An SEIR Mathematical Model for Dog Rabies. Case Study: Bongo District, Ghana.

by

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A Thesis submitted to Department of Mathematics

Kwame Nkrumah University of Science and

Technology

in partial fulfilment of the requirements for the degree

of

Master of Philosophy

College of Science

June 2012

Declaration

I hereby declare that this submission is my own work towards the Master of Philosophy (MPhil.) and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgement has been made in the text.

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Abstract

A SEIR model for rabies between dogs with vaccination effect is formulated. The basic reproduction ratio for this model is derived using the Next Generation Matrix Method. Graphical solutions of the differential equations are produced using Matlab. Stability analysis is performed and the impact of vaccination is analysed. This thesis was written in LaTeX and MATLAB was used for the programming. Appendix contains the Matlab code used in simulating the model.

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Acknowledgements

I am particularly grateful to my two Supervisors, Prof. A. Aidoo and Dr. E. Osei-Frimpong, for guiding and directing the success of this thesis, and I am grateful for his support. It is important to recognize also, Dr. P. Amoako-Yirenkyi for his help and guidance. I am grateful to the Department of Mathematics, Kwame Nkrumah University of Science and Technology, for a part well played in my Education.

Many thanks to Kwame Addo Danso, Anastasia Amissah, George Lamptey Parker, I thank you all for the love you have shown me through my entire studies.

Lastly, to my family for seeing me through to the end of my studies, to mum and dad for their love and support each day. It has been the grace and mercies of God throughout my entire education and I am humbled by that.

Chapter 1

INTRODUCTION

1.1 A Mathematical Model

A mathematical model is a simplified or idealized description of a system or process in mathematical terms, devised to facilitate calculation and prediction. As a schematic description of a system, process or phenomenon, a mathematical model accounts for the known or inferred properties of the system and may be used to further study its characteristics. The usefulness of a model lies in the fact that it allows for the understanding and prediction of a phenomena without the work of performing the complex and expensive experiments (*Allman and John, 2004*).

Understanding the dynamics of disease transmission is essential to addressing them. Mathematical modeling plays an important role in providing this understanding. Once a model that captures the main features of the progression and transmission of a particular disease in a population has been formulated, it can be used to predict the effects of different strategies for disease eradication or control. The world wide eradication of small pox, through a carefully developed vaccination campaign initiated by the World Health Organization in 1967, is a remarkable example of what can be achieved with a well-designed plan (*Allman and John, 2004*).

Infectious disease modeling, though often inexact, has enormous potential to help improve human lives as models aids in the understanding and prediction of the phenomena (*Brauer et al., 2008*).

1.1.1 History and Myths about Rabies

One of the most dreadful diseases that could be transmitted from animals to man is rabies. Rabies was recognized since 2003 BC in ancient Babylonian, early Greek and Egyptian civilizations times. Aristotle, a renowned Greek philosopher first identified rabies as a disease caused by the bite from a rabid dog. It was widely believed at that time that the disease was as a result of eating hot food, the fear of water or the lack of it. It was also attributed sometimes to severe hot weather condition or nervous excitement. Those who believe in stars and planets had influence on life on earth also attributed rabies to the influence of a star which they named DOG STAR.

The first written record of rabies was in the Mesopotamian Codex of Eshnunna which dictated that the owner of a dog showing symptoms of rabies should take preventive measure against bites. Fear of rabies related to methods of transmissions was almost irrational; however, this gave Louis Pasteur ample opportunity to test post-exposure treatments from 1885.

In 1903, a researcher called Negri discovered certain strange particles in the brain cell of a dead rabid animal. The particles were found in the cytoplasm of nerve cells. Today these are referred to as NEGRI BODIES. The presence of Negri bodies in nerve cells has become the criteria for diagnosing rabies worldwide.

It is endemic in most African and Asian countries except for a few countries that have eradicated or remained free of rabies due to their natural protection as islands

or peninsula and by enforcing rigorous quarantine routines e.g. Ireland, Australia, Japan, Fiji, Korea, New Zealand, Sweden, Finland, Hawaii, Singapore etc. It is estimated to cause about 55000 deaths worldwide annually, 56% of these deaths occur in Asia and 44% in Africa (mostly rural areas). 30-50% of reported deaths occur in children below 15 years and an estimated 10 million people receive post-exposure treatment worldwide.

1.1.2 What is Rabies?

Rabies is a Latin word which means madness, fury or rage. It is sometimes called hydrophobia because one of the symptoms of rabies is an inability by the infected individual to swallow water. It is also known in some local Ghanaian languages as Babaso in Akan, Bayinyaa in Dagbani, and Kakru-chuchoo in the Kasena. Rabies is caused by a virus known as rhabdovirus or Rabies virus (RABV). Rabies, which is a neuroinvasive disease, causes acute encephalitis (inflammation of the brain) in mammals. It is mostly caused by a bite from an infected animal but occasionally by other form of contact. It is a natural disease of dogs, cats, bats, raccoons, foxes, skunks, wolves and other warm blooded organisms (*T. Nuertey, 2007; S. Johnson, 2007*).

1.1.3 Mode of Transmission of Rabies

Any warm-blooded animal (including humans) may become infected with the rabies virus and develop symptoms. The virus is transmitted to the victim when virus-laden saliva is introduced through a bite or a scratch from a rabid animal. The virus may be recovered from the Central Nervous System and salivary glands as well as most tissues of infected animals. During this phase, the virus cannot be easily detected within the host, and vaccination may still confer cell-mediated immunity to prevent

symptomatic rabies. In nature it is transmitted from animal to animal by means of a bite introducing the saliva bearing the virus. Rarely, rabies may be transmitted by viral contamination of fresh, already existing wounds. Virus may be present in the saliva and be transmitted by an infected animal several days prior to the onset of clinical signs. Research indicates that the rabies virus can also enter mucous membranes such as those lining the nose and eyes, also people and other mammals can develop rabies after breathing air in caves that house large number of bats which may carry the rabies virus.

The motor pathway to the spinal cord is the primary pathway of spread. Once the rabies virus infects the spinal cord, neuron dessemination proceeds quickly throughout the central nervous system. The brain infection results in behavioural changes, likely due to infection of neurons in limbis areas and this facilitates transmission by biting in rabies vectors. Subsequently, there is spread of the rabies virus away from the central nervous system along neuronal pathways particularly involing the parasympathetic nervous system to multiple organs including the heart, gastrointestinal tract, adrenal medulla, skin and salivary glands. Salivary infection is important in rabies because the rabies virus is secreted in high titer in the saliva which allows transmission to additional hosts by biting. Once the patient becomes symptomatic, treatment is almost never effective and mortality is over 99% Rabies may also inflame the spinal cord, producing transverse myelitis.

RABV progresses through four stages upon introduction into the mammal. These stages are described briefly as;

1.1.3.1 The Incubation period

The period from the time from infection to the onset of symptoms is known as the rabies incubation period. The rabies virus makes its way to the brain from the site of infection by following the peripheral nerves. During the rabies incubation period,

a bite by the infected animal does not carry a risk of rabies because the virus is not yet in the saliva. Only late in the disease, after the rabies virus has reached the brain and multiplied there to cause encephalitis, does the virus move from the brain to the salivary glands and saliva. The incubation period of the disease depends on how far the virus must travel to reach the Central Nervous System. The incubation period varies but generally within 3 to 7 weeks. During this time, the rabies virus is multiplying within the body. The incubation periods usually depends on these factors:

1. Wound severity
2. Wound site in relation to nerve supply
3. Distance from brain
4. Amount and strain of virus
5. Protection provided by clothing
6. Immune status

1.1.3.2 Prodromal stage

At the beginning, there is change of behavior in animal which often may be slight and thus overlooked. It includes shyness, nervousness, difficulty in swallowing and sometimes salivation. There is frequent irritation or stimulation of the urogenital tracts as evidenced by frequent urination, erection in the male and sexual desire. The animal may also stop eating and drinking and seek solitude. The prodromal period last for 1 to 3 days.

1.1.3.3 Excitative stage

This stage which is also known as raging fury or mad-dog syndrome stage is characterized by irrational and vicious aggressiveness, restlessness, excitement and mania

for biting and snapping. The facial expression is one of alertness and anxiety with pupils dilated. Noise invites attack. There is loss of caution and fear of natural enemies. They roam streets and highways biting other animals, people,any moving object and swallowing of foreign bodies etc. During this stage the saliva is highly infectious. As the disease progresses muscular incoordination and convulsive seizures become common that is, the muscles of the body and legs begin to tremble making it unable to walk steadily and breathing becomes very difficult.

1.1.3.4 Dumb or paralytic stage

This appears shortly before death. It includes paralysis of the muscles of the throat, face, trunk and the limbs. There is profuse salivation and inability to swallow, dropping of the lower jaw, rarely attempt or are able to bite. Animals with dumb rabies appear depressed, lethargic and uncoordinated. Gradually they become completely paralysed. If paralysis is prominent, this stage is also called silent fury. Paralysis progresses to all parts of the body with coma and death in a few hours.

1.1.4 Signs and symptoms of Rabies

If the virus enters the spinal cord it induces paralysis whereas if it enters the limbic system it induces transient aggression. Among the first symptoms are pains, burning or numbness at the site of the infection. Soon after, the symptoms expand to slight or partial paralysis, cerebral dysfunction, anxiety, insomnia, confusion, agitation, abnormal behavior, paranoia, terror, hallucinations, progressing to delirium.

The production of large quantities of saliva and tears coupled with an inability to speak or swallow are typical during the later stages of the disease; this can result in hydrophobia, in which the dog has difficulty swallowing because the throat and jaw become slowly paralyzed, shows panic when presented with liquids to drink, and cannot quench his or her thirst. In the final stage, the patient begins to have periods of mania and lethargy and coma. Deaths generally occur due to respiratory

insufficiency. An intention to treat analysis has since found that this protocol has a survival rate of about 8%.

1.1.5 Treatment

The first step in treating a person bitten by any animal is to wash the wound with soap and water. Dangerous as it is, the rabies virus also happens to be one of the most delicate organisms known. It dies in dried saliva within a few hours. It is also killed by ordinary sunlight, heat, household detergent and disinfectants. Pure iodine and hydrogen peroxide, however have no effect on the virus. The animal should either be caged and watched for signs of rabies or killed and its brain tissues watched for signs of rabies.

Because there is no cure and death is almost certain when the symptoms begin to show up, treatment for rabies involves supportive care. However, if a dog or a person is bitten by a rabid animal and has not yet experienced symptoms, there is an extremely effective post-exposure treatment. Most of the time, stitches should not be used for animal bite wounds. There are vaccines that are derived from a variety of tissue culture or chicken embryo origins in live or inactivated forms which are used for treating rabies. Some of these require revaccination,others protect adequately for three years.

1.1.5.1 Rabies vaccine

Rabies research scientists have developed an extremely effective rabies vaccine regimen that provides protection against rabies. This vaccine works in two ways; either after an exposure or for before an exposure. A person who becomes infected with rabies and does not obtain treatment before the symptoms occur, dies in a short period after experiencing convulsions and other violent nervous symptoms. Dogs continue to be the main carrier of rabies in Africa and Asia and are responsible for most of the human rabies deaths worldwide.

1.1.5.2 Pre and Post-Exposure vaccine:

Pre-exposure rabies vaccines are available for dogs, cats, ferrets, horses, sheep, and all other mammals. To be effective, these rabies vaccines must be injected before an animal is exposed to rabies. Although pre-exposure vaccination does not eliminate the need for additional medical attention after a rabies exposure, it simplifies therapy and decreases the number of rabies vaccine doses needed. Secondly, it may enhance immunity in dogs whose post-exposure rabies treatment might be delayed. Finally, it may provide protection to dogs with unapparent exposures to rabies.

If exposed, the dog should get a booster shot. Post-exposure treatment for rabies should begin as soon as possible after an exposure. Administration of rabies vaccine is a medical urgency, not a medical emergency. Post-exposure rabies treatment consists of a regimen of one dose of rabies immune globulin and five doses of rabies vaccine given over a 28-day period. Rabies immune globulin and the first dose of rabies vaccine should be given as soon as possible after exposure.

Rabies immune globulin contains antibodies from blood donors who were given rabies vaccine. The antibodies provide interim protection until the exposed mammal's own antibodies develop in response to the vaccine. In addition, injecting rabies immune globulin at the site of injury reduces the amount of virus that is able to enter the nerve cells and potentially initiate an active infection. The rabies vaccine works by stimulating a mammal's immune system to produce antibodies that neutralize the virus. The mammal develops a protective immune response before the virus reaches the brain and begins to actively replicate.
Possible side effects of the rabies vaccine can include Low-grade fever, Pain, redness, swelling, or itching at the injection site, Headache, Nausea, Abdominal pain, Muscle aches, Dizziness.

1.1.6 Statement of the problem

Although elimination of human rabies transmission from dog-to-dog rabies cycle has been accomplished in most parts of the world, it still exists in some large geographical areas especially in Africa and Asia. World Health Organisation (WHO) statistics indicate that the 55,000 human deaths recorded around the world annually is caused by rabies. Out of this number, children are the most affected victims, according to the reports. Rabies has a human mortality of 100% once symptoms of the disease develop.

Dog rabies is estimated to cause 24,000 human deaths per year in Africa, while Africa is the second continent mostly affected by the disease. However, this estimate is still considered to be conservative. 30 to 60% of dog bite victims in dog-endemic areas are children less than 15 years of age. Unfortunately, the majority of these cases go unreported to parents or health officials.

The viral rabies disease has been within the dog population of Ghana for decades, with the domestic dog *(Canis familiaris)* being the principal vector. In the first 6 months of 1975, canine cases almost doubled over the period average (*Belcher et al., 1976*). Dogs are the most important reservoir for the rabid virus and have been the source of transmitting it to about 99% of all reported human cases. In Ghana, 25 human rabies deaths were recorded between January 2009 and July 2011.

Currently in Ghana, there is an upsurge of rabies among dogs and humans after the free anti-rabies immunisation campaign funded by the Ministry of Food and Agriculture (MOFA) and implemented by the Veterinary Services Department ended in 1998. Rabies is well established in Accra and Bongo in the Upper East Region and there has been no decline in canine or human cases during the past 5 years.

While Ghana recorded 144 rabies deaths countrywide between 1986 and 2003 due

to dog bites, Greater Accra region alone recorded 2,620 dog bites between 2003 and 2008. Out of these bites, 232 which represent 8.9% were detected positive for rabies. From January to August 2009, out of 428 dog bites, 53 were positive. The Bongo hospital has recorded about 101 suspected rabies cases of which 47 have been treated and discharged with five people losing their lives.In December 2011, 19 dog bites were recorded.

In 1993 when Ghana had anti-rabies vaccination campaign there were only five outbreaks as against 41 in 1994 when there was no mass vaccination. From 1998 to 2006 there has not been any mass vaccination the outbreak cases rose to 108. In 2002, 56 rabies positive cases were diagnosed in the laboratories, 2003 (61 cases), 2004 (72 cases), 2005 (78 cases), 2006 (84 cases). The increase in the incidence of rabies in several parts of Ghana lately has been blamed on the unwillingness of pet owners to vaccinate their animals and the continuous presence of too many stray pets whose owners cannot be identified.

Previously, control methods including dog vaccination and stray dog removal have been intermittent and not sustained. Unfortunately, as in several other developing African countries, the patronage of rabies vaccination within the Ghanaian veterinary services is worryingly on the low. Also, unavailability of the vaccine has compelled hospitals and pharmaceutical outlets to sell them at exorbitant prices. Those who are unable to afford it are left to their fate as they are turned away by helpless health officers who can do little about the situation.

The Veterinary Central Laboratory in Accra diagnoses 10 positive cases of rabies in dogs monthly. Globally, about 60,000 people especially, children die of rabies annually, hence the need to address the problem, which seems to be neglected. According to records from the Bongo District Hospital at Bongo in December 2011, dog rabies

is a major disease amongst dogs in the district.

1.1.7 Objectives of the thesis

The main objectives of the study are.

1. To formulate a time-dependent mathematical model that will mimic the behavior of the spread of rabies and simulate the model.
2. To find out the mode of transmission of rabies.
3. To determine the effect of vaccination on the spread of the Rabies disease.

1.1.8 Methodology

The mathematical model will be formulated using differential equations. The computer software Matlab 7.8.0 (R20009a) will be used to simulate the model. The resources to be used are the KNUST school library and the internet.

1.1.9 Justification of the thesis

This thesis will contribute to the research information on Rabies in the country, so that it can help in further work in the further research work in this area.
The thesis seeks to predict whether or not the measures put in place so far to check the spread of Rabies is enough or more still needs to be done in order to prevent it from becoming endemic.

1.1.10 Organization of the thesis

The thesis is organised as follows: Chapter one presents the biological background of the thesis, statement problem, the objectives of the thesis, the methhodology that will

be used for the thesis, thesis justification and organization of the thesis. Chapter two examines the previous work related to the thesis. Chapter three is about the methodology. Chapter four is the discussion of results and analysis of the model. Finally, Chapter five contains the conclusions drawn from the model and reccomendations.

Chapter 2

REVIEW OF RELATED LITERATURE

2.1 Introduction

Mathematical models associated with the study of rabies in various countries have existed over the years. However, mathematical models have not been used to study the spread of the disease in Ghana.

Early models of rabies dynamics followed the SEIR framework where populations were subdivided into specific classes corresponding to susceptible (S), exposed (E), infectious (I), and removed (R) individuals (*Anderson et al., 1991*). The dynamics were encapsulated through the construction of a system of ordinary differencial equations (ODEs) representing either single populatioins or linked metapopulations from which a variety of predictions can be drawn concerning temporal and spatial pattern.

These early models made use of the basic SEIR compartmental framework and these models were used to derive several critical features of disease emergence and spread. The models were used to calculate the critical threshold for epidemic emergence and the basic reproductive number (R_0) for the virus. When R_0 is greater than

1, the infection will spread and an epidermic will result. Using R_0, it is possible to suggest what level of population culling would be necessary in order to bring threshold density below epizootic level.

Although the construction of their model followed the SEIR compartmental framework, they failed to include the R class since there was evidence of natural recovery or development of natural immunity and vaccination which translates susceptibles into the removed category was not considered by then.

