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Superquadratic Functions and The Refinement of Some Classical Inequalities

by

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DOCTOR OF PHILOSOPHY

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Certification

I hereby declare that this submission is my own work towards the PHD and that, to the best of my knowledge, it contains no material previously published by another person, nor material which has been accepted for the award of any other degree of the University, except where due acknowledgment has been made in the
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Abstract

Convexity and inequalities have proved to be useful in many branches of mathematics such as functional analysis, theory of differential and integral equations, optimisation, interpolation, harmonic analysis and probability theory. They have useful applications also in mechanics, physics, economics and other sciences.

Over the last 80 years, inequalities have seen a lot of interest from researchers, which has led to a great number of recent published books and articles. Thus, the theory of inequalities has developed into an independent branch or area of mathematics.

This thesis is dedicated to a new refinement of Jensen's inequality, which permits the use of non-continuous functions considered to be superquadratic. A refinement of Minkowski's inequality is established as a direct consequence of our refined Jensen's inequality.

A lower bound for Young's inequality is also established as an application of superquadratic functions.

Finally a refinement of the local submean inequality for subharmonic functions is also presented as an application of superquadratic functions.

Acknowledgment

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Chapter 1

Introduction

Convexity, a basic notion in geometry, has evolved over the years into an important subject and area of study in mathematics. It plays an important role not only in mathematics but also in other related fields such as economics, science, engineering, technology, physics, chemistry and biology just to mention a few. In mathematics, for instance, the study of convexity(convex sets and convex functions) has proven very useful and important to the study of number theory, linear programming, game theory, polytopes, combinatorial geometry, real analysis, bodies of constant width and the theory of inequalities.

Concerning the latter, two inequalities have proven over the years to be very useful and significant in the evolution and proof of other important theories and inequalities. They are Jensen's and Young's inequalities.

Ever since their introduction to the world of mathematics, Jensen's and Young's inequalities (motivated by the study of convex sets and functions) have proven very useful and arguably two of the most important inequalities the world has ever known.

Jensen's inequality, named after the famous Danish mathematician and engineer Johan Jensen (8 May 1859 5 March 1925), is one of such inequalities derived from the study of convex sets and functions. It relates the value of convex function of an integral to the integral of the convex function. Simply put, Jensen's inequality states that the mean after convex transformation is greater or equal to the convex transformation of the mean(where the converse is true for concave functions).

Young's inequality, named after the English mathematician William Henry Young (20 October 1863 7 July 1942), is an important inequality derived from the study of convex functions. Young's inequality has two different definitions which are used independently based on the subject of study. One definition relates the product of two numbers to their exponentiation whiles the other definition talks about the convolution of two functions.

These inequalities (Jensen's and Young's) along side the smoothness properties of convex functions have been very useful in economics (where together with concave functions are used to explain optimisation problems) and in Probability (in finding expectation in some unique problems). They tend to provide overwhelming results wherever they are applied. They have equally contributed to the proof and derivation of so many important inequalities in mathematics such as the Arithmetic-Geometric Mean Inequality, the Holder's and Minkowski's in-
equalities.

Hence the study of convexity and it's derived inequalities are undoubtedly one of the most important branches of mathematics.

1.1 Justification

Two known definitions have been given for functions considered to be superquadratic. These two definitions are not equivalent, one contains a larger class of functions than the other. The definition of a superquadratic function according to Shoshana Abramovich is : [Abramovich, Jameson, Sinnamon 2004a]A function ϕ : $[0,\infty) \to \mathbb{R}$ is superquadratic provided for all $x \geq 0$ there exists a constant

$P_x \in \mathbb{R}$ such that

$$\phi(y) \geq \phi(x) + P_x(y - x) + \phi(|y - x|)$$

for all $y \geq 0$.

This superquadratic function is a simple modification of the geometric notion of a convex function. For a superquadratic function we require that it lies above its support hyperplane plus a translation of itself.

The Epigraph of a positive superquadratic function coincides with that of the convex function it defines and superquadratic functions increase faster than quadratic functions.

This definition has been used in many research works, for example [Abramovich et al 2004a, 2004b, Banić et al 2008b] which has led to the refinement of many classical inequalities such as Jensen's, Hölder's and Hadamard's inequalities.

The other defintion for a superquadratic function given by [Smajdor 1987] is:

The function $\phi : \mathrm{R} \to \mathrm{R}$ is superquadratic if,

$$\phi(x + y) + \phi(x - y) \geq 2[\phi(x) + \phi(y)]$$

is satisfied for all $x,y \in \mathrm{R}$.

These definitions have given rise to two different classes of functions. The class of functions from Abramovich's definition is contained in that from Smajdor's definition, thus under certain assumptions, functions satisfying Abramovich's definition also satisfy Smajdor's.

However the second definition has not yet been used in the refinement of other inequalities. Refinements given in [Abramovich et al 2004a, 2004b, Banić 2008b] have used Abramovich's definition of superquadratic function.

The main aims and justification of this thesis is to:

1. Expand the domain of functions that could be considered for refinementsvia the second definition of superquadraticity, which allows us to consider non-continuous functions. This would offer a broader scope of application.

2. Obtain a refinement of Jensen's inequality via the second definition of superquadraticity and to apply our result to refining some classical inequalities such as Minkowski's inequality.

3. Obtain a lower bound for Young's inequality.
4. Open up the second definition of superquadraticity in the direction of refinements of other inequalities.
5. Obtain a refinement of the local submean inequality for subharmonic functions.

The first chapter, gives a general view of the scope of the thesis.

In the second chapter, convexity is discussed, where some basic notions, properties and results from convex sets and functions are recalled.

In the third chapter, the two known definitions given for superquadratic functions are discussed. Some properties and examples of superquadratic functions are also given and the relationship between the two given definitions is also established. Chapter 4 talks about the classical Jensen's, Young's and Minkowski's inequalities.

The fifth chapter is devoted to Subharmonic functions: where we looked at some of its properties, results and examples.

The sixth chapter, covers the main results obtained in this research work. These results are refinement of Jensen's, and Minkowski's inequalities, a lower bound for Young's inequality and a refinement of the local submean inequality as an application of superquadratic functions to subharmonic functions.

Chapter 2

Convex Sets and Functions

2.1 Convex Sets

Convexity has proved significantly important in recent years in the study of number theory, extremum problems, linear programming, game theory and theory of inequalities. It draws upon geometry, analysis, linear algebra and topology. In this section we begin by reviewing some definitions and properties of convex sets and functions.

Definition 2.1.1 [Florenzano 2001,Webster 1994] Let C be a subset of $\mathbb{R}^n$ and let x_1,x_2 be arbitrary points of C. C is said to be a *convex set* if

$$\lambda x_1 + \mu x_2 \in C$$

where $\lambda,\mu \in [0,1]$ *and* $\lambda + \mu = 1$.

Generally speaking, a set C is called convex if it contains all of its convex combinations. Thus a set C is convex if $\forall\, x_1,x_2,\cdots,x_n \in C, \exists x \in C$ such that;

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

where $\lambda_1,\lambda_2,\cdots,\lambda_n \in [0,1]$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ are scalars.

Theorem 2.1.2 *[Webster 1994, Eggleston 1958] The operations: intersection, scalar multiplication, closure, interior, vector sum and linear transformation performed on convex sets preserve convexity.*

2.2 Convex Functions

Definition 2.2.1 [Florenzano et al 2001,Wayne 1973, Webster 1994] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function. f is *convex* if the domain of f is a convex set, and $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \forall\, x,y \in$ domain of f and $\alpha \in [0,1]$.

f is *strictly convex* if $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$ for x 6= y and $\alpha \in (0,1)$.

It is important to look at three equivalent definitions of convex functions of one real variable.

Definition 2.2.2 [Dineen 1989] A real-valued measurable function f of a real variable x, is *convex* on the interval (a,b), if $\forall\ x,y \in (a,b)$,

$$f(\frac{x+y}{2}) < \frac{1}{2}[f(x) + f(y)].$$

Definition 2.2.3 [Dineen 1989, Webster 1994] A C^2 function f on (a,b) is *convex* if and only if, $\frac{d^2f}{dx^2} \geq 0 \text{ on } (a,b)$,
where C^2 implies the collection of twice continuously differentiable functions.

Lastly we consider the functions f on (a,b), for which $\frac{d^2f}{dx^2} = 0$. These are continuous affine functions.

Definition 2.2.4 [Dineen 1989] Let g be a function on (a,b), then g is *convex* if and only if, $\forall x,y \in (a,b), x < y$ and for any affine function ϕ, we have $g(x) \leq \phi(x)$ and $g(y) \leq \phi(y)$.

This implies that $g(t) \leq \phi(t)$ for $x \leq t \leq y$.

Definition 2.2.5 [Boyd 2004, Florenzano et al 2001] The *epigraph* of a function $f : I \subseteq \mathrm{R} \to \mathrm{R}$ is the subset of R^2 given by

$\mathrm{Epi}(f) = \{(x,w) : x \in \text{domain of } f, w \in \mathrm{R}, f(x) \leq w\}$.

Theorem 2.2.6 *A function f is convex, if and only if, Epi(f) is a convex set.*
Definition 2.2.7 [Wayne 1973, Niculescu, Persson 2006] A function $f : I \subseteq \mathrm{R} \to \mathrm{R}$ is said to be *midconvex* if

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) \qquad \forall\, x_1, x_2 \in I.$$

An equivalent definition is given for midconvexity using n points, this definition depends on the notion of rational convex combination of points. We give this equivalent definition for midconvexity as a theorem.

Theorem 2.2.8 *[Wayne 1973] f is midconvex on the convex set U ⊆ L if and only if for any rational convex combination of points in U*

$$f\Big(\sum_{i=1}^{n}\alpha_i x_i\Big) \leq \sum_{i=1}^{n}\alpha_i f(x_i), \qquad (2.0)$$

where $\alpha_i \geq 0$ *for* $i = 1, ..., n,\ \alpha_i' s$ *are rational and* $\sum_{i=1}^{n}\alpha_i = 1$.

A convex function is continuous on the interior of its domain, while a midconvex function is not necessarilly continuous and it is clear that convexity of a function implies midconvexity.

Theorem 2.2.9 *[Niculescu, Persson 2006] Let* $f : I \to \mathbb{R}$ *be a continuous function. Then f is convex if and only if f is midconvex, where I is a nondegenerate interval.*

Proof

Let $f : I \to \mathbb{R}$ be a continuous function and suppose f is convex then $\forall x,y \in I$ and $\forall \lambda \in [0,1]$ we have

$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$, Setting $\lambda = \frac{1}{2}$

into the above expression we obtain

$$f\Big(\frac{1}{2}x + \frac{1}{2}y\Big) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

thus f is midconvex.

Conversely: using proof by contradiction:

Suppose f is continuous and midconvex but not convex, then there exists a subinterval $[a,b]$ such that the graph of $f|_{[a,b]}$ is not under the chord joining $(a,f(a))$ and $(b,f(b))$; that is the function

$$F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a), \quad x \in [a,b]$$

verifies $\gamma = \sup\{F(x) : x \in [a,b]\} > 0$.

Clearly F is continuous, since f is continuous and $F(a) = F(b) = 0$.

We claim that the function $F : [a,b] \to \mathbb{R}$ is midconvex.

Proof of claim

Let F be as defined above and let $x_1, x_2 \in [a,b]$, then

$$\begin{aligned} F\Big(\frac{1}{2}x_1 + \frac{1}{2}x_2\Big) &= f\Big(\frac{1}{2}x_1 + \frac{1}{2}x_2\Big) - \frac{f(b)-f(a)}{b-a}\Big[\Big(\frac{1}{2}x_1 + \frac{1}{2}x_2 - a\Big)\Big] - f(a) \\ &\le \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) - \frac{f(b)-f(a)}{b-a}\Big[\Big(\frac{1}{2}x_1 + \frac{1}{2}x_2 - a\Big)\Big] - f(a), \end{aligned}$$

since f is midconvex. So

$$F\Big(\frac{1}{2}x_1+\frac{1}{2}x_2\Big) \le \frac{1}{2}f(x_1)+\frac{1}{2}f(x_2)-\frac{f(b)-f(a)}{b-a}\Big[\Big(\frac{1}{2}x_1+\frac{1}{2}x_2-\frac{1}{2}a-\frac{1}{2}a\Big)\Big]-\frac{1}{2}f(a)-\frac{1}{2}f(a)$$

$$= \frac{1}{2}f(x_1)-\frac{1}{2}\frac{f(b)-f(a)}{b-a}\Big(x_1-a\Big)-\frac{1}{2}f(a)+\frac{1}{2}f(x_2)-\frac{1}{2}\frac{f(b)-f(a)}{b-a}\Big(x_2-a\Big)-\frac{1}{2}f(a)$$

$$= \frac{1}{2}\Big[f(x_1)-\frac{f(b)-f(a)}{b-a}\Big(x_1-a\Big)-f(a)\Big]+\frac{1}{2}\Big[f(x_2)-\frac{f(b)-f(a)}{b-a}\Big(x_2-a\Big)-f(a)\Big],$$

hence we have

$$F\Big(\frac{1}{2}x_1 + \frac{1}{2}x_2\Big) \le \frac{1}{2}F(x_1) + \frac{1}{2}F(x_2)$$

which proves the claim.

Now let $c = \inf\{x \in [a,b] : F(x) = \gamma\}$; then necessarily $F(c) = \gamma$ and $c \in (a,b)$.

By the definition of c, for every $h > 0$ for which $c \pm h \in (a,b)$ we have

$$F(c-h) < F(c) \text{ and } F(c+h) \le F(c)$$

so that

$$F(c) > \frac{F(c-h)+F(c+h)}{2}$$

which contradicts the fact that F is midconvex, hence f is convex.

2.2.1 Operations that preserve the convexity of functions

Theorem 2.2.10 *[Boyd 2004, Niculescu, Persson 2006] The sum of two convex functions is a convex function which is a strictly convex function if one of the two convex functions is strictly convex.*

Proof

Let $f : I \to \mathrm{R}$ and $g : I \to \mathrm{R}$ be two convex functions defined on I then

$\forall, x_1, x_2 \in I$ and $\lambda_1, \lambda_2 \in [0,1] : \lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

and

$$g(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 g(x_1) + \lambda_2 g(x_2).$$

Now defining a new function $h : I \to \mathrm{R}$ such that $\forall x \in I$

$$h(x) = f(x) + g(x),$$

and setting $x = \lambda_1 x_1 + \lambda_2 x_2$, since I is convex then

$$h(\lambda_1 x_1 + \lambda_2 x_2) = f(\lambda_1 x_1 + \lambda_2 x_2) + g(\lambda_1 x_1 + \lambda_2 x_2)$$

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + g(\lambda_1 x_1 + \lambda_2 x_2)$$

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_1 g(x_1) + \lambda_2 g(x_2)$$

since f and g are convex.

So

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 (f(x_1) + g(x_1)) + \lambda_2 (f(x_2) + g(x_2))$$

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 h(x_1) + \lambda_2 h(x_2)$$

follows from the definition of h.

Thus the function h is convex.

Theorem 2.2.11 *[Boyd 2004, Niculescu, Persson 2006] The multiplication of a convex function by a positive scalar preserves the convexity of the function while the multiplication of a convex function by a negative scalar transforms the function into a concave function.*

Proof

Let $f : I \rightarrow \mathrm{R}$ be a convex function and let c be a positive scalar. We also define a function $h : I \rightarrow \mathrm{R}$ such that $\forall x \in I$, and $\forall c \in \mathrm{R}$ (a scalar): $h(x) = cf(x)$.

Now let $x = \lambda_1 x_1 + \lambda_2 x_2$, where $x_1, x_2 \in I$ and $\lambda_1, \lambda_2 \in [0,1] : \lambda_1 + \lambda_2 = 1$, then

$$cf(\lambda_1 x_1 + \lambda_2 x_2) = h(\lambda_1 x_1 + \lambda_2 x_2).$$

If c is a positive scalar, then

$$h(\lambda_1 x_1 + \lambda_2 x_2) = cf(\lambda_1 x_1 + \lambda_2 x_2)$$

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq c(\lambda_1 f(x_1) + \lambda_2 f(x_2))$$

since f is a convex function.

So

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 cf(x_1) + \lambda_2 cf(x_2) \; h(\lambda_1 x_1 +$$

$$\lambda_2 x_2) \leq \lambda_1 h(x_1) + \lambda_2 h(x_2),$$

hence h is convex.

If c is a negative scalar, then we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2),$$

since f is a convex function.

Now multiplying the above inequality by the scalar $c(c < 0)$, we obtain

$$cf(\lambda_1 x_1 + \lambda_2 x_2) \geq c[\lambda_1 f(x_1) + \lambda_2 f(x_2)].$$

Thus

$$h(\lambda_1 x_1 + \lambda_2 x_2) \geq c(\lambda_1 f(x_1) + \lambda_2 f(x_2))\ h(\lambda_1 x_1 +$$

$$\lambda_2 x_2) \geq \lambda_1 cf(x_1) + \lambda_2 cf(x_2)\ h(\lambda_1 x_1 + \lambda_2 x_2) \geq$$

$$\lambda_1 h(x_1) + \lambda_2 h(x_2),$$

hence h is concave.

Theorem 2.2.12 *[Niculescu, Persson 2006] If $f : I \to \mathrm{R}$ is a convex function and $g : \mathrm{R} \to \mathrm{R}$ is also an increasing convex function, then the composition function given by $g \circ f : I \to \mathrm{R}$ defined on the domain I by $g \circ f(x) = g(f(x)) \forall x \in I$*

is also a convex function.

Proof

Let $f : I \to \mathrm{R}$ be a convex function and let $g : \mathrm{R} \to \mathrm{R}$ be an increasing convex function.

Taking $x_1, x_2 \in I$ and $\lambda_1, \lambda_2 \in [0,1] : \lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned}
(g \circ f)(\lambda_1 x_1 + \lambda_2 x_2) &= g(f(\lambda_1 x_1 + \lambda_2 x_2)) \\
&\leq g(\lambda_1 f(x_1) + \lambda_2 f(x_2)) \\
&\leq \lambda_1 g(f(x_1)) + \lambda_2 g(f(x_2)) \\
&= \lambda_1 (g \circ f)(x_1) + \lambda_2 (g \circ f)(x_2),
\end{aligned}$$

hence the composite function $g \circ f$ is convex.

2.2.2 Smoothness Properties Of Convex Functions On Intervals.

Functions, in their generality, are classified according to the properties, behaviours and characteristics of their derivatives. Higher order differentiability corresponds to the existence of more derivatives of higher order and thus of complex behaviour. Functions that have derivatives of all orders (that is: convex functions such as the exponential function e^x) are called free functions.

Hence the discussion on the smoothness properties of a convex function on an interval is based on the characteristics and the behaviour of the convex function with respect to its derivative and its characterisation in terms of slope of the variables secant through arbitrary fixed points of its graph (where secant means the intersection of straight line at a point or two of a curve of the graph).

The study of the smoothness properties of a convex function is a wide and complex area and thus certain basic properties of convex functions such as the continuity and differentiability of functions (convex) must be clearly understood.

Bounds of Convex Functions

Theorem 2.2.13 *[Wayne 1973] Let f be a finite convex function defined on a closed interval* $[a, b]$*. Then f is bounded above by* $M = max\{f(a), f(b)\}$*.*

Proof

Let $c \in [a, b]$. Then c can be expressed as a convex combination of a and b and so:

$$c = \lambda_1 a + \lambda_2 b$$

where λ_1 and λ_2 are scalars such that $\lambda_1 + \lambda_2 = 1$. Thus for any c in $[a, b]$, we have that

$$f(c) = f(\lambda_1 a + \lambda_2 b)$$

$$f(c) \leq \lambda_1 f(a) + \lambda_2 f(b).$$

Let $M = max\{f(a), f(b)\}$ and thus

$$f(c) \leq \lambda_1 M + \lambda_2 M$$

$$f(c) \leq (\lambda_1 + \lambda_2)M \; f(c) \leq$$

$$M$$

since $\lambda_1 + \lambda_2 = 1$.

Thus f is bounded above by M.

Theorem 2.2.14 *[Wayne 1973] If f is a finite convex function defined on the interval* $[a, b]$*, then f is also bounded below.*

Proof

Choose an arbitrary point c in the interval $[a, b]$ and $t > 0$ an arbitrary small real number such that $c = \frac{a+b}{2} + t$. From the definition of a convex function, we have that:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left[\frac{1}{2}\left(\frac{2(a+b)}{2}\right)\right] \\ &= f\left[\frac{1}{2}\left(\frac{a+b}{2} + \frac{a+b}{2} + t - t\right)\right] \\ &= f\left[\frac{1}{2}\left(\frac{a+b}{2} + t\right) + \frac{1}{2}\left(\frac{a+b}{2} - t\right)\right] \\ &\leq \frac{1}{2}f\left(\frac{a+b}{2} + t\right) + \frac{1}{2}f\left(\frac{a+b}{2} - t\right) \end{aligned}$$

since f is a convex function. So

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2}f\left(\frac{a+b}{2} - t\right) \leq \frac{1}{2}f\left(\frac{a+b}{2} + t\right)$$

thus

$$2f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2} - t\right) \leq f\left(\frac{a+b}{2} + t\right).$$

But $\frac{a+b}{2} + t = c$ and so

$$2f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2} - t\right) \leq f(c).$$

By Theorem 2.2.12., we have

$$f\left(\frac{a+b}{2} - t\right) < M$$

so

$$2\Big(\frac{a+b}{2}\Big) - M \leq f(c)$$.

Setting $m = 2f(\frac{a+b}{2}) - M$, we have

$$m \leq f(c),$$

thus *f* is bounded below.

Continuity of Convex function:

From a graphical observation, it is clear that a convex function *f* is continuous at every point *x* in the interior of its domain.

We want to prove that a convex function satisfies the Lipschitz Condition in the interior of its domain. More precisely, we want to show that if $f : I \to \mathrm{R}$ is a convex function defined on the domain *I*, then $\forall x_1, x_2 \in intI$, we have that $| f(x_1) - f(x_2) | \leq k | x_1 - x_2 |$ where *k* is a real constant.

Theorem 2.2.15 *[Gruber 2007, Wayne 1973] If $f : I \to \mathrm{R}$ is convex, then f satisfies a Lipschitz Condition on any closed interval [a,b] contained in the interior of I.*

Consequently, f is absolutely continuous on [a,b] and continuous on the interior of I.

Proof

Let > 0 so that $a - \epsilon$ and $b + \epsilon$ belong to *I* and let *m* and *M* be the lower and upper bounds for f on $[a - \epsilon, b + \epsilon]$.

If *x* and *y* are distinct points of [*a,b*], set

$$z = y + \frac{\epsilon}{|y-x|}(y-x), \quad \lambda = \frac{|y-x|}{\epsilon + |y-x|}.$$

Then $z \in [a - \epsilon, b + \epsilon], y = \lambda z + (1 - \lambda)x,$ and we have

$$f(y) \leq \lambda f(z) + (1 - \lambda) f(x) = \lambda[f(z) - f(x)] + f(x).$$

So

$$f(y) - f(x) \leq \lambda(M - m) < \frac{|y-x|}{\epsilon}(M - m) = K|y - x|$$

where $K = \frac{(M-m)}{\epsilon}$.

