put options have stochastic maturity and that the policyholder can exercise the lapse. Early exercised option, increases the contract value and exposes the insurance company to loss of fees. In the authors work, they analysed specific VAP by focusing on mainly the lapse option, when put options are *out-of-the-money*, rational investors will prefer to lapse the contract. Investors will consider a lapse as rational if: i)it is immediately followed by a re-establishment of a contract with a *better* guarantee ii) the difference between surrender charges and contract policy value is more than the present value of DOB. The author used the famous Black-Scholes pricing option formula which has become the standard valuation technique for traded capital market instruments, to value the GMDBs and the option to lapse in an actual variable annuity contract and take into account the surrender charges schedule, mortality risk, and lapse option using the 1994 VA GMDB mortality data. Also, they did value guaranteed minimum death benefit (GMDB)options in Polaris II variable annuities. The authors' findings gave writers of annuity some kind of realistic indications on the costs of GMBs in their products. Their results also showed that significantly increment of the GMDB option value due to the lapse option. Carole and Christian (2008), considered equity-linked life insurance contracts that give their holder the possibility of individuals to surrender their policy before maturity. Authors of this study used least-square Monte Carlo approach of Longstaff and Schwarts coupled with quasi-Monte Carlo sampling and a control variate in order to construct efficient estimators for the value of such contracts. In the work of Carole and Christian (2008), also showed how to incorporate the mortality risk into pricing risk algorithms without explicitly simulation it. In this paper their focus was on the surrender option, that provides right to the policyholder to opt out of the contract before it matures. This is a financial approach which aimed at obtaining a "market value" of the surrender option in a similar way to what has been done in related works of Andreatta and Corradin (2003) and Bacinello (2003). Carole and Christian (2008), findings include, with a relatively small number of simulations, they got quite precise approximation of the surrender benefit with their methodology. Also, their findings showed how to include mortality risk using constant probabilities that are fixed for everyone. Brennan and Schwarts (1976), are among the first to propose financial approach based on option theory. In Black and Scholes framework Grossen and Jones. (1997), give the optimal exercise barrier by using results from Myneni (1992). Another work done by Shen and Xu. (2005), also have dealt much into surrender options by way of suing partial differential equations. Albizzati and Geman (1994), proposed a model to address the no arbitrage approach to value even the surrender option which has been criticised since these options are not not traded mostly Carole and Christian (2008). The contribution of Conall and Stephen (2013), to mathematical and computational finance, is to extend the application of STS accelerated technology to the two-factor problem of pricing European and American put options under the Heston model which states; $dx_t = rx_t dt + \sqrt{y_t x_t} dz_t$,

$dy_t = \alpha(\beta - y_t) - \lambda\gamma\sqrt{y_t} dw_t$,

$\rho dt = dz_t dw_t$.

Where $x_t$ and $y_t$ are asset price and variance at time t respectively, r is the risk-free rate, $\alpha$ is the mean reversion of the variance, $\beta$ is the long run mean of the variance, $\gamma$ is the volatility of the variance, $\rho$ is the correlation of the asset price and the variance, and $\lambda$ is the market price of risk.

The use of a contingent claims pricing theory is a common method in insurance contracts pricing, and is based on Black and Scholes (1973) work. The application of contingent claims theory in equity-linked life insurance pricing was pioneered by Brennan and Schwarts (1976). Briys and De Varenne (2014) and others have studied participating life insurance contracts.

Briys and De Varenne (1997), presented a contract that contained a point-to-point guarantee, meaning the payments and participation(optional) in the terminal surplus at maturity is guaranteed by the company . Market value of the contract in the model is a function of the guaranteed interest rate which influences the risk of shortfall at maturity. Briys and De Varenne therefore introduced a stochastic interest rates model. In a similar contract by Grossen and Joergensen (2002), without the use of stochastic interest rates, insolvency option of the insurer and impact made by regulatory intervention were considered in addition. Authors of this work; life insurance contracts, featured clique-style annual surplus participation said, in this type of option, either a fraction of the asset return or the guaranteed interest rate, whichever is greater is credited annually to the policy and it automatically becomes part of the guarantee. A bonus account that serves as a smoothing mechanism in asset returns participation is introduced. Grossen and Joergensen (2002), in their study breaks the contract down into a surrender option, a bonus part and a risk-free bond.

Hansen and Miltersen (2002), carried out further studies on contracts embedded with surrender option. These determine contracts with interest rate guaranteed and varying yearly surplus participation schemes are priced fairly. Authors of this paper, proposed a contract pricing method that depends on finite difference

scheme. Person and Miltersen (2003), introduced a participating life insurance contracts model practical in Denmark. Terminal bonus along with a smoothing surplus distribution mechanism like that of Grossen and Joergensen (2000), and interest rate guarantee is provided. In the authors' proposed model, the holder of the policy is allowed to make annual fee payment to either the company or the insurer. Gazert and Kling (2007), introduced a method that increased insurance liability information because, it considered risk measurement and pricing. The authors examined fair valuation effects on the risk situation of the insurer, i.e., (the posibility and extend of a shortfall in demand for fair contracts with equal market value). In identifying key risk drivers the results for different contract types are compared. This includes point-to-point and clique-style guarantees.
Clique-style contracts or options with varying smoothing mechanisms, which are common in the UK, are studied by Haberman, S., Ballota, L. and Wang, N. In this contract type, the liabilities yearly earn either predetermined fraction or guaranteed interest rate whichever is greater. The same authors have another paper published in 2006 which looked at the effect of the default options on pricing fairly. Kling, A., Richer, A. and Rub, J. (2006), presented a framework for clique-style guarantee contracts, quite common in Germany. They were evaluated by considering the regulatory framework of the Germans. Some contract parameters interactions like decision of management regarding guaranteed interest rates and surplus participation rates were analysed.

### 2.2.4 Comorbidity, Multimorbidity and Chronicity

Comorbidity defined by Feinstein (1970), as the combination of additional disease beyond an index disorder. This definition implies that the main interest is on the index condition and possible effects of the other disorders, for instance, on its prognosis. In contrast, multimorbidity is defined as any co-existence of diseases in the same person indicating a shift of interest from a given index condition to individuals who suffer from multiple disorders Alessandra (2009).

However, as the term multimorbidity addresses a wide range of health problems and conditions, the measurement of multimorbidity is particularly complicated. Given the complexity and heterogeneity of chronicity in the elderly, no single definition or operational criteria will serve all research and clinical purpose effectively. Consistently across studies, older persons are more likely to be affected by multimorbidity; in the Italian Longitudinal Study on Aging(ILSA), about 25 % of 65-69 years old subjects, and more than 50 % of persons aged 80-84 years were affected by 2 or more chronic conditions. The study showed the prevalence figures vary widely according to the number of conditions evaluated and the age structure of the study populations. Thousands of persons turn 65 of age every day Alessandra (2009) and it is expected to continue to rise. Researches done in this area showed that the 'compression of morbidity' theory is based on the assumption that mortality at older age will reach a limit beyond which, there can be no further decline and that there is an ongoing increase in the age of disability onset. Under these conditions, there would be a compression of morbidity into a smaller number of years at the end of life. The 'expansion of morbidity' theory and the 'age of delayed degenerative diseases' theory imply that the extension of life for persons with chronic and disabling conditions due to medical progress, without a reduction in the incidence of these conditions will lead to a deterioration in the health of the population Alessandra (2009). Hypothesis of Alessandra (2009), termed 'dynamic equilibrium' states that alongside the reduction in mortality there will also be a reduction in the rate of deterioration of the body's vital organ systems. This could result in more diseases in the population, but the disease will be at a lower level of severity. Michel and Robin (2004), reviewed the main theories on population ageing, and concluded that future trend scenarios (expansion or compression of morbidity and disability depend on four factors:) 1) increase in the survival rates of sick persons; 2) control of the progression of chronic disease; 3) improvement of the health status and health behaviours of new cohorts of elderly people; and 4) emergence of very old frail populations.

However, the contemporary phenomenon of population ageing falls outside the boundaries of theories and models, as the health status of older populations across the globe is experiencing a complex mixture of increased frailty accompanied by reductions in some measures of disability.

A major effort to understand and predict the 'effect of epidemiology' change is the Global Burden of Diseases Study undertaken jointly by the WHO, Harvard University and the World Bank. The study was implemented to stimulate the inclusion of non-fatal health outcomes when quantifying the burden of diseases in worldwide health policy debates. In fact, the Global Burden of Disease Study showed that a chronic disorder was fourth leading worldwide cause of disease burden in 1990. The study also estimated that, by 2020, the burden attributed to non-communicable diseases would rise sharply, with ischemic heart diseases and depression at the top of the leading causes Christopher and Alan (1996).

In response to the worldwide ageing phenomenon and related chronic diseases, many health care planners and governments have promoted further research regarding age-related pathologies. In fact, the cost of health care is highly related to the number of persons treated or monitored for various diseases. However, the majority of the studies have focused on specific illnesses. Dementia, for instance, has been investigated extensively enough to allow the estimation of worldwide occurrence. Several other studies have concentrated on a relatively small number of diseases, such as vascular diseases, diabetes, cancer, and chronic obstructive pulmonary diseases, rather than the whole range of chronic morbidity Christopher and Alan (1996).

### 2.2.5 The Evaluation of different patterns of Comorbidity and Multimorbidity

In spite of the increasing prevalence of multiple chronic conditions with ageing, knowledge concerning how disease co-occur in the same individual is still limited. We have incomplete knowledge concerning comorbidity and multimorbidity because, few studies have attempted to describe the overall pattern of disease with a given population and most of them have used different approaches to address this issue. One of the first problems in evaluating the combinations of clinical conditions is lack of consensus regarding the definition of multimorbidity. Guralnik (2006), defined multimorbidity as the presence of two or more health problems in the individual or person, whether coincidental or not, and comorbidity as the presence of additional conditions given an index disease.

Another problem is the use of different methods to explore the co-occurrence of diseases Guralnik (2006), one possible basic method is the conditional count. Conditional count, is the number of chronic diseases given that the patient has a particular index disease. This approach is useful when studying one particular condition for example arthritis, and its comorbid conditions. The results strongly depend on the number of conditions evaluated. Also, another approach, which has been extensively employed, is the use of indices including both the number and the severity of the individual conditions, such as the Charlson Comorbidity Index Charlson (1987), the Index of Co-existent Disease (ICED), and the Cumulative Illness Rating Scale. Major limitations of these indices are due to the fact that they usually do not cover the overall conditions affecting the population, and often require medical records or skilled researchers. The third method or approach is to assess the proportion of people who have pairs of comorbid diseases. Alessandra (2009), used this approach in the Women Health and Ageing Study and found that the most common comorbid pair was arthritis and visual impairments, with 44% of elderly participants reporting both condition. The same approach used in community-resident individuals aged 55 years and older, and found that arthritis and high blood pressure were the most common comorbid pair (21.1%). This approach, as well as the estimation of the odds ratios, is useful in assessing the degree to which comorbid disease occurrence exceed a level of expected frequency due to chance Verbrugge, L., Lepkowski, L. M. and Imanaka, Y. (1989). Lastly, the cluster analysis is a descriptive technique that considers how variables tend to occur in conjunction with each other. With this method it is possible to go beyond simple comorbid pairs to obtain a general overall picture of how and which diseases are associated in a particular population and where particular diseases of interest appear in the pattern. John, M., Kerby, D. S. and Hennnessy, C. H. (2003), used cluster analysis to describe the distribution of diseases in a sample of old American Indians. They found that diseases aggregated in two major clusters; the cardiopulmonary and the sensory-motor one. Gabriel, A., Richard, T. and Mensah, A. (2014), also used this approach. Gabriel et al. (2014), investigated the imparts of multimorbidity patterns of gastrointestinal low back pain and anxiety disorders (GLAD), cardio-metabolic and pain disorders (CMPD), and cardio-pulmonary disorders (CPD) on the time to death among hospitalised patients and examined how the risk for mortality associated with the multimorbidity patterns moves with time. A total of 61 patients out 0f 154 hospitalized patients of least age 50 years died. 52.5% were with CMPD. 32.7% with CPD and 14.8% with GLAD. From the log-logistic AFT model the time to death is accelerated for patients with CMPD compared to those with GLAD by an estimated factor of o.11 (95% CI: 0.26-0.66). Similarly, among patients with CPD the time to death is accelerated by a factor of 0.40 (95% CI: 0.25-0.63) compared to individuals with GLAD. The authors found that the risk for mortality associated with CMPD and CPD were non-monotonic, in that, they increased over early duration of hospitalized peaking at 0.051 and 0.012 during the 19th day and the 18th day of hospitalisation respectively, following a decreasing trend. For GLAD non-monotonicity of the risk for mortality was less apparent. Also, the CMPD was found to the most life threatening multimorbidity pattern followed by CPD.

### 2.2.6 Numerical Methods Approach

The works of Russel and Collins (1962), applied the Monte Carlo technique to solve the problem of rate making with real problem in the transfer of coverage from one carrier to another by a policyholder who might be in a large deficit position with the original carrier in the field of insurance. According to the author, this position can be avoided if the policyholder is willing to pay an additional charge for a guarantee of an upper limit on the amount of deficit carried forward from previous year to the following years. It is important to know the following expectations; the expected value, the probability value and the variation of claims in excess of a given amount in order to determine the additional charge the policyholder to pay. The author, in his work addressed the problem of determining the frequency distribution of the annual claim cost of a given group of lives for a given year. the author used Monte Carlo method to address this problem. The following properties of the groups used to vary over wide ranges: i) the size of each group, ii) the sex distribution of the groups, iii) the ages distribution of the groups, iv) the total amount of insurance, and v) the distribution of the insurance on individual lives. Bjarke et al. (2001), proposed a model for the valuation of traditional participating life insurance policies in their work. The claims explained to be made up of explicit interest rate guarantees and various embedded option elements, such as surrender and bonus options. With respect to the structure of these contracts, the theory of contingent claims pricing is a particularly well-suited framework for the analysis of their valuations. During the contract period, the pay-off from the contracts are considered important and in particular depend on the history of returns on the insurance company's asset. From literature, path dependence prohibits derivation of closed-form valuation formulas. The author demonstrated that the dimensionality of the problem can be reduced to allow for the development and implementation of a finite difference algorithm for fast and accurate numerical evaluation of the contracts. On mortality risk, the author also demonstrate how fundamental financial model can be extended to allow for mortality risk and they provide a wide range of numerical pricing results. Their work was concluded well by the use of finite difference approach to evaluate the life insurance liabilities. Finite different approaches included the implicit finite difference scheme, the explicit finite difference scheme and the Crank-Nicolson method. Findings of the authors revealed that, Crank-Nicolson method seemed more accurate than implicit finite difference scheme and explicit finite difference scheme because the error associated with the final solution with Crank-Nicolson method seemed smaller as compared to the other finite difference methods used. Li and Hong (2010), used the Hopscotch method and Crank-Nicolson method to solve European option prices. The authors analysed the pricing results from these two methods by comparing result generated from the Black-Scholes model. In their report, they started with an introduction of the numerical approximation of derivatives and applied them to solve the Black-Scholes PDE. Basically, they used explicit and implicit schemes where by the mixture of these two schemes produce the Hopscotch and Crank-Nicolson method that will enhance the accuracy of the result they approximated.

The author's findings showed that, it will be easier to apply the explicit scheme to solve the Black-Scholes PDE by creating and applying these methods and schemes in MATLAB. The Hopscotch and Crank-Nicolson methods combined the benefits of fully explicit and implicit schemes. The methods ensure an accurate outcome, nonetheless, in comparing CPU time the Crank-Nicolson method was found to save computational time than the Hopscotch method.

### 2.2.7 Path-Dependent Option

The option that gives the right but not an obligation, to an individual or an entity to sell or buy an underlying stock at a predetermined price during a specified time period could be said to be a path dependent option, this price is usually not stable but based on the fluctuations in the underlying value during all or part of the contract term. The price of the underlying asset follows a path, which normally, is what a path-dependent option depends on for its pay-off. Simply, path dependence explains how an individual or entity would have to face a set of decisions, therefore any given circumstance is limited by the decisions an individual or entity has made in the past, even though past history of such circumstances may no longer be relevant. Actually, the pay-off of path- dependent option do not merely depend on the final price of the underlying assets, but also, the process that the price was arrived at is important. An example of the path-depended option is the American-style contract, since the holder of the option can exercise the right at any time before expiration and thus ceasing to exist. There are many kinds of path-dependent options, the most popular types are *barrier*, *Asian* and *lookback options.* Others include Russian, Game or Israeli and Cumulative options Xia (2008).

According to Davis (2005), the concept of path dependence originated as an idea that a small initial advantage or a few minor random shocks along the way could alter the course of history. However, the scope of this idea has grown so wide that path dependence has dulled its value and is becoming a trendy way to say that history matters, path dependence no longer provides any analytic leverage. The concept of path-dependence, according to the author, seems almost metaphorical. Path dependence, according to the author simply means that the current and future states, actions or decisions depend on the path of previous states, actions or decisions.