Translating the dynamics protrayed in the flow chart into the following set of ODEs gave them;

$$dS/dt = rS - \gamma SN - \beta SI \tag{2.1}$$

$$dE/dt \quad \beta SI - (\sigma + B + \gamma N)E \tag{2.2}$$

$$dI/dt = \sigma E - (\alpha + B + \gamma N) \tag{2.3}$$

$$N = S + E + I \tag{2.4}$$

where S, E, and I represented densities of susceptible hosts, exposed, and infectious individuals respectively. $r = a - b$, was indicated as the intrinsic per capita growth is with a being the per capita birth rate and 1/b, the mean life expectancy. The rate at which individuals were exposed(E) in the population is proportional to the densities of susceptibles and infectious individual, βSI. Here β is the disease transmission parameter and the average length time a fox remains in the exposed class before becoming infectious is $1/\sigma$.

Anderson et al., 1981 utilized the available estimates from then recent descriptive studies (*Macdonald et al., 1981*). The situation they considered in their models was the introduction of a few rabid foxes into a native population. In order to determine R_0 and the corresponding minimum density of foxes (S_t) necessary for rabies to

spread, they also assumed that the host population prior to the introduction of rabies was at a stable equilibrium which was represented as $K = a/b$. So at the onset of the epidermic, at time t=0, the population size of susceptibles is then $S(t = 0) = K$. They determined that the criteria for epidermic ($dI/dt > 0$), for the equilibrium population size K, at the onset of the first infections is $K > S_t$, where $S_t = (\sigma + a)(\alpha + a)/\beta\sigma$ and the relationship between K and S_t can be reformulated to define R_0:

$$R_0 = \frac{K}{S_t} = \frac{K\beta\alpha}{(\sigma+a)(\alpha+a)}.$$

Based on the available data, *Anderson et al., 1981* determined that the minimum threshold density was S_t 0.99 foxes/km^2. Subsequent to their analysis, it was confirmed that almost all areas of Europe that had seen outbreaks had densities in excess of this mumber. Oral vaccines for rabies had not yet been developed, so the reccommended control stractegy was culling of fox populations in areas with densities above the threshold,S_t.

At the same time of when this model by *Anderson et al., 1981* was developed, fox rabies was continuing to advance southwesternly into France and Switzerland. Descriptive studies then begun to investigate ecological factors that could influence the spatial propagation of virus, such as habit quality or fox densities (*Macdonald et al., 1981;D. MacDonald, 1980*). Following these studies, *Murray et al., 1986* developed a reaction-diffusion model to describe the behaviour of this propagating wave. This model allowed predictive modelling of how a transmission barrier might be implemented at the wave front in order to halt the expansion of the epizootic.

From the work of *Anderson et al., 1981*, a minimum density for preventing epizootic within a population had already been determined. From a practical standpoint, implementation of such large-scale culling or vaccine distribution across Europe ahead

of the wave front would not be possible. However, the model developed by *Murray et al., 1986* allowed for the estimation of movement rates for rabid foxes. It was now possible to suggest how wide and where a break could be implemented in order to halt the spatial propagation of the epidemic.

The framework of the reaction diffusion fromulation used by Murray et al. consisted of coupled partial differncial equations (PDEs) which were one-dimensional reaction diffusion framework identical to the model of (*Anderson et al., 1981*).Their implement density dependence was in terms of an environmental carrying capacity κ, rather than the parameter γ which determined the strength of density dependence. The reaction diffusion famework included a diffusion term which described the movement of infectious foxes across the landscape. It was estimated that the rate of movement of rabid foxes was D $50km^2$/year (*Andral et al., 1982; Murray et al., 1986*). An epidermic wave propagating at a velocity ν in a homogenous environment will maintain the same shape as it traverses space.

Mathematically this allows us to consider a solution in the form $f(x,t) = f(x-\nu t)$ which can give a nontrivial solution as it is important to solve all solutions. Some solutions may describe unrealistic biological scenarios, whereas others may describe the oscillations of standing waves that occur after a significant time has passed.

Smith et al.,2002 developed an interactive network model that incorporated local heterogenities in an attempt to better understand the irregular spread of rabies wave front across Connecticut in the early 1990s. The model used by *Smith et al.,2002* considered the landscape as a network of connected townships where habitat differences among townships could be approximated as variation in local transmission rates between neighboring townships($\lambda_{i,j}$) and global transmission among all townships($\mu_{i,j}$).

The parameters μ_i and $\lambda_{i,j}$ were fixed throughout the course of any simulation, but some degree of stochasticity was implemented since the order in which townships were chosen was based on a uniform random distribution. *Smith et al.,2002* showed convincingly that landscape heterogeneity could help explain the irregular spread of the raccon rabies virus across Connecticut, something which reaction diffusion frameworks had difficulty achieving.

A stochastic spatial model developed by *Smith et al.,2002* described the spread of rabies in Connecticut. Predicting the Spatial Dynamics of Rabies Epidemics on Heterogeneous Landscapes. Results from this model suggested that rivers act as a semipermeable barrier to the spread of rabies resulting in a seven fold reduction in the rate of spread. Analysis of the Connecticut data again, the influence of habitat and long-distance translocation events were used to assess the role of long-distance translocation and spatial heterogeneity in the raccoon rabies epidemic in Connecticut. The results of the reanalysis suggested that rivers interact to further reduce the spatial spread of raccoon rabies (*Smith et al.,2002*).

The stochastic spatial model was used by *Russell et al.* to analyse data from Ohio. Members of this team later authored another paper using an ODE model to show that the spread of rabies may be controlled by distributing vaccine behind barriers such as rivers. This SIR model included the three classes in nine spatial compartments giving a total of 27 ODEs. Results showed that a higher rate of vaccination is needed for a large population and a lower rate with a higher cost.

Optimal control has been recently applied to an epidemic model for rabies in raccoons using an SIR metapopulation model. Space is included through subpopulation arrangement connected by movement. The optimal control vector gives the rate of vaccination in each subpopulation that minimizes the infected class over all subpop-

ulations, accounting as well for the cost of administering the vaccine (*asano et al., 2008*).

Taking into account the actual situation of rabies spreading in China, *Zhang et al., 2011* formulated two mathematical models to study both the spreading dynamics of rabies in dogs and human, and the control strategies. They compared the efficiency of three strategies for controlling rabies: culling, vaccination, culling and vaccination and found that vaccination is the best choice to control rabies. Hong-tao et al established mathematical model of rabies with similar controlling strategies in China. This result emphasis to people infected by exposed dogs, infected dogs and seemingly healthy dogs carrying the virus. Their mathematical analysis and simulation indicated the culling strategy of is the most efficient, vaccination is the intermediate and culling and vaccination is the last effective (*Zhang et al., 2011*).

According to *Zhang et al., 2011*, human rabies is one of the major public-health problems in China. They came out with a model in order to explore effective control and prevention measures we propose a deterministic model to study the transmission dynamics of rabies in China. The model consists of susceptible, exposed, infectious, and recovered subpopulations of both dogs and humans and describes the spread of rabies among dogs and from infectious dogs to humans.

The model simulations agree with the human rabies data reported by the Chinese Ministry of Health. They estimated that the basic reproduction number $R_0 = 2$ for the rabies transmission in China and predict that the number of the human rabies is decreasing but may reach another peak around 2030. They also perform some sensitivity analysis of R_0 in terms of the model parameters and compared the effects of culling and immunization of dogs. Their study demonstrated that reducing dog birth rate and increasing dog immunization coverage rate are the most effective methods for

controlling rabies in China and large scale culling of susceptible dogs can be replaced by immunization of them.

A combined approach is cheaper only when the per capita cost of vaccination is less then 20% of the per capita cost of culling. *Voigt et al., 1985* discussed the global incidence of the disease and list the main animal carriers in the world scene. During the past few hundred years, Europe has been repeatedly subjected to rabies epidermics. It is not known why rabies died out some 50 years or so years before the current epidermic started. The analysis of the models here, however, will provide a possible scenario. The incidence of rabies in man, at least in Europe and America is now rare with only very few deaths a year, but with considerably more in underdeveloped countries. It is particularly horrifying disease for which there is no known cause of a recovery once the disease has reached clinical stage (*Murray et al., 1986; J.D. Murray, 1989*).

The optimization criterion is to minimize the number of infected raccoons while minimizing the cost of distributing the vaccine. Using an optimal control setting, numerical results illustrate strategies for distributing the vaccine depending on the timing of the infection outbreak with respect to the birth pulse (*Tim et al., 2010*). The model takes explicit account of the development of natural immunity to rabies and was used to evaluate culling and vaccination elimination strategies.

For habitats typical of the mid-Atlantic states, and given the assumptions of the model, it was estimated that elimination of rabies in raccoons by culling may involve the annual removal of over 32% of the raccoon population or the yearly vaccination of up to 99% of the susceptible fraction. Assuming a constant marginal cost for both culling and vaccination, the model suggests that, whatever the actual cost of each method, the cheapest strategy will always involve either culling or vaccination alone.

A combined strategy of culling and vaccination will be cheaper than culling alone only when the per capita cost of vaccination is around one-fifth or less the per capita cost of culling.

The models above are other peoples work and do not apply to Ghana. I therefore want to introduce my model which is not as complex as the ones above and also applies to Bongo District in Ghana.

Chapter 3

MATHEMATICAL MODEL

3.1 Introduction

In this chapter, we represent mathematical models that mimics the prominent aspects of epidemiology of rabies in the Bongo district in Ghana. These models will assist in predicting the spread of the disease in the district. Two variations of the standard Susceptible-Exposed-Infected-Removed (SEIR) epidemiological model are utilized to study and analyse the disease.
These are the simple SEIR model to explain the spread of the rabies in Bongo district, followed by the modeling rabies with vaccination in the district.

Bongo District is one of the nine districts in the Upper East Region, with Bongo Township as its district capital. It shares boundaries with Burkina Faso to the north and east, Kassena-Nankana District to the West and Bolgatanga Municipal in the south. The total area is 459 square kilometers. The predominant occupation is subsistence farming along with some handicraft production. In 2000 the population was estimated at 77,885. Making a population density of 1.83 per square kilometers. Below is the map of the Bongo District

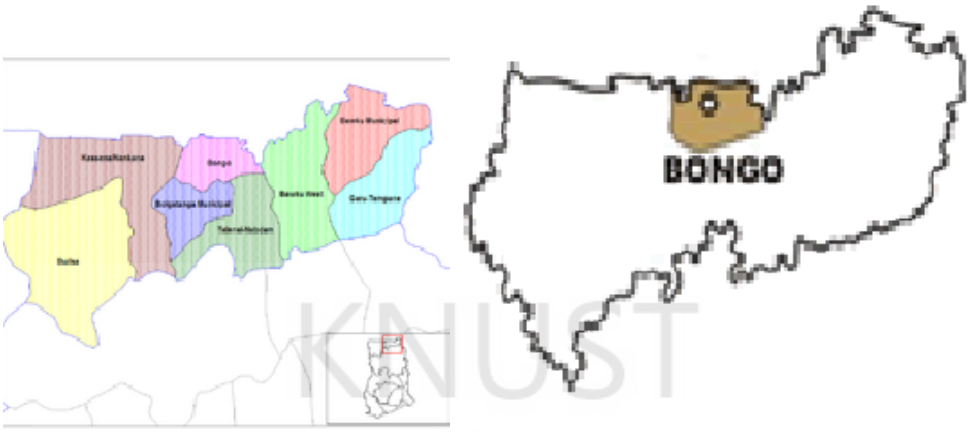

Figure 3.1: Map of Bongo District within the map of Upper East Region of Ghana.

The consumption of the dog meat has gone up in recent times in the district with residents now desiring for it during their leisure times. Dogs suspected to carry rabies among the people are killed and eaten including the heads that are usually that part that is used to examine the presence of rabies. Dog census conducted put the number in the Bongo District at 8,217. The district was choosen for study because it is one of the districts where dog rabies is considered endemic.

3.1.1 Description of SEIR model of Rabies without vaccination

In a standard SEIR model, the population is divided into four compartments. These are the susceptible class(S) which refers to the healthy dogs that have not yet caught the rabies virus but are likely to contract the disease. Dogs that have been bitten by infected dogs but are not infectious make up the exposed class(E). Dogs that are infected with rabies virus and are contagious make up the infective class(I). The removed class(R) constitute dogs which have died from the infection. The proportions of individuals in the compartments S, E, I, R, at time t, is denoted as S(t), E(t), I(t) and R(t) respectively. The flow chart in Fig 3.3 is the flow chart of SEIR model.

Figure 3.2: Flow chart for SEIR model without vaccination

Here,

β = Transmission coefficient between dogs

λ = Latency(incubation) rate in dogs

γ = Death rate in dogs

3.1.2 Model Asumption

1. The dogs mix homogenously. This happens because the dog owners walk or ride freely on their bicycles within the district to their farms so the dogs interact with each other and also for shorter distances between 0-5 miles, dogs run around free-range style on their own for mating and other purposes. Because of the

territorial nature of dogs, they always engage in fights as a new dog enters another's territory. It is therefore easy for an infective to pass the disease on to a susceptible dog.

2. All infected dogs die because dogs showing symptoms of sickness are clubbed to death and their meat are used as meal.

3. Age, sex and type of the dog coupled with the climatic conditions in the district does not affect the probability of a dog being infected.

4. The disease spread in a closed environment; that is there is no emigration or immigration, and there is neither birth nor death in the population, so the total population of dogs in the district remains a constant(N) for all t; that is $S(t) + E(t) + I(t) + R(t) = N$.

   If we let $s(t) = \frac{S(t)}{N}, e(t) = \frac{E(t)}{N}, i(t) = \frac{I(t)}{N}, r(t) = \frac{R(t)}{N}$, then $s(t) + e(t) + i(t) + r(t) = 1$, where s(t), e(t), i(t) and r(t) are susceptible, exposed, infected and recovered fractions of the population respectively.

The rate at which the susceptible class changes is equal to the rate at which infection takes place. Infection occurs when the disease is passed from an infective dog to a susceptible dog. The number of susceptible-infective contacts is proportional to the product of S(t) and I(t). Of these contacts, a proportion will catch the disease. Therefore, the rate of change in the susceptible population of dogs is given by:

$$dS/dt = -\beta SI$$

Where βSI is the force of infection. This term is negative because during infection, the numbers of susceptible dogs decrease. Members of the susceptible class who become exposed to the virus increase the number in the exposed class. The rate at which dogs leave the susceptible class is equal to the rate at which they join the

exposed class. We will assume that the numbers of dogs leaving the exposed class(E) for infective classes are some proportion of the exposed class size. If we let λE denote the proportion of dogs leaving the exposed class(E) for the Infective class(I), the rate of change of the exposed class(E) will be given by:

$$dE/dt = \beta SI \ + \lambda E$$

Assuming the number of dogs leaving the exposed class for the infective class is some proportion of the exposed class size. If we let λE denote the proportion of dogs leaving the exposed class for the infected class and γI to represent the disease induced mortality of the infected dogs, the rate of change of the infective class will be given by:

$$dI/dt = \lambda E \ - \gamma I$$

We also need to consider the number of dogs who die from Rabies in our analysis. R increases at a rate proportional to the proportion of infected dogs that die from the disease. Therefore, the rate of change in the removed population is given by:

$$dR/dt = \gamma I$$

3.2 Model Equations

$$dS/dt = -\beta SI \tag{3.1}$$

$$dE/dt = \beta SI - \lambda E \tag{3.2}$$

$$dI/dt = \lambda E - \gamma I \tag{3.3}$$

$$dR/dt = \gamma I \tag{3.4}$$

3.2.1 SEIR model of Rabies transmission with vaccination

3.2.1.1 Model Assumptions

1. Here, the R compartment constitute dogs which have recovered from the infection upon administration of the rabies vaccine. This we call the Recovered class(R).

2. A portion αS of the susceptibles go to the recovered class(R) directly due to pre-exposure vaccination.

3. A portion αE of the exposed go to the recovered class(R) directly due to post-exposure vaccination.

4. A portion κR of the recovered go to the susceptible class(S) directly due to the waning immunity of the rabies vaccine.

5. All infective dogs die so there is no chance that they will progress to the recovered class(R).

6. The birth rate of dogs is equal to their death rate so the population under consideration is closed.

Here,

$$\begin{aligned}
\delta &= \text{Birth rate of dogs} \\
\beta &= \text{Transmission coefficient between dogs} \\
\lambda &= \text{Latency(incubation) rate in dogs} \\
\mu &= \text{Death rate in dogs} \\
\alpha &= \text{Vaccination rate coefficient} \\
\epsilon &= \text{Disease induced mortality of dogs} \\
\kappa &= \text{Waning immunity in dogs}
\end{aligned}$$

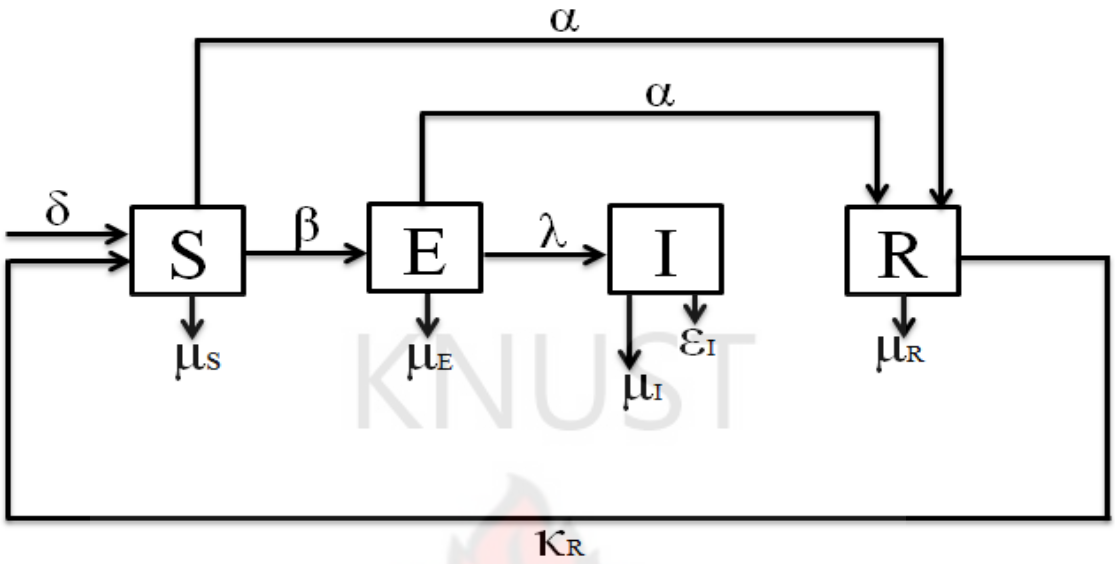

Figure 3.3: Flow chart for SEIR model with vaccination

The model equations are given as follows

$$\frac{dS}{dt} = \delta + \kappa R - \beta SI - \alpha S - \mu S \tag{3.5}$$

$$\frac{dE}{dt} = \beta SI - \alpha E - \mu E - \nu E \tag{3.6}$$

$$\frac{dI}{dt} = \nu E - \mu I - \epsilon I \tag{3.7}$$

$$\frac{dR}{dt} = \alpha S + \alpha E - \mu R - \kappa R \tag{3.8}$$

Here, we also have $S + E + I + R = N$.