Since this is true for any $x,y \in [a,b]$, we conclude that

$$|f(x) - f(y)| \leq K|y - x|.$$

Next we recall that f is absolutely continuous on $[a,b]$ if

$\forall \epsilon > 0, \quad \exists \quad \delta >0$ such that for any collection $\{(a_i,b_i)\}^n{}_i$ of disjoint open subintervals of $[a,b]$ with $\sum_{i=1}^{n}(b_i - a_i) < \delta, \quad \sum_{i=1}^{n}|f(b_i) - f(a_i))| < \epsilon$.

Clearly the choice $\delta = \frac{\epsilon}{K}$ meets the requirement.

Finally the continuity of f on the interior of I is a consequence of the arbitrariness of $[a,b]$.

Derivatives of Convex functions.

A function is said to be differentiable at a point a if there exists an affine map tangent to the function at the point a, and the derivative at a is given by:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Convex functions, specifically, are better described and studied by the analysis of their left and right derivatives defined below:

- The left derivative of a function f at a point a denoted by f'_- is defined by:

$$f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

- The right derivative of a function f at a point a denoted by f'_+ is defined by:

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

The following theorems are important results that best describe the smoothness properties of convex functions.

Theorem 2.2.16 *[Wayne 1973, Webster 1994, Peˇcari´c et al 1992] If $f : I \to \mathbb{R}$ is convex [strictly convex], then $f'_{-}(x)$ and $f'_{+}(x)$ exist and are increasing [strictly increasing] on the interior of I.*

Theorem 2.2.17 *[Gruber 2007, Webster 1994] Let $f : I \to \mathbb{R}$ be a convex function. Let x_1, x_2 and $x_3 \in I$ such that $x_1 < x_3 < x_2$. Then we have that:*

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3}$$

Proof

Let $f : I \to \mathbb{R}$ be a convex function and let x_1, x_2 and $x_3 \in I$ such that $x_1 < x_3 < x_2$. Consider p as a subinterval of I such that p is the line segment between x_1 and x_2 and consider x_3 as a point on p. Then by the definition of convex sets, we have that x_3 can be written as the convex combination of x_1 and x_2, thus $x_3 = \lambda_1 x_1 + \lambda_2 x_2$, where λ_1 and λ_2 are the required constants.

Now since $x_2 - x_1 > x_2 - x_3$ and that $x_2 - x_1 > x_3 - x_1$, we can express the constants λ_1 and λ_2 as the following:

$$\frac{x_2 - x_3}{x_2 - x_1} = \lambda_1 \quad \text{and} \quad \frac{x_3 - x_1}{x_2 - x_1} = \lambda_2$$

because $0 < \frac{x_2 - x_3}{x_2 - x_1} < 1$, $0 < \frac{x_3 - x_1}{x_2 - x_1} < 1$ and that $\frac{x_2 - x_3}{x_2 - x_1} + \frac{x_3 - x_1}{x_2 - x_1} = 1$.

So x_3 can be re-written as the convex combination of x_1 and x_2 as

$$x_3 = \left(\frac{x_2 - x_3}{x_2 - x_1}\right) x_1 + \left(\frac{x_3 - x_1}{x_2 - x_1}\right) x_2 .$$

Thus

$$
\begin{aligned}
f(x_3) &= f\Big[\left(\frac{x_2-x_3}{x_2-x_1}\right)x_1+\left(\frac{x_3-x_1}{x_2-x_1}\right)x_2\Big] \\
&\leq \left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_1)+\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_2) \\
&= \left(\frac{x_2-x_1+x_1-x_3}{x_2-x_1}\right)f(x_1)+\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_2) \\
&= \Big[\left(\frac{x_2-x_1}{x_2-x_1}\right)-\left(\frac{x_3-x_1}{x_2-x_1}\right)\Big]f(x_1)+\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_2) \\
&= \Big[1-\left(\frac{x_3-x_1}{x_2-x_1}\right)\Big]f(x_1)+\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_2) \\
&= f(x_1)-\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_1)+\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_2)
\end{aligned}
$$

$$
\begin{aligned}
f(x_3)-f(x_1) &\leq \left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_2)-\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_1) \\
&= \left(\frac{x_3-x_1}{x_2-x_1}\right)(f(x_2)-f(x_1)),
\end{aligned}
$$

multiplying through the above inequality by $\frac{1}{x_3-x_1}>0$, we have that

$$\frac{f(x_3)-f(x_1)}{x_3-x_1}\leq\frac{f(x_2)-f(x_1)}{x_2-x_1}. \tag{2.1}$$

Now from

$$
\begin{aligned}
f(x_3) &\leq \left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_1)+\left(\frac{x_3-x_1}{x_2-x_1}\right)f(x_2) \\
&= \left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_1)+\left(\frac{-x_2+x_3+x_2-x_1}{x_2-x_1}\right)f(x_2) \\
&= \left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_1)+\left(\frac{-x_2+x_3+x_2-x_1}{x_2-x_1}\right)f(x_2) \\
&= \left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_1)+\Big(-\left(\frac{x_2-x_3}{x_2-x_1}\right)+1\Big)f(x_2) \\
&= \left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_1)+f(x_2)-\left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_2)
\end{aligned}
$$

$$\left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_2)-\left(\frac{x_2-x_3}{x_2-x_1}\right)f(x_1)\leq f(x_2)-f(x_3)$$

$$\left(\frac{x_2-x_3}{x_2-x_1}\right)\Big(f(x_2)-f(x_1)\Big)\leq f(x_2)-f(x_3)$$

and multiplying this through by $\frac{1}{x_2-x_3}>0$, we obtain

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\leq\frac{f(x_2)-f(x_3)}{x_2-x_3}. \tag{2.2}$$

Combining (2.1) and (2.2) we have

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3} \tag{2.3}$$

where $x_1 < x_3 < x_2$.

Corollary 1 *[Webster 1994] If $f : I \to \mathbb{R}$ is a convex function, then we can define another function $g_s : I \setminus \{s\} \to \mathbb{R}$ given by*

$$g_s(x) = \frac{f(x) - f(s)}{x - s}$$

where $s \in I$ and $x \in I \setminus \{s\}$ such that $g_s : I \setminus \{s\} \to \mathbb{R}$ is an increasing function.

Proof

By definition, a function $g_s : I \setminus \{s\} \to \mathbb{R}$ is said to be an increasing function if

$\forall x_1, x_2 \in I \setminus \{s\} : x_1 \leq x_2$, we have that $g_s(x_1) \leq g_s(x_2)$.

Let $f : I \to \mathbb{R}$ be convex and $g_s : I \setminus \{s\} \to \mathbb{R}$ be defined by $g_s(x) = \frac{f(x)-f(s)}{x-s}$ where $x \in I$.

We want to show that $g_s(x_1) \leq g_s(x_2)$. To do this, we let $x_1, x_2 \in I \setminus \{s\}$ such that $x_1 < x_2$. Then for our considerations we have three possible conditions:

Either $s < x_1 < x_2$ or $x_1 < s < x_2$ or $x_1 < x_2 < s$.

Case 1

If $s < x_1 < x_2$ then we have from (2.3) by setting $s = x_1$ and $x_3 = x_1$, that

$$\frac{f(x_1) - f(s)}{x_1 - s} \leq \frac{f(x_2) - f(s)}{x_2 - s} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and hence

$$\frac{f(x_1) - f(s)}{x_1 - s} \leq \frac{f(x_2) - f(s)}{x_2 - s}$$

that is

$$g_s(x_1) \leq g_s(x_2)$$

Hence g_s is an increasing function if $s < x_1 < x_2$.

Case 2

If $x_1 < s < x_2$, then we have from (2.3) by setting $x_3 = s$, that

$$\frac{f(s)-f(x_1)}{s-x_1} \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_2)-f(s)}{x_2-s}$$

and so

$$\frac{f(s)-f(x_1)}{s-x_1} \leq \frac{f(x_2)-f(s)}{x_2-s}$$

that is

$$\frac{-[f(x_1)-f(s)]}{-(x_1-s)} \leq \frac{f(x_2)-f(s)}{x_2-s}$$

thus

$$\frac{f(x_1)-f(s)}{x_1-s} \leq \frac{f(x_2)-f(s)}{x_2-s}$$

so

$$g_s(x_1) \leq g_s(x_2)$$

Hence g_s is an increasing function if $x_1 < s < x_2$. **Case 3** $x_1 < x_2 < s$, then we have from (2.3) by letting $x_3 = x_2$ and $x_2 = s$, that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(s)-f(x_1)}{s-x_1} \leq \frac{f(s)-f(x_2)}{s-x_2}$$

and so which

$$\frac{f(s)-f(x_1)}{s-x_1} \leq \frac{f(s)-f(x_2)}{s-x_2}$$

$$\frac{-[f(x_1)-f(s)]}{-[x_1-s]} \leq \frac{-[f(x_2)-f(s)]}{-[x_2-s]}$$

$$\frac{f(x_1)-f(s)}{x_1-s} \leq \frac{f(x_2)-f(s)}{x_2-s}$$

$$g_s(x_1) \leq g_s(x_2)$$

completes the proof regardless of the value of the variable *s.*

Hence g_s is an increasing function on $I \setminus \{s\}$.

Theorem 2.2.18 *[Pečarić et al 1992, Wayne] Let* $f : (a,b) \to \mathbb{R}$ *be a function, then* f *is convex [strictly convex] if and only if there is an increasing [strictly increasing] function* $g : (a,b) \to \mathbb{R}$ *and a point* $c \in (a,b)$ *such that for all* $x \in (a,b)$,

$$f(x) - f(c) = \int_c^x g(t)dt.$$

Proof

We suppose first that f is convex. Choose $g = f'_+$ which exists and is increasing by Theorem 2.2.15 and let c be any point in (a,b).

By Theorem 2.2.14, f is absolutely continuous on $[c,x]$, $\forall\, x \in (a,b)$.

By an elementary argument for Riemann integrals,

$$f(x) - f(c) = \int_c^x f'_+(t)dt = \int_c^x g(t)dt.$$

Moreover, if f is strictly convex, $g = f'_+$ will be strictly increasing (Theorem 2.2.15)

Conversely, suppose that $f(x) - f(c) = \int_c^x g(t)dt$ holds with g increasing.

Let α,β be positive with $\alpha + \beta = 1$. Then for $x < y$ in (a,b),

$$\alpha f(x) + \beta f(y) - (\alpha + \beta) f(\alpha x + \beta y) = \beta \int_{\alpha x+\beta y}^{y} g(t)dt - \alpha \int_x^{\alpha x+\beta y} g(t)dt.$$

To bound this expression below, we replace both integrands by the constant $g(\alpha x + \beta y)$, this being the smallest value of the first integrand and the largest of the second. We obtain on the right-hand side

$$\beta g(\alpha x + \beta y)[y - (\alpha x + \beta y)] - \alpha g(\alpha x + \beta y)[\alpha x + \beta y - x]$$

which simplifies to 0. Thus

$$\alpha f(x) + \beta f(y) - f(\alpha x + \beta y) \geq 0$$

which is equivalent to the inequality that defines convexity.

Finally, we note that the estimate made above is a strict one when g is strictly increasing.

Theorem 2.2.19 *[Pečarić et al 1992, Wayne, Webster] Suppose f is differentiable on (a,b). Then f is convex [strictly convex] if and only if f' is increasing [strictly increasing].*

Proof

Let f be differentiable, then by Theorem 2.2.15, f' is increasing.

Conversely, suppose f' is increasing [strictly increasing]. Then the fundamental theorem of calculus assures us that

$$f(x) - f(c) = \int_c^x f'(t)dt$$

for any $c \in (a,b)$. f being convex [strictly convex] follows from Theorem 2.2.17.

Theorem 2.2.20 *[Pečarić et al 1992, Wayne 1973] Suppose f'' exists on (a,b). Then f is convex if and only if $f'' \geq 0$. And if $f''(x) > 0$ on (a,b) then f is strictly convex on the interval.*

Proof

Under the given assumption, f' is increasing if and only if f'' is non-negative and f' is strictly increasing when f'' is positive. This combined with Theorem 2.2.18 gives us the result.

Theorem 2.2.21 *[Webster 1994] Let $f : I \to \mathbb{R}$ be a convex function. Then f is continuous and differentiable at each point in the interior of its domain and has left and right derivatives such that $\forall x_1, x_2 \in intI$ with $x_1 < x_2$, we have that*

$$f'_-(x_1) \leq f'_+(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'_-(x_2) \leq f'_+(x_2)$$

Proof

Let $f : I \to \mathbb{R}$ be a real valued convex function defined on I. Let $x_1, x_2, a \in intI$ be such that $x_1 < a < x_2$.

We can thus write from Theorem 2.2.16 that:

$$\frac{f(x_1) - f(a)}{x_1 - a} \leq \frac{f(x_2) - f(a)}{x_2 - a}$$

and thus

$$\lim_{x_1 \to a^-} \frac{f(x_1) - f(a)}{x_1 - a} \leq \lim_{x_2 \to a^+} \frac{f(x_2) - f(a)}{x_2 - a}$$

$$f'_-(a) \leq f'_+(a) \quad (2.4)$$

since $f'_-(a) = \lim_{x_1 \to a^-} \frac{f(x_1)-f(a)}{x_1 - a}$ and that $f'_+(a) = \lim_{x_2 \to a^+} \frac{f(x_2)-f(a)}{x_2 - a}$.

Now we consider the following variables $x_1, x_2, a, b \in intI$ such that $x_1 \leq x_2 \leq a \leq b$ and thus by Theorem 2.2.16, we obtain

$$\frac{f(x_1) - f(a)}{x_1 - a} \leq \frac{f(x_2) - f(a)}{x_2 - a} \leq \frac{f(b) - f(a)}{b - a}$$

and thus

$$\lim_{x_1 \to a^-} \frac{f(x_1) - f(a)}{x_1 - a} \leq \lim_{x_2 \to a} \frac{f(x_2) - f(a)}{x_2 - a} \leq \lim_{b \to a^+} \frac{f(b) - f(a)}{b - a}$$

$$f'_-(a) \leq f'_+(a)$$

which confirms (2.4) and thus $f'_-(a) \leq f'_+(a) \Rightarrow f'_-(x) \leq f'_+(x)$.

Now suppose that $x_1, x_2, a, b \in intI$ such that $x_1 < a < b < x_2 \Rightarrow x_1 < x_2$, then by Theorem 2.2.16, we can deduce that

$$\frac{f(a) - f(x_1)}{a - x_1} \leq \frac{f(b) - f(x_1)}{b - x_1} \leq \frac{f(b) - f(x_2)}{b - x_2},$$

taking the limits as $a \to x_1^+$ and as $b \to x_2^-$, we have that

$$\lim_{a \to x_1^+} \frac{f(a) - f(x_1)}{a - x_1} \leq \lim_{b \to x_2^-} \frac{f(b) - f(x_2)}{b - x_2},$$

since $x_1 < x_2$ then by Corollary 1

$$f'_-(x_1) \leq f'_+(x_2).$$

So we can give a partial conclusion of the Theorem and that is

$$f'_-(x_1) \leq f'_+(x_1) \leq f'_-(x_2) \leq f'_+(x_2) \tag{2.5}$$

$\forall x_1, x_2 \in intI$ such that $x_1 < x_2$.

We now conclude the proof of the Theorem by showing that $f'_+(x_1) \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq f'_-(x_2)$.

Since f is continuous and differentiable and f^0 is increasing on any $x_1, x_2 \in intI$, by the Mean Value Theorem $\exists\, d \in (x_1, x_2) \subset I$, such that

$$f'(d) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and that

$$f'_+(x_1) \leq f'(d) \leq f'_-(x_2)$$

$$f'_+(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'_-(x_2). \tag{2.6}$$

We finally conclude from (2.5) and (2.6) that

$$f'_-(x_1) \leq f'_+(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'_-(x_2) \leq f'_+(x_2)$$

$\forall x_1, x_2 \in intI$ such that $x_1 < x_2$.

Corollary 2 *Let $f : I \to \mathbb{R}$ define a real valued function and let x_1, x_2 be elements in the interior of the domain such that $x_1 < x_2$. If the inequality given by*

$f'_-(x_1) \leq f'_+(x_1) \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq f'_-(x_2) \leq f'_+(x_2)$ *holds for all interior points of the domain, then f is a convex function.*

Proof

Let $f : I \to \mathbb{R}$ define a real valued function and let $x_1, x_2 \in intI$ such that $x_1 < x_2$, so that the expression

$$f'_-(x_1) \leq f'_+(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'_-(x_2) \leq f'_+(x_2)$$

holds.

Let us further assume that $\exists a \in (x_1, x_2) \Rightarrow x_1 < a < x_2$, then a can be written as the convex combination of both x_1 and x_2 by considering λ_1, λ_2 as the appropriate constants such that $a = \lambda_1 x_1 + \lambda_2 x_2$.

From equation (2.6) we had that

$$f'_+(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'_-(x_2).$$

Let us again, suppose that $\exists c,d \in intI$ such that $x_1 \leq c < a < d \leq x_2$ and thus $c \in [x_1,a]$ and $d \in [a,x_2]$.

From the Mean Value Theorem we can write that

$$\frac{f(a) - f(x_1)}{a - x_1} = f'(c) \text{ and } f'(d) = \frac{f(x_2) - f(a)}{x_2 - a}.$$

Since $c < d$ and g defined by $g = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ is an increasing function per Corollary 1, of Theorem 2.2.16 we can deduce the following:

$$f'(c) \leq f'(d)$$

so

$$\frac{f(a) - f(x_1)}{a - x_1} \leq \frac{f(x_2) - f(a)}{x_2 - a}.$$

But $a = \lambda_1 x_1 + \lambda_2 x_2$ and so

$$\frac{f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1)}{(\lambda_1 x_1 + \lambda_2 x_2) - x_1} \leq \frac{f(x_2) - f(\lambda_1 x_1 + \lambda_2 x_2)}{x_2 - (\lambda_1 x_1 + \lambda_2 x_2)}$$

that is

$$\frac{f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1)}{x_1(\lambda_1 - 1) + \lambda_2 x_2} \leq \frac{f(x_2) - f(\lambda_1 x_1 + \lambda_2 x_2)}{x_2(1 - \lambda_2) - \lambda_1 x_1}.$$

In addition $\lambda_1 + \lambda_2 = 1 \Rightarrow \lambda_1 = 1 - \lambda_2$ and so

$$\frac{f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1)}{\lambda_2(x_2 - x_1)} \leq \frac{f(x_2) - f(\lambda_1 x_1 + \lambda_2 x_2)}{\lambda_1(x_2 - x_1)}.$$

Multiplying through by $(x_2 - x_1)$, we obtain:

$$\frac{f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1)}{\lambda_2} \leq \frac{f(x_2) - f(\lambda_1 x_1 + \lambda_2 x_2)}{\lambda_1}$$

$$\lambda_1[f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1)] \le \lambda_2[f(x_2) - f(\lambda_1 x_1 + \lambda_2 x_2)]$$
$$\lambda_1 f(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 f(x_1) \le \lambda_2 f(x_2) - \lambda_2 f(\lambda_1 x_1 + \lambda_2 x_2)$$
$$\lambda_1 f(\lambda_1 x_1 + \lambda_2 x_2) + \lambda_2 f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_2 f(x_2) + \lambda_1 f(x_1)$$
$$f(\lambda_1 x_1 + \lambda_2 x_2)(\lambda_1 + \lambda_2) \le \lambda_1 f(x_1 + \lambda_2 f(x_2).$$

Finally since $\lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

and hence f is a convex function.

Our next characterisation depends on the geometrically evident idea that through any point on the graph of a convex function, there is a line which lies on or below the graph. More formally, we say that a function f defined on I has support at $x_0 \in I$ if there exists an affine function $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \le f(x)$ for every $x \in I$. The graph of the support function A is called a line of support for f at x_0.

Theorem 2.2.22 *[Pečarić et al 1992, Wayne 1973, Webster 1994]* $f : (a,b) \to \mathbb{R}$ *is convex if and only if there is at least one line of support for* f *at each* $x_0 \in (a,b)$.

Proof

Let f be convex and $x_0 \in (a,b)$, choose $m \in [f'_-(x_0), f'_+(x_0)]$. Then

$$\frac{f(x)-f(x_0)}{x-x_0} \ge m \text{ or } \le m$$

according as $x > x_0$ or $x < x_0$. In either case, $f(x) - f(x_0) \ge m(x - x_0)$; that is, $f(x) \ge f(x_0) + m(x - x_0)$.

Conversely, suppose that f has a line of support at each point of (a,b).

Let $x,y \in (a,b)$ and $x_0 = \lambda x + (1-\lambda)y$, $\lambda \in [0,1]$, let $A(x) = f(x_0) + m(x - x_0)$ be the support function for f at x_0. Then

$$f(x_0) = A(x_0) = \lambda A(x) + (1-\lambda)A(y) \le \lambda f(x) + (1-\lambda)f(y)$$

as required

Theorem 2.2.23 *[Gruber 2007, Wayne 1973, Webster 1994] Let $f : (a,b) \to \mathbb{R}$ be convex. Then f is differentiable at x_0 if and only if the line of support for f at x_0 is unique. And in this case,*

$$A(x) = f(x_0) + f'(x_0)(x - x_0)$$

provides this unique support.

Proof

From Theorem 2.2.21 it is clear that for each $m \in [f'_{-}x_0, f'_{+}(x_0)]$, there is a line of support for f at x_0. Uniqueness of the line therefore means $f'_{-}(x_0) = f'_{+}(x_0)$; that is $f'(x_0)$ exists.

Conversely, suppose $f'(x_0)$ exists, then any line of support

$A(x) = f(x_0) + m(x - x_0)$ gives us $f(x) - f(x_0) \geq m(x - x_0)$.

For $x_1 < x_0 < x_2$, we have

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq m \leq \frac{f(x_2) - f(x_0)}{x_2 - x_0}.$$

Taking limits as $x_1 \to x_0$ and $x_2 \to x_0$ gives $f'_{-}(x_0) \leq m \leq f'_{+}(x_0)$, so differentiability at x_0 implies uniqueness of m, hence of the support A at x_0.

Chapter 3

Superquadratic functions

In general a real-valued function f is said to be quadratic if

$$f(x + y) + f(x - y) = 2[f(x) + f(y)] \tag{3.1}$$

for all x,y in the domain of f.

When the "=" is replaced by " ≥" in (3.1), the function f is called superquadratic and subquadratic for "≤".

Two known definitions have been given for functions considered to be superquadratic, one is a modification of the geometrical notion of convex functions, while the other is closely linked to (3.1). These two classes of functions coincide but are not equal as one class contains only continuous functions and the other does not require

continuity.

A convex function has a line of support at each point of its domain and lies above each of its support lines. That is, for each x there exists a slope P_x such that

$$f(y) \geq f(x) + P_x(y - x)$$

for all y.

For a superquadratic function we require that the function lie above its line of support plus a translation of itself.

Definition 3.0.24 [Abramovich et al 2004a, Abramovich 2009] A function ϕ : $[0,\infty) \to \mathbb{R}$ is *superquadratic* provided that for all $x \geq 0$ there exists a constant $P_x \in \mathbb{R}$ such that

$$\phi(y) \geq \phi(x) + P_x(y - x) + \phi(|y - x|) \qquad (3.2)$$

for all $y \geq 0$.

