KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

REVENUE MAXIMIZATION USING THE PRIMAL DUAL METHOD A CASE STUDY OF SUNYANI WEST DISTRICT ASSEMBLY

By

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Declaration

I hereby declare that this submission is my own work towards the MSc. And that, to the best of my knowledge, it contains no material previously published by another person nor materials which has been accepted for the award of any other degree of the university, except where due acknowledgement has been made in the text.

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Dedication

To my lovely Mother Madam Ellen Hoggar and Father, Mr.Harry Hoggar .

Abstract

In this thesis, Primal-dual method, which is one of the interior-point methods, was used to assist the improvement of revenue generation efforts at Sunyani West District Assembly. The data which was collected from the Sunyani West District Assembly was modeled into objective function and subject to constraints. The matrices generated were investigated as well as an implementation of numerical iteration of the models to determine the efficient income generation strategy for the Assembly. THe result shows that the Assembly can raise its revenue to GH¢1358357.28 annually which represents an appreciable 52.12% increase in the Assembly's revenue.

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Chapter 1

Introduction

1.1 Background to the study

Evidence clearly shows that tax efforts rates are much higher for higher-income countries than in low-income countries, supporting the notion that performance and tax mobilization is essential for reaching higher level of income. A low level of government revenue is a constraint on the capacity to finance essential public investment programme and undertake adequate level of spending on social services, which are important for improving living standard.

Kaldor was in fact echoing an earlier call by Sir Arthur Lewis who posited that 'the government of an undeveloped country needs to be able to raise revenue of about 11-19 percent of full in order to give a better than average standard of service' (Martin and Lewis ,1956). The role of government revenue and the capacity of government to raise taxes for the purpose of financing economic development have pre-occupied economists and policy makers for a long time more than 40 year ago. Kaldor (1963), raised the very important question of whether undeveloped countries will 'learn to tax' , with the underlying view that for these counties to reach higher levels of living standard ,they would need to achieve levels of tax effort that are significantly higher than observed at that time.

What is less straight forward is what makes a country or a government capable of achieving high levels of revenue performance. Bird, Vazquez and Torgler (2008) pointed out in their work that, most of the attention in the analysis of tax effort has traditionally been focused on the supply side (or tax handles' in their word), mainly the availability of readily taxable activities such as trade/commerce and natural resources. African Countries have generally performed poorly in tax revenue mobilization.

The average tax-to-GDP ratio in Sub-Saharan Africa increased only reasonably over the past two decades. Two key problems were evident. First, African Countries have been unable to connect natural resource endowment for the purpose of revenue mobilization. Second, African countries have been unable to develop their capacity to mobilize non-resource sources of tax revenue. In the case of resource-reached countries, this is a result of failure to utilize the natural resource bonanza to promote activities outside the natural resource industrial, so as to diversify their production and export base. The problem goes beyond the issue of value addition in the natural resource industry or moving up the value chain. It also lacks capabilities to innovate within and outside the natural resource value chain.

The pool of District financial resources in many developing countries might come from seven main sources. Some of these are Independent revenue sources or own sources (if any) assign to the District, central government financial transfers to the District (which can have different forms), voluntary contributions by community or beneficiary groups, profit from public enterprises or rent from public property etc, financial assistance from donor agencies, short and long term loans and other sources like penalties, selling property (Kroes, 2008).

However, following the decentralization process, District Assemblies in Ghana now have the responsibility to plan and implement their own project or programmes. The Sunyani west District largely depends on internal sources for the day- to- day running of the District administration. These include rates and receipts ,royalties from lands, fees and tolls. Licenses, rent, investments and other miscellaneous activities that accrue as a result of it own effort at revenue mobilization and generation. On the outside, revenue also comes to the District Assembly from the central government in the form of Grant-in-aid and the District Assemblies common fund (Sunyani West District medium term 2011-2014 draft plan). Katentet, (2011) examines the poor revenue generated by most assemblies in Ghana as a result of insufficient revenue base, existence of two institutions working for internally generated funds, poor organizational structure and revenue administrational mechanism. Gap on knowledge and understanding of revenue, weak voluntary compliance as well as revenue leakage.

According to the District medium term draft plan (2011-2014), poor data base on revenue

/ratable items, in adequate qualified revenue collectors, inadequate and poor marketing facilities are some of the challenges in revenue collection. High rate of tax evasion, inadequate logistic to promote Education on the need to pay taxes, lack of permanent internal auditors / local government inspectors, inadequate revenue mobilization capacity and weak tax ,revenue collection mechanism are the major problems of the District revenue mobilization.

The Sunyani West District was carved out of the Sunyani Municipal in November, 2007, through legislative instrument (Li) 1881, 2007. It was inaugurated on the 14th January, 2008. The Administrative Capital of the District is Dumasi. It covers a total land area of 657 square kilometers. According to the 2010 population and housing census, the District has a population of 67,176 and growth rate of 2.8 percentages per annum. The Population is however projected to reach 85,689 in 20014. The predominate occupation in the District is agriculture which employs about 66.5 percent of the active labour force. Services employ 9.4 percent, industrial 5.2 percent and commerce 1.2 percent. The District share common boundaries with Sunyani Municipal to the East, Berekum to the South, Wenchi Municipal to the North.

1.2 Problem Statement

The standard of living in the Sunyani West District keeps on worsening as the Assembly is not able to provide the citizenry with the basic social amenities' such as portable water, better health care facilities, quality education, good roads, improved sanitation, infrastructural development and so on. The reason being that, the District Assembly is not able to mobilize sufficient revenue to execute its project and programme aimed at bettering the lot of its people. The Assembly, since it inception in January 2008 has never met its revenue target and therefore has to rely heavily on the central government for its basic expenditure financing. This problem has been a major headache to the Assembly as it is hampering the effective growth of the District. This research work is basically targeted at developing a mathematical programme that will help the Assembly to optimize its revenue mobilization strategy.

1.3 Objectives of The Study

The study seeks to:

1. Model the Assembly's revenue as linear programming problem
2. Maximize the Assembly's revenue.

1.4 Methodology

The problem of revenue maximization will be modelled as a linear programming problem. Primal-Dual, one of the interior point algorithms will be used to solve the mathematical model. Data would be collected from Sunyani West District Assembly for the research. Software programme on matlab will be developed to run the data.

1.5 Significance of the Study

The Sunyani West District Assembly, since its inception has been under performing in its revenue mobilization efforts. This state of affair has made it difficult for the Assembly to provide basic social services such as schools, healthcare, access roads, places of convenience, portable water supply etc. This project is geared towards finding a lasting solution to help the Assembly to optimize its revenue collection so that it can support it inhabitant to improve upon their standard of living with the provision of many social amenities such as schools, hospitals, provision of portable water etc. It is also envisaged that some other Districts in the country with revenue mobilization challenges can use the findings from this research to improve upon their revenue generation strategy.

1.6 Organization of the Study

The thesis consists of five chapters, including this chapter. Chapter 2 is literature review of the existing theoretical and empirical literature. Chapter 3 deals with methodology that is being used for the study. Chapter 4 deals with data collection, analysis, and discussion of results. The concluding chapter, Chapter 5 summarizes the findings and also provides conclusions and recommendations.

Chapter 2

Literature Review

This chapter reviews the literature on the application of linear programming:

2.1 Linear Programming

Jianq etal., (2004) proposed a linear programming based method to estimate arbitrary motion from two images. The proposed method always finds the global optimal solution of the linearized motion estimation energy function and thus is much more robust than the traditional motion estimation schemes. As well, the method estimates the occlusion map and motion field at the same time, (et al., 2004). To further reduce the complexity-reduced pure linear programming method they presented a two phase scheme to estimating the dense motion field. In the first step, they estimated a relative sparse motion field for the edge pixels using a non-regular sampling scheme, based on the proposed linear programming method. In the second step, they set out a detail-preserving variational method to upgrade the result into a dense motion field. The proposed scheme is much faster than a purely linear programming based dense motion estimation scheme. And, since they used a global optimization method linear programming in the first estimation step, the proposed two-phase scheme was also significantly more robust than a pure variation.

Hoesein and Limantara., (2010) studied the optimization of water supply for irrigation at Jatimlerek irrigation area of 1236 ha. Jatimlerek irrigation scheme was intended to serve more than one district. The methodology consisted of optimization water supply for irrigation with Linear Programming. Results were used as the guidance in cropping pattern and allocating water supply for irrigation at the area.

Linear programming model was applied by Hassan, (2004), to calculate the optimal crop acreage, production and income of the irrigated Punjab. Crops included in the models were wheat, Basmati rice, IRRI rice, cotton, sugarcane, maize, potato, gram and mong / mash. The results showed that the irrigated agriculture in the Punjab was more or less operating at the optimal level. Overall cropped acreage in the Optimal solution decreased by 0.37% as compared to the existing acreage. However, in the optimal cropping pattern some crops like cotton and pluses gained acreage by 9-10% each, while maize and Basmati rice remained unchanged. On the other hand crops like wheat, IRRI rice, potato and sugarcane lost acreage by 4-11%. As a result of optimum croppingpattern income, increased by 1.57%.

Reuter and Deventer, (2003) proposed two linear models, the second being a subset of the first, for the simulation of flotation plants by use of linear programming. The first linear model produced the circuit structure, as well as the optimal flow rates of the valuable element between any number of flotation banks incorporating any number of recycle mills. An optimal grade for the valuable element in the concentrate was given by the second model. Operating conditions in the flotation banks and recycle mills were included as bounds in these models, permitting their possible application in expert systems. The simulated circuit structure, concentrate grade and recoveries closely resembled those of similar industrial flotation plants. The only data required by the simulation models were the feed rates of the species of an element, and their separation factors which were estimated from a multiparameter flotation model.

Becker, (1995) explored the implications of the transformation of the system of water resources allocation to the agricultural sector in Israel from a one in which allotments were allocated to the different users without any permission to trade with water rights. A mathematical planning model was used for the entire Israeli agricultural sector, in which an 'optimal' allocation of the water resources was found and compared to the existing one. The results of the model were used in order to gain insight into the shadow price of the different water bodies in Israel (about eight). These prices could be used to grant property rights to the water users themselves in order to guarantee rational behaviour of water use, since no one could sell their rights at the source itself. From the dual prices of the primal problem they could forecast the equilibrium prices and their implications for the different users. The results showed that there was a potential budgetary benefit of 28 million dollars when capital cost was not included and 64 millions dollars when it was included.

Greenberg et al., (1986) introduced a framework for model formulation and analysis to support operations and management of large-scale linear programs from the combined capabilities of camps and analyze. Both the systems were reviewed briefly and the interface which integrates the two systems was then described. The model formulation, matrix generation, and model management capability of camps and the complementary model and solution analysis capability of analyze were presented within a unified framework. Relevant generic functions were highlighted, and an example was presented in detail to illustrate the level of integration achieved in the current prototype system. Some new results on discourse models and model management support were given in a framework designed to move toward an 'intelligent' system for linear programming modeling and analysis.

To examine how farmers could sustain an economically viable agricultural production in salt-affected areas of Oman, (Naifer et al, 2010), divided a sample of 112 farmers into three groups according to the soil salinity levels, low salinity, medium salinity and high salinity. Linear programming was used to maximize each type of farm's gross margin under water, land and labor constraints. The economic losses incurred by farmers due to salinity were estimated by comparing the profitability of the medium and high salinity farms to the low salinity farm's gross margin. Results showed that when salinity increased from low salinity to medium salinity level the damage was US$1,604 ha^{-1} and US$2,748 ha^{-1} if it increased from medium salinity to high salinity level. Introduction of salt-tolerant crops in the cropping systems showed that the improvement in gross margin was substantial thus attractive enough for medium salinity farmers to adopt the new crops and/or varieties to mitigate the effect of water salinity.

A linear programming model for a river basin was developed by (Avdelas et al, 1992), to include almost all water-related economic activity both for consumers and producers. The model was so designated that the entire basin or basin sub-division could be analyzed. The model included seven sectors, nine objective function criteria, and three river-flow levels. Economic basis for conflicts among sectors over incidence of cost allocation and level of economic activity were traced to some chosen objective. The disposal of untreated household waste water, particularly from the rural household, directly into the river was consistent with maximizing net benefits and minimizing costs. For each of the three industries analyzed separately, paper, wool and tanning, public treatment of industrial waste water was the optimal treatment process in one or more of the solutions.

Cherubini et al., (2009) framed an optimization model which aims at minimizing the maximum link utilization of IP telecommunication networks under the joint use of the traditional IGP protocols and the more sophisticated MPLS-TE technology. The survivability of the network was taken into account in the optimization process implementing the path restoration scheme. This scheme benefits of the Fast Re-Route (FRR) capability allowing service providers to offer high availability and high revenue SLAs (Service Level Agreements). The hybrid IGP/MPLS approach relies on the formulation of an innovative Linear Programming mathematical model that, while optimizing the network utilization, provides optimal user performance, efficient use of network resources, and 100% survivability in case of single link failure. The possibility of performing an optimal exploitation of the network resources throughout the joint use of the IGP and MPLS protocols provides a flexible tool for the ISP (Internet Service Provider) networks traffic engineers. The efficiency of the proposed approach was validated by a wide experimentation performed on synthetic and real networks. The obtained results showed that a small number of LSP tunnels have to be set up in order to significantly reduce the congestion level of the network while at the same time guaranteeing the survivability of the network. They applied this approach to a quadratic-cost single-commodity network design problem, comparing the newly developed algorithm with those based on both the standard continuous relaxation and the two usual variants of PR relaxation.

Mousavi et al., (2004) presented a long-term planning model for optimizing the operation of Iranian Karoon-Dez reservoir system using an interior-point algorithm. The system is the largest multi-purpose reservoir system in Iran with hydropower generation, water supply, and environmental objectives. The focus was on resolving the dimensionality of the problem of optimization of a multi-reservoir system operation while considering hydropower generation and water supply objectives. The weighting and constraints methods of multi-objective programming were used to assess the trade-off between water supply and hydropower objectives so as to find noninferior solutions. The computational efficiency of the proposed approach was demonstrated using historical data taken from Karoon-Dez reservoir system.

Konickova, (2006) proposed a linear programming problem whose coefficients are prescribed by intervals is called strongly unbounded if each linear programming problem obtained by fixing coefficients in these intervals is unbounded. In the main result of the paper a necessary and sufficient condition for strong unboundedness of an interval linear programming problem was described. In order to have a full picture they also showed conditions for strong feasibility and strong solvability of this problem. The necessary and sufficient conditions for strong feasibility, strong solvability and strong unboundedness can be verified by checking the appropriate properties by the finite algorithms. Checking strong feasibility and checking strong solvability are NP-hard. This shows that checking strong unboundedness is NP-hard as well. Optimal solutions of Linear Programming problems may become severely infeasible if the nominal data is slightly perturbed.

Frizzone et al., (1997) developed a separable linear programming model, considering a set of technical factors which might influence the profit of an irrigation project. The model presented an objective function that maximized the net income and specified the range of water availability. It was assumed that yield functions in response to water application were available for different crops and described very well the water-yield relationships. The linear programming model was developed genetically, so that, the rational use of the available water resource could be included in an irrigation project. Specific equations were developed and applied in the irrigation project 'Senator Nilo Coelho' (SNCP), located in Petrolina - Brazil. Based on the water-yield functions considered, cultivated land constraints, production costs and products prices, it was concluded that the model was suitable for the management of the SNCP, resulting in optimal cropping patterns.

