SOLUTION OF INVERSE EIGENVALUE PROBLEM OF CERTAIN SINGULAR HERMITIAN MATRICES

By

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Abstract

We investigate solutions to the Inverse Eigenvalue Problem (IEP) of certain singular Hermitian matrices. Based on a solvability lemma, we propose an algorithm to reconstruct such matrices from their eigenvalues. That is, we develop algorithms and prove that they solve $n \times n$, singular Hermitian matrices of $rank r$. In the case of $n \times n$ matrix, the number of independent matrix elements would reduce to the extent that there would be an isomorphism between the elements and the nonzero eigenvalues. We initiate a differential geometry and numerical analytic interpretation of the Inverse Eigenvalue problem for Hermitian matrices using fibre bundle with structure group $O(n)$. In particular, Newton type algorithm is developed to construct non singular symmetric matrices using certain singular symmetric matrices as the initial matrices for the iteration.

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Introduction

Background: Inverse problems are encountered in several branches of Mathematics, and they have important real-world applications in engineering and the sciences. The direct Eigenvalue Problem in the domain of matrix algebra, may be stated as follows: Given a $n \times n$ matrix A a complex number λ is called an eigenvalue of the matrix if there exists a vector $x \neq 0$ such that $A = \lambda x$. On the other hand, a vector $x \neq 0$ that satisfies $Ax = \lambda x$ for an admissible λ (eigenvalue) is called an eigenvector of A belonging to λ. Thus, when the matrix A is given, we can find an eigenvalue and then the associated eigenvectors. In other words, when the matrix A and λ are given, we can use the equation $Ax = \lambda x$ to find the associated eigenvector. It is clear, therefore, that the number λ becomes an eigenvalue of the square matrix A if and only if it is a root of the characteristic polynomial and satisfies the equation $det(A - \lambda I) = 0$.

From the view point of applications, given a system (represented by a matrix), we can determine its spectrum (represented by the set of eigenvalues of the matrix). However, it is not far fetched to appreciate the fact that many such real life problems also demand that the inverse problem be solved:i.e. given a particular spectrum (eigenvalues) what is the corresponding system (matrix)? ('Can one hear the shape of a drum?' Mac Kack, 1966.) A subset of the set of Hermitian matrices will be our main focus in this study since, being endowed with real eigenvalues, it is, naturally, one of the most important for applications and also among the simplest to deal with.

As mentioned above, the inverse eigenvalue problem (IEP) is concerned with the reconstruction of a physical system from prescribed spectral data. The spectral data may consist of the complete or only partial information of eigenvalues or eigenvectors. To make the problem tractable, the main objective of an inverse eigenvalue problem reduces to the construction of a physical system that maintains a certain specific structure as well as that of the given spectral property. IEP's arise in a remarkable variety of applications which would be discussed in Chapter 2. Even though the setup of an inverse eigenvalue problem seems relatively easy, the solution is non trivial. The tools usually employed to solve such a problem are quite sophisticated, including techniques from orthogonal polynomials, degree theory and optimization.

In dealing with the IEP, we shall be concerned with two main issues, namely, the theory of solvability and the practice of computability. In the theory of solvability, we shall determine a necessary and or sufficient conditions under which the IEP for singular Hermitian matrices has a solution. Related to the solvability is the issue of uniqueness of a solution. Our main concern associated with computability, on the other hand is to develop the procedure by which, knowing a priori that the given spectral data are feasible, we can construct a matrix in a numerically stable fashion. In fact, several authors including (Chu, 2005, Boley and Golub, 1987) have actively used different methods to solve IEP of non singular symmetric matrices (which will be discussed in Chapter 2) and our focus will rather be on IEP of some singular Hermitian matrices. Also most of the previous work on the IEP treated tridiagonal matrices. In this thesis, we concentrate on the IEP of certain entry-wise non vanishing singular Hermitian matrices, which to our knowledge has not yet been considered.

The problem we investigate in this thesis may be stated as follows: We seek a systematic way to solve the IEP in the special case of certain Singular Hermitian Matrices of $rankr$. We define a map between a space of eigenvalues and the space of the corresponding singular Hermitian matrices. Specifically, the consideration of singular matrices is in aid of the need to reduce the number of independent matrix elements from $\frac{n(n+1)}{2}$ to the extent that there is an invertible (isomorphism) map between such matrix elements and the non zero eigenvalues.

Our main objective of the study are as follows: We develop a method, in the context of consistency conditions for solving the direct eigenvalue problem for singular matrices in which eigenvectors are obtained first before their associated non zero eigenvalues. We formulate and solve, based on the above method, the IEP for full small-sized singular Hermitian matrices. That is, we propose and test an algorithm to reconstruct such matrices from their eigenvalues. Finally, we initiate a differential geometric and, hence, via Newton's Method, a numerical analytic interpretation of the problem using a fibre bundle with structural group $O(n)$ (group of orthogonal matrices).