It is noticed that ϕ is subquadratic if and only if $-\phi$ is superquadratic.

Proof

Let $-\phi$ be superquadratic on $[0,\infty)$.

Then $\forall x \in [0,\infty)$, there exists a constant $P_x \in \mathbb{R}$ depending on x such that

$$-\phi(y) \geq -\phi(x) + P_x(y - x) + (-\phi(|y - c|)), \qquad \forall\, y \geq 0.$$

Multiplying the above expression through by −1, we obtain

$$\phi(y) \le \phi(x) - P_x(y - x) + \phi(|y - x|),$$

setting $\hat{P}_x = -P_x \in \mathbb{R}$, which is still a constant depending on x, we have for all $x \ge 0$,

$$\varphi(y) \le \varphi(x) + \hat{P}_x(y - x) + \varphi(|y - x|), \quad \forall y \ge 0.$$

Thus ϕ is subquadratic.

Conversely

Suppose ϕ is subquadratic.

Then $\forall x \ge 0$, there exists a constant $\hat{P}_x \in \mathbb{R}$such that

$$\varphi(y) \le \varphi(x) + \hat{P}_x(y - x) + \varphi(|y - x|).$$

Multiplying the above expression through by −1, we obtain

$$-\varphi(y) \ge -\varphi(x) - \hat{P}_x + (-\varphi(|y - x|),$$

setting $-\hat{P}_x = P_x \in \mathbb{R}$, we have that for all $x \ge 0$,

$$-\phi(y) \ge -\phi(x) + P_x(y - x) + (-\phi(|y - x|)) \qquad \forall y \ge 0.$$

Thus $-\phi$ is superquadratic.

From (3.2) superquadraticity appears to be stronger then convexity, that is ϕ superquadratic ⇒ ϕ convex, but if ϕ takes negative values then superquadraticity may be very weak, that is ϕ not convex but superquadratic.

[Abramovich, Jameson and Sinnamon 2004a] remarked that by setting $P_x = 0$ in Definition 3.0.23, any function ϕ satisfying $-2 \le \phi(x) \le -1$, for all x, is superquadratic.

This is generalized as the following Corollary:

Corollary 3 *[Asare 2009]Any function ϕ satisfying $-2r \le \phi(x) \le -r$, where $r \in \mathbb{R}^+$ for all $x \in \mathbb{R}$ is superquadratic.*

Proof

Let $\phi : [0,\infty) \to \mathbb{R}$ and $\forall x \in [0,\infty)$, let $-2r \le \phi(x) \le -r$, where $r \in \mathbb{R}^+$.

So for y and $(|y-x|) \in [0,\infty)$ we have $-2r \leq \phi(y) \leq -r$ and

$-2r \leq \phi(|y-x|) \leq -r$.

Thus $-4r \leq \phi(x) + \phi(|y-x|) \leq -2r \leq \phi(y) \leq -r$, hence $\phi(y) \geq \phi(x) + \phi(|y-x|)$. Now by setting P_x in the Definition (3.0.23) to

0, this last expression becomes $\phi(y) \geq \phi(x) + \phi(|y-x|) + P_x(y-x)$, this satisfies the definition of ϕ being

superquadratic.

The non-negative superquadratic functions are much better behaved as we can see later.

3.0.3 Properties of superquadratic functions

Lemma 3.0.25 *[Abramovich et al 2004a] Let ϕ be a superquadratic function with P_x as defined above.*

1. *Then $\phi(0) \leq 0$.*
2. *If $\phi(0) = \phi'(0) = 0$, then $P_x = \phi'(x)$ whenever ϕ is differentiable at $x > 0$.*
3. *If $\phi \geq 0$, then ϕ is convex and $\phi(0) = \phi'(0) = 0$.*

Proof (1)

In Definition 3.0.23, let $x = y$, then

$$\phi(x) \geq \phi(x) + \phi(0)$$

so

$$0 \geq \phi(0).$$

Proof of (2)

Let $x > 0$, then we consider two cases where $y < x$ and $y > x$.

For $y < x$, then $|y-x| = (x-y)$ so

$$\lim_{y\to x^-} \left(\frac{\varphi(x)-\varphi(y)}{x-y} + \frac{\varphi(x-y)}{x-y}\right) \leq P_x \quad (3.3)$$

For $y > x$, then $|y - x| = (y - x)$ so

$$\lim_{y\to x^+} \left(\frac{\varphi(y)-\varphi(x)}{y-x} - \frac{\varphi(y-x)}{y-x}\right) \geq P_x \quad (3.4)$$

Now if ϕ is differentiable at 0 and $\phi(0) = \phi^0(0) = 0$ for all $x \in [0,\infty)$ at which $\varphi'(x)$ exists, as y approaches x we have from (3.3), $\phi^0(x) \leq P_x$ and from (3.4), $P_x \leq \phi^0(x)$, hence

$$\phi^0(x) = P_x.$$

Proof (3)

If $\phi \geq 0$, then $\phi(0) \leq 0$ becomes $\phi(0) = 0$ and also as y approaches x then Definition 3.0.23 implies that $\phi(y) - \phi(x) \geq P_x(y - x)$ for all $x,y \geq 0$. If $y_1 < x < y_2$ this yields

$$\frac{\varphi(x)-\varphi(y_1)}{x-y_1} \leq P_x \leq \frac{\varphi(y_2)-\varphi(x)}{y_2-x}.$$

Multiplying the above expression by $(x - y_1)(y_2 - x)$, we obtain

$$(\phi(x) - \phi(y_1))(y_2 - x) \leq (\phi(y_2) - \phi(x))(x - y_1)$$

So

$$\phi(x)[y_2 - x - y_1 + x] \leq \phi(y_2)(x - y_1) + \phi(y_1)(y_2 - x).$$

From which we obtain,

$$\varphi(x) \leq \frac{x-y_1}{y_2-y_1}\varphi(y_2) + \frac{y_2-x}{y_2-y_1}\varphi(y_1),$$

and so ϕ is convex.

We have that for convex functions $\phi : I \to \mathbb{R}$ where I is an interval, ϕ is differentiable at all but a finite number of points. So since $[0,\infty)$ can be covered by countably many intervals $\phi^0(x)$ exists almost everywhere.

From the proof of (2) in Lemma (3.0.24) we can obtain

$$\limsup_{y\to x^-}\left(\frac{\varphi(x)-\varphi(y)}{x-y}+\frac{\varphi(x-y)}{x-y}\right)\le P_x\le \limsup_{y\to x^+}\left(\frac{\varphi(y)-\varphi(x)}{y-x}-\frac{\varphi(y-x)}{y-x}\right).$$

Let $t = x - y$, then it follows that

$$\limsup_{t\to 0^+}\frac{\varphi(t)}{t}\le 0.$$

Since ϕ is non-negative we obtain

$$0\le \liminf_{t\to 0^+}\frac{\varphi(t)}{t}\le \limsup_{t\to 0^+}\frac{\varphi(t)}{t}\le 0$$

so the (one-sided) derivative at zero exists and $\varphi'(0) = 0$. This ends the proof.

Lemma 3.0.26 *[Abramovich et al 2004b] Suppose that ϕ is superquadratic and non-negative. Then ϕ is convex and increasing. Also if P_x is as in Definition 3.0.23, then $P_x \ge 0$.*

Proof

Lemma 3.0.24 gives us the convexity. Together with $\phi(0) = 0$ and $\phi(x) \ge 0$, this implies ϕ is increasing. As mentioned already, we can take $P_x(0) = 0$. For $x > 0$ and $y < x$, we can rewrite (3.2) as

$$P_x\ge \frac{\varphi(x)-\varphi(y)+\varphi(x-y)}{x-y}\ge 0.$$

Lemma 3.0.27 *[Abramovich et al 2004b] If $\phi(0) = \phi'(0) = 0$ and ϕ' is convex (resp. concave) then ϕ is superquadratic (resp. subquadratic).*

Proof

Let ϕ' be convex φ' and(0) = 0, then we have

$$\varphi'(x)=\varphi'\left[\frac{x}{(x+y)}(x+y)\right]\le\left[\frac{x}{x+y}\right]\varphi'(x+y)$$

for $x,y \ge 0$ and hence

$$\phi'(x)+\phi'(y)\le\phi'(x+y).$$

Now let $y > x \ge 0$, then

$$\varphi(y)-\varphi(x)-\varphi(y-x)-(y-x)\varphi'(x) = \int_0^{y-x}[\varphi'(t+x)-\varphi'(t)-\varphi'(x)]dt \geq 0.$$

Similarly for the case $x > y \geq 0$.

Lemma 3.0.28 *[Abramovich et al 2004a] Suppose* $\phi : [0,\infty) \to \mathbb{R}$ *is continuously differentiable and* $\phi(0) \leq 0$*. If* ϕ' *is superadditive or* $\frac{\varphi'(x)}{x}$ *is non-decreasing, then* ϕ *is superquadratic.*

Proof

We first suppose that ϕ' is superadditive and $x \leq y$ then

$$\varphi'(y-x) \leq \varphi'(y)-\varphi'(x),$$

so

$$0 \leq \int_x^y (\varphi'(t)-\varphi'(x)-\varphi'(t-x))dt.$$

Let $v = t - x$, then $dv = dt$, hence

$$\begin{aligned} 0 &\leq \int_x^y \phi'(t)dt - \int_x^y \phi'(x)dt - \int_0^{y-x} \phi'(v)dv \\ &= \phi(y) - \phi(x) - (y-x)\phi'(x) - \phi(y-x) + \phi(0), \\ &\leq \phi(y) - \phi(x) - (y-x)\phi'(x) - \phi(y-x), \end{aligned}$$

since $\phi(0) \leq 0$.

If $y \leq x$, then

$$\varphi'(x-y) \leq \varphi'(x)-\varphi'(y),$$

so

$$0 \leq \int_x^y (\varphi'(x)-\varphi'(x-t)-\varphi'(t))dt.$$

Here we make a substitution similar to that of the one above for the case $x \leq y$ and this yields,

$$0 \le (x-y)\phi'(x) + \phi(0) - \phi(x-y) - \phi(x) + \phi(y),$$

$$\le (x-y)\phi'(x) - \phi(x-y) - \phi(x) + \phi(y).$$

From the two cases shown above, it is clear that for any $x,y \ge 0$,

$$\phi(y) - \phi(x) - \phi(|y-x|) \ge \phi'(x)(y-x).$$

When we set $P_x = \phi'(x)$ we see that ϕ is superquadratic. Let us now suppose that $\frac{\varphi'(x)}{x}$ is non-decreasing then,

$$\varphi'(x+y) = \frac{x\varphi'(x+y)}{x+y} + \frac{y\varphi'(x+y)}{x+y}$$

$$\ge \varphi'(x) + \varphi'(y),$$

and so the second condition reduces to the first, that is: ϕ' is superadditive, completing the proof.

Lemma 3.0.29 *[Abramovich et al 2004a] Suppose ϕ is differentiable and $\phi(0) = \phi'(0) = 0$. If ϕ is superquadratic, then $\frac{\varphi(x)}{x^2}$ is non-decreasing on $(0,\infty)$.*

Proof

From lemma 3.0.24. the constant P_x in the Definition 3.0.23. is necessarily $\phi'(x)$.

When we set $y = 0$ in Definition 3.0.23. we obtain,

$$\phi(0) - \phi(x) - \phi(x) \ge \phi'(x)(0-x),$$

simplifying the above expression we obtain,

$$x\varphi'(x) \ge 2\varphi(x).$$

A function f is said to be increasing on an interval I, if $f'(x) \ge 0$ at each point on the interval I.

Now the first derivative of $\frac{\varphi(x)}{x^2}$, which is given by

$$\frac{d}{dx}\Big(\frac{\varphi(x)}{(x)^2}\Big) = \frac{x\varphi'(x) - 2\varphi(x)}{(x^3)} \ge 0,$$

and it follows that $\frac{\varphi(x)}{(x)^2}$ is non-decreasing.
We now turn our attention to the second definition given for superquadratic functions.

Definition 3.0.30 [Abramovich 2009] Let X be a real vector space. A function ϕ : $X \to \mathrm{R}$ is said to be *superquadratic,* if $\forall x,y \in X$

$$\phi(x+y)+\phi(x-y) \geq 2[\phi(x)+\phi(y)]. \tag{3.5}$$

[Gil´*a*nyi and Troczka 2011] Remarked that a non-negative, superquadratic function defined on a 2-divisible abelian group is Jensen-convex.
When we make the following substitution $x_1 = x + y$ and $x_2 = x - y$, we have that $x = \frac{x_1+x_2}{2}$ and $y = \frac{x_1-x_2}{2}$.
So inequality (3.5) becomes

$$\varphi(x_1)+\varphi(x_2) \geq 2\Big[\varphi\Big(\frac{x_1+x_2}{2}\Big)+\varphi\Big(\frac{x_1-x_2}{2}\Big)\Big]$$

re-arranging it gives,

$$\varphi\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{1}{2}\varphi(x_1)+\frac{1}{2}\varphi(x_2)-\varphi\Big(\frac{1}{2}(x_1-x_2)\Big). \tag{3.6}$$

3.0.4 Some examples of superquadratic functions

1. For $p \geq 2$ the power function $\phi : [0,\infty) \to [0,\infty)$, given by $\phi(t) = t^p$, satisfies Definitions 3.0.23 and 3.0.29.

Proof for Definition 3.0.23

Let $\phi(t) = t^p, p \geq 2$.

We want to show that $\forall x \geq 0$, there exists a constant P_x, depending on x such that

$$\phi(y) \geq \phi(x)+(y-x)P_x+\phi(|y-x|), \forall y \geq 0.$$

Since $\phi(t) = t^p$ is differentiable, the constant $P_t = \phi^0(t) = pt^{p-1}$.

So without loss of generality, let $y \geq x \geq 0$.

We define a function $f(x,y)$ to be:

$$\begin{aligned} f(x,y) &= y^p - x^p - (y-x)px^{p-1} - (y-x)^p \\ &= y^p\Big[1 - \Big(\frac{x}{y}\Big)^p - p\Big(\frac{x}{y}\Big)^{p-1} + p\Big(\frac{x}{y}\Big)^p - \Big(1 - \frac{x}{y}\Big)^p\Big] \end{aligned}$$

Define $\widehat{f}: [0,1] \to \mathrm{R}$ by

$$y^p \widehat{f}\Big(\frac{x}{y}\Big) = f(x,y).$$

Since $y \geq 0$, showing that $f(x,y) \geq 0$ is equivalent to proving $\widehat{f}\Big(\frac{x}{y}\Big) \geq 0$.
Setting $r = \frac{x}{y}$, we have for, $0 \leq r \leq 1$,

$$\begin{aligned} \widehat{f}(r) &= 1 - r^p - pr^{p-1} + pr^p - (1-r)^p \\ \widehat{f}(0) &= 0. \\ \widehat{f}\,'(r) &= -pr^{p-1} - p(p-1)r^{p-2} + p^2 r^{p-1} + p(1-r)^{p-1} \\ &= pr^{p-1}\Big[-1 - (p-1)t^{-1} + p + \Big(\frac{1}{r} - 1\Big)^{p-1}\Big]. \end{aligned}$$

Setting $s = \frac{1}{r}$, we have that $1 \leq s < \infty$ and

$$\widehat{f}\,'(s) = p\Big(\frac{1}{s}\Big)^{p-1}\Big[-1 - (p-1)s + p + (s-1)^{p-1}\Big].$$

Setting $h(s) = (s-1)^{p-1} + (p-1) - (p-1)s$, $1 \leq s < \infty$, we have that $h^0(s) = (p - 1)[(s-1)^{p-2} - 1] \geq 0, 1 \leq s < \infty$.

Thus the function h is increasing on $[1,\infty)$. Since $h(1) = 0$, it follows that $_0$ h is positive and consequently f_b is also positive on $[0,1)$.

Since $f_b(0) = 0$, we have that $\widehat{f}(r) \geq 0, \ \forall r \in [0,1]$.
Hence $\phi(t) = t^p$ is superquadratic for $y \geq x \geq 0$, be Definition 3.0.23

Proof for Definition 3.0.29

Let $f(x,y) = (x+y)^p + (x-y)^p - 2x^p - 2y^p$, $x \geq y \geq 0$. Then

$$f(x,y) = x^p\Big[\Big(1 + \frac{y}{x}\Big)^p + \Big(1 - \frac{y}{x}\Big)^p - 2\Big(\frac{y}{x}\Big)^p - 2\Big].$$

Define $\widehat{f}: [0,1] \to \mathrm{R}$ by

$$x^p \widehat{f}\Big(\frac{y}{x}\Big) = f(x,y).$$

Since $x \geq 0$, showing that $f(x,y) \geq 0$ is equivalent to proving $\widehat{f}(\frac{y}{x}) \geq 0$.

Now set $t = \frac{y}{x}$

$$\widehat{f}(t) = (1+t)^p + (1-t)^p - 2 - 2t^p.$$

So

$$\begin{aligned} \widehat{f}\,'(t) &= p[(1+t)^{p-1} - (1-t)^{p-1} - 2t^{p-1}] \\ &= pt^{p-1}\Big[\Big(1+\frac{1}{t}\Big)^{p-1} - \Big(\frac{1}{t}-1\Big)^{p-1} - 2\Big]. \end{aligned}$$

Setting $s = \frac{1}{t}$, for $0 < t \leq 1$,

$$\widehat{f}\,'\Big(\frac{1}{s}\Big) = ps^{1-p}[(s+1)^{p-1} - (s-1)^{p-1} - 2], \quad s \in [1,\infty).$$

Let $g(s) = (s+1)^{p-1} - (s-1)^{p-1} - 2$, for $s \in [1,\infty)$.

$$g^0(s) = (p-1)[(s+1)^{p-2} - (s-1)^{p-2}] \geq 0, \qquad s \in [1,\infty).$$

Thus the function g is increasing on $[1,\infty)$. Since $g(1) = 2^{p-1} - 2 > 0$, it 0 follows that g is positive and consequently f_b is also positive in $(0,1]$.

Since $f_b(0) = 0$, we have that $f(t) \geq 0$, $\quad \forall\, t \in [0,1]$.

Hence $\phi(t) = t^p$ is superquadratic for $x \geq y \geq 0$.

2. The function $\phi(t) = t^2 \ln t$, for $t > 0$ and $\phi(0) = 0$ is superquadratic by Definition 3.0.23.

Proof of Example 2

Let $\phi(t) = t^2 \ln t$, for $t > 0$ and $\phi(0) = 0$.

We wish to show that $\phi(t) = t^2 \ln t$ is continuously differentiable on $(0,\infty)$.

Now, $\phi(t)$ is continuous on $(0,\infty)$ being the product of two continuous functions on $(0,\infty)$.

$$\phi^0(t) = t[1 + 2\ln t].$$

Clearly the derivative of $\phi(t)$ is also continuous being the product of two continuous functions on $(0,\infty)$.

So $\phi(t) = t^2 \ln t$ is continuously differentiable.

Now we wish to show that $\frac{\varphi'(t)}{t}$ is non-decreasing.

So

$$\frac{d}{dt}\left(\frac{\varphi'(t)}{t}\right) = \frac{d}{dt}(1 + 2\ln t) = \frac{2}{t}.$$

Thus $\frac{\varphi'(t)}{t}$ is non-decreasing on $(0,\infty)$.

Hence by Lemma 3.0.27, $\phi(t) = t^2 \ln t$ is superquadratic on $(0,\infty)$.

3. For $x \geq y \geq 2$, the exponential function $\phi : [2,\infty) \to [0,\infty)$, given by $\phi(t) = e^t$, satisfies Definition 3.0.29.

   For the proof of Example 3, see Proposition 6.2.1.

4. Let $\phi : \mathrm{R} \to \mathrm{R}^-$ be defined by

$$\varphi(t) = \begin{cases} -3 & \text{if } t = 0 \\ -1 & \text{if } t \neq 0 \end{cases}$$

   Then $\phi(t)$ is superquadratic according to Definition 3.0.29.

   Proof of Example 4

   We wish to show that for all $x,y \in \mathrm{R}$

$$\phi(x + y) + \phi(x - y) \geq 2[\phi(x) + \phi(y)].$$

   Proof by Cases is empolyed for this prove.

   Case 1 *When $x = y = 0$.*

   Let $f(x,y) = \phi(x + y) + \phi(x - y) - 2[\phi(x) + \phi(y)]$.

   Then

$$\begin{aligned} f(0,0) &= \phi(0) + \phi(0) - 2[\phi(0) + \phi(0)] \\ &= -3 - 3 - 2[-3 - 3] \\ &= -6 + 12 = 6 \\ &> 0, \end{aligned}$$

thus $f(x,y) > 0$, *for* $x = y = 0$.

Hence $\phi(x + y) + \phi(x - y) > 2[\phi(x) + \phi(y)]$.

Case 2 *When* $x \neq 0, y \neq 0$ *but* $x = y$.

Then

$$\begin{aligned} f(x,y) &= \phi(2x) + \phi(0) - 2[\phi(x) + \phi(y)] \\ &= -1 - 3 - 2[-1 - 1] \\ &= -4 + 4 = 0, \end{aligned}$$

thus $f(x,y) = 0$.

Hence $\phi(x + y) + \phi(x - y) = 2[\phi(x) + \phi(y)]$ *is satisfied.*

Case 3 *When* $x \neq 0, y \neq 0, x \neq y$ *but* $x + y = 0$.

$$\begin{aligned} f(x,y) &= \phi(0) + \phi(x - y) - 2[\phi(x) + \phi(y)] \\ &= -3 - 1 - 2[-1 - 1] \\ &= -4 + 4 = 0, \end{aligned}$$

thus $f(x,y) = 0$.

Hence $\phi(x + y) + \phi(x - y) = 2[\phi(x) + \phi(y)]$ *is satisfied.*

Case 4 *When* $x = 0$ *and* $y \neq 0$.

Then

$$f(0,y) = \phi(y) + \phi(-y) - 2[\phi(0) + \phi(y)]$$

$$= -1 - 1 - 2[-3 - 1] = 6$$

$$> 0,$$

thus $f(x,y) \geq 0$.

Hence $\phi(x + y) + \phi(x - y) > 2[\phi(x) + \phi(y)]$.

Case 5 *When* $x 6= 0$ *and* $y = 0$.

Then

$$f(x,0) = \phi(x) + \phi(x) - 2[\phi(x) + \phi(0)]$$

$$= -6 - 2[-1 - 3] = 2$$

$$> 0,$$

thus $f(x,y) > 0$.

Hence $\phi(x + y) + \phi(x - y) > 2[\phi(x) + \phi(y)]$.

This concludes the fact that Example 5 is superquadratic according to Definition 3.0.29

Chapter 4

Jensen's Inequality and other related inequalities

In this section we shall take a close look at Jensen's, Young's and Minkowski's inequalities and a refinement of the Jensen's inequality via Definition 3.0.23.

As an important tool, Jensen's inequality has proved very useful in the derivation of many other inequalities such as Young's, H¨*o*lder's, Minkowski's, Chebyshevs inequalities. These inequalities have proved useful in many areas of application, such as in statistics, information theory, Rao-Blackwell theorem.