Ali (2013), described a dynamic process that produces outcomes at discrete time intervals indexed by the integers, $t = 1, 2, ....$ He denotes the outcome at time t as $x_t$. In addition to the outcome, there are other information, opportunities, or events that may arise in a given period which he described as the environment at time t. This contains exogenous factors that influence outcomes. A history at time T, $h_t$ is the combination of all outcomes $x_t$ up through time $(T - 1)$ and all other factors, the $y_t$, through time $T$.

Financial derivatives derive their value from an underlying asset that is traded as financial security, whose price is modelled by some stochastic process. In general form, the option pay-off is path dependent since it depends on the entire future path to its current state traversed by the underlying security. Path-dependent options are defined using either discrete or continuous price sampling. Closed form solutions are often available for continuous sample, but in practice, most traded path-dependent options are discretely sampled. It is known that, the application of these closed form solutions leads to substantial pricing errors for discretely sampled options.

## 2.3 SURVIVAL RATES

Computing risk models often arise very often in several fields: biostatistics, reliability, finance, economics etc. They are relevant when two or more causes of failure act simultaneously, but the smallest failure and its type only are observed. In other words, each failure time is potentially right censored by every other failure times. A key point to note is that, all these failures are dependent on a priority. Thus, these can be dealt with by the standard arguments of random censoring models Jean (1991).

According to Stephen (2005),the length of a spell for a subject (person, firm, etc.) is a realisation of a continuous random variable T with a *cumulative distribution function* (cdf), F(t), and *probability density function* (pdf), f(t) - F(t) is also known in the survival analysis literature as the *failure function.* The survivor function is $S(t) \equiv 1 - F(t)$; t is the elapsed time since entry to the state at time 0.

Failure function(CDF);

$Pr(T \leq t) = F(t)$, which implies, for the survivor function: $Pr(T > t) = 1 - F(t) \equiv S(t)$. The PDF is the slope of the CDF (Failure) function:

$$f(t) \lim_{\Delta \longrightarrow 0} \frac{Pr(t \leq T \leq t + \Delta t)}{\Delta t} = \frac{\partial F(t)}{\partial t} = -\frac{\partial S(t)}{\partial t}.$$

Where $\Delta t$ is very small (infinitesimal) interval of time. The f(t)$\Delta t$ is akin to the unconditional probability of having a spell of length exactly t, i.e. leaving state in tiny time interval of time [t, t + $\Delta t$]. The survivor function S(t) and the Failure time function F(t) are each probabilities, and therefore inherit the properties of probabilities.

In literature, survival rates have been estimated from annual surveys by tracking the abundance of one or more cohorts, as measured by catch per unit of sampling effort, from time to time (yearly). John and Todd (2007), work showed that data from several years can be analysed simultaneously to get a single estimate of survival under the assumption that survival is constant over the period analysed. This method requires that only a single cohort be identified and separated from the other age groups. The author applied $S_t = \frac{N \geq a+1, t+1}{N \geq a, t} = \frac{I \geq a+1, t+1}{I \geq a, t}$ to a catch rate to obtain annual estimates of survival rate and then convert these to estimates of the instantaneous rates of total mortality (Z) according to the formula; $Z = -log_e(S)$ using a data from the 1963 and 1964. Arithmetic mean results were calculated from their formula applied over the periods of stable mortality identified by Gedamke and Hoenig (2006), from their analysis of mean sizes.

From the literature review, it observed that researchers have not addressed issues of surrender of policyholders who might to surrender their insurance policy (s) due to coexistence of chronic diseases in them. This has necessitated me to combine multimorbidity with insurance contract as a research topic hence, valuation of surrender option of policyholders likely to be multimorbid and wish to surrender for a value (surrender value).

# CHAPTER 3

# METHODOLOGY

## 3.1 Introduction

This chapter consist of the methodology used in the valuation of surrender value of the insured, who is likely (probability) of developing multimorbidity condition and has the right to surrender the contract before expiration (maturity) in a Life Insurance contract. Using the Crank-Nicolson and Hopscotch methods to solve the modified model, where a new parameter known as survival rate (S) is incorporated into the Black-Scholes model and S is simulated using the R-software and under Exponential and Weibull distributions.

## 3.2 Probability Space

The modern theory of probability stems from Kolmogorov (1956). Kolmogorov associate a random experiment with a probability space, which is a triplet, $(\Omega, f, P)$, consisting of the set of outcomes, $\Omega$, a $\sigma - field$, $f$, with Boolean algebra properties, and a probability measure, P.

### Definition 3.1 (Sample Space)

When an experiment is performed, the set of all possible outcomes is called the sample space, denoted $\Omega$. All subsets of the sample space $\Omega$ form a set denoted by $2^{\Omega}$.

### Definition 3.2 (Events and Probability)

The set parts $2^{\Omega}$ satisfies the following properties:

1. It contains the empty set $2^{\Omega}$

2. If it contains a set A, then it also contains its complement $\bar{A}= \Omega/A$

3. It is closed with regard to unions, i.e...is a sequence of sets, $A_1uA_2$...also belongs to $2^{\Omega}$.

Any subset $f$ of $2^{\Omega}$ that satisfies the three properties is called a $\delta$-field. The set belonging to $f$ are called events. this way, the complement of an event, or the union f an event is also an event. We say that an event occurs if the outcome of an experiment is an element of that sub-set. The chance of occurrence of an event is measured by a probability function P:$\rightarrow$ $[0, 1]$ which satisfies the following two properties:

1. P($\Omega$)=1;

2. For any mutually disjoint events $A_1, A_2$,...$\in f$,

$P(A_1uA_2uA...) = P(A_1) + (A_2) + ...$ The triplet $(\Omega, f, P)$ is called a probability space. This is the main set up in which the probability theory works.

## 3.3 Dynamics of Derivative Prices

### 3.3.1 Stochastic Process

A stochastic process on the probability space $(\Omega, f, P)$, is a family of random variables X parametrized by t $\in$ T, where T$\subset$ $B$. If T is an interval we say that $X_t$ is a stochastic process in continuous time. If T=1,2,3..., we shall say that $X_t$ is stochastic in discrete time.

Consider that all the information accumulated until time t is contained by the $\sigma$-field $F_t$. This means that, $F_t$ contains the information of which events have already occurred until time t, and which did not. Since the information is growing in time, we have:

$$F_s \subset F_t \subset F$$

For any s,t $\in$ T with $s \leq t$. The family $F_t$ is called filtration. A stochastic process $X_t$ is called adopted to filtration $F_t$ if $X_t$ is $F_t$-predictable, for any $t \in T$

### Definition 3.3 (Markov Process)

A Markov process is a process for which everything that we know about its future is summarised by its current value. A continuous time stochastic process $X = \{X_t,\, t \geq 0\}$ is Markov process if

$$Prob[X_t \leq x|X_u, 0 \leq u \leq s] = Prob[X_t \leq x \mid X_s] fors < t$$

## 3.3.2 Brownian Motion

The observation made first by Botanist Robert Brown in 1827, that small pollen grains suspended in water have irregular and unpredictable state of motion, led to the definition of the Brownian Motion, which is formalised in the following;

### Definition 3.4 (Brownian motion)

A Brownian motion is a stochastic process $B_t$, $T \geq 0$ which satisfies;

1. The process starts at the origin, $B_o = 0$
2. $B_t$ has stationary, independent, increments.
3. The process $B_t$ is continuous in t.
4. The increments $B_t - B_s$ are normally distributed with mean zero and variance $|t - 1|$, $B_t - B_s \sim N\,(0, t-1)$.

The process $X_t = x + B_t$ has all the properties of Brownian motion that starts at x. Since $B_t - B_s$ is stationary, its distribution function depends only on the time interval $t - s$; i.e. is $P(B_t - B_s < a) = P(B_t - B_0 < a) = P(B_t < a)$ .

It is worth noting that even if $B_t$ is continuous, it is nowhere differentiable. From condition 4 we get that $B_t$ is normally distributed with mean$E\left[B_t\right] = 0$ and $\mathrm{var}[B_t] = t$. i.e. $B_t \sim N(0,t)$.
This implies that the second moment is $E\left[B_t^2\right] = t$. Let $0 < s < t$, and since the increments are independent, we can write;

$$E[B_t B_S] = E[(B_s - B_0)(B_t - B_s) + B_s^2] = E[B_s - B_o]E[B_t - B_s] + E[B_s^2] = s.$$

Consequently, $B_s$ and $B_t$ are not independent.
A Brownian Motion Process $B_t$: Is said to a martingale based on this set of information $F_t = \delta(B_s \leq t)$

### Definition 3.5 A Weiner-process $(W_t)$:

Is a process adopted to filtration $F_t$ such that ;

1. The process starts at the origin, $W_o = 0$
2. $W_t$ is an $F_t$-martingale with $E[W_t^2] < \infty$ for all $t \geq 0$ and

   $E[(W_t - W_s)^2] = t - s,\ s < t;$
3. The process $W_t$ is continuous in t. Since $W_t$ is a martingale, it's increments are unpredictable and hence $E[W_t - W_s] = 0$ ; in particular $[W_t] = 0$

   and $Var[W_t] = t$.

If $W_t$ is a Weiner process with respect to the information set $f_t$, then $Y_{_t} = W_t^2 - t$ is a martingale. Hence, $E[W_t^2 - t/F_t] = W_s^2 - s$, for $s < t$ .

## 3.3.3 Brownian Motion With Drift

The process $Y_t = \mu t + W_t,\ t \geq 0$ is called Brownian motion with drift. This process $Y_t$ tends to drift off at the a rate $\mu$. It starts at $Y_{_0} = 0$ and it is a

Gaussian process with mean

$$E[Y_t] = \mu t + E[W_t] = \mu t$$

and variance;

$$Var[Y_t] = Var[\mu t + W_t] = Var[W_t] = t$$

## Martingales

In simple terms a martingale is a stochastic process for which its current value is the optimal estimator of its final value. The features of a martingale rely on the application of its final value. Let $(S_t)$ denotes observed FRV, where time is said to be non-discrete over an interval *[0,T]*. $0 = t_0 < t_1 ... < t_{k-1} < t_k = T$ and $\{I_s, t\epsilon[0,T]\}$ as periods T and filtration respectively. $\{S_t, 0, t\epsilon[0,\infty)\}$ is said to be adopted to $I_t, t\epsilon[0,\infty\}$ if at some time t the price process value $(S_t)$ is included in the set of information $I_t$ for $t \leqslant 0$, where $S_t$ is known when set of information about $I_t$ is given Davis (2005).

## Definition 3.6

The process $M_t, t \geq 0$ is said to be a martingale regarding the set information $I_t$ and probability Q, for all $t \geqslant 0$.

a. $E_Q[\mid M_t \mid] < \infty$.

b. When $0 \leq l < t$, we have $E_Q[W_t \mid I_s] = M_s$

Martingale; (1)present value of conditional and expected value of the future by the process of martingale is known. (2) martingale is expected to drift but not upwards and hence this denotes a fair game. (3) probability measure and set of information what the definition of martingale is based on Davis (2005).

**Definition 3.7 Conditional Expectation**

Conditional expectation of a random Variable X, given Y = y, is defined to be the mean of the conditional of X given Y = y, denoted by $E[X/Y = y]$. As y varies, so too will $E[X/Y = y]$ and we get the random variable $E[X/Y]$.

**Definition 3.8 Martingale in Discrete time**

A discrete-time stochastic process $X_o, X_1, X_2, ...$ is said to be a martingale if;

- $E[|X_n|] < \infty$ for all n.
- $E[X_n|X_{0,}X_1..., X_m] = X_m$ for all $m < n$.

In words, the current $X_m$ of a martingale is the estimator of its future value $X_n$. In this setting a martingale is a stochastic process such that;

- $E[|X_n|] < \infty$ for all n
- $E[X_n/F_s] = X_s$ for all $s < t$

## 3.4 Differential Equations

**Definition 3.9 Differential Equation:**

A differential equation is an equation involving the unknown function $Y = f(t)$, together with its derivatives $y', y'', ..., y^{(n)}$.

Mathematically a differential equation may be expressed implicitly as:

$$F(t, y', y'', ..., y^{(n)}) = 0 \tag{3.1}$$

Explicitly, the general form of a differential equation can be written as:

$$y^{(n)} = f(t, y', y'', ..., y^{(n-1)}) \tag{3.2}$$

### Definition 3.1.0: Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation involving an known function of a single variable together with one or more of its derivatives.

### Definition 3.1.1: Order of Differential Equations

**A first order** differential equation is of the form:

$$y' = f(t, y) \tag{3.3}$$

and the equation is to be said to be in **normal form.** A differential equation of **order n** is of the form:

$$f(t, y, y', y'', ...y^{(n)}) = 0 \tag{3.4}$$

and is also said to be in **normal form** A typical $n^{th}$ order linear equation is given by

$$y^{(n)} + a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + ... + a_{(n-1)}(t)y' + a_n(t)y = f(t) \tag{3.5}$$

### Definition 3.1.2: Partial Differential Equation (PDE)

A *partial differential equation* (PDE) is an equation that involves an unknown function (the dependent variable) and some of its partial derivatives with respect to two or more independent variables. Mathematically, PDE is of the form;

$$F(t_1, ..., t_n, u, \frac{\partial u}{\partial t_1}, ..., \frac{\partial u}{\partial t_n}, \frac{\partial^2}{\partial t_1 \partial t1}, ..., \frac{\partial^2 u}{\partial t_1 \partial t_n}, ...) \tag{3.6}$$

If F is a linear function of u and its derivatives, then the PDE is called *linear*. An $n^{th}$-order PDE has the highest order derivatives of order n. A simple PDE is

$$\frac{\partial u}{\partial t}(t, y) = 0 \tag{3.7}$$

This relation implies that the function $u(t, y)$ is independent of t. However the equation gives no information on the function's dependence on the variable y. Hence the general solution of this equation is

$$u(t, y) = f(y) \tag{3.8}$$

where f is an arbitrary function of y.

General linear second order PDE is of the form

$$a(t, y)u_{tt} + 2b(t, y)u_{ty} + c(t, y)u_{yy} + d(t, y)u_t + e(t, y)u_y + g(t, y)u = f(t, y) \tag{3.9}$$

where $(t, y) \in \Omega$ is a domain in t - y coordinates.

### Definition 3.1.3: Stochastic Differential Equation (SDE)

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process Davis (2005). In probability theory, a stochastic process or sometimes random process (widely used) is a collection of random variables; this is often used to represent evolution of some random value, or system over time. This is the probabilistic counter part to a deterministic process (or deterministic system). Stochastic Differential Equation is used to model randomness of the underlying asset in valuing insurance liabilities. For example, the asset price behaviour in an interval $\Delta_t$ can be denoted by the SDE given;

$$\Delta S_t = \alpha(A_t, t)\Delta t + \sigma(A_t, t)\Delta w_t, \quad for \quad t \in [0, \infty) \tag{3.10}$$

## 3.4.1 Finite Difference Equation

Let a region $\Omega$ in the $X_t$-plain be covered by a rigid $(x_n, t_j)$. If all the derivatives in the PDE are replaced by difference quotients, the result is the finite difference equation. i.e., $L[u] = f(x,t)$ in $\Omega$ is the PDE is equal to

$$D[U_{nj}] = f_{nj}(x_n, t_j) \quad in \quad \Omega \tag{3.11}$$

The amount by which the solution to $[U] = f$ fails to satisfy the difference equation is called the local truncation error. It can be represented by

$$T_{ij} = D[U_{nj}] - f_{nj} \tag{3.12}$$

The difference Equation

$$D[u_{nj}] = f_{nj}$$

is said to be consistent with PDE $L[U] = f$ is

$$\lim_{h,t \to 0} T_{nj} = 0 \tag{3.13}$$

If $U_{nj}$ is the exact solution to $D[u_{nj}] = f_{nj}$ and $u_{nj}$ is the solution of L[U] $=$ f(PDE) evaluated at $(x_n, t_j)$, the destigmatization error is defined as $U_{nj} - u_{nj}$. The difference method is said to be convergent if

$$\lim_{h,t \to 0} | U_{nj} - u_{nj} |= 0, (x_n, t) \quad in \quad \Omega. \tag{3.14}$$

### 3.4.2 Finite Difference Approximation

Finite difference method seeks to give solution to partial difference equation by a system of algebraic equations. It proceeds by replacing the derivatives in the equation by finite differences. They serve as ways of obtaining numerical solutions to partial differential equations. Types of difference methods are classified according to how we approximate the partial derivative with respect to time. In formulating finite difference method involves the following steps:

- Partial difference equation

- Area of space-time on which the partial difference equation is based must be met.

- Initial boundary conditions to be satisfied.

### 3.4.3 Types of Solving PDE's

The common types of finite difference method for solving PDE's are explicit method, implicit method, and Crank-Nicolson which are types of difference equation. Depending on how we approximate the PDE with respect to time, we have:

1. Explicit finite difference scheme, when we use the forward difference formula

2. Implicit finite scheme, when we use the backwards difference formula

3. Crank-Nicolson finite difference scheme, when we use the centred difference formula.That is finding the average of Explicit and implicit schemes.