Chung et al., (2008) considered a municipal water supply system over a 15-year planning period with initial infrastructure and possibility of construction and expansion during the first and sixth year on the planning horizon. Correlated uncertainties in water demand and supply were applied on the form of the robust optimization approach of Bertsimas and Sim to design a reliable water supply system. Robust optimization aims to find a solution that remains feasible under data uncertainty. It was found that the robust optimization approach addressed parameter uncertainty without excessively affecting the system. While they applied their methodology to hypothetical conditions, extensions to real-world systems with similar structure were straightforward. Therefore, their study showed that this approach was a useful tool in water supply system design that prevented system failure at a certain level of risk.

Khan et al., (2005) used Linear Programming Model to calculate the crop acreage, production and income of cotton zone. This was carried out in the three districts of the Bahawalpur. These three districts were collected by purposive sampling technique. The study was conducted on 4652 acres of the irrigated areas from the three districts. Crops included in the model were wheat, basmati rice, IRRI rice, cotton and sugar cane. The results showed that the cotton was the only crop, which gained acreage by about 10% at the expense of all other crops. Overall optimal crop acreage decreased by 1.76%, while optimal income was increased by 3.28% as compared to the existing solutions. The study reported that Bahawalpur division was more or less operating at the optimal level.

Matthews., (2005) evaluated and optimized the utility of the nurse personnel at the Internal Medicine Outpatient Clinic of Wake Forest University Baptist Medical Center. Linear programming (LP) was employed to determine the effective combination of nurses that would allow for all weekly clinic tasks to be covered while providing the lowest possible cost to the department. A specific sensitivity analysis was performed to assess just how sensitive the outcome was to the stress of adding or deleting a nurse to or from the payroll. The nurse employee cost structure in this study consisted of five certified nurse assistants (CNA), three licensed practicing nurses (LPN), and five registered nurses (RN). The LP revealed that the outpatient clinic should staff four RNs, three LPNs, and four CNAs with 95 percent confidence of covering nurse demand on the floor.

Kumar and Khepar, (980) in their study demonstrated the usefulness of alternative levels of water use over the fixed yield approach when there is a constraint on water. In the multi-crop farm models used, a water production function for each crop was included so that one had the choice of selecting alternative levels of water use depending upon water availability. Water production functions for seven crops, viz. wheat, gram, mustard, berseem, sugarcane, paddy and cotton, based on experimental data from irrigated crops were used. The fixed yield model was modified incorporating the stepwise water production functions using a separable programming technique. The models were applied on a selected canal command area and optimal cropping patterns determined. Sensitivity analysis for land and water resources was also conducted. The water production function approach gave better possibilities of deciding upon land and water resources.

Heidari, (2007) formulated and solved ground water management model based on the linear systems theory using linear programming. The model maximized the total amount of pound water that could be pumped from the system subject to the physical capability of the system and institutional constraints. The results were compared with analytical and numerical solutions. Then, this model was applied to the Pawnee Valley area of south-central Kansas. The results of this application supported the previous studies about the future ground water resources of the Valley. These results provided a guide for the ground water resources management of the area over the next ten years.

Vimonsatit et al., (2003) proposed a linear programming (LP) formulation for the evaluation of the plastic limit temperature of flexibly connected steel frames exposed to fire. Within a framework of discrete models and piecewise linearized yield surfaces, the formulation was derived based on the lower-bound theorem in plastic theory, which lead to a compact matrix form of an LP problem. The plastic limit temperature was determined when the equilibrium and yield conditions were satisfied. The plastic mechanism can be checked from the dual solutions in the final simplex tableau of the primal LP solutions. Three examples were presented to investigate the effects of the partial-strength beam-to-column joints. Eigenvalue analysis of the assembled structural stiffness matrix at the predicted limit temperature was performed to check for structural instability. The advantage of the proposed method is that it is simple, computationally efficient, and its solutions provide the necessary information at the limit temperature. The method can be used as an efficient tool to a more refined but computationally expensive step-by-step historical deformation analysis.

Banks and Fleck, (2010) applied Linear programming techniques to ground-water- flow model in order to determine optimal pumping scenarios for 14 extraction wells located downgradient of a landfill and upgradient of an estuary. The model was used to simulate flow as well as the effects of a pump-and-treat remediation system designed to capture contaminated ground water from the water-table aquifer before it reached the adjacent estuary. The objective function involved varying pumping rates and frequencies to maximize capture of ground water from the water-table aquifer. At the same time, the amount of water extracted and needing treatment was minimized. The constraints placed on the system insured that only ground water from the landfill was extracted and treated. To do this, a downward gradient from the disposal area toward the extraction wells was maintained.

Khaled, (2004) developed four models of optimal water allocation with deficit irrigation in order to determine the optimal cropping plan for a variety of scenarios. The first model (Dynamic programming model (DP)) allocated a given amount of water optimally over the different growth stages to maximize the yield per hectare for a given crop, accounting for the sensitivity of the crop growth stages to water stress. The second model (Single Crop Model) tried to find the best allocation of the available water both in time and space in order to maximize the total expected yield of a given crop. The third model (Multi crop Model) was an optimization model that determined the optimal allocation of land and water for different crops. It showed the importance of several factors in producing an optimal cropping plan. The output of the models was prepared in a readable form to the normal user by the fourth model (Irrigation Schedule Model).

Turgeon, (1986) developed a parametric mixed-integer linear programming (MILP) method for selecting the sites on the river where reservoirs and hydroelectric power plants were to be built and then determining the type and size of the projected installations. The solution depended on the amount of money the utility was willing to invest, which itself was a function of what the new installations would produce. This method was used based on the fact that the branch-and-bound algorithm for selecting the sites to be developed (and consuming most of the computer time) was solved a minimum number of times. Between the points where the MILP problem was solved, LP parametric analysis was applied.

Belotti et al., (2005) proposed to tackle large-scale instances of Maximum feasible subsystem using randomized and thermal variants of the classical relaxation method for solving systems of linear inequalities. They established lower bounds on the probability that these methods identify an optimal solution within a given number of iterations. These bounds, which are expressed as a function of a condition number of the input data, imply that with probability one these randomized methods identify an optimal solution after finitely many iterations. Computational results obtained for medium - to large-scale instances arising in the design of linear classifiers, in the planning of digital video broadcasts and in the modeling of the energy functions driving protein folding, indicate that an efficient implementation of such a method perform very well in practice.Industrial switching involves moving materials on rail cars within or between industrial complexes and connecting with other rail carriers. Planning tasks include the making up of trains with a minimum shunting effort, the feasible and timely routing through an in-plant rail network on short paths, and assigning and scheduling of locomotives under safety and network capacity aspects. A human planner must often resort to routine and simple heuristics, not least for the reason of unavailability of computer aided suggestions.

Tsakiris and Spiliotis, (2004) treated the Systems Analysis formulation problem of water allocation to various users as a linear programming problem with the objective of maximizing the total productivity. This was intended to solve one of the basic problems of Water Resource Management in the allocation of water resources to various users in an optimal and equitable way respecting the constraints imposed by the environment. In this work a fuzzy set representation of the unit revenue of each use together with a fuzzy representation of each set of constraints, were used to expand the capabilities of the linear programming formulation. Numerical examples were presented for illustrative purposes and useful conclusions are derived.

Yoshito, (2004) considered the problem of finite dimensional approximation of the dual problem in abstract linear programming approach to control system design. A constraint qualification that guarantees the existence of a sequence of finite dimensional dual problems that computes the true optimal value. The result is based on the averaging integration by a probability measures. A matrix is sought that solves a given dual pair of systems of linear algebraic equations. Necessary and sufficient conditions for the existence of solutions to this problem were obtained, and the form of the solutions was found. The form of the solution with the minimal Euclidean norm was indicated. Conditions for this solution to be a rank one matrix were examined. On the basis of these results, an analysis was performed for the following two problems: modifying the coefficient matrix for a dual pair of linear programs (which can be improper) to ensure the existence of given solutions for these programs, and modifying the coefficient matrix for a dual pair of improper linear programs to minimize its Euclidean norm. Necessary and sufficient conditions for the solvability of the first problem were given, and the form of its solutions was described. For the second problem, a method for the reduction to a nonlinear constrained minimization problem was indicated, necessary conditions for the existence of solutions were found, and the form of solutions was described (Matthews, 2005).

The primal-dual method for approximation algorithms considers a primal integer programming formulation of the problem in question and the dual of a linear programming relaxation of the integer program. The method above is modified by relaxing complementary slackness conditions related to dual variables; that is, we relax the condition that if $X_j > 0$ the corresponding primal constraint must be met with equality. As we will see below, relaxing this constraint in appropriate ways leads to provably good algorithms for NP-hard problems in combinatorial optimization. The method yields a solution to the primal integer problem that costs no more than α times the value of the feasible dual solution constructed, which implies that the primal solution is within a factor of α of optimal. The value of the dual solution is always within some factor of α of the value of the primal solution, but may from instance to instance be much closer; by comparing the value of the primal and dual solutions generated, we can give a guarantee for the instance which might be better than α. The performance guarantee of an algorithm using the primal-dual method is thus connected with the integrality gap of the integer programming formulation of the problem. The integrality gap of a formulation is the worst-case ratio over all instances of the value of the integer program to the value of the corresponding linear programming relaxation. Since the performance guarantee of an algorithm using the primal-dual method is proven by comparing the value of a primal solution against the value of a feasible dual, its performance guarantee can never be shown to be better than the integrality gap of the formulation used. Conversely, a proof of a performance guarantee of α obtained in this way implies that the integrality gap is no more than α. So far the primal-dual method for approximation algorithms usually leads to dual-ascent algorithm sin which dual variables are never decreased, (Williamson, 2002).

Chapter 3

Methodology

3.1 Introduction

This chapter reviews the methodology used for developing revenue mobilization model. The first phase of this chapter talks about some procedures, the linear programming model, theoretical method used in solving it and software for solving linear programming.

3.2 Linear Programming

Linear programming is a mathematical technique that deals with the optimization (maximizing or minimizing) of a linear function known as objective function subject to a set of linear equations or inequalities known as constraints. It is a mathematical technique which involves the allocation of scarce resources in an optimum manner, on the basis of a given criterion of optimality. The technique used here is linear because the decision variables in any given situation generate straight line when graphed. It is also programming because it involves the movement from one feasible solution to another until the best possible solution is attained. A variable or decision variables usually represent things that can be adjusted or controlled. An objective function is a mathematical expression that combines the variables to express your goal and the constraints are expressions that combine variables to express limits on the possible solutions.

Linear programs can be expressed in the form:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

where x represents the vector of variables (to be determined), c and b are vectors of

(known) coefficients and A is a (known) matrix of coefficients. The expression to be maximized or minimized is called the objective function ($c^T x$ in this case). The equations $Ax \leq b$ are the constraints which specify a convex polytope over which the objective function is to be optimized.
Linear programming can be applied to various fields of study. It is used most extensively in business and economics, but can also be utilized for some engineering problems. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.

3.3 Standard Form

The Standard form is the usual and most intuitive form of describing a linear programming problem. It consists of the following four parts:

- A linear function
- Problem constraints
- Non-negative variables
- Non-negative right hand side constants

Given an m-vector $b = (b_1, ..., b_m)^T$, an n-vector $c = (c_1, .., c_n)^T$ and an $m \times n$ matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{pmatrix}$$

of real numbers, the standard form can be described as

$$\text{maximize} \;\; F = C^T X$$

Subject to

$$Ax \leq b$$

$$\text{where } x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

3.4 Methods of Solving Linear programming

Basically, there are several methods of solving a linear programming problem. These are

i. The graphical (Geometrical) Method

ii. The simplex (Algebraic) Method

iii. Revised simplex method

iv Interior point Methods

3.4.1 A Unique Optimal Solution

This is where the solution to the problem occurs at one and only one extreme point of the feasible region. That is, the combination that gives the highest contribution or profit or the minimum cost or time depending on the problem at hand.

Example:

$$\text{Max } Z = 6x_1 + 8x_2$$

$$\begin{aligned} \text{Sub} \quad 5x_1 + 10x_2 &\leq 60 \\ 4x_1 + 4x_2 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution: Resulting Equalities

$$
\begin{aligned}
5x_1 + 10x_2 + S_1 &= 60 \\
4x_1 + 4x_2 + S_2 &= 40
\end{aligned}
$$

Now using the Gauss Jordan elimination method. Let $x_1, x_2 = 0$ in both constraints

$$
\begin{aligned}
5(0) + 10(0) + S_1 &= 60 \\
4(0) + 4(0) + S_2 &= 40
\end{aligned}
$$

The Basic solution $x_1 = 0, x_2 = 0, s_1 = 60, s_2 = 40$ Another Basic solution using Gauss Jordan elimination. Set $s_1, s_2 = 0$

$$
\begin{aligned}
5x_1 + 10x_2 + s_1(0) &= 60 \\
4x_1 + 4x_2 + s_2(0) &= 40
\end{aligned}
$$

Basic solution $x_1 = 8, x_2 = 2, s_1 = 0, s_2 = 0$

From the solution, 4 basic solution is attained, which shows that feasible solution is achieved and also, optimality conditions were reached after a few iteration. The optimal solution occurred at a single extreme point.

3.5 Simplex Method

It is a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function. Simplex usually starts at the corner that represents doing nothing. It moves to the neighboring corner that best improves the solution. It does this over and over again, making the greatest possible improvement each time. When no more improvement can be made, the most attractive corner corresponding to the optimal solution has been found.

3.5.1 The Standard Maximum Form for s Linear Program

A standard maximum problem is a linear program in which the objective is to maximize an objective function of the form:

$$Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$$

$$\begin{aligned} \text{Sub to:} \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\ \vdots &\leq \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m \end{aligned}$$

where $x_1, x_2, \ldots, x_n \geq 0$ and $b_j \geq 0$ for $j = 1, 2, \ldots, m$

3.6 The Simplex Tableau

To set up the simplex tableau for a given objective function and its constraints, add none negative slack variable s_i to the constraints. This is to convert the constraints into equations. The constraints therefore become:

$$
\begin{aligned}
\text{Sub to: } \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 &= b_2 \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_n &= b_m
\end{aligned}
$$

where $x_i \geq 0$ for $i = 1, 2, \ldots, n$

Table 3.1: Table Showing the Formulation of the Simplex Tableau

	C_j	c_1	c_2	$\ldots$	c_n	0	0	$\ldots$	0	
C_B	B_v	x_1	x_2	$\ldots$	x_n	s_1	s_2	$\ldots$	s_n	RHS
0	s_1	a_{11}	a_{12}	$\ldots$	a_{1n}	1	0	$\ldots$	0	b_1
0	s_2	a_{21}	a_{22}	$\ldots$	a_{2n}	0	1	$\ldots$	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
0	s_m	a_{m1}	a_{m2}	$\ldots$	a_{mn}	0	0	$\ldots$	1	b_m
	Z_j	0	0	$\ldots$	0	0	0	$\ldots$	0	0
	$C_j - Z_j$	c_1	c_2	$\ldots$	c_n	0	0	$\ldots$	0	

C_B is objective function coefficients for each of the basic variables.

Z_j is the increase the value of the objective function that will result if one unit of the variable corresponding to the j^{th} column of the matrix formed from the coefficients of the variables in the constraints is brought into the basis (thus if the variable is made a basic variable with a value of one)

$C_j - Z_j$ is called the Net Evaluation Row, is the net change in the value of the objective function if one unit of the variable corresponding to the j^{th} column of the matrix (formed from the coefficient of the variables in the constraints), is brought into solution.

From the $C_j - Z_j$ we locate the column that contains the largest positive number and this becomes the Pivot Column. In each row we now divide the value in the RHS by the positive entry in the pivot column (ignoring all zero or negative entries) and the smallest one of these ratios gives the pivot row. The number at the intersection of the pivot column and the pivot row is called the PIVOT.

We then divide the entries of that row in the matrix by the pivot and use row operation to reduce all other entries in the pivot column, apart from the pivot, to zero.

