Cosmology in $(2+1)$-dimensions

by

Samuel Amoh Gyampoh, BSc. Mathematics (Hons.)

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Declaration

I hereby declare that this submission is my own work towards the MSc. and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgement has been made in the text.

Samuel Amoh Gyampoh	....	30/03/2010
Student	Signature	Date

Certified by:

Dr. A. K. Agyei	....	30 March, 2010
Supervisor	Signature	Date

Certified by:

Dr. S. K. Amponsah	....	7/04/10
Head of Department	Signature	Date

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Abstract

In this project, we consider a $(2 + 1)$-dimensional universe. We want to find out how a 2-dimensional creature living on a two-dimensional surface and having no sense of a third dimension will see its universe. We construct a metric in $(2 + 1)$-dimensions, which turns out be analogous to the Robertson-Walker metric. To do this, we consider the two-dimensional surface to be a space embedded in a 3-dimensional hypersurface. (Likewise in the $(3+1)$-dimensions, the Robertson-Walker metric can be derived by considering a 3-dimensional space as a space embedded in a 4-dimensional hypersurface.) The metric thus obtained is used to solve the Einstein field equations, which allows us to formulate the corresponding Friedmann models for the case of comoving pressure-free "matter or dust particles". The results are compared with the results obtained with the Robertson-Walker metric in $(3 + 1)$-dimensions. All the results obtained have their analogies in $(3 + 1)$-dimensions, except that we found that the 2-D universe is always expanding irrespective of the curvature of the space. Specifically, the 2-D universe expands linearly in time forever. In 3-D universe, the expansion is generally nonlinear in time and under certain situations contraction is possible.

Contents

List of Figures

List of Figures

Chapter 1

Introduction

Recently there has been much interest in $(2+1)$-dimensional theory of cosmology, as it is supposed to be a useful model for understanding $(3+1)$-dimensional theory of cosmology since it is of a lower dimensionality. Until recently, cosmology was studied in dimensions other than 2-dimensions but now a lot of research is going on in the area of $(2+1)$-dimensional theory of cosmology to ascertain its usefulness in the study of the origin and evolution of the universe. The study of cosmology in $(3+1)$-dimensions has a particular difficulty which is;

- Computations involved in the solutions of Einsteins field equations are very tedious.

This difficulty motivated us to undertake this study. That is, to investigate how a 2-dimensional creature will see its universe. Because 3-dimensional creatures see their universe to be curved. We will also compare the solutions of $(3+1)$-dimensional theory of cosmology to those of $(2+1)$-dimensional theory of cosmology to see if the results in $(2+1)$-dimensions can be carried over into the study of $(3+1)$-dimensional theory of cosmology. If it's possible, then we can use the $(2+1)$-dimensional theory of cosmology as a prototype for the $(3+1)$-dimensional theory of cosmology.

The outline of the project is as follows. In Chapter I, we have a brief introduction of the project and also the aims and objectives of the project. In Chapter II, we briefly look at the fundamentals of cosmology and the study of galaxies. Chapter III, go on to discuss tensors and its properties and also how the general expressions for computing the non-vanishing Christoffel symbols are derived. We briefly discuss some particular tensors which include, the Kronecker symbol, the Levi-Civita tensor, the Generalized Kronecker delta and the Metric tensor. Riemannian spaces, the curvature tensor, the Ricci tensor, the Curvature scalar, the Einstein tensor and the energy momentum tensor are discussed to conclude the third Chapter.

The derivation of the field equations and solutions to them are discussed in chapter four. The Robertson-Walker metric is the working metric of $(3 + 1)$-dimensional cosmology of a homogenous, isotropic universe. It is derived in Chapter V. Solutions to the Einstein field equations for $(2 + 1)$-D space are discussed in the sixth chapter. Finally, Chapter VII gives the conclusion and discussions, i.e. comparison between the solutions of the field equations of the Robertson-Walker metric in $(3 + 1)$-dimensions and that of Robertson - Walker metric in $(2 + 1)$-dimensions. It also talks about how a 2-dimensional creature will see its universe.

Chapter 2

Study of the Universe

2.1 Cosmology

Cosmology is the scientific study of large scale properties of the universe as a whole or the study of the history of the universe from the perspective of physics. It endeavours to use scientific methods to understand the origin and evolution of the universe. Cosmology involves the construction of theories of, or models for, the universe that make specific verifiable predictions about observed phenomena. It addresses questions like:

- What is the origin of the universe?
- How old is the universe?
- Is the universe expanding?
- What is the ultimate fate of the universe?
- What is the structure of the universe?

The universe is the sum of everything that exists in the cosmos, including time and space itself. Cosmos is an orderly or harmonious system. The shape of the universe is determined by a struggle between the momentum of expansion and the pull of gravity.

The rate of expansion is expressed by the Hubble constant, H_0, while the strength of gravity depends on the density and pressure of the matter in the universe. If the pressure of the matter is low, then the fate of the universe is governed by the density. If the density of the universe is less than the "critical density"which is proportional to the square of the Hubble constant, then the universe will expand forever. Gravity might slow the expansion rate down over time, but for densities below the critical density, there isnt enough gravitational pull from the material to ever stop or reverse the outward expansion. This is also known as the "Big Chill"or "Big Freeze"because the universe will slowly cool as it expands until eventually it is unable to sustain any life. If the density of the universe is greater than the critical density, then the universe will collapse back on itself, the so called "Big Crunch". In this situation, there is sufficient mass in the universe to slow the expansion to a stop, and then eventually reverse it. Recent observations have suggested that the expansion of the universe is actually accelerating or speeding up.

Classical cosmology theory is based on the cosmological principle, which states that, on large scale, the universe is homogenous and isotropic, (Andrew Liddle[1]). That is to say that the universe looks the same from each point and in all directions. These do not automatically imply one another. For example, a universe with a uniform magnetic field is homogeneous, as all points are the same, but it fails to be isotropic because directions along the field lines can be distinguished from those perpendicular to them.

For example, Matt Roos[2] considered an observer A in an inertial frame who measures the density of ~~stars and~~ their velocities in the space around him. Because of the homogeneity and isotropy of space, he would see the same mean density of stars (at one time t) in the two different directions r and r',

$$\varphi_A(\vec{r},t) = \varphi_A(\vec{r'},t)$$

Another observer B in another inertial frame, looking in the direction $\vec{r'}$ from his location would also see the same mean density of stars,

$$\varphi_B(\vec{r'},t) = \varphi_A(\vec{r},t)$$

The velocity distribution of stars would also look the same to both observers, in fact in all directions, for instance in the $\vec{r'}$ direction,

$$V_B(\vec{r'},t) = V_A(\vec{r'},t)$$

Hence, we conclude that the universe is homogeneous and isotropic.

Space is defined by a set of coordinate axes. We have two types of spaces, namely the Euclidean space, which is a space described by a set of rectangular coordinates and the non - Euclidean space or Riemannian space, which is a space that cannot be covered with a set of rectangular coordinate system. Spacetime is any mathematical model that combines space and time into a single construct called the spacetime continuum. Thus, 4-dimensional spacetime being three dimensional space plus time which plays the role of the fourth dimension. Dimensions are components of a coordinate grid typically used to locate a point in space or on the globe. However, with spacetime, the coordinate grid is used to locate "events"(rather than just points in space), so time is added as another dimension to the grid.

2.1.1 The Galaxies

William Herschel (1738 - 1822) in his study of stars gave a first understanding of what the Milky Way is. The Milky Way is a distribution of stars in a disc spread all around the sun. Such a distribution of stars on a large scale is now known as a galaxy. The Milky Way is also called the Galaxy. The structure of the universe as viewed face on shows the spiral structure of the Galaxy whereas the edge on picture demonstrates that it is a disc with a central bulge. The disc is also referred to as the galactic plane. Below are some examples of pictures of the Milky Way, as described by William Herschel and the types of galaxies.

In 1926 Hubble classified the various types of galaxy according to their shape. There are four types of galaxies. They are Spiral, Barred spiral, Elliptical and Irregular galaxies. Our galaxy is an example of a spiral galaxy.

Spiral galaxies are the most numerous amongst the various types of bright galaxies. A typical spiral galaxy contains about one hundred thousand billion stars and is approximately one hundred thousand light years across. Spiral galaxies are characterized by large and thick central bulge and spiral arms (see Fig. 2.2.) The spiral arms are made up of millions of relatively young stars in a constant orbit around the center of the galaxy. Our galaxy the Milky Way is an example of such a spiral galaxy.

A barred spiral galaxy is characterized by a bar running through its nucleus. The arms of a barred spiral galaxy originate not from the nucleus but from bars running through the galaxys nucleus. Like the spiral galaxy, the barred spiral galaxy has a central bulge containing the majority of mass in the galaxy (see Fig. 2.3.)

An elliptical galaxy is ellipsoidal in shape and they are the most numerous amongst all galaxies. They exhibit very little rotation and have very little gas and dust (see Fig. 2.4.)

Irregular galaxies can come in many different types. They are put in this group because they dont fit into any of Hubbles other classification (see Fig. 2.5.)

In 1929, Edwin Hubble announced that his observation of galaxies outside our own Milky Way showed that they are systematically moving away from us with a speed proportional to their distance from us. The more distant the galaxy, the faster it was receding from us. The specific form of Hubble's expansion law is important: the speed of recession is proportional to the distance for not too large cosmic distances.

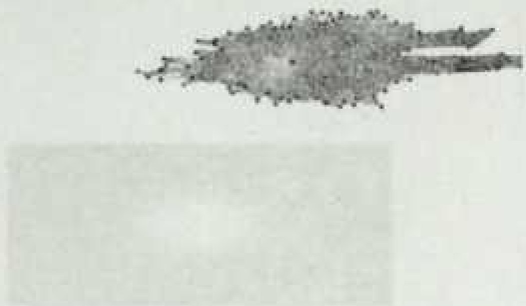

Figure 2.1: "The shape of the Milky way as deduced from star counts by William Herschel in 1785; the solar system was assumed to be near the centre."

Figure 2.2: NGC 4414, a typical spiral galaxy in the constellation Coma Berenices, is about 17,000 parsecs in diameter and approximately 20 million parsecs distant.Credit: Hubble space, Telescope/NASA/ESA.

Figure 2.3: NGC 1300, an example of a barred spiral galaxy. Credit: Hubble Space Telescope/NASA/ESA

Figure 2.4:. An example of an elliptical galaxy. Credit: Hubble Space Telescope/NASA/ESA

Figure 2.5: An example of an irregular galaxy. Credit: Hubble Space Telescope/NASA/ESA

Chapter 3

Mathematical Tools of Mathematical Cosmology: Tensor Analysis

3.1 Tensors

3.1.1 Definition

A tensor is any quantity that transforms between coordinate systems according to a particular transformation law. Physical laws must be independent of any particular coordinate systems used in describing them mathematically, if they are to be valid. A study of the consequences of this requirement leads to tensor analysis. The discussions in this project involve tensor analysis. A tensor of rank zero is called a scalar and a tensor of rank one is known as a vector. Since the physical laws are independent of any particular coordinate system, it should be emphasized that, if all the components of any tensor vanish in a given coordinate system, they will vanish in any other coordinate system[3].

The set of N^r quantities $T^{i_1,i_2,i_3,\ldots,i_r}$ is said to constitute the components of a contravariant tensor of rank r at a point P in an N-dimensional space if under the coordinate transformation $\overline{x}^j = \overline{x}^j(x^i)$, these quantities transform according to the law

$$\overline{T}^{i_1,i_2,i_3,\ldots,i_r} = \frac{\partial \overline{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{j_r}} T^{j_1,j_2,j_3,\ldots,j_r} \tag{3.1}$$

The set of N^s quantities $T_{i_1,i_2,i_3,\ldots,i_s}$ is said to constitute the components of a covariant tensor of rank s at a point P in N-dimensional space, if under the coordinate transformation $\overline{x}^j = \overline{x}^j(x^i)$, these quantities transform according to the law

$$\overline{T}_{i_1,i_2,i_3,\ldots,i_s} = \frac{\partial x^{j_1}}{\partial \overline{x}^{i_1}} \cdots \frac{\partial x^{j_s}}{\partial \overline{x}^{i_s}} T_{j_1,j_2,j_3,\ldots,j_s} \tag{3.2}$$

A set of N^{r+s} quantities $T^{i_1,i_2,i_3,\ldots,i_r}{}_{j_1,j_2,j_3,\ldots,j_s}$ is said to constitute the components of a mixed tensor of rank r + s, contravariant of order or rank r and covariant of order or rank s, at a point P in an N-dimensional space, if under the coordinate transformation $\overline{x}^j = \overline{x}^j(x^i)$, these quantities transform according to the law

$$\overline{T}^{i_1,i_2,i_3,\ldots,i_r}{}_{j_1,j_2,j_3,\ldots,j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \overline{x}^{j_s}} T^{k_1,k_2,k_3,\ldots,k_r}{}_{l_1,l_2,l_3,\ldots,l_s} \tag{3.3}$$

Next we go on to discuss some algebraic properties of tensors.

3.1.2 Tensor Algebra

Addition

The sum of two or more tensors of the same rank and type (i.e. same number of contravariant indices and same number of covariant indices) is also a tensor of the same rank and type. Addition of tensors is commutative and associative. For example, $A^{pq}{}_r$ and $B^{pq}{}_r$ are tensors; i.e.

$$\overline{A}^{jk}{}_l = \frac{\partial \overline{x}^j}{\partial x^p}\frac{\partial \overline{x}^k}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^l}A^{pq}{}_r$$

$$\overline{B}^{jk}{}_l = \frac{\partial \overline{x}^j}{\partial x^p}\frac{\partial \overline{x}^k}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^l}B^{pq}{}_r$$

where Einstein's summation convention is implied. Then

$$(\overline{A}^{jk}{}_l + \overline{B}^{jk}{}_l) = \frac{\partial \overline{x}^j}{\partial x^p}\frac{\partial \overline{x}^k}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^l}(A^{pq}{}_r + B^{pq}{}_r)$$

Hence $A^{pq}{}_r + B^{pq}{}_r$ is a tensor of the same rank and type as $A^{pq}{}_r$ and $B^{pq}{}_r$.
The difference between $A^{jk}{}_l$ and $B^{jk}{}_l$ can be written as $A^{jk}{}_l + (-B^{jk}{}_l)$.