3.2.2 Basic Reproductive Ratio(R_0) of Rabies Transmission Without Vaccination Using The Next Generation Matrix Approach

The basic reproductive number, R_0, is defined as the expected number of secondary cases produced by a single infection in a completely susceptible population. R_0 is a dimensionless number. If more than one secondary infection is produced from one primary infection, that is, $Ro > 1$, then an epidemic occurs. When $Ro < 1$, then there is no epidemic, meaning the disease dies out. When $Ro = 1$, then the disease becomes endemic, meaning the disease remains in the population at a constant rate, as one infected dog transmits the disease to one susceptible (*H. W. Hethcote, 2006*).

We can calculate our R_0 using the Next Generation Matrix Approach. The Next Generation Matrix comprises two matrices F and V. The elements in matrix F constitute the new infections that will arise, while that of matrix V constitute the transfer of infections from one compartment to another. R_0 here is a dorminant eigen value of the matrix $G = FV^{-1}$

Linearizing about the disease-free equilibrium, re-ordering the states E, I, S, R and separating new infections F from other transitions V. We get

$$\begin{array}{rcll} \frac{dE}{dt} & = & \beta\left(\frac{S}{N}\right)I - \nu E & ..... \quad A(E,I,S,R) \\ \frac{dI}{dt} & = & \nu E - \gamma I & ..... \quad B(E,I,S,R) \\ \frac{dS}{dt} & = & -\beta\left(\frac{S}{N}\right)I - \nu E & ..... \quad C(E,I,S,R) \\ \frac{dR}{dt} & = & \gamma I & ..... \quad D(E,I,S,R) \end{array} \tag{3.9}$$

Linearization of the SEIR model gives the Generation matrix (G) evaluated at

the Disease Free Equilibrium.

$$\text{G}=\begin{bmatrix} A_E & A_I & A_S & A_R \\ B_E & B_I & B_S & B_R \\ C_E & C_I & C_S & C_R \\ D_E & D_I & D_S & D_R \end{bmatrix}$$

According to the Next Generation Matrix Approach, the G-matrix above can be divided into four 2x2 submatrices. Elements in the top left submatrix is said to be F-V, the upper right submatrix is always a zero matrix, Elements in the lower left submatrix gives us J_1 and the lower right submatrix is termed J_2.

$$\text{G} \quad =\begin{bmatrix} F-V & 0 \\ J_1 & J_2 \end{bmatrix}$$

From equation (3.11) above we get our Generation matix as

$$\text{G} \quad =\begin{bmatrix} -\nu & \beta & 0 & 0 \\ \nu & -\gamma & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & \gamma & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
F - V &= \begin{bmatrix} -\nu & \beta \\ \nu & -\gamma \end{bmatrix} \\
&= \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \nu & 0 \\ -\nu & \gamma \end{bmatrix} \\
F &= \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \quad and \ V = \begin{bmatrix} \nu & 0 \\ -\nu & \gamma \end{bmatrix}
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
V^{-1} &= \frac{1}{\nu\gamma} \begin{bmatrix} \gamma & 0 \\ \nu & \nu \end{bmatrix} \\
V^{-1} &= \begin{bmatrix} \frac{1}{\nu} & 0 \\ \frac{1}{\gamma} & \frac{1}{\gamma} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
FV^{-1} &= \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\nu} & 0 \\ \frac{1}{\gamma} & \frac{1}{\gamma} \end{bmatrix} \\
FV^{-1} &= \begin{bmatrix} \frac{\beta}{\gamma} & 0 \\ 0 & 0 \end{bmatrix} \\
R_0 &= \frac{\beta}{\gamma}
\end{aligned} \tag{3.11}
$$

$$If\ \ \beta > \gamma,\ \text{then}\ \ R_0 > 1$$

$$If\ \ \gamma < \beta,\ \text{then}\ \ R_0 < 1$$

3.2.3 Basic reproductive ratio(R_0) of Rabies transmission with vaccination

Using the Next Generation Matrix Approach, we re-order the states E, I, S, R for the model equations for rabies transmission with vaccination and linearize the model equations to get

$$\begin{aligned}
\frac{dE}{dt} &= \beta\left(\frac{S}{N}\right)I - (\alpha+\mu+\nu)E \quad ..... \quad A(E,I,S,R)\\
\frac{dI}{dt} &= \nu E - (\mu+\epsilon)I \quad ..... \quad B(E,I,S,R)\\
\frac{dS}{dt} &= \delta + \kappa R - \beta\left(\frac{S}{N}\right)I - (\alpha+\mu)S \quad ..... \quad C(E,I,S,R)\\
\frac{dR}{dt} &= \alpha S + \alpha E - (\mu+\kappa)R \quad ..... \quad D(E,I,S,R)
\end{aligned}$$

This gives us the Generation matrix

$$\mathrm{G} = \begin{bmatrix}
-\alpha-\mu-\nu & \beta & 0 & 0\\
\nu & -\mu-\epsilon & 0 & 0\\
0 & -\beta & -\alpha-\mu & \kappa\\
\alpha & 0 & \alpha & -\mu-\kappa
\end{bmatrix}$$

$$
\begin{aligned}
F - V &= \begin{bmatrix} -\alpha-\mu-\nu & \beta \\ \nu & -\mu-\epsilon \end{bmatrix} \\
&= \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \alpha+\mu+\nu & 0 \\ -\nu & -\mu+\epsilon \end{bmatrix} \\
F &= \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \\
V &= \begin{bmatrix} \alpha+\mu+\nu & 0 \\ -\nu & -\mu+\epsilon \end{bmatrix} \\
V^{-1} &= \frac{1}{(\alpha+\mu+\nu)(\mu+\epsilon)} \begin{bmatrix} \mu+\epsilon & 0 \\ \nu & \alpha+\mu+\nu \end{bmatrix} \qquad (3.12) \\
V^{-1} &= \begin{bmatrix} \frac{1}{\alpha+\mu+\nu} & 0 \\ \frac{\nu}{\alpha+\mu+\nu} & \frac{1}{\mu+\epsilon} \end{bmatrix} \\
FV^{-1} &= \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha+\mu+\nu} & 0 \\ \frac{\nu}{(\alpha+\mu+\nu)(\mu+\epsilon)} & \frac{1}{\mu+\epsilon} \end{bmatrix} \\
FV^{-1} &= \begin{bmatrix} \frac{\beta\nu}{(\alpha+\mu+\nu)(\mu+\epsilon)} & \frac{\beta}{\mu+\epsilon} \\ 0 & 0 \end{bmatrix} \\
R_0 &= \frac{\beta\nu}{(\alpha+\mu+\nu)(\mu+\epsilon)}
\end{aligned}
$$

3.3 Equilbrium points of Rabies Model with vaccination

In order to determine the stability of the model, we first evaluate the equilibrium points or steady states of the ordinary differential equations (3.7), (3.8), (3.9) and (3.10). The points to be determined are disease-free (where $I = 0$) and endemic (where $I \neq 0$). We set the right hand side of equations (3.7), (3.8), (3.9) and (3.10) to zero and solve for the values of S, E, I and R. At the steady state, $dS/dt = 0$, $dE/dt = 0$, $dI/dt = 0$, $dR/dt = 0$

This makes

$$\delta + \kappa R - \beta SI - \alpha S - \mu S = 0 \tag{3.13}$$

$$\beta SI - \alpha E - \mu E - \nu E = 0 \tag{3.14}$$

$$\nu E - \mu I - \epsilon I = 0 \tag{3.15}$$

$$\alpha S + \alpha E - \mu R - \kappa R = 0 \tag{3.16}$$

From (3.17),

$$I = \left(\frac{\nu}{\mu + \epsilon}\right) E \tag{3.17}$$

From equation(3.16),

$$\begin{aligned} \beta SI &= (\alpha + \mu + \nu)E \\ SI &= \left(\frac{\alpha + \mu + \nu}{\beta}\right) E \\ \Rightarrow S &= \frac{\alpha + \mu + \nu}{\beta I} \end{aligned}$$

insert I into S above

$$S = \left(\frac{\alpha + \mu + \nu}{\beta}\right) \times \left(\frac{\mu + \epsilon}{\nu}\right)$$

$$\Rightarrow S = \frac{(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} \tag{3.18}$$

From (3.18),

$$R = \frac{\alpha S + \alpha E}{\mu+\kappa}$$

Insert (3.20) into R above

$$R = \frac{\left(\frac{\alpha(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu}\right) + \alpha E}{\mu+\kappa}$$

$$R = \frac{\alpha(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu(\mu+\kappa)} + \frac{\alpha E}{\mu+\kappa} \tag{3.19}$$

we know $\beta SI = (\alpha+\nu+\mu)E$ so we then insert (3.20) into (3.15)

$$\Rightarrow \delta + \kappa R - (\alpha+\mu+\nu)E - \frac{(\alpha+\mu)(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} = 0$$

$$\begin{aligned}(\alpha+\mu+\nu)E - \kappa R &= \delta - \frac{(\alpha+\mu)(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} \\ &= \frac{\delta\beta\nu - (\alpha+\mu)(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu}\end{aligned}$$

Insert (3.21) into the above equation and grouping like terms, we get

$$(\alpha+\mu+\nu)E - \frac{\alpha E}{\mu+\kappa} = \frac{\delta\beta\nu - (\alpha+\mu)(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} + \kappa\frac{\alpha(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu}$$

$$\begin{aligned}((\mu+\kappa)(\alpha+\mu+\nu) - \alpha)\,E &= \delta + (\kappa\alpha - \mu - \alpha)\left(\frac{(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu}\right) \\ &= \delta - \frac{(\alpha(\kappa-1)+\mu)(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} \\ &= \frac{\delta\beta\nu - (\alpha(\kappa-1)+\mu)(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu}\end{aligned}$$

$$E = \frac{\delta\beta\nu - (\alpha(\kappa-1)+\mu)(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu(\alpha+\mu+\nu)(\mu+\epsilon) - \alpha\beta\nu} \tag{3.20}$$

From (3.19)

$$I = \left(\frac{\nu}{\mu+\epsilon}\right) E$$

$$I = \frac{\delta\beta\nu^2 - (\alpha\nu(\kappa-1)+\mu\nu)(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\beta\nu(\alpha+\mu+\nu)(\mu+\epsilon)-\alpha\beta\nu)} \tag{3.21}$$

Insert E into (3.21)

$$R = \frac{\alpha(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\kappa)(\beta\nu)} + \frac{\alpha}{\mu+\kappa}\left(\frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu(\alpha+\mu+\nu)(\mu+\epsilon)-\alpha\beta\nu}\right)$$

$$R = \frac{1}{(\mu+\kappa)(\beta\nu)}\left(\alpha(\alpha+\mu+\nu)(\mu+\epsilon) + \frac{\alpha\delta\beta\nu}{(\mu+\kappa)(\alpha+\mu+\nu)-\alpha} - \frac{\mu\alpha(\alpha+\mu+\nu)(\mu+\epsilon)}{(\alpha+\mu+\nu)(\mu+\kappa)}\right)$$

$$R = \frac{1}{(\mu+\kappa)(\beta\nu)}\left(\alpha(\mu+\epsilon)(\alpha+\mu+\nu-\frac{\mu}{\mu+\kappa}) + \frac{\alpha\delta\beta\mu}{(\mu+\kappa)(\alpha+\mu+\nu)-\alpha}\right)$$

$$R = \frac{1}{(\mu+\kappa)(\beta\nu)}\left(\alpha(\mu+\epsilon)(\frac{(\alpha+\mu+\nu)(\mu+\kappa)-\mu}{\mu+\kappa}) + \frac{\alpha\delta\beta\mu}{(\mu+\kappa)(\alpha+\mu+\nu)-\alpha}\right)$$

$$R = \frac{\alpha\mu + \alpha\epsilon(\alpha+\mu+\nu)(\mu+\kappa)-\mu}{(\mu+\kappa)\beta\nu} + \frac{\alpha\delta\beta\nu}{\beta\nu(\mu+\kappa)(\alpha+\mu+\nu)-\alpha(\mu+\kappa)\beta\nu}$$

$$R = \frac{1}{\mu+\kappa}\left(\frac{\alpha(\mu+\epsilon)(\alpha+\mu+\nu)-\mu}{\beta\nu(\mu+\kappa)} + \frac{\alpha\delta}{(\mu+\kappa)(\alpha+\mu+\nu)-\alpha}\right) \tag{3.22}$$

The equilibrium point is given as $(S^*, E^*, I^*, R^*) =$

$$\left(\tfrac{(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu}, \tfrac{\delta\beta\nu-\mu(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu(\alpha+\mu+\nu)(\mu+\epsilon)-\alpha\beta\nu}, \tfrac{\delta\beta\nu-\mu\nu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\beta\nu(\alpha+\mu+\nu)(\mu+\epsilon)-\alpha\beta\nu)}, \tfrac{1}{\mu+\kappa}\left(\tfrac{\alpha(\mu+\epsilon)(\alpha+\mu+\nu)-\mu}{\beta\nu(\mu+\kappa)} + \tfrac{\alpha\delta}{(\mu+\kappa)(\alpha+\mu+\nu)-\alpha}\right)\right)$$

3.4 The Disease-Free Equilibrium point of Rabies model with vaccination

At the disease-free equilibrium, we consider the case where there is no infection. Since there are no new infections, we let $E = 0$ and $I = 0$. Putting $dS/dt = 0$ and $dR/dt = 0$, we solve for the values of S and R.

$dS/dt = 0$

$$\begin{aligned} \delta + \kappa R - \beta SI - (\alpha + \mu)S &= 0 \\ \delta + \kappa R - \beta S(0) - (\alpha + \mu)S &= 0 \\ \delta + \kappa R - (\alpha + \mu)S &= 0 \\ \kappa R - (\alpha + \mu)S &= -\delta \\ -(\alpha + \mu)S + \kappa R &= -\delta \end{aligned}$$

$$(\alpha + \mu)S - \kappa R = \delta \tag{3.23}$$

$dR/dt = 0$

$$\alpha S + \alpha E - (\mu + \kappa)R = 0$$

$$\alpha S + \alpha E(0) - (\mu + \kappa R) = 0$$

$$\alpha S - (\mu + \kappa)R = 0 \tag{3.24}$$

We proceed to solve (3.25) and (3.26) simulteneously. From (3.25),

$$S = \left(\frac{\mu + \kappa}{\alpha}\right) R \tag{3.25}$$

insert (3.26) into (3.24)

$$\begin{aligned}(\alpha+\mu)\left(\frac{\mu+\kappa}{\alpha}\right)R-\kappa R &= \delta \\ (\alpha+\mu)(\mu+\kappa)R-\alpha\kappa R &= \alpha\delta \\ R\left(\alpha+\mu)(\mu+\kappa)\right)-\alpha\kappa) &= \alpha\delta\end{aligned}$$

$$R = \frac{\alpha\delta}{\mu^2+(\alpha+\kappa)\mu} \tag{3.26}$$

insert (3.27) into (3.26)

$$\begin{aligned}S &= \left(\frac{\mu+\kappa}{\alpha}\right)\left(\frac{\alpha\delta}{(\mu^2+(\alpha+\kappa)\mu)}\right) \\ S &= \left(\frac{(\mu+\kappa)\delta}{(\mu^2+(\alpha+\kappa)\mu)}\right)\end{aligned}$$

At Disease-Free Equilibrium,

$$(S,E,I,R)=\left(\frac{(\mu+\kappa)\delta}{(\mu^2+(\alpha+\kappa)\mu)},\ 0\ ,\ 0\ ,\frac{\alpha\delta}{(\mu^2+(\alpha+\kappa)\mu)}\right)$$

3.5 Stability analysis of Disease-Free Equilibrium point of Rabies transmission with vaccination

To determine the stability of the system at the disease-free equilibrium, we will consider the linearized system of equations below about the equilibrium point.

$$\begin{aligned}
\frac{dS}{dt} &= \delta + \kappa R - \beta\left(\frac{S}{N}\right) I - (\alpha+\mu)S \\
\frac{dE}{dt} &= \beta\left(\frac{S}{N}\right) I - (\alpha+\mu+\nu)E \\
\frac{dI}{dt} &= \nu E - (\mu+\epsilon)I \\
\frac{dR}{dt} &= \alpha S + \alpha E - (\mu+\kappa)R
\end{aligned}$$

The Jacobian is therefore given by

$$J = \begin{bmatrix} -\alpha-\mu-\beta I & 0 & -\beta S & \kappa \\ \beta I & -\alpha-\mu-\nu & \beta S & 0 \\ 0 & \nu & -\mu-\epsilon & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa \end{bmatrix}$$

Since S=1 and I=0 at disease-free equilibrium, the Jacobian matrix becomes

$$J = \begin{bmatrix} -\alpha-\mu & 0 & -\beta & \kappa \\ 0 & -\alpha-\mu-\nu & \beta & 0 \\ 0 & \nu & -\mu-\epsilon & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa \end{bmatrix}$$

$$J - \lambda I = \begin{bmatrix} -\alpha-\mu & 0 & -\beta & \kappa \\ 0 & -\alpha-\mu-\nu & \beta & 0 \\ 0 & \nu & -\mu-\epsilon & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$J - \lambda I = \begin{bmatrix} -\alpha-\mu-\lambda & 0 & -\beta & \kappa \\ 0 & -\alpha-\mu-\nu-\lambda & \beta & 0 \\ 0 & \nu & -\mu-\epsilon-\lambda & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa-\lambda \end{bmatrix}$$