This thesis is organized as follows: In Chapter 1, we give preliminary definitions and basic theorems that would be used in our work. We discuss some of the various forms of the inverse eigenvalue problems so far solved in chapter 2. We review the IEP for the quadratic pencil, the IEP for both the Jacobi and periodic Jacobi matrices and the PIEP. In Chapter 3, we discuss the theorems on the solvability of different types of IEPs and computability of the different methods that we discuss about the IEP. The main results are presented in Chapters 4 and 5. In Chapter 4 we discuss the inverse eigenvalue problem of singular Hermitian matrices. We develop algorithm that generates singular Hermitian matrices when their eigenvalues are given. We devote Chapter 5 for the numerical interpretation of the IEP using fibre bundles with structural group $O(n)$. Given the eigenvalues and a singular symmetric matrix as an initial matrix for direct iteration, we generate a non singular symmetric matrix.

CHAPTER 1

Preliminary Definitions and Basic Theorems

1.1 Introduction

We give the notations, definitions of terms and basic results that will appear throughout this thesis in this chapter. We thus present the basic theory of the eigenvalue problem that will be fundamental to our discussions and analyses.

Definition:(Matrix)

Complex matrices are members of $C^{m\times n}$. Thus a vector space over the field of complex numbers. Notation: We represent matrices by upper case letters A, B, C, subscripted lower case letters a_{ij} represent elements of a matrix in ith row and jth column. We also write $A = [a_{ij}]$, $i = 1, ..., m$ $j = 1, ..., n$ so that $A \in C^{m\times n}$ and $a_{ij} \in C$. Types of Matrices: See for example (Strang, 1980 and Kreyszig, 1999)

1. Column vector; $x \in C^{m\times 1}$
2. Row vector: $y \in C^{1\times m}$
3. Square matrix; $A \in C^{n\times n}$
4. Hermitian matrix: $A = \bar{A}^T$
5. Anti Hermitian matrix $A = -\bar{A}^T$

Operations on Matrices: As a vector space $C^{m\times n}$, the following operations hold naturally:

1. Addition; $A + B = [a_{ij} + b_{ij}]$.
2. Scalar multiplication; $kA = [ka_{ij}], k \in C$.

Remark:(Properties of Hermitian and Anti-Hermitian matrices)

See for example (Strang, 1980, Leon, 1990 Kreyszig, 1999, and Hardy, 2005)

1. The diagonal elements of a Hermitian matrix are real. Indeed, $a_{ij} = \bar{a}_{ji} \rightarrow a = \bar{a}_{ii}$, where $(a_{ii}) = \frac{1}{2}(a_{ii} - \bar{a}_{ii}) = 0$.

2. The diagonal elements of an anti-Hermitian are pure imaginary. Indeed, $a_{ij} = -\bar{a}_{ji} \Rightarrow a_{ii} = -\bar{a}_{ii} \Rightarrow Re(a_{ii}) = \frac{1}{2}(a_{ii} + \bar{a}_{ii}) = 0$.

3. It is also clear that any square matrix can be decomposed as the sum of Hermitian and anti-Hermitian matrices. Indeed, $A = \frac{1}{2}(A + \bar{A}^T) + \frac{1}{2}(A - \bar{A}^T)$ where $A - \bar{A}^T$ is Hermitian and $A - \bar{A}^T$ is anti-Hermitian.

4. $C^{m\times n}$ may constitute a ring provided multiplication (inner product) is defined as follows: $C = AB \Leftrightarrow D_{ij} = \Sigma_{i=1}^{p} a_{ik}b_{kj}$ where $A \in C^{m\times p}, B \in C^{p\times n}$ and $D \in C^{m\times n}$

Invertibility:

If for $A \in C^{n\times n}$ there exists $A^{-1} \in C^{n\times n}$ such that $AA^{-1} = A^{-1}A = I_n$ where $I = [\delta_{ij}]$ and

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

then A is said to be invertible and I is said to be a unit matrix. δ_{ij} is known as the Kronecker delta.

The Matrix Equation:

$Ax = H$ is known as a non homogeneous system of equations if $H \neq 0$ and is a homogeneous system of equations if $H = 0$.

Remark:

A is invertible if and only if $X = A^{-1}H$ is a unique solution. Thus if A is non invertible for (singular) then X is not unique. In particular, for a homogeneous system $X = 0$ is a unique solution if and only if A is invertible. Therefore $X = 0$ is non trivial if and only if A is singular.

Determinant:

The determinant function for an $n \times n$ square matrix is given by;

$$|A| = \Sigma_{\pi \in S_n} sgn\pi a_{1\pi 1} a_{2\pi 2} ....a_{n\pi n} \text{ where } sgn\{\pi\} = \begin{cases} 1, & \pi \text{ is even} \\ 0, & \text{otherwise} \\ -1, & \pi \text{ is odd} \end{cases} .$$

A is non singular if and only if $|A| \neq 0$

Eigenvalue Problem:

Consider $BX = 0.....\star$ where $X \neq 0$ is required. Then from the foregoing B is singular, $|B| = 0$. Now suppose $B = A - \lambda I$, then the problem $\star$ is said to be an eigenvalue problem, where admissible values of λ are referred to as eigenvalues and the corresponding solutions X_λ are called eigenvectors belonging to the eigenvalues.