Multiplication: The Outer (or Direct) Product Theorem.

The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product which involves ordinary multiplication of the components of the tensor is called the outer product. Let $R^k{}_{ij}$ and S^{lm} be tensors, whose transformation laws are respectively given by

$$\overline{R}^k{}_{ij} = \frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial x^t}{\partial \overline{x}^j}R^r{}_{st}$$

$$\overline{S}^{lm} = \frac{\partial \overline{x}^l}{\partial x^p}\frac{\partial \overline{x}^m}{\partial x^q}S^{pq}$$

Therefore

$$\overline{R}^k{}_{ij}\overline{S}^{lm} = \frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial x^t}{\partial \overline{x}^j}\frac{\partial \overline{x}^l}{\partial x^p}\frac{\partial \overline{x}^m}{\partial x^q}R^r{}_{st}S^{pq}$$

$$= \frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial \overline{x}^l}{\partial x^p}\frac{\partial \overline{x}^m}{\partial x^q}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial x^t}{\partial \overline{x}^j}R^r{}_{st}S^{pq}$$

This is the transformation law for a tensor of type (3, 2).

Contraction

The contraction of a tensor of type (r, s) over a pair of indices, one contravariant and the other covariant, results in a tensor of type $(r-1, s-1)$. For example, consider the tensor $R^k{}_{ij}$. Let k = j

$$\overline{R}^j{}_{ij} = \frac{\partial \overline{x}^j}{\partial x^r}\frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial x^m}{\partial \overline{x}^j}R^r{}_{lm} = \frac{\partial \overline{x}^j}{\partial x^r}\frac{\partial x^m}{\partial \overline{x}^j}\frac{\partial x^l}{\partial \overline{x}^i}R^r{}_{lm}$$

$$= \delta^m_r \frac{\partial x^l}{\partial \overline{x}^i}R^r{}_{lm} = \frac{\partial x^l}{\partial \overline{x}^i}R^m{}_{lm}$$

which is the transformation law for covariant tensor of the type (0, 1).

The Inner Product Theorem

The inner product of two tensors of type (r_1, s_1) and (r_2, s_2) is a tensor of type $(r_1 + r_2 - 1, s_1 + s_2 - 1)$ provided that the contraction is over a pair of indices, one contravariant and the other covariant. For example, let T^{ij} be a contravariant tensor of the second rank, so that it transforms according to the law:

$$\overline{T}^{ij} = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s}T^{rs}$$

And let C_p and F_q be two covariant vectors. They transform according to these transformation laws:

$$\overline{C_p} = \frac{\partial x^l}{\partial \overline{x}^p}C_l$$

$$\overline{F_q} = \frac{\partial x^m}{\partial \overline{x}^q}F_m$$

$$\overline{T}^{ij}\overline{C}_p\overline{F}_q = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \overline{x}^p}\frac{\partial x^m}{\partial \overline{x}^q}T^{rs}{}_{lm}$$

Let us now contract over the indices j, p

$$\overline{T}^{ij}\overline{C}_j\overline{F}_q = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial x^m}{\partial \overline{x}^q}T^{rs}{}_{lm}$$

$$= \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^m}{\partial \overline{x}^q}\delta^l_s T^{rs}{}_{lm}$$

$$= \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^m}{\partial \overline{x}^q}T^{rs}{}_{sm}$$

which is the transformation law for a tensor of type $(1, 1)$.

Criteria for Tensor Character: The Quotient Theorem

Suppose it is not known whether a quantity X is a tensor or not. If an inner or outer product of X with an arbitrary tensor is itself a tensor, then X is also a tensor. This is called the quotient theorem.

Symmetric and Skew-Symmetric Tensors

Now, we look at symmetric and skew-symmetric tensors. A tensor is called symmetric with respect to two contravariant or two covariant indices if its components remain unaltered upon interchange of the indices. Thus if $A^{mpr}{}_{qs} = A^{pmr}{}_{qs}$ the tensor is symmetric in m and p. If a tensor is symmetric with respect to any two contravariant and any two covariant indices, it is called symmetric.

A tensor is called skew-symmetric with respect to two contravariant or two covariant indices if its component change sign upon interchange of the indices. Thus if $A^{mpr}{}_{qs} = -A^{pmr}{}_{qs}$ the tensor is skew-symmetric in m and p. If a tensor is skew-symmetric with respect to any two contravariant and any two covariant indices it is called skew-symmetric. It should also be emphasized that all symmetry and skew-symmetry properties of tensors are independent of the choice of the coordinate system.

3.1.3 Some Particular Tensors

The Kronecker Symbol

It is defined by,

$$\delta_i^j = \begin{cases} 1 & if \quad j = i \\ 0 & if \quad j \neq i \end{cases} \tag{3.4}$$

Clearly

$$\overline{\delta}_i^j = \frac{\partial \overline{x}^j}{\partial x^r}\frac{\partial x^s}{\partial \overline{x}^i}\delta_s^r = \frac{\partial \overline{x}^j}{\partial x^r}\frac{\partial x^r}{\partial \overline{x}^i} = \frac{\partial \overline{x}^j}{\partial \overline{x}^i}$$

Hence $\overline{\delta}_i^j$ is type $(1,1)$ tensor and it is the same in all coordinate systems.

The Levi-Civita Symbols

The Levi-Civita symbols also known as permutation symbols are defined by,

$$\varepsilon_{h_1 \ldots h_n} = \delta_{h_1 \ldots h_n}^{12 \ldots n}$$

and

$$\varepsilon^{j_1 \ldots j_n} = \delta_{12 \ldots n}^{j_1 \ldots j_n}$$

The general form of the Levi-Civita symbol is given by,

$$\varepsilon_{h_1 \ldots h_n} = \varepsilon^{h_1 \ldots h_n} = \begin{cases} +1 & if \quad h_1 \ldots h_n \quad \text{is an even permutation of 1,2,}\ldots\text{,n} \\ -1 & if \quad h_1 \ldots h_n \quad \text{is an odd permutatiion of 1,2,}\ldots\text{,n} \\ 0 & \qquad\qquad otherwise \end{cases} \tag{3.5}$$

The Generalized Kronecker Delta

The generalized Kronecker delta, denoted by $\delta_{h_1 \ldots h_r}^{j_1 \ldots j_r}$, is one of the most important numerical or isotropic tensors, namely, tensors that have identical components in all coordinate systems. It possesses an equal number of sub- and superscripts (r) and is defined in terms of an r x r determinant as follows:

$$\delta^{j_1 \cdots j_r}_{h_1 \ldots h_r} = \begin{vmatrix} \delta^{j_1}_{h_1} & \delta^{j_1}_{h_2} \ldots \delta^{j_1}_{h_r} \\ \delta^{j_2}_{h_1} & \delta^{j_2}_{h_2} \ldots \delta^{j_2}_{h_r} \\ \ldots\ldots & \ldots\ldots \\ \delta^{j_r}_{h_1} & \delta^{j_r}_{h_2} \ldots \delta^{j_r}_{h_r} \end{vmatrix} \tag{3.6}$$

which means that it is skew-symmetric under interchange of any two sub- or superscripts and that if any two sub- or superscripts coincide, the corresponding $\delta^{j_1 \cdots j_r}_{h_1 \ldots h_r}$, vanishes identically. Clearly the Kronecker delta δ^j_h and the Levi-Civita symbols, $\varepsilon_{i_1 i_2 i_3}$ and $\varepsilon^{j_1 j_2 j_3}$, are special cases of the generalized Kronecker delta, and since, as shown above, the Kronecker delta is a type (1, 1) tensor, it follows immediately that the generalized Kronecker delta (3.6) is a type (r, r) tensor.

The Metric Tensor

The metric tensor is undoubtedly the most important tensor in mathematical physics. It prescribes the element of length in a space to which the notion of distance applies. Consider an N-dimensional Euclidean space defined by the rectangular coordinates $x^i (i = 1, 2, \ldots, N)$. By analogy with 3-D Euclidean space, the element of distance in such a space is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^N)^2$$

Now, requiring that this distance be invariant under coordinate transformation $\overline{x^i} = \overline{x^j}(x^i)$, and that the coordinate elements dx^k transforms like the components of a contravariant vector, i.e.

$$d\overline{x}^k = \sum_i \frac{\partial \overline{x}^k}{\partial x^i} dx^i = \frac{\partial \overline{x}^k}{\partial x^i} dx^i$$

(by Einsteins summation convention), we have

$$
\begin{aligned}
ds^2 = d\bar{s}^2 &= (d\bar{x}^1)^2 + (d\bar{x}^2)^2 + \cdots + (d\bar{x}^N)^2 \\
&= \sum_{k=1}^{N} d\bar{x}^k d\bar{x}^k \\
&= \sum_{k=1}^{N} \frac{\partial \bar{x}^k}{\partial x^i} dx^i \frac{\partial \bar{x}^k}{\partial x^j} dx^j \\
&= \sum_{k=1}^{N} \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^k}{\partial x^j} dx^i dx^j
\end{aligned}
$$

or

$$ds^2 = g_{ij} dx^i dx^j \tag{3.7}$$

where

$$g_{ij} = \sum_{k=1}^{N} \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^k}{\partial x^j}$$

It is easy to show that the metric tensor are the components of a symmetric second-rank covariant tensor. This tensor is called the metric tensor.

We postulate that any space for which the distance between two points in the space can be defined is endowed with a metric g_{ij} giving the element of length or line element of the space in the form (3.7). In general, g_{ij} is not diagonal and its determinant is not positive definite or non-zero, but it must be symmetric and invariant under any coordinate transformation. A space endowed with a metric with a positive definite determinant is called a Riemannian space, while a space endowed with a metric whose determinant vanishes identically or is not positive definite is called pseudo-Riemannian. Euclidean spaces are therefore Riemannian, whereas Minkowski space is pseudo-Riemannian. If a metric is a non-null tensor, i.e. the determinant is non-zero then it has an inverse. g_{ij} has an inverse of g^{ij}. Let $g = |g_{ij}|$ denote the determinant with elements g_{ij} and $g \neq 0$. Then $g^{ij} = \frac{G_{ij}}{g}$ where G_{ij} are co-factors of g_{ij}; i.e. $G_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} are the minors of the matrix. g^{ij} is a contravariant tensor of rank two. g^{ij} raises indices whiles g_{ij} lowers indices. i.e.

$$g_{ij}T^{jk}{}_{lm} = T_i^{k}{}_{lm}$$
$$g^{il}T^{jk}{}_{lm} = T^{jki}{}_{m}$$

3.1.4 Differentiation

The derivative of a tensor is generally not a tensor, but we can use it to construct a tensor and the tensor thus obtained is known as the covariant derivative of the tensor. As is to be expected, the results for covariant and contravariant tensors are not the same. The covariant derivatives of the covariant tensor A_p with respect to x^q is a covariant tensor of rank two, denoted by $A_{p;q}$ and given by

$$A_{p;q} = \frac{\partial A_p}{\partial x^q} - \Gamma_p{}^s{}_q A_s$$

where the $\Gamma_p{}^s{}_q$ are quantities known as the affine connection. The covariant derivative of the contravariant tensor A^p with respect to x^q is denoted by $A^p{}_{;q}$ and is given by

$$A^p{}_{;q} = \frac{\partial A^p}{\partial x^q} + \Gamma_s{}^p{}_q A^s$$

The above results can be extended to covariant derivatives of higher rank tensors. Thus

$$A^{j_1\dots j_r}{}_{i_1\dots i_s;k} = \frac{\partial A^{j_1\dots j_r}{}_{i_1\dots i_s}}{\partial x^k} - \sum_{\alpha=1}^{s}\Gamma_{i_\alpha}{}^{m}{}_{k}A^{j_1\dots j_r}{}_{i_1\dots i_{\alpha-1}mi_{\alpha+1}\dots i_s} + \sum_{\beta=1}^{r}\Gamma_{m}{}^{j_\beta}{}_{k}A^{j_1\dots j_{\beta-1}mj_{\beta+1}\dots j_r}{}_{i_1\dots i_s}$$

The structure of this expression is governed by the following rules:

- Apart from the partial derivatives, there is a negative term, known as an affine term, for each covariant index and a positive affine term for each contravariant index.

- The second subscript of the Γ symbol is always the differentiation index.

- Each index in $A^{j_1 \ldots j_r}{}_{i_1 \ldots i_s}$ is transferred in turn to the unoccupied spot of the same character (subscript or superscript) on the Γ symbol and its place is taken by a dummy index which also occupies the remaining spot on the Γ symbol.

Thus,

$$A^r{}_{ij;k} = \frac{\partial A^r{}_{ij}}{\partial x^k} - \Gamma_i{}^m{}_k A^r{}_{mj} - \Gamma_j{}^m{}_k A^r{}_{im} + \Gamma_m{}^r{}_k A^m{}_{ij}.$$

It can be shown that the covariant derivative of the metric tensor g_{ij} vanishes identically, i.e.