   Another way of solving PDE Numerically is the Hopscotch method.

4. Hopscotch, is a method which involves the combination of the explicit and implicit schemes.

These methods differs in terms of stability, accuracy and execution speed.

### 3.4.4 Finite Difference formula of Ordinary Differential Equations (ODE)

There are three commonly used finite difference formulas to approximate first order derivative of a function $f(x)$. They are forward finite difference, backward finite difference and central finite difference.

In this work, the central difference method and Hopscotch method would be used.

Let's consider **Taylor's series** expansion of a function $f(x)$ in the neighbourhood of $x = x_i$

$$f_{i+1} = f_i + \Delta x f_i' + \frac{(\Delta x)^2}{2!} f_i'' + \frac{(\Delta x)^3}{3!} f_i''' + \frac{(\Delta x)^4}{4!} f_i''' + ... \quad (3.15)$$

where $\Delta x = x_{i+1} - x_i$ solving equation 3.15 for $f_i$, we have

$$f_i' = \frac{f_{i+1} - f_i}{\Delta x} - \frac{(\Delta x)^2}{2!} f_i'' - \frac{(\Delta x)^3}{3!} f_i''' - ... \quad (3.16)$$

Using the mean value theorem, equation 3.16 becomes

$$f_i' = \frac{f_{i+1} - f_i}{\Delta x} - \frac{\Delta x}{2} f''(\xi); x_i < \xi < x_{i+1} \quad (3.17)$$

where $0(\Delta x) = -\frac{\Delta x}{2} f''(\xi)$, the order $\Delta x$, indicates the error is proportional to to the step length $(\Delta x)$and also a second derivative of $f$. Hence

$$f_i' \approx \frac{f_{i+1} - f_i}{\Delta x} \quad (3.18)$$

This equation (3.17) is called the **Forward Difference Formula**.

Also,

$$f_{i-1} = f_i - \Delta x f_i' + \frac{(\Delta x)^2}{2!} f_i'' - \frac{(\Delta x)^3}{3!} f_i''' + ... \quad (3.19)$$

This is given by

$$f_i' \approx \frac{f' - fi - 1}{\Delta x} \quad (3.20)$$

with the error term $0(\Delta x) = \frac{\Delta x}{2} f''(\xi)$. Equation 3.20 is called **Backward Difference Formula**.

Finally, subtracting equations 3.19 from 3.15, we get the **central difference formula**. Given by

$$f_i' \frac{f_{i+1} - f_{i-1}'}{2\Delta x} \quad (3.21)$$

with the error $0(\Delta x) = -\frac{(\Delta x)^2}{2} f'''(\xi)$

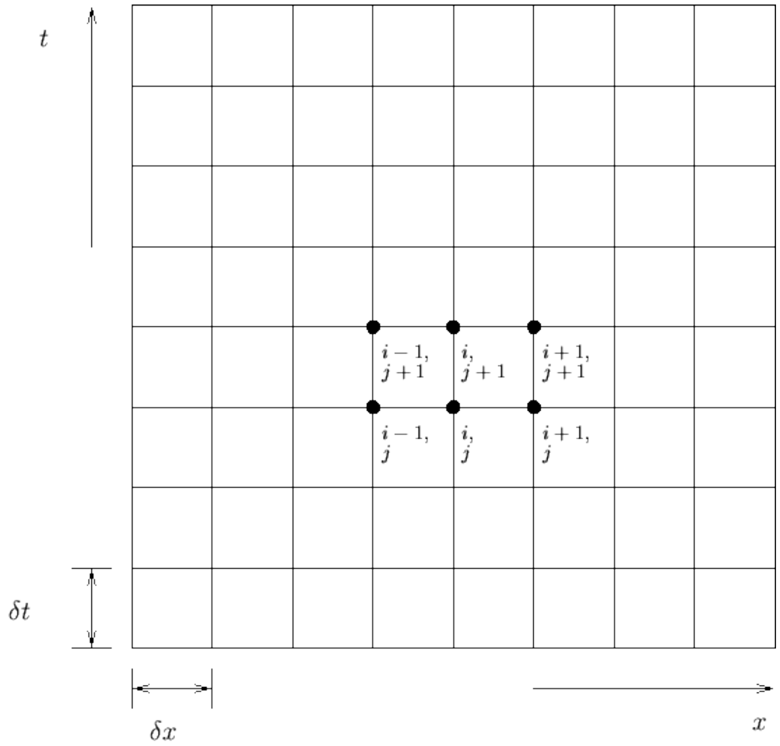


Figure 3.1: Two dimensional grid

### 3.4.5 Finite Difference Approximation for Partial Differential Equations(PDE)

Partial derivatives denotes the local variation of a function with respect to a particular independent variable while all other independent variables are held constant, finite difference approximation of ordinary derivatives can be adapted for the partial derivatives. If there are two independent variables, we use the notation (i,j) to designate the pivot point, and if there are three independent variables, (i,j,k) are used where i,j and k are the counters in the x, y and z directions. Since in many financial and engineering problems, the function $f$ depends on two or more independent variables, hence the need for finite-difference approximation of partial derivatives.

Figure 3.1 above is a two-dimensional finite-difference grid. If we consider the function $f(x, y)$, then the finite-difference approximation for the partial derivative $\dfrac{\partial f(x, y)}{\partial x}$ at $x = x_i$, $y = y_i$ can be found by fixing the value of y at $y_i$ and treating $f(x, y_i)$ as a one-variable function. The forward, backward and central difference of $\dfrac{\partial f}{\partial x}$ can be express as:

$$\frac{\partial f}{\partial x}\mid_{i,j}\approx \frac{f(x_i+\Delta x, y_j)-f(x_i,y_j)}{\Delta x}$$

$$\frac{\partial f}{\partial x}\mid_{i,j}\approx \frac{f(x_i+\Delta x, y_j)-f(x_i,y_j)}{\Delta x} \tag{3.22}$$

$$\frac{\partial f}{\partial x}\mid_{i,j}\approx \frac{f(x_i,y_j)-f(x_i\Delta x, y_j)}{\Delta x} \tag{3.23}$$

$$\frac{\partial f}{\partial x}\mid_{i,j}\approx \frac{f(x_i+\Delta x, y_j)-f(x_i-\Delta x,y_j)}{2\Delta x} \tag{3.24}$$

## Central-Difference Approximation of Second Partial Derivatives

The central-difference approximation of second partial derivatives at $(x_i, y_j)$ can be derived as

$$\frac{\partial^2 f}{\partial x^2}\mid_{i,j}\approx \frac{f(x_i+\Delta x, y_j)-2f(x_i,y_j)+f(x_i-\Delta x,y_j)}{(\Delta x)^2} \tag{3.25}$$

$$\frac{\partial^2 f}{\partial y^2}\mid_{i,j}\approx \frac{f(x_i, y_j+\Delta y)-2f(x_i,y_j)+f(x_i,y_j-\Delta y)}{(\Delta y)^2} \tag{3.26}$$

and

$$\frac{\partial^2 f}{\partial x \partial y}\mid_{i,j}\approx \frac{f(x_i+\Delta x, y_j+\Delta y)-f(x_i+\Delta x,y_j-\Delta y)-f(x_i-\Delta x,y_j+\Delta y)+f(x_i-\Delta x,y_j}{4\Delta x\Delta y} \tag{3.27}$$

### Error of finite-difference approximation of partial derivatives

To find the error associated with finite-difference approximation of partial derivatives, we use Taylor series expansion of f(x,y) around the point $(x_i, y_j)$. That is,

$$f_i\pm 1, j = f_i, j \pm \Delta x\frac{\partial f}{\partial x}\mid_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2}\mid (i,j) \pm \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3}\mid_{i,j} + ... \tag{3.28}$$

$$f_i, j \pm 1 = f_i, j \pm \Delta y \frac{\partial f}{\partial y} |_{i,j} + \frac{(\Delta y)^2}{2!} \frac{\partial^2 f}{\partial y^2} | (i,j) \pm \frac{(\Delta y)^3}{3!} \frac{\partial^3 f}{\partial y^3} |_{i,j} + ... \tag{3.29}$$

Truncating equation 3.28 after the nth order, we have the error

$$R_{x,n} \simeq (-1)^{n+1} \frac{(\Delta x)^{n+1}}{(n+1)!} \frac{\partial^{n+1} f(x,y)}{\partial x^{n+1}} |_{i,j} \tag{3.30}$$

and truncating equation 3.30 after the nth order gives the error

$$R_{y,n} \simeq (-1)^{n+1} \frac{(\Delta y)^{n+1}}{(n+1)!} \frac{\partial^{n+1} f(x,y)}{\partial y^{n+1}} |_{i,j} \tag{3.31}$$

### 3.4.6 Finite difference approximation for two dimensional PDEs

Let's consider a two-dimensional PDE

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = g(x,y) \tag{3.32}$$

such that $a \leq x \leq b$ and $c \leq y \leq d$. If we let $U(a,y) = U_a$, $U(b,y) = U_b$, $U(x,c) = U_c$ and $U(x,d) = U_d$, where $U_a$, $U_b$, $U_c$ and $U_d$ are the boundary conditions at y and x respectively. Note that, $\Delta x$ is not necessarily equal to $\Delta y$, but for this case we let $\Delta x = \Delta y = h$. Let's consider the grid below:

At the generic points

$$\frac{\partial^2 U}{\partial x^2} |_{i,j} + \frac{\partial^2 U}{\partial y^2} |_{i,j} = g_{i,j} \tag{3.33}$$

Using the central difference scheme we have

$$\frac{\partial^2 U_{i,j}}{\partial x^2} \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} \tag{3.34}$$

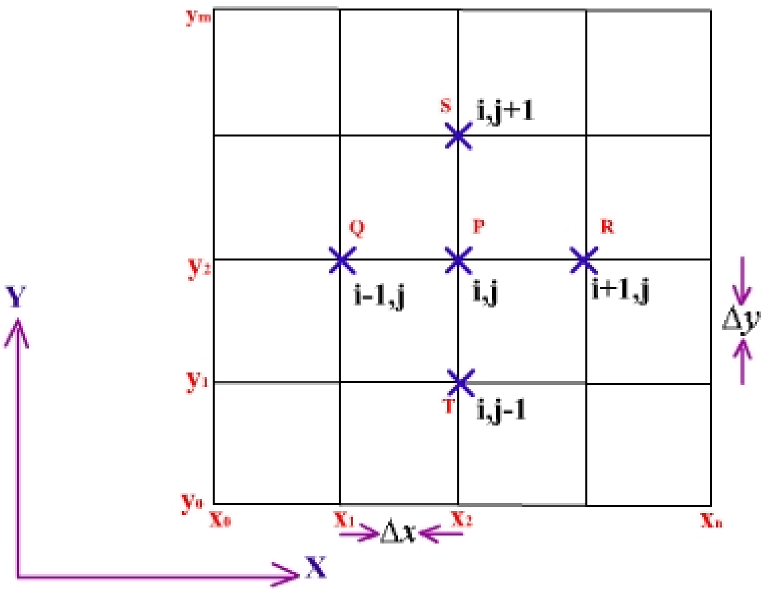


Figure 3.2: Simplified two dimensional grid

and have

$$\frac{\partial^2 U_{i,j}}{\partial y^2} \approx \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2} \tag{3.35}$$

Adding equations 3.34 and 3.35, we have

$$\frac{\partial^2 U_{i,j}}{\partial y^2} + \frac{\partial^2 U_{i,j}}{\partial y^2} \approx \frac{U_{i-1,j} + U_{i+1,j} - 2U_{i,j} + U_{i,j-1} + U_{i,j+1}}{h^2} = g_{i,j} \tag{3.36}$$

$$\Rightarrow U_{i-1,j} + U_{i+1,j} - 4U_{i,j} + U_{i,j-1} + U_{i,j+1} = h^2 g_{i,j} \tag{3.37}$$

At $P_1$: i = 1, j=1

$\Rightarrow U_{0,1} + U_{2,1} - 4U_{1,1} + U_{1,0} + U_{1,2} = h^2 g_{1,1}$

but $U_{0,1} = U_a$, and $U_{1,0} = U_c$

$\Rightarrow -4U_{1,1} + U_{2,1} + U_{1,2} = h^2 g_{1,1} - U_a - U_c$

$$\Rightarrow -4P_1 + P_2 + P_4 = h^2 g_{1,1} - U_a - U_c \tag{3.38}$$

At $P_2 : i = 2, j = 1$

$$P_1 - 4P_2 + P_3 + P_5 = h^2 g2, 1 - U_c \tag{3.39}$$

Using the computational model blow,

Writing the above systems in matrix, we obtain

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 9 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \end{bmatrix}$$

where $b_1 = h^2 g_{1,1} - U_a - U_b, b_2 = h^2 g_{2,1} - U_c, b_3 = h^2 g_{2,1} - U_c, b_4 = h^2 g_{3,1} - U_b, b_4 = h^2 g_{1,2} - U_a, b_5 = h^2 g_{2,2}, b_6 = h^2 g_{3,2} - U_a - U_b, b_7 = h^2 g_{1,3} - U_a - U - d, b_8 = h^2 g_{2,3} - U_d$ and $b_9 = h^2 g_{3,3} - U_b - U_d$

$$If\ \ we\ \ let\ \ A = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} and \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.40}$$

Also $B_1 = b_1 : b_3, B_2 = b_4 : b_6, B_3 = b_7 : b_9, X_1 = P_1 : P_3, X_2 = P_4 : P_6$ and

$X_3P_7 : P_9$, we obtain the matrix

$$\begin{bmatrix} A & I & O \\ I & A & I \\ O & I & A \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{3.41}$$

which is simplified in the form $HX = B$ When systems are expressed in the form $HX = B$, we have several solution techniques in solving it.

### 3.4.7 Solution Techniques

There exist different types of solution techniques. Notable among them are the LU and QR decomposition, Gauss-Jordan Elimination, Gaussian Elimination and iterative methods. The iterative methods include Gauss-Seidel Jacobi and relation methods (Successive Under Relaxation and Successive Over Relaxation-SOR).

#### Iterative Methods

As stated earlier, the common iterative techniques for solving linear systems are Gauss-Seidel, Jacobi and SOR method. The basic idea is solve the $i^{th}$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \tag{3.42}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \tag{3.43}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \tag{3.44}$$

$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \tag{3.45}$$

Solving for $x_1$,$x_2$,$x_3$,$x_4$ in equations 3.42 to 3.45, we have

$$x_1 = -\frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \frac{a_{14}}{a_{11}}x_4 + \frac{b_1}{a_{11}} \tag{3.46}$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \frac{a_{24}}{a_{22}}x_4 + \frac{b_2}{a_{22}} \tag{3.47}$$

$$x_3 = -\frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \frac{a_{34}}{a_{33}}x_4 + \frac{b_3}{a_{33}} \tag{3.48}$$

$$x_4 = -\frac{a_{41}}{a_{44}}x_1 - \frac{a_{42}}{a_{44}}x_2 - \frac{a_{43}}{a_{44}}x_3 + \frac{b_4}{a_{44}} \tag{3.49}$$

Iterative methods are stopped at certain conditions. Below are two possibilities:

1. Iterations are stopped when the norm of the change in the solution vector x from iteration to the next is sufficiently small or

2. When the norm of the residual vector, $\| Ax - b \|$, is below a specified tolerance.

### 3.4.8 Stability, Consistency and Convergence of PDE's

We have a finite difference Scheme produced, when the partial derivatives in the partial differential equation governing a phenomenon are replaced by a finite difference approximation. A partial difference equation is an equation that involves both a function and its partial derivatives.

## Consistency

A finite difference scheme operator is consistent if the operator reduces to the original differential equation as the small increments in the independent variables fades out i.e. $(\Delta_s, \Delta_t \longrightarrow 0)$. For us to get a specific solution to a partial differential equation, additional conditions must be imposed on the solution function. Typically, these conditions occur in the form of boundary values that are prescribed on all/part of the perimeter of the region in which the solution is sought. The nature of the boundary and boundary values are usually the determining factors in setting up an appropriate numerical scheme for obtaining the approximate solution.

## Stability

Stability in this case means the error that is caused as a result of small perturbation in the numerical solution remains bound.

## Convergence

Converge in this work could mean that, the finite -difference solution approaches the true solution as the partial differential equation as the increments $\Delta_x$, $\Delta_t$ go to zero. The basic idea of converges and stability analysis for a linear PDE consist in writing the solution to the equation as a complete Furrier Series and analysing a generic component of the solution. For stability in PDE, we need to get a boundary condition and initial condition.

### 3.4.9 Stability: for a PDE with a bounded solution,

The difference method $D[U_{nj}] = f_{nj}$ is said to be stable if the $E_{nj}$ is the error coming from the computations of the difference equation as one progresses.i.e. If for some constant M and some positive integer $Y \mid E_{nj} \mid < M, (j, Y]$.

**Lax Equivalence theorem:** Given a well-posed initial boundary value problem and a finite difference problem consistent with it, stability is both necessary and sufficient for convergence.