3.6.1 The Stopping Criterion

The optimal solution to the linear program problem is reached when all the entries in the net evaluation row, that is, $C_j - Z_j$ are all negative or zero.

3.6.2 Infinite Many Solution

This is where the optimal solution to the problem is obtained at more than one extreme point. This implies that there is no unique solution to the problem. When this happens, the assumption made is that the graph of the objective function is parallel to at least one of the constraints binding the feasible region. Thus two or more different points may give the same value. Thus all points on this line will give an optimal solution.

Example:

$$\text{Max } Z = 4x_1 + 3x_2$$

$$\begin{aligned} \text{Sub to:} \quad 8x_1 + 6x_2 &\leq 25 \\ 3x_1 + 4x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

$$\text{Max } Z = 4x_1 + 3x_2 + s_1 + s_2$$

$$\begin{aligned} \text{Sub to:} \quad 8x_1 + 6x_2 + s_1 &= 25 \\ 3x_1 + 4x_2 + s_2 &= 15 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

Table 3.2: Iteration One

	C_j	4	3	0	0	
C_B	B_v	x_1	x_2	S_1	S_2	RHS
0	S_1	8	6	1	0	25
0	s_2	3	4	0	1	15
	Z_j	0	0	0	0	
	$C_j - Z_j$	4	3	0	0	

Table 3.3: Iteration Two

	C_j	4	3	0	0	
C_B	B_v	x_1	x_2	S_1	S_2	RHS
4	x_1	1	$\frac{3}{4}$	$\frac{1}{8}$	0	$\frac{25}{8}$
0	s_2	0	$\frac{7}{4}$	$\frac{3}{8}$	1	$\frac{45}{8}$
	Z_j	4	3	$\frac{1}{2}$	0	$\frac{100}{8}$
	$C_j - Z_j$	0	0	$-\frac{1}{2}$	0	

As shown in table 3. 3, (in iteration two) optimality is reached but the variable x_2 is not in bases, which has its $(C_j - Z_j)$ value zero. There is the need for another iteration to complete optimality (Iteration 3).

Table 3.4: Iteration Three

	C_j	4	3	0	0	
C_B	B_v	x_1	x_2	S_1	S_2	RHS
4	x_1	1	0	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{5}{7}$
0	x_2	0	1	$-\frac{3}{8}$	$\frac{4}{7}$	$\frac{45}{14}$
	Z_j	4	3	$\frac{1}{2}$	0	$\frac{100}{8}$
	$C_j - Z_j$	0	0	$-\frac{1}{2}$	0	

3.6.3 Unbounded Solution

This is a situation where the feasible region is not enclosed by constraints. In such situation, there may or may not be an optimal solution. However, in all cases if the feasible region is unbounded, then there exists no maximum solution but rather a minimum solution. To illustrate unbounded solution, let us consider a numerical example.

Example:

$$\text{Maz } Z = 4x_1 + 3x_2$$

$$\begin{aligned} \text{Sub to: } x_1 - 6x_2 &\leq 5 \\ 3x_1 &\leq 11 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

$$\text{Max } Z = 4x_1 + 3x_2 + 0s_1 + 0s_2$$

$$\begin{aligned} \text{Sub to:} \quad x_1 - 6x_2 + s_1 &= 5 \\ 3x_1 + s_2 &= 11 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

Table 3.5: Iteration Four

	C_j	4	3	0	0	
C_B	B_v	x_1	x_2	S_1	S_2	RHS
0	s_1	1	-6	1	0	5
0	s_2	3	0	0	1	11
	Z_j	0	0	0	0	0
	$C_j - Z_j$	4	3	0	0	

Table 3.6: Iteration Five

	C_j	4	3	0	0	
C_B	B_v	x_1	x_2	S_1	S_2	RHS
0	s_1	0	-6	1	$-\frac{1}{3}$	$\frac{4}{3}$
4	x_1	1	0	0	$\frac{1}{3}$	$\frac{11}{3}$
	Z_j	4	0	0	$\frac{4}{3}$	
	$C_j - Z_j$	0	3	0	$\frac{-4}{4}$	

Unboundedness occurs in this solution, because there is an entering variable in the second iteration but there is no leaving variable in the same iteration.

3.6.4 No Solution

There may also be a situation where there is no solution to the problem at hand. In such case, there will be no feasible region hence; the bounded area will be empty.

3.6.5 Minimizing the Objective Function

Standard form of LP problem consists of a maximizing objective function. Simplex method is described based on the standard form of LP problems. If the problem is a minimization type, the objective function is multiplied through by -1 so that the problem becomes maximization one.

$$\text{Min } F = -\text{Max } F$$

3.6.6 Constraints of the $\geq$ Type

The LP problem with 'greater-than-equal-to' ($\geq$) constraint is transformed to its standard form by subtracting a non negative surplus variable from it:

$$a_i x \geq b_i$$

is equivalent to

$$a_i x - s_i = b_i \text{and } s_i \geq 0$$

3.6.7 Constraints with Negative Right Hand Side Constants

Multiply both side of the constraint by -1 and add either an artificial variable or a surplus and artificial variable as required. Assuming we have the constraint:

$$-2x_1 + 7x_2 \leq -10$$

Multiplying both sides by a negative gives:

$$2x_1 - 7x_2 \geq 10$$

To convert the new constraint into equality, we add both a surplus and artificial variable as follow:

$$2x_1 - 7x_2 - 1s_1 + 1A_1 = 10$$

where s_1 and A_1 are surplus and artificial variables respectively.

3.6.8 Unconstrained Variables

If some variable x_j is unrestricted in sign, replace it everywhere in the formulation by $x_j^l - x_j^{ll}$, where $x_j^l \geq$ and $x_j^{ll} \geq 0$.

Example

$$\text{Max } Z = 100x_1 + 200x_2$$

$$\begin{aligned} \text{Sub to:} \quad 5x_1 + 7x_2 &\leq 30 \\ 5x_1 + 2x_2 &\geq 5 \end{aligned}$$

Where x_1 and x_2 unconstrained

Solution:

To Solve the unconstrained, let $x_1 = x_1^l - x_1^{ll}$ and $x_2 = x_2^l - x_2^{ll}$

$$\text{Max } Z = 100x_1^l - 100x_1^{ll} + 200x_2^l - 200x_2^{ll}$$

$$\begin{aligned} \text{Sub to:} \quad 5x_1^l - 5x_1^{ll} + 7x_2^l - x_2^{ll} &\leq 30 \\ 5x_1^l - 5x_1^{ll} + 2x_2^l - x_2^{ll} &\geq 5 \\ x_1^l, x_1^{ll}, x_2^l, x_2^{ll} &\geq \end{aligned}$$

This reformulated problem can now be solved using the simplex tableau. The initial and the final simplex tableaus are presented in tables 3.7 and 3.8

Table 3.7: Iteration 1

	C_j	100	-100	200	200	0	0	-M	
C_v	B_v	x_1^l	x_1^{ll}	x_2^l	x_2^{ll}	s_1	s_2	A_1	RHS
0	s_1	5	-5	7	-7	1	0	0	30
-M	A_1	5	-5	2	-2	0	-1	1	5
	Z_j	-5M	5M	-2M	2M	0	M	-M	-5M
	$C_j - Z_j$	100+5M	100+5M	200+2M	-200-2M	0	-M	0	

Table 3.8: Iteration 2

	C_j	100	-100	200	200	0	0	-M	
C_v	B_v	x_1^l	x_1^{ll}	x_2^l	x_2^{ll}	s_1	s_2	A_1	RHS
-100	x_1^{ll}	-1	1	0	0	0.08	0.28	-0.28	1
200	x_2^l	0	0	1	-1	0.20	0.20	-0.20	5
	Z_j	100	-100	200	-200	32	12	-12	900
	$C_j - Z_j$	0	0	0	0	-32	-12	-M+12	

However, to determine the optimal solution to the original problem, these variables must be reconnected to their original.

$$x_1 = x_1^l - x_1^{ll}$$
$$x_2 = x_2^l - x_2^{ll}$$

Thus the solution to the original problem does indeed have one variable with negative value (i.e $x_1 = x_1^l - x_1^{ll} = -1$) including that the production rate should decrease.

3.6.9 The Central Path Method

The guiding principle in primal-dual interior-point algorithms is to follow the so-called central path toward an optimal solution. The central path is a smooth curve connecting an initial point and a complementary solution. Consider a primal linear programme in standard form:

$$\begin{array}{ll} \text{Minimize} & C^T X \\ \text{subject to} & AX = b \end{array}$$

We denote the feasible region of this problem by F_P. We assume that Is nonempty and the optimal solution set of the problem is bounded. Associated with this problem, we define for $\mu \geq 0$ the barrier problem.

$$\begin{array}{ll} \text{(BP) Minimize} & C^T X - \mu \sum_{j=1}^{n} \log X_j \\ \text{subject to} & AX = b \\ & X \geq 0 \end{array}$$

As μ is varied continuously toward 0 , there is a path $X(\mu)$ define by the solution to (BP), this path $X(\mu)$ is termed the primal central path. As any $\mu > 0$, Langrange multiplier vector y is introduce for the Linear equality constraints to form the Langrangian

$$C^T X - \mu \sum_{j=1}^{n} \log X_j - y^T(AX - b)$$

The derivatives with respect to the X_{js} are set to zero leading the conditions

$$C_j - \frac{\mu}{X_j} - y_i a_j = 0 \text{ for each } j$$

or equivalent

$$\mu X^{-1} 1 + A^T y = C$$

Where as before a_j is the j^{th} column of A , I is the vector of 1s and X is the diagonal matrix whose diagonal entries are the component of $X > 0$ setting $S_j = \frac{\mu}{X_j}$ the complete set of conditions can be written as $X \cdot S = \mu$

$$\begin{aligned} AX &= b \\ A^T y + S &= C \end{aligned}$$

Note that y is a dual feasible solution.

Consider the problem of maximizing within the unit square $S = [0,1]^2$. The problem is formulated as

$$\text{Minimize } Z = -X$$

$$\begin{aligned} \text{Subject to} \quad x_1 + x_3 &= 1 \\ x_2 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Here x_3 and x_4 are the slack variables for the original problem to put it in standard form.The optimality conditions for $X(\mu)$ consist of the original two linear constraints equation and the four equations.

$$\begin{aligned} x_1 + s_1 &= 1 \\ x_2 + s_2 &= 0 \\ x_1 + s_3 &= 0 \\ x_2 + s_4 &= 0 \end{aligned}$$

Together with the relation $s_i = \frac{\mu}{x_i}$ for $i = 1, 2, \ldots 4$. These equations are readily solved with a series of elementary variable elimination. Find

$$\begin{aligned} X_1(\mu) &= \frac{1 - 2\mu \pm \sqrt{1 + 4\mu^2}}{2} \\ X_2 &= \frac{1}{2} \end{aligned}$$

Here the central path in this case the analytical center of the optimal face (at $\mu \to 0$)

3.7 The Revised Simplex Method

The revised simplex method is a succinct and efficiently implementable algebraic representation of the simplex method. Only a small part of the condensed tableau is actually calculated. These entries are needed to completely determine a pivot step and the resulting economy of computation has proved the key to practical software implementation of the simplex method. Instead of representing the whole tableau explicitly, we manipulate the basic and nonbasic variables. Let us consider the following LP problem.

$$\begin{aligned} &\text{Maximize} \quad & Z = CX \\ &\text{Subject to:} \quad & AX \leq b \end{aligned}$$

Initial constraints become (standard form).

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X_s \\ X_s \end{bmatrix} = b$$

X_s = slack variables. Let $B = I$

Where I is the identity matrix that appeared in the solution of a given problem.

X_B = basic variable value

$$X_B = \begin{bmatrix} X_{B1} \\ \vdots \\ \vdots \\ \vdots \\ X_{BM} \end{bmatrix}$$

At any iteration non-basic variable = 0

$$BX_B = b$$

Therefore $X_B = B^{-1}b$

At any iteration given the original b vector and the inverse matrix, X_B (current R.H.S) can be calculated.

$$Z = C_B X_B$$

Where C_B = objective coefficient of basic variables.

3.7.1 Steps in Revised Simplex Method

Step 1: Determine entering variable x_j ,with associated vector p_j

i. compute $Y = C_B B^{-1}$

ii. compute $Z_J - C_J = Y p_j - C_J$ for all non basic variables

iii. choose largest negative value (maximization). If non stop.

Step 2: Determine the leaving variable X_r, with associated vector P_r

i. compute $bX_B = B^{-1}$ (current RHS)

ii. compute current constraints coefficient of entering variable $\alpha^j = B^{-1}P_J$

$X_r$is associated with $\theta = \left\{ \frac{(X_B)_K}{\alpha_K^j}, \alpha_k^j > 0 \right\}$

Step 3: Determine next basis (calculate B^{-1})

Go to step 1

Example:

$$\text{Max } Z = 3x_1 + 5x_2$$

$$\begin{aligned} \text{Subject to} \quad x_1 &\leq 4 \\ 2x_1 + 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} x_1 + s_1 &= 4 \\ 2x_1 + 2x_2 + s_2 &= 12 \\ 3x_1 + 2x_2 + s_3 &= 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$X_B = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

$$C_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$Z = C_B X_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0$$

Step 1: Determine entering variable x_j, with associated vector p_j

$$Y = C_B B^{-1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$Z_J - C_J = Yp_j - C_J$ for all non-basis

$$Z_1 - C_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - 3 = -3$$

$$Z_2 - C_2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} - 5 = -5$$

Therefore x_2 is the entering variable.

Step 2: Determine the leaving variable X_r, with associated vector P_r $X_B = B^{-1}$; $X_B = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$

$$\alpha^2 = B^{-1}P_J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Therefore s_2 leaving the basis

Step 3: Determine new B^{-1} $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$; $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Solution after one iteration :

$$X_B = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

Go To Step 1

Step 1: Second Iteration Compute $Y = C_B B^{-1}$

$$Y = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{2} & 0 \end{bmatrix}$$

Compute $Z_J - C_J = Yp_j - C_J$ for all non- basis (variable X_1 and S_1)

$$X_1 = Z_1 - C_1 = \begin{bmatrix} 0 & \frac{5}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - 3 = -3$$

$$X_2 = Z_4 - C_4 = \begin{bmatrix} 0 & \frac{5}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - = \frac{5}{2}$$

Therefore X_1 enters the basis

Step 2: Determine the leaving variable

$$X_B = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} \qquad \alpha^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$\theta = \min\ \left\{\frac{4}{1}, -, \frac{6}{3}\right\}$

Therefore S_3 leaves the basis

Step 3: Determine new B^{-1}

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Solution after two iteration

$$X_B = B^{-1}b = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

Step 1: Compute $Y = C_B B^{-1}$

$$Y = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 0 & \frac{3}{2} & 1 \end{bmatrix}$$

Compute $Z_J - C_J = Yp_j - C_J$ for all non- basis (variable S_2 and S_3)

$$S_2 = Z_4 - C_4 = \begin{bmatrix} 0 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = \frac{3}{2}$$

$$S_3 = Z_5 - C_5 = \begin{bmatrix} 0 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = 1$$

No negative therefore stop. Optimal solution.

$$S_1^* = 2$$

$$X_2^* = 6$$

$$X_1^* = 2$$

$$Z^* = C_B X_B = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36$$

3.7.2 The Interior-Point Method

Interior point methods are certain class of algorithms used to solve linear and nonlinear convex optimization problems. They follow a path through the interior of the feasible region until the final solution is attained.