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma_i{}^m{}_k g_{mj} - \Gamma_j{}^m{}_k g_{im} = 0 \tag{3.8}$$

Performing a cyclic permutation of the indices i, j, k, we obtain two other formulae:

$$\frac{\partial g_{jk}}{\partial x^i} - \Gamma_j{}^m{}_i g_{mk} - \Gamma_k{}^m{}_i g_{jm} = 0 \tag{3.9}$$

$$\frac{\partial g_{ki}}{\partial x^j} - \Gamma_k{}^m{}_j g_{mi} - \Gamma_i{}^m{}_j g_{km} = 0 \tag{3.10}$$

Now adding (3.9) and (3.10) and subtracting (3.8), we obtain on using the symmetry properties of g_{ij} and $\Gamma_i{}^m{}_k$:

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} - 2\Gamma_i{}^m{}_j g_{km} = 0$$

$$2g_{km}\Gamma_i{}^m{}_j = \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$

Solving for $\Gamma_i{}^m{}_j$, by multiplying both sides by $\frac{1}{2}g^{nk}$ and using the relation $g^{nk}g_{mk} = \delta^n_m$, we obtain for the components of the affine connection the expression

$$\Gamma_i{}^n{}_j = \frac{1}{2}g^{nk}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \tag{3.11}$$

In particular, for a reference frame with a diagonal metric tensor g_{ij}, the non-zero components of the affine connection are given by the following expressions;

(i) $\Gamma_j{}^i{}_k = 0 \quad for \quad i \neq j \neq k$

(ii) $\Gamma_{j}{}^{i}{}_{j} = \frac{1}{2}g^{ik}\left[\frac{\partial g_{jk}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^k}\right]$

$$= \frac{1}{2}g^{ii}\left[\frac{\partial g_{ji}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i}\right] = -\frac{1}{2}g^{ii}\frac{\partial g_{jj}}{\partial x^i}$$

$$\therefore \quad \Gamma_{j}{}^{i}{}_{j} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i} \quad for \quad i \neq j$$

(iii) $\Gamma_{j}{}^{i}{}_{i} = \Gamma_{i}{}^{i}{}_{j} = \frac{1}{2}g^{ik}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k}\right]$

$$= \frac{1}{2}g^{ii}\left[\frac{\partial g_{ii}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^i}\right] = \frac{1}{2}g^{ii}\frac{\partial g_{ii}}{\partial x^j}$$

$$\therefore \quad \Gamma_{j}{}^{i}{}_{i} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^j}$$

(iv) $\Gamma_{i}{}^{i}{}_{i} = \frac{1}{2}g^{ik}\left[\frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^k}\right]$

$$= \frac{1}{2}g^{ii}\left[\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right] = \frac{1}{2}g^{ii}\frac{\partial g_{ii}}{\partial x^i}$$

$$\therefore \quad \Gamma_{i}{}^{i}{}_{i} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i}$$

3.1.5 Repeated Covariant Differentiation: The Curvature Tensor

Consider the tensor X^j. Its covariant derivative is, as seen below, given by

$$X^j{}_{;h} = \frac{\partial X^j}{\partial x^h} + \Gamma_{l}{}^{j}{}_{h}X^l$$

This is a type (1, 1) ~~tensor, which~~ can be differentiated to obtain

$$X^j{}_{;h;k} = \frac{\partial (X^j{}_{;h})}{\partial x^k} + \Gamma_{m}{}^{j}{}_{k}X^m{}_{;h} - \Gamma_{h}{}^{l}{}_{k}X^j{}_{;l}$$

$$= \frac{\partial}{\partial x^k}\left[\frac{\partial X^j}{\partial x^h} + \Gamma_l{}^j{}_h X^l\right] + \Gamma_m{}^j{}_k\left[\frac{\partial X^m}{\partial x^h} + \Gamma_l{}^m{}_h X^l\right] - \Gamma_h{}^l{}_k X^j{}_{;l}$$

$$= \frac{\partial^2 X^j}{\partial x^k \partial x^h} + \frac{\partial \Gamma_l{}^j{}_h}{\partial x^k} X^l + \Gamma_l{}^j{}_h \frac{\partial X^l}{\partial x^k} + \Gamma_m{}^j{}_k \frac{\partial X^m}{\partial x^h} + \Gamma_m{}^j{}_k \Gamma_l{}^m{}_h X^l - \Gamma_h{}^l{}_k X^j{}_{;l} \quad (3.12)$$

Similarly,

$$X^j{}_{;k;h} = \frac{\partial^2 X^j}{\partial x^h \partial x^k} + \frac{\partial \Gamma_l{}^j{}_k}{\partial x^h} X^l + \Gamma_l{}^j{}_k \frac{\partial X^l}{\partial x^h} + \Gamma_m{}^j{}_h \frac{\partial X^m}{\partial x^k} + \Gamma_m{}^j{}_h \Gamma_l{}^m{}_k X^l - \Gamma_k{}^l{}_h X^j{}_{;l} \quad (3.13)$$

Assuming that $\frac{\partial^2 X^j}{\partial x^k \partial x^h} = \frac{\partial^2 X^j}{\partial x^h \partial x^k}$, and noting that the indices l and m in the third and fourth terms in (3.12) and (3.13) are dummy indices, we have

$$X^j{}_{;h;k} - X^j{}_{;k;h} = \left(\frac{\partial \Gamma_l{}^j{}_h}{\partial x^k} - \frac{\partial \Gamma_l{}^j{}_k}{\partial x^h}\right) X^l + (\Gamma_m{}^j{}_k \Gamma_l{}^m{}_h - \Gamma_m{}^j{}_h \Gamma_l{}^m{}_k) X^l - (\Gamma_h{}^l{}_k - \Gamma_k{}^l{}_h) X^j{}_{;l}$$

$$= \left(\frac{\partial \Gamma_l{}^j{}_h}{\partial x^k} - \frac{\partial \Gamma_l{}^j{}_k}{\partial x^h} + \Gamma_m{}^j{}_k \Gamma_l{}^m{}_h - \Gamma_m{}^j{}_h \Gamma_l{}^m{}_k\right) X^l - (\Gamma_h{}^l{}_k - \Gamma_k{}^l{}_h) X^j{}_{;l}$$

$$= K^j{}_{lhk} X^l - S^l{}_{hk} X^j{}_{;l},$$

where

$$K^j{}_{lhk} = \frac{\partial \Gamma_l{}^j{}_h}{\partial x^k} - \frac{\partial \Gamma_l{}^j{}_k}{\partial x^h} + \Gamma_m{}^j{}_k \Gamma_l{}^m{}_h - \Gamma_m{}^j{}_h \Gamma_l{}^m{}_k$$

$$S^l{}_{hk} = \Gamma_h{}^l{}_k - \Gamma_k{}^l{}_h$$

The quantities $K^j{}_{lhk}$ are components of a type (1, 3) tensor, called the curvature tensor of the space X_N and the quantities $S^l{}_{hk}$ constitute the components of a type (1, 2) tensor called the torsion tensor. In a space endowed with a symmetric affine connection, i.e. $\Gamma_l{}^m{}_k = \Gamma_k{}^m{}_l$, the torsion tensor vanishes identically but the curvature tensor does not in general vanish, which means that in general the results of repeated covariant differentiation ~~depend~~ on the order of differentiation. Thus,

$$X^j{}_{;h;k} - X^j{}_{;k;h} = K^j{}_{lhk} X^l - S^l{}_{hk} X^j{}_{;l}, \quad (3.14)$$

and we can similarly show that

$$Y_{j;h;k} - Y_{j;k;h} = -K^l{}_{jhk} Y_l - S^l{}_{hk} Y_{j;l}, \tag{3.15}$$

$$T^{jl}{}_{;h;k} - T^{jl}{}_{;k;h} = K^j{}_{mhk} T^{ml} + K^l{}_{mhk} T^{jm} - S^m{}_{hk} T^{jl}{}_{;m}, \tag{3.16}$$

and

$$T_{jl;h;k} - T_{jl;k;h} = -K^m{}_{jhk} T_{ml} - K^m{}_{lhk} T_{jm} - S^m{}_{hk} T_{jl;m} \tag{3.17}$$

The relations (3.14) to (3.17) are often referred to as the Ricci identities. There are spaces for which the curvature tensor vanishes identically. Such spaces are called flat spaces.

3.2 Riemannian Spaces

3.2.1 The Curvature Tensor

Riemannian spaces are spaces endowed with a symmetric metric and such that $\det(g_{ij}) \neq 0$. The curvature tensor is denoted by $R^j{}_{lhk}$ in Riemannian space. In this case, the components of the affine connection are called Christoffel symbols and are denoted by $\gamma_h{}^j{}_k$, i.e. $\Gamma_h{}^j{}_k = \gamma_h{}^j{}_k$. Here the Christoffel symbols $\gamma_h{}^j{}_k$ are symmetric in their subscripts h and k, so that the corresponding torsion tensor vanishes identically, and we have

$$R^j{}_{lhk} = \frac{\partial \gamma_l{}^j{}_h}{\partial x^k} - \frac{\partial \gamma_l{}^j{}_k}{\partial x^h} + \gamma_m{}^j{}_k \gamma_l{}^m{}_h - \gamma_m{}^j{}_h \gamma_l{}^m{}_k \tag{3.18}$$

$$S^l{}_{hk} \equiv 0$$

$$\therefore X^j{}_{;h;k} - X^j{}_{;k;h} = R^j{}_{lhk} X^l$$

$$Y_{j;h;k} - Y_{j;k;h} = -R^l{}_{jhk} Y^l$$

$$T^{jl}{}_{;h;k} - T^{jl}{}_{;k;h} = R^{j}{}_{mhk}T^{ml} + R^{l}{}_{mhk}T^{jm}$$

$$T_{jl;h;k} - T_{jl;k;h} = -R^{m}{}_{jhk}T_{ml} - R^{m}{}_{lhk}T_{jm}$$

$R^{j}{}_{lhk}$is known as the Riemannian-Christoffel curvature tensor. Some of the properties of the curvature tensor are as follows:

i) The curvature tensor is skew-symmetric on the last pair of covariant indices. i.e.

$$R^{j}{}_{lhk} = -R^{j}{}_{lkh}.$$

ii)

$$R^{j}{}_{lhk} + R^{j}{}_{hkl} + R^{j}{}_{klh} = 0, \tag{3.19}$$

which is obtained by cyclically permuting the indices l, h, k in (3.18) and adding the two results to (3.18).

iii)

$$R^{l}{}_{jhk;p} + R^{l}{}_{jkp;h} + R^{l}{}_{jph;k} = 0$$

iv) A type (0, 4) tensor R_{jlhk} can be constructed from it by taking its inner product with the metric tensor g_{lm}, i.e.

$$R_{jlhk} = g_{lm}R^{m}{}_{jhk}.$$

This new type (0, 4) tensor is usually called the covariant curvature tensor.

It possesses the following properties:

i) It is symmetric in jl and hk, i.e. $R_{jlhk} = R_{hkjl}$

ii)The covariant curvature tensor is skew-symmetric in the first two subscripts as well as the last two subscripts. i.e. $R_{jlhk} = -R_{ljhk} = -R_{jlkh} = R_{ljkh}$

iii)The inner product of (3.19) with the metric tensor g_{ij} yields

$$g_{ij}(R^{j}{}_{lhk} + R^{j}{}_{hkl} + R^{j}{}_{klh}) = 0,$$

i.e. $$R_{ilhk} + R_{ihkl} + R_{iklh} = 0$$

iv). $R_{jlhk;p} + R_{jlkp;h} + R_{jlph;k} = 0$. This is known as the Bianchi identity

3.2.2 The Ricci Tensor and The Curvature Scalar

From the covariant curvature tensor R_{jlhk} we can construct the symmetric tensor

$$R_{lk} = g^{jh} R_{jlhk} = g^{jh} R_{hkjl} = R_{kl} \tag{3.20}$$

and the scalar

$$R \equiv g^{jh} g^{lk} R_{jlhk} = g^{jh} R_{jh}. \tag{3.21}$$

Then tensor R_{lk} is known as the Ricci tensor, while the scalar R is known as the curvature scalar. They are the cornerstones of General Relativity, as will be seen presently.

3.2.3 The Einstein Tensor

Consider the Bianchi identity

$$R_{jlhk;p} + R_{jlkp;h} + R_{jlph;k} = 0$$

Contracting this identity with g^{jh}, we obtain

$$g^{jh} R_{jlhk;p} + g^{jh} R_{jlkp;h} + g^{jh} R_{jlph;k} = 0$$

$$R_{lk;p} + g^{jh} R_{jlkp;h} - g^{jh} R_{jlhp;k} = 0$$

$$R_{lk;p} + g^{jh} R_{jlkp;h} - R_{lp;k} = 0$$

Contracting again with g^{lk}, we have

$$g^{lk} R_{lk;p} + g^{lk} g^{jh} R_{jlkp;h} - g^{lk} R_{lp;k} = 0$$

$$R_{;p} - g^{lk} g^{jh} R_{ljkp;h} - g^{lk} R_{lp;k} = 0$$

$$R_{;p} - g^{jh} R_{jp;h} - g^{lk} R_{lp;k}$$

$$R_{;p} - R^h{}_{p;h} - R^k{}_{p;k} = 0,$$

or, since h and k are dummy indices,

$$R_{;p} - 2R^h{}_{p;h} = 0$$

or

$$\left(R^h{}_p - \frac{1}{2}\delta_p^h R\right)_{;h} = 0,$$

where δ_p^h is the Kronecker delta.

The tensor

$$G_p^h = R^h{}_p - \frac{1}{2}\delta_p^h R \tag{3.22}$$

is known as the Einstein tensor.

3.3 Energy Momentum Tensor of Matter in the Universe

The galaxies being small compared to the distances between them are considered to be particles: dust particles, relativistic particles or particles of a fluid. Dust particles are the simplest situation. In this situation, particles of matter move without any relative motion as cited by Jayant V. Narlikar[4]. The four - velocity U^i in the co-moving coordinate system is given by $U^i = (1, 0, 0, 0)$. The only non - zero component of the energy tensor is

$$T^{11} = \sum_a m_a c^2 = \rho c^2$$

where the summation is over a unit volume. Here ρ is the rest mass density of the dust. Hence for dust particles, the energy momentum tensor is of the form;

$$T^{ik} = diag(\rho c^2, 0, 0, 0) \tag{3.23}$$

For relativistic particles and fluid, the particles are in random motion through space. Using the fact that the particles are moving randomly, we find that the energy momentum tensor is given by

$$T^{ik} = diag(\rho c^2, p, p, p), \tag{3.24}$$

i.e. that $T^{22} = T^{33} = T^{44}$, where p is the pressure exerted by the particles, and $T^{11} = \rho c^2$. Now if ε is the kinetic energy density of the randomly moving particles, then $p = \frac{1}{3}\varepsilon$, and we have

$$T^{ik} = diag\left(\rho c^2, \frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon\right)$$

The factor $\frac{1}{3}$ comes from randomizing in all the three space directions.