In statistics, under the multiple comparisons among mean vectors, the ANOVA problem is to test whether all the means are equal and to identify the nature of departure from the test in case of its rejection. The F and T tests are the best known tests used for this purpose, these tests and the associated multiple comparisons can be obtained using Roy's union-intersection approach and a modification of H¨older's inequality; see [Subbaiah, P and Mudholkar G.S 1983].

The discrete version of Jensen's inequality for convex functions is given as

Theorem 4.0.31 Jensen's inequality *[Bani´c et al 2008b, Cloud 1998]*

$$\phi\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i), \tag{4.1}$$

where φ *is a convex function defined on an interval* I *in* $\mathbb{R}$, $(x_1,...,x_n) \in I^n (n \ge 2)$ *and* $(p_1,...,p_n)$ *is any non-negative n-tuple satisfying* $P_n = \sum_{i=1}^{n} p_i$.

Proof

We use an inductive argument.

For $n = 2, P_2 = p_1 + p_2 = \sum_{i=1}^{2} p_i$, so

$$\begin{aligned}\phi\Big(\frac{1}{P_2}\sum_{i=1}^{2} p_i x_i\Big) &= \phi\Big(\frac{p_1 x_1}{P_2} + \frac{p_2 x_2}{P_2}\Big) \\ &\le \frac{p_1}{P_2}\phi(x_1) + \frac{p_2}{P_2}\phi(x_2).\end{aligned}$$

by the convexity of φ, hence the statement is true for $n = 2$ Fix $k \in \mathbb{N}$, $k > 2$ and suppose (4.1) is true, that is

$$\phi\Big(\frac{1}{P_k}\sum_{i=1}^{k} p_i x_i\Big) \le \frac{1}{P_k}\sum_{i=1}^{k} p_i \phi(x_i).$$

Now for $k + 1 \in \mathbb{N}$, we have

$$\begin{aligned}\phi\Big(\frac{1}{P_{k+1}}\sum_{i=1}^{k+1}p_ix_i\Big) &= \phi\Big(\frac{1}{P_{k+1}}\sum_{i=1}^{k}p_ix_i+\frac{p_{k+1}x_{k+1}}{P_{k+1}}\Big)\\ &= \phi\Big(\frac{P_k}{P_{k+1}P_k}\sum_{i=1}^{k}p_ix_i+\frac{p_{k+1}x_{k+1}}{P_{k+1}}\Big)\\ &= \phi\Big(\frac{P_k}{P_{k+1}}\Big(\frac{1}{P_k}\sum_{i=1}^{k}p_ix_i\Big)+\frac{p_{k+1}x_{k+1}}{P_{k+1}}\Big)\\ &\le \frac{P_k}{P_{k+1}}\phi\Big(\frac{1}{P_k}\sum_{i=1}^{k}p_ix_i\Big)+\frac{p_{k+1}}{P_{k+1}}\phi(x_{k+1})\end{aligned}$$

by the convexity of φ.
But $\phi(\frac{1}{P_k}\sum_{i=1}^{k}p_ix_i)\le\frac{1}{P_k}\sum_{i=1}^{k}p_i\phi(x_i)$, from the inductive hypothesis,
so

$$\begin{aligned}\phi\Big(\frac{1}{P_{k+1}}\sum_{i=1}^{k+1}p_ix_i\Big) &\le \frac{P_k}{P_{k+1}}\Big(\frac{1}{P_k}\sum_{i=1}^{k}p_i\phi(x_i)\Big)+\frac{p_{k+1}}{P_{k+1}}\phi(x_{k+1})\\ &= \frac{1}{P_{k+1}}\sum_{i=1}^{k+1}p_i\phi(x_i),\end{aligned}$$

which completes the proof.

A refinement of Jensen's inequality is obtained when the convex function in (4.1) is replaced by a function which satisfies Definition 3.0.23.

The refined inequality is given as follows

Theorem 4.0.32 Refined Jensen's inequality*[Abramovich 2009, Bani´c et al 2008a]*

Suppose ϕ is superquadratic and for any two non-negative $n-$tuples $(x_i,...,x_n)$ *and* $(p_i,...,p_n)$ *such that* $P_n=\sum_{i=1}^{n}p_i$*. Then*

$$\varphi\Big(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\Big)\le\frac{1}{P_n}\sum_{i=1}^{n}p_i\varphi(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\varphi\Big(\Big|x_i-\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\Big|\Big). \qquad (4.2)$$

Proof

Let $\nu(i)=\frac{p_i}{P_n}$ be a discrete measure on $\{1,...,n\}$ and let p_i,P_n be as stated above.
Setting $y=x_i$ and $x=\overline{x}=\frac{1}{Pn}\sum_{j=1}^{n}p_jx_j$ in Definition 3.0.23, we have

$$\phi(x_i)\ge\phi(\overline{x})+\phi(|x_i-\overline{x}|)+P_x(x_i-\overline{x}),$$

taking the weighted average of both sides, that is,

$$\sum_{i=1}^{n} \nu i\varphi(x_i) \ge \sum_{i=1}^{n} \nu i\varphi(\overline{x}) + \sum_{i=1}^{n} \nu i\varphi(|x_i - \overline{x}|) + P_x \sum_{i=1}^{n} \nu i(x_i - \overline{x}),$$

which simplifies to

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i\varphi(x_i) \ge \frac{1}{P_n} \sum_{i=1}^{n} p_i\varphi(\overline{x}) + \frac{1}{P_n} \sum_{i=1}^{n} p_i\varphi(|x_i - \overline{x}|),$$

since

$$\sum_{i=1}^{n} \nu i\varphi(|x_i - \overline{x}|) = 0.$$

So the above expression becomes

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i\varphi(x_i) \ge \varphi(\overline{x}) + \frac{1}{P_n} \sum_{i=1}^{n} p_i\varphi(|x_i - \overline{x}|),$$

re-arranging the above expression gives the desired inequality.

The equivalent continuous version of the refined Jensen's inequality is given in [Abramovich et al 2004a] as

$$\phi\left(\int f d\mu\right) \le \int [\phi(f(s)) - \phi(|f(s) - \textstyle\int f d\mu|)] d\mu(s), \tag{4.3}$$

for all probability measures μ and all non-negative, μ-integrable functions f.

Theorem 4.0.33 *[Bani´c et al 2008a] For the function $\phi : [0,\infty) \to \mathbb{R}$ the following conditions are equivalent:*

1. *The function ϕ is a superquadratic function; that is, there exists a constant P_x such that $\forall x \ge 0$*

   $$\phi(y) \ge \phi(x) + P_x(y - x) + \phi(|y - x|)$$

   $\forall y \ge 0$.

2. *For any two non-negative $n-$tuples $(x_1,...,x_n)$ and $(p_1,...,p_n)$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ the following inequality holds*

   $$\varphi\Big(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\Big) \le \frac{1}{P_n} \sum_{i=1}^{n} p_i\varphi(x_i) - \frac{1}{P_n} \sum_{i=1}^{n} p_i\varphi\Big(\Big|x_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\Big|\Big).$$

3. *The following inequality*

$\phi(\lambda y_1+(1-\lambda)y_2) \leq \lambda\phi(y_1)+(1-\lambda)\phi(y_2)-\lambda\phi[(1-\lambda)|y_1-y_2|]-(1-\lambda)\phi[\lambda|y_1-y_2|]$

holds for all $y_1,y_2 \geq 0$ *and* $\lambda \in [0,1]$.

4. *For all* $x,y_1,y_2 \geq 0$ *such that* $y_1 < x < y_2$ *we have*

$$\varphi(x) \leq \frac{y_2 - x}{y_2 - y_1}[\varphi(y_1) - \varphi(x - y_1)] + \frac{x - y_1}{y_2 - y_1}[\varphi(y_2) - \varphi(y_2 - x)]$$

or equivalently

$$\frac{\varphi(y_1) - \varphi(x) - \varphi(x - y_1)}{y_1 - x} \leq \frac{\varphi(y_2) - \varphi(x) - \varphi(y_2 - x)}{y_2 - x}$$

Proof

(1) =⇒ (2) see proof of Theorem 4.0.31.

(2) =⇒ (3)

Suppose (2) holds for all non-negative n−tuples $(x_1,...,x_n)$ and $(p_1,...,p_n)$ such that $P_n = \sum_{i=1}^{n} p_i > 0$.

In the special case for $n = 2$, $p_1 = \lambda \in [0,1]$, $p_2 = (1-\lambda)$ and for $x_1 = y_1$, $x_2 = y_2$ we have

$$\phi(\lambda y_1 + (1 - \lambda)y_2) \leq \lambda\phi(y_1) + (1 - \lambda)\phi(y_2) - \lambda\phi[|y_1 - (\lambda y_1 + (1 - \lambda)y_2)|] - (1 - \lambda)\phi[|y_2 - (\lambda y_1 + (1 - \lambda)y_2)|],$$

which simplifies to

$$\phi(\lambda y_1 + (1 - \lambda)y_2) \leq \lambda\phi(y_1) + (1 - \lambda)\phi(y_2) - \lambda\phi[(1 - \lambda)|y_1 - y_2|] - (1 - \lambda)\phi[\lambda|y_1 - y_2|].$$

(3) =⇒ (4)

Suppose that (3) holds for all $y_1,y_2 \geq 0$ and $\lambda \in [0,1]$.

Let $x,y_1,y_2 \geq 0$ be such that $y_1 < x < y_2$. Then there exists $\lambda \in [0,1]$ such that $x = \lambda y_1 + (1 - \lambda)y_2$. Hence we have

$$\lambda = \frac{y_2 - x}{y_2 - y_1} \text{ and } 1 - \lambda = \frac{x - y_1}{y_2 - y_1}.$$

Substituting these expressions in (3) we obtain,

$$\varphi(x) \leq \frac{y_2 - x}{y_2 - y_1}\varphi(y_1) + \frac{x - y_1}{y_2 - y_1}\varphi(y_2) - \frac{y_2 - x}{y_2 - y_1}\varphi\Big(\frac{x - y_1}{y_2 - y_1}\Big|y_1 - y_2\Big|\Big) -$$

$$\frac{x - y_1}{y_2 - y_1}\varphi\Big(\frac{y_2 - x}{y_2 - y_1}\Big|y_1 - y_2\Big|\Big),$$

$$\varphi(x)(y_2 - y_1) \leq (y_2 - x)\varphi(y_1) + (x - y_1)\varphi(y_2) - (y_2 - x)\varphi(x - y_1) -$$

$$(x - y_1)\phi(y_2 - x),$$

$$\phi(x)(y_2 - x + x - y_1) \leq (y_2 - x)\phi(y_1) + (x - y_1)\phi(y_2) - (y_2 - x)\phi(x - y_1) -$$

$$(x - y_1)\phi(y_2 - x),$$

$$(y_2 - x)[\phi(x) - \phi(y_1) + \phi(x - y_1)] \leq (x - y_1)[\phi(y_2) - \phi(x) - \phi(y_2 - x)].$$

Dividing the last inequality by $(x - y_1)(y_2 - x) > 0$ we obtain (4).

(4) =⇒ (1)

Suppose that (4) holds for all $x, y_1, y_2 \geq 0$ such that $y_1 < x < y_2$. By fixing $y_1 \in (0, x)$, we obtain a lower bound that shows that

$$P_x \equiv \inf_{y_2 > x} \frac{\varphi(y_2) - \varphi(x) - \varphi(y_2 - x)}{y_2 - x}.$$

Taking $y_2 = y$,

$$P_x(y - x) \leq \phi(y) - \phi(x) - \phi(y - x), (y > x). \tag{4.4}$$

From (4) for $y_1 < x$,

$$\sup_{y_1 < x} \frac{\varphi(y_1) - \varphi(x) - \varphi(x - y_1)}{y_1 - x} \equiv P_x$$

exists.

Thus

$$\inf_{y_2 > x} \frac{\varphi(y_2) - \varphi(x) - \varphi(y_2 - x)}{y_2 - x} = P_x = \sup_{y_1 < x} \frac{\varphi(y_1) - \varphi(x) - \varphi(x - y_1)}{y_1 - x}.$$

Now taking $y_1 = y$,

$$\phi(y) - \phi(x) - \phi(x - y) \geq P_x(y - x), (y < x). \tag{4.5}$$

From inequalities (4.4) and (4.5),

$$\phi(y) - \phi(x) - \phi(|x - y|) \geq P_x(y - x), (\forall x,y > 0),$$

thus ϕ satisfies (1).

From inequality (4.2) when we set $n = 2$ and $p_1 = p_2 = 1$, we obtain

$$\varphi\Big(\frac{x_1 + x_2}{2}\Big) \leq \frac{1}{2}\varphi(x_1) + \frac{1}{2}\varphi(x_2) - \varphi\Big(\Big|\frac{x_1 - x_2}{2}\Big|\Big). \tag{4.6}$$

From (4.6) one can clearly see that all functions satisfying Definition 3.0.23 also satisfy Definition 3.0.29 but the converse is not always true even if we restrict the domain of ϕ in Definition 3.0.29 to $\mathbb{R}^+$, see [Abramovich 2009].

4.1 Young's inequality

Theorem 4.1.1 *[Cloud 1998, Gruber 2007, Niculescu, Persson 2006] Let* $x,y \geq 0$ *and* $p,q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}. \tag{4.7}$$

Proof

Let x,y,p and q be as given then

$$\begin{aligned} xy &= e^{\ln xy} \\ &= e^{\left(\ln x^{\frac{1}{p}p} + \ln y^{\frac{1}{q}q}\right)} \\ &= e^{\left(\frac{1}{p}\ln x^p + \frac{1}{q}\ln y^q\right)}. \end{aligned}$$

Since the exponential function is convex we have that,

$$\begin{aligned} xy &\leq \frac{1}{p}e^{\ln x^p} + \frac{1}{q}e^{\ln y^q} \\ &= \frac{1}{p}x^p + \frac{1}{q}y^q. \end{aligned}$$

4.2 Minkowski's inequality

Theorem 4.2.1 *[Cloud 1998, Jiˇr´r et al 2000, Niculescu, Persson 2006] Assume that* $x_1,...,x_n$ *and* $y_1,...,y_n$ *are real numbers, and let* $p \geq 1$. *Then*

$$\Big[\sum_{i=1}^{n}\Big(x_i+y_i\Big)^p\Big]^{\frac{1}{p}} \leq \Big(\sum_{i=1}^{n}x_i^p\Big)^{\frac{1}{p}}+\Big(\sum_{i=1}^{n}y_i^p\Big)^{\frac{1}{p}}. \tag{4.8}$$

Proof

Let $(x_1,...,x_n)$ and $(y_1,...,y_n)$ be real numbers and let $p \in \mathbb{N}$.

Consider the expression

$$\left(\frac{x_i+y_i}{(\sum_{i=1}^n x_i^p)^{\frac{1}{p}}+(\sum_{i=1}^n y_i^p)^{\frac{1}{p}}}\right)^p =$$

$$\left(\frac{x_i}{(\sum_{i=1}^n x_i^p)^{\frac{1}{p}}+(\sum_{i=1}^n y_i^p)^{\frac{1}{p}}}+\frac{y_i}{(\sum_{i=1}^n x_i^p)^{\frac{1}{p}}+(\sum_{i=1}^n y_i^p)^{\frac{1}{p}}}\right)^p$$

Suppose that $z = (\sum_{i=1}^n x_i^p)^{\frac{1}{p}}$ and that $z' = (\sum_{i=1}^n y_i^p)^{\frac{1}{p}}$ and thus the above expression becomes

$$\left(\frac{x_i+x_i'}{z+z'}\right)^p = \left(\frac{x_i}{z+z'}+\frac{x_i'}{z+z'}\right)^p$$

$$= \Big[\frac{z}{z+z'}\Big(\frac{x_i}{z}\Big)+\frac{z'}{z+z'}\Big(\frac{y_i}{z'}\Big)\Big]^p.$$

Since the power function is a convex function for all $n \geq 1$, we have by Jensen's inequality the inequality

$$\left(\frac{x_i+y_i}{z+z'}\right)^p \leq \frac{z}{z+z'}\Big(\frac{x_i}{z}\Big)^p+\frac{z'}{z+z'}\Big(\frac{y_i}{z'}\Big)^p.$$

Summing the above expression over the i^0s running from $i = 1,2,...,n-1,n$, we have

$$\sum_{i=1}^{n}\left(\frac{x_i+y_i}{z+z'}\right)^p \leq \sum_{i=1}^{n}\Big[\frac{z}{z+z'}\Big(\frac{x_i}{z}\Big)^p+\frac{z'}{z+z'}\Big(\frac{y_i}{z'}\Big)^p\Big]$$

$$= \frac{z}{z+z'}\Big[\sum_{i=1}^{n}\Big(\frac{x_i}{z}\Big)^p\Big]+\frac{z'}{z+z'}\Big[\sum_{i=1}^{n}\Big(\frac{y_i}{z'}\Big)^p\Big]$$

$$= \frac{z}{z+z'}\Big[\sum_{i=1}^{n}\frac{x_i^p}{z^p}\Big]+\frac{z'}{z+z'}\Big[\sum_{i=1}^{n}\frac{y_i^p}{z'^p}\Big].$$

But since it is only the $x^0{}_is$ and $y_i{}^0s$ that are dependent on i, we have that

$$\sum_{i=1}^{n}\left(\frac{x_i+y_i}{z+z'}\right)^p \leq \frac{z}{z+z'}\Big[\sum_{i=1}^{n}\frac{x_i^p}{z^p}\Big] + \frac{z'}{z+z'}\Big[\sum_{i=1}^{n}\frac{y_i^p}{z^p}\Big]$$

$$= \frac{z}{z+z''}\Big[\frac{\sum_{i=1}^{n} x_i^p}{z^p}\Big] + \frac{z'}{z+z'}\Big[\frac{\sum_{i=1}^{n} y_i^p}{z'^p}\Big].$$

Hence

$$\sum_{i=1}^{n}\left(\frac{x_i+y_i}{z+z'}\right)^p \leq \frac{z}{z+z'} + \frac{z'}{z+z'} = 1$$

and so

$$\sum_{i=1}^{n}\left(\frac{x_i+y_i}{z+z'}\right)^p \leq 1,$$

that is which says

$$\frac{\sum_{i=1}^{n}(x_i+y_i)^p}{(z+z')^p} \leq 1$$

$$\sum_{i=1}^{n}(x_i+y_i)^p \leq (z+z^0)^p.$$

Finally replacing z and z^0, we conclude that

$$\sum_{i=1}^{n}(x_i+y_i)^p \leq \Big[\Big(\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} y_i^p\Big)^{\frac{1}{p}}\Big]^p$$

$$\Big[\sum_{i=1}^{n}(x_i+y_i)^p\Big]^{\frac{1}{p}} \leq \Big(\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n} y_i^p\Big)^{\frac{1}{p}}.$$

Chapter 5

Subharmonic functions

Subharmonic functions are a larger class of functions than the harmonic functions, but they have a lot of similar useful properties. Before we go into the details of subharmonic functions, let us first look at holomorphic functions briefly.

5.1 Holomorphic functions

Definition 5.1.1 A *holomorphic* function is defined on an open subset U of the complex plane C, with values in C and is complex-differentiable at every point in U.

Complex- differentiability is a much stronger condition than real- differentiability and this implies that holomorphic functions are infinitely differentiable and can be expressed as Taylor series.

The term analytic function is often used interchangeably with holomorphic functions, further more, the class of analytic functions coincides with the class of holomorphic functions. Holomorphic functions are sometimes called regular functions. An entire function is a function which is holomorphic on the whole of C. A complex function $f(x+iy) = u(x,y)+iv(x,y)$ is holomorphic, if both u and v have first partial derivatives, with respect to x and y, which satisfy the Cauchy Riemann equations,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

If continuity of the partial derivatives is not given, the converse is not necessarily true.

Every holomorphic function can be separated into it's real and imaginary parts, and each of these is a solution of Laplace's equation on $\mathbb{R}^2$, thus if we write a holomorphic function $f(z) = u(x,y) + iv(x,y)$, both u and v are harmonic functions.

A function of several complex variables is holomorphic, if and only if, it satisfies the Cauchy- Riemann equations and is locally square- integrable.

Definition 5.1.2 [Toshio 2000] Let f be a complex-valued function defined on a domain D in $\mathbb{C}^n$. If f satisfies the following two conditions:

(i) f is continuous in D and

(ii) f, has partial derivatives $\frac{\partial f}{\partial z_j}(j = 1, 2, ..., n)$ in D,

then f is a *holomorphic* function on D.

The infinite dimensional theory of holomorphic functions started with the works of M. Fr´echet and R. Gaˆteaux, their works led to two important definitions of holomorphic functions, thus the *Strong* and *Weak* definitions. The strong definition was due to Fr´echet and the weak definition associated with Gaˆteaux.

Let X and Y be complex Banach spaces and let W be an open subset of X.

Without loss of generality we can take W as an open ball of radius r centered on x.

Definition 5.1.3 (Fr´echet definition) A function $f : W \to Y$ is *holomorphic* if for each $x \in W$ there exists a continuous complex- linear mapping $Df(x) : X \to Y$ such that

$$\lim_{y\to 0} \frac{||f(x+y) - f(x) - Df(x)y||}{||y||} = 0.$$

This shows that functions that are holomorphic in the above sense are continuous and locally bounded.

Definition 5.1.4 (Gaˆteaux definition) A function $f : W \to Y$ is *holomorphic* if it is locally bounded and for each $x \in W, y \in W$ and linear functional $` \in Y^*$, the function

$$h(\lambda) = `(f(x + \lambda y))$$

is *holomorphic* at $\lambda = 0$, where λ is a complex variable.

5.2 Upper Semicontinuous Functions

There are functions that are not continuous, but exhibit a weaker property that ensures some of the properties of continuous functions.

Definition 5.2.1 (Upper and Lower Semicontinuous Functions) [Ransford 1995 , Armitage 2001, Dineen 1989, Laurent-Thi´ebant 2011]
Let X be a topological space and $u : X \to [-\infty, \infty)$.

1. We say that u is *upper semicontinuous* if for each $\alpha \in \mathbb{R}$, the set $\{x \in X : u(x) < \alpha\}$ is open.
2. We say that u is *lower semicontinuous* if for each $\alpha \in \mathbb{R}$, the set $\{x \in X : u(x) > \alpha\}$ is open.

We know that u is continuous if for all open intervals $(\alpha, \beta) \in \mathbb{R}, u^{-1}(\alpha, \beta)$ is open in X, thus $\{x \in X : \alpha < u(x) < \beta\}$ is open.

An equivalent formulation of upper and lower semicontinuous functions is.