To find the characteristic equation of the matrix, we set the determinant of J-λI to zero.

$$det(J - \lambda I) = \begin{bmatrix} -\alpha-\mu-\lambda & 0 & -\beta & \kappa \\ 0 & -\alpha-\mu-\nu-\lambda & \beta & 0 \\ 0 & \nu & -\mu-\epsilon-\lambda & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa-\lambda \end{bmatrix} = 0$$

To compute the determinant of the above matrix, we divide the above matrix into four 3X3 matrices and find their determinants.

$$d_1 = \begin{bmatrix} -\alpha-\mu-\nu-\lambda & \beta & 0 \\ \nu & -\mu-\epsilon-\lambda & 0 \\ \alpha & 0 & -\mu-\kappa-\lambda \end{bmatrix} \quad d2 = \begin{bmatrix} 0 & \beta & 0 \\ 0 & -\mu-\epsilon-\lambda & 0 \\ \alpha & 0 & -\mu-\kappa-\lambda \end{bmatrix}$$

$$d_3 = \begin{bmatrix} 0 & -\alpha-\mu-\nu-\lambda & 0 \\ 0 & \nu & 0 \\ \alpha & \alpha & -\mu-\kappa-\lambda \end{bmatrix} \quad d_4 = \begin{bmatrix} 0 & -\alpha-\mu-\nu-\lambda & \beta \\ 0 & \nu & -\mu-\epsilon-\lambda \\ \alpha & \alpha & 0 \end{bmatrix}$$

$$\begin{aligned} det(J-\lambda I) &= (-\alpha-\mu-\lambda)\times d_1 - (0\times d_2) - (\beta \times d_3) + (\kappa \times d_4) \\ &= (-\alpha-\mu-\lambda)\times d_1 + (\kappa \times d_4) \end{aligned}$$

Solving for the determinant of the four submatices above, we get

$$\begin{aligned} d_1 &= ((-\alpha-\mu-\nu-\lambda)(-\mu-\epsilon-\lambda)(-\mu-\kappa-\lambda) - \beta\nu(-\mu-\kappa-\lambda)) \\ d_2 &= 0 \\ d_3 &= 0 \\ d_4 &= (\alpha+\mu+\nu+\lambda)\alpha(-\mu-\epsilon-\kappa) + \beta\alpha\nu \end{aligned}$$

We proceed by inserting the values of d_1, d_2, d_3 and d_4 into the above formula for finding the determinant of a 4×4 matrix to get $(-\alpha-\mu-\lambda)\times d_1 = (-\alpha-\mu-\lambda)\,((-\alpha-\mu-\nu-\lambda)(-\mu-\epsilon-\lambda)(-\mu-\kappa-\lambda) - \beta\nu(-\mu-\kappa-\lambda))$

$$\kappa \times d_4 = \kappa \times (\alpha+\mu+\nu+\lambda)\alpha(\mu-\epsilon-\kappa) - \kappa\beta\alpha\nu$$

Adding the two equations above, we get
$(-\alpha-\mu-\lambda)((-\alpha-\mu-\nu-\lambda)(-\mu-\epsilon-\lambda)(-\mu-\kappa-\lambda) - \beta\lambda(-\mu-\kappa-\lambda)) + (\alpha\kappa \times (\alpha+\mu+\nu+\lambda)\alpha(\mu+\epsilon+\kappa) - \beta\alpha\lambda)$

We now factorize $(-\mu-\epsilon-\lambda)$ out, we will be left with
$((-\alpha-\mu-\lambda)(-\alpha-\mu-\nu-\lambda)(-\mu-\kappa-\lambda)-\beta\lambda(-\mu-\kappa-\lambda))+(\alpha\kappa(\alpha+\mu+\nu-\lambda)+\alpha\kappa\beta\nu)$
to expand and group the like terms.

Expanding the first term, we get
$(-\alpha-\mu-\lambda)(\alpha\mu+\alpha\kappa+\alpha\lambda+\mu^2+\mu\kappa+\mu\lambda+\mu\nu+\nu\kappa+\nu\lambda+\lambda\mu+\lambda\kappa+\lambda^2+\beta\nu\kappa+\beta\nu\mu+\beta\nu\lambda)$

Multiplying $(-\alpha-\mu-\lambda)$ through, we get
$(-\alpha^2\mu-\alpha^2\kappa-\alpha^2\lambda-\alpha\mu^2-\alpha\mu\kappa-\alpha\mu\lambda-\alpha\nu\mu-\alpha\nu\kappa-\alpha\nu\lambda-\alpha\lambda\mu-\alpha\lambda\kappa-\alpha\lambda^2-\alpha\beta\nu\mu-\alpha\beta\nu\kappa-\alpha\beta\nu\lambda)+(-\alpha\mu^2-\alpha\mu\kappa-\alpha\mu\lambda-\mu^3-\mu^2\kappa-\mu^2\lambda-\nu\mu^2-\mu\nu\kappa-\nu\lambda\mu-\lambda\mu^2-\mu\lambda\kappa-\mu\lambda^2-\beta\nu\mu^2-\mu\beta\nu\kappa-\mu\beta\lambda)+(-\lambda\alpha\mu-\alpha\lambda\kappa-\alpha\lambda^2-\lambda\mu^2-\lambda\mu\kappa-\lambda^2-\lambda\nu\mu-\lambda\mu\kappa-\nu\lambda^2-\lambda^2\mu-\lambda^2\kappa-\lambda^3-\lambda\beta\nu\mu-\lambda\beta\nu\kappa-\beta\nu\lambda^2)$

Expanding the second term, we get
$(\alpha^2\kappa+\alpha\kappa\mu+\alpha\kappa\nu-\alpha\kappa\lambda+\beta\alpha\kappa\nu)$

Adding the two terms and grouping like terms we finall get
$\lambda^3 + (3\mu+2\alpha+\beta\nu+\nu+\kappa)\lambda^2 + (3\mu^2+(4\alpha+3\kappa+2\nu-\beta\nu)\mu+2\alpha\kappa+\alpha^2+\alpha\nu+\alpha\beta\nu+\beta\nu\kappa)\lambda + (\mu^3+(2\alpha+\beta\nu+\kappa+\nu)\mu^2+(\alpha^2+\alpha\nu+\alpha\beta\nu+\alpha\kappa+\nu\kappa+\beta\nu\kappa)\mu)$
$\det(J-\lambda I)=0$, so

$(-\mu-\epsilon-\lambda)(\lambda^3+(3\mu+2\alpha+\beta\nu+\nu+\kappa)\lambda^2+(3\mu^2+(4\alpha+3\kappa+2\nu-\beta\nu)\mu+2\alpha\kappa+\alpha^2+\alpha\nu+\alpha\beta\nu+\beta\nu\kappa)\lambda+(\mu^3+(2\alpha+\beta\nu+\kappa+\nu)\mu^2+(\alpha^2+\alpha\nu+\alpha\beta\nu+\alpha\kappa+\nu\kappa+\beta\nu\kappa)\mu))=0$

To solve the cubic equation, we let

$$\begin{aligned} p &= (3\mu + 2\alpha + \beta\nu + \nu + \kappa) \\ q &= (3\mu^2 + (4\alpha + 3\kappa + 2\nu - \beta\nu)\mu + 2\alpha\kappa + \alpha^2 + \alpha\nu + \alpha\beta\nu + \beta\nu\kappa) \\ r &= (\mu^3 + (2\alpha + \beta\nu + \kappa + \nu)\mu^2 + (\alpha^2 + \alpha\nu + \alpha\beta\nu + \alpha\kappa + \nu\kappa + \beta\nu\kappa)\mu) \end{aligned}$$

Inserting p, q and r into the above cubic equation gives us $\lambda^3 + p\lambda^2 + q\lambda + r$. Since $\det(J - \lambda I) = \lambda^3 + p\lambda^2 + q\lambda + r = 0$, it follows that $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. Solving the cubic equation gives us three eigen values $(\lambda_2, \lambda_3 and \lambda_4)$. The above equation has discriminant $\Delta = p^2q^2 - 4q^3 - 4p^3r - 27r^2 + 18pqr$
The following cases need to be considered

1. If $\Delta > 0$, then the equation has three distinct real roots.
2. If $\Delta = 0$, then the equation has a multiple root and all its roots are real.
3. If $\Delta < 0$, then the equation has one real root and two nonreal complex conjugate roots.

To find the roots of the cubic equation, let the coefficient of λ^3 which in this case is equal to 1 be a. Our new cubic equation now becomes $a\lambda^3 + p\lambda^2 + q\lambda + d = 0$. If the cubic equation $a\lambda^3 + p\lambda^2 + q\lambda + d = 0$ with integer coefficients has a rational real root, it can be found using the rational root test.If r is any root of the cubic, then we may factor out $(\lambda - r)$ using polynomial long division to obtain $(\lambda - r)(a\lambda^2 + (p + a\lambda)\lambda + q + pr + ar^2) = a\lambda^3 + p\lambda^2 + q\lambda + d$. Hence if we know one root we can find the other two by using the quadratic formula to solve the quadratic $(a\lambda^2 + (p + a\lambda)\lambda + q + pr + ar^2) = a\lambda^3 + p\lambda^2 + q\lambda + d$ giving $\frac{-p-ra\pm\sqrt{p^2-4aq-2apr-3a^2r^2}}{2a}$ for the other two roots.

3.5.1 Geometric interpretation of the roots

3.5.1.1 Three real roots

Here arccos $(\frac{3q}{2p}\sqrt{\frac{-3}{p}})$ is an angle in the unit circle; taking $\frac{1}{3}$ of that angle corresponds to taking a cube root of a complex number; adding $-k\frac{2\pi}{3}$ for k = 1, 2 finds the other cube roots; and multiplying the cosines of these resulting angles by $2\sqrt{-\frac{p}{3}}$ corrects for scale.

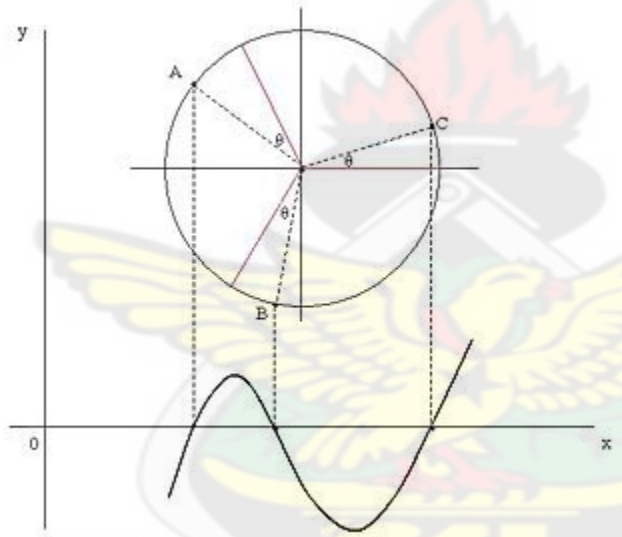

Figure 3.4: For the cubic function with three real roots, the roots form an equilateral triangle with vertices A, B, and C in the circle.

3.5.2 One real and two complex roots

In the Cartesian plane,if a cubic is plotted in the Cartesian plane, the real root can be seen graphically as the horizontal intercept of the curve. But further, if the complex conjugate roots are written as $g + hi$, then g is the abscissa (the positive or negative

Figure 3.5: Graph of a cubic function with 3 real roots (where the curve crosses the horizontal axis-where y = 0). It has 2 critical points

horizontal distance from the origin) of the tangency point of a line that is tangent to the cubic curve and intersects the horizontal axis at the same place as does the cubic curve; and $|h|$ is the square root of the tangent of the angle between this line and the horizontal axis. In the complex plane, with one real and two complex roots, the three roots can be represented as points in the complex plane, as can the two roots of the cubic's derivative.

The points in the complex plane representing the three roots serve as the vertices of an isosceles triangle. (The triangle is isosceles because one root is on the horizontal (real) axis and the other two roots, being complex conjugates, appear symmetrically above and below the real axis.) Marden's Theorem says that the points representing the roots of the derivative of the cubic are the foci of the Steiner inellipse of the triangle—the unique ellipse that is tangent to the triangle at the midpoints of its sides. If the angle at the vertex on the real axis is less than $\frac{\pi}{3}$ then the major axis of the ellipse lies on the real axis, as do its foci and hence the roots of the derivative. If that angle is greater than $\frac{\pi}{3}$, the major axis is vertical and its foci, the roots of the derivative, are complex. And if that angle is $\frac{\pi}{3}$, the triangle is equilateral, the Steiner inellipse is

simply the triangle's incircle, its foci coincide with each other at the incenter, which lies on the real axis, and hence the derivative has duplicate real roots.

3.5.2.1 Omar Khayym's solution

As shown in this graph, to solve the third-degree equation $x^3+a2x = b$ Omar Khayym constructed the parabola $x^2 = ay$, a circle with diameter $\frac{b}{a^2}$, and a vertical line through an intersection point. The solution is given by the length of the horizontal line segment from the origin to the intersection of the vertical line and the x-axis.

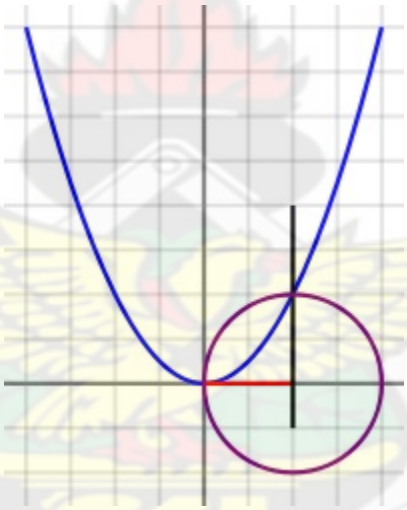

Figure 3.6: Omar Khayym's geometric solution of a cubic equation

We can clearly see that $(-\mu - \epsilon - \lambda) = 0$. This implies $\lambda_1 = -\mu - \epsilon < 0$. If $\Delta > 0$ and all the three distinct roots are negative,we say the system is asymptotically stable.

On the other hand, the system is unstable if any of the other three roots of the cubic equation is not negative, then the disease free equilibrium is said to be unstable.

3.6 Stability Analysis of Disease-Free Equilibrium point of Rabies transmission without vaccination

linearizing the model equations of the rabies transmission without vaccination we get,

$$\begin{aligned}
\frac{dS}{dt} &= -\beta(\frac{S}{N})I \\
\frac{dE}{dt} &= \beta(\frac{S}{N})I - \lambda E \\
\frac{dI}{dt} &= \lambda E - \gamma I = 0 \\
\frac{dR}{dt} &= \gamma I = 0
\end{aligned}$$

The Jacobian matrix is given as

$$J = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & -\lambda & \beta & 0 \\ 0 & \lambda & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{bmatrix}$$

$det(J - \lambda I)$=0, So

$$det(J - \lambda I) = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & -\lambda & \beta & 0 \\ 0 & \lambda & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = 0$$

$$det(J - \lambda I) = \begin{bmatrix} 0-\lambda & 0 & -\beta & 0 \\ 0 & -\lambda-\lambda & \beta & 0 \\ 0 & \lambda & -\gamma-\lambda & 0 \\ 0 & 0 & \gamma & 0-\lambda \end{bmatrix} = 0$$

$$det(J-\lambda I) = \begin{bmatrix} -\lambda & 0 & -\beta & 0 \\ 0 & -2\lambda & \beta & 0 \\ 0 & \lambda & -\gamma-\lambda & 0 \\ 0 & 0 & \gamma & -\lambda \end{bmatrix}$$

dividing the above matrix into four 3×3 square submatrices, we get

$$d_1 = \begin{bmatrix} -2\lambda & \beta & 0 \\ \lambda & -\gamma-\lambda & 0 \\ 0 & \gamma & -\lambda \end{bmatrix}$$

$$d_2 = \begin{bmatrix} 0 & \beta & 0 \\ 0 & -\gamma-\lambda & 0 \\ 0 & \gamma & -\lambda \end{bmatrix}$$

$$d_3 = \begin{bmatrix} 0 & -2\lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$d_4 = \begin{bmatrix} 0 & -2\lambda & \beta \\ 0 & -\lambda & -\gamma-\lambda \\ 0 & 0 & \gamma-\lambda \end{bmatrix}$$

$det(J-\lambda I) = -\lambda \times |d_1| - 0 \times |d_2| - \beta \times |d_3| - 0 \times |d_4|$. Since d_2 and d_4 are being multiplied by zero, we find only the determinants for the submatrices d_1 and d_3 only.

$$\begin{aligned} |d_1| &= -2\lambda((-\gamma-\lambda)\lambda - 0) - \beta(\lambda^2 - 0) \\ &= -2\lambda(-\gamma\lambda - \lambda^2) - \beta(\lambda^2) \\ &= 2\gamma\lambda^2 + 2\lambda^3 - \beta\lambda^2 \end{aligned}$$

$|d_3| = 0(-\lambda^2 - 0) + 2\lambda(0-0) + 0(0-0)$=0

$det(J - \lambda I) = -\lambda(2\gamma\lambda^2 + 2\lambda^3 - \beta\lambda^2)$

$det(J - \lambda I) = -2\gamma\lambda^3 - 2\lambda^4 + \beta\lambda^3$

$det(J - \lambda I) = -2\lambda^4 - (2\gamma - \beta)\lambda^3$

$det(J - \lambda I) = \lambda^4 + \frac{2\gamma-\beta}{2}\lambda^3$

Since $det(J-\lambda I)$=0, it follows that $\lambda^4 + \frac{2\gamma-\beta}{2}\lambda^3$=0. To solve our quartic equation, lets represent $\lambda^4 + \frac{2\gamma-\beta}{2}\lambda^3 = 0$ as $\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda = 0$. Then,

$b = \frac{2\gamma-\beta}{2}$

$c = 0$

$d = 0$

$e = o$

We can solve the above by the Factorization Into Quadratics Method which factors it into a product of two quadratics.