1.2 Nonsingular and singular matrices

An $n \times n$ (square) matrix A is called invertible or nonsingular or non-degenerate, if there exists an $n \times n$ matrix B such that $AB = BA = I_n$ where I_n denotes the $n \times n$ identity matrix and the multiplication used is the ordinary matrix multiplication. If this is the case, then the matrix B is uniquely determined by A and is called the inverse of A, denoted by A^{-1}.

Theorem 1.2.1. *Inverses are unique. That is if A has inverses B and C, then $B = C$*

Proof. Let $AB = CA = I$. But $B = IB = (CA)B = C(AB) = CI = C$

Also if A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$. □

Theorem 1.2.2. *Let A be an $n \times n$ matrix with the property that the homogeneous system $AX = 0$ has only the trivial solution $X = 0$. Then A is non-singular. Conversely, if A is singular, then the homogeneous system $AX = 0$ has a non-trivial solution.*

Proof. : If A is $n \times n$ and the homogeneous system $AX = 0$ has only the trivial solution, then it follows that the reduced row-echelon form B of A cannot have zero rows and must therefore be I_n. Hence A is non-singular. □

The system of equation $Ax = 0$ has a trivial solution if $x = 0$ is the solution in that case A is non singular and A^{-1} exists. The system $Ax = 0$ has a nontrivial solution if A is singular and $x \neq 0$.

Non-square matrices ($m \times n$ matrices for $m \pm n$) do not have an inverse. However, in some cases such a matrix may have a left inverse or right inverse. If A is $m \times n$ and the rank is equal to n, then A has a left inverse:an $n \times m$ matrix B such that $BA = I$. If A has rank m, then it has a right inverse an $n \times m$ matrix B such that $AB = I$.

Definition: (Eigenvalues and Eigenvectors) Let A be any square matrix, real or complex. A number λ is an eigenvalue of A if the equation $Ax = \lambda x$ is true for some nonzero vector x, where λ is real or complex number. The vector x is an eigenvector associated with the eigenvalue λ. The eigenvector may also be complex.

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

We see that:

$Ax_1 = 0x_1, Ax_2 = -4x_2$, and $Ax_3 = 3x_3$ where

$$x_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, x_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \text{ and } x_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}.$$

Hence x_1 is an eigenvector of A associated to the eigenvalue 0 and x_2 is an eigenvector of A associated to the eigenvalue -4 while x_3 is an eigenvector of A associated to the eigenvalue 3. The eigenspace corresponding to one eigenvalue of a given matrix is generated by the set of all eigenvectors of a matrix with that eigenvalue.

1.2.1 Characteristic Polynomial

When a transformation is represented by a square matrix A, the eigenvalue equation can be arranged as $Ax - \lambda Ix = 0$. If there exists an inverse $(A - \lambda I)^{-1}$ then both sides can be left multiplied by the inverse to obtained the trivial solution $x = 0$. Thus we require there to be no inverse by assuming from linear algebra that the determinant equals zero. $det(A - \lambda I) = 0$. The determinant requirement is called the Characteristic Equation of A and the left-hand side is called the Characteristic Polynomial. When expanded, this gives a polynomial equation for λ. This definition $det(A - \lambda I) = 0$ of an eigenvalue does not directly involve the corresponding eigenvector. The degree of the polynomial is the order of the matrix. This implies that an $n \times n$ matrix has n eigenvalues, counting multiplicities.

1.3 Eigenvalue Problem for Hermitian Matrices

:

Theorem 1.3.1. *The eigenvalue of a Hermitian matrices are real.*

Proof. Let A be a square matrix, then we have $(A - \bar{A}^T)X = 0 \rightarrow \lambda - \bar{\lambda} = 0$ □

We state the following theorem without proof.

Theorem 1.3.2. *Eigenvalues belonging to distinct eigenvalues of Hermitian matrices are orthogonal.*

Definition:(Normal Matrices)

A matrix A is said to be normal if $A\bar{A}^T = \bar{A}^T A$. A matrix which has a complete orthonormal set of eigenvectors is normal.

Theorem 1.3.3. *A matrix A is normal if and only if A possesses a complete orthonormal set of eigenvectors.*

See Leon, (1994) for the proof.

1.4 Hermitian matrices

A Hermitian matrix (or self adjoint matrix) is a square matrix with complex entries which is equal to its own conjugate transpose-that is, the element in the ith row and jth column is equal to the complex conjugate of the element in the jth row and ith column, for all indexes i and j, $a_{i,j} = \bar{a_{ji}}$ or $A = \bar{A}^t$

Hermitian matrices can be considered as the complex extension of real symmetric matrices. A matrix that has only real entries is Hermitian if and only if it is symmetric matrix, i.e., if it is symmetric with respect to the main diagonal. Thus a real, symmetric matrix is simply a special case of a Hermitian matrix. Every Hermitian matrix is normal, and the finite-dimensional spectral theorem applies. This says that any Hermitian matrix can be diagonalized by a unitary matrix, and that the resulting diagonal matrix has only real entries. This means that all eigenvalues of a Hermitian matrix are real, and, moreover, eigenvectors with distinct eigenvalues are orthogonal. It can be easily proved that the sum of any two Hermitian matrices is Hermitian, and the inverse of an invertible Hermitian matrix is Hermitian as well. However, the product of two Hermitian matrices A and B will only be Hermitian if they commute, i.e., if $AB = BA$. Thus A is Hermitian if A^n is Hermitian for any integer n. The matrices A and B below are examples of complex and real Hermitian matrices.