## Von-Newman Stability Criterion:

A difference method for an initial boundary value problem with a bounded solution is Von-Newman stable if extended solution to $D[U_{nj}] = 0$. If the form $U_{nj} = \xi^j e^{kn\Delta x}$ has the property $\mid \xi \mid \leq 1$.

Theorem: For two level difference methods, Von-Newman stability is both necessary and sufficient for stability.

Consider an initial-boundary value problem with N nodes in the x-direction and define a column vector f errors at level j $E_n = (E_{1j}, E_{2j}, ...E_{nj})^T$ for two level difference methods, the errors at levels j and j+1 are related by $E_{j+1} = CE_j$, where $c = n \times n$ matrix.
Let p(c), the spectral radius of c, denote the maximum of the magnitudes of the eiganvalues of c.

### Matrix Stability Criterion:

A two-level difference method for an initial boundary value problem with a boundary solution is matrix stable if $p(c) \leq 1$
Matrix stability criterion is a necessary condition for for stability of two level-level method.
Theorem: let C be a symmetric or similar to a symmetric matrix, where by all eigenvalues of C are C. Then matrix stability is necessary and sufficient for stability.

### 3.4.10 Stock Price and Contract Dynamic Model

If we let $A_t$ denotes market value of the insurer's asset portfolio, $L_t$ denotes the policyholder's account balance and $B_t = A_t - L_t$ is the bonus reserve at time t. This basically describes a simplified form of the liability and asset situation in relation to a given contract but not necessarily a company's balance sheet.
It is assumed that the insurance companies operates in a frictionless complete and arbitrage-free financial market over a time interval [0,T], where time T corresponds to the expiration date of the insurance contract. As the insurance contract expires at time T, the insurance company closes and its assets are liquidated and distributed to stakeholders Chunli and Jing (2014). Since charges are disregarded, the insured's account balance at time zero, $L_o$ equals the single up-front premium $P$; $L_o = P$. If the contract is lapsed at time $vo \in 1, ..., T$, the insured (policyholder) receives the current account balance $L_{vo}$. Also, the insured

may surrender his/her policy at any time during the contract and is entitled to a surrender value and share holders are assumed to be paid dividends during the anniversaries as compensations for the adopted risk. According Bjarke et al. (2001), stocks and bonds are are liquid assets insurance companies largely invest in making the observation of market prices easy. For this reason, asset(A) is traded. L(t) is the policy account balance, which can be considered to be the funds set aside to cover the insurance contract liability of a distributed reserve. B(t) is the buffer, which protect the policy reserve from unfavourable fluctuations in the asset base. The dynamic asset side is modelled when considering the policy interest rate.

### 3.4.11 Asset Dynamic Models

Under this section the geometric Brownian motion with deterministic interest rate and a geometric Brownian motion with stochastic interest rate Metescu et al. (2013), where classical Black-Scholes set up is used. The asset process evolves according to stochastic differential equation under the risk-free measure Q.

$$dA_t = rA_t dt + \sigma_A A_t dW_t, A_o = P(1 + x_o) \tag{3.50}$$

where r is the constant short rate, $\sigma_A$ is the volatility of the asset process A and W is the standard Brownian motion under Q(martingale).
Asset prices move randomly because of the efficient market hypothesis. The hypothesis give two basic and important informations:

- In the present price is reflected fully the past history and holds no further information.
- An asset new information is responded immediately to by the markets.

The arrival of new information about asset prices as time goes on can be said to be asset price modelling. Anticipated prices of assets follows the Markov process based on the above assumptions by Wilmott, P., Howison, S. and Dewyne, J. (1995).

## 3.5 Black-Scholes set up for the Valuation model

Black and Scholes in 1973, tried their best to formulate PDE's that governs contingent claims behaviour. They also, solved the partial differential equation, which today has brought changes in the general picture of how to price derivatives as financial instruments.

Lets consider the concept of arbitrage and Hedging, which allow the establishment of relationship between prices and hence determine these prices when using the Black-Scholes set up to the valuation model.

### 3.5.1 Hedging

Hedging is defined as risk trading carried out in financial markets. Businesses do not want market-wide risk considerations which they can not control and to interfere with their economic activities. Any market parameter who sells derivatives on his own account will say that hedging is key to pricing. If a contract is not hedged, one can sell it at any price, even the right one, and still lose money. The price of the contract must be the cost of the hedge, plus margin, and the profit/loss of the deal will depend crucially on the hedge being effective. Hence hedging, is said to be a financial strategy used to reduce the risk of investing in financial markets. This suggest that as one is in business, hedging is an important aspect to consider when one really wants to make a business success. Delta hedging is one very important hedging strategy. The delta, $\Delta$, of the option is defined as the change of the option price with respect to the change in the price of the underlying asset. In other words, it is the first derivative of the option price with respect to the stock price:

$$\Delta = \frac{\partial V}{\partial A}$$

### 3.5.2 Arbitrage

The basic concept that underlays the theory of financial pricing and hedging is arbitrage. Also, described as a way of offsetting potential loss or gains incurred by a companion investment. Finance theory has assumed that investments that give guarantee returns are based on risk-free- existence with no default Ali (2013) cited Wilmott et al. (1995). The returns made on the highest risk-free portfolio (assets) is the same return from a bank if same amount were put in the bank. Taking the advantage of a price difference between two or more markets: striking a combination of matching deals that capitalize upon the imbalance, the profit margin being the difference between the market prices. In theory, an arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state: in simple terms, it is the possibility of a risk-free profit at zero cost (Ali,2013).

### 3.5.3 The Black-Scholes Analysis

When developing a model for the price of an asset, it will be important to do the modelling for the price of an asset itself. Theory of Economics and data show that, returns made on assets consists of two parts; First, values of the asset increases with time at the drift rate (r). Secondly, values of an asset as time changes depends on a lot of influential factors. These changes are expressed by a random variable X with unique properties. A price change of a risk-less asset as time goes is:

$$\Delta A = Ar\Delta t \tag{3.51}$$

but no stock is risk-less. Risk is modelled by the stochastic term X with properties:

a. $\Delta X = \phi\sqrt{\Delta t}$, $\phi \sim N(0,1)$

b. The value of $\Delta X$ in the time interval $(\Delta t)$ does not depend on $\Delta X$.

For any time steps.

$$E\left[\sum_{i=1}^{n} \Delta x_i\right] = 0 \tag{3.52}$$

$$Var\left[\sum_{i=1}^{n} \Delta A_i\right] = n\Delta t \tag{3.53}$$

Hence, it could be possible to model an asset behaviour as:

$$\Delta A = rA\Delta t + \sigma A \Delta X, \tag{3.54}$$

AS $\Delta t \to 0$.

$$dA = rAdt + \sigma AdX \tag{3.55}$$

Equation (3.55) is the **asset price model** and we make **returns** with each asset price variability, which is defined as the variability over (divided) by the original value. Considering a small subsequent time interval dt, during which A changes to A + dA as shown, the return on the asset $\frac{dA}{A}$ is modelled.

If $\sigma = 0$

$\frac{dA}{A} = rdt$

$$\int_0^t \frac{dA}{A} = \int_t^T rdt \tag{3.56}$$

$In(A_t - A_0) = r(T-t)$

$In(\frac{A_t}{A_0}) = r(T-t)$

$\frac{A_t}{A_0} = e^{r(T-t)}$

$$A_t = A_0 e^{r(T-t)} \tag{3.57}$$

Where $A_0$ is the value of the asset at time t=0, thus $\sigma = 0$, the asset is totally deterministic and the future price of the asset can be predicted with certainty.

From equation (3.55), the Asset A follows an *Itô* process according to Wilmott et al. (1995).

Black-Scholes analysis assumes that the prices of asset behave as just demonstrated and the following are the assumptions it follows:

1. Asset price follows log normal random walk. i.e. the stock price can go up or down with the same probability. Also, the stock price in time $t+1$ is independent from the price in time t.

2. Risk-free interest rate r and the asset volatility $\sigma$ are known functions of time over the life of the option/contract.

3. There is no transaction cost associated with hedging a portfolio.

4. Underlying asset has no dividends during the life of the option/contract.

5. There are no arbitrage possibilities.

6. The Black-Scholes model assumes European style options which can only be exercised on the expiration date. American style options can be exercised at any time during the life of the option. Thus, making American style option the more valuable due to their greater flexibility.i.e. trading of the underlying asset can take place continuously.

7. Short selling is permitted and the assets are divisible: The Black-Scholes model assumes that markets are perfectly liquid and it is possible to purchase or sell any amount of assets or options or their fractions at any given time(liquidity).

Insurers are actually selling a naked put option to the buyer of the insurance. Therefore, the method of finding the value of put options can be applied in the valuation of the life insurance contract.

Consider a constant V(A, t), where V is not necessary a call or put but the value of the whole portfolio of different contract. Computations from stochastic calculus ($It\hat{o}, s$ process) . Since the portfolio is a function of the value of the underlying asset A and time t; both expected drift and volatility can change over time. A n-dimensional Itô process, is a process that can be represented by

$$X_t = X_0 + \int_0^t asds + \int_0^t bsdWs. \tag{3.58}$$

In which a and b are functions of the value of the underlying asset (A) and time (t).

Where W is an m-dimensional standard Brownian motion and a and b are n-dimensional $(n \times m)$-dimensional $F_t$-adapted process respectively.

$X_t = X_0 + a_t t + b_t W_t$

$$dX_t = a_t dt + b_t W_t \quad where \quad X_0 = 0 \tag{3.59}$$

where the n-dimensional stochastic differential equation has the form

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t \tag{3.60}$$

We can represent the above equation as $X_t = X + \int_0^t a(X_s, S)ds + \int_0^t (X_s, t)dWs$. where $(X_s, S)$ is the function of the stock price and time.Hence suppose x follows a general $It\hat{o}$ process

$$dx = a(x, t)dt + b(x, t)dz \tag{3.61}$$

Tailor's expansion of solving diffusion process $f(x_t)$ is given by

$$df(x_t) = f'(x_t)dx_t + \frac{1}{2}f''(x_t)(dx_t)^2 + ... \tag{3.62}$$

all terms beyond the second order is zero.

Dividing by $d_t$ and let $d_t \to 0$ gives

$$\frac{df(X_t)}{dt} = f'(X_t)\frac{dX_t}{dt} + \lim_{dt \longrightarrow 0} \frac{1}{2}f''(X_t)\frac{(dX_t)^2}{dt}$$

since

$$\lim_{dt \longrightarrow 0} \frac{(dX_t)^2}{dt} = \frac{dX_t}{dt}(\lim_{dt \longrightarrow 0} dX_t) = 0$$

.

the second term on the right hand side must vanish given the chain rule

$$\frac{df(X_t)}{dt} = f'\left(X_t \frac{dX_t}{dt}\right)$$

but when we replace the second term by a non-differentiable Brownian motion ($B_t^2$). Hence, the Taylor's theorem to the second-order is

$$df(B_t) = f'(B_t)\partial B_t + f''(B_t)(\partial B_t)^2$$

taking limit $\partial t \to 0$ effectively involves replacing $\partial$ by d ignoring the second order and higher order terms. However with Brownian motion, it turns out that the second-order term ($dB_t^2$) cannot be ignored and it must be changed to dt.

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2$$

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

Next we want to substitute in for $\partial x_t$ in terms of $\partial_t$ and $\partial B_t$. Let's introduce one final complication,that is the form of $It\hat{o}$'s lemma that we will consider is for functions not just for a diffusion process but also function that explicitly depend on time. In other words, function of the form $f(t, x_t)$. Using the chain rule:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

now using chain rule for $f(t, x_t)$ then

$$\partial(t, x_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2.}$$

Consider a contract of $V(A, t)$, where V is not necessarily a call or a put but the value of the whole portfolio of different contracts. We use $It\hat{o}, s$ lemma, that states that if x follows a general $It\hat{o}$ process

$$dx = a(x, t)dt + b(x, t)dz \tag{3.63}$$

and $f = f(x, t)$ then

$$df = (\frac{\partial f}{\partial x}a(x, t) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2})(b(x, t)^2)dt + \frac{\partial f}{\partial x}b(x, t)dz \tag{3.64}$$

Applying $It\hat{o}, s$ lemma to the value of the whole portfolio $V(A, t)$, we have

$$dv = (\frac{\partial V}{\partial A}rA + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)dt + \frac{\partial V}{\partial A}\sigma A dz \tag{3.65}$$

This expression is difficult to solve due to the presence of $dz$ (the stochastic term). The main idea behind Black-Scholes model is for us to create a portfolio which consist of shares of assets and derivatives that is instantaneously risk-less and thus, the noisy part eliminated in equation (3.63). Portfolio at any time consist of one long position in the derivative and a short position of exactly $\frac{\partial V}{\partial A}$ shares of the underlying assets. The portfolio value is calculated by

$$\Pi = V - \frac{\partial V}{dA}A. \tag{3.66}$$

The instantaneous change of the portfolio $\Pi$ is given as:

$$d\Pi = \partial V - \frac{dV}{\partial A}dA \tag{3.67}$$

putting equations (3.59),(3.65), and (3.66) together, we have

$$d\Pi = -\frac{\partial V}{\partial A}dA + (\frac{\partial V}{\partial A}rA + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)dt + \frac{\partial V}{\partial A}\sigma A dZ$$

$$d\Pi = -\frac{\partial V}{\partial A}(rAdt + \sigma AdZ) + (\frac{\partial V}{\partial A}rA + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)dt + \frac{\partial V}{\partial A}\sigma AdZ$$

$$\Rightarrow d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)dt \tag{3.68}$$

The instantaneously change in the risk-less portfolio is independent of the stochastic term dZ as in equation (3.68). To maintain a portfolios risk-less property, then at every point in time t must be balanced. $\frac{\partial V}{\partial A}$ cannot be maintained for different values of t.

The return on the amount $\Pi$ invested in risk-less assets would see growth of $r\Pi$ dt in time dt considering the concept of arbitrage, supply and demand with assumptions that there are no transaction cost. Since it is a risk-free portfolio, the assumption that there are no arbitrage opportunities shows that it must attract exactly the risk-free rate. That is

$$\Pi = r\Pi t$$

$$\Rightarrow d\Pi = r\Pi dt. \tag{3.69}$$

we have:

$$\partial\Pi = r\Pi dt = r(V - \frac{\partial V}{\partial A}A)dt = (\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)dt$$

By simplifying gives the the Black-Scholes partial differential equation(PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2\frac{\partial^2 V}{\partial A} - rV = 0 \tag{3.70}$$

In analysing contract/options on a path-dependent quantity, such as the the average asset price, Black-Scholes approach become inadequate. This is because there are many realisations of the asset prices random walk leading to the current value, any two of these give a different value for the path-dependent Wilmott et al. (1995). This led to the introduction of the variable S in addition, A and t which will measure the relevant path-dependent quantity. since we use continuously sampled quantities for the pay off of average strike option, our average will depend on a time integral.

To look at the time integrals of the random walk, consider European option/contract with pay-off depending on A and on

$$\int_0^T f(A(T),T)dT \tag{3.71}$$

Note: $f$ is said to be function in terms of A and t. The pay-off at expiring for average strike call is

$$max(A - \frac{1}{T}\int_0^T A(T)dT, 0) \tag{3.72}$$

We have $f(A,t) = A$. Let

$$P = \int_0^t f(A(T),T)dT \tag{3.73}$$

We treat P, A and t as independent variables since the history of the asset price is independent of the current price. Note that, P varies depending on the variation of the random walk. The pay-off depends on both P and A, the value of an exotic path-dependent contract is written as V(A,P,t). This means that, the value of the option depends on the current asset price A, the time t and history of the asset P. The changes in P due to small changes in t and A is given by the stochastic differential equation

$$P(t+dt) = P + dp = \int_0^{t+dt} f(A(T),T)dt \tag{3.74}$$

After simplifying equation (3.74); the order of dt, we have

$$P + dp = \int_0^t f(A(T),T)dt + f(A(t),t)dt \tag{3.75}$$

where dp = f(A,t)dt.

The above equation (3.75) is a Stochastic Differential Equation (SDE) of P without random the random component. Depending on A,t and P, to value a contract, we apply $It\hat{o}'s$ assumption to V(A,P,t) and this gives

$$dV = \sigma\frac{\partial V}{\partial A}dX + (\frac{1}{2}\sigma^2 A^2\frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} + \frac{\partial V}{\partial t} + f(A,t)\frac{\partial V}{\partial P})dt \tag{3.76}$$

Since dp introduces no new source of risk, it is anticipated that the option can be hedge using the underlying asset only. Considering arbitrage leads to

$$\frac{\partial V}{\partial t} + f(A,t)\frac{\partial V}{\partial p} + \frac{1}{2}\sigma^2 A^2\frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} - rV = 0 \tag{3.77}$$

Note that, the path-dependent quantity P is updated discretely and is therefore constant between sampling dates. The PDE for the option value between sampling dates becomes just the basic Black-Scholes equation with P treated as a parameter. So in valuing the path-dependent option with discrete sampling, we start from the expiry date, when the option value is known (i.e. equal to the payoff) and work backwards.
Hence we have:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2\frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} - rV = 0 \tag{3.78}$$

### 3.5.4 Definition 3.1.6 Portfolio

A $portfolio$ is a position in the market that consists in long and short positions in one or more stocks and other securities. The value of a portfolio could be denoted algebraically as a linear combination of stock prices and other securities' values:

$$P = \sum_{j=1}^{n} a_j S_j + \sum_{k=1}^{m} b_k F_k.$$

The market participant holds $a_j$ units of stock $S_j$ and $b_k$ units in derivatives $F_k$. The coefficients are positive for long positions and negative for short positions. For instance, a portfolio given by 2F -3S means that we buy 2 securities and sell

3 units of stock (a position with 2 securities long and 3 stocks short).