$0 = \lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e = (\lambda^2 + p\lambda + q)(\lambda^2 + r\lambda + s)$

$= \lambda^4 + (p + r)\lambda^3 + (q + s + pr)\lambda^2 + (ps + qr)\lambda + qs$

By equating coefficients, this results in the following set of simultaneous equations:

$b = p + r$

$c = q + s + pr$

$d = ps + qr$

$e = qs$

This can be simplified by starting again with a depressed quartic where b = 0, which can be obtained by substituting $(\frac{-b}{4})$ for λ, then $r = -p$, and: $c + p^2 = s + q$

$d/p = s - q$

$e = sq$

Elimination of s and q gives us: $(c + p^2)^2 - (d/p)^2 = (s + q)^2 - (s - q)^2$

$= 4sq$

=4e

If we set $P = p^2$, then this equation turns into the resolvent cubic equation $P^3 + 2cP^2 + (c^2 - 4e)P - d^2 = 0$

Then $r = -p$

$2s = c + p2 + d/p$

$2q = c + p^2 + \frac{d}{p}$

There are three roots of the cubic, corresponding to the three ways that a quartic can be factored into two quadratics, and choosing positive or negative values of p for the square root of P merely exchanges the two quadratics with one another. The above solution shows that the quartic polynomial with a zero coefficient on the cubic term is factorable into quadratics with rational coefficients if and only if the resolvent cubic $P^3+2cP^2+(c^2-4e)P-d^2$ has a root which is the square of a rational.

Depending on the roots, we can determine if the system is stable if all the roots are negative else we consider the system unstable.

3.6.1 Endemic equilibrium point of Rabies transmission with vaccination and its stability analysis

If $R_0 > 1$, then the system has an endemic infection because of the introduction of those with secondary infection. Here, we consider the case where $I \neq 0$. The Jacobian of the linearised matrix gives us

$$\begin{aligned}
\frac{dS}{dt} &= \delta + \kappa R - \beta\left(\frac{S}{N}\right) I - (\alpha+\mu)S \\
\frac{dE}{dt} &= \beta\left(\frac{S}{N}\right) I - (\alpha+\mu+\nu)E \\
\frac{dI}{dt} &= \nu E - (\mu+\epsilon)I
\end{aligned}$$

$$\text{J}=\begin{bmatrix} -\beta I-(\alpha+\mu) & 0 & -\beta S \\ \beta I & -(\alpha+\mu+\nu) & \beta S \\ 0 & \nu & -(\mu+\epsilon) \end{bmatrix}$$

$$\text{J-}\lambda I = \begin{bmatrix} -\beta I-(\alpha+\mu)-\lambda & 0 & -\beta S \\ \beta I & -(\alpha+\mu+\nu)-\lambda & \beta S \\ 0 & \nu & -(\mu+\epsilon)-\lambda \end{bmatrix}$$

$$\det(J-\lambda I) = \begin{bmatrix} -\beta I-(\alpha+\mu)-\lambda & 0 & -\beta S \\ \beta I & -(\alpha+\mu+\nu)-\lambda & \beta S \\ 0 & \nu & -(\mu+\epsilon)-\lambda \end{bmatrix}$$

We solve for the eigen values then we substitute the values of S and I to get the eigen values.

$= -\beta I-(\alpha+\mu)-\lambda((-\alpha-\mu-\nu)-\lambda)(((-\mu-\epsilon)-\lambda)-\nu\beta S)+0-\beta S(\nu\beta I)$

$= \lambda+(\beta I+\alpha+\mu)\left[\lambda^2+(2\mu+\epsilon+\alpha+\nu)\lambda+(\mu+\epsilon)(\alpha+\mu+\nu)-\nu\beta S\right]-\nu\beta^2 SI$

$= \lambda^3 + (2\mu+\epsilon+\alpha+\nu)\lambda^2 + ((\mu+\epsilon)(\alpha+\mu+\nu) - \nu\beta S)\lambda + (\beta I + (\alpha+\mu)\lambda^2 + (\beta I + \alpha + \mu)(2\mu+\epsilon+\alpha+\nu)\lambda + (\beta I + \alpha + \mu)(\mu+\epsilon)(\alpha+\mu+\nu) - \nu\beta S)) - \nu\beta^2 SI$

$= \lambda^3 + (3\mu+2\alpha+\nu+\epsilon+\beta I)\lambda^2 + ((\mu+\epsilon)(\alpha+\mu+\nu) - \nu\beta S + (\beta I + (\alpha+\mu))(2\mu + \epsilon+\alpha+\nu)\lambda + (\alpha+\mu)[(\mu+\epsilon)(\alpha+\mu+\nu) - \nu\beta S] + \beta I(\mu+\epsilon)(\alpha+\mu+\nu) + \nu\beta^2 - \nu\beta^2$ (3.29)

We know

$$\begin{aligned} S &= \frac{(\mu+\epsilon)(\alpha+\nu+\mu)}{\beta\nu} \\ \beta S &= \frac{(\mu+\epsilon)(\alpha+\nu+\mu)}{\nu} \\ \nu\beta S &= (\mu+\epsilon)(\alpha+\nu+\mu) \end{aligned}$$

Inserting $\nu\beta S$ into (3.25) gives us

$\lambda^3 + (3\mu+2\alpha+\nu+\epsilon+\beta I)\lambda^2 + ((\beta I + \alpha + \mu)(2\mu+\epsilon+\alpha+\nu))\,\lambda + \beta I(\mu+\epsilon)(\alpha+\mu+\nu)$ (3.26)

Also,

$$I = \frac{\delta\beta\nu - \mu\nu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\beta\nu(\alpha+\mu+\nu)(\mu+\epsilon) - \alpha\beta\nu)}$$

So

$$\beta I = \frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa) - \alpha}$$

Insert βI into (3.26),our quadratic equation becomes

$\lambda^3 + \left(3\mu + 2\alpha + \nu + \epsilon + \frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa)-\alpha}\right)\lambda^2 + \left(\left(\frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa)-\alpha} + \alpha + \mu\right)(2\mu+\epsilon+\alpha+\nu)\right)\lambda + \left(\left(\frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa)-\alpha}\right)(\mu+\epsilon)(\alpha+\mu+\nu)\right)$

let

$$\begin{aligned} p &= \left(3\mu + 2\alpha + \nu + \epsilon + \frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa) - \alpha}\right) \\ q &= \left(\left(\frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa) - \alpha} + \alpha + \mu\right)(2\mu + \epsilon + \alpha + \nu)\right) \\ r &= \left(\left(\frac{\delta\beta\nu - \mu(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa) - \alpha}\right)(\mu+\epsilon)(\alpha+\mu+\nu)\right) \end{aligned}$$

where $(\mu+\epsilon)(\alpha+\mu+\nu)(\mu+\kappa) - \alpha \neq 0$

Giving us $\lambda^3 + p\lambda^2 + q\lambda + r$.

Since $\det(J - \lambda I) = \lambda^3 + p\lambda^2 + q\lambda + r = 0$, it follows that $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. Solving the quadratic equation gives us three eigen values $(\lambda_1, \lambda_2 \, and \, \lambda_3)$.

The above equation has discriminant $\Delta = p^2q^2 - 4q^3 - 4p^3r - 27r^2 + 18pqr$

The following cases need to be considered

- If $\Delta > 0$, then the equation has three distinct real roots.
- If $\Delta = 0$, then the equation has a multiple root and all its roots are real.
- If $\Delta < 0$, then the equation has one real root and two nonreal complex conjugate roots.

Using the Factor Method, we can find the roots of the cubic equation. If all the roots are negative, we say the system is stable else its unstable.

3.7 Endemic Equilibrium point of Rabies transmission without vaccination and its stability analysis

If $R_0 > 1$, then the system has an endemic infection because of the introduction of those with secondary infection. Here, we consider the case where $I \neq 0$. The Jacobian of the linearised matrix gives us

$$J = \begin{bmatrix} -\beta I & 0 & -\beta S & 0 \\ \beta I & -\lambda & \beta S & 0 \\ 0 & \lambda & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{bmatrix} \tag{3.27}$$

$$det(J - \lambda I) = \begin{bmatrix} -\beta I & 0 & -\beta S & 0 \\ \beta I & -\lambda & \beta S & 0 \\ 0 & \lambda & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$det(J - \lambda I) = \begin{bmatrix} -\beta I & 0 & -\beta S & 0 \\ \beta I & -2\lambda & \beta S & 0 \\ 0 & 0 & -\gamma - \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}$$

To find the determinant of the above matrix, we divide it into four 3×3 square submatrices which we denote as d_1,d_2,d_3,d_4.

$$d_1 = \begin{bmatrix} -2\lambda & \beta S & 0 \\ 0 & -\gamma - \lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$d_2 = \begin{bmatrix} \beta I & \beta S & 0 \\ 0 & -\gamma - \lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$d_3 = \begin{bmatrix} \beta I & -2\lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$d_4 = \begin{bmatrix} \beta I & -2\lambda & \beta S \\ 0 & 0 & -\gamma - \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

$det(J - \lambda I) = (-\beta I - \lambda) \times |d_1| - 0 \times |d_2| - \beta S \times |d_3| - 0 \times |d_4|$. Since $d_2$and d_4 are both being multiplied by zero, we find the determinants of d_1 and d_4.

$$|d_1| = -2\lambda((-\gamma - \lambda)(-\lambda) - 0) - \beta S(0 - 0)$$
$$|d_1| = -2\lambda(\gamma\lambda + \lambda^2)$$
$$|d_1| = -2\gamma\lambda^2 - 2\lambda^3$$

$$|d_3| = \beta I(0 - 0) + 2\lambda(0 - 0) + 0$$
$$|d_3| = 0 + 0 + 0$$
$$|d_3| = 0$$

$$det(J - \lambda I) = -\beta I - \lambda(-2\gamma\lambda^2 - 2\lambda^3)$$
$$det(J - \lambda I) = 2\gamma\beta I\lambda^2 + 2\beta I\lambda^3 + 2\gamma\lambda^3 + 2\lambda^4$$
$$det(J - \lambda I) = 2\lambda^4 + 2\gamma\lambda^3 + 2\beta I\lambda^3 + 2\gamma\beta I\lambda^2$$
$$det(J - \lambda I) = \lambda^4 + \gamma\lambda^3 + \beta I\lambda^3 + \gamma\beta I\lambda^2$$
$$det(J - \lambda I) = \lambda^4 + (\gamma + \beta I)\lambda^3 + \gamma\beta I\lambda^2$$

From the equilibrium point,I=0. So
$det(J - \lambda I) = \lambda^4 + \gamma\lambda^3 = 0$. This implies that $\lambda^4 + \gamma\lambda^3 = 0$. We then proceed to

find its roots. Using the Factorization Into Quadratics Method, the system is said to be stable if all the roots are negative, else, it is said to be unstable.

3.7.1 Herd Immunity Ratio for Rabies transmission with vaccination

This is the percentage of the population that needs to be immuned to control transmission of the disease. The equation given by Diekmann and Heesterbeek for the Herd Immunity Threshold denoted by H_1 is

$$\begin{aligned} H_1 &= 1 - \frac{1}{R_0} \\ &= 1 - \frac{1}{\frac{\beta\nu}{(\alpha+\mu+\nu)(\mu+\epsilon)}} \\ &= 1 - \frac{(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} \\ H_1 &= \frac{\beta\nu - (\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} \end{aligned} \tag{3.30}$$

Chapter 4

MODEL ANALYSIS AND RESULTS

4.1 Introduction

Currently in Bongo District, there has been an upsurge of rabies positive cases among dogs after the free anti-rabies immunization campaign ended in 1998.

In this thesis, standard values for the ODE parameters used were obtained from Ghana Veterinary Medical Association Report, 2010. The simulations and analysis made are based on these standard values which are displayed below in Table 4.1.

PARAMETER	DESCRIPTION	STANDARD VALUE
β	Transmission coefficient	$3.0417 \times 10^{-3}(\text{Dogsmonth})^{-1}$
ν	Latency Rate	$2.1429 \times 10^{-3}\text{month}^{-1}$
γ	Death Rate	$2.293 \times 10^{-3}\text{month}^{-1}$
δ	Birth rate	$0.1975\text{Kits/female month}^{-1}$
ϵ	Disease Induced Mortality Rate	$4.9167 \times 10^{-3}\text{month}^{-1}$
κ	Wanning Immunity Rate	$1.9177 \times 10^{-3}\text{month}^{-1}$
α	Vaccination rate	$2.975 \times 10^{-3}\text{month}^{-1}$
μ	Death rate	$2.293 \times 10^{-3}\text{month}^{-1}$

Table 4.1: Parameter description for the ODEs

4.1.1 Estimating the Basic reproductive ratio(R_0) of Rabies transmission without vaccination

From the SEIR model equation in chapter three, we had the Basic Reproductive ratio of Rabies Transmission without vaccination as

$$\begin{aligned} R_0 &= \frac{\beta}{\gamma} \\ &= \frac{3.0417 \times 10^{-3}}{2.293 \times 10^{-3}} \\ &= 1.3267 > 1 \end{aligned}$$

Since $R_0 > 1$, the prevalence of Rabies is considered an epidemic. This is because the transmission coefficient between dogs exceeds the death rate in dogs.

4.1.2 Estimating the Bacic reproductive ratio(R_0) of Rabies transmission with vaccination

According to this model, the Basic Reproductive Ratio of Rabies transmission with vaccination is

$$\begin{aligned} R_0 &= \frac{\beta\nu}{(\alpha+\mu+\nu)(\mu+\epsilon)} \\ &= \frac{(3.0417\times10^{-3})(2.1429\times10^{-3})}{(2.975\times10^{-3}+2.293\times10^{-3}+2.9750\times10^{-3})(2.293\times10^{-3}+4.9167\times10^{-3})} \\ &= \frac{6.5181\times10^{-6}}{(7.4109\times10^{-3})(2.3422\times10^{-3})} \\ &= \frac{6.51881\times10^{-6}}{1.7358\times10^{-5}} \\ &= 0.3755 < 1 \end{aligned}$$

4.1.3 Stability analysis of equilibrium points

4.1.3.1 Stability Analysis Of The Disease Free Equilibrium Point of Rabies Transmission With Vacciantion

$$(S,E,I,R) = \left(\frac{(\mu+\kappa)\delta}{(\mu^2+(\alpha+\kappa)\mu)},\ 0\ ,\ 0\ ,\frac{\alpha\delta}{(\mu^2+(\alpha+\kappa)\mu)}\right)$$

$$= \left(\frac{(2.293\times10^{-3}+1.9177\times10^{-6})0.1975}{(2.293\times10^{-6}+(2.9752.975\times10^{-3}+1.9177\times10^{-3})2.293\times10^{-3})},0,0,\frac{(2.9752.975\times10^{-3})(0.1975)}{(2.293\times10^{-6}+(2.9752.975\times10^{-3}+1.9177\times10^{-3})2.293\times10^{-3})}\right)$$

$$\begin{aligned} &= \left(\frac{4.5325\times10^{-4}}{1.2084\times10^{-5}},\quad 0\quad ,\quad 0\quad ,\frac{5.8756\times10^{-4}}{1.2084\times10^{-5}}\right) \\ &= (37.5085,\quad 0\quad ,\quad 0\quad ,48.6233) \end{aligned}$$

From Chapter three, we know that the Jacobian was given as

$$J = \begin{bmatrix} -\alpha-\mu & 0 & -\beta & \kappa \\ 0 & -\alpha-\mu-\nu & \beta & 0 \\ 0 & \nu & -\mu-\epsilon & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa \end{bmatrix}$$

$$det(J-\lambda I) = \begin{vmatrix} -\alpha-\mu-\lambda & 0 & -\beta & \kappa \\ 0 & -\alpha-\mu-\nu-\lambda & \beta & 0 \\ 0 & \nu & -\mu-\epsilon-\lambda & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa-\lambda \end{vmatrix} = 0$$

Inserting the values of $\alpha, \mu, \kappa, \beta, \epsilon$ and ν into the above,we get

$$det(J-\lambda I) = \begin{vmatrix} -5.268\times10^{-3}-\lambda & 0 & -3.0417\times10^{-3} & 1.9177\times10^{-3} \\ 0 & -7.4109\times10^{-3}-\lambda & 3.0417\times10^{-3} & 0 \\ 0 & 2.1429\times10^{-3} & -2.3421\times10^{-3}-\lambda & 0 \\ 2.975\times10^{-3} & 2.975\times10^{-3} & 0 & -2.2949\times10^{-3}-\lambda \end{vmatrix} = 0$$

We divide the above matrix into four 3×3 submatrices d_1, d_2, d_3 and d_4

$$d_1 = \begin{vmatrix} -7.4109\times10^{-3}-\lambda & 3.0417\times10^{-3} & 0 \\ 2.1429\times10^{-3} & -2.3421\times10^{-3}-\lambda & 0 \\ 2.975\times10^{-3} & 0 & -2.2949\times10^{-3}-\lambda \end{vmatrix}$$

$$d_2 = \begin{vmatrix} 0 & 3.0417\times10^{-3} & 0 \\ 0 & -2.3421\times10^{-3}-\lambda & 0 \\ 2.975\times10^{-3} & 0 & -2.2949\times10^{-3}-\lambda \end{vmatrix}$$

$$d_3 = \begin{vmatrix} 0 & -7.4109 \times 10^{-3} - \lambda & 0 \\ 0 & 2.1429 \times 10^{-3} & 0 \\ 2.975 \times 10^{-3} & 2.975 \times 10^{-3} & -2.2949 \times 10^{-3} - \lambda \end{vmatrix}$$

$$d_4 = \begin{vmatrix} 0 & -7.4109 \times 10^{-3} - \lambda & 3.0417 \times 10^{-3} \\ 0 & 2.1429 \times 10^{-3} & -2.3421 \times 10^{-3} - \lambda \\ 2.975 \times 10^{-3} & 2.975 \times 10^{-3} & 0 \end{vmatrix}$$

We now determine the determinants of all the submatrices

$d_1 = (-7.4109 \times 10^{-3} - \lambda)(-2.3421 \times 10^{-3} - \lambda)(-2.2949 \times 10^{-3} - \lambda) -$

. $(3.0417 \times 10^{-3})(2.1429 \times 10^{-3})(2.2949 \times 10^{-3} - \lambda)$

$= (-7.4109 \times 10^{-3} - \lambda)(-2.3421 \times 10^{-3} - \lambda)(-2.2949 \times 10^{-3} - \lambda) -$