Theorem 5.2.2 *[Krantz 2000, Armitage 2001, Conway 1978]*

1. *u is upper semicontinuous iff* $\forall x \in X, \limsup_{y \to x} u(y) \le u(x)$.
2. *u is lower semicontinuous iff* $\forall x \in X, \liminf_{y \to x} u(y) \ge u(x)$.
3. *u is lower semicontinuous iff $-u$ is upper semicontinuous.*
4. *u is continuous iff it is both upper and lower semicontinuous.*

Proof of (1)

Suppose first that u is upper semicontinuous and let $x \in X$. Suppose also that $u(x)$ is finite. Let $\epsilon > 0$, then $\{y \in X : u(y) < u(x) + \epsilon\}$ is open and contains x, so contains an open neighbourhood U of x. So

$$y \in U \Rightarrow u(y) < u(x) + \epsilon.$$

Thus for each > 0, there exists a neighbourhood U of x, with this property. By definition

$$\limsup_{y \to x} u(y) \leq u(x) + \epsilon.$$

As > 0 is arbitrary, we have

$$\limsup_{y \to x} u(y) \leq u(x) \quad (5.1)$$

A similar approach is used for the case where $u(x) = -\infty$, when we consider the set $\{y \in X : u(y) < -L\}$, with large positive L.

Conversely suppose that (5.1) holds. Let $\alpha \in \mathbb{R}$ and consider the set

$A = \{x \in X : u(x) < \alpha\}$. We must show that the set A is open. Let $x \in A$, so that $u(x) < \alpha$. We choose > 0 so small that $u(x) < \alpha - 2\epsilon$, then as (5.1) holds, there exists a neighbourhood U of x such that

$$y \in U \Rightarrow u(y) < u(x) + \epsilon < \alpha - \epsilon$$

$$\Rightarrow y \in A$$

$$\Rightarrow U \subset A.$$

That is, each point of A is contained in an open neighbourhood inside A. So, A is open, as required.

The Proof of (2) is similar to that of (1).

Proof of (3)

First suppose that $-u$ is upper semicontinuous, then from Definition 5.2.1, the set $\{x \in X : -u(x) < \alpha\}$ is open.

Thus $\{x \in X : u(x) > -\alpha\}$ is open.

Denoting $-\alpha$ by $\beta \in \mathbb{R}$, then $\{x \in X : u(x) > \beta\}$ is open, and this satisfies (2) of Definition 5.2.1, hence u is lower semicontinuous.

The proof of the converse is obtained by reversing the argument.

Proof of (4)

First suppose that u is continuous, then by definition of continuity we have $\forall x \in X$,

$$\lim_{y \to x} u(y) = u(x),$$

so

$$\liminf_{y \to x} u(y) = \limsup_{y \to x} u(y) = u(x).$$

Hence u is both upper and lower semicontinuous.

Conversely if u is both upper and lower semicontinuous, then

$$\limsup_{y \to x} u(y) \leq u(x) \leq \liminf_{y \to x} u(y).$$

But by definition,

$$\limsup_{y \to x} u(y) \geq \liminf_{y \to x} u(y).$$

Hence

$$\limsup_{y \to x} u(y) = u(x) = \liminf_{y \to x} u(y).$$

Thus

$$\lim_{y \to x} u(y) = u(x).$$

Therefore u is continuous as required.

One of the most important properties of an upper semicontinuous function is that it attains its maxima.

Theorem 5.2.3 *[Ransford 1995] Let u be an upper semicontinuous function on a topological space X and let C be a compact subset of X. Then u is bounded above on C and attains its supremum on C.*

Proof

The set $U_n = \{x \in X : u(x) < n\}$ where $(n \geq 1)$ is open and the set $\{U_n : n \in \mathbb{N}\}$ covers C. Since C is compact, there exists a finite subcover for C. As n increases these sets also increase, i.e $U_n \subseteq U_{n+1}$. It follows that just one of these sets (with sufficiently large n) contains C . So u is bounded on C. Let

$$H = \sup\{u(x) : x \in C\}.$$

The open sets $\{x \in X : u(x) < H - \frac{1}{n}\}$ cannot cover C . For if they did, there would be a finite subcover. Then for some large n,

$$u(x) < H - \frac{1}{n} \in C$$

and this contradicts the definition of H. It follows that there exists at least a $y \in C$ with

$$y \notin \bigcup_{n=1}^{\infty}\{x \in X : u(x) < H - \frac{1}{n}\},$$

so

$$u(y) \geq H.$$

But by definition, $u(y) \leq H$.

That is, $u(y) = H = \sup \{u(x) : x \in C\}$, and u attains its least upper bound on C.

An upper semicontinuous function is the limit of a decreasing sequence of continuous functions.

Theorem 5.2.4 *[Ransford 1995, Krantz 2000] Let u be an upper semicontinuous function on a metric space (X,d) and suppose that u is bounded above on X. Then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions, $\varphi_n : X \to \mathbb{R}$ such that, $\varphi_n \geq \varphi_{n+1}$ on X and $\lim_{n\to\infty} \varphi_n = u$ on X.*

Proof

Suppose that u is not identically $-\infty$. For $n \geq 1$, define $\varphi_n : X \to \mathbb{R}$ by

$$\varphi_n(x) = \sup_{y\in X}\{u(y) - nd(x,y)\}.$$

We show that φ_n so defined is continuous.

If $x \in X$,

$$\varphi_n(x) = \sup_{y\in X}\{u(y) - nd(z,y) + nd(z,y) - nd(x,y)\}.$$

By the triangle inequality, we obtain

$$\begin{aligned}\varphi_n(x) &\leq \sup_{y\in X}\{u(y) - nd(z,y) + nd(z,x)\}\\ &= \sup_{y\in X}\{u(y) - nd(z,y)\} + nd(z,x)\\ &= \varphi_n(z) + nd(z,x).\end{aligned}$$

That is

$$\varphi_n(x) - \varphi_n(z) \leq nd(z,x).$$

Interchanging x and z we have

$$|\varphi_n(x) - \varphi_n(z)| \leq nd(x,z).$$

Hence φ_n is continuous on X.

The next thing to show is that φ_n is a monotonic decreasing sequence that converges to u.

For $n = 1$ we have that

$$\varphi_1 = \sup_{y\in X}\{u(y) - d(x,y)\}$$

and for $n = 2$ we have

$$\varphi_2 = \sup_{y\in X}\{u(y) - 2d(x,y)\}.$$

Now $\varphi_1 - \varphi_2$ gives us,

$$\begin{aligned}&= \sup_{y\in X}\{u(y) - d(x,y) - u(y) + 2d(x,y)\}\\ &= \sup_{y\in X}\{d(x,y)\} \geq 0.\end{aligned}$$

Going through this inductively, it is clearly seen that φ_n is a monotone decreasing sequence.

Now if $y = x$, then

$$\varphi_n(x) \geq (u(x) - nd(x,x)) = u(x).$$

That is

$$\lim_{n\to\infty} \varphi_n(x) \geq u(x).$$

In the other direction, let $\rho > 0$ and let us denote $\{y \in X : d(x,y) < \rho\}$ by B(x, ρ).

We consider the cases where y is inside or outside B(x, ρ).

We have

$$\varphi_n \leq \max\{\sup_{B(x,\rho)} u, \sup_X u - n\rho\}.$$

So

$$\begin{aligned}\lim_{n\to\infty} \varphi_n(x) &\leq \limsup_{n\to\infty}\{\max\{\sup_{B(x,\rho)} u, \sup_X u - n\rho\}\} \\ &\leq \sup\{\sup_{B(x,\rho)} u, -\infty\} = \sup_{B(x,\rho)} u.\end{aligned}$$

Using upper semicontinuity, and letting $\rho \to 0$ gives

$$\lim_{n\to\infty} \varphi_n(x) \leq u(x).$$

Combining this and the earlier inequality gives

$$\lim_{n\to\infty} \varphi_n(x) = u(x).$$

If u is identically $-\infty$, we set $\varphi_n = -n, n \geq 1$.

Clearly φ_n so defined is continuous, since for each n, it is a constant map, and $(\varphi_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence.

Furthermore $\lim_{n\to\infty} \varphi_n = -\infty$.

5.3 Subharmonic functions

Definition 5.3.1 [Dineen 1989, Ransford 1995] Subharmonic and Superharmonic Functions

Let $U \subset \mathbb{C}$ be open and $u : U \to [-\infty,\infty)$. We say u is *subharmonic if*

1. u is upper semicontinuous, and
2. u satisfies the submean inequality : that is, given $w \in U$, there exists $\rho > 0$ such that for $0 \leq r \leq \rho$,

$$u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})d\theta.$$

u is *superharmonic* if $-u$ is subharmonic.

The integral is well defined, since we have seen that, an upper semicontinuous function is locally the limit of a monotone decreasing sequence of continuous functions. So by Lebesgue's Monotone Convergence Theorem, u is then Lebesgue measurable, and

$$\int_0^{2\pi} u(w + re^{i\theta})d\theta = \lim_{n\to\infty} \int_0^{2\pi} \phi_n(w + re^{i\theta})d\theta.$$

5.3.1 The Maximum Principle

In this section we are going to look at one of the properties that subharmonic functions derive from the harmonic functions, which is the maximum principle for harmonic functions. This is also refered to as the maximum modulus principle for analytic functions. Then we shall derive some useful results from it, such as the Phragmen-Lindelof principle.

Theorem 5.3.2 *[Armitage 2001, Ransford 1995]* **Maximum Principle** *Let u be a subharmonic function on a domain D in* $\mathbb{C}$*.*

1. *If u attains a global maximum on D, then u is constant.*

2. *If* $\forall \varsigma \in \partial D$,

$$\limsup_{z \to \varsigma} u(z) \leq 0,$$

then $u \leq 0$ *on* D. *(If* D *is unbounded, this boundary includes* $\varsigma = \infty$*).*

Proof of (1)

Suppose that u attains a maximum value M on D. Let us define two sets A and B as

$$A = \{z : u(z) < M\};\ B = \{z : u(z) = M\}.$$

Then A is open, since u is upper semicontinuous and the set B is nonempty, by hypothesis.

Let $p \in B$, u satisfies the submean inequality, since u is subharmonic.

So there exists $\epsilon_p > 0$, such that for $0 \leq \epsilon < \epsilon_p$,

$$\begin{aligned} M &= u(p) \leq \frac{1}{2\pi}\int_0^{2\pi} u(p + \epsilon e^{i\theta})d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} M d\theta, \\ &= M. \end{aligned}$$

Therefore $u(p + \epsilon e^{i\theta}) = M$ for all $0 \leq \theta \leq 2\pi$. Since this holds for all $0 < \epsilon < \epsilon_p$, with $D(p, \epsilon_p) \subseteq B$, B is open.

Let $z \in B$. Then by the submean inequality, for small enough $r > 0$,

$$M = u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})dt.$$

But if at some $t_0 \in [0, 2\pi]$, we have

$$u(z + re^{it_0}) < M,$$

then by upper semicontinuity,

$$\limsup u(w) \leq u(z + re^{it_0}) < M.$$

So for some $\epsilon > 0$ such that $|t - t_0| \leq \epsilon$ we can have, $w \to z + re^{it_0}$

$$u(z + re^{it_0}) \leq M - \epsilon.$$

Then

$$\begin{aligned} M &= u(z) \leq \frac{1}{2\pi}\int_{[0,2\pi]\cap[t_0-\epsilon,t_0+\epsilon]}(M-\epsilon)dt + \frac{1}{2\pi}\int_{[0,2\pi]\setminus[t_0-\epsilon,t_0+\epsilon]}Mdt. \\ &\leq M - \frac{\epsilon^2}{\pi}. \end{aligned}$$

This contradicts our choice of *M*, thus for small enough $r > 0$ and all $t \in [0,2\pi]$,

$$u(z + re^{it}) = M,$$

and *B* is open.

Finally, *B* is non-empty, while *A,B* are open, and their union is connected *D*.

Hence $B = D$, while *A* is empty. So *u* is constant on *D*.

Proof of (2)

We extend *u* to ∂D by defining, for each $\varsigma \in \partial D$,

$$u(\varsigma) = \limsup_{z \to \varsigma} u(z).$$

Our extension ensures that *u* is upper semicontinuous on ∂D, and subharmonic on *D*. Thus *u* is upper semicontinuous on $\overline{D}$, which is compact (on the Riemann sphere), so by Theorem 5.3.2, *u* attains a

maximum on $\overline{D}$. Let *w* be a point at which the maximum is attained. If $w \in \partial D$, then by assumption

$$\max_{\overline{D}} u = u(w) \leq 0.$$

That is, $u \leq 0$ in *D*.

If $w \in D$, then by (1), *u* is constant on *D*, and hence on $\overline{D}$. Again $u \leq 0$ in *D*.

Remark 5.3.3

In (1), u can attain a local maximum, or a global minimum, without being constant on D. For example, let D = C and

$$u(z) = \max\{Re(z), 0\}.$$

This is subharmonic. Moreover, u attains a global minimum (of 0) at any point in the closed left-half plane Re $z \leq 0$. Moreover, it attains the local minimum 0 at any point in the left-half plane Re $z < 0$. But u is not constant. We now weaken the requirements on u at ∞, in (2) above. The main idea of the PhragmenLindelof principle is that if u is bounded by 0 on the finite parts of the boundary of D and does not grow too fast at ∞, then $u \leq 0$ everywhere in D. The rate at which u is allowed to grow at ∞ depends on the geometry of the domain D.

Theorem 5.3.4 *[Ransford 1995]* **Phragmen-Lindelof Principle** *Let D be an unbounded domain in* C*, and u be a subharmonic function on D such that* $\forall \varsigma \in \partial D \backslash \{\infty\}$,

$$\limsup_{z \to \varsigma} u(z) \leq 0.$$

Suppose also that there exists a finite valued superharmonic functon v on D such that,

$$\liminf_{z \to \infty} v(z) > 0$$

and

$$\limsup_{z \to \infty} \frac{u(z)}{v(z)} \leq 0.$$

Then $u \leq 0$ *on D.*

Proof

Let us first consider the case where $v > 0$ in D.

Let > 0 and set

$$u_\epsilon = u - \epsilon v.$$

Clearly u is subharmonic on D, since $(-\epsilon v)$ will be subharmonic and for all

$\varsigma \in \partial D \backslash \{\infty\}$,

$$\limsup_{z \to \varsigma} u_\epsilon(z) \leq 0,$$

while for $\varsigma = \infty$,

$$\limsup_{z\to\infty} u_\epsilon(z) = \limsup_{z\to\infty} v(z)(\tfrac{u}{v}(z) - \epsilon) \leq 0.$$

So by (2) of the Maximum Principle,

$$u_\epsilon(z) = u(z) - \epsilon v(z) \leq 0$$

for all $z \in D$. Since this holds for all > 0, and since $v(z)$ is finite, the result follows.

Let us consider the case where v is not necessarily positive in D.

Let $\lambda > 0$ and set

$$G_\lambda = \{z \in D : u(z) \geq \lambda\}.$$

This is a closed set, it's complement $\{z \in D : u(z) < \lambda\}$, is open. Since v is lower semicontinuous, and

$$\liminf_{z\to\infty} v(z) > 0,$$

it follows that v is bounded below on G_λ. We can choose r so large that $v(z) > 0, z \in D$ such that $|z| > r$. So $v(z)$ is bounded below by 0 for large $|z|$. On the other hand, $\{z \in D : |z| \leq r \text{ and } u(z) \geq \lambda\} = G_\lambda \cap \{z : |z| \leq r\}$ is closed, bounded and so compact. Then v, being lower semicontinuous, attains its minimum on this compact set, so is bounded below. Thus v is bounded below on G_λ. Adding a constant to v, we can suppose that

$$v > 0 \in G_\lambda.$$

Now let

$$V = \{z \in D : v(z) > 0\}.$$

We now apply the case proved above to the function $u - \lambda$ on each (connected) component of V. First we claim that

$$\varsigma \in \partial V \backslash \{\infty\} \Rightarrow \limsup_{z \to \varsigma} (u - \lambda)(z) \le 0 . \tag{5.2}$$

We know that $V = D \cap \{z : v(z) > 0\}$, so $\varsigma \in \partial V \backslash \{\infty\}$ implies that either $\varsigma \in \partial D \backslash \{\infty\}$ or $\varsigma \in D$, but $v(\varsigma) \le 0$ (for V is open as v is lower semicontinuous).

Proof of (5.2)

For $\varsigma \in \partial D \backslash \{\infty\}$, here we use the hypothesis and $\lambda > 0$ to deduce

$$\limsup_{z \to \varsigma} (u - \lambda)(z) \le \limsup_{z \to \varsigma} u(z) - \lambda \le \limsup_{z \to \varsigma} u(z) \le 0 .$$

For $\varsigma \in D$ with $v(\varsigma) \le 0$.

We recall that $v > 0$ on G_λ, so $\varsigma \notin G_\lambda$. Then by definition of G_λ, we have

$$u(\varsigma) < \lambda.$$

By upper semicontinuity of u,

$$\limsup_{z \to \varsigma} (u - \lambda)(z) = \limsup_{z \to \varsigma} u(z) - \lambda \le u(\varsigma) - \lambda < 0 .$$

So (5.2) is true.

Moreover

$$\limsup_{z \to \infty} \frac{u(z) - \lambda}{v(z)} \le 0 .$$

Then applying the case of the theorem proved above to the function $u - \lambda$ on each component of V, we obtain

$$u - \lambda \le 0 \text{ on } V .$$

As $G_\lambda \subset V$, (for $v > 0$ on G_λ), it follows that $u - \lambda \le 0$ on G_λ, while $u - \lambda < 0$ on $D \backslash G_\lambda$. Thus

$$u - \lambda \le 0 \text{ on } D.$$

As this is true for each $\lambda > 0$, we have $u \le 0$ on D.

From this principle, a number of results can be derived one of which says that if u does not grow too fast at ∞, and is bounded above by 0 at a finite number of points of the boundary of D, then $u \leq 0$ in D.

5.3.2 Conditions for Subharmonicity

In this section, we prove a number of conditions for a function to be subharmonic. We shall show that a function is subharmonic, if it satisfies the submean value inequality for every disk in the domain discussed.

Theorem 5.3.5 *[Ransford 1995, Krantz 2000] Let* $U \subset \mathbb{C}$ *be open, and let* $u : U \to [-\infty,\infty)$ *be an upper semicontinuous function. The following are equivalent:*

1. *u is subharmonic on U.*
2. *Whenever D is a relatively compact subdomain of U, and h is a harmonic function on D , if* $\forall \varsigma \in \partial D$,

$$\limsup_{z\to\varsigma}(u-h)(z) \leq 0, \tag{5.3}$$

then $u \leq h$ *in D.*

3. *Whenever* $\overline{B}(w,\rho) \subset U$*, then for* $r < \rho$ *and* $0 \leq t \leq 2\pi$,

$$u(w+re^{it}) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\theta-t)+r^2}u(w+\rho e^{i\theta})d\theta, \tag{5.4}$$

where $\frac{1}{2\pi}\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\theta-t)+r^2} = P(r,t)$ *is the Poisson kernel.*

The relatively compact subdomain D means that D is compact in the relative topology, thus every open cover of D by open subsets of U has a finite subcover. We use the following result to prove Theorem 5.3.5.

Theorem 5.3.6 *If* $B = B(w,\rho)$ *and* $\chi : \partial B \to \mathbb{R}$ *is a Lebesgue-integrable function and its Poisson integral* $P_B\chi : B \to \mathbb{R}$ *is defined by*

$P_B\chi(w+re^{it}) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\theta-t)+r^2}\chi(w+re^{i\theta})d\theta,\ for\ r<\rho$ *and*

$0 \leq t \leq 2\pi$ *then,*

1. *$P_B\chi$ is harmonic on B and*

2. *if χ is continuous at $t_0 \in \partial B$, then* $\lim_{z\to t_0} P_B\chi(z) = \chi(t_0)$.

Proof Theorem 5.3.5

(1) ⇒ (2)

Let u is subharmonic in D and suppose h is harmonic in D, then $u - h$ is subharmonic in D, since $-h$ is subharmonic.

Since inequality (5.3) holds for all $\varsigma \in \partial D$, the maximum principle for subharmonic functions gives

$$u - h \leq 0 \text{ in } D.$$

Proof (2) ⇒ (3)

Suppose $\overline{B} = \overline{B}(w,\rho) \subset U$. Then u attains it maximum on this disk, so u is bounded above there.

By Theorem 5.2.4, there exists continuous functions $\varphi_n : \partial B \to \mathbb{R}$ such that φ_n decreases to u on ∂B as $n \to \infty$.

From Theorem 5.3.6, each $P_B\varphi_n$ is harmonic on B, also $\lim_{z\to t_0} P_B\varphi_n(z) = \varphi_n(t_0)$ for all $t_0 \in \partial B$, and hence

$$\limsup_{z\to t_0}(u - P_B\varphi_n)(z) \leq u(t_0) - \varphi_n(t_0) \leq 0.$$

From (2) it follows that $u \leq P_B\varphi_n$ on B. Letting $n \to \infty$ and using the monotone convergence theorem gives the desired inequality, since φ_n is bounded above and monotonic.

Proof (3) ⇒ (1)

Suppose (3) holds and let $r = 0$, then for $r < \rho$, we have

$$u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} u(w+\rho e^{i\theta})d\theta.$$

That is, u satisfies the submean inequality in each disk contained in D.

Theorem 5.3.7 *[Ransford 1995]* **Integrability of Subharmonic Functions** *Let u be a subharmonic function on a domain D in $\mathbb{C}$, with $u \neq -\infty$ on D. Then u is locally integrable on D, that is*

$$\int_K |u|dA < \infty,$$

for each compact subset K of D, where dA denotes two-dimensional Lebesgue measure.

Theorem 5.3.8 *[Ransford 1995, Laurent-Thiébant 2011] Let $-\infty \leq a < b \leq \infty$, and let $u : U \to [a,b)$ be a subharmonic function on an open set U in $\mathbb{C}$. Let $\varphi : (a,b) \to \mathbb{R}$ be an increasing convex function. Then φ o u is subharmonic on U, where we define*

$$\varphi(a) = \lim_{t\to a}\varphi(t).$$

Proof

Choose a sequence (a_n) in (a,b) that decreases to a as $n \to \infty$. Let

$$u_n = \max\{u,a_n\}, n > 1.$$

Then u_n is subharmonic in U. So φ o u_n is upper semicontinuous, being the composition of continuous increasing φ with upper semicontinuous u_n. Indeed,

$$\begin{aligned}\limsup_{y\to x}\varphi(u_n(y)) &= \varphi(\limsup_{y\to x} u_n(y))\\ &\leq \varphi(u_n(x)).\end{aligned}$$

Also, if $\overline{B}(w,\rho) \subset U$, then as u_n satisfies the submean inequality and φ is increasing,

$$\begin{aligned}(\phi \ o \ u_n)(w) &\leq \phi(\tfrac{1}{2\pi}\textstyle\int_0^{2\pi} u_n(w+\rho e^{i\theta}) \ d\theta)\\ &= \phi(\textstyle\int_0^{2\pi} u_n(w+\rho e^{i\theta}) \ \frac{d\theta}{2\pi})\end{aligned}$$

$$\leq \int_0^{2\pi} (\phi \ o \ u_n)(w + \rho e^{i\theta}) \frac{d\theta}{2\pi},$$

by Jensen's inequality applied with the measure $\frac{d\theta}{2\pi}$ on $[0,2\pi]$. (Note that this is a measure of total mass 1). So $\varphi \ o \ u_n$ satisfies the submean inequality on U and is then subharmonic there. Since $\varphi \ o \ u_n$ decreases to $\varphi \ o \ u$ as n increases, and from the Lebesgue monotone convergence theorem we have that a decreasing sequence of subharmonic functions has subharmonic limit, thus $\varphi \ o \ u$ is subharmonic on U.