In $(2+1)$-dimensions, since the particle move in only two directions, the relationship between the pressure, P and energy density, ε, is given by

$$P = \frac{1}{2}\varepsilon \tag{3.25}$$

For the purpose of this project, we will consider the simplest case. That is, the case of dust particles. Hence the appropriate energy momentum tensor is given by

$$T^{11} = \rho c^2; \qquad T^{22} = T^{33} = T^{44} = 0 \tag{3.26}$$

for 3D space and

$$T^{11} = \rho c^2; \qquad T^{22} = T^{33} = 0 \tag{3.27}$$

for 2D space.

Chapter 4

Introduction to General Relativity

4.1 Einstein Field Equations

4.1.1 Derivation of the Field Equations

In a weak static field produced by a nonrelativistic mass density ρ, the time - time component of the metric tensor is approximately given by

$$g_{00} = -(1 + 2\phi)$$

Here ϕ is the Newtonian potential, determined by Poisson's equation

$$\nabla^2 \phi = \frac{4\pi G}{c^4}\rho$$

where G is Newton's constant. Furthermore the energy density T_{00} for the nonrelativistic matter is just equal to its mass density:

$$T_{00} = \rho$$

Combining the above equations, we have then

$$\nabla^2 g_{00} = -\frac{8\pi G}{c^4} T_{00} \tag{4.1}$$

Equation (4.1) leads us to guess that the weak-field equations for a general distribution $T_{\alpha\beta}$ of energy and momentum take the form

$$G_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta}$$

where $G_{\alpha\beta}$ is a linear combination of the metric and its first and second derivatives. It follows then from the principle of equivalence that the equations which govern gravitational fields of arbitrary strength must take the form

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{4.2}$$

where $G_{\mu\nu}$ is a tensor which reduces to $G_{\alpha\beta}$ for weak fields.

Considering the left hand side of (4.2), we require $G_{\mu\nu}$ to take the form

$$G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R \tag{4.3}$$

because $G_{\mu\nu}$ is a tensor and by assumption,$G_{\mu\nu}$ contains only terms with $N = 2$ derivatives of the metric, where c_1 and c_2 are constants. Using the Bianchi identity $(R^{\mu}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta}R)_{;\mu}$ gives the covariant divergence of $G_{\mu\nu}$ as

$$G^{\mu}_{\nu;\mu} = \left(\frac{c_1}{2} + c_2\right) R_{;\nu}$$

Since $G^{\mu}_{\nu;\mu} = 0$ allows two possibilities, either $c_2 = -\frac{c_1}{2}$ or $R_{;\nu}$ vanishes everywhere. We can reject the second possibility, because (4.2) and (4.3) give

$$G^{\mu}_{\mu} = (c_1 + 4c_2)R = -\frac{8\pi G}{c^4}T^{\mu}_{\mu}$$

Thus if $R_{;\nu}$ vanishes, then so must $T^{\mu}_{\mu\,;\nu}$, and this is not the case in the presence of inhomogeneous nonrelativistic matter. We conclude then that $c_2 = -\frac{c_1}{2}$, so (4.3) becomes

$$G_{\mu\nu} = c_1(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)$$

Setting $c_1 = 1$, we obtain

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

With (4.2), this gives the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{4.4}$$

where $R_{\mu\nu}$ is the Ricci tensor,R is the curvature scalar,$g_{\mu\nu}$ the metric tensor and $T_{\mu\nu}$ the energy momentum tensor. The constant κ is called the Einstein constant (of gravitation). The above form of the Einstein field equation is for the $-+++$ metric sign convention. Using the $+---$ metric sign convention leads to an alternate form of the Einstein field equation which is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{4.5}$$

where $\kappa = \frac{8\pi G}{c^4}$, π is Archimedes' constant, G, Newton's general gravitational constant and c the speed of light. The change of sign on the right hand side occurs because the values of $T_{\mu\nu}$ have signs which are determined by the kind of energy momentum tensor being used.

Although the Einstein field equations were initially formulated in the context of a four-dimensional theory, the equations can be seen to hold in n dimensions. Because of this,[Cornish and Frankel[5]] studied Einstein gravity in (d + 1)-dimensions and have shown that, in any dimension Einstein's equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{d-1}{d-2}G_d S_d T_{\mu\nu} \tag{4.6}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ is the solid angle of a sphere in the space in question, $G_d \equiv G_3 l^{d-3}$, being some fundamental length. One can write the Einstein field equations in a more compact form in terms of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R; \tag{4.7}$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{4.8}$$

The expression on the left represents the curvature of spacetime as determined by the metric and the expression on the right represents the matter/energy content of spacetime. The Einstein field equations can then be interpreted as a set of equations dictating how the curvature of spactime is related to the matter/energy content of the universe.

In (3 + 1)-dimensions,

$$S_3 = \frac{2\pi^{1/2}}{\Gamma(3/2)} = 4\pi, \qquad G_3 = G_3 l^0 = G, \quad \textit{Newton's gravitational constant},$$

and the Einstein equations takes the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{4.9}$$

The singular nature of the coupling constant κ in d = 1, 2 spatial dimensions demands individual consideration. When d = 1, $G_{\mu\nu} = 0$, this is reflected in this equation;

$$G_{\mu\nu} = \kappa T_{\mu\nu} \Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{d-1}{d-2}G_d S_d T_{\mu\nu}$$

where $\kappa \to 0$ when $d \to 1$. The coupling constant κ diverges when $d \to 2$. This is an unacceptable situation as all terms are infinite and the theory as a whole is unworkable. A more satisfactory solution is to renormalize the gravitational constant in two spatial dimensions via $\frac{G_d}{d-2} \to G_d$. Hence, when d = 2, Einstein's equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\pi G T_{\mu\nu} \tag{4.10}$$

because

$$S_2 = \frac{2\pi}{\Gamma(1)} = 2\pi \qquad \frac{G_d}{d-2} \to G_d \approx G$$

4.1.2 Properties of the Field Equations

By the definition of the Einstein field equation, $G_{\mu\nu}$ is a tensor, because it manifestly covariance.

Since the energy momentum tensor $T_{\mu\nu}$ is symmetric, so is $G_{\mu\nu}$.

An important consequence of the Einstein field equations is the conservation of energy and momentum. The divergence or covariant differentiation of the energy momentum tensor is equal to zero. i.e.

$$\nabla_\nu T^{\mu\nu} = T^{\mu\nu}{}_{;\nu} = 0$$

This is because

$$G^{\mu\nu}{}_{;\lambda} \equiv 0$$

4.2 Solution of the Field Equations

The solutions of the Einstein field equations are metrics of spacetime. These metrics describe the structure of the spacetime including the inertial motion of objects in the spacetime. As the field equations are non-linear, they cannot always be completely solved (i.e. without making approximations). In practice, it is usually possible to simplify the problem by replacing the full set of equations of state with a simple approximation. Some common approximations are:

Vacuum: $T_{\mu\nu} = 0$

Fluid: $T_{\mu\nu} = (p + \rho c^2)u^\mu u^\nu - p\eta^{\mu\nu}$

Here ρ and p are the density and pressure of the fluid respectively

Dust: $T_{\mu\nu} = \rho_0 c^2 u^\mu u^\nu$

As stated earlier, we considered the dust particles.

4.3 The Cosmological Constant

We can modify the Einstein field equation by introducing a term proportional to the metric.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{4.11}$$

The constant Λ is called the cosmological constant. Since Λ is a constant, the energy conservation law is unaffected. The cosmological constant term was originally introduced by Einstein to allow for a static universe (i.e. one that is not expanding or contracting). This effort was unsuccessful for two reasons: the static universe described by this theory was unstable, and observation of distant galaxies by Hubble a decade later confirmed that our universe is, in fact, not static but expanding. So Λ was abandoned, with Einstein calling it the "biggest blunder he ever made", as discussed by Gamow George[6]. For many years the cosmological constant was almost universally considered to be zero.

Chapter 5

Cosmology in (3+1)-dimensions

5.1 The Robertson-Walker Metric

The Einstein equations form the basis of the mathematical study of cosmology. All that is needed is the construction of the appropriate metric $g_{\mu\nu}$. The simplest universe is one which appears isotropic to a set of co-moving observers because each observer sees the other as moving along with the overall cosmic expansion. Coordinates are called co-moving coordinates when two objects at different spatial coordinates can remain at those coordinates at all times, while the proper distance between them changes with time. The most important thing is that there are sites from which the universe would appear isotropic (there is no preferred direction) if there were anyone there to observe. The time measured by each co-moving observer equipped with a standard clock is the same because they are all controlled by the same rules of physics. Then, we can say that the universe is homogeneous (the same at all points). In a homogeneous and isotropic universe, the interval between events can be described by the Robertson-Walker Metric.

Hubble observed that the universe seems to be an orderly structure in which the galaxies, ~~considered~~ as ~~basic units~~, are moving apart from one another. This intuitive picture of regularity is often expressed formally as the Weyl postulate, after the early work of the mathematician Hermann Weyl. The postulate states that the world lines of galaxies designated as fundamental observers form a 3 - bundle of nonintersecting geodesics orthogonal to a series of spacelike hyperspaces, (see Jayant V. Narlikar[4]).

Einstein also believed that the universe has so much matter as to close the space. Einstein assumed homogeneity and isotropy in his cosmological problem. He further assumed that space is static. That led him to choose a time coordinate t such that the line element of spacetime could be described by

$$ds^2 = c^2dt^2 - \alpha_{\mu\nu}dx^\mu dx^\nu \tag{5.1}$$

where $\alpha_{\mu\nu}$ are functions of space coordinates $(\mu, \nu = 0, 1, 2, 3)$ only.

For a space of constant positive curvature, the easiest way of deriving the Robertson-Walker metric is to consider a 3-dimensional space as a space embedded in a 4-dimensional hypersurface given by;

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = S^2 \tag{5.2}$$

where S is in general a function of t.
In such a space, the line element is given by

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \tag{5.3}$$

Setting

$$x^1 = S\sin\chi\cos\theta$$

$$x^2 = S\sin\chi\sin\theta\cos\phi$$

$$x^3 = S\sin\chi\sin\theta\sin\phi$$

$$x^4 = S\cos\chi, \tag{5.4}$$

Computing the $(dx^i)^2$, substituting into (5.3), and simplifying, we obtain

$$d\sigma^2 = S^2[d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2]$$

If we further set

$$r = \sin\chi, \qquad d\chi^2 = \frac{dr^2}{\cos^2\chi} = \frac{dr^2}{1-r^2},$$

we find

$$d\sigma^2 = S^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{5.5}$$

The quantity S is called the radius of the universe or the expansion factor. The line element is therefore given by

$$ds^2 = c^2dt^2 - d\sigma^2 \tag{5.6}$$

$$= c^2dt^2 - S^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{5.7}$$

This is the Robertson-Walker metric for a space of constant positive curvature.

For a space of constant negative curvature, the Robertson-Walker metric is most easily derived by considering a 3-dimensional space embedded in a 4-dimensional hypersurface given by;

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = -S^2 \tag{5.8}$$

where S is in general a function of the time.
The line element in this pseudo-Euclidean space is given by

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$$

Now setting

$$x^1 = S\sinh\chi\cos\theta$$

$$x^2 = S \sinh\chi \sin\theta \cos\phi$$

$$x^3 = S \sinh\chi \sin\theta \sin\phi$$

$$x^4 = S \cosh\chi, \tag{5.9}$$

we find

$$d\sigma^2 = S^2[d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi \sin^2\theta d\phi^2]$$

If we further set

$$r = \sinh\chi, \qquad d\chi^2 = \frac{dr^2}{\cosh^2\chi} = \frac{dr^2}{1+\sinh^2\chi} = \frac{dr^2}{1+r^2}$$

we obtain

$$d\sigma^2 = S^2\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{5.10}$$

The quantity S is called the radius of the universe or the expansion factor. The line element

$$ds^2 = c^2dt^2 - d\sigma^2 \tag{5.11}$$

then takes the form

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{5.12}$$

Equations (5.7) and (5.12) can be combined into a single expression by introducing a parameter k that takes values ± 1, as well as the value 0:

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1-kr^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{5.13}$$

The $k = 0$ case gives the line element of a 3-D Euclidean space, expressed in spherical coordinates:

$$ds^2 = c^2dt^2 - S^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{5.14}$$

We can consider this line element as describing a flat space. Hence the most general line element satisfying the Weyl postulate and the cosmological principle is given by

$$ds^2 = c^2dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{5.15}$$

The above equation is known as the Robertson - Walker metric in (3+1)-dimensional spacetime. Three different surfaces can be embedded in this space. They are; the surface of zero curvature, the surface of positive curvature and the surface of negative curvature.

When $k = 0$, we have a space of zero curvature, such a space, examples of which are Euclidean spaces like the plane and pseudo-Euclidean spaces like Minkowski space called a flat space. For $k = 1$ we have a space of positive curvature. This is necessarily a closed space, spherical and ellipsoidal spaces being prime examples. Finally, the $k = -1$ case describes a space of negative curvature. As the embedding equation (5.3) shows, it is a saddle-shaped, and therefore an open space.

As noted above, the scale factor $S(t)$ is often called the expansion factor. The reason is that it in effect determines the rate of expansion of the universe and its determination is arguably the basic problem of mathematical cosmology. Below we touch upon its determination for the various cases of spatial curvature. The solutions are known as the Friedmann models of the universe.