$$A = \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 5 & -3 \\ -2i & -3 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 6 \\ 4 & 6 & 5 \end{pmatrix}$$

1.4.1 Eigendecomposition

Theorem: (Spectral Theorem)

The spectral theorem for matrices can be stated as follows: Let A be an $n \times n$ matrix. Let $q_1, ..., q_k$ be an eigenvector basis, i.e. an index set of k linearly independent eigenvectors, where k is the dimension of the space spanned by the eigenvectors of A. If $k = n$, then A can be written as

$$A = Q \wedge Q^{-1} \tag{1.1}$$

where Q is the square $n \times n$ matrix whose ith column is the basis eigenvector q_i of A and $\wedge$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e. $\wedge_{ii} = \lambda_i$.

Definition:

Let A and B be two $n \times n$ matrices. We say that A is similar to B if there exists an invertible matrix P such that $A = PBP^{-1}$.

Theorem 1.4.1. *If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.*

Proof. : See for example, (Larson and Falvo, 2010)

Because A and B are similar, there exists an invertible matrix P such that $B = P^{-1}AP$. By the properties of determinant, it follows that

$$\begin{aligned}
|\lambda I - B| &= |\lambda I - P^{-1}AP| \\
&= |P^{-1}\lambda IP - P^{-1}AP| && (1.2)\\
&= |P^{-1}(\lambda I - A)P| && (1.3)\\
&= |P^{-1}||\lambda I - A||P| && (1.4)\\
&= |P^{-1}||P||\lambda I - A| && (1.5)\\
&= |P^{-1}P||\lambda I - A| && (1.6)\\
&= |\lambda I - A| && (1.7)
\end{aligned}$$

□

Theorem 1.4.2. *An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.*

Proof. :See for example, (Larson *et al*, 2010)

First, assume A is diagonalizable. Then there exists an invertible matrix P such that $P^{-1}AP = D$ is diagonal. Letting the main entries of D be $\lambda_1, \lambda_2 ....., \lambda_n$ and the column vectors of P be $p_1, p_2, ..., p_n$ produces

$$PD = [p_1 \vdots p_2 \vdots \cdots \vdots p_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{1.8}$$

$$= [\lambda_1 p_1 \vdots \lambda_2 p_2 \vdots \cdots \vdots \lambda_n p_n] \tag{1.9}$$

Because $P^{-1}AP = D, AP = PD$, it implies

$$[Ap_1 \vdots Ap_2 \vdots \cdots \vdots Ap_n] = [\lambda_1 p_1 \vdots \lambda_2 p_2 \vdots \cdots \vdots \lambda_n p_n]$$

In other words, $Ap_i = \lambda_i p_i$ for each column vector $p_i, i = 1, 2, ..., n$. This means that the column vectors p_i of P are eigenvectors of A. Moreover, because P is invertible, its column vectors are linearly independent. So, A has n linearly independent eigenvectors. Conversely, assume A has n linearly independent eigenvectors $p_1, p_2, \cdots, p_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$. Let P be the matrix whose columns are these n eigenvectors. That is, $P = [p_1 \vdots p_2 \vdots \cdots \vdots p_n]$. Because each p_i is an eigenvector of A, we have $Ap_i = \lambda_i p_i$ and

$$AP = A[p_1 \vdots p_2 \vdots \cdots \vdots p_n] = [\lambda_1 p_1 \vdots \lambda_2 p_2 \vdots \cdots \vdots \lambda_n p_n]$$

. The right-hand matrix in this equation can be written as the matrix product below.

$$\begin{aligned} AP &= [p_1 \vdots p_2 \vdots \cdots \vdots p_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 \cdots & \lambda_n & \end{bmatrix} \\ &= PD \end{aligned} \tag{1.10}$$

Finally, because the vectors $p_1, p_2 \cdots, p_n$ are linearly independent, P is invertible and we can write the equation $AP = PD$ as $P^{-1}AP = D$, which means that A is diagonalizable. $\square$

Theorem 1.4.3. *For a symmetric matrix, the matrix X such that $X \Lambda X^{-1}$ is orthogonal.*

Proof. A is real and $A = A^T$,then its eigenvalue decomposition is $A = X \wedge X^T$, with $X^T X = I = XX^T$. On the other hand if A is complex and $A = \bar{A}^T$, then its eigenvalue decomposition is

$$A = X \wedge \bar{X}^T,$$

with $\wedge$ real and $\bar{X}^T X = I = X\bar{X}^T$. $\square$

1.5 Consistency Conditions for Systems of Linear Equations

We discuss the consistency conditions which can be used to solve systems of linear equations of the form $Ax = h$

Definition:

A system of linear equations is a collection of linear equations involving the same set of variables. A general system of m equations with n unknowns can be written as:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \quad \vdots \quad \vdots \quad \vdots &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

In the above equations, $x_1, x_2, \cdots, x_n$ are the unknown variables and $a_{11}, a_{12} \cdots, a_{mn}$ are the coefficients of the system and $b_1, b_2 \cdots, b_{mn}$ are constants. A solution of the above system of equations is an assignment of values to the variables $x_1, x_2, \cdots x_n$ such that each of the equations is satisfied. The set of all possible solutions is called the solution set.