5.3.3 Some examples of subharmonic functions

1. If h is holomorphic on $D \subseteq \mathbb{C}^n$, then $|h|^p$ is subharmonic for all $p > 0$.

2. If h is holomorphic on $D \subseteq \mathbb{C}^n$, then $\log^+ |h|$ is subharmonic, where $\log^+_x = \max\{0, \log x\}, x > 0.$

3. Let $U \subset \mathbb{C}$ be open and $u : U \to [0,\infty)$. Then $\log u$ is subharmonic on U iff $u|e^p|$ is subharmonic on U for every polynomial p (with complex coefficients).

Chapter 6

Main results

6.1 Refinement of Jensen's inequality for two points

In this section, we prove a new refinement of the Jensen's inequality by using Definition 3.0.29. of superquadraticity, where ϕ is from $D \subset [0,\infty)$ to $[0,\infty)$, where D is considered to be a 2-divisible abelian gorup.

[Gil´$a$nyi and Troczka 2011] Shown in their paper that a non-negative, superquadratic function defined on a 2-divisible abelian group is Jensen-convex.

In the next four (4) propositions, we consider the ϕ in inequality (3.5) to be non-negative and defined on a 2-divisible abelian group $D \subset$ R, which gives rise to inequality (3.6).

The inequality (3.6) is employed in the proof of the propositions 6.1.1, 6.1.2, 6.1.8 and 6.1.9

Proposition 6.1.1 *[Asare and Prempeh 2015a] For* $\lambda_1 = \frac{1}{2^n}, \lambda_2 = \frac{2^n-1}{2^n}$ *and for all* $n \in$ N

$$\varphi(\lambda_1 a + \lambda_2 b) \leq \lambda_1 \varphi(a) + \lambda_2 \varphi(b) - \Big[\sum_{i=1}^{n} (\lambda_1 2^i) \varphi\Big(\frac{b-a}{2^i}\Big)\Big] \tag{6.1}$$

where ϕ satisfies Definition 3.0.29 and a,b ∈ D.

Proof

We establish (6.1) using induction.

For n = 1, (6.1) is (3.6).

Fix $k \in$ N,k > 1 and suppose (6.1) is true, that is
$\varphi(\frac{a}{2^k} + \frac{2^k-1}{2^k} b) \leq \frac{1}{2^k}\varphi(a) + \frac{2^k-1}{2^k}\varphi(b) - [\sum_{i=1}^{k}(\frac{1}{2^{k-i}})\varphi(\frac{b-a}{2^i})]$.

Now

$$\begin{aligned} \varphi\Big(\frac{a}{2^{k+1}} + \frac{2^{k+1}-1}{2^{k+1}} b\Big) &= \varphi\Big(\frac{1}{2}\Big(\frac{a}{2^k} + \frac{2^{k+1}-1}{2^k} b\Big)\Big) \\ &= \varphi\Big(\frac{1}{2}\Big(\frac{a + (2^k-1)b}{2^k} + b\Big)\Big) \end{aligned}$$

and so by inequality (3.6),

$$\begin{aligned} \varphi\Big(\frac{1}{2}\Big[\frac{a + (2^k-1)b}{2^k} + b\Big]\Big) &\leq \frac{1}{2}\varphi\Big(\frac{a + (2^k-1)b}{2^k}\Big) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{1}{2}\frac{a + (2^k-1)b}{2^k} - b\Big) \\ &= \frac{1}{2}\varphi\Big(\frac{a + (2^k-1)b}{2^k}\Big) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{b-a}{2^{k+1}}\Big). \end{aligned}$$

But $\varphi(\frac{a}{2^k} + \frac{2^k-1}{2^k} b) \leq \frac{1}{2^k}\varphi(a) + \frac{2^k-1}{2^k}\varphi(b) - [\ \sum_{i=1}^{k}(\frac{1}{2^{k-i}})\varphi(\frac{b-a}{2^i})]$, from the inductive hypothesis.

So

$$\varphi\Big(\tfrac{a}{2^{k+1}}+\tfrac{2^{k+1}-1}{2^{k+1}}b\Big)$$
$$\leq \tfrac{1}{2^{k+1}}\varphi(a)+\tfrac{2^k-1}{2^{k+1}}\varphi(b)-\tfrac{1}{2}\Big[\sum_{i=1}^{k}\Big(\tfrac{1}{2^{k-i}}\Big)\varphi\Big(\tfrac{b-a}{2^i}\Big)\Big]+\tfrac{1}{2}\varphi(b)-\varphi(\tfrac{b-a}{2^{k+1}}).$$

Thus

$$\varphi\Big(\frac{a}{2^{k+1}}+\frac{2^{k+1}-1}{2^{k+1}}b\Big)\leq\frac{1}{2^{k+1}}\varphi(a)+\frac{2^{k+1}-1}{2^{k+1}}\varphi(b)-\Big[\sum_{i=1}^{k+1}\Big(\frac{1}{2^{k+1-i}}\Big)\varphi\Big(\frac{b-a}{2^i}\Big)\Big],$$

which completes the proof.

Proposition 6.1.2 *[Asare and Prempeh 2015a]*

$$\varphi\Big(\frac{\alpha}{2^n}a+\frac{\beta}{2^n}b\Big)$$
$$\leq\frac{\alpha}{2^n}\varphi(a)+\frac{\beta}{2^n}\varphi(b)-\Big[\sum_{i=1}^{2}\Big(\frac{1}{2^{n-i}}\Big)\varphi\Big(\frac{b-a}{2^i}\Big)+\sum_{i=3}^{n}\Big(\frac{1}{2^{n-i}}\Big)\varphi\Big(\frac{\epsilon_{(n+1-i)}}{2^i}[b-a]\Big)\Big], \tag{6.2}$$

where α,β ∈ N such that α + β = 2^n*, n − 2, is the number of splits and* $\epsilon_{(n+1-i)}$*is the minimum of {α, β} before the (n + 1 − i) split.*

Definition 6.1.3 For $\lambda_1 = \frac{\alpha}{2^n}, \lambda_2 = \frac{\beta}{2^n}$ we define *a split of* $\varphi(\frac{\alpha}{2^n}a + \frac{\beta}{2^n}b)$ to be

$$\varphi\Big(\frac{1}{2}\Big[\frac{\alpha a + (\beta - 2^{n-1})b}{2^{n-1}} + b\Big]\Big).$$

To demonstrate the techniques in the proof of proposition 6.1.2, we first consider some examples for given values of *n,α* and *β*.

Example 6.1.4 *We consider the case where n* = 3*, α* = 3 *and β* = 5*, that is*

$$\begin{aligned}\varphi\Big(\frac{3a}{8} + \frac{5b}{8}\Big) &= \varphi\Big(\frac{1}{2}\Big[\frac{3a}{4} + \frac{5b}{4}\Big]\Big) \\ &= \varphi\Big(\frac{1}{2}\Big[\Big(\frac{3a}{4} + \frac{b}{4}\Big) + b\Big]\Big) \\ &\leq \frac{1}{2}\varphi\Big(\frac{3a+b}{4}\Big) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{3}{8}[b-a]\Big),\end{aligned}$$

using the split and (3.6).
From inequality (6.1)

$$\varphi\Big(\frac{3a}{4} + \frac{b}{4}\Big) \leq \frac{3}{4}\varphi(a) + \frac{1}{4}\varphi(b) - \Big[\frac{1}{2}\varphi\Big(\frac{b-a}{2}\Big) + \varphi\Big(\frac{b-a}{4}\Big)\Big].$$

So

$$\varphi\Big(\frac{3a}{8} + \frac{5b}{8}\Big) \leq \frac{3}{8}\varphi(a) + \frac{1}{8}\varphi(b) + \frac{1}{2}\varphi(b) - \frac{1}{2}\Big[\frac{1}{2}\varphi\Big(\frac{b-a}{2}\Big) + \varphi\Big(\frac{b-a}{4}\Big)\Big] - \varphi\Big(\frac{3}{8}[b-a]\Big)$$

$$\varphi\Big(\frac{3a}{8} + \frac{5b}{8}\Big) \leq \frac{3}{8}\varphi(a) + \frac{5}{8}\varphi(b) - \Big[\frac{1}{4}\varphi\Big(\frac{b-a}{2}\Big) + \frac{1}{2}\varphi\Big(\frac{b-a}{4}\Big) + \varphi\Big(\frac{3}{8}[b-a]\Big)\Big]$$

Example 6.1.5 *We consider the case where n* = 4*, α* = 5 *and β* = 11*, that is*

$$\begin{aligned}\varphi\Big(\frac{5a}{16} + \frac{11b}{16}\Big) &= \varphi\Big(\frac{1}{2}\Big[\frac{5a}{8} + \frac{11b}{8}\Big]\Big) \\ &= \varphi\Big(\frac{1}{2}\Big[\Big(\frac{5a+3b}{8} + b\Big]\Big) \\ &\leq \frac{1}{2}\varphi\Big(\frac{5a+3b}{8}\Big) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{5}{16}[b-a]\Big)\end{aligned}$$

,

using the split and (3.6).

Now from Example 6.1.4,

$$\varphi\Big(\frac{5a+3b}{8}\Big) \le \frac{5}{8}\varphi(a) + \frac{3}{8}\varphi(b) - \Big[\frac{1}{4}\varphi\Big(\frac{b-a}{2}\Big) + \frac{1}{2}\varphi\Big(\frac{b-a}{4}\Big) + \varphi\Big(\frac{3}{8}[b-a]\Big)\Big].$$

So

$$\begin{aligned}\varphi\Big(\frac{5a}{16}+\frac{11b}{16}\Big) \;\le\; & \frac{5}{16}\varphi(a) + \frac{3}{16}\varphi(b) - \Big[\frac{1}{8}\varphi\Big(\frac{b-a}{2}\Big) + \frac{1}{4}\varphi\Big(\frac{b-a}{4}\Big)\Big] \\ - \; & \frac{1}{2}\varphi\Big(\frac{3}{8}[b-a]\Big) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{5}{16}[b-a]\Big)\end{aligned}$$

This simplifies to

$$\varphi\Big(\tfrac{5a}{16}+\tfrac{11b}{16}\Big)$$

$$\le \frac{5}{16}\varphi(a)+\frac{11}{16}\varphi(b)-\Big[\frac{1}{8}\varphi\Big(\frac{b-a}{2}\Big)+\frac{1}{4}\varphi\Big(\frac{b-a}{4}\Big)\Big]-\Big[\frac{1}{2}\varphi\Big(\frac{3}{8}[b-a]\Big)+\varphi\Big(\frac{5}{16}[b-a]\Big)\Big].$$

Proof of Proposition 6.1.2

If α and β are both even, say $2\delta = \alpha$ and $2\sigma = \beta$ where $\delta, \sigma \in Z^+$, then

$\delta + \sigma = 2^{n-1}$ and so we only need to prove the result for α, β both odd.

Let $\alpha = 2\delta + 1$ and $\beta = 2\sigma + 1$, where $\delta, \sigma \in Z^+ \cup \{0\}$, so

$2(\delta + \sigma + 1) = 2^n$ and therefore $\sigma = 2^{n-1} - \delta - 1$.

Without loss of generality let $\sigma \ge \delta$.

We consider the two special cases, $n = 1$ and $n = 2$.

Case 1 where $n = 1$, we have $\delta = \sigma = 0$ and $\alpha = \beta = 1$, $(n - 2)$ is -1. But we cannot have a negative number of splits, so we ignore the last expression in (6.2) and summing i from 1 to 1 since $n = 1$, (6.2) reduces to

$$\varphi\Big(\frac{1}{2}a + \frac{1}{2}b\Big) \le \frac{1}{2}\varphi(a) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{b-a}{2}\Big)\Big],$$

which is precisely inequality (3.6).

Case 2 where $n = 2$, the number of splits is zero and we have only two odd numbers 1 and 3, thus we have the expression $\varphi\Big(\frac{a+3b}{4}\Big)$ to expand. This is (6.1) a specal case of (6.2) for $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{3}{4}$, giving

$$\varphi\Big(\frac{a+3b}{4}\Big) \le \frac{1}{4}\varphi(a) + \frac{3}{4}\varphi(b) - \Big[\frac{1}{2}\varphi\Big(\frac{1}{2}[b-a]\Big) + \varphi\Big(\frac{1}{4}[b-a]\Big)\Big]$$

$$= \frac{1}{4}\varphi(a) + \frac{3}{4}\varphi(b) - \Big[\sum_{i=1}^{2}\Big(\frac{1}{2^{2-i}}\Big)\varphi\Big(\frac{b-a}{2^i}\Big)\Big],$$

hence inequality (6.2) is true for the special cases where $n = 1$ and $n = 2$.

We consider $n = 2$ as our base point since the number of splits is zero

For any pair of odd numbers α,β such that $\alpha + \beta = 2^n, \forall n \in \mathbb{N}, n \geq 2$, the pair α,β is split recursively until the pair (1,3) is obtained, which is the base point. Therefore we use the principle of mathematical induction to establish the result for the split for $n \geq 3$. For $n = 3$, there is only one split.

$n = 3, 2^n = 8$ and we have the 4 odd numbers {1,3,5,7}, we consider those pairs whose sum is 8, that is (1,7) and (3,5).

It is noted that the pair (3,5) is the case of Example 6.1.4. above. Therefore we consider the pair (1,7) to obtain

$$\begin{aligned}\varphi\Big(\frac{a+7b}{8}\Big) &= \varphi\Big(\frac{1}{2}\Big(\frac{a+7b}{4}\Big)\Big)\\ &= \varphi\Big(\frac{1}{2}\Big(\Big[\frac{a+3b}{4}+b\Big]\Big)\Big)\\ &\leq \frac{1}{2}\varphi\Big(\frac{a+3b}{4}\Big)\varphi(a) + \frac{1}{2}\varphi(b) - \varphi\Big(\frac{1}{8}[b-a]\Big),\end{aligned}$$

using the split and (3.6).

using the base point $\varphi\Big(\frac{a+3b}{4}\Big)$, we have

$$\varphi\Big(\frac{1}{4}a + \frac{3}{4}b\Big) \leq \frac{1}{4}\varphi(a) + \frac{3}{4}\varphi(b) - \Big[\frac{1}{2}\varphi\Big(\frac{1}{2}[b-a]\Big) + \varphi\Big(\frac{1}{4}[b-a]\Big)\Big].$$

Hence

$$\varphi\Big(\frac{a+7b}{8}\Big) \leq \frac{1}{2}\Big[\frac{1}{4}\varphi(a)+\frac{3}{4}\varphi(b)-\Big[\frac{1}{2}\varphi\Big(\frac{1}{2}[b-a]\Big)+\varphi\Big(\frac{1}{4}[b-a]\Big)\Big]\Big]+\frac{1}{2}\varphi(b)-\varphi\Big(\frac{1}{8}[b-a]\Big),$$

which simplifies to

$$\varphi\Big(\frac{a+7b}{8}\Big) \leq \frac{1}{8}\varphi(a) + \frac{7}{8}\varphi(b) - \Big[\frac{1}{4}\varphi\Big(\frac{1}{2}[b-a]\Big) + \frac{1}{2}\varphi\Big(\frac{1}{4}[b-a]\Big) + \varphi\Big(\frac{1}{8}[b-a]\Big)\Big],$$

$$= \frac{1}{8}\varphi(a)+\frac{7}{8}\varphi(b)-\Big[\sum_{i=1}^{2}\Big(\frac{1}{2^{3-i}}\Big)\varphi\Big(\frac{b-a}{2^i}\Big)+\sum_{i=3}^{3}\Big(\frac{1}{2^{3-i}}\Big)\varphi\Big(\frac{\epsilon_{(4-i)}(\alpha,\ \beta)}{2^i}[b-a]\Big)\Big].$$

Fix $k \in \mathbb{N}, k > 3$ and suppose (6.2) is true, that is

$$\varphi\left(\frac{\alpha}{2^k}a+\frac{\beta}{2^k}b\right)$$
$$\leq \frac{\alpha}{2^k}\varphi(a)+\frac{\beta}{2^k}\varphi(b)-\sum_{i=1}^{2}\left(\frac{1}{2^{k-i}}\right)\varphi\left(\frac{b-a}{2^i}\right)-\sum_{i=3}^{k}\left(\frac{1}{2^{k-i}}\right)\varphi\left(\frac{\epsilon_{(k+1-i)}(\alpha,\beta)}{2^i}[b-a]\right)\Big].$$

For $k+1 \in \mathrm{N}$, we have $k-1$ splits. Setting $\alpha = (2\delta+1)$ and $\beta = (2\sigma+1)$,

$$\begin{aligned}\varphi\Big[\frac{(2\delta+1)}{2^{k+1}}a+\frac{(2\sigma+1)}{2^{k+1}}b\Big] &= \varphi\Big(\frac{1}{2}\Big[\frac{(2\delta+1)}{2^k}a+\frac{(2\sigma+1)}{2^k}b\Big]\Big)\\ &= \varphi\Big(\frac{1}{2}\Big[\frac{(2\delta+1)a+(2^k-2\delta-1)b}{2^k}+b\Big]\Big),\end{aligned}$$

so
$$\varphi\left[\frac{(2\delta+1)}{2^{k+1}}a+\frac{(2\sigma+1)}{2^{k+1}}b\right]$$
$$\leq \frac{1}{2}\varphi\left(\frac{(2\delta+1)a+(2^k-2\delta-1)b}{2^k}\right)+\frac{1}{2}\varphi(b)-\varphi\left(\frac{(2\delta+1)}{2^{k+1}}[b-a]\right), \text{ using (3.6).}$$

But from the inductive hypothesis, the first term of the immediate inequality becomes:
$$\varphi\left(\frac{(2\delta+1)a+(2^k-2\delta-1)b}{2^k}\right)$$
$$\leq \frac{(2\delta+1)}{2^k}\varphi(a)+\frac{(2^k-2\delta-1)}{2^k}\varphi(b)-\sum_{i=1}^{2}\left(\frac{1}{2^{k-i}}\right)\varphi\left(\frac{b-a}{2^i}\right)-$$
$$\sum_{i=3}^{k}\left(\frac{1}{2^{k-i}}\right)\varphi\left(\frac{\epsilon_{(k+1-i)}(\alpha,\beta)}{2^i}[b-a]\right).$$

Therefore
$$\varphi\left(\frac{(2\delta+1)}{2^{k+1}}a+\frac{(2^{k+1}-2\delta-1)}{2^{k+1}}b\right)$$
$$\leq \frac{(2\delta+1)}{2^{k+1}}\varphi(a)+\frac{(2^{k+1}-2\delta-1)}{2^{k+1}}\varphi(b)-\sum_{i=1}^{2}\left(\frac{1}{2^{k+1-i}}\right)\varphi\left(\frac{b-a}{2^i}\right)-$$
$$\sum_{i=3}^{k}\left(\frac{1}{2^{k+1-i}}\right)\varphi\left(\frac{\epsilon_{(k+1-i)}(\alpha,\beta)}{2^i}[b-a]\right)-\varphi\left(\frac{(2\delta+1)}{2^{k+1}}[b-a]\right),$$
which simplifies to
$$\varphi\left(\frac{(2\delta+1)}{2^{k+1}}a+\frac{(2^{k+1}-2\delta-1)}{2^{k+1}}b\right)$$
$$\leq \frac{(2\delta+1)}{2^{k+1}}\varphi(a)+\frac{(2^{k+1}-2\delta-1)}{2^{k+1}}\varphi(b)-\sum_{i=1}^{2}\left(\frac{1}{2^{k+1-i}}\right)\varphi\left(\frac{b-a}{2^i}\right)-$$

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$$\sum_{i=3}^{k+1}\left(\frac{1}{2^{k+1-i}}\right)\varphi\left(\frac{\epsilon_{(k+2-i)}(\alpha,\beta)}{2^i}[b-a]\right),$$

as required.

Remark 6.1.6 $\frac{(2\delta+1)}{2^{k+1}}$ is the required minimum before the first split.

Hence inequality (6.2) is true for all N.

Remark 6.1.7 [Asare and Prempeh 2015a] An inequality similar to that of inequality (6.2) is obtained, when $\alpha + \beta$ 6= 2^n.

Proof

Let α,β and $m \in$ N such that $\alpha + \beta = m$ 6= 2^n, where $n \in$ N. Choose $s \in$ N

such that $2^{s-1} < m < 2^s$ then

$$\begin{aligned}
\varphi\Big(\frac{\alpha x_1 + \beta x_2}{m}\Big) &= \varphi\Big[\frac{1}{2^s}\Big(\alpha x_1 + \beta x_2 + (2^s - m)\Big(\frac{\alpha x_1 + \beta x_2}{m}\Big)\Big)\Big] \\
&= \varphi\Big[\frac{1}{2}\Big(\frac{\alpha x_1 + B_1 x_2}{2^{s-1}}\Big) + \frac{1}{2}\Big(\frac{B_2 x_2 + (2^s - m)\Big(\frac{\alpha x_1+\beta x_2}{m}\Big)}{2^{s-1}}\Big)\Big] \\
&\leq \frac{1}{2}\varphi\Big(\frac{\alpha x_1 + B_1 x_2}{2^{s-1}}\Big) + \frac{1}{2}\Big(\frac{B_2 x_2 + (2^s - m)\Big(\frac{\alpha x_1+\beta x_2}{m}\Big)}{2^{s-1}}\Big) - \varphi\Big(\frac{1}{2}[A - E]\Big)
\end{aligned}$$

where $B_1,B_2 \in$ N such that $B_1 + B_2 = \beta$, $A = \Big(\frac{B_2 x_2+(2^s-m)\Big(\frac{\alpha x_1+\beta x_2}{m}\Big)}{2^{s-1}}\Big)$ and $E = \Big(\frac{\alpha x_1+B_1 x_2}{2^{s-1}}\Big)$.

But $\varphi\Big(\frac{\alpha x_1+B_1 x_2}{2^{s-1}}\Big)$ simplifies to either inequality (6.1) or (6.2) depending on the values of α and B_1, as shown in Propositions 6.1.1 and 6.1.2.

However for the expression $\varphi\Big(\frac{B_2 x_2+(2^s-m)\Big(\frac{\alpha x_1+\beta x_2}{m}\Big)}{2^{s-1}}\Big)$ we have,

$$\varphi\Big(\frac{B_2 x_2+(2^s-m)\Big(\frac{\alpha x_1+\beta x_2}{m}\Big)}{2^{s-1}}\Big) \leq \frac{B_2}{2^{s-1}}\varphi(x_2) + \frac{(2^s-m)}{2^{s-1}}\varphi\Big(\frac{\alpha x_1+\beta x_2}{m}\Big) - D,$$

where D is the appropriate extra terms.