5.2 The Friedmann Models

The Friedmann models are the solutions of the Einstein field equation with the Robertson - Walker metric for k = 0, ±1. The metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-S^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -S^2r^2 & 0 \\ 0 & 0 & 0 & -S^2r^2\sin^2\theta \end{pmatrix}$$

Let $\quad dx^1 = cdt, \qquad dx^2 = dr, \qquad dx^3 = d\theta, \qquad dx^4 = d\phi$

The non-vanishing Christoffel symbols are:

$$\gamma_{2}{}^{2}{}_{2} = \frac{kr}{1-kr^2} \qquad \gamma_{2}{}^{2}{}_{1} = \frac{\dot{S}}{cS} \qquad \gamma_{3}{}^{3}{}_{1} = \frac{\dot{S}}{cS} \qquad \gamma_{4}{}^{4}{}_{1} = \frac{\dot{S}}{cS}$$

$$\gamma_{3}{}^{3}{}_{2} = \frac{1}{r} \qquad \gamma_{4}{}^{4}{}_{2} = \frac{1}{r} \qquad \gamma_{4}{}^{4}{}_{3} = \cot\theta \qquad \gamma_{2}{}^{1}{}_{2} = \frac{S\dot{S}}{c(1-kr^2)}$$

$$\gamma_{3}{}^{1}{}_{3} = \frac{S\dot{S}r^2}{c} \qquad \gamma_{4}{}^{1}{}_{4} = \frac{S\dot{S}r^2\sin^2\theta}{c} \qquad \gamma_{3}{}^{2}{}_{3} = -r(1-kr^2)$$

$$\gamma_{4}{}^{2}{}_{4} = -r(1-kr^2)\sin^2\theta \qquad \gamma_{4}{}^{3}{}_{4} = -\sin\theta\cos\theta$$

Giving the following non-vanishing components of the Ricci tensor

$$R_{ik} = \frac{\partial^2 \ln(-g)^{1/2}}{\partial x^i \partial x^k} - \frac{\partial \gamma_{i}{}^{l}{}_{k}}{\partial x^l} + \gamma_{i}{}^{m}{}_{n}\gamma_{k}{}^{n}{}_{m} - \gamma_{i}{}^{l}{}_{k}\frac{\partial \ln(-g)^{1/2}}{\partial x^l}; \tag{5.16}$$

where

$$g = -\frac{S^6 r^4 \sin^2\theta}{1-kr^2}$$

is the determinant of the metric:

$$R_{11} = \frac{3\ddot{S}}{c^2 S}$$

$$R_{22} = -\frac{1}{c^2}\left(\frac{S\ddot{S} + 2\dot{S}^2 + 2kc^2}{1-kr^2}\right)$$

$$R_{33} = -\frac{r^2}{c^2}(S\ddot{S} + 2\dot{S}^2 + 2kc^2)$$

$$R_{44} = -\frac{r^2\sin^2\theta}{c^2}(S\ddot{S} + 2\dot{S}^2 + 2kc^2).$$

From these components ~~we easily~~ compute the curvature scalar

$$R = R_i{}^i = g^{ik}R_{ik} = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44}.$$

It is given by

$$R = \frac{6}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) \tag{5.17}$$

The Einstein's equation is given by;

$$G_{ik} \equiv R_{ik} - \frac{1}{2}g_{ik}R = -\kappa T_{ik} = -\frac{8\pi G}{c^4}T_{ik} \tag{5.18}$$

$$G_{11} \equiv R_{11} - \frac{1}{2}g_{11}R = -\frac{8\pi G}{c^4}T_{11}$$

$$\frac{3\ddot{S}}{c^2 S} - \frac{1}{2}\left[\frac{6}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)\right] = -\frac{8\pi G}{c^4}\rho$$

$$\frac{3\ddot{S}}{c^2 S} - \frac{3}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) = -\frac{8\pi G}{c^4}\rho$$

$$G_{11} \equiv \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3}\rho \tag{5.19}$$

$$G_{22} \equiv R_{22} - \frac{1}{2}g_{22}R = -\frac{8\pi G}{c^4}T_{22}$$

$$G_{22} \equiv -\frac{1}{c^2}\left(\frac{S\ddot{S} + 2\dot{S}^2 + 2kc^2}{1 - kr^2}\right) - \frac{1}{2}\left(-\frac{S^2}{1 - kr^2}\right)\cdot\frac{6}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) = 0$$

$$G_{22} \equiv \frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0 \tag{5.20}$$

Due to symmetry:

$$G_{22} \equiv G_{33} \equiv G_{44} \equiv \frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0$$

As stated earlier on,

$$T_{11} = \epsilon, \qquad T_{22} = T_{33} = T_{44} = -p$$

From equation (5.19), we have

$$S(\dot{S}^2 + kc^2) = \frac{8\pi G}{3} S^3 \rho$$

Differentiating both sides with respect to t, we obtain;

$$\frac{d}{dt}[S(\dot{S}^2 + kc^2)] = \frac{8\pi G}{3} \frac{d}{dt}(\rho S^3)$$

$$S(2\dot{S}\ddot{S}) + \dot{S}(\dot{S}^2 + kc^2) = \frac{8\pi G}{3} \frac{d}{dt}(\rho S^3)$$

$$\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3S^2} \frac{d}{dS}(\rho S^3)$$

From equation (5.20)

$$\frac{d}{dS}(\rho S^3) = 0$$

$$S^3 \rho = {S_0}^3 \rho_0,$$

where ρ_0 and S_0 are the density of matter in the universe in the present epoch and the expansion factor at the present epoch respectively.

Thus

$$\rho = \frac{{S_0}^3}{S^3} \rho_0 \tag{5.21}$$

Now the friedmann models for dust particles are given by;

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0 \tag{5.22}$$

$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3} \rho_0 \frac{{S_0}^3}{S^3} \tag{5.23}$$

5.2.1 Euclidean Sections(k=0)

For Euclidean sections, i.e. when k = 0, at the present epoch;

$$S = S_0 \left(\frac{t}{t_0}\right)^{2/3} \tag{5.24}$$

Where the age of the universe at the present epoch is,

$$t_0 = \frac{2}{3H_0} \tag{5.25}$$

where

$$H_0 = \left(\frac{\dot{S}}{S}\right)\Bigg|_{t_0} \tag{5.26}$$

is the Hubbles constant. In the Euclidean section, the universe continue to expand forever.

5.2.2 Closed Sections(k=1)

For closed sections, i.e. when k = 1, at the present epoch, the solutions are obtained in parametric form. They are;

$$S = \frac{1}{2}\alpha(1 - \cos\theta)$$

$$ct = \frac{1}{2}\alpha(1 - \sin\theta) \tag{5.27}$$

where α is a constant.
In this section, the universe will expand to a particular limit and will start to contract. Thus, at some point in the future the "galaxies" will stop receding from each other and begin to approach each other as the universe collapses on itself.

5.2.3 Opened sections(k=-1)

For opened sections, i.e. when k = -1, at the present epoch, the solutions are also obtained in parametric form but in a hyperbolic space. They are;

$$S = \frac{1}{2}\beta(\cosh\psi - 1)$$

$$ct = \frac{1}{2}\beta(\sinh\psi - \psi) \tag{5.28}$$

where β is a constant.

Like the Euclidean section, this model will continue to expand forever.

Chapter 6

Cosmology in (2 + 1)-Dimensions

6.1 Robertson-Walker Metric in (2 + 1)-Dimensions: The 2-D Coordinate Subspaces and their Curvature.

In this chapter, we will first construct a Robertson-Walker like metric in (2 + 1)-dimensional space. The easiest way of doing this is to consider a 2-dimensional coordinate space as a space embedded in a 3-dimensional "hypersurface". For a space of constant negative curvature, the hypersurface is described by

$$(x^1)^2 + (x^2)^2 - (x^3)^2 = -S^2 \tag{6.1}$$

where S is independent of the coordinates $x^i (i = 1, 2, 3)$, but may depend on time (see Jayant V. Narlikar[4]), and the metric

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2 \tag{6.2}$$

Clearly Eq. 6.1 admits of the transformation

$$x^1 = S\sinh\chi\cos\theta, \qquad x^2 = S\sinh\chi\sin\theta, \qquad x^3 = S\cosh\chi$$

From which we find

$$(dx^1)^2 = S^2[\cosh^2\chi\cos^2\theta d\chi^2 + \sinh^2\chi\sin^2\theta d\theta^2 - 2\cosh\chi\sinh\chi\cos\theta\sin\theta d\chi d\theta].$$

$$(dx^2)^2 = S^2[\cosh^2\chi\sin^2\theta d\chi^2 + \sinh^2\chi\cos^2\theta d\theta^2 + 2\cosh\chi\sinh\chi\cos\theta\sin\theta d\chi d\theta].$$

$$(dx^3)^2 = S^2\sinh^2\chi d\chi^2$$

and

$$d\sigma^2 = S^2[(\cosh^2\chi\cos^2\theta + \cosh^2\chi\sin^2\theta - \sinh^2\chi)d\chi^2 + (\sinh^2\chi\sin^2 + \sinh^2\chi\cos^2\theta)d\theta^2]$$

$$= S^2[d\chi^2 + \sinh^2\chi d\theta^2]$$

Now let us set

$$r = \sinh\chi$$

Then

$$dr = \cosh\chi d\chi$$

$$d\chi^2 = \frac{dr^2}{\cosh^2\chi} = \frac{dr^2}{1+\sinh^2\chi} = \frac{dr^2}{1+r^2},$$

and

$$d\sigma^2 = S^2\left[\frac{dr^2}{1+r^2} + r^2 d\theta^2\right]$$

The line element for the (2 + 1) spacetime is therefore given by

$$ds^2 = c^2dt^2 - d\sigma^2$$

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1+r^2} + r^2 d\theta^2\right] \tag{6.3}$$

For a space of constant positive curvature, the Robertson-Walker metric is derived by considering a 2-dimensional coordinate space as a space embedded in the 3-dimensional hypersurface described by the equation

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = S^2 \tag{6.4}$$

(where S is a constant) and the metric

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{6.5}$$

Setting

$$x^1 = S\sin\chi\cos\theta, \qquad x^2 = S\sin\chi\sin\theta, \qquad x^3 = S\cos\chi,$$

we have

$$(dx^1)^2 = S^2[\cos^2\chi\cos^2\theta d\chi^2 + \sin^2\chi\sin^2\theta d\theta^2 - 2\cos\chi\sin\chi\cos\theta\sin\theta d\chi d\theta],$$

$$(dx^2)^2 = S^2[\cos^2\chi\sin^2\theta d\chi^2 + \sin^2\chi\cos^2\theta d\theta^2 + 2\cos\chi\sin\chi\cos\theta\sin\theta d\chi d\theta],$$

$$(dx^3)^2 = S^2\sin^2\chi d\chi^2,$$

and

$$d\sigma^2 = S^2[(\cos^2\chi\cos^2\theta + \cos^2\chi\sin^2\theta + \sin^2\chi)d\chi^2 + (\sin^2\chi\sin^2\theta + \sin^2\chi\cos^2\theta)d\theta^2]$$

$$= S^2[d\chi^2 + \sin^2\chi d\theta^2].$$

If we further substitute;

$$r = \sin\chi$$

$$dr = \cos\chi d\chi$$

$$d\chi^2 = \frac{dr^2}{\cos^2\chi} = \frac{dr^2}{1-\sin^2\chi} = \frac{dr^2}{1-r^2}$$

or

$$d\sigma^2 = S^2\left[\frac{dr^2}{1-r^2} + r^2 d\theta^2\right].$$

where

$$r = \sin\chi.$$

The line element of (2 + 1)-dimensional spacetime is therefore given by

$$ds^2 = c^2dt^2 - d\sigma^2$$

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1-r^2} + r^2 d\theta^2\right] \tag{6.6}$$

Equations (6.3) and (6.6) can be combined into a single expression by introducing a parameter k that takes values ± 1:

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right] \tag{6.7}$$

In analogy with the Robertson-Walker metric, the above metric can be called the Robertson-Walker metric in (2 + 1)-dimensional spacetime. The metric also describes a 2-dimensional coordinate space of zero curvature, (ie a flat 2-dimensional space), which is the case for k = 0.

Thus, when k = 0, we have a space of zero curvature, ie a flat space, k = 1 is a space of positive curvature, which is a closed space and k = -1 describes a space of negative curvature, which is an open space. Examples of such surfaces are shown below.

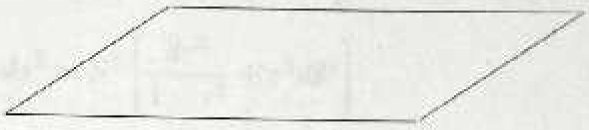

Figure 6.1: Example of surfaces of zero curvature, i.e. a plane

Figure 6.2: Example of surfaces of positive(non-zero) curvature, i.e. a spherical surface in 3-D space

Figure 6.3: Example of ~~surfaces of~~ negative curvature, i.e. surface of a saddle in 3-D space

To determine the curvature of each surface, we use the spatial part of the Robertson-Walker metric. The various surfaces and their curvatures are discussed below.