A linear system may behave in any one of three possible ways:

1. The system has a single unique solution.
2. The system has infinitely many solutions.
3. The system has no solution.

1. Usually, a system with fewer equations than unknowns has infinitely many solutions or sometimes unique sparse solutions. Such a system is also known as an under-determined system.

2. A system with the same number of equations and unknowns usually has a single unique solution.

3. Lastly, a system with more equations than unknowns usually has no solution. Such a system is also known as an over-determined system.

1.5.1 Linear Equation-Conditions for Consistency

The system of equations:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

1. is consistent with unique solution, if $a_1/a_2 \neq b_1/b_2$. That is , the lines formed by these equations intersect and hence not parallel.

2. consistent with infinitely many solutions, if $a_1/a_2 = b_1/b_2 = c_1/c_2$. Thus, the lines represented by these equations coincide. For a two variable system of equations to be consistent, the lines formed by the equations have to meet at some point. But for three variable system of equations to be consistent, the planes formed by the equations must meet two conditions;

a. All three planes must be parallel.

b. Any two of the planes have to be parallel and the third must meet one of the planes at some point and other at another point. Three variable system of equations with no solution arise when the planes formed by the equations in the system neither meet at a point nor are parallel.

3. inconsistent, if $a_1/a_2 = b_1/b_2 \neq c_1/c_2$. Here the lines represented by the equations are parallel and non-coincident.

1.6 The Trace of a matrix

The trace of an $n \times n$ square matrix A is defined to be the sum of the elements of the main diagonal (the diagonal from the upper left to the lower right) of $A = [a_{ij}]$ i.e

$$tr(A) = a_{11} + a_{22}..... + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Proposition 1.6.1. *The trace of an $n \times n$ square matrix A is the sum of the eigenvalues.*

Proof. :

Let's consider the matrix

$$A = P^{-1}DP$$

where D is a diagonal matrix where the diagonal elements are the eigenvalues of A and P an invertible matrix whose column vectors are the eigenvectors of the matrix A. Then since taking the trace as multiplicative and $Tr(P^{-1}) = \frac{1}{trP}$ We have $Tr(A) = Tr(P^{-1}DP) = Tr(P^{-1}) \cdot Tr(D) \cdot Tr(P) = Tr(D)$.

Thus $Tr(D)$ is the sum of the eigenvalues by definition of D. □

We state the following proposition without proof;

Proposition 1.6.2. *: If a square matrix A has one row (column) as a scalar multiple of another row (column), then A is a singular matrix and $detA = 0$*

1.7 Lie Group

Definition: Manifold. (Worst, 2007)

An n-dimensional manifold is a space that is equipped with a set of local Cartesian coordinates so that points in a neighbourhood of any fixed point can be parameterized by n-tuples of real numbers. In other words a smooth manifold is a topological space such that in a neighbourhood of each point on it there are smooth coordinates. An example of a manifold is a surface in space.

Definition: (Conlon, 2001)

Let X be a set. A topology S for X is a collection of subsets of X satisfying:

1. $\emptyset$ and X are in S.
2. The intersection of two members of S is in S.
3. The union of any member of members of S is in S.

The set X with S is called a topological space. The members of $U \in S$ are called the open sets.

A Lie group is a nonempty subset G which satisfies the following conditions:

a. G is a group.

b. G is a smooth manifold. This means that G is a differentiable manifold.

c. G is a topological group. In particular, the group operation of multiplication, $\mu : G \times G \longrightarrow G$

$\mu : (gh) \longrightarrow gh$ and the inverse map

$i : G \longrightarrow G$

$i : g \longrightarrow g^{-1}$ are differentiable maps (smooth).

In particular, the 2×2 real invertible matrices form a group under multiplication, denoted by $GL_2(R)$:

$$GL_2(R) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | detA = ad - cb \neq 0\}$$

is a Lie group. Similarly, the rotation matrices form a group of $GL_2(R)$, denoted by $SO_2(R)$.

$$SO_2(R) = \{\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} | \theta \in R/2\pi Z\}$$

is also a Lie Group.

Other Lie groups are $GL(n, C)$,$SL(n, R)$ and $U(n)$. The orthogonal group $O_n(R)$, consisting of all $n \times n$ orthogonal matrices with real entries of dimension $\frac{n(n-1)}{2}$ is a Lie group.