### Definition 3.1.7 Risk-less portfolios

A portfolio P is called $risk-less$ if the increments dP are completely predictable. In this case the increments' value dP should equal the interest earned in the time interval dt on the portfolio P. This can be written as

$$dP = rPdt$$

Where r denotes the risk-free rate and for the seek of simplicity the rate r is usually assumes as a constant.

## 3.6 Dividend Paying Asset

Equation (3.78) known as the Black-Scholes equation, is said not earn dividends as its assumption as far as the contract is valid.
Let $\Phi$ be a known constant continuous divided yield. This means the policyholder receives dividend $(\Phi A\Delta t)$ in the range of the time interval $\Delta t$. After the dividend, the share value is lowered making the expected rate return r be $(r - \Phi)$. So the geometric Brownian motion model in equation (3.59) becomes.

$$dA = (r - \Phi)Adt + \sigma AdX \tag{3.79}$$

and the Black-Scholes equation becomes:

$$\frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + (r - \Phi)A\frac{\partial V}{\partial A} - rV = 0 \tag{3.80}$$

From above equation 3.80, it is considered that at constant rate dividends are paid continuously Hull (2003).

## 3.7 Survival Rates and The Black-Scholes Model

The prevalence of long term diseases that co-exist simultaneously in the same individual over time commonly known as multimorbidity has increased globally, partly due to noticeable improvements in health systems and ageing populations. This phenomenon is gradually becoming a clinical representation of the elderly population.

For this reason, most studies are largely configured to the prevalence and impact of multimorbidity. The prevalence of multimorbidity has often been investigated in few countries, particularly Australia, Sweden and Canada. A systematic review of various studies on the prevalence of multimorbidity in different countries published between 1980 and September 2010 revealed that, the prevalence of multimorbidity varies from 3.5% to 98.5% in primary care and 13.1% to 71.8% in general population. In Australia, the overall prevalence of multimorbidity was estimated as 37.1%. 29.0% 0f patients who attended a general practice and 25.5% of the general population. Moreover, the overall prevalence of multimorbidity in Dutch population was estimated as 13% and among those older than 55 years the prevalence was estimated as 37% Gabriel et al. (2014).

Multimorbidity increases the rate of mortality and a variety of adverse health outcomes. The state of a person with multimorbidity over over time depends on the efficacy of medication given and estimated survivorship may be non-monotonic. Parametric survival distributions that allow for non-monotonic hazards can be utilised under the assumption that risk changes over time. Although there is a growing body of knowledge regarding multimorbidity, only few studies have analysed non random chronic disease clusters, often referred to as multimorbidity patterns or clusters Gabriel et al. (2014).

To model and analyse the probability of a policyholder having the chance of developing the multimorbidity condition, the exponential distribution is employed for the data to be fitted. Exponential distribution model is used because it accommodates the non-monotonicity of a hazard function and does not depend on the "memory". The survival survival rate of a policyholder is obtained from the relationship between a survival function and a hazard function.

## 3.8 Parametric Survival distribution

There are some distributions that have been used frequently in the literature of survival analysis, such as the Log-Logistic, Log-Normal, Gamma, Weibull and Exponential distributions.

To model and analyse the probability of a policyholder developing multimorbidity condition, the exponential and Weibull distributions are employed for the data to be fitted. Weibull distribution model is used because it accommodates the non-monotonicity of a hazard function and does not depend on the "memory" of the policyholder's condition. The survival rate of a policyholder is obtained from the relationship between a survival function and a hazard function

### 3.8.1 Exponential Distribution

The simplest distribution for survival time is the Exponential distribution, with density function as

$$p(t_i/\lambda_i) = \lambda_i exp(-\lambda_i t_i).$$

Exponential distribution is said to be probability distribution that describes the time between events in a Poisson process (a process in which events occur continuously and independently at a constant average rate) with a function exp(n, rate = 1) in R.The exponential distribution has a unique property of "lack of memory", because of its constant hazard rate $\lambda$. The probability to failure within a particular time interval depends only on the length, not on the location of this interval. In real-world application, the assumption of a constant rate is rarely satisfied.

### 3.8.2 Log-logistic AFT Model

The log-logistic model is a parametric model which accommodates the acceleration failure time (AFT) assumption. The log-linear form of the AFT model is given by this equation;

$$log(T_i) = X_i\beta + \sigma \in_i$$

Where $log(T_i)$ is the log of failure time. $\beta$ is the vector of model parameters corresponding to the covariate vector , $X_i, \in$ is a random error term, and $\sigma$ is a scale parameter. If the errors in the model are assumed to follow a logistic distribution, then the resulting model is the log-logistic. The log-logistic model has a survival function S(t) of the form;

$$S(t) = \frac{1}{1 + (\lambda t)^{\gamma}}$$

and the corresponding hazard function h(t) is given by

$$h(t) = \frac{\lambda\gamma(\lambda t)^{\gamma - y}}{1 + (\lambda t)^{\gamma}}$$

Where the shape parameter $\gamma > 0$ and $\gamma = \frac{1}{\sigma}$. The log-logistic distribution allows for non-monotonic hazards, i.e. those that can increase initially and then decrease. Specifically, if $\gamma > 1$, the hazard increases with duration to a maximum point and then decreases over time. On the other hand, if $\gamma \leq 1$ the hazard decrease with time or duration.

After incorporating the survival rate (S) into the Black-Scholes model, then we have:

$$\frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + (r - S)A\frac{\partial V}{\partial A} - rV = 0$$

## 3.9 Continuous Dividends

To construct a models in which dividends could be paid continuously might sound unreasonable for a single stock, but will not be unreasonable for options index funds. Let's consider you purchased some number of shares in a company 700 funds, it is expected that you continue to receive dividends at many different times in a year. Suppose at constant rate r the asset pays dividend (dividend yield), that is during time $dt$, $rAdt$ dividends are received. Considering the well known stochastic model, we have: $dA = rAdt + \sigma AdX - rAdt = (r - S)Adt + \sigma AbX$. Proceeding in the same fashion as in the derivation of the Black-Scholes PDE, lets treat $r - s$ in place of $r - \phi$ as in equation 3.78 and we have

$$\frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + (r - S)A\frac{\partial V}{\partial A} - rV = 0 \tag{3.81}$$

Assumptions of the model are:

- The survival rate S is lies between 0 and 1
- The survival rate is the median survival rate for all rates as time changes or whiles the contract still active.

**Note: For this model (continuous dividend paying paying asset), replace** $r$ with $r$ - $\phi$

### 3.9.1 Numerical Methods

A closed form solutions does not exist for American and Asian options, the only way market participant will be able to obtain a price is by using an appropriate numerical method. Some of these numerical methods are Monte-Carlos Simulation, Binomial tree methods, finite difference method and Risk-neutral valuation methods.

In this paper, we will compare the Crank-Nicolson and Hopscotch methods as the explicit method is conditionally stable in the valuation of life insurance contract embedded with surrender option.

## 3.10 Finite Difference Approximation for Black-Scholes DE

By way of approximating the DE over the integrating area by systems of equations; obtaining numerical solutions to the Black-Scholes partial differential equation is finite difference methods. The explicit method, the implicit method the Crank-Nicolson method and the Hopscotch method are the well known methods for finding solution to the Black-Scholes Partial differential equations. In formulating a PDE problem, three components are considered, these are:

1. The PDE.

2. Space-time in which the PDE is desired to be satisfied.

3. Initial conditions and boundary(auxiliary) to be satisfied.

   The finite difference methods differ in stability, accuracy and execution speed though they seem related. This work will consider the Crank-Nicolson method and the Hopscotch method of solving the Black-Scholes partial differential equation.

### Discretization of Black-Scholes Equation

Finite difference method requires the discretization of the pricing of the partial differential equation and the boundary conditions using a forward difference, a backward differential or central difference approximation. **The Black-Scholes PDE in terms of dividend paying asset and the surrender rate** is written as:

$$\frac{\partial V(A_t,t)}{\partial t}+\frac{\sigma^2 A^2 \partial^2 V(A_t,t)}{2\partial A_t^2}+\frac{(r-S)A_t \partial V(A_t,t)}{\partial A_t}$$

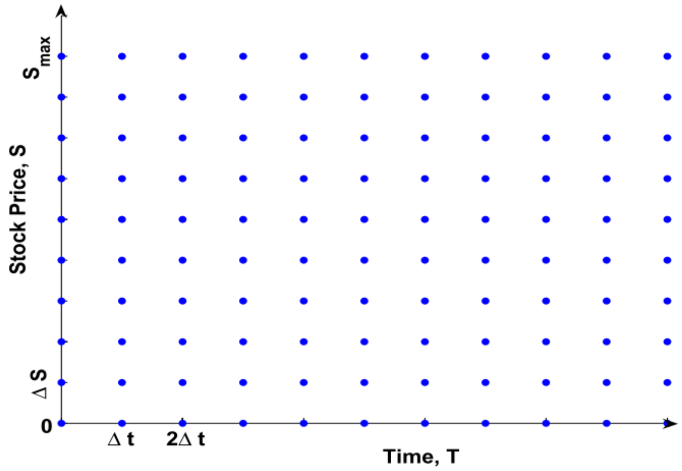


Figure 3.3: The mesh points for the finite difference approximation

in a simplified form is written as:

$$\frac{\partial V}{\partial t} + \frac{\partial^2 A^2 \partial^2 V}{2\partial A^2} + \frac{(r-S)A\partial V}{\partial A} = rV \tag{3.82}$$

**Note: For continuous dividend paying asset and the surrender value, replace** r with $r - S$

The equation is discretized with time (t), and with respect to the price of the underlying asset(A). The (A,t) plane is divided into a grid form using approximate infinitesimal steps ($\Delta A$) and $\Delta t$ by small fixed finite steps. An array of N+1 equally spaced grid points $t_0, t_1, ..., t_N$ is used to discretize the time $t_{i+1} - t_i = \Delta_t and \Delta_t = T/N$. Also, since asset price cannot be negative and it is assumed that, $A_{max} = 2A_0$. We also have M+1 equally spaced mesh points $A_0 \, to \, A_M$ and this used to discretize price of the asset derivative with $A_{j+1} - A_j = \Delta_j$ and $\Delta A = \frac{A_{max}}{M}$. We are able to compute the solution at discrete points with a total grid points of (M+1)(N+1). Using the grid coordinates (i,j), based on a rectangular region on $(A, t)$ plane with sides $(0, S_{max})$ and $(0, T)$ we have the (i,j) points on the grid corresponds to time $i\Delta t \;\; for \;\; i = 0 \;\; to \;\; N$ and the asset price $j\Delta A$ for j=0 to M. As shown in Figure 3.3 below where

Representing (A,t) in the grid by $V_i, j$, their respective expansions of $V(A+\Delta A, t)$

and $V(A-\Delta A,t)$ in Taylor series are:

$$V(A+\Delta A,t) = V + \frac{\partial V}{\partial A}\Delta A + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\Delta A^2 + 0(\Delta A^3) \quad and \tag{3.83}$$

$$V(A-\Delta A,t) = V - \frac{\partial V}{\partial A}\Delta A + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\Delta A^2 - 0(\Delta A^3) \tag{3.84}$$

Changing the derivatives to difference equation, then equation (3.88) gives the forward difference equation:

$$\frac{\partial V}{\partial A} = \frac{V(A+\Delta A,t) - V(A,t)}{\Delta A} + 0(\Delta A)$$

$$\frac{\partial V}{\partial A} \approx \frac{V_{i,j+1} - Vi,j}{\Delta A} \tag{3.85}$$

and equation 3.78 gives the backward difference equation:

$$\frac{\partial V}{\partial A} = \frac{V(A,t) - V(A-\Delta A,t)}{\partial A} + 0(\Delta A)$$

$$\frac{\partial V}{\partial A} \approx \frac{V_{i,j} - V_{i,j-1}}{\partial A} \tag{3.86}$$

Subtracting equation (3.84) from equation (3.83) gives the central difference:

$$\frac{\partial V(A,t)}{\partial A} = \frac{V(A+\Delta A,t) - V(A-\Delta A,t)}{2\Delta A} + 0(\Delta A^2)$$

$$\frac{\partial V(A,t)}{\partial A} \approx \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta A} \tag{3.87}$$

To estimate the second order partial derivatives, we use the central approximation. By adding equations (3.83) and (3.84), we get

$$\frac{\partial^2 V}{\partial A^2} = \frac{V(A+\Delta A,t) - 2V(A,t) + V(A-\Delta A,t)}{\Delta A^2}$$

$$\frac{\partial^2 V}{\partial A^2} \approx \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta A^2} \tag{3.88}$$

Expanding $V(A, t + \Delta t)$ in Taylor series, we obtain:

$$\frac{\partial V}{\partial t} = \frac{V(A, t + \Delta t) - V(A, t)}{\partial t} + 0(\Delta t)$$

$$\frac{\partial V}{\partial t} \approx \frac{V_{i+i,j} - V_{i,j}}{\partial t} \tag{3.89}$$

## Boundary and Initial Conditions

The solution to Black-Scholes PDE can either have uncountable of solutions or no solution because of boundary or initial conditions. Hence, the need to state the boundary and initial conditions for a contract like the European style contract, whose value (payoff) is given by $maximum(K - A_T, 0)$. When an asset is lost it value, a put is worth its strike price K. This is

$$V_{i,0} = K \tag{3.90}$$

for $i = 0, 1, ..., N$ The value of the contract approaches zero (0) as the price of the asset increases. Hence $A_{max} = A_M$ and this means

$$V_{i,M} = 0 \tag{3.91}$$

$fori = 0, 1, ..., N$

Since the value of the contract is known at time T, we can find the initial condition

$$V_{N,j} = maximum(K - j\Delta A, 0) \tag{3.92}$$

$forj = 0, 1, ..., M$

The initial condition results in the value of the contract V at the end of the period of the condition and not the beginning, implying a backward move from maturity to time zero. The American style is also handled almost the same way,

$$V_{N,j} = maximum(j\Delta A - K, 0) \tag{3.93}$$

$for j = 0, 1, ..., M$

### 3.10.1 Approaches of Finite Difference Scheme

We will consider four approaches of the finite difference: Implicit finite difference method, explicit finite difference method, Crank-Nicolson method and Hopscotch method. Let's consider the European contract stated in equation (3.87), suppose that T is the maturity of the asset and $A_{max}$ is the maximum asset price. Let $M\Delta A = A_{max}$ and $N\Delta T = TV_{I,J}$ denotes the asset value at $(i\Delta t, j\Delta t)$.

#### Explicit Finite Difference Method

We can have an expression giving the subsequent value next value $V_{i,j}$ explicitly in terms of $V_{i+1,j-1}$ and $V_{i+1,j+1}$. since we know the value of the contract at maturity time. We therefore discretize Black-Scholes partial differential equation (PDE) in equation (3.86) by denoting the forward difference for time and central difference for the asset price discretization. We have;

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A}[V_{i+1,j+1} - V_{i+1,j-1}] + \frac{\sigma^2 j^2 A^2}{2\Delta A^2}[V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}] = rV_{i,j} \tag{3.94}$$

Now making $V_{i,j}$ the subject, we obtain

$$V_{i,j} = \frac{1}{1 + r\Delta t}[\alpha_j V_{i+1,j-1} + \beta_j V_{i+1,j} + \gamma_j V_{i+1,j+1}] \quad \text{for} \quad i = 0, 1, ..., N \quad and \quad j = 1, 2, ..., M \tag{3.95}$$

Where the weights $\alpha_j, \beta_j \, and \gamma_j$ are given by

$$\left\{ \begin{array}{c} \alpha_j = \dfrac{\sigma^2 j^2 \Delta t}{2} - \dfrac{(r-S)j\Delta t}{2} \\ \beta_j = 1 - \sigma^2 j^2 \Delta t \\ \gamma_j = \dfrac{(r-S)j\Delta t}{2} + \dfrac{\sigma^2 j^2 \Delta t}{2} \end{array} \right\} \tag{3.96}$$

Since the finite difference for the discretization of the time is accurate to $0(\Delta t)$ and that of the central difference of the asset discretisation is $0(\Delta t, \Delta t^2)$.