. $(6.5181 \times 10^{-6})(2.2949 \times 10^{-3} - \lambda)$

$$\begin{aligned} &= (-7.4109 \times 10^{-3} - \lambda)(-2.3421 \times 10^{-3} - \lambda)(-2.2949 \times 10^{-3} - \lambda) + 1.4958 \times 10^{-8}\lambda \\ &= (-7.4109 \times 10^{-3} - \lambda)(\lambda^2 + 4.637 \times 10^{-3}\lambda + 5.3749 \times 10^{-6}) + 1.4958 \times 10^{-8}\lambda \\ &= (-\lambda^3 - 0.01205\lambda^2 - 3.9734 \times 10^{-5}\lambda - 3.9822 \times 10^{-5}) + 1.4958 \times 10^{-8}\lambda \\ &= -\lambda^3 - 0.01205\lambda^2 - 3.9719 \times 10^{-5}\lambda - 3.9822 \times 10^{-5} \end{aligned}$$

$d_2 = 0$

$d_3 = 0$

$d_4 = (7.4109 \times 10^{-3} + \lambda)(2.975 \times 10^{-3})(-2.3421 \times 10^{-3} - \lambda) +$

. $(3.0417 \times 10^{-3})(2.1429 \times 10^{-3})(2.975 \times 10^{-3})$

$$\begin{aligned} &= (2.975 \times 10^{-3})(7.4109 \times 10^{-3} + \lambda)(-2.3421 \times 10^{-3} - \lambda) + 1.9391 \times 10^{-8} \\ &= (2.975 \times 10^{-3})(-\lambda^2 - 9.744 \times 10^{-3}\lambda - 1.7357 \times 10^{-5}) + 1.9391 \times 10^{-8} \\ &= -2.975 \times 10^{-3}\lambda^2 - 2.899 \times 10^{-5}\lambda - 3.225 \times 10^{-8} \end{aligned}$$

The determinant is given by $(-5.268 \times 10^{-3} - \lambda)d_1 + (1.9177 \times 10^{-6})d_2$.

$$\begin{aligned} &= (-5.268 \times 10^{-3} - \lambda)d_1 \\ &= (-5.268 \times 10^{-3} - \lambda)(-\lambda^3 - 0.01205\lambda^2 - 3.9719 \times 10^{-5}\lambda - 3.9822 \times 10^{-5}) \\ &= \lambda^4 + 0.0155\lambda^3 + 1.0320 \times 10^{-4}\lambda^2 + 4.004 \times 10^{-5}\lambda + 2 - 0984 \times 10^{-7} \end{aligned} \tag{4.1}$$

$$\begin{aligned} &= (1.9177 \times 10^{-6})d_2 \\ &= (1.9177 \times 10^{-6})(-2.975 \times 10^{-3}\lambda^2 - 2.899 \times 10^{-5}\lambda - 3.225 \times 10^{-8}) \\ &= 5.7052\lambda^2 + 5.5594 \times 10^{-3}\lambda + 6.1845 \times 10^{-11} \\ &= \lambda^2 + 9.7444 \times 10^{-3}\lambda + 0.0108 \end{aligned} \tag{4.2}$$

Adding equation(4.1) to equation(4.2), we end up with

$$det(J - \lambda I) = \lambda^4 + 0.0155\lambda^3 + 1.0001\lambda^2 + 9.7844 \times 10^{-3}\lambda + 0.0108 = 0$$

Using the Factorization Into Quadratics Method, we end up with

$(-2.835 \times 10^{-3} + 0.9945i - \lambda_1)(-4.91 \times 10^{-3} + 0.1044i - \lambda_2)(-4.91 \times 10^{-3} - 0.1044i -$

$\lambda_3)(-2.8353 - 0.9945i - \lambda_4)$

$\lambda_1 = -2.835 \times 10^{-3} + 0.9945i$

$\lambda_2 = -4.91 \times 10^{-3} + 0.1044i$

$\lambda_3 = -4.91 \times 10^{-3} - 0.1044i$

$\lambda_4 = -2.8353 - 0.9945i$

Since all the roots are complex, we therefore say that the equilibrium point is therefore unstable.

4.1.3.2 Stability Analysis of Disease-Free Equilibrium Point of Rabies Transmission Without Vaccination

(S, E, I, R)= $(\frac{\gamma}{\beta}, 0, 0, 0)$ and (0 , 0 , 0 , 0).

$$= \left(\frac{2.293 \times 10^{-3}}{3.0417 \times 10^{-3}}, 0, 0, 0\right)$$

$$= \left(\frac{2.293 \times 10^{-3}}{3.0417 \times 10^{-3}}, 0, 0, 0\right)$$

$$= (0.7539, 0, 0, 0)$$

From chapter three, we know that

$$J = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & -\lambda & \beta & 0 \\ 0 & \lambda & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{bmatrix}$$

$$det(J - \lambda I) = \begin{bmatrix} -\lambda & 0 & -\beta & 0 \\ 0 & -2\lambda & \beta & 0 \\ 0 & \lambda & -\gamma - \lambda & 0 \\ 0 & 0 & \gamma & -\lambda \end{bmatrix} = 0$$

inserting the values of λ, β, γ into the above, we get

$$det(J - \lambda I) = \begin{bmatrix} -\lambda & 0 & -3.0417 \times 10^{-3} & 0 \\ 0 & -2\lambda & 3.0417 \times 10^{-3} & 0 \\ 0 & \lambda & -2.293 \times 10^{-3} - \lambda & 0 \\ 0 & 0 & 2.293 \times 10^{-3} & -\lambda \end{bmatrix} = 0$$

dividing the above Jacobian matrix into four 3×3 submatrices gives us

$$d_1 = \begin{vmatrix} -2\lambda & 3.0417 \times 10^{-3} & 0 \\ \lambda & -2.293 \times 10^{-3} - \lambda & 0 \\ 0 & 2.293 \times 10^{-3} & -\lambda \end{vmatrix} \qquad d_2 = \begin{vmatrix} 0 & 3.0417 \times 10^{-3} & 0 \\ 0 & -2.293 \times 10^{-3} - \lambda & 0 \\ 0 & 2.293 \times 10^{-3} & -\lambda \end{vmatrix}$$

$$d_3 = \begin{vmatrix} 0 & -2\lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \qquad d_4 = \begin{vmatrix} 0 & -2\lambda & 3.0417 \times 10^{-3} \\ 0 & -\lambda & -2.293 \times 10^{-3} - \lambda \\ 0 & 0 & 2.293 \times 10^{-3} - \lambda \end{vmatrix}$$

$det(J - \lambda I) = -\lambda \times |d_1| - 0 \times |d_2| - \beta \times |d_3| - 0 \times |d_4|$. Since d_2 and d_4 are being multiplied by zero, we find only the determinants for the submatrices d_1 and d_3 only.

$$\begin{aligned} |d_1| &= -2\lambda((-2.293 \times 10^{-3} - \lambda)\lambda - 0) - 3.0417 \times 10^{-3}(\lambda^2 - 0) \\ &= -2\lambda(-2.293 \times 10^{-3}\lambda - \lambda^2) - 3.0417 \times 10^{-3}(\lambda^2) \\ &= 2(2.293 \times 10^{-3})\lambda^2 + 2\lambda^3 - 3.0417 \times 10^{-3}\lambda^2 \\ &= 4.586 \times 10^{-3}\lambda^2 + 2\lambda^3 - 3.0417 \times 10^{-3}\lambda^2 \end{aligned}$$

$|d_3| = 0(-\lambda^2 - 0) + 2\lambda(0 - 0) + 0(0 - 0)$=0

$det(J - \lambda I) = -\lambda(4.586 \times 10^{-3}\lambda^2 + 2\lambda^3 - 3.0417 \times 10^{-3}\lambda^2)$

$det(J - \lambda I) = -4.586 \times 10^{-3}\lambda^3 - 2\lambda^4 - 3.0417 \times 10^{-3}\lambda^3$

$det(J - \lambda I) = -2\lambda^4 - (4.586 \times 10^{-3} + 3.0417 \times 10^{-3})\lambda^3$

$det(J - \lambda I) = -2\lambda^4 - (7.6277 \times 10^{-3})\lambda^3$

$det(J - \lambda I) = \lambda^4 + 3.8139 \times 10^{-3}\lambda^3$

Since $det(J - \lambda I)$=0, it follows that $\lambda^4 + 3.8139 \times 10^{-3}\lambda^3$=0

Using the Factorization Into Quadratics Method, we get

$(8.6118 \times 10^{-9} - \lambda_1)(-4.306 \times 10^{-9} + 4.405 \times 10^{-9}i - \lambda_2)(-4.306 \times 10^{-9} - 4.405 \times 10^{-9}i - \lambda_3)(-3.814 \times 10^{-3} - \lambda_4)$

$\lambda_1 = 8.6118 \times 10^{-9}$

$\lambda_2 = -4.306 \times 10^{-9} + 4.405 \times 10^{-9}i$

$\lambda_3 = -4.306 \times 10^{-9} - 4.405 \times 10^{-9}i$

$\lambda_4 = -3.814 \times 10^{-3}$

Since all the roots of the equation are not negative,we therefore say that the system is unstable.

4.1.3.3 Stability Analysis Of The Endemic Equilibrium Point of Rabies Transmission With Vaccination

The determinants of the Jacobian matrix of the endemic eqiulibrium in Chapter Three was given as

$$\det(J-\lambda I)=\begin{vmatrix} -\beta I-(\alpha+\mu)-\lambda & 0 & -\beta S \\ \beta I & -(\alpha+\mu+\nu)-\lambda & \beta S \\ 0 & \nu & -(\mu+\epsilon)-\lambda \end{vmatrix}=0$$

We know from equation(3.19) that

$$\begin{aligned} I &= \frac{\delta\beta\nu^2-(\alpha\nu(\kappa-1)+\mu\nu)(\alpha+\mu+\nu)(\mu+\epsilon)}{(\mu+\epsilon)(\beta\nu(\alpha+\mu+\nu)(\mu+\epsilon)-\alpha\beta\nu)} \\ I &= \frac{(2.7586\times10^{-9}+1.4614\times10^{-6})(1.7357\times10^{-5})}{(2.3421\times10^{-3})(1.1314\times10^{-10})-(1.9391\times10^{-8})} \\ &= \frac{(1.4642\times10^{-6})(1.735\times10^{-5})}{2.6499\times10^{-13}-1.9391\times10^{-8}} \\ &= \frac{(2.5411\times10^{-11})}{-1.9391\times10^{-8}} \\ &= -1.3104\times10^{-3} \\ \beta I &= (3.0417\times10^{-3})(-1.3104\times10^{-3}) \\ \beta I &= -3.9859\times10^{-6} \end{aligned}$$

We also know that

$$\begin{aligned} S &= \frac{(\alpha+\mu+\nu)(\mu+\epsilon)}{\beta\nu} \\ S &= \frac{(7.4109\times10^{-3})(2.342\times10^{-3})}{6.5181\times10^{-6}} \\ &= \frac{1.7358\times10^{-5}}{6.5181\times10^{-6}} \\ &= 2.6630 \\ \beta S &= (3.0417\times10^{-3})(2.6630) \\ \beta S &= 8.1\times10^{-3} \end{aligned}$$

Insert βI and βS into the Jacobian matrix

$$\det(J-\lambda I) = \begin{bmatrix} -5.264\times10^{-3}-\lambda & 0 & -8.1\times10^{-3} \\ -3.9859\times10^{-6} & -7.4109\times10^{-3}-\lambda & -8.1\times10^{-3} \\ 0 & 2.1429\times10^{-3} & -2.3422\times10^{-3}-\lambda \end{bmatrix}$$

$= (-5.264\times10^{-3}-\lambda)((-7.4109\times10^{-3}-\lambda)(-2.3422\times10^{-3}-\lambda) -$

. $(2.1429\times10^{-3})(8.1\times10^{-3}) - (8.1\times10^{-3})(-3.9859\times10^{-6})$

$$\begin{aligned} &= (-5.264\times10^{-3}-\lambda)((-7.4109\times10^{-3}-\lambda)(-2.3422\times10^{-3}-\lambda) - 1.7357\times10^{-5}) \\ &= (-5.264\times10^{-3}-\lambda)(\lambda^2 + 9.7531\times10^{-3}\lambda + 1.0\times10^{-9}) \\ &= -\lambda^3 - 0.0150\lambda^2 - 5.1340\times10^{-3}\lambda - 5.1340\times10^{-5} \\ &= \lambda^3 + 0.0150\lambda^2 + 5.1340\times10^{-3}\lambda + 5.1340\times10^{-5} \end{aligned}$$

$\lambda^3 + 0.0150\lambda^2 + 5.1340\times10^{-3}\lambda + 5.1340\times10^{-5} = 0$, since $\det(J-\lambda I) = 0$

$(-1.0253\times10^{-9}-\lambda)(-7.8\times10^{-3}-\lambda)^2 = 0$

$(-1.0253\times10^{-9}-\lambda_1)(-7.8\times10^{-3}-\lambda_2)(-7.8\times10^{-3}-\lambda_3) = 0$

$\lambda_1 = -1.0253\times10^{-9}$

$\lambda_2 = -7.8\times10^{-3}$

$\lambda_3 = -7.8\times10^{-3}$

Since all the roots of the equation are negative, our equilibrium point is said to be stable.

4.1.3.4 Stability Analysis Of The Endemic Equilibrium Point of Rabies Transmission Without Vaccination

Equations(3.7) to (3.10) in Chapter three gives us the Jacobian,

$$J = \begin{bmatrix} -\beta I & 0 & -\beta S & 0 \\ \beta I & -\lambda & \beta S & 0 \\ 0 & \lambda & -\gamma & 0 \\ 0 & 0 & \gamma & 0 \end{bmatrix}$$

$$det(J - \lambda I) = \begin{vmatrix} -\beta I & 0 & -\beta S & 0 \\ \beta I & -2\lambda & \beta S & 0 \\ 0 & 0 & -\gamma - \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$det(J - \lambda I) = \begin{vmatrix} 3.0417 \times 10^{-3} I - \lambda & 0 & 3.0417 \times 10^{-3} S & 0 \\ 3.0417 \times 10^{-3} I & -2\lambda & 3.0417 \times 10^{-3} S & 0 \\ 0 & 0 & -2.293 \times 10^{-3} - \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

To find the determinant of the above matrix, we divide it into four 3×3 square submatrices which we denote as d_1,d_2,d_3,d_4.

$$d_1 = \begin{bmatrix} -2\lambda & 3.0417 \times 10^{-3} S & 0 \\ 0 & -2.293 \times 10^{-3} - \lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$d_2 = \begin{bmatrix} 3.0417 \times 10^{-3} I & 3.0417 \times 10^{-3} S & 0 \\ 0 & -2.293 \times 10^{-3} - \lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$d_3 = \begin{bmatrix} 3.0417 \times 10^{-3}I & -2\lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$d_4 = \begin{bmatrix} 3.0417 \times 10^{-3}I & -2\lambda & 3.0417 \times 10^{-3}S \\ 0 & 0 & -2.293 \times 10^{-3} - \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

$det(J-\lambda I) = (-3.0417\times10^{-3}I-\lambda)\times|d_1|-0\times|d_2|-3.0417\times10^{-3}S\times|d_3|-0\times|d_4|$.
Since $d_2$and d_4 are both being multiplied by zero, we find the determinants of d_1 and d_4.

$|d_1| = -2\lambda((-2.293 \times 10^{-3} - \lambda)(-\lambda) - 0) - 3.0417 \times 10^{-3}S(0 - 0)$
$|d_1| = -2\lambda(2.293 \times 10^{-3}\lambda + \lambda^2)$
$|d_1| = -2(2.293 \times 10^{-3})\lambda^2 - 2\lambda^3$
$|d_1| = -4.586 \times 10^{-3}\lambda^2 - 2\lambda^3$

$|d_3| = 3.0417 \times 10^{-3}I(0 - 0) + 2\lambda(0 - 0) + 0$
$|d_3| = 0 + 0 + 0$
$|d_3| = 0$

$det(J - \lambda I) = (-3.0417 \times 10^{-3}I - \lambda)(-4.586 \times 10^{-3}\lambda^2 - 2\lambda^3)$
$det(J - \lambda I) = (1.3945 \times 10^{-6}I\lambda^2 + 6.0834 \times 10^{-3}I\lambda^3 I + 4.586 \times 10^{-3}\lambda^3 + 2\lambda^4)$
$det(J - \lambda I) = 2\lambda^4 + (6.0834 \times 10^{-3}I + 4.586 \times 10^{-3})\lambda^3 + 1.3945 \times 10^{-6}I\lambda^2$
$det(J - \lambda I) = \lambda^4 + (3.0417 \times 10^{-3}I + 2.293 \times 10^{-3})\lambda^3 + 6.9725 \times 10^{-7}I\lambda^2$

From the equilibrium point,I=0. So
$det(J - \lambda I) = \lambda^4 + 2.293 \times 10^{-3}\lambda^3 = 0$. This implies that $\lambda^4 + 2.293 \times 10^{-3}\lambda^3 = 0$.

We then proceed to find its roots. Using the Factorization Into Quadratics Method, we get

$(8.6736 \times 10^{-19} - \lambda_1)(-4.3368 \times 10^{-19} - \lambda_2)(-4.3368 \times 10^{-19} - \lambda_3)(-0.002393 - \lambda 4)$

$\lambda_1 = 8.6736 \times 10^{-19}$

$\lambda_2 = -4.3368 \times 10^{-19}$

$\lambda_3 = -4.3368 \times 10^{-19}$

$\lambda_4 = -0.002393$

All the roots here are not negative so our equilibrium point here is said to be unstable.

4.1.4 The Herd Immunity Threshold(H_1) estimation

Herd Immunity Theory proposes that in contagious diseases that are transmitted from individual to individual, chains of infection are likely to be disrupted when large numbers of a population are immune or less susceptible to the disease. Using the equation given by Diekmann and Heesterbeek for the Herd Immunity Threshold, we estimate our H_1.

$$\begin{aligned} H_1 &= 1 - \frac{1}{R_0} \\ &= 1 - \frac{1}{1.3267} \\ &= 1 - 0.7537 \\ &= 0.2463 \end{aligned}$$

Therefore, to control the epidemic, about 24.63% of the population need to be vaccinated.