All interior point algorithms are based on the general framework which is summarized below:

3.7.3 General Optimization Algorithm

- Given an iterate x^k , find the search direction $\triangle\, x$ by solving the linear system

$$\nabla f(x^k) \triangle x = -f(x^k)$$

- Find the step size α_k

- Update x^k to $x^k = x^k + \alpha_k \triangle$

The symbol ∇f represents the derivative, gradient, or Jacobian of the function f depending on the definition of the function f.

Starting Point

The choice of starting point depends on two requirements: the centrality of the point and the magnitude of the corresponding infeasibility. These conditions are met by solving two least squares problems which aim to satisfy the primal and dual constraints:

$$\begin{aligned} &\text{(P)} \quad \text{Minimize} && c^T x \\ &\text{Subject to} && Ax = b,\ x \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{(D)} \quad \text{Maximize} && b^T y \\ &\text{Subject to} && A^T y + s = c,\ s \geq 0 \end{aligned}$$

These problems have solution:

$$\tilde{x} = A^T(AA^T)^{-1}b,\ \tilde{y} = (AA^T)^{-1}Ac,\ \tilde{s} = c - A^T\tilde{y}$$

The solution is further shifted inside the positive octant to obtain the starting point as:

$$w^0 = (\tilde{x} + \delta_x e,\ \tilde{y},\ \tilde{s} + \delta_s e)$$

where δ_x and δ_s are positive quantities.

Search Direction

It is $(\nabla x, \nabla y, \nabla z)$. It is obtained by solving the Newton's equation:

$$\triangledown f(x, y, s) \begin{bmatrix} \nabla x \\ \nabla y \\ \nabla z \end{bmatrix} = -f(x^k, y^k, s^k)$$

Step Size

The choice of step-size is essential in proving good convergence properties of interior point methods. The step size is chosen so that the positivity of x and s are preserved when updated. $\alpha^{\max}$ is the maximum step size that is chosen until one of the variables becomes zero (0). $\alpha^{\max}$ is calculated as follows:

$$\alpha_P^{\max} = \min\left\{-\frac{x_i}{(dx)_i} : (dx)_i < 0,\ i = 1, \ldots, n\right\}$$

$$\alpha_D^{\max} = \min\left\{-\frac{s_i}{(ds)_i} : (ds)_i < 0,\ i = 1, \ldots, n\right\}$$

$$\alpha^{\max} = \min\{\alpha_P^{\max}, \alpha_D^{\max}\}$$

Since none of the variables is allowed to be zero (0), $\alpha = \max\{1, \theta\alpha^{\max}\}$ is taken, where $\theta \in (0, 1)$. The usual choice is $\theta = 0.9$ or $\theta = 0.95$.

Termination Criteria

Due to the presence of the barrier term that keeps the iterates away from the boundary, they can never produce an exact solution. Feasibility and complementarily can therefore be attained only within a certain level of accuracy.

For these reasons, termination criteria for the algorithm to be used has to be decided on. Some common termination criteria used in practice are as follows: $\frac{\|Ax-b\|}{1+\|x\|_\infty} \leq 10^{-p}$ $\frac{\|A^Ty+s-c\|}{1+\|s\|_\infty} \leq 10^{-p}$ $\frac{\|c^Tx-b^Ty\|}{1+\|b^Ty\|_\infty} \leq 10^{-q}$

The values of p and q required depend on the specific application.

3.7.4 Primal-Dual Methods

It is one of the three main categories of the interior point methods. The primal dual algorithm operates simultaneously on the primal and the dual linear programming. They find the solutions

(x^*, y^*, s^*) of

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S^k & 0 & A^k \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix} = \begin{bmatrix} r_P^k \\ r_D^k \\ -X^k S^k + \gamma\mu_k e \end{bmatrix}$$

by applying variants of Newton's method to the above and modifying the search directions and the step lengths so that inequalities $(x, s) \geq 0$ are satisfied strictly at every iteration. $X, S \in R^{n\times n}$ are diagonal matrices of x_i, s_i respectively and $e \in R^n$ is a vector of ones.

3.7.5 Karmakar:

Karmarkar's algorithm falls within the class of interior point methods: the current guess for the solution does not follow the boundary of the feasible set as in the simplex method, but it moves through the interior of the feasible region, improving the approximation of the optimal solution by a definite fraction with every iteration, and converging to an optimal solution with rational data.

Karmarkar method is applied to a Linear Programme in the following.

$$\begin{aligned} \text{Min} \quad Z &= CX \\ \text{Subject to:} \quad AX &= 0 \\ X_1, X_2, \ldots X_n &= 1 \\ X &\geq 0 \end{aligned}$$

where $X = \{X_1, X_2, \ldots, X_n\}^T$. A is an $m \times n$ matrix, $C = [C_1, C_2, \ldots C_n]$ and 0 is an n-dimensional column vector of zeros. The LP must also satisfy $\left[\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right]^T$ is feasible, optimal Z value $= 0$

Karmakar Algorithm

1. Begin at the feasible point $X^0 = \left[\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right]$ and set $k = 0$
2. Stop if $CX^k < \varepsilon$, If not go to step 3

3. Find the new point $y^{k+1} = \left[y_1^{k+1}, y_2^{k+1}, \ldots, y_n^{k+1}\right]^T$ in the transformed unit simplex given by
$$y^{k+1} = \left[\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right]^T - \frac{\theta(I - P^T(PP^T)^{-1}P)\left[Diag(X^k)C^T\right]}{\|C_p\|\sqrt{n(n-1)}}$$

Here $\|C_p\|$ = length of $(I - P^T(PP^T)^{-1}P)\left[Diag(X^k)C^T\right]$, P is the $(m+1)\times n$ matrix whose whose first m rows are $A\left[Diag(X^k)\right]$ and whose last row is a vector of one's and $0 < \theta < 1$ chosen to ensure convergence of the algorithm, $\theta = \frac{1}{4}$ is known to ensure convergence.

Example:

$$\text{Min } Z = x_1 + 3x_2 - 3x_3$$

$$\begin{aligned} \text{Sub to:} \quad x_1 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

Solution: Let $K = 0$, given $\varepsilon = 0.10, n =$ number of variables, $A =$ coefficient matrix of constraint diagonal matrix

Let $X^K = X^0$

$X^0 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T$ X^0 yields $Z = \frac{1}{3} > 0.10$

$$A = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$$

$$D(X^0) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}; \; AD(X^0) = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}; \; PP^T = \begin{bmatrix} \frac{2}{9} & 0 \\ 0 & 3 \end{bmatrix}; \; (PP^T)^{-1} = \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

But

$$(I - P^T(PP^T)^{-1}P) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & -3 \end{bmatrix};\ D(X^0)C^T = \begin{bmatrix} \frac{1}{3} \\ 1 \\ -1 \end{bmatrix}$$

But

$$(I - P^T(PP^T)^{-1}P)(D(X^0)C^T) = \begin{bmatrix} \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \end{bmatrix}$$

Now using $\theta = 0.25$

$$y^1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T - 0.25 \frac{\begin{bmatrix} \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \end{bmatrix}}{\sqrt{3(2) \| \frac{2}{9} \quad -\frac{1}{9} \quad -\frac{1}{9} \|}}$$

Because $\| \frac{2}{9} \quad -\frac{1}{9} \quad -\frac{1}{9} \| = \sqrt{\frac{4}{18} + \frac{1}{18} + \frac{1}{18}} = \frac{\sqrt{6}}{9}$ We obtain

$$\begin{aligned} y^1 &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T - \begin{bmatrix} \frac{6}{72} & -\frac{3}{72} & -\frac{3}{72} \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}^T \end{aligned}$$

For $X^1 = Z = \frac{1}{4} + 3\left(\frac{3}{8}\right) - \left(\frac{3}{8}\right)$

Therefore $\frac{1}{4} < \frac{1}{3}$. Hence converges

3.7.6 The Primal-Dual

Given the linear programming problem in the standard form:

$$\begin{aligned} &\text{(P)} \quad \text{Minimize} && c^T x \\ &\text{Subject to} && Ax = b,\ x \geq 0 \end{aligned}$$

Where $c \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$ are given data, and x is the decision variable. The dual (D) to the primal (P) can be written as:

$$\begin{aligned} &\text{(D)} \quad \text{Maximize} && b^T y \\ &\text{Subject to} && A^T y + s = c,\ s \geq 0 \end{aligned}$$

with variables $y \in R^m$ and $s \in R^n$

3.7.7 Fundamental Steps in the Primal-Dual Method

The use of primal-dual algorithm to solve linear programs is based on three steps:

Step 1: The application of the Lagrange multiplier approach of classical calculus to transform an equality constrained optimization problem into an unconstrained one

Step 2: The transformation of an inequality constrained optimization problem into a sequence of unconstrained problems by incorporating the constraints in a logarithmic barrier function that imposes a growing penalty as the boundary ($x_j = 0, z_j = 0$ for all j) is approached.

Step 3: The solution of a set of nonlinear equations using Newton's method, thereby arriving at a solution to the unconstrained optimization problem.

When solving the sequence of unconstrained problems, as the strength of the barrier function is decreased, the optimum follows a well-defined path that ends at the optimal solution to the original problem.

Finding the Lagrangian of the Function

A well-known procedure for determining the minimum or maximum of a function subject to equality constraints is the Lagrange multiplier approach.
Consider the general problem

$$\begin{aligned} &\text{Maximize} && f(x) \\ &\text{Subject to} && g_i(x) = 0,\ i = 1, \ldots, m \end{aligned}$$

Where $f(x)$ and $g_i(x)$ are scalar functions of the $n-$dimensional vector x The Lagrangian for this problem is

$$L(x, y) = f(x) - \sum_{i=1}^{n} y_i g_i(x)$$

Where the variables $y = (y_1, y_2, \ldots, y_m)$ are the Lagrange multipliers.

Necessary conditions for a stationary point (Maximum or minimum) of the constrained optimization of $f(x)$ are that the partial derivatives of the Lagrangian with respect to the components of x and y be zero; i.e

$$\frac{\partial L}{\partial x} = 0,\ i = 1, 2, \ldots, n \text{ and } \frac{\partial L}{\partial y} = 0,\ j = 1, 2, \ldots, n$$

For linear constraints $(a_i X - b_i = 0)$, the conditions are sufficient for a maximum if the function $f(X)$ is concave and sufficient for a minimum if $f(X)$ is convex.

Constructing a Barrier in the Interior Region

The idea of the barrier approach is to start from a point in the strict interior of the inequalities ($x_j > 0, s_j > 0$ for all j) and construct a barrier that prevents any variable from reaching a boundary . (e.g.,$x_j = 0$). Adding "$\log(x_j)$" to the objective function of the primal, for example, will cause the objective function to decrease without bound as x_j approaches 0. The difficulty with this idea is that if the constrained optimum is on the boundary (that is , one or more $x_j^* = 0$, which is always the case in linear programming), then the barrier will prevent the optimum from being reached on the boundary. To get around this difficulty, a barrier parameter μ is added to balance the contribution of the true objective function with that of the barrier term. This is shown in the table below;

Table 3.9: Primal and Dual barrier problems

(P) Maximize $B_p(\mu) = cx + \mu \sum_{j=1}^{n} \log(X_j)$	(D) Minimize $B_p(\mu) = yb - \mu \sum_{j=1}^{n} \log(Z_j)$
Subject to $Ax = b$	Subject to $yA - z = c$

The parameter μ is required to be positive and controls the magnitude of the barrier term. Because the function $\log(x)$ takes on very large negative values as x approaches zero from above, as long as x remains positive the optimal solution to the barrier problem will be interior to the nonnegative octants (x_j and $z_j > 0$ for all). The barrier term is added to the

objective function to have nonlinear objective functions with linear equality constraints, and can be solved with the Lagrange technique for $\varepsilon > 0$ fixed. The solution to these problems will approach the solution to the original problem as μ approaches zero.
Table 3.10 shows the development of the necessary optimal conditions for the barrier problems. These conditions are also sufficient because the primal Lagrangian is concave and the dual Lagrangian is convex. Note that the dual variables are the Lagrange multipliers of the primal, and the primal variables X are the Lagrange multipliers of the dual.

Table 3.10: Necessary conditions for the barrier problems (complementary slackness)

Lagrangian	
$L_p(\mu) = cx + \mu \sum_{j=1}^{n} \log \mu(X_j) - y(Ax - b)$	$L_D(\mu) = yb - \mu \sum_{j=1}^{n} \log(Z_j) - (yA - Z - C)X$
$\frac{\partial L_P}{\partial x_j} = 0$	$\frac{\partial L_P}{\partial x_j} = 0$
$C_j - \sum_{j=1}^{m} a_{ij} yj + \frac{\mu}{x_j} = 0$	
$-Z + \frac{\mu}{x} = 0$	$\frac{\mu}{Z} + X_j = 0$
$Z_j X_j = \mu,\ j = 1, \ldots, n$	$Z_j X_j = \mu,\ j = 1, \ldots, n$
($\mu -$ Complementary Slackness)	($\mu -$ Complementary Slackness)

Thus the optimal conditions are nothing more than primal feasibility, dual feasibility, and complementary slackness satisfied to within a tolerance of ε. Theory shows that when μ goes to zero the solution to the original problem would be attained; however, we cannot just set μ to zero because that would destroy the convergence properties of algorithm. To facilitate the process, a two $n \times n$ diagonal matrices containing the components of x and z, respectively are defined. That is;

$$X = \text{diag}\{x_1, x_2, \ldots, x_n\}$$
$$Z = \text{diag}\{z_1, z_2, \ldots, z_n\}$$

Also , let $e = (1, 1, \ldots, 1)^T$ be a column vector of size n. Using this notation, the necessary and sufficient conditions derived in Table 3.10 for the simultaneous solution of both the primal and dual barrier problems can be written as:

Primal feasibility: $Ax - b = 0$ (m linear equations)

Dual feasibility : $A^T y^T - z - c = 0$ (n linear equations)

μ - Complementary Slackness: $XZe - \mu e = 0$ (n non linear equations)

There is therefore the need to solve this set of nonlinear equations for variables (x, y, z)

Finding the Stationary Solution Using Newton's Method

Newton's method is an iterative procedure for numerically solving a set of nonlinear equations. For instance; consider a single variable problem of finding h to satisfy the nonlinear equation $f(h) = 0$ where f is once continuously differentiable. Let h^* be the unknown solution. At some point h^*, one can calculate the functional value, $f(h^k)$, and the first derivative , $f^I(h^k)$. Using the derivative as a first order approximation for how the function changes with y, one can predict the amount of change $\nabla = h^{k+1} - h^k$ required to bring the function to zero.

Taking the first order Taylor series expansion of $f(h)$ around h^* gives

$$f(h^(k+1)) \equiv f(h^k) + \nabla f^I(h^k).$$

Setting the approximation of $f(h^{k+1})$ to zero and solving for ∇ gives

$$\nabla = -f(h^k)/f^I(h^k)$$

The point $h^{k+1} = h^k + \nabla$ is an approximation solution to the equation. It can be shown that if one starts at a point h^o sufficiently close to h^*, the value of h^kwill approach h^* as $k \to \infty$

The method extends to multidimensional functions. Consider the general problem of finding the $r-$dimensional vector h that solves the set of r equations $f_1(h) = 0, i = 1, \ldots r$ **or** $f(h) = 0$ Let the unknown solution to the equations be h^*. The $n \times n$ Jacobian matrix describes the first order variations of these with the components of h. The Jacobian at h^k is

$$J(h^k) = \begin{bmatrix} \frac{\partial f_1}{\partial h_1} & \frac{\partial f_1}{\partial h_2} & \cdots & \frac{\partial f_1}{\partial h_n} \\ \frac{\partial f_2}{\partial h_1} & \frac{\partial f_2}{\partial h_2} & \cdots & \frac{\partial f_2}{\partial h_n} \\ & \vdots & \ddots & \\ \frac{\partial f_n}{\partial h_1} & \frac{\partial f_n}{\partial h_2} & \cdots & \frac{\partial f_n}{\partial h_n} \end{bmatrix}$$

All the partial derivatives are evaluated at h^k. Now, taking the first order Taylor series expansion around the point h^k, and setting it to zero gives $f(y^k) + J(y^k)d = 0$ where $d = h^{k+1} - h^k$ is an $n-$dimensional vector whose components represent the change of position for the $k + 1st$ iteration. Solving for d leads to

$$d = -J(h)^{k-1} f(h^k)$$

The point $h^(k + 1) = h^k + d$ is an approximation for the solution to the set of equations. Once again, if one starts at an initial point h^0 sufficiently close to h^*, the value of h^k will approach h^* for large values of k.