So

$$\varphi\Big(\frac{\alpha x_1 + \beta x_2}{m}\Big) \leq \frac{\alpha}{2^s}\varphi(x_1) + \frac{B_1}{2^s}\varphi(x_2) + \frac{B_2}{2^s}\varphi(x_2) + \frac{(2^s - m)}{2^s}\varphi\Big(\frac{\alpha x_1 + \beta x_2}{m}\Big) -$$

$$\sum_{i=1}^{s-1-r}\Big(\frac{1}{2^{s-i}}\Big)\varphi\Big(\frac{x_2-x_1}{2^i}\Big)-\sum_{i=s-r}^{s-1}\Big(\frac{1}{2^{s-i}}\Big)\varphi\Big(\frac{\epsilon_{(s-i)}}{2^i}[x_2-x_1]\Big)-D,$$

$$(\frac{m}{2^s})\varphi\Big(\frac{\alpha x_1+\beta x_2}{m}\Big)\leq\frac{\alpha}{2^s}\varphi(x_1)+\frac{\beta}{2^s}\varphi(x_2)-\sum_{i=1}^{s-1-r}\Big(\frac{1}{2^{s-i}}\Big)\varphi\Big(\frac{x_2-x_1}{2^i}\Big)-$$

$$\sum_{i=s-r}^{s-1}\Big(\frac{1}{2^{s-i}}\Big)\varphi\Big(\frac{\epsilon_{(s-i)}}{2^i}[x_2-x_1]\Big)-D.$$

Thus

$$\varphi\Big(\frac{\alpha x_1+\beta x_2}{m}\Big)\leq\frac{\alpha}{m}\varphi(x_1)+\frac{\beta}{m}\varphi(x_2)-\frac{2^s}{m}\sum_{i=1}^{s-1-r}\Big(\frac{1}{2^{s-i}}\Big)\varphi\Big(\frac{x_2-x_1}{2^i}\Big)-$$

$$\frac{2^s}{m}\sum_{i=s-r}^{s-1}\Big(\frac{1}{2^{s-i}}\Big)\varphi\Big(\frac{\epsilon_{(s-i)}}{2^i}[x_2-x_1]\Big)-\frac{2^s}{m}D.$$

6.1.1 Generalized refined Jensen's inequality

Proposition 6.1.8 *Let* $\phi:[0,\infty)\to[0,\infty)$ *be superquadratic,* $n,m\in\mathbb{N}, n=2^m$ *and let* $x_i's\in D\subset[0,\infty)$ *then*

$$\varphi\Big(\frac{1}{2^m}\sum_{i=1}^{2^m}x_i\Big)\leq\frac{1}{2^m}\sum_{i=1}^{2^m}\varphi(x_i)-$$

$$\sum_{j=1}^{m}\Big[\frac{1}{2^{m-j}}\sum_{i=1}^{2^{m-j}}\varphi\Big(\frac{1}{2^j}\Big[(x_{2^j(i-1)+1}+\ldots+x_{2^ji-2^{j-1}})-(x_{2^ji-2^{j-1}+1}+\ldots+x_{2^ji})\Big]\Big)\Big].\quad(6.3)$$

Proof

We note that the right hand side of the inequality remains non-negative as it is a sum over 2^{m-1} pairs x_1,x_2 of non-negative terms like the right hand side of (3.6).

We establish the proof using mathematical induction.

For $m=1$, we obtain inequality (3.6).

In the sequel the following notations are used:

$$T^+: \;=\; x_{2^j(i-1)+1}+\ldots+x_{2^ji-2^{j-1}}$$

$$U^+: \;=\; x_{2^ji-2^{j-1}+1}+\ldots+x_{2^ji}$$

$$T^+_{2^k}: \;=\; x_{2^j(i-1)+2^k+1}+\ldots+x_{2^ji-2^{j-1}+2^k}$$

$$U^+_{2^k}: \;=\; x_{2^ji-2^{j-1}+2^k+1}+\ldots+x_{2^ji+2^k}$$

Fix $k\in\mathbb{N}, k>1$ and suppose (6.3) is true, so the inductive hypothesis says

$$\varphi\Big(\frac{1}{2^k}\sum_{i=1}^{2^k} x_i\Big) \le \frac{1}{2^k}\sum_{i=1}^{2^k}\varphi(x_i) - \sum_{j=1}^{k}\Big[\frac{1}{2^{k-j}}\sum_{i=1}^{2^{k-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big].$$

Now

$$\begin{aligned}\varphi\Big(\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} x_i\Big) &= \varphi\Big[\frac{1}{2}\Big(\frac{1}{2^k}\sum_{i=1}^{2^k} x_i\Big) + \frac{1}{2}\Big(\frac{1}{2^k}\sum_{i=2^k+1}^{2^{k+1}} x_i\Big)\Big]\\ &\le \frac{1}{2}\varphi\Big(\frac{1}{2^k}\sum_{i=1}^{2^k} x_i\Big) + \frac{1}{2}\varphi\Big(\frac{1}{2^k}\sum_{i=1+2^k}^{2^{k+1}} x_i\Big)\\ &- \qquad \varphi\Big(\frac{1}{2^{k+1}}\Big[\sum_{i=1}^{2^k} x_i - \sum_{i=2^k+1}^{2^{k+1}} x_i\Big]\Big)\end{aligned}$$

by inequality (3.6).

Applying the inductive hypothesis to the first two terms on the right hand side of the above expression we have

$$\varphi\Big(\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} x_i\Big) \le \frac{1}{2^{k+1}}\sum_{i=1}^{2^k}\varphi(x_i) + \frac{1}{2^{k+1}}\sum_{i=1+2^k}^{2^{k+1}}\varphi(x_i) - \sum_{j=1}^{k}\Big[\frac{1}{2^{k+1-j}}\sum_{i=1}^{2^{k-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big]$$

$$- \sum_{j=1}^{k}\Big[\frac{1}{2^{k+1-j}}\sum_{i=1}^{2^{k-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+_{2^k} - U^+_{2^k}\Big]\Big)\Big] - \varphi\Big(\frac{1}{2^{k+1}}\Big[\sum_{i=1}^{2^k} x_i - \sum_{i=2^k+1}^{2^{k+1}} x_i\Big]\Big).$$

Thus

$$\varphi\Big(\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} x_i\Big) \le \frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}}\varphi(x_i) - \sum_{j=1}^{k}\Big[\frac{1}{2^{k+1-j}}\sum_{i=1}^{2^{k+1-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big] -$$

$$\varphi\Big(\frac{1}{2^{k+1}}\Big[\sum_{i=1}^{2^k} x_i - \sum_{i=2^k+1}^{2^{k+1}} x_i\Big]\Big).$$

Rewriting the last term as

$$\sum_{j=k+1}^{k+1}\Big[\frac{1}{2^{k+1-j}}\sum_{i=1}^{2^{k+1-j}}\varphi\Big(\frac{1}{2^{k+1}}\Big[T^+ - U^+\Big]\Big)\Big],$$

we have

$$\varphi\Big(\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} x_i\Big) \le \frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}}\varphi(x_i) - \sum_{j=1}^{k+1}\Big[\frac{1}{2^{k+1-j}}\sum_{i=1}^{2^{k+1-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big].$$

which completes the proof.

Proposition 6.1.9 *Let ϕ : [0,∞) → [0,∞) be superquadratic, n,m* ∈ N : *n* 6= 2^m

and let $x^0_i s \in D \subset [0,\infty)$ *then*

$$\varphi\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \leq \frac{1}{n}\sum_{i=1}^{n}\varphi(x_i) - \frac{2^s}{n}\sum_{j=1}^{s}\Big[\frac{1}{2^{s-j}}\sum_{i=1}^{2^{s-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big] \qquad (6.4)$$

for some s ∈ N *satisfying* $2^{s-1} < n < 2^s$.

Proof

Since ϕ is a superquadratic function on [0,∞), setting $y = 0$ in (3.5) yields $\phi(0) \leq 0$.

But since ϕ maps into [0,∞) we have $\phi(0) = 0$.

Now if *n* 6= 2^m then there exists $s \in$ N : $2^{s-1} < n < 2^s$, so for $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$\begin{aligned}\varphi\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) &= \varphi\Big(\frac{1}{2^s}[x_1 + \ldots + x_n + (2^s - n)\overline{X}_n]\Big)\\ &= \varphi\Big(\frac{1}{2}\Big[\frac{x_1 + \ldots + x_{2^{s-1}}}{2^{s-1}}\Big] + \frac{1}{2}\Big[\frac{x_{2^{s-1}+1} + \ldots + x_n + (2^s - n)\overline{X}_n}{2^{s-1}}\Big]\Big),\end{aligned}$$

From inequality (3.6) the above expression becomes,

$$\varphi\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \leq \frac{1}{2}\varphi\Big[\frac{x_1 + \ldots + x_{2^{s-1}}}{2^{s-1}}\Big] + \frac{1}{2}\varphi\Big[\frac{x_{2^{s-1}+1} + \ldots + x_n + (2^s - n)\overline{X}_n}{2^{s-1}}\Big] -$$

$$\varphi\Big(\frac{1}{2}\Big(\Big[\frac{x_1 + \ldots + x_{2^{s-1}}}{2^{s-1}}\Big] - \Big[\frac{x_{2^{s-1}+1} + \ldots + x_n + (2^s - n)\overline{X}_n}{2^{s-1}}\Big]\Big)\Big).$$

From Proposition 6.1.8. we have that,

$$\varphi\Big[\frac{x_1 + \ldots + x_{2^{s-1}}}{2^{s-1}}\Big] \leq \frac{1}{2^{s-1}}\sum_{i=1}^{2^{s-1}}\varphi(x_i) - \sum_{j=1}^{s-1}\Big[\frac{1}{2^{s-(j+1)}}\sum_{i=1}^{2^{s-(j+1)}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big],$$

and

$$\varphi\Big[\frac{x_{2^{s-1}+1} + \ldots + x_n + (2^s - n)\overline{X}_n}{2^{s-1}}\Big] \leq \frac{1}{2^{s-1}}\sum_{i=1+2^{s-1}}^{n}\varphi(x_i) + \frac{2^s - n}{2^{s-1}}\varphi(\overline{X}_n) -$$

$$\sum_{j=1}^{s-1}\Big[\frac{1}{2^{s-(j+1)}}\sum_{i=1}^{2^{s-(j+1)}}\varphi\Big(\frac{1}{2^j}\Big[T^+_{2^{s-1}} - U^+_{2^{s-1}}\Big]\Big)\Big],$$

where we define $x_{n+1} =,\ldots,= x_{2^s} = \overline{X}_n$ so that

$$\sum_{i=n+1}^{2^s} x_i = (2^s - n)\overline{X}_n.$$

So

$$\varphi\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \le \frac{1}{2^s}\sum_{i=1}^{n}\varphi(x_i) + \frac{2^s-n}{2^s}\varphi(\overline{X}_n) - \sum_{j=1}^{s-1}\Big[\frac{1}{2^{s-(j+1)}}\sum_{i=1}^{2^{s-(j+1)}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big] -$$

$$\sum_{j=1}^{s-1}\Big[\frac{1}{2^{s-(j+1)}}\sum_{i=1}^{2^{s-(j+1)}}\varphi\Big(\frac{1}{2^j}\Big[T^+_{2^{s-1}} - U^+_{2^{s-1}}\Big]\Big)\Big] -$$

$$\varphi\Big(\frac{1}{2}\Big|\Big[\frac{x_1 + \dots + x_{2^{s-1}}}{2^{s-1}}\Big] - \Big[\frac{x_{2^{s-1}+1} + \dots + x_n + (2^s - n)\overline{X}_n}{2^{s-1}}\Big]\Big|\Big),$$

which becomes

$$\frac{n}{2^s}\varphi\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \le \frac{1}{2^s}\sum_{i=1}^{n}\varphi(x_i) - \sum_{j=1}^{s-1}\Big[\frac{1}{2^{s-(j+1)}}\sum_{i=1}^{2^{s-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big] -$$

$$\sum_{j=s}^{s}\Big[\frac{1}{2^{s-j}}\sum_{i=1}^{2^{s-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big).$$

This simplifies to

$$\varphi\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \le \frac{1}{n}\sum_{i=1}^{n}\varphi(x_i) - \frac{2^s}{n}\sum_{j=1}^{s}\Big[\frac{1}{2^{s-j}}\sum_{i=1}^{2^{s-j}}\varphi\Big(\frac{1}{2^j}\Big[T^+ - U^+\Big]\Big)\Big],$$

which completes the proof.

Example 6.1.10 *Consider the case where* $n = 10,\ \overline{X}_{10} = \frac{1}{10}\sum_{i=1}^{10} x_i$.

Then $4 \in \mathbb{N}$ *and* $8 = 2^{s-1} < n = 10 < 2^s = 16$.

Now

$$\varphi\Big(\frac{1}{10}\sum_{i=1}^{10} x_i\Big) = \varphi\Big[\frac{1}{16}(x_1 + \dots + x_{10} + 6\overline{X}_{10})\Big]$$

$$= \varphi\Big[\frac{1}{2}\Big(\frac{x_1 + \dots + x_8}{8}\Big) + \frac{1}{2}\Big(\frac{x_9 + x_{10} + 6\overline{X}_{10}}{8}\Big)\Big]$$

$$\le \frac{1}{2}\varphi\Big(\frac{1}{8}\sum_{i=1}^{8} x_i\Big) + \frac{1}{2}\varphi\Big(\frac{x_9 + x_{10} + 6\overline{X}_{10}}{8}\Big) - \varphi\Big(\frac{1}{16}\Big[\sum_{i=1}^{8} x_i - \Big(\sum_{i=9}^{10} x_i + 6\overline{X}_{10}\Big)\Big]\Big)$$

by inequality (3.6).

From Proposition 6.1.8. we have that

$$
\begin{aligned}
\varphi\Big(\frac{1}{8}\sum_{i=1}^{8} x_i\Big) \;\; &\leq \;\; \frac{1}{8}\sum_{i=1}^{8}\varphi(x_i) \\
&- \;\; \frac{1}{4}\Big[\varphi\Big(\frac{1}{2}\Big[x_1 - x_2\Big]\Big) + \varphi\Big(\frac{1}{2}\Big[x_3 - x_4\Big]\Big) + \varphi\Big(\frac{1}{2}\Big[x_5 - x_6\Big]\Big) + \varphi\Big(\frac{1}{2}\Big[x_7 - x_8\Big]\Big)\Big] \\
&- \;\; \frac{1}{2}\Big[\varphi\Big(\frac{1}{4}\Big[(x_1 + x_2) - (x_3 + x_4)\Big]\Big) + \varphi\Big(\frac{1}{4}\Big[(x_5 + x_6) - (x_7 + x_8)\Big]\Big)\Big] \\
&- \;\; \varphi\Big(\frac{1}{8}\Big[\sum_{i=1}^{4} x_i - \sum_{i=5}^{8} x_i\Big]\Big),
\end{aligned}
$$

which is precisely

$$
\varphi\Big(\frac{1}{8}\sum_{i=1}^{8} x_i\Big) \leq \frac{1}{8}\sum_{i=1}^{8}\varphi(x_i) - \sum_{j=1}^{3}\Big[\frac{1}{2^{3-j}}\sum_{i=1}^{2^{3-j}}\varphi\Big(\frac{1}{2^j}[T - U]\Big)\Big].
$$

From Proposition 6.1.9. we have

$$
\varphi\Big(\frac{x_9 + x_{10} + 6\overline{X}_{10}}{8}\Big) \leq \frac{1}{8}\sum_{i=9}^{10}\varphi(x_i) + \frac{6}{8}\varphi(\overline{X}_{10}) - \frac{1}{4}\Big[\varphi\Big(\frac{1}{2}\Big[x_9 - x_{10}\Big]\Big)\Big] -
$$

$$
\frac{1}{2}\varphi\Big(\frac{1}{4}\Big[(x_9 + x_{10}) - 2\overline{X}_{10}\Big]\Big) - \varphi\Big(\frac{1}{8}\Big[(x_9 + x_{10} + 2\overline{X}_{10}) - 4\overline{X}_{10}\Big]\Big),
$$

which is precisely

$$
\varphi\Big(\frac{x_9 + x_{10} + 6\overline{X}_{10}}{8}\Big) \leq \frac{1}{8}\sum_{i=9}^{10}\varphi(x_i) + \frac{6}{8}\varphi(\overline{X}_{10}) - \sum_{j=1}^{3}\Big[\frac{1}{2^{3-j}}\sum_{i=1}^{2^{3-j}}\varphi\Big(\frac{1}{2^j}\Big[T_{2^3} - U_{2^3}\Big]\Big)\Big].
$$

So

$$
\varphi\Big(\frac{1}{10}\sum_{i=1}^{10} x_i\Big) \leq \frac{1}{16}\sum_{i=1}^{10}\varphi(x_i) + \frac{6}{16}\varphi(\overline{X}_{10}) -
$$

$$
\frac{1}{8}\Big[\varphi\Big(\frac{1}{2}\Big[x_1 - x_2\Big]\Big) + \varphi\Big(\frac{1}{2}\Big[x_3 - x_4\Big]\Big) + \varphi\Big(\frac{1}{2}\Big[x_5 - x_6\Big]\Big) + \varphi\Big(\frac{1}{2}\Big[x_7 - x_8\Big]\Big) + \varphi\Big(\frac{1}{2}\Big[x_9 - x_{10}\Big]\Big)\Big] -
$$

$$
\frac{1}{4}\Big[\varphi\Big(\frac{1}{4}\Big[(x_1 + x_2) - (x_3 + x_4)\Big]\Big) + \varphi\Big(\frac{1}{4}\Big[(x_5 + x_6) - (x_7 + x_8)\Big]\Big) + \varphi\Big(\frac{1}{4}\Big[(x_9 + x_{10}) - 2\overline{X}_{10}\Big]\Big)\Big] -
$$

$$
\frac{1}{2}\Big[\varphi\Big(\frac{1}{8}\Big[\sum_{i=1}^{4} x_i - \sum_{i=5}^{8} x_i\Big]\Big) + \varphi\Big(\frac{1}{8}\Big[(x_9 + x_{10} + 2\overline{X}_{10}) - 4\overline{X}_{10}\Big]\Big)\Big] -
$$

$$
\varphi\Big(\frac{1}{16}\Big[\sum_{i=1}^{8} x_i - (x_9 + x_{10} + 6\overline{X}_{10})\Big]\Big),
$$

which simplifies to

$$\varphi\Big(\frac{1}{10}\sum_{i=1}^{10}x_i\Big) \le \frac{1}{10}\sum_{i=1}^{10}\varphi(x_i) -$$

$$\frac{1}{5}\Big[\varphi\Big(\frac{1}{2}\Big[x_1-x_2\Big]\Big)+\varphi\Big(\frac{1}{2}\Big[x_3-x_4\Big]\Big)+\varphi\Big(\frac{1}{2}\Big[x_5-x_6\Big]\Big)+\varphi\Big(\frac{1}{2}\Big[x_7-x_8\Big]\Big)+\varphi\Big(\frac{1}{2}\Big[x_9-x_{10}\Big]\Big)\Big]-$$

$$\frac{2}{5}\Big[\varphi\Big(\frac{1}{4}\Big(x_1+x_2)-(x_3+x_4)\Big]\Big)+\varphi\Big(\frac{1}{4}\Big[(x_5+x_6)-(x_7+x_8)\Big]\Big)+\varphi\Big(\frac{1}{4}\Big[(x_9+x_{10})-2\overline{X}_{10}\Big]\Big)\Big]-$$

$$\frac{4}{5}\Big[\varphi\Big(\frac{1}{8}\Big[\sum_{i=1}^{4}x_i-\sum_{i=5}^{8}x_i\Big]\Big)+\varphi\Big(\frac{1}{8}\Big[(x_9+x_{10}+2\overline{X}_{10})-4\overline{X}_{10}\Big]\Big)\Big]-$$

$$\frac{16}{10}\varphi\Big(\frac{1}{16}\Big[\sum_{i=1}^{8}x_i-(x_9+x_{10}+6\overline{X}_{10})\Big]\Big),$$

which is precisely

$$\varphi\Big(\frac{1}{10}\sum_{i=1}^{10}x_i\Big) \le \frac{1}{10}\sum_{i=1}^{10}\varphi(x_i) - \frac{2^4}{10}\sum_{j=1}^{4}\Big[\frac{1}{2^{4-j}}\sum_{i=1}^{2^{4-j}}\varphi\Big(\frac{1}{2^j}\Big[T-U\Big]\Big)\Big].$$

6.2 A lower bound on Young's inequality

In this next section we obtain a lower bound for the product of two real numbers, hence a lower bound for Young's inequality and finally we obtain a refinement of Minkowski's inequalities using particular superquadratic functions.

We now state example 3 as a proposition and give the proof.

Proposition 6.2.1 *For $x \ge y \ge 2$, the function $\phi : [2,\infty) \to [0,\infty)$, given by $\phi(t) = e^t$, satisfies inequality (3.5).*

Proof

Let $f(x,y) = e^{x+y} + e^{x-y} - 2e^x - 2e^y$, $\quad x \ge y \ge 2.$

For $x = y = 2$,

$$f(2,2) = e^4 + 1 - 4e^2 = e^2[e^2 - 4] + 1,$$

hence $f(2,2) \ge 0$, since $2 < e < 3$.

Let f_1 and f_2 be the partial derivatives of $f(x,y)$ with respect to x and y respectively.

So

$$
\begin{aligned}
f_1(x,y) &= e^x e^y + e^x e^{-y} - 2e^x \\
&= e^x[e^y + e^{-y} - 2] = \\
&\quad 2e^x[\cosh y - 1],
\end{aligned}
$$

but $\cosh y > 1$ for $y \geq 2$, thus $f_1(x,y)$ is increasing for $x \geq y \geq 2$.

$$
\begin{aligned}
f_2(x,y) &= e^x e^y - e^x e^{-y} - 2e^y \\
&= e^x[e^y - e^{-y}] - 2e^y = 2[e^x \\
&\quad \sinh y - e^y].
\end{aligned}
$$

For $y \geq 2$, $\sinh 2 > 3$ and $e^x \geq e^y$, so $e^x \sinh y - e^y > 0$.

So $f_2(x,y)$ is also increasing for $x \geq y \geq 2$.
Thus the function $f(x,y) = e^{x+y} + e^{x-y} - 2e^x - 2e^y > 0$, $x \geq y \geq 2$.
Hence $\phi(t) = e^t$ is superquadratic for $x \geq y \geq 2$.

Since the exponential function is superquadratic on $[2,\infty)$, we can obtain a lower bound on the product of two real numbers and hence the Young's inequality as follows:

Proposition 6.2.2 *For* $x \geq y \geq 8$, *the product* xy *is bounded below by* $2(x + y)$.

Proof Let $x \geq y \geq 8$,

then

$$
\begin{aligned}
xy &= e\ln(xy) \\
&= e(\ln x + \ln y).
\end{aligned}
$$

From proposition 6.2.1, we have that the exponential function is superquadratic on $\ln x \geq \ln y \geq 2$, that is

$$
\begin{aligned}
e^{(\ln x + \ln y)} &\geq 2[e^{\ln x} + e^{\ln y}] - e^{(\ln x - \ln y)} \\
xy &\geq 2[x+y] - e^{\ln\left(\frac{x}{y}\right)} \\
&= 2[x+y] - \frac{x}{y},
\end{aligned}
$$

thus

$$2[x+y] - \frac{x}{y} \leq xy,$$

hence

$2[x + y] \leq xy$, since $\frac{x}{y} \geq 1$ for $x \geq y \geq 8$.