When we consider linear Lie groups, the tangent space to G at the identity, $g = T_1G$, plays an important role. In particular, this vector space is equipped with a multiplication operation, the Lie bracket, that makes g into a Lie algebra. Definition: (Lie Algebra. Rossman, 2002)

A Lie algebra g is a vector space over a field F together with a binary operation

$[\cdot,\cdot]$

$[\cdot,\cdot] : g \times g \longrightarrow g$

which is called a Lie bracket satisfies the following axioms:

a. Bilinearity:

$[ax+by,z] = a[x,z]+b[y,z][z,ax+by] = a[z,x]+b[z,y]$ for all scalars $a,b \in F$ and all elements $x,y,z \in g$.

b. Alternating on g.

$[x,x] = 0$ for all $x \in g$.

The Jacobi identity:

$[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ for all $x,y,z \in g$.

We have to note that the bi-linearity and the alternating properties of g imply anti-commutativity, that is, $[x,y] = -[y,x]$ for all $x,y \in g$, while anti-commutativity only implies the alternating property if the field's characteristic is not 2.

Every Lie group has an associated Lie algebra whose underlying vector space is a tangent space of G at the identity element where G is a Lie group, which completely captures the local structure of the group. We can informally say that the elements of the Lie algebra are elements of the group that are "infinitesimally" close to the identity, and the Lie bracket is something to do with the commutator of two such infinitesimal elements. We give two examples of Lie bracket.

1. The Lie algebra of a vector space R^n is just R^n with the Lie bracket given by $[X,Y] = 0$. The Lie algebra is abelian. In general, the Lie bracket of a connected Lie group is always 0 if and only if the Lie group is abelian.

2. The Lie algebra of the general linear group $GL_n(R)$ of invertible matrices is the vector space $M_n(R)$ of square matrices with Lie bracket given by $[X,Y] = XY - YX$.

Proposition 1.7.1. *For any $X,Y \in g$ there is a unique element $[X,Y] \in g$ so that*

$$[X,Y]_l = X_l Y_l - Y_l X_l. \tag{1.11}$$

The operation $(X,Y) \longrightarrow [X,Y]$ makes g into a Lie algebra.

Proof. : See for example, (Rossmann, 2002)

As operators, $X_l = U$, and $Y_l = V$ commute with right translation a_r, hence so does $[U,V] = UV - VU$, and we may define $[X,Y]$ by the requirement that $[X,Y]_l = [X_l, Y_l]$. The Jacobi Identity is a formal consequence of $[U,V] = UV - VU$. □

Definitions:(Conlon, 2001, Lang, 1972 and Worst, 2007).

Tangent Space: The tangent space at a point on a surface is just the set of vectors tangent to the surface at that point.

Tangent Bundle: The tangent bundle is the union of all the tangent spaces.

The Lie algebra $so(n,R)$ of the Lie group $SO(n,R)$ consists of real skew symmetric $n \times n$ matrices, is the corresponding set of infinitesimal rotations. The geometric link between the Lie group and its corresponding Lie algebra is the fact that the Lie algebra is viewed as the tangent space to the Lie group at the identity. The exponential map is a map from the tangent space to the Lie group, i,e., $exp : so(n,R) \longrightarrow SO(n,R)$. The Lie algebra is considered as a linearization of the Lie group at the identity element and the exponential map provides "delinearization," that is it takes us back to the Lie group. This shows that there is an automorphism between the Lie algebra and the Lie group (Jean Ballier, 2011).

Given an $n \times n$ (real or complex) matrix $A = (a_{ij})$, we define the exponential e^A of A as the sum of the series

$$e^A = I_n + \Sigma_{p \geq 1} \frac{A^p}{p!} = \Sigma_{p \geq 0} \frac{A^p}{p!},$$

letting $A^0 = I_n$. This problem is well-defined as shown in the lemma below. (Jean Ballier, 2011)

Lemma 1.7.1. *:*

Let $A = (a_{ij})$ be a real or complex $n \times n$ matrix, and let $\mu = max\{|a_{ij}|, |1 \leq i; j \leq n\}$.

If $A^p = (a_{ij}^{(p)})$, then $|a_{ij}^{(p)}| \leq (n\mu)^p$ for all $i, j, 1 \leq i, j \leq n$. As a consequence, the n^2 series

$$\Sigma_{p\geq 0}\frac{a^{(p)}}{p!}$$

converges absolutely, and the matrix

$$e^A = \Sigma_{p\geq 0}\frac{A^p}{p!}$$

is a well-defined matrix.

Proof. :

We prove by induction on p. For $p = 0$, we have $A^0 = I_n, (n\mu)^0 = 1$, and the lemma is obvious. Assume that

$$|a_{ij}^{(p)}| \leq (n\mu)^p$$

for all $i, j, 1 \leq i, j \leq n$. Then we have

$$|a_{ij}^{(p+1)}| = | \Sigma_{k=1}^{n} a_{ik}^{(p)} a_{kj} | \leq \Sigma_{k=1}^{n} |a_{ik}^{(p)}||a_{kj}| \leq \mu\Sigma_{k=1}^{n}|a_{ik}^{(p)}| \leq n\mu(n\mu)^p = (n\mu)^{p+1},$$

for all $i, j, 1 \leq i, j \leq n$. For every pair (i, j) such that $1 \leq i, j \leq n$, since

$$|a_{ij}^{(p)}| \leq (n\mu)^p,$$

the series

$$\Sigma_{p\geq 0}\frac{|a_{ij}^{(p)}|}{p!}$$

is bounded by the convergent series

$$e^{n\mu} = \Sigma_{p\geq 0}\frac{(n\mu)^p}{p!},$$

and thus it is absolutely convergent. This shows that

$$e^A = \Sigma \frac{A^k}{k!}$$

is well defined. □

We illustrate the definition of the exponential of a matrix with an example of the exponential of the real skew symmetric matrix