Using Newton's Method for Solving Barrier Problems

The stage is now set for Newton's method to be used to solve the optimality conditions for the barrier problems given in Table 3.10) for a fixed value of μ. For $h = (x, y, z)$ and $r = 2n + m$, the corresponding equations and Jacobian are:

$$\begin{aligned} Ax - b &= 0 \\ A^T - z - C^T &= 0 \end{aligned}$$

$$J(h) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & -1 \\ Z & 0 & X \end{bmatrix}$$

$$XZe - \mu e = 0$$

Assuming that a starting point (X^0, y^0, Z^0) satisfying $X^0 > 0, y^0 > 0, Z^0 > 0$ and denoted by

$$\delta_p = b - Ax^0$$

$$\delta_D = C^T - A^T(y^0)^T + Z^0$$

Are the primal and dual residual vectors at this starting point. The Optimality conditions can be written as

$$J(y)d = -f(y)$$

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & -1 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \delta_P \\ \delta_D \\ \mu e - XZe \end{bmatrix}$$

Where the $(2n + m)$-dimensional vector $d = (dx, dy, dx)^T$ is called the Newton direction. The d will now be solved.

In explicit form, the above system is

$$\begin{aligned} Adx &= \delta_p \\ A^T dy - dz &= \delta_D \\ Zdx + Xdz &= \mu e - XZe \end{aligned}$$

The first step is to find dy. In making dy a subject the following equation is obtained;

$$(AZ^{-1}XA^T)dy = -b + \mu AZ^{-1}e + AZ^{-1}X\delta_D \text{ or}$$

$$dy = (AZ^{-1}XAZ)^{-1}(-b + \mu AZ^{-1}e + AZ^{-1}X\delta_D \tag{3.1}$$

It is worth noting that $Z^{-1} = \text{diag}\frac{1}{Z_1}, \frac{1}{Z_2}, \ldots, \frac{1}{Z_n}$ and is trivial to compute. Further multiplications and substitutions give

$$dz = -\delta_D + A^T d \tag{3.2}$$

and

$$dx = Z^{-1}(\mu e - Xze - Xdz) \tag{3.3}$$

From these results, it is obvious in part why it is necessary to remain in the interior of the feasible region. In particular, if either Z^{-1} or X^{-1} does not exist the procedure breaks down. Once the Newton direction has been computed, dx is used as a search direction in the $x-$space and (dy, dz) as a search direction in the (y, z)-space. That is, the iterant moves from the current point (x^0, y^0, z^0) to a new point (x^1, y^1, z^1) by taking a step in the direction (dx, dy, dz). The step sizes,∂_p and ∂_D , are chosen in the two spaces to preserve $x > 0$ and $y > 0$.

This requires a ratio test similar to that performed in the simplex algorithm. The simplex approach is to use

$$\begin{aligned}
\alpha_P &= \gamma \text{min} \left\{ \frac{-X_j^k}{(dx)_j^k} : (dx)_j^k < 0 \right\} \\
\alpha_z &= \gamma \text{min} \left\{ \frac{-Z_j^k}{(dz)_j^k} : (dz)_j^k < 0 \right\}
\end{aligned}$$

Where γ is the step size factor that keeps the iterant from actually touching the boundary. Typically, $\gamma = 0.995$. the notation $(dx)_j$ refers to the jth component of the vector dx. The new point is

$$\begin{aligned}
x^1 &= x^0 + \alpha_p dx \\
y^1 &= y^0 + \alpha_p dy \\
z^1 &= z^0 + \alpha_p dz
\end{aligned}$$

Which completes one iteration. Ordinarily, one would now resolve equation (1)-(3) at (x^1, y^1, Z^1) to find a new Newton direction and hence a new point. Rather than iterating in this manner until the system converges for the current value of μ, it is much more efficient to reduce μ after every iteration. The primal-dual method itself suggests how to update μ. It is straight forward to show that the Newton step reduces the duality gap (θ), which is the difference between the dual and primal objective values at the current point. Assume that is primal feasible and (x^0, Z^0) is dual feasible, then in general case let $\theta(0)$ denote the current duality gap,

$$\begin{aligned} \theta(0) &= y^0 b - cx^0 \\ &= y^0(AX^0) - (y^0 A - Z^0)^T x^0 \text{ (primal and dual feasibility)} \\ &= (Z^0)^T x^0 \end{aligned}$$

If we let $\alpha = min\alpha_p, \alpha_D$ then $\theta(\alpha) = (Z^0 + \alpha dz)^T (X^0 + \alpha dx)$ and with a little algebra, it can be shown that $\theta(\alpha) < \theta(0)$ as long as $\mu < \frac{\theta(0)}{n}$

The following formula was used in the computations made;

$$\mu^k = \frac{\theta(\alpha^k)}{n^2} = \frac{(Z^k)^T X^k}{n^2}$$

Which indicates that the value of μ^k is proportional to the duality gap, (θ).

Termination Criteria

Due to the presence of the barrier term that keeps the iterant away from reaching the boundary, they can never produce an exact solution. Feasibility and complementary can therefore be attained only within a certain level of accuracy.

For this reason, termination criteria for the algorithm to be used have to be decided on. The most common criterion is the use of the duality gap. That is , at optimality the duality gap is zero (0)

Iterative procedure for Newton's method

Step 1: In summarizing the basic steps of the algorithm the following inputs are assumed:

1. The data of the problem (a, b, c) ,where the $m \times n$ matrix A has full row rank

2. Initial primal and dual feasible solutions $x^0 > 0, Z^0 > 0, y^0 > 0$

3. The optimality tolerance $\varepsilon > 0$ and the step size parameter $\gamma \in (0, 1)$.

Step 2: (Initialization) Start with some feasible point $X^0 > 0, y^0 > 0, Z^0 > 0$

Chose (X^0, y^0, z^0) such that $(X^0, y^0, z^0) > 0$ and set the iteration counter $k = 0$.

Step 3 (Optimality test) .If $(z^k)^T X^k < \varepsilon$ stop; otherwise, go to step 4.

Step 4: (Compute Newton direction). Let

$$
\begin{aligned}
X^k &= \operatorname{diag}\left\{X_1^k, X_1^k, \ldots, X_n^k\right\} \\
X^k &= \operatorname{diag}\left\{X_1^k, X_1^k, \ldots, X_n^k\right\}
\end{aligned}
$$

$$\mu^2 = \frac{(Z^k)^T X^k}{n^2}$$

Solve the following linear system equivalent to (7) to get d_x^k, d_y^k and d_z^k

$$
\begin{aligned}
Adx &= 0 \\
A^T dy - dz &= 0 \\
Zdx + Xdz &= \mu e - XZe
\end{aligned}
$$

Note that $\delta_p = 0$ and $\delta_D = 0$ due to the feasibility of initial point.

Step 5: (Find step lengths). Let

$$
\begin{aligned}
\alpha_P &= \gamma \min\left\{\frac{-X_j^k}{(dx)_j^k} : (dx)_j^k < 0\right\} \\
\alpha_z &= \gamma \min\left\{\frac{-Z_j^k}{(dz)_j^k} : (dz)_j^k < 0\right\}
\end{aligned}
$$

Step 6: (Update solution) . Take a step in the Newton direction to get;

$$X^{k+1} = X^k + \alpha_p (dx)^k y^{k+1} = y^k + \alpha_p (dy)^k Z^{k+1} = Z^k + \alpha_p (dz)^k$$

Table 3.11: Table showing standard minimum and dual maximum constraints

	x_1	x_2	$\cdots$	x_n	
y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\geq b_1$
y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\geq b_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\geq b_n$
	$\leq c_1$	$\leq c_2$	$\cdots$	$\leq c_n$	

Put $k \to k+1$ and go to step 2.

3.7.8 The centering parameter (σ)

It balances the movement towards the central path against the movement toward optimal solutions. If $\sigma = 1$, then the updates move towards the center of the feasible region. If $\sigma = 0$, then the update step is in the direction of the optimal solution.

3.7.9 The duality Gap (μ)

It is the difference between the primal and dual objective functions. Theoretically, these two quantities are equal and so give a result of zero (0) at optimality. In practice however, the algorithm drives the result down to a small amount. This is given by the equation:

$$\mu = \frac{1}{n}(x^T s) = c^T x - b^T y$$

While $\mu \geq \varepsilon$, Newton's method is applied until $\mu \geq \varepsilon$ when the algorithm terminates. ε is a positive fixed number.

The general standard minimum problem and the dual standard maximum problem may be together illustrated as:

3.7.10 Formulation of the Dual From the Primal of a Linear Programming Method

To illustrate this, let us consider the problem:

$$\text{Minimize } \; x_1 + x_2$$

$$\begin{aligned} \text{Subject to } \; x_1 + x_2 &\geq 4 \\ 4x_1 + 4x_2 &\geq 12 \\ -x_1 + x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

As a first step, a matrix A is formed from the coefficients of the primal objective function and its constraints as:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 2 & 12 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

A second matrix B is formed from the transpose of A. That is:

$$B = \begin{bmatrix} 1 & 4 & -1 & 1 \\ 2 & 2 & 1 & 1 \\ 4 & 12 & 1 & 0 \end{bmatrix}$$

The dual problem is formulated as follows:

$$\text{Maximize } \; 4y_1 + 12y_2 + y_3$$

$$\begin{aligned} \text{Subject to } 4y_1 + 4y_2 - y_3 &\leq 1 \\ 2y_1 + 2y_2 + y_3 &\leq 1 \\ y_1, y_2, y_3 \geq 0 & \end{aligned}$$

It must be noted that if the objective function in the primal is to be minimize, then its dual objective function will maximize and vice versa.

3.7.11 The Primal-Dual Algorithm

Initialization

Step 1: Determine (x^0, y^0, z^0) such that $(x^0, s^0) > 0$ and $\|x^0 s^0 - \mu_0 e\| \leq \beta\mu_0$ where $\mu_0 = \frac{(x^0)^T s^0}{n}$

Then choose $\beta, \gamma \in (0, 1)$ and $(\varepsilon_P, \varepsilon_D, \varepsilon_G) > 0$

Step 2: Set $k = 0$

Step 3: Set $r_P^k = b - Ax^k$, $r_D^k = c - Ak^T y^k - s^k$, $\mu_k = \frac{(x^k)^T s^k}{n}$

Step 4: Check the termination. If $\|r_P^k\| \leq \varepsilon_P$, $\|r_D^k\| \leq \varepsilon_P$, $(x^k)^T s^k \leq \varepsilon_G$ then terminate.

Step 5: Compute the direction by solving the system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S^k & 0 & A^k \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} r_P^k \\ r_D^k \\ -X^k S^k + \gamma\mu_x e \end{bmatrix}$$

Step 6: Compute the step size

$$\alpha = \max\{\alpha' : \|X(\alpha)s(\alpha) - \mu(\alpha)e\| \leq \beta(\alpha), \forall\alpha \in [0, \alpha']\}$$

where

$$x(\alpha) = x^k + \alpha d_x, \; s(\alpha) = s^k + \alpha d_s, \text{ and } \mu(\alpha) = \frac{x^T(\alpha)s(\alpha)}{n}$$

Step 7: Update

$$x^{k+1} = x^k + \alpha_k d_x, \; y^{k+1} = y^k + \alpha_k d_y, \; s^{k+1} = s^k d_s$$

Step 8: Set $k = k + 1$, and go to Step 3.

NUMERICAL EXAMPLE

$$\text{Minimize } 2x_1 + 1.5x_2$$

$$\begin{aligned} \text{Subject to } 12x_1 + 24x_2 &\geq 120 \\ 16x_1 + 16x_2 &\geq 120 \\ 30x_1 + 12x_2 &\geq 120 \\ x_1 &\leq 15 \\ x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3.7.12 Standard non-negative equations

$$\text{Minimize } 2x_1 + 1.5x_2$$

$$\begin{aligned} \text{Subject to } 12x_1 + 24x_2 - x_3 &= 120 \\ 16x_1 + 16x_2 - x_4 &= 120 \\ 30x_1 + 12x_2 - x_5 &= 120 \\ x_1 + x_6 &= 15 \\ x_2 + x_7 &= 15 \\ x_1, \ldots, x_7 &\geq 0 \end{aligned}$$

Substituting $(\tilde{x_1}, \tilde{x_2}) = (10, 10)$ into the equations above and solving for the slack variables $(\tilde{x_3}, \ldots, \tilde{x_7})$ give:

$$\tilde{x} = \begin{bmatrix} 10 & 10 & 240 & 200 & 300 & 5 & 5 \end{bmatrix}^T > 0$$

For an initial dual iterate, the algorithm requires a $\tilde{y}$ such that $\tilde{s} = c - A^T y > 0$. Writing these explicitly give:

$$\begin{aligned}
\tilde{s_1} &= 2 - 12\tilde{y_1} - 16\tilde{y_2} - 30\tilde{y_3} - 1\tilde{y_4} > 0 \\
\tilde{s_2} &= 1.5 - 24\tilde{y_1} - 16\tilde{y_2} - 12\tilde{y_3} \quad - 1\tilde{y_5} > 0 \\
\tilde{s_3} &= 0 + 1\tilde{y_1} \quad > 0 \\
\tilde{s_4} &= 0 \quad + 1\tilde{y_2} > 0 \\
\tilde{s_5} &= 0 \quad + 1\tilde{y_3} > 0 \\
\tilde{s_6} &= 0 \quad - 1\tilde{y_4} > 0 \\
\tilde{s_y} &= 0 \quad - 1\tilde{y_5} > 0
\end{aligned}$$

The above inequalities are satisfied by putting in $\tilde{y_1} = \tilde{y_2} = \tilde{y_3} = 1$ and $\tilde{y_4} = \tilde{y_5} = -60$. This gives:

$$\tilde{y} = \begin{bmatrix} 1 & 1 & 1 & -60 & -60 \end{bmatrix}, \tilde{s} = \begin{bmatrix} 4 & 9.5 & 1 & 1 & 1 & 60 & 60 \end{bmatrix} > 0$$

The matrices generated are as follows:

$$A = \begin{bmatrix} 12 & 24 & -1 & 0 & 0 & 0 & 0 \\ 16 & 16 & 0 & -1 & 0 & 0 & 0 \\ 30 & 12 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 120 \\ 120 \\ 120 \\ 15 \\ 15 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1.5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With $\alpha = 0.95$ and complementarity tolerance $\varepsilon = 0.00001$, the algorithm will stop when all $x_j s_j < 0.00001$.

$$\left[\begin{array}{ccccccc|ccccc}
-0.4000 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 16 & 30 & 1 & 0 \\
0 & -0.9500 & 0 & 0 & 0 & 0 & 0 & 24 & 16 & 12 & 0 & 1 \\
0 & 0 & -0.0042 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.0050 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0033 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 & 0 & 1 \\
\hline
12 & 24 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 16 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 12 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\triangle x_1 \\ \triangle x_2 \\ \triangle x_3 \\ \triangle x_4 \\ \triangle x_5 \\ \triangle x_6 \\ \triangle x_7 \\
\hline
\triangle y_1 \\ \triangle y_2 \\ \triangle y_3 \\ \triangle y_4 \\ \triangle y_5
\end{array}\right]$$

$$= \left[\begin{array}{ccccccc|ccccc} 4 & 9.5 & 1 & 1 & 1 & 60 & 60 & 0 & 0 & 0 & 0 & 0 \end{array}\right]^T$$