For positive curvature, the metric of the spatial part is given by;

$$d\sigma^2 = S^2\left[\frac{dr^2}{1-r^2} + r^2 d\theta^2\right] \tag{6.8}$$

Then

$$g_{ij} = \begin{pmatrix} \frac{S^2}{1-r^2} & 0 \\ 0 & S^2 r^2 \end{pmatrix},$$

where (i, j) =(r, θ)=(1, 2) Computation of the non-vanishing Christoffel symbols, yields

$$\gamma_{i}{}^{i}{}_{i} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^i}$$

$$\gamma_{1}{}^{1}{}_{1} = \frac{1}{2g_{11}} \cdot \frac{\partial g_{11}}{\partial x^1}$$

$$\gamma_{1}{}^{1}{}_{1} = \frac{1}{2\left(\frac{S^2}{1-r^2}\right)} \cdot \frac{\partial\left(\frac{S^2}{1-r^2}\right)}{\partial r}$$

$$\gamma_{1}{}^{1}{}_{1} = \frac{r}{1-r^2}$$

$$\gamma_{i}{}^{i}{}_{j} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^j}$$

$$\gamma_{2}{}^{2}{}_{1} = \frac{1}{2g_{22}} \cdot \frac{\partial g_{22}}{\partial x^1}$$

$$\gamma_{2}{}^{2}{}_{1} = \frac{1}{2S^2r^2} \cdot \frac{\partial(S^2r^2)}{\partial r}$$

$$\gamma_{2\ 1}^{\ 2} = \frac{1}{r}$$

$$\gamma_{j\ j}^{\ i} = \frac{-1}{2g_{ii}} \cdot \frac{\partial g_{jj}}{\partial x^i}$$

$$\gamma_{2\ 2}^{\ 1} = \frac{-1}{2g_{11}} \cdot \frac{\partial g_{22}}{\partial x^1}$$

$$\gamma_{2\ 2}^{\ 1} = \frac{-1}{2\left(\frac{S^2}{1-r^2}\right)} \cdot \frac{\partial (S^2 r^2)}{\partial r}$$

$$\gamma_{2\ 2}^{\ 1} = -r(1-r^2)$$

For a 2-D space, the only non-zero components of the curvature tensor

$$R_{jlhk} = \frac{1}{2}\left(\frac{\partial^2 g_{jh}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{lh}}{\partial x^k \partial x^j} + \frac{\partial^2 g_{lk}}{\partial x^h \partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^h \partial x^l}\right) + g_{rs}[\gamma_{j\ h}^{\ r}\gamma_{l\ k}^{\ s} - \gamma_{j\ k}^{\ r}\gamma_{l\ h}^{\ s}] \tag{6.9}$$

are the components R_{1212} and R_{2121}, with $R_{1212} = R_{2121}$. Using the Christoffel symbols obtained above and the formula above we had

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} + \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2}\right) + g_{11}[\gamma_{1\ 1}^{\ 1}\gamma_{2\ 2}^{\ 1} - \gamma_{1\ 2}^{\ 1}\gamma_{2\ 1}^{\ 1}] + g_{22}[\gamma_{1\ 1}^{\ 2}\gamma_{2\ 2}^{\ 2} - \gamma_{1\ 2}^{\ 2}\gamma_{2\ 1}^{\ 2}]$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\right) + g_{11}(\gamma_{1\ 1}^{\ 1}\gamma_{2\ 2}^{\ 1}) - g_{22}(\gamma_{1\ 2}^{\ 2}\gamma_{2\ 1}^{\ 2})$$

$$= \frac{1}{2}\left(\frac{\partial^2 (S^2 r^2)}{\partial r \cdot \partial r}\right) + \frac{S^2}{1-r^2} \cdot \left(\left(\frac{r}{1-r^2}\right)(-r(1-r^2))\right) - S^2 r^2 \left(\frac{1}{r^2}\right)$$

$$= \frac{1}{2}(2S^2) - \frac{S^2 r^2}{1-r^2} - S^2$$

$$R_{1212} = -\frac{S^2 r^2}{1-r^2} = R_{2121}$$

Now the curvature of a 2-D space described by the metric g_{ij} is given by the Gaussian curvature

$$K = -\frac{R_{1212}}{g} \tag{6.10}$$

where g is the determinant of the metric. In this case

$$g = \frac{S^4 r^2}{1 - r^2}$$

and

$$K = \frac{S^2 r^2}{1 - r^2} \cdot \frac{1 - r^2}{S^4 r^2} = \frac{1}{S^2},$$

which proves that the space indeed has a positive curvature.

For a 2-D space with k = -1, we have

$$d\sigma^2 = S^2 \left[\frac{dr^2}{1 + r^2} + r^2 d\theta^2\right] \tag{6.11}$$

and

$$g_{ij} = \begin{pmatrix} \frac{S^2}{1+r^2} & 0 \\ 0 & S^2 r^2 \end{pmatrix}$$

If (i, j) = (r, θ)=(1, 2), the non-vanishing Christoffel symbols are

$$\gamma_i{}^i{}_i = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^i}$$

$$\gamma_1{}^1{}_1 = \frac{1}{2g_{11}} \cdot \frac{\partial g_{11}}{\partial x^1}$$

$$\gamma_1{}^1{}_1 = \frac{1}{2\left(\frac{S^2}{1+r^2}\right)} \cdot \frac{\partial\left(\frac{S^2}{1+r^2}\right)}{\partial r}$$

$$\gamma_1{}^1{}_1 = -\frac{r}{1 + r^2}$$

$$\gamma_i{}^i{}_j = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^j}$$

$$\gamma_2{}^2{}_1 = \frac{1}{2g_{22}} \cdot \frac{\partial g_{22}}{\partial x^1}$$

$$= \frac{1}{2S^2r^2} \cdot \frac{\partial (S^2r^2)}{\partial r}$$

$$\gamma_2{}^2{}_1 = \frac{1}{r}$$

$$\gamma_j{}^i{}_j = -\frac{1}{2g_{ii}} \cdot \frac{\partial g_{jj}}{\partial x^i}$$

$$\gamma_2{}^1{}_2 = -\frac{1}{2g_{11}} \cdot \frac{\partial g_{22}}{\partial x^1}$$

$$= -\frac{1}{2\left(\frac{S^2}{1+r^2}\right)} \cdot \frac{\partial (S^2r^2)}{\partial r}$$

$$\gamma_2{}^1{}_2 = -r(1+r^2)$$

and the only non-zero independent component of the Riemann curvature tensor is given by

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} + \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2}\right) + g_{11}[\gamma_1{}^1{}_1\gamma_2{}^1{}_2 - \gamma_1{}^1{}_2\gamma_2{}^1{}_1] + g_{22}[\gamma_1{}^2{}_1\gamma_2{}^2{}_2 - \gamma_1{}^2{}_2\gamma_2{}^2{}_1]$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\right) + g_{11}(\gamma_1{}^1{}_1\gamma_2{}^1{}_2) - g_{22}(\gamma_1{}^2{}_2\gamma_2{}^2{}_1)$$

$$= \frac{1}{2}\left(\frac{\partial^2(S^2r^2)}{\partial r\cdot\partial r}\right) + \frac{S^2}{1+r^2}\cdot\left(\left(-\frac{r}{1+r^2}\right)(-r(1+r^2))\right) - S^2r^2\left(\frac{1}{r^2}\right)$$

$$= \frac{1}{2}(2S^2) + \frac{S^2r^2}{1+r^2} - S^2$$

Thus,

$$R_{1212} = \frac{S^2r^2}{1+r^2},$$

which yields for the Gaussian curvature

$$K = -\frac{g_{1212}}{g},$$

where

$$g = \frac{S^4r^2}{1+r^2}$$

is the determinant of the metric, and, hence, the curvature of the space, the expression

$$K = -\frac{S^2r^2}{1+r^2}\cdot\frac{1+r^2}{S^4r^2} = -\frac{1}{S^2}$$

Hence a 2-D space with k = -1 is indeed a space with negative curvature.

Finally, for a 2-D space with k = 0, ie one with the line element

$$d\sigma^2 = S^2[dr^2 + r^2d\theta^2] \tag{6.12}$$

and metric

$$g_{ij} = \begin{pmatrix} S^2 & 0 \\ 0 & S^2r^2 \end{pmatrix}, \tag{6.13}$$

The non-vanishing Christoffel symbols are

$$\gamma_i{}^i{}_j = \frac{1}{2g_{ii}}\cdot\frac{\partial g_{ii}}{\partial x^j}$$

$$\gamma_2{}^2{}_1 = \frac{1}{2g_{22}}\cdot\frac{\partial g_{22}}{\partial x^1}$$

$$= \frac{1}{2S^2r^2} \cdot \frac{\partial(S^2r^2)}{\partial r}$$

$$\gamma_2{}^2{}_1 = \gamma_1{}^2{}_2 = \frac{1}{r}$$

$$\gamma_j{}^i{}_j = -\frac{1}{2g_{ii}} \cdot \frac{\partial g_{jj}}{\partial x^i}$$

$$\gamma_2{}^1{}_2 = -\frac{1}{2g_{11}} \cdot \frac{\partial g_{22}}{\partial x^1}$$

$$= -\frac{1}{2S^2} \cdot \frac{\partial(S^2r^2)}{\partial r}$$

$$\gamma_2{}^1{}_2 = -r$$

the only non-zero independent component

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} + \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2}\right) +$$
$$g_{11}[\gamma_1{}^1{}_1\gamma_2{}^1{}_2 - \gamma_1{}^1{}_2\gamma_2{}^1{}_1] + g_{22}[\gamma_1{}^2{}_1\gamma_2{}^2{}_2 - \gamma_1{}^2{}_2\gamma_2{}^2{}_1]$$
$$= \frac{1}{2}\left(\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\right) - g_{22}(\gamma_1{}^2{}_2\gamma_2{}^2{}_1)$$
$$\frac{1}{2}\left(\frac{\partial^2(S^2r^2)}{\partial r \partial r}\right) - S^2r^2\left(\frac{1}{r^2}\right)$$
$$\frac{1}{2}(2S^2) - S^2 = 0,$$

and the Gaussian curvature

$$K = -\frac{R_{1212}}{g},$$

where

$$g = S^4r^2$$

is the determinant of the metric (6.13), is clearly equal to zero.

Hence the 2-D space with k = 0 is indeed a flat space.

6.2 The Einstein Field Equations for (2 + 1)-D Spacetime

Let us return to the (2 + 1)-D spacetime. As shown above, the Robertson-Walker line element for this Riemannian space is

$$ds^2 = c^2dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2\right],$$

and the corresponding Robertson-Walker metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{S^2}{1-kr^2} & 0 \\ 0 & 0 & -S^2r^2 \end{pmatrix}, \quad or \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1-kr^2}{S^2} & 0 \\ 0 & 0 & -\frac{1}{S^2r^2} \end{pmatrix}$$

Setting $(t, r, \theta) = (x^1, x^2, x^3)$, and using these metrics, we easily compute the non-vanishing Christoffel symbols:

$$\gamma_i{}^i{}_i = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^i}$$

$$\gamma_2{}^2{}_2 = \frac{1}{2g_{22}} \cdot \frac{\partial g_{22}}{\partial x^2}$$

$$= -\frac{1-kr^2}{2S^2} \cdot \frac{\partial}{\partial r}\left(-\frac{S^2}{1-kr^2}\right)$$

$$\gamma_2{}^2{}_2 = \frac{kr}{1-kr^2}$$

$$\gamma_i{}^i{}_j = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^j}$$

$$\gamma_2{}^2{}_1 = \frac{1}{2g_{22}} \cdot \frac{\partial g_{22}}{\partial x^1}$$

$$= -\frac{1-kr^2}{2S^2} \cdot \frac{\partial}{c\partial t}\left(-\frac{S^2}{1-kr^2}\right)$$

$$\gamma_{2}{}^{2}{}_{1} = \frac{\dot{S}}{cS}$$

$$\gamma_{3}{}^{3}{}_{1} = \frac{1}{2g_{33}} \cdot \frac{\partial g_{33}}{\partial x^{1}}$$

$$= -\frac{1}{2S^{2}r^{2}} \cdot \frac{\partial}{c\partial t}(-S^{2}r^{2})$$

$$\gamma_{3}{}^{3}{}_{1} = \frac{\dot{S}}{cS}$$

$$\gamma_{3}{}^{3}{}_{2} = \frac{1}{2g_{33}} \cdot \frac{\partial g_{33}}{\partial x^{2}}$$

$$= -\frac{1}{2S^{2}r^{2}} \cdot \frac{\partial}{\partial r}(-S^{2}r^{2})$$

$$\gamma_{3}{}^{3}{}_{2} = \frac{1}{r}$$

$$\gamma_{j}{}^{i}{}_{j} = -\frac{1}{2g_{ii}} \cdot \frac{\partial g_{jj}}{\partial x^{i}}$$

$$\gamma_{2}{}^{1}{}_{2} = \frac{1}{2g_{11}} \cdot \frac{\partial g_{22}}{\partial x^{1}}$$

$$= -\frac{1}{2} \cdot \frac{\partial}{c\partial t}\left(-\frac{S^{2}}{1-kr^{2}}\right)$$

$$\gamma_{2}{}^{1}{}_{2} = \frac{S\dot{S}}{c(1-kr^{2})}$$

$$\gamma_{3}{}^{1}{}_{3} = \frac{1}{2g_{11}} \cdot \frac{\partial g_{33}}{\partial x^{1}}$$

$$= -\frac{1}{2} \cdot \frac{\partial}{c\partial t}(-S^{2}r^{2})$$

$$\gamma_3{}^1{}_3 = \frac{S\dot{S}r^2}{c}$$

$$\gamma_3{}^2{}_3 = \frac{1}{2g_{22}} \cdot \frac{\partial g_{33}}{\partial x^2}$$

$$= \frac{(1-kr^2)}{2S^2} \cdot \frac{\partial}{\partial r}(-S^2r^2)$$

$$\gamma_3{}^2{}_3 = -r(1-kr^2)$$

Hence the non-vanishing Christoffel symbols are:

$$\gamma_2{}^2{}_2 = \frac{kr}{1-kr^2} \qquad \gamma_2{}^2{}_1 = \frac{\dot{S}}{cS} \qquad \gamma_3{}^3{}_1 = \frac{\dot{S}}{cS}$$

$$\gamma_3{}^3{}_2 = \frac{1}{r} \qquad \gamma_2{}^1{}_2 = \frac{S\dot{S}}{c(1-kr^2)} \qquad \gamma_3{}^1{}_3 = \frac{S\dot{S}r^2}{c}$$

$$\gamma_3{}^2{}_3 = -r(1-kr^2)$$

Using these results, we easily compute the non-vanishing Ricci tensor components using the formula below;