4.2 Sensitivity Analysis

Sensitivity analysis is the study of how the uncertainty in the output of a model can be apportioned to different sources of uncertainty in the model input.
It is a technique used to determine how different values of an independent variable will impact a particular dependent variable under a given set of assumptions. This technique is used within specific boundaries that will depend on one or more input variables. In more general terms uncertainty and sensitivity analysis investigate the robustness of a study.

4.2.1 Sensitivity Analysis of the R_0 of Rabies Transmission without Vaccination

1. If β is reduced and γ remains the same, that is $\beta = 1.908 \times 10^3$ and $\gamma = 2.293 \times 10^{-3}$

   $R_0 = \frac{\beta}{\gamma} = \frac{1.908 \times 10^{-3}}{2.293 \times 10^{-3}} = 0.8321 < 1$

   That is to say that keeping γ the same, any number for β that is less than 2.293×10^{-3} will make our $R_0 < 1$

2. If γ is increased and β remains the same, that is $\beta = 3.0417 \times 10^3$ and $\gamma = 3.9293 \times 10^{-3}$

   $R_0 = \frac{\beta}{\gamma} = \frac{3.0417 \times 10^{-3}}{3.9293 \times 10^{-3}} = 0.0.7741 < 1$

   That is to say that keeping β the same, any number for γthat is greater than 3.0417×10^{-3} will make our $R_0 < 1$.

4.2.2 Sensitivity Analysis of the R_0 of Rabies Transmission with Vaccination

Keeping the values of $\alpha, \mu, \nu, \epsilon$ the same, if β is increased to 8.802×10^{-3},

$$\begin{aligned} R_0 &= \frac{\beta\nu}{(\alpha+\mu+\nu)(\mu+\epsilon)} \\ &= \frac{(8.802\times10^{-3})(2.1429\times10^{-3})}{(2.975\times10^{-3}+2.293\times10^{-3}+2.9750\times10^{-3})(2.293\times10^{-3}+4.9167\times10^{-3})} \\ &= \frac{1.8915\times10^{-6}}{(7.4109\times10^{-3})(2.3422\times10^{-3})} \\ &= \frac{1.8915\times10^{-6}}{1.7358\times10^{-5}} \\ &= 1.0897 > 1 \end{aligned}$$

That is to say that when values of $\alpha, \mu, \nu, \epsilon$ are kept the same, any value of $\beta \geq 8.102 \times 10^{-3}$ will make our $R_0 \geq 1$

4.2.2.1 Sensitivity Analysis of Disease-Free Equilibrium Point of Rabies Transmission With Vaccination

We know the Jacobian of the disease-free equilibrium point as given in Chapter three as

$$J = \begin{bmatrix} -\alpha-\mu & 0 & -\beta & \kappa \\ 0 & -\alpha-\mu-\nu & \beta & 0 \\ 0 & \nu & -\mu-\epsilon & 0 \\ \alpha & \alpha & 0 & -\mu-\kappa \end{bmatrix}$$

Decoupling the above Jacobian Matrix, we get

$$J = \begin{bmatrix} -\alpha-\mu & 0 & -\beta \\ 0 & -\alpha-\mu-\nu & \beta \\ 0 & \nu & -\mu-\epsilon \end{bmatrix}$$

$$det(J-\lambda I) = \begin{vmatrix} -\alpha-\mu-\lambda & 0 & -\beta \\ 0 & -\alpha-\mu-\nu-\lambda & \beta \\ 0 & \nu & -\mu-\epsilon-\lambda \end{vmatrix} = 0$$

Inserting the values of $\alpha, \mu, \beta, \epsilon$ and ν into the above,we get

$$det(J-\lambda I) = \begin{vmatrix} -5.268\times10^{-3}-\lambda & 0 & -3.0417\times10^{-3} \\ 0 & -7.4109\times10^{-3}-\lambda & 3.0417\times10^{-3} \\ 0 & 2.1429\times10^{-3} & -2.3421\times10^{-3}-\lambda \end{vmatrix} = 0$$

$$(det(J-\lambda I) = -5.268\times10^{-3}-\lambda((-7.4109\times10^{-3}-\lambda)(2.3421\times10^{-3}-\lambda) - 2.1429\times10^{-3})(3.0417\times10^{-3}) - 0 + 0$$

$$(det(J-\lambda I) = -5.268\times10^{-3}-\lambda(1.7357\times10^{-5} + 7.4109\times10^{-3}\lambda + 2.3421\times10^{-3} + \lambda^2 - 6.5181\times10^{-3})$$

$$(det(J-\lambda I) = -5.268\times10^{-3}-\lambda(\lambda^2 + 9.753\times10^{-3}\lambda + 1.0839\times10^{-5})$$

$$(det(J-\lambda I) = (-\lambda^3 - 9.753\times10^{-3} - 1.0839\times10^{-5} - 5.268\times10^{-3}\lambda^2 - 5.8000\times10^{-5}\lambda)$$

$$(det(J-\lambda I) = (-\lambda^3 - 0.015021\lambda^2 - 6.2228\times10^{-3}\lambda - 5.8000\times10^{-8})$$

$$(det(J-\lambda I) = (\lambda^3 + 0.015021\lambda^2 + 6.2228\times10^{-3}\lambda + 5.8000\times10^{-8})$$

$$(det(J - \lambda I) = (-1.4741 \times 10^{-3} - \lambda)(-6.7734 \times 10^{-3} - \lambda)(-6.77 \times 10^{-3} - \lambda)$$

$\lambda_1 = -1.4741 \times 10^{-3}$

$\lambda_2 = -6.7734 \times 10^{-3}$

$\lambda_3 = -6.7734 \times 10^{-3}$

Since all the roots are negative, we conclude that the Disease-Free equilibrium point for Rabies transmission with vaccination is stable

4.2.3 Sensitivity Analysis of Rabies Transmission Without Vaccination by Simulation

We proceed in this thesis to simulate our model with Matlab 7.8.0 (R2009a) using the data displayed in Table 4.1. The values of S, E, I and R are altered and the changes that occur in model are observed.

Considering a period of six months, we plot graphs for each compartment of the rabies model without vaccination for $S = 500$, $E = 0$, $I = 0$ and $R = 0$, as displayed in Figure 4.1.

The number of susceptible dogs is observed to remain at 500 throughout the six months period. The graphs for the number of exposed, infected and removed dogs in Figure 4.1 remain at zero during the same period.

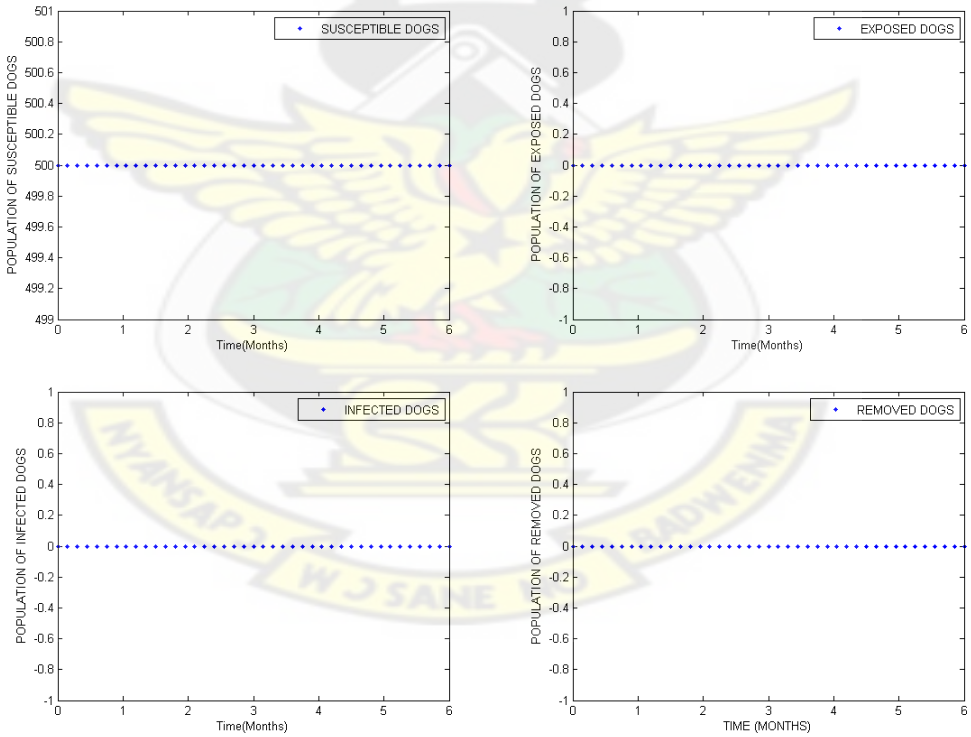

Figure 4.1: Graphs for S=500, E=0, I=0, R=0

Introducing one (1) infective dog into our system, the number of susceptibles as shown Figure

4.2 decrease to zero within a period of one month.

The number of exposed dogs increase in Figure 4.2 form 0 to 450 dogs where it reaches its peak within the first month and starts reducing. By the begining of the sixth month, the number of exposed dogs will be about 160 dogs.

The number of infected dogs as seen in Figure 4.2, increases gradually for the first fifteen days then it rises rapidly to about the fourth month and starts increasing gradually till it reachs a peak of about 170 dogs by the fifth month then it starts reducing.

The removed dogs in Figure 4.2, increase gradually within the first month then it rises rapidly to about 170 dogs by the begining of the first month.

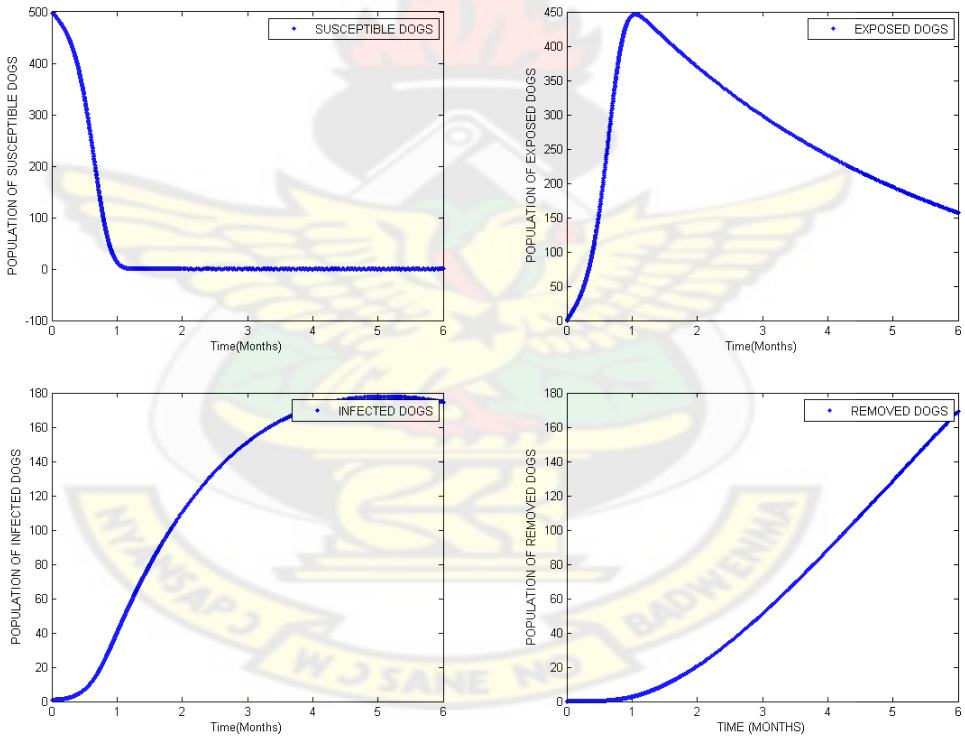

Figure 4.2: Graphs for S, E, I and R representing the model after the introduction of one(1) infective.

The number of infectives is increased to ten (10) in Figure 4.3. The number of exposed rise rapidly to about 470 dogs within the first ten days where it arrives at its peak then it starts reducing

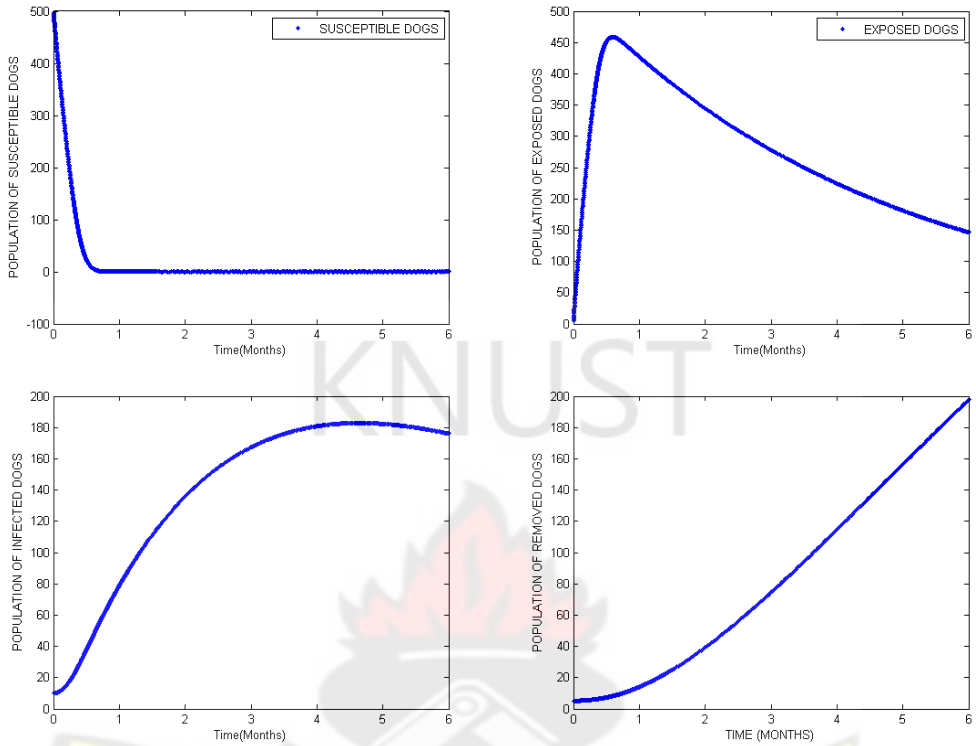

Figure 4.3: Graphs for S, E, I and R after the introduction of ten(10) infectives

to about 150 dogs by the begining of the sixth month.

The number of infectives rise continuously about 182 dogs by the 17^{th} and starts decreasing. The removed dogs increase to about 200 dogs by the begining of the sixth month.

4.2.4 Sensitivity Analysis of Rabies Transmission With Vaccination by Simulation

Figure 4.4 gives us the graphs for all four compartments (S, E , I and R) assuming the number susceptible dogs in our population is 500 with no exposed dogs and no infective dogs.

You will observe that the number of susceptibles remain at 500 through the time period. The number of exposed and infective dogs remain at zero through out the entire period.

Assuming 100 dogs in the population have been vaccinated, one (1) infective dog is introduced. Graphs for the various compartments as shown in Figure 4.5 shows a rise in the number of exposed

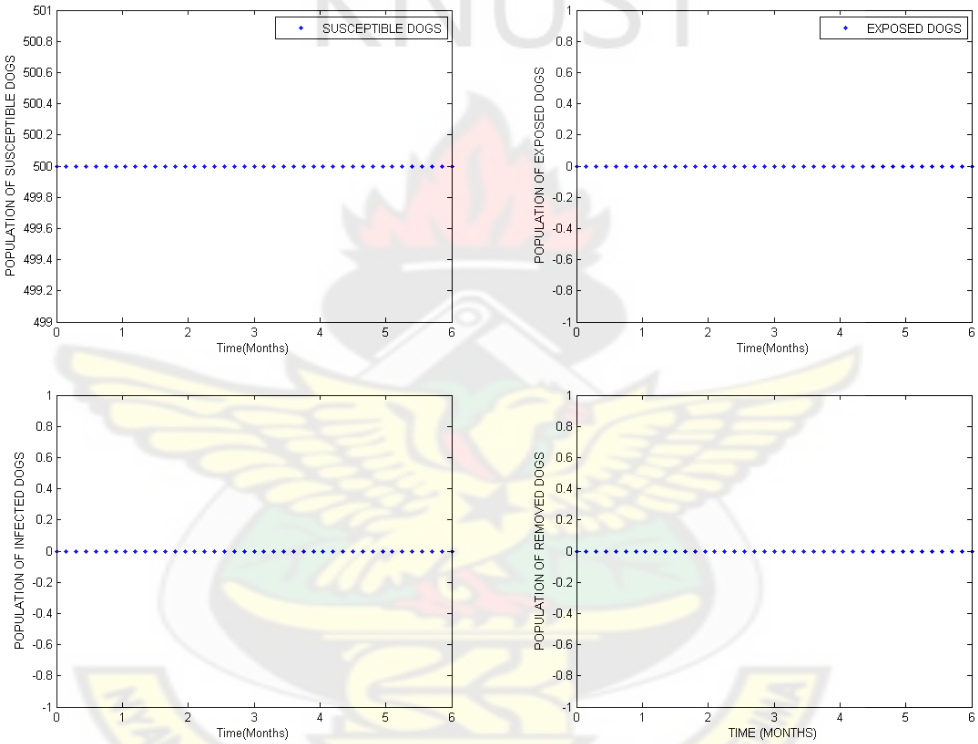

Figure 4.4: Graphs for S=500, E=0, I=0, R=0

to about 350 dogs by the first month and reduces to about 120 by the begining of the sixth month.

The number of infectives rise continuously to about 140 by the begining of the 6^{th} month and starts decreasing. It is important to observe the number of dogs which have recovered increase to about 232 dogs by the begining of the sixth month.