6.3 Refined Minkowski's inequality

We obtain a refinement of (4.8) through Definition 20.0.25.

Proposition 6.3.1 *Assume that $x_1,...,x_n$ and $y_1,...,y_n$ are positive real numbers, and let* $p \geq 2, \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} = \overline{X}$ *and* $\left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}} = \overline{Y}$. *Then*

$$
\begin{aligned}
\left(\sum_{i=1}^{n}\left(x_i+y_i\right)^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}} \\
&- (\overline{X}+\overline{Y})\left[\frac{1}{p}(A+B) - \frac{1-p}{2p^2}(A+B)^2 + ...\right]. \qquad (6.5)
\end{aligned}
$$

where

$$A = \sum_{i=1}^{n}\left[\sum_{j=1}^{2}\frac{1}{2^{n-j}}\left(\frac{1}{2^j}\left[\frac{x_i}{\overline{X}} - \frac{y_i}{\overline{Y}}\right]\right)^p\right]$$

and

$$B = \sum_{i=1}^{n}\left[\sum_{j=3}^{n}\frac{1}{2^{n-j}}\left(\frac{\epsilon_{n+1-i}}{2^j}\left[\frac{x_i}{\overline{X}} - \frac{y_i}{\overline{Y}}\right]\right)^p\right]$$

.

Proof

We set $\frac{\overline{X}}{\overline{X}+\overline{Y}} = \frac{\alpha}{2^n}$ and $\frac{\overline{Y}}{\overline{X}+\overline{Y}} = \frac{\beta}{2^n}$, where $\alpha + \beta = 2^n, \forall n \in \mathbb{N}$. Since the power function is superquadratic from (6.2) with $a = \frac{x_i}{\overline{X}}$ and $b = \frac{y_i}{\overline{Y}}$, we have

$$\begin{aligned}\left[\frac{x_i+y_i}{\overline{X}+\overline{Y}}\right]^p &= \left[\frac{x_i+y_i}{2^n}\right]^p\\ &= \left[\frac{\alpha}{2^n}\Big(\frac{x_i}{\overline{X}}\Big)+\frac{\beta}{2^n}\Big(\frac{y_i}{\overline{Y}}\Big)\right]^p\\ &\le \frac{\alpha}{2^n}\Big(\frac{x_i}{\overline{X}}\Big)^p+\frac{\beta}{2^n}\Big(\frac{y_i}{\overline{Y}}\Big)^p-\sum_{j=1}^{2}\frac{1}{2^{n-j}}\Big(\frac{1}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p\\ &- \sum_{j=3}^{n}\frac{1}{2^{n-j}}\Big(\frac{\epsilon_{n+1-j}}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p,\end{aligned}$$

Summing the above inequality over *i* running from 1 to *n*, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\Big(\frac{x_i+y_i}{\overline{X}+\overline{Y}}\Big)^p &\le 1-\sum_{i=1}^{n}\Big[\sum_{j=1}^{2}\frac{1}{2^{n-j}}\Big(\frac{1}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p\Big]\\ &- \sum_{i=1}^{n}\Big[\sum_{j=3}^{n}\frac{1}{2^{n-j}}\Big(\frac{\epsilon_{n+1-j}}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p\Big].\end{aligned}$$

Which becomes

$$\begin{aligned}\sum_{i=1}^{n}\Big(x_i+y_i\Big)^p &\le (\overline{X}+\overline{Y})^p-(\overline{X}+\overline{Y})^p\sum_{i=1}^{n}\Big[\sum_{j=1}^{2}\frac{1}{2^{n-j}}\Big(\frac{1}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p\Big]\\ &- (\overline{X}+\overline{Y})^p\sum_{i=1}^{n}\Big[\sum_{j=3}^{n}\frac{1}{2^{n-j}}\Big(\frac{\epsilon_{n+1-j}}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p\Big].\end{aligned}$$

Let $A=\sum_{i=1}^{n}\Big[\sum_{j=1}^{2}\frac{1}{2^{n-j}}\Big(\frac{1}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p\Big]$ and
$B=\sum_{i=1}^{n}\Big[\sum_{j=3}^{n}\frac{1}{2^{n-j}}\Big(\frac{\epsilon_{n+1-i}}{2^j}\Big[\frac{x_i}{\overline{X}}-\frac{y_i}{\overline{Y}}\Big]\Big)^p\Big]$, then

$$\sum_{i=1}^{n}\Big(x_i+y_i\Big)^p\le(\overline{X}+\overline{Y})^p\Big[1-(A+B)\Big],$$

which becomes

$$\begin{aligned}\Big(\sum_{i=1}^{n}\Big(x_i+y_i\Big)^p\Big)^{\frac{1}{p}} &\le (\overline{X}+\overline{Y})\Big[1-(A+B)\Big]^{\frac{1}{p}}\\ &= (\overline{X}+\overline{Y})\Big[1-\frac{1}{p}(A+B)+\frac{1-p}{2p^2}(A+B)^2+\ldots\Big].\end{aligned}$$

Hence

$$\begin{aligned}\Big(\sum_{i=1}^{n}\Big(x_i+y_i\Big)^p\Big)^{\frac{1}{p}} &\le \Big(\sum_{i=1}^{n}x_i^p\Big)^{\frac{1}{p}}+\Big(\sum_{i=1}^{n}y_i^p\Big)^{\frac{1}{p}}\\ &- (\overline{X}+\overline{Y})\Big[\frac{1}{p}(A+B)-\frac{1-p}{2p^2}(A+B)^2+\ldots\Big].\end{aligned}$$

Proposition 6.3.2 *For* $x\ge 4$ *and* $y\le 0$ *such that* $2-x\le y\le 0$, *the exponential* *function satisfy the inequality (3.6).*

Proof

Let

$$f(x,y) = e^{\left(\frac{x+y}{2}\right)} + e^{\left(\frac{x-y}{2}\right)} - \frac{1}{2}(e^{\ln x} + e^{\ln y})$$
$$= e^{\left(\frac{x}{2}\right)}\left[2\cosh\left(\frac{y}{2}\right)\right] - \frac{1}{2}(e^{x} + e^{y}.)$$

For $x = 4$, we have that $-1 \leq \frac{y}{2} \leq 0$,

thus

$$f(4,y) = e^{2}\left[2\cosh\left(\frac{y}{2}\right)\right] - \frac{1}{2}[e^{4} + e^{y}],$$

but

$$2 \leq \cosh\left(\frac{y}{2}\right) \leq 2\cosh(-1),$$

so we have that,

$$e^{2}\left[2\cosh(-1) - \frac{1}{2}e^{2}\right] < 0,$$

since $0 < e^{y} \leq 1$.

Let f_1 and f_2 be the partial derivatives of $f(x,y)$ with respect to x and y respectively.

So

$$f_1(x,y) = \frac{1}{2}e^{\left(\frac{x+y}{2}\right)} + \frac{1}{2}e^{\left(\frac{x-y}{2}\right)} - \frac{1}{2}e^{x}$$
$$= \frac{1}{2}\left[2e^{\left(\frac{x}{2}\right)}\cosh\left(\frac{y}{2}\right) - e^{x}\right]$$
$$= \frac{1}{2}e^{\left(\frac{x}{2}\right)}\left[2\cosh\left(\frac{y}{2}\right) - e^{\left(\frac{x}{2}\right)}\right].$$

Since $1 - \frac{x}{2} \leq \frac{y}{2} \leq 0$, we have that

$$2 \leq 2\cosh\left(\frac{y}{2}\right) \leq 2\cosh\left(1 - \frac{x}{2}\right).$$

So

$$2\cosh\left(1 - \frac{x}{2}\right) - e^{\left(\frac{x}{2}\right)} = e^{\left(1 - \frac{x}{2}\right)} + e^{\left(\frac{x}{2} - 1\right)} - e^{\left(\frac{x}{2}\right)}.$$

For $x = 4$, we have

$$e^{-1} + e^{1} - e^{2} < 0.$$

As $x \to \infty$, we have that $-e^{\left(1-\frac{x}{2}\right)} \to 0$,

thus

$$e^{\left(\frac{x}{2}-1\right)} - e^{\frac{x}{2}} < 0,$$

since $\frac{x}{2} > \frac{x}{2} - 2$,

hence $f_1(x,y) < 0$.

$$\begin{aligned} f_2(x,y) &= \frac{1}{2}e^{\left(\frac{x+y}{2}\right)} - \frac{1}{2}e^{\left(\frac{x-y}{2}\right)} - \frac{1}{2}e^{y} \\ &= \frac{1}{2}\left[2e^{\left(\frac{x}{2}\right)}\sinh\left(\frac{y}{2}\right) - e^{y}\right]. \end{aligned}$$

Now for $1 - \frac{x}{2} \le \frac{y}{2} \le 0$, we have that $\sinh\left(\frac{y}{2}\right) < 0$, hence $f_2(x,y) < 0$.

Thus the function $f(x,y) = e^{\left(\frac{x+y}{2}\right)} + e^{\left(\frac{x-y}{2}\right)} - \frac{1}{2}(e^{\ln x} + e^{\ln y}) < 0$, for $x \ge 4$ and $2 - x \le y \le 0$.

Hence $\phi(t) = e^{(t)}$ satisfy inequality (3.6).

6.4 Application of refined Jensen's inequality to subharmonic functions

Proposition 6.4.1 *[Asare 2009, Asare and Prempeh 2015b]*

Let $0 \le a < b \le \infty$ *and let* $u : U \to (a,b)$ *be a subharmonic function on an open set* $U \in \mathbb{C}$. *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be a non-negative superquadratic function. Then* $\phi \circ u$ *is subharmonic on U.*

Proof

By Lemma 3.0.25, ϕ is an increasing function on $[0,\infty)$.

Let u be subharmonic on U, then

$$\limsup_{y \to x} u(y) \le u(x).$$

So

$$\limsup_{y\to x} \phi(u(y)) = \phi(\limsup_{y\to x} u(y)) \leq \phi(u(x)),$$

since ϕ is continuous and increasing on $[0,\infty)$.

Thus ϕ o u is upper semicontinuous, being the composition of an increasing continuous ϕ with upper semicontinuous u.

We next show that ϕ o u satisfies the submean inequality.

(See Definition 5.3.1 (2))

Now

$(\varphi \;\; o \;\; u)(z) \leq \varphi(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta)$, since u is subharmonic and ϕ is increasing.

So by the refined Jensen's inequality (4.2) we obtain,

$$(\varphi \;\; o \;\; u)(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(u(z+re^{i\theta}))d\theta(z) - \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(\left|u(z+re^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta\right|\right)d\theta(z). \tag{6.8}$$

Since $\phi \geq 0$, then

$$(\varphi \;\; o \;\; u)(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(u(z+re^{i\theta}))d\theta(z),$$

which shows that ϕ o u satisfies the local submean inequality and is thus subharmonic.

The inequality (6.8) is a refinement of the submean inequality.

We give a neccessary and sufficient conditions for a function to be superquadratic.

Definition 6.4.2 A function $k : [0,\infty) \to \mathrm{R}$ is *superadditive* provided $k(x+y) \geq k(x) + k(y)$.

Lemma 6.4.3 *[Asare and Prempeh 2015b]*

Suppose ϕ is continuously differentiable and $\phi(0) = \phi^0(0) = 0$. Then ϕ is superquadratic if and only if $\frac{\varphi'}{Id}$ is non-decreasing or ϕ^0 is superadditive on $(0,\infty)$.

Proof

Suppose ϕ is superquadratic, then from Lemma 3.0.24, the constant P_x in the Definition 3.0.23. is necessarily $\phi'(x)$.

When we set $y = 0$ in Definition 3.0.23. we obtain,

$$\varphi(0) - \varphi(x) - \varphi(x) \geq \varphi'(x)(0 - x),$$

since $x \geq 0$ and so,

$$x\varphi'(x) \geq 2\varphi(x).$$

Dividing through the above expression by $2x^2$, we have

$$\frac{\varphi'(x)}{2x} \geq \frac{\varphi(x)}{x^2}$$

and

$$\frac{\varphi'(x)}{x} \geq \frac{\varphi'(x)}{2x} \geq \frac{\varphi(x)}{x^2}, \forall x \in (0, \infty).$$

But from Lemma 3.0.28. we have that $\frac{\varphi(x)}{x^2}$ is non-decreasing on $(0,\infty)$, hence $\frac{\varphi'(x)}{x}$ is non-decreasing on $(0,\infty)$.

By assumption ϕ is continuously differentiable and superquadratic so for $x \leq y$, we have, by (3.2)

$$\varphi(y) - \varphi(x) - (y - x)\varphi'(x) - \varphi(y - x) \geq 0,$$

the left hand side of which is

$$\int_x^y [\varphi'(t) - \varphi'(x) - \varphi'(t - x)]dt,$$

since $\phi(0) = 0$.

Let $h(t) = \phi'(t) - \phi'(x) - \phi'(t - x)$ and we define a function g given by

$$g(y) = \int_x^y h(t)dt, 0 \leq x \leq y < \infty.$$

Thus $g(y) \geq 0$ and by definition the function h is continuous $\forall x,t \in [0,\infty)$, so by the Fundamental Theorem of Calculus we have that,

$$g'(y) = \phi'(y) - \phi'(x) - \phi'(y - x) \geq 0.$$

Thus for $x \le y$, we have $\phi^0(y) - \phi^0(x) \ge \phi^0(y-x)$.

Interchanging the roles of x and y in the above discussion we have that, $\phi^0(x) - \phi^0(y) \ge \phi^0(x-y)$, for $y \le x$.

Hence $\forall x,y \in (0,\infty)$, $\phi^0(x+y) \ge \phi^0(x) + \phi^0(y)$, So ϕ^0 is superadditive.

The converse is reported in [Abramovich et al 2004a, Asare 2009]

If ϕ^0 is superadditive, then for $x \le y$ we have,

$$0 \le \int_x^y [\phi^0(t) - \phi^0(x) - \phi^0(t-x)]dt$$

$$= \phi(y) - \phi(x) - (y-x)\phi^0(x) - \phi(y-x).$$ For $y \le x$,

we have that

$$\begin{aligned} 0 &\le \int_y^x [\varphi'(x) - \varphi'(x-t) - \varphi'(t)]dt \\ &= \varphi(y) - \varphi(x) + (x-y)\varphi'(x) - \varphi(x-y). \end{aligned}$$

Thus $\forall x,y \ge 0$ we have that,

$$\phi(y) \ge \phi(x) + \phi^0(x)(y-x) + \phi(|y-x|).$$

Setting $\phi^0(x) = P_x$, we have that ϕ is superquadratic.

If ϕ is continuously differentiable and $\frac{\varphi'(x)}{x}$ is non-decreasing then $\forall x, y \ge 0,\ \frac{\varphi'(x+y)}{x+y} \ge \frac{\varphi'(x)}{x}$.

Now

$$\begin{aligned} \varphi'(x+y) &= x\frac{\varphi'(x+y)}{x+y} + y\frac{\varphi'(x+y)}{x+y} \\ &\ge \varphi'(x) + \varphi'(y). \end{aligned}$$

Hence ϕ^0 is superadditive on $(0,\infty)$, and so the argument reduces to the first case.

Proposition 6.4.4 *[Asare and Prempeh 2015b]*

Let $\vartheta : [0,\infty) \to [0,\infty)$ *be an increasing convex function such that* $\vartheta(0) = 0$*, let* $\phi : [0,\infty) \to [0,\infty)$ *be continuously differentiable and* $\phi(0) = 0$ *and let* $u : U \to (a,b)$ *be subharmonic, where* $0 \le a \le b \le \infty$ *and* $U \subset \mathbb{C}$*. If* ϕ^0 *is superadditive or* $\frac{\varphi'}{Id}$ *is non-decreasing, then* $(\vartheta \circ \phi \circ u)$ *is subharmonic.*

Proof

Since ϕ satisfies all the conditions in Lemma 6.4.3, then ϕ is superquadratic. From Proposition 6.4.1, $\phi \circ u$ is subharmonic and therefore $(\vartheta \circ \phi \circ u)$ is upper semicontinuous, since ϑ is an increasing convex function.

We have by the refined Jensen's inequality that

$$(\vartheta(\varphi(u)))(z)$$

$$\le \vartheta(\frac{1}{2\pi}\Big[\int_0^{2\pi} \varphi(u(z+re^{i\theta})) - \int_0^{2\pi} \varphi\Big(\Big|u(z+re^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta\Big|\Big)\Big]d\theta(z)),$$

and

$$\vartheta(\frac{1}{2\pi}\Big[\int_0^{2\pi} \varphi(u(z+re^{i\theta})) - \int_0^{2\pi} \varphi\Big(\Big|u(z+re^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta\Big|\Big)\Big]d\theta(z))$$

$$\le \vartheta(\frac{1}{2\pi}\int_0^{2\pi} \varphi(u(z+re^{i\theta}))d\theta(z)) - \vartheta(\frac{1}{2\pi}\int_0^{2\pi} \varphi\Big(\Big|u(z+re^{i\theta}) - \int_0^{2\pi} u(z+re^{i\theta})d\theta\Big|\Big)d\theta(z)),$$

since ϑ is superadditive.

From the Jensen's inequality we have that

$$\vartheta(\frac{1}{2\pi}\int_0^{2\pi} \varphi(u(z+re^{i\theta}))d\theta) \le \frac{1}{2\pi}\int_0^{2\pi} \vartheta[\varphi(u(z+re^{i\theta}))]d\theta,$$

since ϑ is convex.

So $(\vartheta(\phi(u)))(z)$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \vartheta[\varphi(u(z+re^{i\theta}))]d\theta(z) - \vartheta\Big(\frac{1}{2\pi}\int_0^{2\pi} \varphi\Big(\Big|u(z+re^{i\theta}) - \int_0^{2\pi} u(z+re^{i\theta})d\theta\Big|\Big)\Big)d\theta(z) \tag{6.9}$$

Now ϑ maps into $[0,\infty)$, that is, $\vartheta\Big(\frac{1}{2\pi}\int_0^{2\pi} \varphi\Big(\Big|u(z+re^{i\theta}) - \int_0^{2\pi} u(z+re^{i\theta})d\theta\Big|\Big)\Big) \ge 0$.

So

$$\vartheta(\varphi(u))(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \vartheta[\varphi(u(z+re^{i\theta}))]d\theta(z)$$

Thus $(\vartheta(\phi(u)))$ satisfies the local submean inequality, hence subharmonic.

The inequality (6.9) is a refinement of the local submean inequality.

Lemma 6.4.5 *[Asare and Prempeh 2015b]*

Every positive superquadratic function ϕ *defined on* $[0,\infty)$ *is superadditive.*

Proof

From lemma 3.0.24, we have that ϕ is convex and $\phi(0) = 0$, so for all $0 \leq \alpha \leq 1$, $\phi(\alpha x) \leq \alpha\phi(x), \forall x \in [0,\infty)$. Now for any $x,y \in [0,\infty)$

$$\begin{aligned}
\varphi(x)+\varphi(y) &= \varphi(\frac{x+y}{x+y}x)+\varphi(\frac{x+y}{x+y}y) \\
&= \varphi(\frac{x}{x+y}(x+y))+\varphi(\frac{y}{x+y}(x+y)) \\
&\leq \frac{x}{x+y}\varphi(x+y)+\frac{y}{x+y}\varphi(x+y) \\
&= \varphi(x+y),
\end{aligned}$$

hence for all $x,y \in [0,\infty)$, ϕ is superadditive.

6.5 Application of the generalized refined Jensen's inequality

Let $\phi(x) = |x^3|, -\infty < x < \infty$. Then ϕ satisfies Definition 3.0.29. [Abramovich 2009]

Proof

Let $\phi(x) = |x^3|$ on R.

For $x \geq 0$, $\phi(x) = x^3$. So for $x \geq 0$, $\phi(x)$ is a power function with $p = 3$, hence it follows that $\phi(x) = |x^3|$ is superquadratic on $[0,\infty)$, See page 34.

Now for $x < 0$, $\phi(x) = -x^3$, our aim is to show that:

$f(x,y) = -(x+y)^3 - (x-y)^3 + 2x^3 + 2y^3 \geq 0$, on $x \leq y \leq 0$.

Define $\hat{f}: [0,1] \to \mathbb{R}$ by

$$x^3 \hat{f}\left(\frac{y}{x}\right) = f(x,y).$$

Since $x \leq 0$, showing that $f(x,y) \geq 0$ is equivalent to proving $\hat{f}\left(\frac{y}{x}\right) \leq 0$.
Now set $t = \frac{y}{x}$, where $0 \leq t \leq 1$

$$\hat{f}(t) = -(1+t)^3 - (1-t)^3 + 2 + 2t^3.$$

So

$$\begin{aligned} \hat{f}'(t) &= -3(1+t)^2 + 3(1-t)^2 + 6t^2 \\ &= 6t(t-2). \end{aligned}$$

Thus $\hat{f}(t)$ is decreasing on $0 \leq t \leq 1$, since $\hat{f}'(t) \leq 0$ for $0 \leq t \leq 1$.

Since $\hat{f}(0) = 0$, we have that $\hat{f}(t) \leq 0, \forall t \in [0,1]$.
Hence $\phi(x) = -x^3$ is superquadratic on $x \leq 0$.

Thus $\phi(x) = |x^3|$ is superquadratic on $\mathbb{R}$.

Let $x_1,\ldots,x_n \in \mathbb{R}, n \in \mathbb{N} : n = 2^m$, for some $m \in \mathbb{N}$, then by Proposition 6.1.8 we have:

$$\left|\left(\frac{1}{2^m}\sum_{i=1}^{2^m} x_i\right)^3\right| \leq \frac{1}{2^m}\sum_{i=1}^{2^m} |x_i^3| - \sum_{j=1}^{m}\left[\frac{1}{2^{m-j}}\sum_{i=1}^{2^{m-j}}\left|\left(\frac{1}{2^j}(T^+ - U^+)\right)^3\right|\right].$$

For the case where $n \neq 2^m$, applying Proposition 6.1.9 gives:

$$\left|\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^3\right| \leq \frac{1}{n}\sum_{i=1}^{n} |x_i^3| - \frac{2^s}{n}\sum_{j=1}^{s}\left[\frac{1}{2^{s-j}}\sum_{i=1}^{2^{s-j}}\left|\left(\frac{1}{2^j}(T^+ - U^+)\right)^3\right|\right],$$

where T^+ and U^+ are as defined in the proof of Proposition 6.1.8.

Example 6.5.1 *For* $m = 2, x_1 = -2, x_2 = 5, x_3 = 8$ *and* $x_4 = 3$, *we obtain a refinement of the Jensen's inequality for* $\phi(x) = |x^3|$ *as follows:*

$$\left|\left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)^3\right| \leq \frac{1}{4}\sum_{i=1}^{4} |x_i^3| - \sum_{j=1}^{2}\frac{1}{2^{2-j}}\sum_{i=1}^{2^{2-j}}\left|\left(\frac{1}{2^j}(T^+ - U^+)\right)^3\right|$$

$$42.875 \leq 168 - 37.25 = 130.75.$$

The original Jensen's gives 168, whiles our refinement reduces the value to 130.75, thus our extra terms reduce the error by about 22%*.*

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