Solving the system yields:

$$\triangle x = \begin{bmatrix} -0.1017 \\ -0.0658 \\ -2.7997 \\ -2.6803 \\ -3.8414 \\ 0.1017 \\ 0.0658 \end{bmatrix} \quad \text{and} \quad \triangle y = \begin{bmatrix} -0.9883 \\ -0.9866 \\ -0.9872 \\ 61.2208 \\ 60.7895 \end{bmatrix}$$

$\triangle s$ is obtained by setting:

$$\triangle s = -\tilde{s} - \tilde{x}^{-1} \tilde{\sum} \triangle x$$

$$= \begin{bmatrix} 4 \\ 9.5 \\ 1 \\ 1 \\ 1 \\ 60 \\ 60 \end{bmatrix} - \begin{bmatrix} 0.4000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0042 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0050 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0033 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} -0.1017 \\ -0.0658 \\ -2.7997 \\ -2.6803 \\ -3.8414 \\ 0.1017 \\ 0.0658 \end{bmatrix} = \begin{bmatrix} -3.9593 \\ -9.4375 \\ -0.9883 \\ -0.9866 \\ -0.9872 \\ -61.2208 \\ -60.7895 \end{bmatrix}$$

The ratio $\frac{\tilde{x}_j}{-\triangle x_j}$ is computed for each of the five $\triangle x_j < 0$, and θ_x is set to the smallest:

$$\begin{aligned}
\triangle x_1 < 0 : \frac{x_1}{-\triangle x_1} &= \frac{10}{0.1017} = 98.3284 \\
\triangle x_2 < 0 : \frac{x_2}{-\triangle x_2} &= \frac{10}{0.0658} = 152.0034 \\
\triangle x_3 < 0 : \frac{x_3}{-\triangle x_3} &= \frac{240}{2.7997} = 85.7242 \\
\triangle x_4 < 0 : \frac{x_4}{-\triangle x_4} &= \frac{200}{2.6803} = 74.6187 = \theta_x \\
\triangle x_5 < 0 : \frac{x_5}{-\triangle x_5} &= \frac{300}{3.8414} = 78.0972
\end{aligned}$$

The ratios $\frac{\tilde{s}_j}{-\triangle s_j}$ are computed in the same way to determine θ_s:

$$\begin{aligned}
\triangle s_1 < 0 : \frac{s_1}{-\triangle s_1} &= \frac{4}{3.9593} = 1.0103 \\
\triangle s_2 < 0 : \frac{s_2}{-\triangle s_2} &= \frac{9.5}{9.4375} = 1.0066 \\
\triangle s_3 < 0 : \frac{s_3}{-\triangle s_3} &= \frac{1}{0.9883} = 1.0118 \\
\triangle s_4 < 0 : \frac{s_4}{-\triangle s_4} &= \frac{1}{0.9866} = 1.0136 \\
\triangle s_5 < 0 : \frac{s_5}{-\triangle s_5} &= \frac{1}{0.9872} = 78.0972 \\
\triangle s_6 < 0 : \frac{s_6}{-\triangle s_6} &= \frac{60}{61.2208} = 0.9801 = \theta_s \\
\triangle s_7 < 0 : \frac{s_7}{-\triangle s_7} &= \frac{60}{60.7895} = 0.9870
\end{aligned}$$

The step length is given by:

$$\begin{aligned}\theta &= \min(1, \alpha\theta_x, \alpha\theta_s) \\ &= \min(1, 0.995 \cdot 74.6187, 0.995 \cdot 0.9801) = 0.975159\end{aligned}$$

The iteration ends with the computation of the next iterate as:

$$\tilde{x} = \tilde{x} + \theta(\triangle x) = \begin{bmatrix} 10 \\ 10 \\ 240 \\ 200 \\ 300 \\ 5 \\ 5 \end{bmatrix} + 0.97519 \begin{bmatrix} -0.1017 \\ -0.0658 \\ -2.7997 \\ -2.6803 \\ -3.8414 \\ 0.1017 \\ 0.0658 \end{bmatrix} = \begin{bmatrix} 9.9008 \\ 9.9358 \\ 237.2699 \\ 197.3863 \\ 296.2541 \\ 5.0992 \\ 5.0642 \end{bmatrix}$$

The algorithm carries out a total of 9 iterations before reaching a solution that satisfies the stopping condition.

The optimal solution for the first two variables are $x_1 = 1.6667$ and $x_2 = 5.8333$ with stopping condition of $0.000004 < 0.00001$

Table 3.12: Table Showing Iterations of first two variables out of seven in the problem

Iter.	x_1	x_2	θ	$\max x_j s_j$
0	10.0000	10.0000	...	300.000000
1	9.9008	9.9358	0.975159	11.058316
2	6.9891	9.2249	0.423990	6.728827
3	3.2420	8.5423	0.527256	2.878729
4	1.9735	6.6197	0.697264	1.156341
5	2.0266	5.4789	0.693037	0.189301
6	1.8769	5.7796	0.841193	0.027134
7	1.7204	5.7796	0.841193	0.027134
8	1.6683	5.8317	0.979129	0.000836
9	1.6667	5.8333	0.994501	0.000004

Chapter 4

Data Analysis and Results

4.1 Data Collection

The Sunyani West District Assembly has its way of collecting revenue on taxable items using a policy called The Fee Fixing Policy. This Fee Fixing Policy is a document on rates and fixing resolutions and it focuses on taxes such as; Rates, Fees and Fines, Licenses and Lands. Each category has sub taxes which constitute to the group. For instance Rates which is one of the main categories is constituted by basic and property rates. Similarly, registration of building plots, stool land revenue, building permit and revenue from concession constitute the category of lands.

4.1.1 Type of Data and Source

The data for this project work is a secondary quarterly data obtained from the offices of the Sunyani West District Assembly in the Brong Ahafo region of Ghana, and it spans between 1st quarter 2010 and 4th quarter 2012.

4.1.2 The Raw Data

The tables 4.1 to 4.4 share similar characteristics. The 1st column in table 4.1 made up of the tax item number whiles the 2nd column is made up of the revenue sub-heads. The remaining columns form the estimated revenue (E.R) and the actual revenue (A.R) for the various quarters and their respective averages as shown in the respective tables below:

The rest of the raw data up to the 4th quarter of 2012 can be found at the appendix A, table 4.1

Table 4.1: Table Showing the (E.R) and the (A.R) generated by the assembly for the four quarters of 2010

Tax Item No.	Revenue Sub-Head	Q1(E.R)	Q1(A.R)	Q2(E.R)	Q2(A.R)	Q3(E.R)	Q3(A.R)	Q4(E.R)	Q4(A.R)	MEAN (E.R)	MEAN (A.R)
1	Basic Rate	8000.00	30.00	8000.00	46.00	8000.00	52.00	8000.00	52.00	8000.00	45.00
1	Property Rate	21,000.00	15,000.00	20,000.00	17,000.00	20,000.00	20,000.00	21,000.00	22,000.00	20,500.00	18,500.00
3	Stool Land Revenue	35,000.00	0.00	35,000.00	7,000.00	53,000.00	10,000.00	35,000.00	20,000.00	35,000.00	9,250.00
4	Building Permit	4,050.00	400.00	4,050.00	647.00	4,050.00	1,272.00	4,050.00	1,410.00	4,050.00	932.25
5	Market Tools	2,420.00	1,540.00	2,420.00	4,890.00	2,242.00	6,880.00	2,420.00	6,880.00	2,420.00	5047.60
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
30	Registration 0f Chainsaws	220.00	0.00	220.00	0.00	220.00	0.00	220.0	0.00	220.00	0.00

4.2 Data Analysis

Steps involved in Processing the Raw Data

Step 1: The averages of the raw data were determined to get the estimated revenue average (E.R) and actual revenue average (A.R) for the data.

Step 2: The various revenue sub-heads were assigned variable names.

Step 3: The summation of the actual revenue and expected revenue became the constraint column matrix.

Step 4: The unit charge for each of the items was also determined from the Fee-fixing Policy Document given by the Assembly. This formed the coefficient matrix.

Step 5: The coefficient of the objective function was formulated by using the unit charge for each of the items from the fee-fixing police document by the Assembly. These values formed the coefficient of the objective function.

Table 4.2: Table Showing the E.R and A.R generated by the assembly for the 12 quarters of the assembly

	REVENUE SUB-HEAD	E.R (AVERAGE)	A.R (AVERAGE)
1	Basic Rate	8,733.33	2,331.66
2	Property Rate	22,000.00	18750.00
3	Stool land Revenue	36,666.66	9916.66
4	Building Permit	6,050.00	1057.25
5	Market Tolls	3,420.00	5297.56
⋮	⋮	⋮	⋮
29	Traditional Caterers	443.33	160.58
30	Registration of Chainsaws	373.33	0.00

Table 4.3 This table is made up of 2 columns. Column 1 forms the decision variables whiles column 2 is made up of the revenue sub-heads. The remaining part of the table can be found in appendix E.

Table 4.3: Table showing the number of people paying for each category of tax as a variable, x_j, $j = 1, 2, 3 \ldots 30$.

DECISION VARIABLE (X_i)	REVENUE SUB-HEAD
x_1	Basic Rate
x_2	Property Rate
x_3	Stool land Revenue
x_4	Building Permit
x_5	Market Tolls
$\vdots$	$\vdots$
x_{29}	Traditional Caterers
x_{30}	Registration of Chainsaws

Table 4.4: This table serves as the pivot for the whole problem formulation .It is made up of 7 columns. The first column deals with the tax item number. The 2nd column talks about the broad Revenue Heads. These include the Rates, Lands, Fees and Fines, Licenses and Rent. This broad category has been sub-grouped into the next basic unit called revenue sub-heads in the 3rd column. The 4th column is made up of the decision variables (X_j) . Column 5 is made up of the unit charge which forms the coefficient matrix, A. The right hand side (R.H.S) matrix is obtained from column 6 and 7 of the table. Finally, the coefficient of the objective function C_j is obtained from column 3 and 4 of table 4.4. The complete form of the table can be found in the appendix F.

Table 4.4: Table represents the tax payers (x_j), average (A.R), average (E.R) and the unit charge from the fee-fixing policy document of the assembly

TAX ITEM NO.	REVENUE HEAD	REVENUE SUB-HEAD	TAX PAYERS' VARIABLE (X_j)	UNIT CHARGE (A)	AVERAGE A.R FOR THE PAST 12 QUARTERS	AVERAGE E.R FOR THE PAST 12 QUARTERS
1	Rates	Basic Rate	x_1	0.10	2,331.66	8733.33
2	Rates	Property Rate	x_2	10.00	18750.00	22000.00
3	Lands	Stool land Revenue	x_3	10.00	9916.66	36666.66
4	Lands	Building Permit	x_4	35.62	1057.25	6050.00
5	Fees and Fines	Market Tolls	x_5	0.20	5297.56	1,059.51
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
29	Licenses	Traditional Caterers	x_{29}	9.67	160.58	433.33
30	Licenses	Registration of Chainsaws	x_{30}	48.33	200.00	373.33

4.3 Data Input Format

In expressing the above information in terms of matrices the following matrix equation will be obtained; $Ax = b$; where A is the coefficient matrix of the taxpayers function x, b is the constraint column matrix and c is the coefficient of the objective function. The complete form of the data in A, c and x can be found in table 4.3 of appendix F.

$$A = \begin{bmatrix} 0.1 & 10 & 0 & 0 & 0 & \cdots & 10 & \cdots & 0 \\ 0 & 0 & 3 & 36.62 & 0 & \cdots & 0.2 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 & \cdots & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 46980.53 \\ 10386.86 \\ 83897.46 \\ 20892.91 \\ 197634.10 \\ 11548.16 \\ 149734.64 \\ 1982438.24 \\ 3281.00 \\ 712.20 \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{30} \end{bmatrix} ; X_1, X_2, X_3 \ldots X_{30} \geq 0$$

$$c = \begin{pmatrix} 0.1 & 10 & 10 & 36.62 & 0.2 & \ldots & 9.67 & 48.33 \end{pmatrix}$$

4.4 Model Formulation

The various taxes collected by the district assembly are broadly categorized into five groups. The groups are Rates, Lands, Fees and Fines, Licenses and Rent. These broad categories are shown in column 2 of table 4.3. The unit charge is obtained from the fee- fixing policy of the district. For instance, each member of the Sunyani West district is required by law to pay a basic rate of GHs 0.10 per annum.

4.4.1 Formulation of the Objective Function

At this point we seek to maximize revenue from an objective function generated from the data collected. As stated in chapter three, the function $f(x)$ being maximized is called the objective function and conditions associated with the problem are called the constraints. In using the variable representing number of people paying each tax (X_j) and each unit charge (C_j), we model an objective function represented by z from table 4.4 as follows:

$$Z = \sum_{j=1}^{30} C_j X_j$$

$$Z = (0.1X_1 + 10X_2 + 10X_3 + 36.62X_4 + \cdots + 48.33X_{30}) \tag{4.1}$$

The C_j represents the coefficients of the objective function. The full data can be found in column 5 of table 4.3 in appendix F.

From the table 4.3, ten constraints are generated for the objective function, Z, based on the broad categories of the taxes collected. Two constraint are generated each for Rates, Licenses, Fees and Fines, Rent and Lands respectively.

4.4.2 Formulation of the Constraints

The right hand side of each of the constraints represents the respective sum of the actual revenue (A.R) and expected revenue (E.R) generated by the respective variables. This information can be found in column 6 and 7 of table 4.4. Under listed below shows the constraints formed by the broad category of the revenue collected.