$$R_{ik} = -\frac{\partial \gamma_i{}^l{}_k}{\partial x^l} + \frac{\partial^2(\ln(g)^{1/2})}{\partial x^i \partial x^k} + \gamma_i{}^m{}_n \gamma_k{}^n{}_m - \frac{\partial}{\partial x^n}(\ln(g)^{1/2}) \cdot \gamma_i{}^n{}_k \tag{6.14}$$

where

$$g = \frac{S^4r^2}{1-kr^2}$$

is the determinant of the metric (6.7). We find

$$R_{11} = -\frac{\partial \gamma_1{}^l{}_1}{\partial x^l} + \frac{\partial^2(\ln(g)^{1/2})}{(\partial x^1)^2} + \gamma_1{}^m{}_n \gamma_1{}^n{}_m - \frac{\partial}{\partial x^n}(\ln(g)^{1/2}) \cdot \gamma_1{}^n{}_1$$

$$= \frac{\partial^2}{c^2 \partial t^2}\left[\ln\left(\frac{S^2 r}{\sqrt{1-kr^2}}\right)\right] + (\gamma_1{}^2{}_2)^2 + (\gamma_1{}^3{}_3)^2$$

$$= \frac{\partial^2}{c^2\partial t^2}\left[\ln S^2 + \ln\frac{r}{\sqrt{1-kr^2}}\right] + ({\gamma_1}^2{}_2)^2 + ({\gamma_1}^3{}_3)^2$$

$$= \frac{\partial^2}{c^2\partial t^2}\left[\ln S^2\right] + ({\gamma_1}^2{}_2)^2 + ({\gamma_1}^3{}_3)^2$$

since r does not explicitly depend on t. Thus

$$R_{11} = \frac{2\ddot{S}}{c^2 S} - \frac{2\dot{S}^2}{c^2S^2} + \frac{2\dot{S}^2}{c^2S^2}$$

$$R_{11} = \frac{2\ddot{S}}{c^2 S} \tag{6.15}$$

$$R_{22} = -\frac{\partial {\gamma_2}^l{}_2}{\partial x^l} + \frac{\partial^2(\ln(g)^{1/2})}{(\partial x^2)^2} + {\gamma_2}^m{}_n{\gamma_2}^n{}_m - \frac{\partial}{\partial x^n}(\ln(g)^{1/2})\cdot{\gamma_2}^n{}_2$$

$$= -\left[\frac{\partial {\gamma_2}^1{}_2}{\partial x^1} + \frac{\partial {\gamma_2}^2{}_2}{\partial x^2}\right] + \frac{\partial^2(\ln(g)^{1/2})}{(\partial x^2)^2} + {\gamma_2}^m{}_1{\gamma_2}^1{}_m + {\gamma_2}^m{}_2{\gamma_2}^2{}_m + {\gamma_2}^m{}_3{\gamma_2}^3{}_m$$
$$-\frac{\partial}{\partial x^1}(\ln(g)^{1/2})\cdot{\gamma_2}^1{}_2 - \frac{\partial}{\partial x^2}(\ln(g)^{1/2})\cdot{\gamma_2}^2{}_2$$

$$= -\left[\frac{\partial}{c\partial t}\left(\frac{S\dot{S}}{c(1-kr^2)}\right) + \frac{\partial}{\partial r}\left(\frac{kr}{1-kr^2}\right)\right] + \frac{\partial^2}{\partial r^2}\left[\ln\left(\frac{S^2 r}{\sqrt{1-kr^2}}\right)\right] + {\gamma_2}^2{}_1{\gamma_2}^1{}_2 + {\gamma_2}^1{}_2{\gamma_2}^2{}_1$$
$$+{\gamma_2}^2{}_2{\gamma_2}^2{}_2 + {\gamma_2}^3{}_3{\gamma_2}^3{}_3 - \frac{\partial}{c\partial t}\left[\ln\left(\frac{S^2 r}{\sqrt{1-kr^2}}\right)\right]\cdot\frac{S\dot{S}}{c(1-kr^2)} - \frac{\partial}{\partial r}\left[\ln\left(\frac{S^2 r}{\sqrt{1-kr^2}}\right)\right]\cdot\frac{kr}{1-kr^2}$$

$$= -\frac{\dot{S}^2 + S\ddot{S}}{c^2(1-kr^2)} - \frac{\partial}{\partial r}\left(\frac{kr}{1-kr^2}\right) + \frac{\partial^2}{\partial r^2}\left[\ln\left(\frac{r}{\sqrt{1-kr^2}}\right)\right] + \frac{2\dot{S}^2}{c^2(1-kr^2)} + \left(\frac{kr}{1-kr^2}\right)^2$$
$$+\frac{1}{r^2} - \frac{\partial}{\partial t}[\ln S^2]\cdot\frac{S\dot{S}}{c(1-kr^2)} - \frac{\partial}{\partial r}\left[\ln\left(\frac{r}{\sqrt{1-kr^2}}\right)\right]\cdot\frac{kr}{1-kr^2}$$

$$= -\frac{\dot{S}^2 + S\ddot{S}}{c^2(1-kr^2)} - \frac{\partial}{\partial r}\left(\frac{kr}{1-kr^2}\right) - \frac{1}{r^2} + \frac{\partial}{\partial r}\left(\frac{kr}{1-kr^2}\right) + \frac{2\dot{S}^2}{c^2(1-kr^2)}$$
$$+\left(\frac{kr}{1-kr^2}\right)^2 + \frac{1}{r^2} - \frac{2\dot{S}^2}{c^2(1-kr^2)} - \frac{k}{1-kr^2} - \left(\frac{kr}{1-kr^2}\right)^2$$

$$= -\frac{\dot{S}^2 + S\ddot{S}}{c^2(1-kr^2)} - \frac{k}{1-kr^2}$$

$$R_{22} = -\frac{1}{c^2}\left(\frac{S\ddot{S} + \dot{S}^2 + kc^2}{1-kr^2}\right) \tag{6.16}$$

$$R_{33} = -\frac{\partial {\gamma_3}^l{}_3}{\partial x^l} + \frac{\partial^2(\ln(g)^{1/2})}{(\partial x^3)^2} + {\gamma_3}^m{}_n {\gamma_3}^n{}_m - \frac{\partial}{\partial x^n}(\ln(g)^{1/2}) \cdot {\gamma_3}^n{}_3$$

$$= -\left[\frac{\partial {\gamma_3}^1{}_3}{\partial x^1} + \frac{\partial {\gamma_3}^2{}_3}{\partial x^2}\right] + \frac{\partial^2(\ln(g)^{1/2})}{(\partial x^3)^2} + {\gamma_3}^m{}_1 {\gamma_3}^1{}_m + {\gamma_3}^m{}_2 {\gamma_3}^2{}_m$$
$$+ {\gamma_3}^m{}_3 {\gamma_3}^3{}_m - \frac{\partial}{\partial x^1}(\ln(g)^{1/2}) \cdot {\gamma_3}^1{}_3 - \frac{\partial}{\partial x^2}(\ln(g)^{1/2}) \cdot {\gamma_3}^2{}_3$$

$$= -\left[\frac{\partial}{c\partial t}\left(\frac{S\dot{S}r^2}{c}\right) + \frac{\partial}{\partial r}(-r(1-kr^2))\right] + {\gamma_3}^3{}_1 {\gamma_3}^1{}_3 + {\gamma_3}^3{}_2 {\gamma_3}^2{}_3 + {\gamma_3}^1{}_3 {\gamma_3}^3{}_1 + {\gamma_3}^2{}_3 {\gamma_3}^3{}_2$$
$$- \frac{\partial}{c\partial t}\left[\ln\left(\frac{S^2 r}{\sqrt{1-kr^2}}\right)\right] \cdot \frac{S\dot{S}r^2}{c} - \frac{\partial}{\partial r}\left[\ln\left(\frac{S^2 r}{\sqrt{1-kr^2}}\right)\right] \cdot (-r(1-kr^2))$$

$$= -\frac{\dot{S}^2 r^2 + S\ddot{S}r^2}{c^2} + 1 - 3kr^2 + \frac{2\dot{S}^2 r^2}{c^2} - 2(1-kr^2) - \frac{\partial}{c\partial t}[\ln S^2] \cdot \frac{S\dot{S}r^2}{c}$$
$$+ \frac{\partial}{\partial r}\left[\ln\left(\frac{r}{\sqrt{1-kr^2}}\right)\right] \cdot r(1-kr^2)$$

$$= -\frac{\dot{S}^2 r^2 + S\ddot{S}r^2}{c^2} + 1 - 3kr^2 + \frac{2\dot{S}^2 r^2}{c^2} - 2(1-kr^2) - \frac{2\dot{S}^2 r^2}{c^2} + \left[\frac{1}{r} + \frac{kr}{1-kr^2}\right] \cdot r(1-kr^2)$$

$$= -\frac{\dot{S}^2 r^2 + S\ddot{S}r^2}{c^2} - kr^2$$

$$R_{33} = -\frac{r^2}{c^2}(S\ddot{S} + \dot{S}^2 + kc^2) \tag{6.17}$$

Thus

$$R_{11} = \frac{2\ddot{S}}{c^2 S},$$

$$R_{22} = -\frac{1}{c^2}\left(\frac{S\ddot{S} + \dot{S}^2 + kc^2}{1 - kr^2}\right),$$

$$R_{33} = -\frac{r^2}{c^2}(S\ddot{S} + \dot{S}^2 + kc^2)$$

These results allow us to compute the curvature scalar

$$R = R_i^{\ i} = g^{ik}R_{ik} = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \tag{6.18}$$

We have

$$g^{11}R_{11} = \frac{2\ddot{S}}{c^2 S}$$

$$g^{22}R_{22} = -\frac{(1 - kr^2)}{S^2} \cdot -\left(\frac{S\ddot{S} + \dot{S}^2 + kc^2}{c^2(1 - kr^2)}\right)$$

$$= \frac{1}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

$$g^{33}R_{33} = -\frac{1}{S^2 r^2} \cdot -\frac{r^2}{c^2}(S\ddot{S} + \dot{S}^2 + kc^2)$$

$$= \frac{1}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

$$R = \frac{2\ddot{S}}{c^2 S} + \frac{1}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) + \frac{1}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

$$= \frac{2\ddot{S}}{c^2 S} + \frac{2}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

$$= \frac{4\ddot{S}}{c^2 S} + \frac{2}{c^2}\left(\frac{\dot{S}^2 + kc^2}{S^2}\right)$$

$$R = \frac{2}{c^2}\left(\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

The Einstein field equations for the metric (6.7), given by

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = -\kappa T_{ik} \tag{6.19}$$

$$G_{11} \equiv R_{11} - \frac{1}{2} g_{11} R = -\kappa T_{11}$$

$$G_{11} \equiv \frac{2\ddot{S}}{c^2 S} - \frac{1}{2} \cdot \frac{2}{c^2}\left(\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) = -\frac{2\pi G}{c^4}\epsilon$$

$$G_{11} \equiv \frac{\dot{S}^2 + kc^2}{S^2} = 2\pi G\rho \tag{6.20}$$

$$G_{22} \equiv R_{22} - \frac{1}{2} g_{22} R = -\kappa T_{22}$$

$$G_{22} \equiv -\frac{1}{c^2}\left(\frac{S\ddot{S} + \dot{S}^2 + kc^2}{1 - kr^2}\right) - \frac{1}{2}\left(-\frac{S^2}{1 - kr^2}\right) \cdot \frac{2}{c^2}\left(\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) = 0$$

$$= -\frac{S\ddot{S} + \dot{S}^2 + kc^2}{c^2(1 - kr^2)} + \frac{2S\ddot{S} + \dot{S}^2 + kc^2}{c^2(1 - kr^2)} = 0$$

$$\ddot{S} = 0$$

$$G_{33} \equiv R_{33} - \frac{1}{2} g_{33} R = -\kappa T_{33}$$

$$G_{33} \equiv -\frac{r^2}{c^2}(S\ddot{S} + \dot{S}^2 + kc^2) + \frac{1}{2}(S^2 r^2) \cdot \frac{2}{c^2}\left(\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) = 0$$

$$\ddot{S} = 0$$

$$G_{33} \equiv G_{22} \equiv \ddot{S} = 0 \tag{6.21}$$

6.3 The Friedmann Model in (2 + 1)-D Spacetime

In this project, we considered the time when the 2-D universe is dust-dominated. The dust dominated universe is described by the equations

$$\frac{\dot{S}^2 + kc^2}{S^2} = 2\pi G\rho \tag{6.22}$$

$$\ddot{S} = 0 \tag{6.23}$$

The solution to (6.23) is clearly

$$S(t) = at + b,$$

where a and b are constants.
Assuming that $S = 0$ at $t = 0$, we have $b = 0$, and

$$S(t) = at \tag{6.24}$$

At that reference epoch, Eq. 6.22 becomes;

$$\dot{S}^2 = 2\pi G\rho_0 {S_0}^2 - kc^2,$$

or

$$a^2 = 2\pi G\rho_0 {S_0}^2 - kc^2$$

$$a = \sqrt{2\pi G\rho_0 {S_0}^2 - kc^2}$$

where ρ_0 and S_0 are the density of matter in the universe at the $t = t_0$ or reference epoch and the expansion factor at the reference epoch respectively.

From Eq. 6.24 we finally obtain the expansion factor for (2 + 1)-D spacetime as,

$$S(t) = \sqrt{(2\pi G\rho_0 {S_0}^2 - kc^2)t} \tag{6.25}$$

For Euclidean section, i.e. when $k = 0$, from (6.22) at the reference epoch, we have;

$$\left(\frac{\dot{S}}{S}\right)^2\Bigg|_{t_0} = {H_0}^2 = 2\pi G\rho_0 \tag{6.26}$$

where H_0 is the Hubbles constant(see Eq. 5.26)

6.3.1 Model for 2-D Coordinate Euclidean space.

For $k = 0$, i.e. for flat or Euclidean space, (6.25) becomes,

$$S(t) = \sqrt{(2\pi G\rho_0 {S_0}^2)t}$$

In terms of the Hubble constant, this equation can be rewritten as

$$S(t) = S_0 H_0 t \tag{6.27}$$

or

$$S(t) = S_0\left(\frac{t}{t_0}\right) \tag{6.28}$$

where

$$t_0 = \frac{1}{H_0} \tag{6.29}$$

estimates the age of the universe.