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

We need to find an inductive formula expressing the powers A^n. Let us observe that

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^2 = -\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, letting

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have

$$A^{4n} = \theta^{4n} I_2, \tag{1.12}$$

$$A^{4n+1} = \theta^{4n+1} J, \tag{1.13}$$

$$A^{4n+2} = -\theta^{4n+2} I_2, \tag{1.14}$$

$$A^{4n+3} = -\theta^{4n+3} J, \tag{1.15}$$

and so

$$e^A = I_2 + \frac{\theta}{1!}J - \frac{\theta^2}{2!}I_2 - \frac{\theta^3}{3!}J + \frac{\theta^4}{4!}I_2 + \frac{\theta^5}{5!}J...$$

Rearranging the order of terms, we have

$$e^A = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...)I_2 + (\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...)J.$$

We recognize the power series for $cos\theta$ and $sin\theta$, and thus $e^A = cos\theta I_2 + sin\theta J$, that is

$$e^A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

where θ is any real number in $[0, 2\pi)$. Thus, e^A is a rotation matrix. This is a general fact. If A is a skew symmetric matrix, then e^A is an orthogonal matrix of determinant $+1$. The theorem below confirms that there is an onto mapping between the Lie algebra $(so(n))$ and the Lie group $(SO(n))$. The Lie algebra is a linearization of the Lie group while the exponential map takes it back to the Lie group (delinearization).

Theorem 1.7.1. *The exponential map*

$$exp : s0(n) \longrightarrow SO(n)$$

is well-defined and surjective.

Proof. : (See for example, Gallier, 2011)

First, we need to prove that if A is a skew symmetric matrix, then e^A is a rotation matrix. For this check that

$$(e^A)^t = e^{A^t}.$$

Then, since $A^t = -A$, we get

$$(e^A)^t = e^{A^t} = e^{-A},$$

and so

$$(e^A)^t e^A = e^{-A} e^A = e^{-A+A} = e^{0_n} = I_n,$$

and similarly,

$$e^A (e^A)^t = I_n,$$

showing that e^A is orthogonal. Also

$$det(e^A) = e^{tr(A)},$$

and since A is real skew symmetric, its diagonal entries are 0, i.e.,$tr(A) = 0$, and so $det(e^A) = \pm 1$ $\square$

We omit the proof for surjectivity.

When $n = 3$ (and A is skew symmetric), we can work out for an explicit formula for e^A. For any real skew symmetric matrix A, we have,

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

and letting $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

we have the following result known as $Rodrigues's formula(1840)$, (J. Gallier, 2011).

Lemma 1.7.2. *The exponential map* $exp : s0(3) \longrightarrow SO(3)$ *is given by*

$$e^A = cos\theta I_3 + \frac{sin\theta}{\theta} A + \frac{(1 - cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{sin\theta}{\theta}A + \frac{(1 - cos\theta)}{\theta^2}A^2$$

if $\theta \neq 0$, *with* $e^{0_3} = I_3$.

Proof. (sketch):

First, we prove that $A^2 = -\theta^2 I + B$,

$AB = BA = 0$.

From the above, we deduce that $A^3 = -\theta^2 A$,

and for any $k \geq 0$,

$A^{4k+1} = \theta^{4k}A$,

$A^{4k+2} = \theta^{4k}A^2$,

$A^{4k+3} = -\theta^{4k+2}A$,

$A^{4k+4} = -\theta^{4k+2}A^2$.

Finally, we prove the desired result by writing the powers for e^A and regrouping terms so that the power series cos and sin show up. □

We state the following lemma without proof. The lemma shows that A is a rotation matrix while the matrix B is a skew matrix obtained from A.

Lemma 1.7.3. *For every symmetric matrix B, the matrix* e^B *is symmetric positive definite. For every symmetric positive definite matrix A, there is a unique symmetric matrix B such that* $A = e^B$.

1.8 Fibre Bundle

A fibre bundle consists of data (E, B, π, F), where E, B,and F are topological spaces and $\pi : E \longrightarrow B$ is a continuous map such that every point of B has an open neighbourhood U such that there is a homeomorphism $\varphi : \pi^{-1}U \longrightarrow U \times F$,such that $\pi^{-1}U$,$U \times F$ and U commute. The space B is called the base space of the bundle, E the total space, and F the fibre. We assume that the base space B is connected (Gallier, 2011).

A smooth fibre bundle is a fibre bundle in the category of smooth manifolds. That is E, B, and F are required to be smooth manifolds and all the functions above are required to be smooth maps.