Rates:

The constraint for the rate is obtained from 3 decision variables, X_1, X_2 and X_{27} with corresponding R.H.S values of 10386.86 and 46980.53.

$$0.10X_1 + 36.62X_2 + 10X_{27} \leq 46980.53$$

$$0.10X_1 + 36.62X_2 + 10X_{27} \geq 10386.86$$

Lands:

The constraint for lands is obtained from the decision variables, X_3, X_4 and X_{23} and with respective R.H.S values of 20892.91 and 83877.46.

$$10X_3 + 36.62X_4 + 450X_{23} \leq 83877.46$$

$$10X_3 + 36.62X_4 + 450X_{23} \geq 20892.91$$

Fees and Lines:

The constraint for fees and fines is obtained from 7 decision variables of $X_5, X_6, X_7,$ $X_8, X_{10}, X_{11},$ and $, X_24$ and with R.H.S values of 11548.16 and 197634.10

$$0.2X_5 + 10X_6 + 0.89X_7 + 30.33X_8 + 14.78X_{10} + 3.78X_{11} + 24.67X_{24} \leq 197634.10$$

$$0.2X_5 + 10X_6 + 0.89X_7 + 30.33X_8 + 14.78X_{10} + 3.78X_{11} + 24.67X_{24} \geq 11548.16$$

Licenses:

The constraint for licenses is made up of 17 decision variables of $X_{12}, X_{13}, X_{14}, X_{15}, X_{17},$ $X_{18}, X_{19},$ up to X_{30} excluding X_{22}, X_{23}, X_{24} and X_{27} and the R.H.S values of 149734.64 and 1982438.24 .

$$67.50X_{12}+67.50X_{13}+352.22X_{14}+\cdots+3.0X_{22}+20X_{25}+300X_{26}+0.28X_{28}+\cdots+48.33X_{30} \leq 1982438.24$$

$$67.50X_{12}+67.50X_{13}+352.22X_{14}+\cdots+3.0X_{22}+20X_{25}+300X_{26}+0.28X_{28}+\cdots+48.33X_{30} \geq 149734.64$$

Rent:

The constraint for the rent is obtained from a unit decision variable, X_{22} and with corresponding R.H.S values of 712.20 and 3281.00.

$$3X_{22} \leq 3281.00$$

$$3X_{22} \geq 712.20$$

4.4.3 Formulation of the Problem

$$\text{Maximize} \qquad Z = (0.1X_1 + 10X_2 + 10X_3 + 36.62X_4 + \cdots + 48.33X_{30})$$

$$\begin{aligned}
\textbf{Subject to:} \quad 0.10X_1 + 36.62X_2 + 10X_{27} &\leq 46980.53 \\
0.10X_1 + 36.62X_2 + 10X_{27} &\geq 10386.86 \\
10X_3 + 36.62X_4 + 450X_{23} &\leq 83877.46 \\
10X_3 + 36.62X_4 + 450X_{23} &\geq 20892.91 \\
0.2X_5 + 10X_6 + 0.89X_7 + 30.33X_8 + & \\
14.78X_{10} + 3.78X_{11} + 24.67X_{24} &\leq 197634.10 \\
0.2X_5 + 10X_6 + 0.89X_7 + 30.33X_8 + & \\
14.78X_{10} + 3.78X_{11} + 24.67X_{24} &\geq 11548.16
\end{aligned}$$

$$
\begin{aligned}
67.50X_{12} + 67.50X_{13} + 352.22X_{14} + \cdots + 3.0X_{22} + & \\
20X_{25} + 300X_{26} + 0.28X_{28} + \cdots + 48.33X_{30} &\leq 1982438.24 \\
67.50X_{12} + 67.50X_{13} + 352.22X_{14} + \cdots + 3.0X_{22} + & \\
20X_{25} + 300X_{26} + 0.28X_{28} + \cdots + 48.33X_{30} &\geq 149734.64 \\
3X_{22} &\leq 3281.00 \\
3X_{22} &\geq 712.20
\end{aligned}
$$

4.4.4 Problem Formulation involving the Slacks

Expressing the above inequalities in the equality form we have the following equations:

$$
\begin{aligned}
0.10X_1 + 36.62X_2 + 10X_{27} + s_1 &= 46980.53 \\
0.10X_1 + 36.62X_2 + 10X_{27} + s_2 &= 10386.86 \\
10X_3 + 36.62X_4 + 450X_{23} + s_3 &= 83877.46 \\
10X_3 + 36.62X_4 + 450X_{23} + s_4 &= 20892.91 \\
0.2X_5 + 10X_6 + 0.89X_7 + 30.33X_8 + & \\
14.78X_{10} + 3.78X_{11} + 24.67X_{24} + s_5 &= 197634.10 \\
0.2X_5 + 10X_6 + 0.89X_7 + 30.33X_8 + & \\
14.78X_{10} + 3.78X_{11} + 24.67X_{24} + s_6 &= 11548.16 \\
67.50X_{12} + 67.50X_{13} + 352.22X_{14} + \cdots + 3.0X_{22} + & \\
20X_{25} + 300X_{26} + 0.28X_{28} + \cdots + 48.33X_{30} + s_7 &= 1982438.24 \\
67.50X_{12} + 67.50X_{13} + 352.22X_{14} + \cdots + 3.0X_{22} + & \\
20X_{25} + 300X_{26} + 0.28X_{28} + \cdots + 48.33X_{30} + s_8 &= 149734.64 \\
3X_{22} + s_9 &= 3281.00 \\
3X_{22} + s_{10} &= 712.20
\end{aligned}
$$

$X_j, S_j \geq 0$ for $j = 1, 2, 3, \ldots 30$

4.4.5 Iterative Primal-Dual Interior-Point Algorithm

In summarizing the basic steps of the algorithm the following inputs are assumed:

Step 0: **(i)** The data of the problem (A, b, c), where the $m \times n$ matrix A has full row rank,

(ii) Initial primal and dual feasible solutions $X^0 > 0, Z^0 > 0, y^0 > 0$.

(iii) The optimality tolerance $\epsilon > 0$ and the step size parameter $\gamma \in (0, 1)$.

Step 1: (Initialization). Start with some feasible point $X^0 > 0, Z^0 > 0, y^0 > 0$. Choose (X^0, Z^0, y^0) such that $(X^0, Z^0, y^0) > 0$ and set the iteration counter $k = 0$.

Step 2: (Optimality test). If $(Z^k)^T X^k < \epsilon$ stop; otherwise, go to Step 3.

Step 3: (Compute Newton directions). Let

$$\begin{aligned} X^k &= \operatorname{diag}\{X_1^k, X_2^k, \ldots, X_n^k\} \\ Z^k &= \operatorname{diag}\{Z_1^k, Z_2^k, \ldots, Z_n^k\} \\ \mu^k &= \frac{(Z^k)^T X^k}{n^2} \end{aligned}$$

Solve the following linear system equivalent to step 5 in section 3.7.10 to get $d_x^k, d_\pi^k$and d_z^k

$$\begin{aligned} Ad_x &= 0 \\ A^T d_\pi - d_z &= 0 \\ Zd_x + Xd_z &= \mu e - XZe \end{aligned}$$

Note that $\delta_P = 0$ and $\delta_D = 0$ due to the feasibility of the initial point.

Step 4: (Find step lengths). Let

$$\alpha_P = \gamma \min_j \left\{ \frac{-x_j^k}{(d_x)_j^k} : (d_x)_j^k < 0 \right\} \text{ and } \alpha_D = \gamma \min_j \left\{ \frac{-z_j^k}{(d_z)_j^k} : (d_z)_j^k < 0 \right\}$$

Step 5 (Update solution). Take a step in the Newton direction to get

$$\begin{aligned} X^{k+1} &= X^k + \alpha_P (d_x)^k \\ y^{k+1} &= y^k + \alpha_P (d_y)^k \\ Z^{k+1} &= Z^k + \alpha_D (d_z)^k \end{aligned}$$

Put $k = k + 1$ and go to Step 2.

4.5 Computational Method

The coefficients of the tax functions, left-hand side constraint inequalities and right-hand side constants were written in matrices form. Matlab program software was used for coding the primal-dual algorithm.

The matrices were inputted in the Matlab program code and ran on AMD Athlon(tm)X2 DualCore QL-66 CPU 2.20GHz , 32-bit operating system, Windows7 HP laptop computer. The code ran successfully on ten trials with hundred iterations for each trial.

4.6 Result

The result below gives the primal solution and the dual solution. The $X_{(}j); j = 1, 2, 3 \ldots 30$ gives the total amount that each revenue item contributes in arriving at the optimal solution. After 10 successful trials with 24 iterations for each of them, an optimal value of $f = 1359357.28$ was achieved. The results of the final test run for the total revenue generated after 24 iterations are shown below:

4.7 Discussion

The data collected from the Sunyani West District Assembly which was used for this research work reveals that the average total annual revenue by the Assembly for the past three years has been GH¢893,608.52. A total of GH¢1358357.28 annually, based upon the primal-dual

x_1 to x_{10}	x_{11} to x_{20}	x_{21} to x_{30}
7948	752	2076
795	263	906
2462	263	55
672	0	115
1421	351	751
284	287	504
3195	255	795
94	5	2710
4976	2971	2047
192	1401	277

y	Value
y_1	8
y_2	7
y_3	5
y_4	7
y_5	2
y_6	3
y_7	0
y_8	2
y_9	2
y_{10}	8

algorithm code, would be obtained Hence with this research work, the Assembly can raise its revenue to GH¢1358357.28 annually which represents an appreciable 52.12% increase in the Assembly's revenue collection.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

The revenue data collected from the Sunyani West District Assembly was modeled into Linear Programming Problem. An optimal revenue mobilization strategy was then developed out of the Linear Programming Problem. The data was then run on a matlab software code. The analysis done in chapter four using primal-dual interior-point algorithm showed that average annual revenue generated by the Assembly between 2008 and 2011 was GH¢893,608.52. Based upon this research work, the Assembly can raise its revenue to GH¢1358357.28 which represents 52.12% increase in the Assembly's revenue. The results also revealed that the tax item which performed well was the Burial Fees, Self Employed Artisans, Kiosks and Financial Institutions and the tax item which performed badly was the Registration of Chainsaws, Revenue from Concession, Development Levy and Lotto Operators.

5.2 Recommendations

The Sunyani West District Assembly as aforementioned in the problem statement has not been performing well in revenue mobilization. This state of affairs has contributed immensely in the Assembly's inability in providing basic social amenities such as schools, hospitals, portable water, improved sanitation facilities etc. This research work has come at an opportune time and it is a sigh of relief for most of the indigenes in the Sunyani West District Assembly. This reason stems from the fact that internally generated revenue which has being the assembly's major headache can now be addressed by this research work. I hereby recommend the following results and findings of this thesis to the Sunyani West District Assembly:

1. The work should serve as basis for further research works in improving revenue mobiliza-

tion strategy by the Assembly and other District Assemblies in Ghana.

2. The research work also reveals that the contribution of basic rate showed a significant impact on the overall revenue generation, but many of the citizens' default in its payment. It is however recommended that this tax will be linked up with the national health insurance registration and renewal. This will take care of citizens who evade this tax, and will also widen the tax bracket. It is also recommended that the basic rate should be increased from its current form of GH¢0.10 to GH¢0.50 with attractive commission for the tax collectors.

3. The researcher is of the view that the assembly will benefit a lot by way of addressing revenue leakages if they can acquire automated tax collection machines for tax collection by the assembly.

4. The model did not include certain tax item due to the fact that Sunyani West District was one of the newly created District prior to when this research was conducted. Therefore the availability of certain tax items was a problem. We therefore recommend that further research is to be extended to include these items, since when included can help modelers predict the severity revenue leakages in the country.

The study was carried out on secondary data obtained from the Assembly spanning between 2010 and 2012. The focus of the subject area of the study was internally generated revenue of the district.

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Appendix A

Table 5.1: Table Showing Tax No. and the associated Revenue Sub-Head.

Tax (Item N0.)	Revenue Sub-Head
1	Basic Rate
2	Property Rate
3	Stool Land Revenue
4	Building Permit
5	Market Tools
6	Court Fines
7	Farm Produce
8	Marriage and Divorce
9	Toilet Management Revenue
10	Burial Fees
11	Lorry parks
12	Petroleum Products
13	General Goods
14	Financial Institutions
15	Kiosks
16	Chemical Sellers
17	Sale of Bid Documents
18	Adverts/Bill Boards
19	Lotto Operators
20	Self Employed Artisans
21	Clod Stores
22	Market Stores
23	Revenue from Concession
24	Registration of Building Plots
25	Registration of Business
26	Awards of Contracts
27	Development Levy
28	Hawkers
29	Traditional Caterers
30	Registration 0f Chainsaws

Table 4.1: Table Showing the (E.R) and the (A.R) generated by the assembly for the four quarters of 2010

TAX	Q1(E.R)	Q1(A.R)	Q2(E.R)	Q2(A.R)	Q3(E.R)	Q3(A.R)	Q4(E.R)	Q4(A.R)
1	8000.00	30.00	8000.00	46.00	8000.00	52.00	8000.00	52.00
2	21,000.00	15,000.00	20,000.00	17,000.00	20,000.00	20,000.00	21,000.00	22,000.00
3	35,000.00	0.00	35,000.00	7,000.00	53,000.00	10,000.00	35,000.00	20,000.00
4	4,050.00	400.00	4,050.00	647.00	4,050.00	1,272.00	4,050.00	1,410.00
5	2,420.00	1,540.00	2,420.00	4,890.00	2,242.00	6,880.00	2,420.00	6,880.00
6	400.00	220.00	400.00	340.00	400.00	380.00	400.00	410.00
7	2,270.00	810.00	2,270.00	1,110.00	2,270.00	2,140.00	2,270.00	2,800.00
8	500.00	0.00	500.00	0.00	500.00	0.00	500.00	0.00
9	1,000.00	90.00	1,000.00	100.00	1,000.00	100.00	1,000.00	100.00
10	1,500.00	350.00	1,500.00	1,690.00	1,500.00	4,080.00	1,500.00	4,900.00
11	550.00	80.00	550.00	2,080.00	550.00	5,090.00	550.00	5,506.00
12	100.00	180.00	100.00	510.00	100.00	530.00	100.00	550.00
13	190.00	0.00	190.10	0.00	190.00	200.10	190.00	200.10
14	40.00	22.00	40.00	240.00	40.00	530.00	40.00	574.00
15	500.00	0.00	500.00	590.00	500.00	592.00	500.00	590.00
16	100.00	46.00	100.00	78.00	100.00	410.00	100.00	640.00
17	600.00	0.00	600.00	0.00	600.00	370.00	600.00	1,275.00
18	52.00	0.00	52.00	0.00	52.00	0.00	52.00	0.00
19	950.00	0.00	950.00	0.00	950.00	0.00	950.00	0.00
20	2,500.00	106.00	2,500.00	250.50	2,500.00	305.00	2,500.00	447.00
21	35.60	0.00	35.60	54.00	35.60	78.00	35.60	147.00
22	150.00	60.00	150.00	80.00	150.00	250.00	150.00	300.00
23	300.00	0.00	300.00	0.00	300.00	0.00	300.00	0.00
24	200.00	0.00	200.00	0.00	200.00	0.00	200.00	0.00
25	50.00	0.00	50.00	0.00	50.00	0.00	50.00	0.00
26	1,500.00	0.00	1,500.00	0.00	1,500.00	0.00	1,500.00	0.00
27	0.00	0.00	0.00	.00	0.00	0.00	0.00	0.00
28	30.00	0.00	30.00	0.00	30.00	12.00	30.00	20.00
29	310.00	0.00	310.00	0.00	310.00	145.00	310.0	404.00
30	220.00	0.00	220.00	0.00	220.00	0.00	220.0	0.00

Appendix B

Table 4.2: Table showing the estimated revenue (E.R) and the actual revenue (A.R) generated by the assembly for the four quarters of 2011