From (6.28) we see that if the 2-D coordinate space were flat, the (2 + 1)-D spacetime would expand forever.

6.3.2 Model for Closed 2-D Coordinate space.

For k = 1, i.e. for 2-D coordinate spaces of positive curvature, we have from (6.25)

$$S(t) = (2\pi G\rho_0 S_0 - c^2)^{1/2}t$$

As noted above, ρ_0 is the density of matter in the 2-D universe at the reference epoch. If $\rho_0 > \frac{c^2}{2\pi GS_0}$, then the universe is expanding linearly in time, and since in the steady state the total amount of matter in the universe must be a constant, ρ_0 will decrease from epoch to epoch and will approach $\rho_t = \frac{c^2}{2\pi GS_0}$ asymptotically.

In other words, the universe will expand in any given epoch, but the rate of expansion will decrease from epoch to epoch until the terminal density $\rho_t = \frac{c^2}{2\pi GS_0}$ is reached. If the density is reached abruptly, the expansion will stop, and all phenomena, like the redshift, associated with the expansion will cease to exist; otherwise the expansion will continue asymptotically and the 2-D universe will expand indefinitely.

However, an abrupt increase in the density of matter ρ_0, it will kick start a contraction; which will cause the universe to contract and the rate of contraction will decrease from epoch to epoch until the terminal density $\rho_t = \frac{c^2}{2\pi GS_0}$ is reached, then the contraction will stop, and all the phenomena, like the redshift will cease to exist.

It can be seen from the graph below that the rate of expansion keeps on decreasing as the density of matter decreases from epoch to epoch.

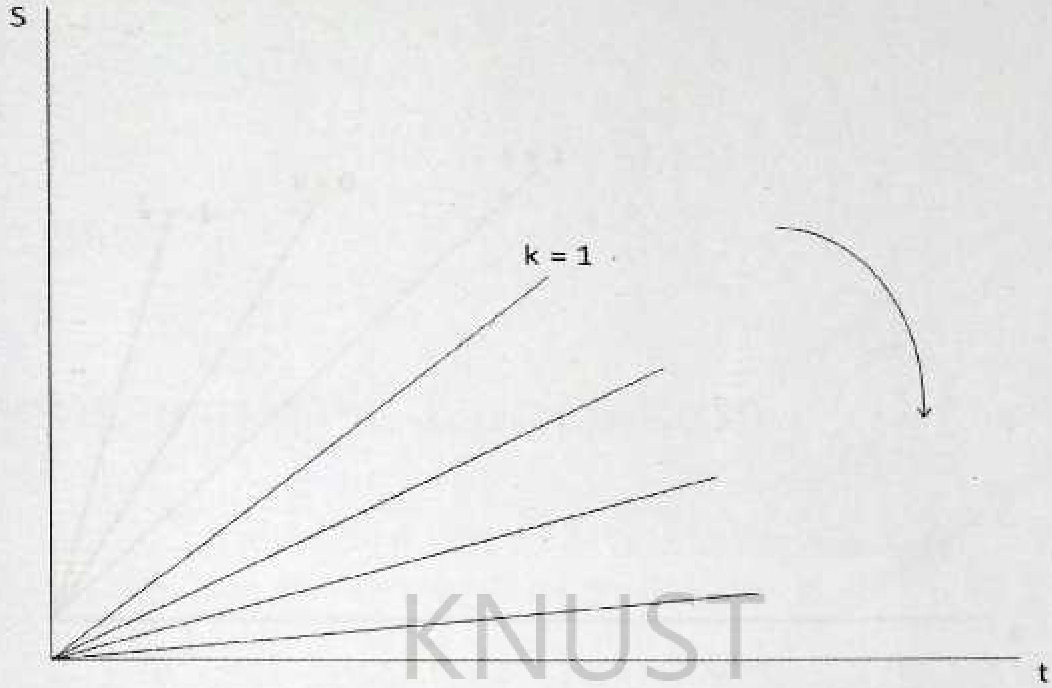

Figure 6.4: Solutions of Einstein's equations for a Robertson-Walker (2 + 1)-D universe with curvature k=1

6.3.3 Model for Opened 2-D Coordinate space.

For k = -1, i.e. for 2-D coordinate spaces of negative curvature, we have from (6.25)

$$S(t) = (2\pi G\rho_0 S_0 + c^2)^{1/2} t$$

From the equation above, it is seen clearly that, the 2-D universe will continue to expand forever. Similarly, since the totality of matter in this universe is constant, when the universe starts expanding it will start to decrease in size. When the matter in the 2-D universe continues to decrease till its content becomes very small, this universe will continue to expand but this time at the velocity of light. This shows that, in all situations this universe will expand forever.

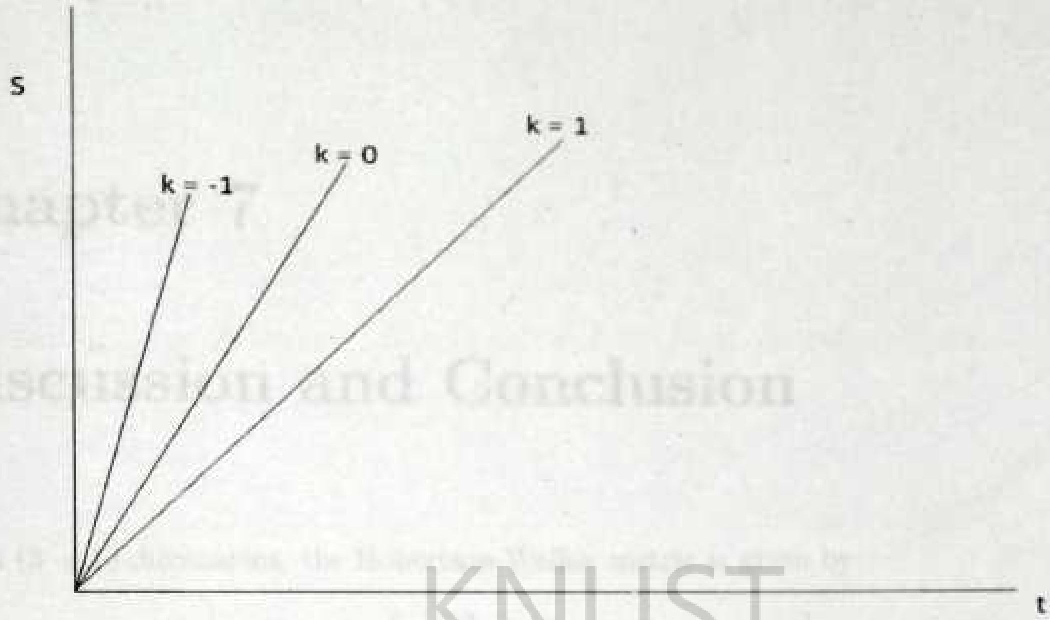

Figure 6.5: Solutions of Einstein's equations for a Robertson-Walker (2 + 1)-D universe with curvatures k=1, k=0 and k=-1 (The slopes are not drawn to scale)

Putting the three sections together we obtain the graph above.

From the graph, we can see clearly that the gradient of k = -1 is the greatest, followed by k = 0 then followed by k = 1

In analogy with the Friedmann models in (3 + 1)dimensional spacetime we can refer to these models as the Friedmann models for (2 + 1)dimensional spacetime.

Chapter 7

Discussion and Conclusion

In (3 + 1)-dimensions, the Robertson-Walker metric is given by

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{7.1}$$

We observed that in (2 + 1)-dimensions, the Robertson-Walker metric is given by

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2\right] \tag{7.2}$$

Comparing Eq.(7.1) to Eq.(7.2) we can see that the fourth term in equation (7.1) is missing in equation (7.2). Equation (7.2) was not obtained by just cancelling the fourth term or setting $\varphi = 0$ in Eq.(7.1) but by considering transformations of 2-dimensional space as a space embedded in 3-dimensional hypersurface. Whiles solid figures are embedded in 4-dimensional hypersurface, 2-dimensional surfaces are embedded in 3-dimensional hypersurface.

In (3 + 1)-dimensions, due to the complex nature of the equations, the expansion factor cannot be given by a single equation but we can obtain the various sections, ie. the Euclidean, closed and opened sections. For Euclidean sections, the expansion factor is given by;

$$S = S_0\left(\frac{t}{t_0}\right)^{2/3} \tag{7.3}$$

where the age of the universe is,

$$t_0 = \frac{2}{3H_0}$$

and

$$H_0 = \left(\frac{\dot{S}}{S}\right)_{t_0}$$

For closed sections, the solution is given by the parametric equations below;

$$S = \frac{1}{2}\alpha(1 - \cos\theta)$$

$$ct = \frac{1}{2}\alpha(\theta - \sin\theta) \tag{7.4}$$

where α is a constant.

For opened sections, the solution is given by the parametric equations in hyperbolic space as shown below;

$$S = \frac{1}{2}\beta(\cosh\psi - 1)$$

$$ct = \frac{1}{2}\beta(\sinh\psi - \psi) \tag{7.5}$$

where β is a constant.

In (2 + 1)-dimensions, the expansion factor was obtained as a linear function of time(t) given by;

$$S(t) = \sqrt{(\frac{4\pi G}{2}\rho_0 S_0^2 - kc^2)t} \tag{7.6}$$

We observed that in (2 + 1)-dimensions, the expansion factor for the Euclidean section ($k = 0$) is given by;

$$S = S_0\left(\frac{t}{t_0}\right) = S_0\left(\frac{t}{t_0}\right)^{2/2} \tag{7.7}$$

where the age of the universe is given by;

$$t_0 = \frac{1}{H_0}$$

Comparing (7.3) to (7.7), we conjectured that in the Euclidean section the expansion factor is given by;

$$S = S_0 \left(\frac{t}{t_0}\right)^{2/d} \tag{7.8}$$

where

$$t_0 = \frac{2}{dH_0}$$

and d denotes spatial dimension.

It was observed that whiles the expansion factor in the (3 + 1)-dimensional spacetime was exponential, the expansion factor in the (2 + 1)-dimensional spacetime is linear. It was also observed that, in both (3 + 1) and (2 + 1)-dimensions, the Euclidean section will expand forever.

It was also observed that in the opened sections for both dimensions, ie. (3 + 1) and (2 + 1)-dimensions, the universe expands forever. Whiles the solution for opened section in (3 + 1)-dimensions is given in a parametric form, that of the (2 + 1)-dimensions is obtained as a linear function of time(t).

For closed sections (ie. $k = 1$), the universe expands to a certain limit and a contraction may begin in both (3 + 1) and (2 + 1)-dimensions. The solution in (3 + 1)-dimensions is given in a parametric form in hyperbolic space whiles the expansion factor is a linear function of time(t) in (2 + 1)-dimensions.

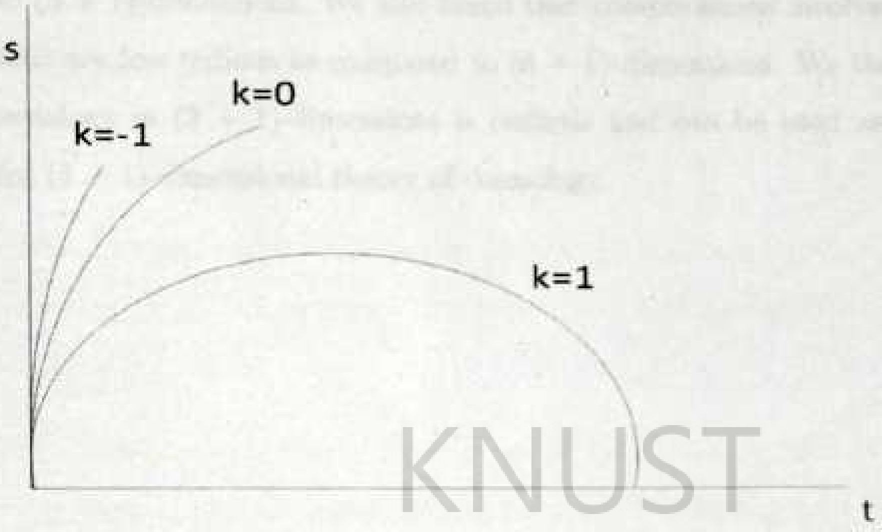

Figure 7.1: The solutions of Einstein's equations for a Robertson-Walker metric in (3 + 1)-dimensions with curvature k = 1, k = 0 and k = -1.

Comparing Fig. 6.5 to Fig. 7.1, we can clearly see that, in Fig. 6.5, the gradient of k =-1 is the greatest, followed by k = 0 then k = 1. Likewise in fig. 7.1, the gradient of k =-1 is the greatest, followed by k = 0 then k = 1. Also, since the solutions in (3 + 1)-dimensions are a bit complex and parametric in nature, the graphs are curved but they are straight in (2 + 1)-dimensions because the expansion factor for the various models is a linear function if time(t).

In conclusion, comparing the solutions in (2 + 1)-dimensions to (3 + 1)-dimensions, we found that all the results obtained have their analogies (3 + 1)-dimensions, except that we found that the 2-D universe is always expanding irrespective of the curvature of the space. Specifically, the 2-D universe expands linearly in time forever. In 3-D universe, the expansion is generally nonlinear in time and under certain situations contraction is possible. Hence, the results in $(2 + 1)$-dimensions can be carried over into $(3 + 1)$-dimensions. We also found that computations involved in $(2 + 1)$-dimensions are less tedious as compared to $(3 + 1)$-dimensions. We then concluded that cosmology in $(2 + 1)$-dimensions is realistic and can be used as a prototype model for $(3 + 1)$-dimensional theory of cosmology.

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