The weights, which happens to be the risk-free probabilities of the (3) assets prices $A - \Delta A, A \quad and \quad A + \Delta A \quad at \quad t + \Delta t$ adds up to one (1) and $\dfrac{1}{1 + r\Delta t}$ is the discounted factor. But we can get negative probabilities unless further restrictions are imposed on $\Delta t$ and $\Delta A$. This gives results which would not converge to the solution of PDE and this shows the explicit method is unstable unless those restrictions are imposed on $\Delta t$ and $\Delta A$. The conditions to have non negative probabilities is $\sigma^2 j^2 \Delta t < 1$ **and** $r < \sigma^2 j$ Hull (2003). This system is represented in the matrix form as

$$\begin{bmatrix} \beta_0 & \gamma_0 & 0 & \dots & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{M-1} & \beta_{M-1} & \gamma_{M-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_M & \beta_M \end{bmatrix} \begin{bmatrix} V_{i+1,0} \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M} \end{bmatrix} = \begin{bmatrix} V_{i,0} - \alpha_0 \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} - C_M \end{bmatrix} \tag{3.97}$$

These series of equations can be put in the form; $AV_{i+1,j} = V_{i,j} \quad$ for $\quad j = 0\, to\, M$ and the error ones (terms) are ignored since the boundary conditions cater for them.

The vectors of the stock prices $V_{i+1,j}$ is known at time T from the initial condition we solve for $V_{i,j}$ by working backward using the matrix above; made up of the probabilities $\alpha_j, \beta_j \quad and \quad \gamma_j$ which are known probabilities and the backward iteration leads to the contract value obtained at time-zero.

## Stability of Finite Difference Scheme

In the asset price and time discretization are two fundamental sources of truncation error. A numerical scheme is characterized by consistency, stability and convergence. These fundamental factors are linked by **Lax Equivalence theorem** which states; that for a *given a posed linear initial value problem and a consistent finite difference scheme, stability is the important and sufficient condition for the convergence* (Smith, 1985). The eigenvalues $\lambda_i$ of $n \times n$ matrix

$$\begin{bmatrix} y & z & & & \\ x & y & z & & \\ \cdots & \cdots & \cdots & & \\ & x & y & z \\ & & x & y \end{bmatrix}$$

is given by$\lambda_i = y + 2(\sqrt{xz})cos\dfrac{i\pi}{N+1}$ for i= 1,2,...,N where x, y and z may be real or complex number. The system is stable if $\mid \lambda_i \mid \leq 1$ (Smith, 1985).

### 3.10.2 Stability of the Explicit Finite Difference Scheme

Analysing the stability of Explicit Difference Method, we can use the matrix (A), which is symmetric. When $\lambda_i$ is the ith eiganvalue of the matrix (A), we get;

$\parallel A \parallel_2 = \rho(A) = maximum \mid \lambda_i \mid$

Then eigenvalues $\lambda_i$ are produced by

$$\lambda_i = \beta_j + 2(\sqrt{\alpha_j \gamma_j})cos\frac{i\pi}{N} \quad for\, i = 1, 2, ...N-1 \tag{3.98}$$

Substituting $\alpha, \beta\, and\, \gamma$ into equation (3.100).

**Note**, the the identity;

$$Cos\frac{i\pi}{N} = \left[1 - 2sin^2\frac{i\pi}{N}\right]$$

$$\Rightarrow \lambda_i = 1 - \sigma^2 j^2 \Delta t + 2\sqrt{\left(\frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2}\right)\left(\frac{\sigma^2 j^2 \Delta t}{2} + \frac{rj\Delta t}{2}\right)} Cos\frac{i\pi}{N}$$

$$\beta_j + 2(\sqrt{\alpha_j \gamma_j}) = 1 - \sigma^2 j^2 \Delta t + 2\sqrt{\left(\frac{\sigma^2 j^2 \Delta t}{2}\right)^2 - \left(\frac{rj\Delta t}{2}\right)^2}$$

$$= 1 - \sigma^2 j^2 \Delta t + 2\sqrt{\left(\frac{\sigma^2 j^2 \Delta t}{2}\right)^2 \left(1 - \frac{0.25r^2 j^2 \Delta t^2}{0.25\sigma^4 j^4 \Delta t^2}\right)}$$

$$= 1 - \sigma^2 j^2 \Delta t + \frac{2\sigma^2 j^2 \Delta t}{2}\sqrt{\left(1 - \frac{r^2}{\sigma^4 j^2}\right)}$$

$$= 1 - \sigma^2 j^2 \Delta t + \sigma^2 j^2 \Delta t \left(1 - \frac{r^2}{\sigma^2 j^2}\right)^{1/2}$$

$$\Rightarrow \lambda_i = 1 - \sigma^2 j^2 \Delta t + \sigma^2 j^2 \Delta t \left(1 - \frac{r^2}{\sigma^2 j^2}\right)^{1/2} \left(1 - 2sin^2\frac{i\pi}{N}\right)$$

We expand $\left(1 - \frac{r^2}{\sigma^4 j^2}\right)^{1/2}$ using binomial expansion; where $n = 1/2$, $x = \frac{-r^2}{\sigma^2 j^2}$

We obtain

$1 + 1/2\left(\frac{-r^2}{\sigma^4 j^2}\right) + ... + ...$ Ignoring other terms; $\lambda_i \approx 1 - 2\sigma^2 j^2 \Delta t sin^2\frac{i\pi}{2N}$

**Note** $\quad Sin^2\frac{i\pi}{N} = 1$

The scheme is stable when

$\parallel A \parallel_2 = max \mid -1 \leq 1 - 2\sigma^2 j^2 \Delta t sin^2\frac{i\pi}{2N} \leq 1$

$\Rightarrow = -1 \leq 1 - 2\sigma^2 j^2 \Delta t sin^2\frac{i\pi}{2N} \leq 1 \quad for \quad i = 1, 2, ..., N-1$

As $\Delta t \longrightarrow 0$, $N \longrightarrow \infty \quad and \quad sin^2\frac{(N-1)\pi}{2N} \longrightarrow 1$

Now we make $1 - 2\sigma^2 j^2 \Delta t$ stable; by solving

$0 \leq \sigma^2 j^2 \Delta t \leq 1$

Hence $0 \leq \sigma^2 j^2 \Delta t \leq 1$.

Therefore the scheme stability, convergence and consistency for $0 \leq \sigma^2 j^2 \Delta t \leq 1$.

Therefore, the explicit finite difference method is conditionally stable.

## The Implicit Finite Difference Method

let's substitute equations 3.91, 3.92 and 3.33 into equation (3.86) and express $V_{i+1,j}$ implicitly in terms of the unknowns $V_{i,j-1}, V_{i,j}$ and $V_{i,j+1}$. That is, we discretize Black-Scholes PDE in equation 3.86 using FD for time and central difference for the asset price. We have

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A}[V_{i,j+1} - V_{i,j-1}] + \frac{\sigma^2 j^2 \Delta A^2}{2\Delta A^2}[V_{i,j+1} - 2V_{i,j} + V_{i,j-1}] = rV_{i+1,j} \tag{3.99}$$

making $V_{i+1,j}$ the subject in equation 3.99

$$V_{i+1,j} = \frac{1}{1 - r\Delta t}[x_j V_{i,j-1} + y_j V_{i,j} + z_j V_{i,j+1}] \tag{3.100}$$

for i = 0,1,...,N and j = 1,2,...,M - 1.

Similarly to the explicit method, the implicit method is accurate to $0(\Delta t, \Delta A^2)$. The weights $x$, $y$ and $z$ are given by

$$\left\{ \begin{array}{c} x_j = \dfrac{(r-S)j\Delta t}{2} - \dfrac{\sigma^2 j^2 \Delta t}{2} \\ y_j = 1 + \sigma^2 j^2 \Delta t \\ z_j = -\dfrac{\sigma^2 j^2 \Delta t}{2} - \dfrac{(r-S)j\Delta t}{2} \end{array} \right\} \tag{3.101}$$

The system of equations in tridiagonal matrix form is

$$\begin{bmatrix} V_{i+1,0} - x_0 \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M} - z_M \end{bmatrix} = \frac{1}{1 - r\Delta t} \begin{bmatrix} y_0 & z_0 & 0 & \dots & 0 & 0 & 0 \\ x_1 & y_1 & z_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{M-1} & y_{M-1} & y_{M-1} \\ 0 & 0 & 0 & \dots & 0 & x_M & y_M \end{bmatrix} \begin{bmatrix} V_{i,0} \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} \end{bmatrix} \tag{3.102}$$

The system is written as $AV_{i,j} = V_{i+1,j}$ for $j = 0, 1, ..., M$. The matrix A has $y_j = 1 + \sigma^2 j^2 \Delta t$ is positive in its diagonal form. A matrix produced can be non-singular if the diagonal elements product are also producing non-zero results. We can solve by working out for the inverse of the matrix A, $A^{-1}$. Applying the boundary conditions with (3.104) changes the element $y_o, y_m = 1$ and $z_0, x_M = 0$ in the matrix A.

## Implicit Finite Difference Method Stability

The eigenvalue is produced by

$$\lambda_i = y_j + 2\sqrt{(x_j, z_j)} cos\frac{i\pi}{N} \quad for \quad i = 1, 2, ..., N-1 \tag{3.103}$$

Substituting for x, y and z in (3.107) and simplifying, then we obtain

$$\lambda_i = 1 + \sigma^2 j^2 \Delta t + \sigma^2 j^2 \Delta t \left[1 - \frac{r^2}{\sigma^4 j^2}\right]^{\frac{1}{2}} \left[1 - 2sin^2\frac{i\pi}{2N}\right]$$

we then have

$$\lambda_i \approx 1 + 2\delta^2 j^2 \Delta t - 2\sigma^2 j^2 \Delta t sin^2\frac{i\pi}{2N} \tag{3.104}$$

The change from cos to sin is based on the truncation of the binomial expansion. This scheme stability is achieved when

$$\parallel A \parallel_2 max \mid 1 + 2\sigma^2 j^2 \Delta t sin^2\frac{i\pi}{2N} \mid \leq 1$$

this implies that

$$-1 \leq 1 + 2\sigma^2 j^2 \Delta t sin^2\frac{i\pi}{2N} \leq 1 \tag{3.105}$$

As $\Delta t \longrightarrow 0, N \longrightarrow \infty$ and $sin^2\frac{(N-1)\pi}{2N} \longrightarrow 1, \mid \lambda_i \mid \leq 1$.

Therefore the scheme is unconditionally stable, convergence and consistent.

### 3.10.3 The Crank Nicolson Method

This method is defined as the **hybrid** between the implicit finite difference method and explicit finite difference method. When we apply the Crank-Nicolson idea to the Black-Scholes model (finding average of Implicit and Explicit schemes), we figure out the following grid equation:

$$\frac{V_{i+1,j}-V_{i,j}}{\Delta t}+\frac{rj\Delta A}{2\Delta A}\left[V_{i+1,j+1}-V_{i+1,j-1}+V_{i,j+1}-V_{i,j-1}\right]$$

+

$$\frac{\sigma^2 j^2 \Delta A_2}{4\Delta A^2}\left[V_{i,j-1}-2V_{i,j}+V_{i,j+1}+V_{i+1,j-1}-2V_{i+1,j}+V_{i+1,j+1}\right]$$

$$=\frac{1}{2}\left[rV_{i,j}+rV_{i+1,j}\right] \tag{3.106}$$

From above, re-arranging will give

$$_jV_{i,j-1}+\beta_j V_{i,j}+\gamma_j V_{i,j+1}=x_j V_{i+1,j-1}+y_j V_{i+1,j}+z_j V_{i+1,j+1} \tag{3.107}$$

for i = 0,1,...,N and j = 1,2,...,M-1.

Where our parameters in equation (3.07); $\alpha_j,\beta_j,\gamma_j,x_j,y_j \quad and \quad z_j$ are given by

$$\left\{\begin{array}{r}
\alpha_j=\dfrac{(r-S)j\Delta t}{4}-\dfrac{\sigma^2 j^2\Delta t}{4}\\
\beta_j=1+\dfrac{(r-S)\Delta t}{2}+\dfrac{\sigma^2 j^2\Delta t}{2}\\
\gamma_j=-\dfrac{\sigma^2 j^2\Delta t}{4}-\dfrac{(r-S)j\Delta t}{4}\\
x_j=\dfrac{\sigma^2 j^2\Delta t}{4}-\dfrac{(r-S)j\Delta t}{4}\\
y_j=1-\dfrac{(r-S)\Delta t}{2}+\dfrac{\sigma^2 j^2\Delta t}{2}\\
z_j=\dfrac{(r-S)j\Delta t}{4}+\dfrac{\sigma^2 j^2\Delta t}{4}
\end{array}\right\} \tag{3.108}$$

Equation (3.108) system of equations could be expressed as $C\vec{V_i} = D\vec{V}_{i+1}$, resulting in triadiagonals and this will give

$$\begin{bmatrix} \beta_0 & \gamma_0 & 0 & \dots & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{M-1} & \beta_{M-1} & \gamma_{M-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_M & \beta_M \end{bmatrix} \begin{bmatrix} V_{i+1,0} \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M} \end{bmatrix} =$$

$$\begin{bmatrix} y_0 & z_0 & 0 & \dots & 0 & 0 & 0 \\ x_1 & y_1 & z_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{M-1} & y_{M-1} & y_{M-1} \\ 0 & 0 & 0 & \dots & 0 & x_M & y_M \end{bmatrix} \begin{bmatrix} V_{i,0} \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} \end{bmatrix} \quad (3.109)$$

## Solution to System

The are elements of vectors $V_{i+1}$ known at time T. We can show the system in equation (3.109) as $V_i = C^{-1}DV_{i+1}$. We obtain the value of V as the value of the life insurance contract by iterating many times from time T to time zero. The diagonal entries of the matrix C is $\beta_j 1 + \frac{r\Delta t}{2} + \frac{\sigma^2 j^2 \Delta t}{2}$ and its positive with non-zero diagonal elements. Hence, the matrix is non-singular as the diagonal entries are non-zero.

### 3.10.4 Accuracy of Crank-Nicolson Method

The Crank-Nicolson method with the accuracy $0(\Delta t^2, \Delta A^2)$, making it more accurate than the explicit and implicit method. From equating the central and symmetric central differences and expand $V_{i+1,j}$ by Tailor series at $V_{i+\frac{1}{2},j}$ we have:

$$V_{i+\frac{1}{2},j} = V(t+\frac{\Delta t}{2}, A).$$

$$\Rightarrow V_{i+1,j} = V_{i+\frac{1}{2},j} + \frac{\partial V}{2\partial}\Delta t + 0(\Delta t^2) \tag{3.110}$$

Expanding $V_{i,j}$ at $V_{i+\frac{1}{2}}, j$ gives

$$V_{i,j} = V_{i,\frac{1}{2},j} - \frac{\partial V}{2\partial t}\Delta t + 0(\Delta t^2) \tag{3.111}$$

Adding equations (3.110 and 3.111) and finding the average, we have

$$\frac{V_{i,j} + V_{i+1,j}}{2} = V_{i+\frac{1}{2},j} + 0(\Delta t^2) \tag{3.112}$$

This implies that

$$V_{i+\frac{1}{2},j-1} - 2V_{i,+\frac{1}{2},j} + V_{i+\frac{1}{2},j+1} = \frac{1}{2}[V_{i,j-1} - 2V_{i,j} + V_{i,j+1}] + \frac{1}{2}[V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}] + 0(\Delta t^2) \tag{3.113}$$

From above equation (3.113), the right-hand side of the equation represents the difference central at i and i + 1. Diving by $\Delta A^2$ we obtain

$$\frac{\partial^2 V(t+\frac{\Delta t}{2}, \Delta T)}{\partial A^2} = \frac{1}{2}\left[\frac{\partial_2 V(t,A)}{\partial A^2} + \frac{\partial^2 V(t+\Delta t, A)}{\partial A^2}\right] + 0(\Delta t^2, \Delta A^2) \tag{3.114}$$

This is the symmetric CDA (3.114). The subscript j is arbitrary and we deduce the central DA as:

$$V_{i+\frac{1}{2},j+1} - V_{i+\frac{1}{2},j-1} = \frac{1}{2}[V_{i,j+1} - V_{i,j-1}] + \frac{1}{2}[V_{i+1,j+1} - V_{i+1,j-1}] + 0(\Delta t^2) \tag{3.115}$$

Dividing by $2\Delta A$, we get

$$\frac{\partial V(t+\frac{\Delta t}{2}, \Delta T)}{\partial A} = \frac{1}{2}\left[\frac{\partial V(t,A)}{\partial A} + \frac{\partial V(t+\Delta t, A)}{\partial A}\right] + 0(\Delta t^2, \Delta A) \tag{3.116}$$

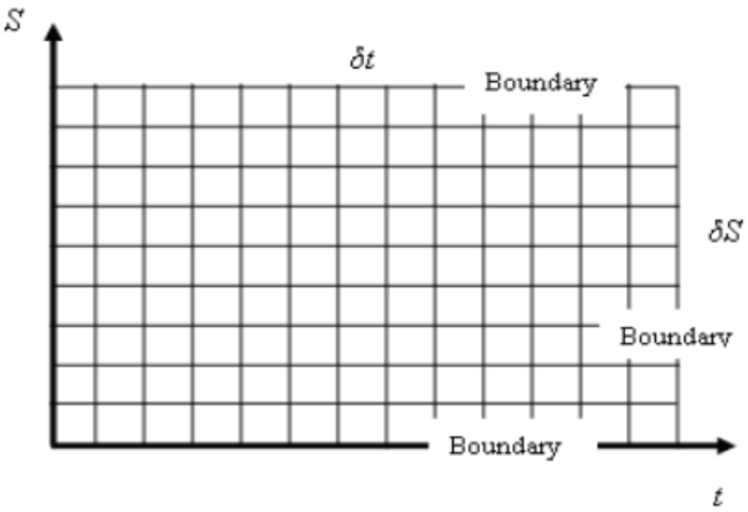


Figure 3.4: Partial Difference in grid form when solved

and this is referred to as the first order central difference approximation. Subtracting equation (3.115) from (3.114), we obtain the approximation of $\frac{\partial V}{\partial t}$ at $(t+\frac{1}{2}\Delta t, A)$, that is:

$$\frac{\partial V(t+\frac{\Delta t}{2}, \Delta T)}{\partial A} = \frac{V_{i+1,j}-V_{i,j}}{\Delta t} + 0(\Delta t^2) \tag{3.117}$$

Therefore, the Black-Scholes formula centred at $(t+\dfrac{1}{2}\Delta t, A)$ has a FDA as

$$\frac{V_{i+1,j}-V_{i,j}}{\Delta t}+\frac{(r-\lambda)j\Delta A}{4\Delta A}[V_{i,j+1}-V_{i,j-1}+V_{i+1,j+1}-V_{i+1,j-1}]+\frac{\sigma^2 j^2\Delta A^2}{4\Delta A^2}[V_{i,j-1}-2V_{i,j}+V_{i,j+1}+V_{i+1} \tag{3.118}$$

Kerman (2002), re-arranging equation (3.122) we get equation of the form (3.111)which is the exact Crank-Nicolson scheme. the scheme has a leading order $0(\Delta t^2, \Delta A^2)$.