TAX	Q1(E.R)	Q1(A.R)	Q2(E.R)	Q2(A.R)	Q3(E.R)	Q3(A.R)	Q4(E.R)	Q4(A.R)
1	8,200.00	190.00	8,200.00	3,406.00	8,200.00	4,152.00	8000.00	6,652.00
2	24,000.00	17,000.00	24,000.00	18,000.00	24,000.00	20,000.00	24,000.00	20,000.00
3	35,000.00	0.00	35,000.00	8000.00	35,000.00	11,000.00	35,000.00	21,000.00
4	4,050.00	500.00	4,050.00	747.00	4,050.00	1,472.00	4,050.00	1,610.00
5	3,420.00	1,540.40	3,420.00	4,890.00	3,420.00	7,880.00	3,420.00	8,880.00
6	400.00	220.00	400.00	340.00	400.00	380.00	400.00	410.00
7	2,570.00	910.00	2,570.00	1,310.00	2,570.00	2,340.00	2,570.00	2,800.00
8	500.00	0.00	500.00	0.00	500.00	0.00	500.00	
9	1,000.00	90.00	1,000.00	100.00	1,000.00	100.00	1,000.00	100.00
10	1,800.00	450.00	1,800.00	1,890.00	1,800.00	4,580.00	1,800.00	5,900.00
11	550.00	80.00	550.00	2,080.00	550.00	5,090.00	550.00	5,506.00
12	200.00	280.00	200.00	550.00	200.00	630.00	200.00	750.00
13	300.00	0.00	300.10	100.00	300.00	200.10	300.00	200.10
14	400.00	50.00	400.00	260.00	400.00	560.00	400.00	674.00
15	500.00	0.00	500.00	590.00	500.00	592.00	500.00	590.00
16	400.00	100.00	400.00	120.00	400.00	400.00	400.00	640.00
17	600.00	0.00	600.00	120.00	600.00	370.00	600.00	1,275.00
18	0.00	100.00	0.00	100.00	0.00	100.00	0.00	100.00
19	950.00	0.00	950.00	0.00	950.00	0.00	950.00	0.00
20	2,800.00	120.00	2,800.00	350.50	2,800.00	405.00	2,800.00	547.00
21	35.60	0.00	35.60	54.00	35.60	78.00	35.60	147.00
22	250.00	80.00	250.00	100.00	250.00	350.00	250.00	400.00
23	300.00	0.00	300.00	0.00	300.00	0.00	300.00	0.00
24	200.00	0.00	200.00	0.00	200.00	0.00	200.00	0.00
25	50.00	0.00	50.00	0.00	50.00	80.00	50.00	100.00
26	2,000.00	0.00	2,000.00	50.00	2,000.00	50.00	2,000.00	50.00
27	8,328.60	00.00	8,328.60	0.00	8,328.60	0.00	8,328.60	0.00
28	30.00	0.00	30.00	0.00	30.00	12.00	30.00	20.00
29	510.00	80.00	510.00	100.00	510.00	145.00	510.0	404.00
30	400.00	0.00	400.00	0.00	400.00	0.00	400.00	0.00

Appendix C

Table 4.3: Table showing the estimated revenue (E.R) and the actual revenue (A.R) generated by the assembly for the four quarters of 2012

TAX	Q1(E.R)	Q1(A.R)	Q2E.R	Q2A.R	Q3E.R	Q3A.R	Q4E.R	Q4A.R
1	10,000.00	2000.00	10,000.00	3000.00	10,000.00	4000.00	10,000.00	5000.00
2	25,000.00	17,000.00	25,000.00	18,000.00	25,000.00	20,000.00	25,000.00	22,000.00
3	40,000.00	5000.00	40,000.00	7000.00	40,000.00	10,000.00	40,000.00	20,000.00
4	10,050.00	600.00	10,050.00	1,247.00	10,050.00	1,372.00	10,050.00	1,410.00
5	4,420.00	1,540.40	4,420.00	4,890.00	4,420.00	6,880.00	4,420.00	6,880.00
6	500.00	220.00	500.00	340.00	500.00	380.00	500.00	410.00
7	4,270.00	910.00	4,270.00	1,310.00	4,270.00	2,240.00	4,270.00	3,800.00
8	1000.00	60.00	1000.00	80.00	1000.00	150.00	1000.00	180.00
9	1,000.00	150.00	1,000.00	400.00	1,000.00	500.00	1,000.00	600.00
10	4,500.00	1,700.00	4,500.00	1,800.00	4,500.00	4,080.00	4,500.00	4,900.00
11	850.00	100.00	850.00	2,880.00	850.00	6,090.00	850.00	6,506.00
12	500.00	220.00	500.00	510.00	500.00	530.00	500.00	550.00
13	4000.00	100.00	400.10	120.00	4000.00	300.10	400.00	400.10
14	440.00	22.00	440.00	240.00	440.00	530.00	440.00	574.00
15	500.00	100.00	500.00	590.00	500.00	592.00	500.00	590.00
16	550.00	120.00	550.00	150.00	550.00	410.00	550.00	640.00
17	800.00	150.00	800.00	200.00	800.00	370.00	800.00	1,275.00
18	150.00	0.00	150.00	0.00	150.00	40.00	150.00	60.00
19	1,000.00	0.00	1,000.00	0.00	1,000.00	0.00	1,000.00	0.00
20	3,500.00	220.00	3,500.00	1,250.50	3,500.00	1,305.00	3,500.00	2,447.00
21	35.60	0.00	35.60	54.00	35.60	78.00	35.60	147.00
22	350.00	80.00	350.00	100.00	350.00	350.00	350.00	400.00
23	300.00	0.00	300.00	0.00	300.00	0.00	300.00	0.00
24	200.00	0.00	200.00	0.00	200.00	0.00	200.00	0.00
25	100.00	0.00	100.00	40.00	100.00	60.00	100.00	140.00
26	2,000.00	150.00	2,000.00	200.00	2,000.00	200.00	2,000.00	240.00
27	8,328.60	00.00	8,328.60	0.00	8,328.60	0.00	8,328.60	0.00
2	800.00	0.00	800.00	80.00	800.00	90.00	800.00	220.00
29	510.00	0.00	510.00	100.00	510.00	145.00	510.0	404.00
30	500.00	0.00	500.00	0.00	500.00	0.00	500.00	0.00

Appendix D

Table 4.4: Table showing the average E.R and A.R generated by the assembly for the past 12 quarters of the assembly.

TAX NO. ITEM	REVENUE SUB-HEAD	E.R (AVERAGE)	A.R (AVERAGE)
1	Basic Rate	8,733.33	2,331.66
2	Property Rate	22,000.00	18750.00
3	Stool land Revenue	36,666.66	9916.66
4	Building Permit	6,050.00	1057.25
5	Market Tolls	3,420.00	5297.56
6	Court Fines	433.33	337.50
7	Farm Produce	3036.66	1873.33
8	Marriage and Divorce	666.67	39.16
9	Toilet Management Revenue	1000.00	202.5
10	Burial Fees	2600.00	3,026.66
11	Lorry parks	650.00	3424.00
12	Petroleum Products	266.67	482.50
13	General Goods	296.66	151.75
14	Financial Institutions	293.33	356.00
15	Kiosks	500.00	549.77
16	Chemical Sellers	350.00	313.00
17	Sale of Bid Documents	666.66	440.41
18	Adverts/Bill Boards	100.66	8.33
19	Lotto Operators	966.66	0.00
20	Self Employed Artisans	2933.33	646.08
21	Clod Stores	47.46	69.75
22	Market Stores	250.00	212.50
23	Revenue from Concession	300.00	0.00
24	Registration of Building Plots	200.00	0.00
25	Registration of Business	66.67	35.00
26	Awards of Contracts	1833.33	78.33
27	Development Levy	5552.4	0.00
28	Hawkers	286.66	37.83
29	Traditional Caterers	443.33	160.58
30	Registration of Chainsaws	373.33	0.00

Appendix E

Table 4.5: Table showing the number of people paying for each category of tax as a variable $X_j, j = 1, 2, 3, \cdots 30$

DECISION VARIABLE (X_i)	REVENUE SUB-HEAD
x_1	Basic Rate
x_2	Property Rate
x_3	Stool land Revenue
x_4	Building Permit
x_5	Market Tolls
x_6	Court Fines
x_7	Farm Produce
x_8	Marriage and Divorce
x_9	Toilet Management Revenue
x_{10}	Burial Fees
x_{11}	Lorry parks
x_{12}	Petroleum Products
x_{13}	General Goods
x_{14}	Financial Institutions
x_{15}	Kiosks
x_{16}	Chemical Sellers
x_{17}	Sale of Bid Documents
x_{18}	Adverts/Bill Boards
x_{19}	Lotto Operators
x_{20}	Self Employed Artisans
x_{21}	Clod Stores
x_{22}	Market Stores
x_{23}	Revenue from Concession
x_{24}	Registration of Building Plots
x_{25}	Registration of Business
x_{26}	Awards of Contracts
x_{27}	Development Levy
x_{28}	Hawkers
x_{29}	Traditional Caterers
x_{30}	Registration of Chainsaws

Appendix F

Table 4.6: Table representing the tax items and the unit charges used to model the L.P for the problem. Tax Payer's Variable x_i and Unit Charge c_i for $i = 1, 2, \ldots 30$

TAX NO.	REVENUE HEAD	REVENUE SUB-HEAD	(x_i)	(c_i)	AV. A.R FOR 12 QUARTERS	RATIO OF A.R TO (C_i)
1	Rates	Basic Rate	x_1	0.10	2,331.66	233.16
2	Rates	Property Rate	x_2	10.00	18750.00	18,7500
3	Lands	Stool land Revenue	x_3	10.00	9916.66	99,166.6
4	Lands	Building Permit	x_4	35.62	1057.25	37,659.24
5	Fees and Fines	Market Tolls	x_5	0.20	5297.56	1,059.51
6	Fees and Fines	Court Fines	x_6	10.00	337.50	3370.50
7	Fees and Fines	Farm Produce	x_7	0.89	1873.33	1667.26
8	Fees and Fines	Marriage and Divorce	x_8	30.33	39.16	1,187.72
9	Toilet Management Revenue	Toilet Management Revenue	x_9	17.50	202.5	3543.75
10	Fees and Fines	Burial Fees	x_{10}	14.78	3,026.66	4,4734.03
11	Fees and Fines	Lorry parks	x_{11}	3.78	3424.00	12,942.72
12	Licenses	Petroleum Products	x_{12}	67.50	482.50	32,568.75
13	Licenses	General Goods	x_{13}	67.50	151.75	10,243.12
14	Licenses	Financial Institutions	x_{14}	352.22	356.00	125,390.32
15	Licenses	Kiosks	x_{15}	3.94	549.77	2,166.09
16	Licenses	Chemical Sellers	x_{16}	25.00	313.00	7,825
17	Licenses	Sale of Bid Documents	x_{17}	76.67	440.41	33766.23
18	Licenses	Adverts/ Bill Boards	x_{18}	33.13	8.33	275.97
19	Licenses	Lotto Operators	x_{19}	25.50	0.00	0.00
20	Licenses	Self Employed Artisans	x_{20}	4.73	646.08	3055.95
21	Licenses	Clod Stores	x_{21}	9.83	69.75	685.64
22	Rent	Market Stores/ Stalls	x_{22}	3.00	212.50	637.5
23	Lands	Revenue from Concession	x_{23}	450.00	0.00	0.00
24	Fees and Fines	Registration of Building Plots	x_{24}	21.67	0.00	0.00
25	Licenses	Registration of Business	x_{25}	20.00	35.00	700.00
26	Licenses	Awards of Contracts	x_{26}	300.00	78.33	23499.00
27	Rates	Development Levy	x_{27}	10.00	0.00	0.00
28	Licenses	Hawkers	x_{28}	0.28	37.83	10.59
29	Licenses	Traditional Caterers	x_{29}	9.67	160.58	1552.80
30	Licenses	Registration of Chainsaws	x_{30}	48.33	0.00	0.00

Appendix G

Number of iteration run for the optimal value.

iter 1: $mu = 3.52e + 014$, resid $= 7.42e + 009$

iter 2: $mu = 6.47e + 013$, resid $= 1.79e + 009$

iter 3: $mu = 5.12e + 012$, resid $= 1.20e + 008$

iter 4: $mu = 3.28e + 011$, resid $= 7.78e + 006$

iter 5: $mu = 2.85e + 010$, resid $= 6.82e + 005$

iter 6: $mu = 3.58e + 009$, resid $= 1.09e + 005$

iter 7: $mu = 7.09e + 008$, resid $= 1.92e + 004$

iter 8: $mu = 9.59e + 007$, resid $= 2.39e + 003$

iter 9: $mu = 2.71e + 007$, $resid = 6.52e + 002$

iter 10: $mu = 5.27e + 006$, resid $= 1.30e + 002$

iter 11: $mu = 1.58e + 006$, resid $= 3.92e + 001$

iter 12: $mu = 1.58e + 005$, resid $= 3.86e + 000$

iter 13: $mu = 2.98e + 004$, resid $= 8.12e - 001$

iter 14: $mu = 3.00e + 003$, resid $= 9.71e - 002$

iter 15: $mu = 3.06e + 002$, resid $= 6.72e - 003$

iter 16: $mu = 1.17e + 002$, resid $= 2.64e - 003$

iter 17: $mu = 1.17e + 001$, resid $= 2.51e - 004$

iter 18: $mu = 1.18e + 000$, resid $= 2.51e - 005$

iter 19: $mu = 1.18e - 001$, resid $= 2.52e - 006$

iter 20: $mu = 1.19e - 002$, resid $= 2.53e - 007$

iter 21: $mu = 1.19e - 003$, resid $= 2.53e - 008$

iter 22: $mu = 1.20e - 004$, resid $= 2.54e - 009$

iter 23: $mu = 1.45e - 006$, resid $= 3.08e - 011$

iter 24: $mu = 2.12e - 010$, resid $= 4.51e - 015$

Appendix H

Matlab Code for the Algorithm

```
function [x,y,s,f] = pdip(A,b,c)

% primal-dual interior-point method for problem

%

% min c'x s.t. Ax=b, x¿=0,

%

% whose dual is

%

% max b'y s.t. A'y+s=c, s¿=0.

%

% calling sequence:

%

% [x,y,s,f] = pdip(A,b,c)

%

% input: A is an m x n SPARSE constraint matrix.

% b is an m x 1 right-hand side vector

% c is an n x 1 cost vector.

%

% output: x is the n x 1 solution of the primal problem

% y is the m x 1 dual solution

% s is the n x 1 vector of "dual slack"

% f is the optimal objective value

if margin  = 3

error('must have three input arguments');

end


if  issparse(A)
```

```
error('first input argument A must be a SPARSE matrix; possibly use sparse() to convert');
end

t0=cputime;
[m,n] = size(A);
if m ¡= 0 or n ¡= 0
error('input matrix A must be nontrivial');
end

if n  = length(c)
error('size of vector p must match number of columns in A');
end
if m  = length(b)
error('size of vector b must match number of rows in A');
end

% set initial point, based on largest element in (A,b,c)
bigM = max(max(abs(A)));
bigM = max([norm(b,inf), norm(p,inf), bigM]);
x = 100*bigM*ones(n,1); s = x; y = zeros(m,1);

    % find row/column ordering that gives a sparse Cholesky % factorization of ADA'
ordering = symmmd(A*A');
bp = 1+max([norm(b), norm(c)]);

for iter=1:100

% compute residuals
Rd = A'*y+s-c;
```

```
Rc = A*x-b;
Rp = x.*s;
mu = mean(Rp);
relResidual = norm([Rd;Rc;Rp])/bp;
% fprintf('iter %2i: mu = %9.2e, resid = %9.2e n', iter, mu, relResidual);
fprintf('iter %2i: mu = %9.2e, resid = %9.2e n', iter, full(mu), ...
full(relResidual));
if(relResidual ¡= 1.e-7 & mu ¡= 1.e-7) break; end;
Rp = Rp - min(0.1,100*mu)*mu;

    % set up the scaling matrix, and form the coefficient matrix for
% the linear system
d = min(5.e+15, x./s);
B = A*sparse(1:n,1:n,d)*A'; % use the form of the Cholesky routine "cholinc" thatś best
% suited to interior-point methods
R = cholinc(B(ordering,ordering),'inf');

% set up the right-hand side
t1 = x.*Rd-Rp;
t2 = -(Rc+A*(t1./s));

% solve it and recover the other step components
dy = zeros(m,1);
dy(ordering) = R  (R'2̂(ordering));
dx = (x.*(A'*dy)+t1)./s;
ds = -(s.*dx+Rp)./x;
tau = max(.9995,1-mu);
ac= -1/min(min(dx./x),-1);
ad = -1/min(min(ds./s),-1);
```

```
ac = tau*ac;
ad = tau*ap;
x = x + ac*dx;
s = s + ad*ds;
y = y + ad*dy;
end

f = c'*x;

% convert x,y,s to full data structures
x=full(x); s=full(s); y=full(y);

fprintf('Done! t[m n] = [%g %g] tCPU = %g n', m, n, cputime-t0);
return;
```