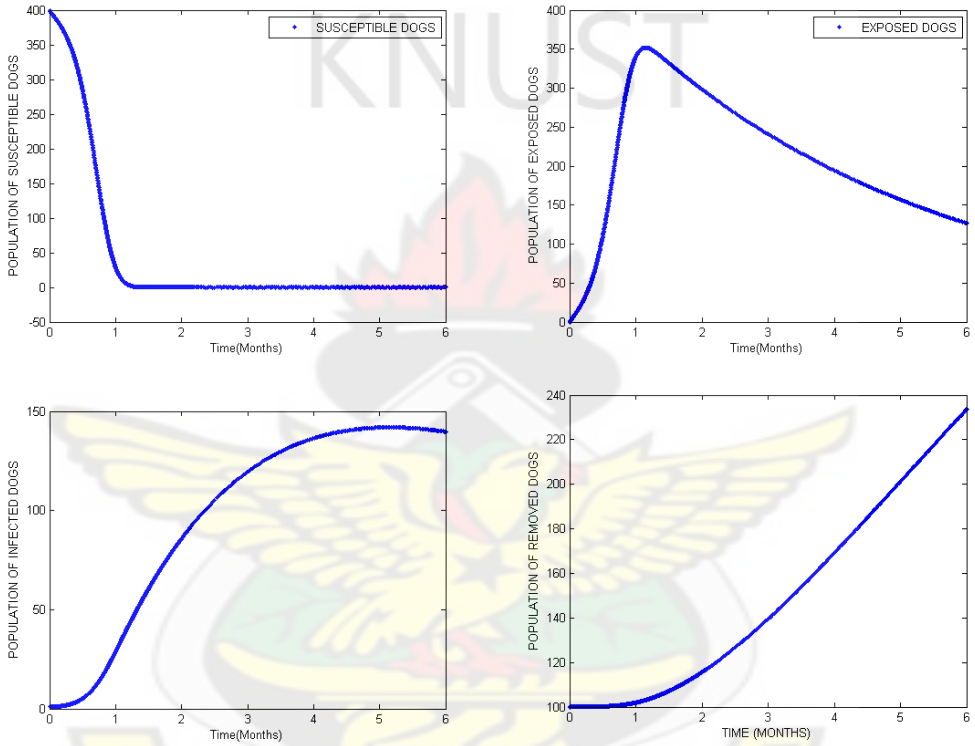

Figure 4.5: Graphs for S, E, I and R with 100 vaccinated dogs and one(1) infective dog

Increasing the number dogs which have been vaccinated to 200 dogs, it is observed in Figure 4.6 that by a period of 6 weeks, the number of exposed dogs decrease form 350 dogs in our previous simulation to 250 dogs within the six weeks and starts reducing. By the begining of the sixth month, the number of exposed dogs will be about 100 dogs.

The number of infected dogs decrease from 140 dogs as seen in Figure 4.5 to about 105 dogs

and the number of recovered dogs increase from 232 dogs in Figure 5 to about 298 dogs.

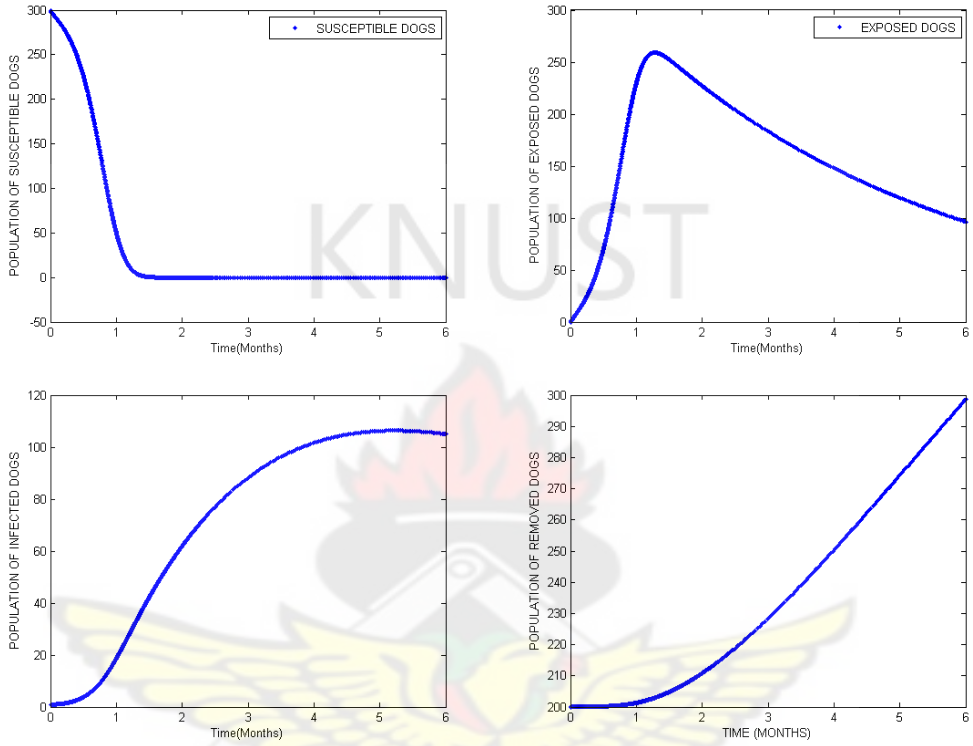

Figure 4.6: Graphs for S, E, I and R with 200 vaccinated dogs and one(1) infective dog

Increasing the number of vaccinated dogs to 400, with still an infective introduced in the system, it is observed in Figure 4.7 that it takes about two(2) months for the number of susceptibles to decreased to zero.

The number of exposed dogs decrease form 250 dogs in Figure 6 to 80 dogs within two months. By the begining of the sixth month, the number of exposed dogs will have decreased to about 35 dogs. The recovered dogs increase form 298 dogs in Figure 6 to about 450 dogs.

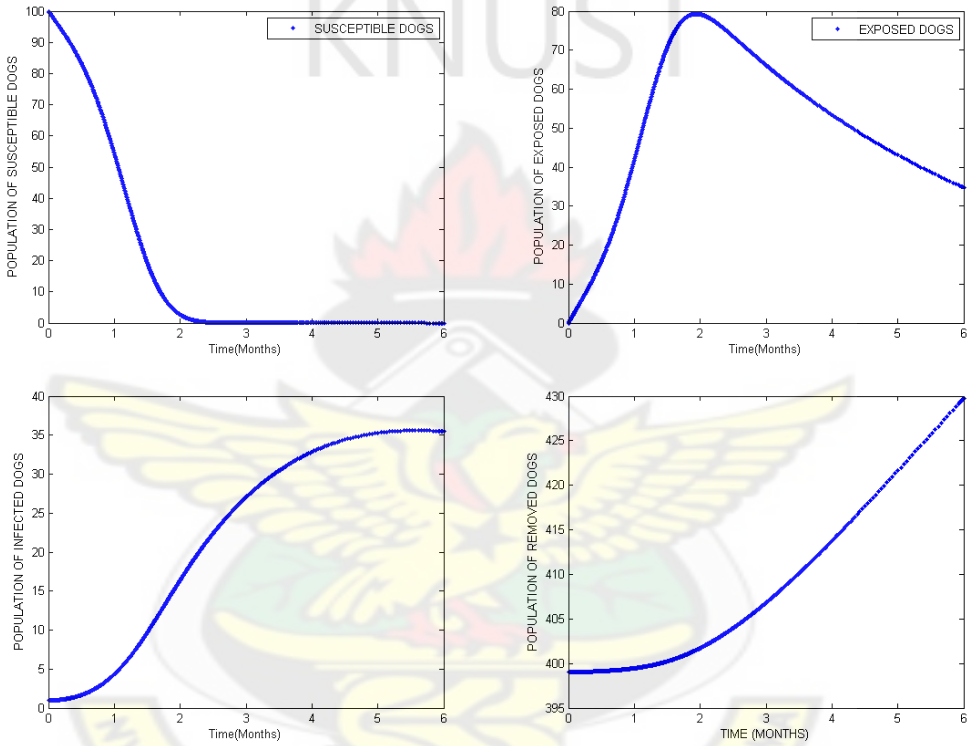

Figure 4.7: Graphs for S, E, I and R with 400 vaccinated dogs and one(1) infective dog

4.2.5 Discussion of results

In this thesis, we attempted to use standard SEIR differential equation model to predict the spread of Rabies in Bongo District. We discussed the existence and stability of the disease free and disease endemic equilibria of the model and performed sensitivity analysis of the model by varying the numbers of the number of infectious dogs which are introduced into the model and the number of dogs which have been vaccinated. We also considered Herd immunity as the sole immunization stractegy in the thesis.

Based on the data displayed in Table 4.1, we estimated the basic reroductive number of the rabies transmission without vaccination to be $R_0 = 1.3267$. This indicates an epidemic.
The basic reproductive number (R_0) of rabies transmission with vaccination was estimated to be $R_0 = 0.3755$. The decrease in the value of the R_0 is due to the introduction of vaccination in the model. Therefore if vaccination is intensified, it will further reduce the spread of rabies.

In the sensitivity analysis of basic reproductive number of rabies transmission without vaccination, it is obsereved that keeping γ constant, any value of β which is less than 2.293×10^{-3} will make $R_0 < 1$. Also, keeping β constant, any value of γ which is greater than 3.0417×10^{-3} will make our $R_0 > 1$. That is to say that if dogs are kept within the confinement of their households, the interaction between infective and suscetible dogs will decrease.

Increasing the number of infectives directly reduces the number of susceptibles but the number of exposed and infected dogs also increase alongside. That is to conclude that the disease will spread if the stakeholders of the nation fail to put proper measures in place to curb it and if the vaccination programs are intensified, throughout the whole nation, the disease will eventually die out.

According to the estimated Herd Immunity Threshold in the thesis, 24.63% of the population need to be vaccinated in order to control the spread of the disease. Unvaccinated individuals are indirectly protected by vaccinated individuals, as latter will not contract and transmit the disease between infected and susceptible individuals.

Since only a fraction of the population would be left unvaccinated for this method to be effective, it is considered best left for dogs which cannot safely receive vaccines because of their location within the District. No rabies vaccine offers permanent immunity, but the spread of disease form dog to dog and on a serious note, from dog to man is much higher in those who remain unvaccinated (*De la*

Sen et al., 1992). If the proportion of immmune individuals exceed the Herd Immuntity Threshold level by mass vaccination, the disease will die out. Thus 24.63% represents the minimum proportion of the population that must be immunised regularly for the infection to die out in the population.

If the vaccine used is insufficiently effective or the required coverage cannot be reached, the programme may not be able to exceed the herd immunity threshold, it can, however, disturb the balance of the infection without eliminating it. This change occurs simply because there are now fewer susceptibles in the population who can be infected. On the other hand, if the vaccination exercise causes the proportion of immune dogs in a population to exceed the Herd Immunity Threshold for a significant length of time, transmission of the rabies disease in that population will gradually come to a halt. This is known as elimination of the infection (*T. J. John and R. Samuel, 2004*).

Also from our simulations in this thesis, it was found that when the number of vaccinated dogs are increased, the number of dogs that will attained a level of immunity also increase. If vaccination is done on regular basis, then we are sure to have a lot of rabies immuned dogs in our system thereby decreasing the spread rabies amongst dogs and humans at large.

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In conclusion, we found out that Rabies is an acute fatal disease caused by a virus and the sole mode of transmission of rabies in dogs is through the bite of a rabid animal. It is characterised by disturbed consciousness, increased nerve irritability and subsequent symptoms of paralysis.

In this thesis, when sensitivity analysis was performed on the rabies transmission with vaccination, we saw that increasing the use of rabies vaccine had a significant impact on the rate of spread of Rabies transmission by increasing the number of recovered in the model and reducing the use of the rabies vaccine increased the number of recovered in the model.

Increasing the Rabies vaccination coverage in Ghana will decrease the prevalence and spread of Rabies even if the number of infected dogs in a particular locality should increase. Decreasing the vaccination coverage will increase in the rate of transmission of rabies in that locality and Ghana as a whole.

Even though this model did not consider mass vaccination as one of the methods to prevent the prevalence of Rabies in Ghana but concentrated on the herd immunity due to the huge sum of money that needs to be spent in carrying it out, the results from the herd immunity can also be

used as a tool to aid the introduction of rabies vaccination in the district.

Due to the waning immunity of the rabies vaccine, it is important to re-vaccinate all dogs in the district at the right time since vaccinated dogs lose their immunity with time and any encounter with a rabid dog thereafter will lead to the spread of the disease.

5.1.1 Recomendation

To eradicate rabies from Ghana we recommend that

1. Government should be urged to reintroduce the free anti-rabies vaccination program to undertake a mass vaccination exercise which should be followed by the consistent re-vaccination of dogs in the Bongo district.
2. Government should commit funds to procure anti-rabies vaccines which is cheaper instead of importing millions of doses of post-exposure rabies vaccines in anticipation of an exposure.
3. There should be enforcement of laws on dog owners to ensure regular vaccination of their dogs.

Appendix

5.1.1.1 M-File for Rabies Model without Vaccination

```
function dy=model(t,y,beta,gamma,nu)
dy=zeros(4,1);
dy(1)= -beta*y(1)*y(3);
dy(2)= beta*y(1)*y(3)-nu*y(2);
dy(3)= nu*y(2)-gamma*y(3);
dy(4)= gamma*y(3);
```


5.1.1.2 M-File for Rabies Model with Vaccination

```
function dy=model(t,y,beta,gamma,nu,delta,kappa,alpha,mu,epsilon)
dy=zeros(4,1);
dy(1)= delta+kappa*y(4)-beta*y(1)*y(3)-alpha*y(1)-mu*y(1);
dy(2)= beta*y(1)*y(3)-alpha*y(2)-mu*y(2)-nu*y(2);
dy(3)= nu*y(2)-mu*y(3)-epsilon*y(3);
dy(4)= alpha*y(1)+alpha*y(2)-mu*y(4)-kappa*y(4);
```

5.1.1.3 Scripts Used in Calling the M-Files for Rabies Models

```
delta=0.1975;
kappa = 1.977 × 10^-3;
alpha = 2.975 × 10^-3;
mu = 2.293 × 10^-3;
epsilon = 4.9167 × 10^-3;
beta = 3.0417 × 10^-3;
gamma = 0.0948;
nu = 2.1429 × 10^-3;
```

options = odeset('RelTol',1e-9,'AbsTol',1e-9); [T,Y] = ode45(@seir,[0 6],[500 5 10 300],options, beta,gamma,nu);

```
figure(1)
```

```
plot(T,Y(:,1),'.')
legend('SUSCEPTIBLE DOGS')
xlabel('Time(Months)');ylabel('POPULATION OF SUSCEPTIBLE DOGS');
figure(2)
plot(T,Y(:,2),'.')
legend('EXPOSED DOGS')
xlabel('Time(Months)');ylabel('POPULATION OF EXPOSED DOGS');
figure(3)
plot(T,Y(:,3),'.')
legend('INFECTED DOGS')
xlabel('Time(Months)');ylabel('POPULATION OF INFECTED DOGS');
figure(4)
plot(T,Y(:,4),'.')
legend('REMOVED DOGS')
xlabel('TIME (MONTHS)');ylabel('POPULATION OF REMOVED DOGS');
```

Bibliography

Bibliography

[1] E. S. Allman and A. R. John. Mathematical Models in Biology - An Introduction. Cambridge University Press, New York, 2004.

[2] R.M. Anderson and R.M. May. Regulation and stability of host-paradite population interactions in destabilising processes. *Nature*, 280:47, 249-268, 1978.

[3] R. Anderson, H. Jackson, R. May, and A. Smith. Population dynamics of fox rabies in Europe. *Nature*, 289:765-771, 1981.

[4] L. Andral, A. M. and J. Blancou. Radio-tracking of rabid foxes. *Comparative Immunology, Microbiology and Infectious Diseases*, 5:285-291, 1982

[5] E. Asano, L. Gross, S. Lenhart, and L. A. Real. Optimal control of vaccine distribution in a rabies metapopulation model. *Math. Biosci. Eng*, 5:219, 2008.

[6] D. Belcher, F. Wurapa, and D. Atuora. Endemic rabies in ghana. *Epidemiology and control measures*, 1976.

[7] T. Claytona, S. Duke-Sylvesterb, L. J. Grossc, S. Lenhartd, and L. A. Real. Journal of biological dynamics, 2010.

[8] D.Smith, B. Lucey, Waller, C. J., and L. Real. Predicting the spatial dynamics of rabies epidermics on heterogenous landscapes. *Proc. Natl. Acad. Sci, USA*, 99:3668-3672, 2002.

[9] M. De la Sen, A. Ibeas, and S. Alonso-Quesada. Vaccination rules for a true-mass action SEIR epidemic model based on an observer synthesis.

[10] H. W. Hethcote. The basic epidemiology models: Models expressios for r0, parameter estimation and applicataions.

[11] T. J. John and R. Samuel. Herd immunity and herd efect: new insights and definitions, 2000.

[12] S. Johnson. Rabies. Ghana Veterinary Medical Association Newsletter, 6:18-19, 2007.

[13] C. M. Kahn and S. Line. The Merck Veterinary Manual, 5th edition.

[14] D. MacDonald. Rabies and Wildlife: A Biologist's Perspective. Oxford University Press, Oxford, 1980.

[15] D. MacDonald, R. Bruce, and P. Bacon. Fox population, habitat characterization and rabies control. *J. Biogeogr*, 8:145-151, 1981.

[16] J. Murray, E. Stanley, and D. Brown. On the spatial spread of rabies among foxes. *Proc. R. Soc. Lond. B Biol. Sci*, 229:111-150, 1986.

[17] J. D. Murray. Mathematical Biology. Springer-Verlag, 1989.

[18] T. Nuertey. Human rabies. Ghana Veterinary Medical Association Newsletter, 6:16-17, 2007.

[19] Vijay G. Panjeti and leslie A. Real. The advances in virus research. Mathematical Models for Rabies, 79: 377-395, 2011.

[20] D. Voigt, R. Tinline, and L. Broekhoven. A spatial simulation model for rabies control. *Popul.Dyn. Rabies Wildl.*,311-349, 1985.

[21] Fred Brauer, Pauline van den Driessche and Jianhong Wu. Mathematical Epidemiology. Springer-Verlag Berlin Heidelberg, 2008.

[22] Eti Dwi Wiraningsih and Widodo, Lina Aryati and Syamsuddin, Toaha and Suzanne Lenhart. Optimal Control for SEIR Rabies Model between Dogs and Human with Vaccination Effect in dogs. *Proceedings of the 6th IMT-GT Conference on Mathematics, Statistics and its Applications (ICMSA2010) Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia*, 6:1161-1175, 2010.

[23] Changjin Xu and Maoxin liao. Stability and Bifurcation Analysis in A SEIR Epidemic Model with Nonlinear Incidence Rates. *IAENG International Journal of Applied Mathematics*

[24] J. Zhang, Z. Jinm, G.Q. Sun, T. Zhou, and S. Ruan. Analysis of rabies in China: Transmission dynamics and control, July 2011.