## Hopscotch Method

After solving the PDE, we then create mesh (or grid) as in figure 3.4 above.

If we combine the forward- and backwards difference and place the nodes as in the figure 3.5 above.

Calculations of explicit and implicit are done as we move from one node to

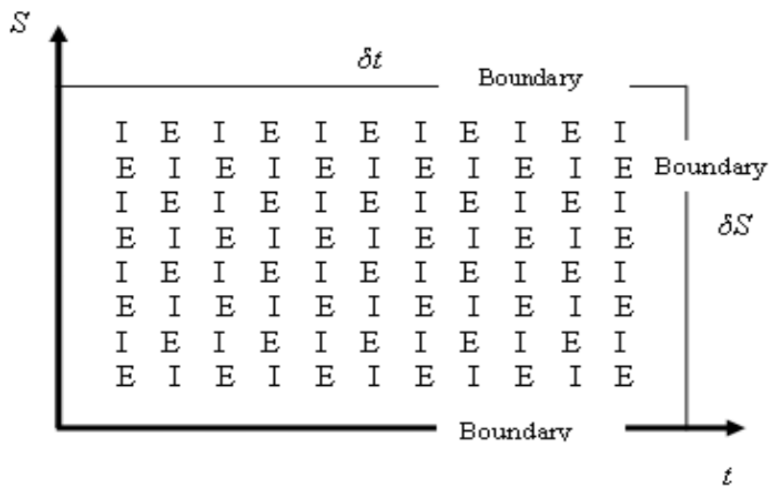


Figure 3.5: Combined and Backward difference placed in the nodes

the other but we alternate these calculations as we move from node to node. Making sure that at each time, we first of all do the calculations at the 'explicit nodes' in the usual way. We then do calculations at the 'implicit nodes' without solving a set of simultaneous equations because the values at the adjacent nodes would have been calculated. Furthermore, mixing the nodes in this way, we get almost the same same accuracy as in the Crank-Nicolson scheme. That is: the Hopscotch method, as well as the Crank-Nicolson method, can avoid the numerical instability.

*Hopscotch* method can be used in finding solutions to parabolic and EPDEs in two or more state variables Zhao (2006). Financial applications regarding the utility of Hopscotch has not been realised yet. The idea is to divide the mesh points in the two-dimensional x-y mesh (ih, jh) as follows:

$i + j$ odd

$i + j$ even

The Hopscotch consists of two 'sweeps'. In the first sweep (and subsequent odd-numbered sweeps) the mesh points $i + j$ is odd, are calculated based on current values (time level n) at the neighbouring points. This is defined as:

$$\frac{V_{ij}^{n+1} - V_{ij}^{n}}{K} = \Delta_x^2 V_{ij}^n + \Delta_y^2 V_{ij}^n \quad for \quad (i+j) \quad odd. \tag{3.119}$$

For the second sweep at the same time $n+1$ the same calculation is based at the next node(even). This second sweep is fully implicit, the scheme is:

$$\frac{V_{ij}^{n+1} - V_{ij}^{n}}{K} = \Delta_x^2 V_{ij}^{n+1} + \Delta_y^2 V_{ij}^{n} \quad for \quad (i+j) \quad even. \tag{3.120}$$

In the second and subsequent even-numbered time steps, the roles of the implicit(I) and explicit (E) are interchanged.

Lets consider the grid form of the Hopscotch Scheme in two steps:

a) An Explicit Scheme

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{\Delta A}[V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}] = rV_{i,j} \tag{3.121}$$

For even values of (i,j) i.e. either both be even or odd, this leads to the difference scheme;

$$V_{i,j} = \frac{1}{1+r\Delta t}[a_j V_{i+1,j-1} + b_j V_{i+1,j} + c_j V_{i+1,j+1}] \tag{3.122}$$

for i = 0,1,2,..., N and j = 1,2,...,M.

b) An Implicit Scheme

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A}[V_{i,j+1} - V_{i,j-1}] + \frac{\sigma^2 j^2 \Delta A^2}{2\Delta A}[V_{i,j+1} - 2V_{i,j} + V_{i,j-1}] = rV_{i+1,j} \tag{3.123}$$

for odd values of (i,j) i.e. either both be even or odd, this leads to the difference scheme;

$$V_{i+1,j} = \frac{1}{1-r\Delta t}[\alpha j V_{i,j-1} + \beta j V_{i,j} + \gamma j V_{i,j+1}] \tag{3.124}$$

for i = 0,1,2,...,N and j = 1,2,...,M-1.

# CHAPTER 4

# ANALYSIS

## ANALYSIS AND RESULTS

## 4.1 Introduction

In this chapter, we look at application of the modified Black-Scholes partial differential equation, Crank-Nicolson and Hopscotch difference schemes in the valuation of life insurance contract. This chapter compares and contrasts the convergence of the modified model and Black- Scholes based on the assumptions of the models used in this work.

## 4.2 Matlab Implementation

The matrices that are obtained by using the Crank-Nicolson and Hopscotch finite difference schemes are generally large tridiagonal matrices and requires more computational time. For this reason, R-studio and Matlab were used to enable me find the solutions to the systems. See Appendix I for R-studio codes and, and appendix I and II for Matlab codes for Crank-Nicolson and Hopscotch method respectively. R-studio codes were implemented for the survival function and Matlab codes was also implemented for Crank-Nicolson and Hopscotch method. See appendix III and IV for Matlab codes.

## 4.3 Stability of Crank-Nicolson Method and Hopscotch

The table below shows the eigenvalues of the matrix of the scheme as $N \to \infty$.

Table 4.1: The eigenvalues of the Crank-Nicolson method as $N \to \infty$

| N = 100 | | N = 500 | | N = 1000 | | N = 2000 | | N = 4000 | |
|---|---|---|---|---|---|---|---|---|---|
| j | $\lambda_j$ | j | $\lambda$ | j | $\lambda$ | j | $\lambda$ | j | $\lambda$ |
| 93 | 1.0572 | 493 | 1.0119 | 993 | 1.0060 | 1993 | 1.0030 | 3993 | 1.0015 |
| 94 | 1.0398 | 494 | 1.0088 | 994 | 1.0044 | 1994 | 1.0022 | 3994 | 1.0011 |
| 95 | 1.0284 | 495 | 1.0062 | 995 | 1.0016 | 1995 | 1.0016 | 3995 | 1.0008 |
| 96 | 1.0189 | 496 | 1.0040 | 996 | 1.0020 | 1996 | 1.0010 | 3996 | 1.0005 |
| 97 | 1.0111 | 497 | 1.0023 | 997 | 1.0012 | 1997 | 1.0006 | 3997 | 1.0003 |
| 98 | 1.0055 | 498 | 1.0011 | 998 | 1.0006 | 1998 | 1.0003 | 3998 | 1.0001 |
| 99 | 1.0020 | 499 | 1.0004 | 999 | 1.0002 | 1999 | 1.0001 | 3999 | 1.0000 |

Source:Ali (2013)

The table 4.1 indicates that $N \to \infty$, the eigenvalues approaches one (1) showing the stability of the Crank-Nicolson's method. Also the this method is with an accuracy of $0(\Delta t^2, \Delta A^2)$ and that also indicates how accurate the results is to the actual value. Note: The stability of Hopscotch is done in the same manner.

## 4.4 Comparing the convergence of the Crank-Nicolson and Hopscotch methods

The data from a company used are as follows: Asset price, A= 50, strike price, K = 52, risk-free interest rate, r = 0.05, surrender period t = 2 years, maturity period, T= 30 years, and $\sigma = 0.02231$. The surrender value of the life insurance contract is 5.4650 with the value at maturity being 8.220 for non-dividend paying asset (see table 4.2).

In the tables, the values in the bracket are the difference between the actual values and values obtained from the various numerical methods.

## 4.5 Simulation of survival(S) using R-studio package under exponential distribution.

The rate of being multimorbid is obtained from simulation of the survival rate with sample size of 60. The simulation is done 10,000 times with the help of the R software from which the values of lower and upper confidence intervals were obtained. The lower and upper values for maximum, minimum, mean and median were gotten, from which a random value was picked to be fixed into the Black-Scholes model. See tables (4.5 and 4.6). The last last values of both lower and confidence intervals are picked and the following were calculated from it;

### For Exponential Distribution

Random values were picked between 0.234057 and 0.04544183 for minimum survival rate, 0.6914513 and 0.4209266 for maximum survival rate, 0.4539849 and 0.2148305 for mean survival rate, and finally 0.4574076 and 0.2158757 for median survival rate.

### For Weibull Distribution

Random values were picked between 0.3387227 and 0.5783236 for minimum survival rate, 0.8723033 and 0.9792022 for maximum survival rate, 0.5981787 and 0.8293827 for mean survival rate, and finally 0.5931416 and 0.83586830 for median survival rate.

**Note**: where survival rate is defined as the rate of developing multimorbidity condition or being multimorbid.

Table 4.2: Comparison of the two methods in the valuation of surrender option with no survival rate (S). Surrender value at t =2 years. Expected value = 5.4650

| No. of steps | Crank-Nicolson | Hopscotch |
|---|---|---|
| 30 | 5.4204(.0446) | 5.4387(.0302) |
| 90 | 5.4465(.0185) | 5.4483(.0167) |
| 150 | 5.4503(.0147) | 5.4512(.0138) |
| 270 | 5.4541(.0109) | 5.4534(.0116) |
| 330 | 5.4546(.0104) | 5.4545(.0105) |
| 450 | 5.4552(.0098) | 5.4549(.0101) |
| 570 | 5.4556(.0094) | 5.4550(.0100) |
| 630 | 5.4557(.0093) | 5.4552(.0098) |
| 720 | 5.4559(.0091) | 5.4543(.0107) |
| 780 | 5.4559(.0091) | 5.4543(.0107) |
| 810 | 5.4560(.0090) | 5.4554(.0096) |
| 870 | 5.4560(.0090) | 5.4554(.0096) |

Table 4.3: Comparison of the two methods in the valuation of surrender option with minimum rate (S) = 0.04 of being multimorbid. Where S is the minimum of the simulated survival rate of the upper and lower confidence intervals (under exponential distribution) Surrender value at t =2 years.

| No. of steps | Crank-Nicolson | Hopscotch |
|---|---|---|
| 30 | 6.9215 | 7.7545 |
| 90 | 6.9305 | 7.7762 |
| 150 | 6.9317 | 7.7807 |
| 270 | 6.9335 | 7.7838 |
| 330 | 6.9337 | 7.7845 |
| 450 | 6.9337 | 7.7853 |
| 570 | 6.9339 | 7.7858 |
| 630 | 6.9339 | 7.7860 |
| 720 | 6.9340 | 7.7862 |
| 780 | 6.9340 | 7.7863 |
| 810 | 6.9340 | 7.7863 |
| 870 | 6.9340 | 7.7864 |

Table 4.4: Comparison of the two methods in the valuation of surrender option with maximum rate (S) = 0.5 of being multimorbid. Where S is the maximum of the simulated survival rate of the upper and lower confidence intervals (under exponential distribution). Surrender value at t =2 years.

| No. of steps | Crank-Nicolson | Hopscotch |
|---|---|---|
| 30 | 77.9123 | 80.9370 |
| 90 | 77.9100 | 80.8274 |
| 150 | 77.9098 | 80.8247 |
| 270 | 77.9097 | 80.8274 |
| 330 | 77.9097 | 80.8289 |
| 450 | 77.9097 | 80.8290 |
| 570 | 77.9097 | 80.8294 |
| 630 | 77.9097 | 80.8296 |
| 720 | 77.9097 | 80.8302 |
| 780 | 77.9097 | 80.8304 |
| 810 | 77.9097 | 80.8303 |
| 870 | 77.9097 | 80.8305 |

Table 4.5: Comparison of the two methods in the valuation of surrender option with mean rate (S) = 0.3 of being multimorbid. Where S is the mean of the simulated survival rate of the upper and lower confidence intervals(under exponential distribution). Surrender value at t =2 years.

| No. of steps | Crank-Nicolson | Hopscotch |
|---|---|---|
| 30 | 36.0721 | 38.3565 |
| 90 | 36.0962 | 38.4275 |
| 150 | 36.0981 | 38.4394 |
| 270 | 36.0992 | 38.4470 |
| 330 | 36.0994 | 38.4488 |
| 450 | 36.0994 | 38.4501 |
| 570 | 36.0995 | 38.4510 |
| 630 | 36.0995 | 38.4514 |
| 720 | 36.0995 | 38.4520 |
| 780 | 36.0995 | 38.4522 |
| 810 | 36.0995 | 38.4522 |
| 870 | 36.0995 | 38.4525 |

Table 4.6: Comparison of the two methods in the valuation of surrender option with minimum rate (S) = 0.3 of being multimorbid. Where S is the minimum of the simulated survival rate of the upper and lower confidence intervals (under Weibull distribution) Surrender value at t =2 years.

| No. of steps | Crank-Nicolson | Hopscotch |
|---|---|---|
| No. of steps | Crank-Nicolson | Hopscotch |
| 30 | 36.0721 | 38.3565 |
| 90 | 36.0962 | 38.4275 |
| 150 | 36.0981 | 38.4394 |
| 270 | 36.0992 | 38.4470 |
| 330 | 36.0994 | 38.4488 |
| 450 | 36.0994 | 38.4501 |
| 570 | 36.0995 | 38.4510 |
| 630 | 36.0995 | 38.4514 |
| 720 | 36.0995 | 38.4520 |
| 780 | 36.0995 | 38.4522 |
| 810 | 36.0995 | 38.4522 |
| 870 | 36.0995 | 38.4525 |

Table 4.7: Comparison of the two methods in the valuation of surrender option with minimum rate (S) = 0.8 of being multimorbid. Where S is the minimum of the simulated survival rate of the upper and lower confidence intervals (under Weibull distribution) Surrender value at t =2 years.

| No. of steps | Crank-Nicolson | Hopscotch |
|---|---|---|
| No. of steps | Crank-Nicolson | Hopscotch |
| 30 | 182.9764 | 184.1583 |
| 90 | 183.0559 | 183.6542 |
| 150 | 183.0507 | 183.6136 |
| 270 | 183.0487 | 183.6037 |
| 330 | 183.0484 | 183.6049 |
| 450 | 183.0482 | 183.6026 |
| 570 | 183.0480 | 183.6019 |
| 630 | 183.0480 | 183.6017 |
| 720 | 183.0480 | 183.6018 |
| 780 | 183.0480 | 183.6021 |
| 810 | 183.0480 | 183.6018 |
| 870 | 183.0479 | 183.6021 |

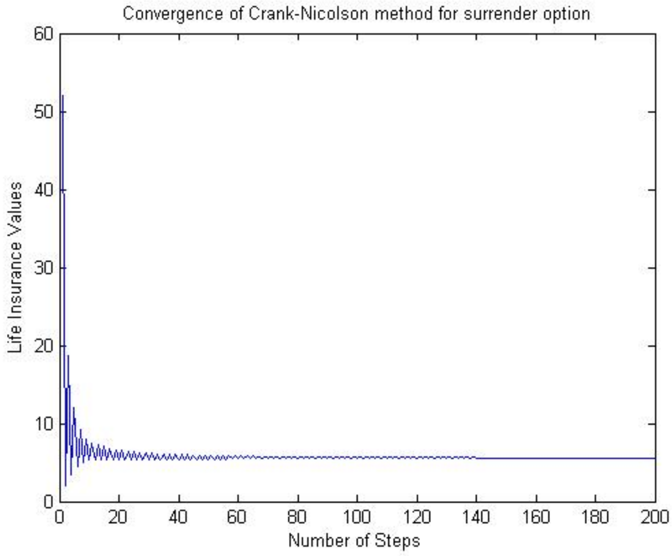


Figure 4.1: Chart on Crank-Nicolson Method for valuation Surrender Option

Table 4.8: Comparison of the two methods in the valuation of surrender option with minimum rate (S) = 0.6 of being multimorbid. Where S is the minimum of the simulated survival rate of the upper and lower confidence intervals (under Weibull distribution) Surrender value at t =2 years.

| No. of steps | Crank-Nicolson | Hopscotch |
|---|---|---|
| No. of steps | Crank-Nicolson | Hopscotch |
| 30 | 106.2292 | 109.1596 |
| 90 | 106.2197 | 108.9223 |
| 150 | 106.2184 | 108.9084 |
| 270 | 106.2179 | 108.9072 |
| 330 | 106.2178 | 108.9083 |
| 450 | 106.2177 | 108.9077 |
| 570 | 106.2177 | 108.9078 |
| 630 | 106.2177 | 108.9083 |
| 720 | 106.2177 | 108.9083 |
| 780 | 106.2177 | 108.9085 |
| 810 | 106.2177 | 108.9084 |
| 870 | 106.2177 | 108.9086 |

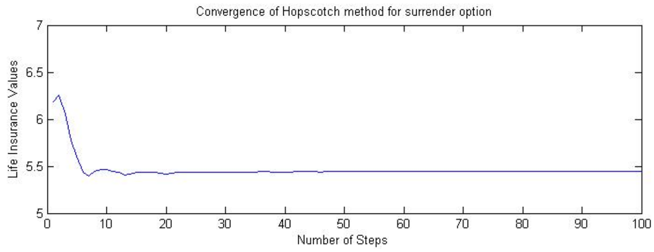


Figure 4.2: Chart on Hopscotch Method for the valuation of Surrender Option

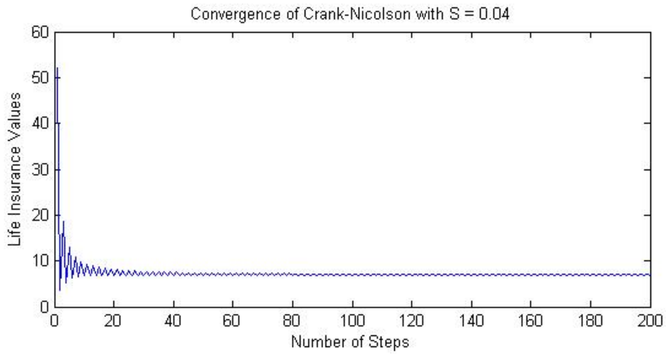


Figure 4.3: Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.04

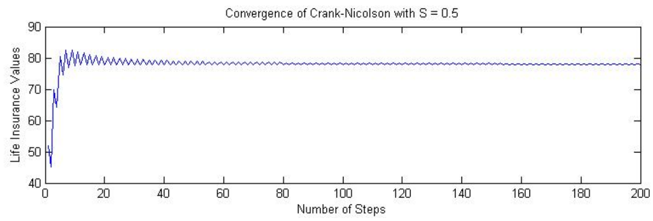


Figure 4.4: Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.5

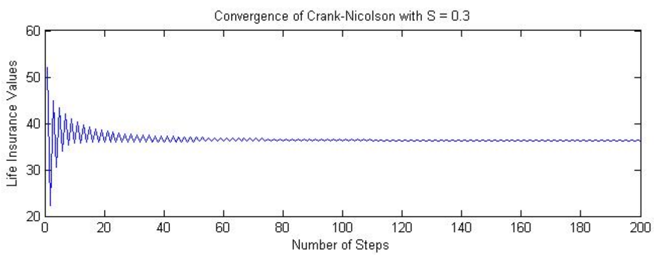


Figure 4.5: Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.3

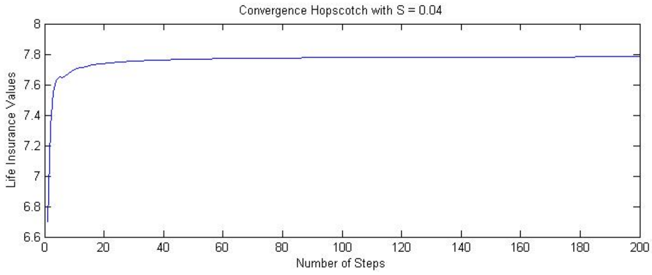


Figure 4.6: Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.04

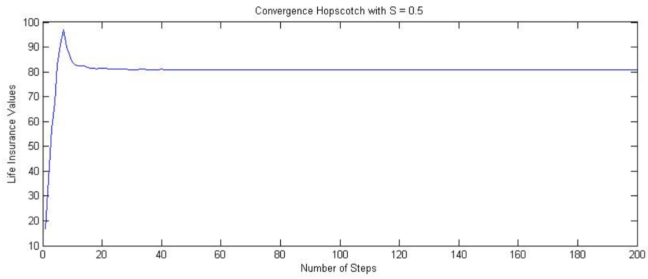


Figure 4.7: Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.5

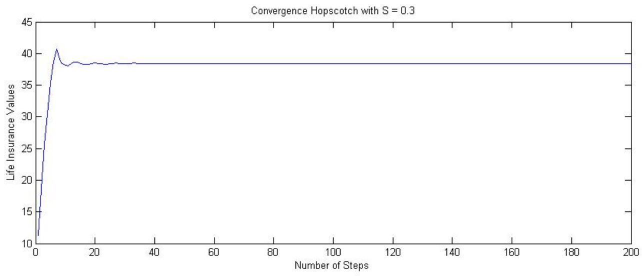


Figure 4.8: Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.3

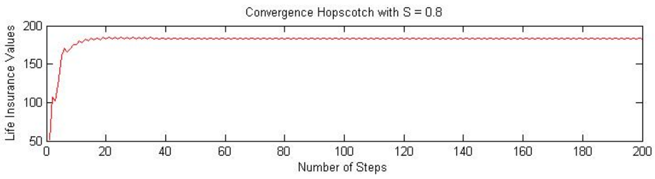


Figure 4.9: Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.8

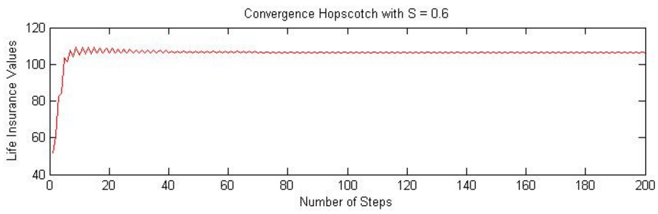


Figure 4.10: Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.6

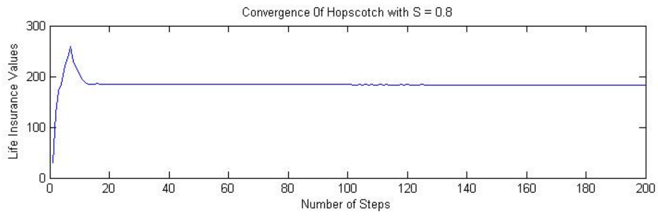


Figure 4.11: Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.8

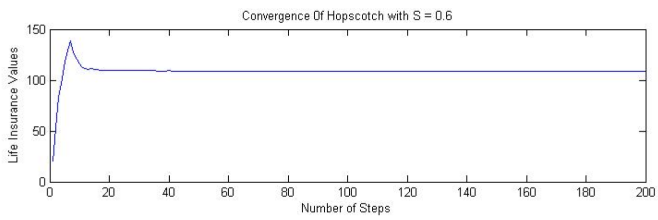


Figure 4.12: Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multimorbid at S = 0.6

# CHAPTER 5

# CONCLUSION AND RECOMMENDATION

## 5.1 Introduction

The chapter looks at the survey of the results obtained from the analysis, the conclusions drawn and some recommendations in relation to the methods used.

## 5.2 Summary of Results

From the analysis of this work, it was realised that, asset price discretization and time discretization are two fundamental sources of error. Checking for consistency, stability and convergence which are the fundamental factors that characterized a numerical scheme, Lax Equivalence theorem was used.

The study used the eigenvalue to check how stable the two finite difference methods would be. The results showed that the Crank-Nicolson and Hopscotch methods were unconditionally stable (see Table 4.1).

The results from the tables showed that values of Crank-Nicolson were closer to the expected values than that of Hopscotch values. Hence, making Crank-Nicolson method better than Hopscotch method as far as faster convergence to the expected value is concerned (see Table 4.2).

The simulated survival rates that were deducted from the rate of returns also showed that, the higher the rate of developing the co-morbidity condition (the higher the S value)the greater the payoff value to the insured.

The Hopscotch method gave higher values than the Crank-Nicolson method when the survival rates were incorporated into the Black-Scholes model. Hence, making the insured receive more payment with the Hopscotch method than the Crank-Nicolson method. On the part of the insurance company, can lose more money when Hopscotch method is used to determine the surrender value (see Tables; 4.3,4.5, 4.6, 4.7 and 4.8). The Crank-Nicolson method was found to give more accurate and consistent results for life insurance contract containing surrender options in Ghana than the Black-Scholes partial differential equation. In the case of the modified model, the hopscotch method gave a little higher values than that of Crank-Nicolson making it converge faster.

## 5.3 Conclusion

The Crank-Nicolson method converges faster than the Hopscotch method when these schemes are used in solving the Black-Scholes partial differential equation. That is Crank-Nicolson method gives more accurate results than the Hopscotch method.

Initially, the values of Hopscotch were higher than the Crank-Nicolson method. As the step sizes were increased, (mesh sizes) for both methods, Crank-nicolson started converging faster than the Hopscotch method.

When the survival rates (S) were deducted from the rate of returns in the Black-Scholes model, the Hopscotch method gave higher payoffs (values) than the Crank-Nicolson method. This means, an insurance company could lose more money when Hopscotch method is used to determine the surrender value (payoff); this method favours the insured than the Crank-Nicolson method.

## 5.4 Recommendation

In finding the value of the American styled life insurance contracts, the Crank-Nicolson method gives more accurate results than the Hopscotch method. In the case where the modified model is going to be used, then the Hopscotch method converges faster and gives more accurate results than the Crank-Nicolson.

## 5.5 Further Studies

Further work could look at adding a parameter to my modified model as a penalty parameter, which penalises the policyholder for early termination of the contract. Also, further studies could look at how dividends could be paid at different surrender dates for multimorbidity patterns.

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# APPENDIX A-D

## Appendix A: Matlab Code for Crank-Nicolson Method

```
function[P]=CrankNicolsonFDBS(S,K,r,sigma,T,N,M,dividend_yield)
% If no dividend payment was made, enter zero for the dividend_yield
% S is the asset price
% K is the strike price
% T is the maturity period
% N is the number of iterations in the time direction
% M is the number of iterations in the asset direction
 % sigma  is the volatility
 lambda=dividend_yield ;
 dt=T/N ;
 ds=2*S/M ;
 A=zeros(M+1,M+1) ;
 f=max(K-(0:M)*ds,0)' ;
 for  m=1:M-1
     A(m+1,m)=((r-lambda)*m*dt-sigma.^2*m.^2*dt)/4 ;
     A(m+1,m+1)=1+0.5*(r-lambda)*dt+0.5*sigma.^2*m.^2*dt ;
     A(m+1,m+2)=(-(r-lambda)*m*dt-sigma.^2*m.^2*dt)/4 ;
end
 A(1,1)=1 ;
 A(M+1,M+1)=1 ;
 A ;
 for m=1:M-1
     B(m+1,m)=(-(r-lambda)*m*dt+sigma.^2*m.^2*dt)/4 ;
     B(m+1,m+1)=1-0.5*(r-lambda)*dt-0.5*sigma.^2*m.^2*dt ;
     B(m+1,m+2)=((r-lambda)*m*dt+sigma.^2*m.^2*dt)/4 ;
```

```
end
 B(1,1)=1 ;
 B(M+1,M+1)=1 ;
 B ;
 for  i=N:-1:1
     f=A^(-1)*(B*f) ;
     f=max(f,(K-(0:M)*ds)') ;
end
 f ;
 P=f(round((M+1)/2)) ;
```

## APPENDIX B: Matlab Code for Crank-Nicolson Method With dividend

```
function price = HopPut(S0,K,r,T,sigma,Smax,dS,dt)


M=round(Smax/dS);
dS=Smax/M;
N=round(T/dt);
dt=T/N;
matval=zeros(M+1,N+1);
vetS=linspace(0,Smax,M+1)';
veti=0:M;
vetj=0:N;
%set up boundary conditions
matval(:,N+1) = max(K-vetS,0);
matval(1,:) = K*exp(-r*dt*(N-vetj));
matval(M+1,:) = 0;


for j=N:-1:1
```

```
    for i=2:M
        if mod(j+i,2)==1
            %Use E
            a=0.5*dt*(sigma^2*veti-r).*veti;
            b=1-dt*(sigma^2*veti.^2+r);
            c=0.5*dt*(sigma^2*veti+r).*veti;
            matval(i,j)=a(i)*matval(i-1,j+1)+b(i)...
            *matval(i,j+1)+c(i)*matval(i+1,j+1);
        end
    end
    for i=2:M
        if mod(j+i,2)==0
            %Use I

            x=0.5*(r*dt*veti-sigma^2*dt*(veti.^2));
                y=1+sigma^2*dt*(veti.^2)+r*dt;
                z=-0.5*(r*dt*veti+sigma^2*dt*(veti.^2));
                matval(i,j)=(1/y(i))*matval(i,j+1)-(z(i)/y(i))...
                *matval(i+1,j)-(x(i)/y(i))*matval(i-1,j);
        end
    end
end
price=interp1(vetS,matval(:,1),S0);
```

## Appendix C: Method Matlab Code for Hopscotch With dividend

```
\subsection*{Appendix C: Matlab Code for Hopscotch Method with dividend}
 \begin{verbatim}
```

```
%function price = HopPut(S0,K,r,T,sigma,Smax,Dividend,dS,dt)
dS=Smax/M;
dt=T/N;
matval=zeros(M+1,N+1);
vetS=linspace(0,Smax,M+1)';
veti=0:M;
vetj=0:N;
%set up boundary conditions
matval(:,N+1) = max(K-vetS,0);
matval(1,:) = K*exp(-(r-s)*dt*(N-vetj));
matval(M+1,:) = 0;

for j=N:-1:1
    for i=2:M
        if mod(j+i,2)==1
            %Use E
            a=0.5*dt*(sigma^2*veti-(r - S)).*veti;
            b=1-dt*(sigma^2*veti.^2+(r - S));
            c=0.5*dt*(sigma^2*veti+(r - S)).*veti;
            matval(i,j)=a(i)*matval(i-1,j+1)+b(i)*...
            matval(i,j+1)+c(i)*matval(i+1,j+1);
        end
    end
    for i=2:M
        if mod(j+i,2)==0
            %Use I

            x=0.5*(r -S)*dt*veti-sigma^2*dt*(veti.^2);
                y=1+sigma^2*dt*(veti.^2)+(r -S)*dt;
```

```
            z=-0.5*((r - S)*dt*veti+sigma^2*dt*(veti.^2));
            matval(i,j)=(1/y(i))*matval(i,j+1)-(z(i)/y(i))*...
            matval(i+1,j)-(x(i)/y(i))*matval(i-1,j);
        end
    end
end
price=interp1(vetS,matval(:,1),S0)
```

## Appendix D: R Codes for the Simulation for Survival Rate

```
y<-function(n)
{
  r<-matrix(0,nrow=n, ncol=2)
  for(i in 1:n){
    lifetimes<-rexp(60,rate=1/15)
    censtimes<-15+5*runif(60)
    ztimes<-pmin(lifetimes,censtimes)
    status<-as.numeric(censtimes>lifetimes)
    m<-summary(survfit(Surv(lifetimes,status)~1))
    st<-length(m$lower)
    g<-m$lower[t]
    h<-m$upper[t]
    d<-cbind(g,h)
    r[i,]<-d
    max(y(10000)[,1])
    max(y(10000)[,2])
    min(y(10000)[,1])
    min(y(10000)[,2])
    mean(y(10000)[,1])
```

```
    mean(y(10000)[,2])

    median(y(10000)[,1])

    median(y(10000)[,2])

  }

  r

}
```