Definition (Tangent Bundle):

The tangent bundle of a differentiable manifold M is the disjoint union of the tangent spaces of M. These tangent spaces are given by the relation;

$$TM = \amalg_{x \in M} T_x M = \cup_{x \in M} x \times T_x M \tag{1.16}$$

where $T_x M$ denotes the tangent space to M at a point x. An element of TM can therefore be thought as a pair (x, v), where x is a point in M and v is a tangent to M at x. There is a natural projection $\Pi : TM \longrightarrow M$ defined by $\Pi(x, v) = x$.

This projection maps each tangent space $T_x M$ to the single point x. The tangent bundle to a manifold is the prototypical example of a vector bundle (a fibre bundle whose fibres are vector spaces). The main role of the tangent bundle is to provide a domain and range for the derivative of a smooth function. That is, if we consider the function; $f : M \longrightarrow N$, as a smooth function, where M and N are smooth manifolds, its derivative is a smooth function.

$Df : TM \longrightarrow TN$

These tangent spaces under consideration are lie groups which are differentiable manifolds with the property that the operations are compatible with the smooth structure. We have similarity transformation that transform symmetric matrices into other symmetric matrices(including diagonal ones) involves orthogonal matrices. These orthogonal matrices constitute lie group which forms a differentiable manifold. The tangent or derivative of which are anti-symmetric matrices which give rise to the associated lie algebra.

1.9 Parameterized Inverse Eigenvalue Problem

When we consider the unknown entries of a matrix to be constructed as parameters, we can say that an IEP is generally a parameter estimation problem. By "parameterized" IEP, we mean the ways by which these parameters regulate the problem or how the structural constraint is regulated by a set of parameters. Although every IEP can be regarded as parameter estimation, the emphasis in this work is on the meticulous way that these parameters regulate them.

A generic $PIEP$ can be described as: Given a family of matrices $A(c) \in M$ where M is a family of symmetric matrices with $c = [c_1, c_2, ..., c_m] \in F^m$ and the scalars $\lambda_1, \lambda_2, ..._{,n} \subset F$, we can find a parameter c such that,

$\theta(A(c)) = \lambda_1, \lambda_2, ..., \lambda_n.$

We have to note that the number m, of parameters in c, are different from n, where n determines the order of the matrix. In the PIEP, the family of matrices in the affine subspace is given by;

$$A(c) = A_0 + \sum_{i=1}^{n} c_1 A_i, \tag{1.17}$$

where $A_i \in S_n$, see for example, (Moody and Golub, 2001).

1.9.1 Symmetric Nonnegative Inverse Eigenvalue Problem

A real $n \times n$ matrix is said to be nonnegative if each of the entries is nonnegative. A related problem is the symmetric nonnegative inverse eigenvalue problem $(SNIEP)$. Given a list of some scalars λ_i, $i = 1, 2, .., n$, we can randomly construct an $n \times n$ symmetric matrices by these eigenvalues. For the solvability to Inverse real symmetric, see for example(G. Sun, 1986) An algorithm is developed for the construction of such matrices using eigenvalue-eigenvector decomposition as follows: Given a list of real eigenvalues $\lambda = \lambda_1, \lambda_1, ..., \lambda_1$, let M denote the set of all real symmetric matrices with real eigenvalues λ such that

$$M = A \in S_n | A = V \wedge V^T, \tag{1.18}$$

where V is an orthogonal $n \times n$ matrix and $\wedge$ a diagonal matrix. We choose the set of symmetric nonnegative matrices denoted by N. We can construct the matrix X where $X \in M \cap N$ by an iterative process. The matrix X is chosen such that it is the best approximant in N to A, (Orsi and Yang, 2006).

1.10 Power Method (Power Iteration)

The power iteration is an eigenvalue algorithm: given a square matrix A, the algorithm will produce a number λ (the eigenvalue) and a nonzero vector x (the eigenvector) such that $Ax = \lambda x$. The power iteration algorithm starts with a vector x_0, which may be an approximation to the dominant eigenvector or a random vector. The method is implemented by the iteration,

$$x_{k+1} = \frac{Ax_k}{|Ax_k|}.$$

At every iteration, the vector x_k is multiplied by the matrix A and normalized. Thus, the chosen initial vector x is repeatedly multiplied by the matrix A, iteratively calculating $Ax, A^2x, ..., A^nx$.

The matrix A and the vector x generate the corresponding eigenvalues. We have

$$\lambda = \frac{Ax \cdot x}{x \cdot x}$$

which is called the Rayleigh quotient. With the initial vector x_0, we compute Ax_0 and scale it such that $x_1 = \frac{Ax_0}{|Ax_0|}$. We compute $\frac{Ax_1 \cdot x_1}{x_1 \cdot x_1}$ which gives λ_1. The process continues until it converges to the dominant eigenvalue.

1.10.1 Lanczos Method

During the process of applying the power method for finding the eigenvalues of a square matrix A, in order to obtain the ultimate eigenvector $A^{n-1}v$, we also obtain series of vectors A^iv, $i = 0, 1, 2, .., n-2$ which were eventually discarded. The Lanczos iteration is therefore used to save this information and use the Gram-Schmidt process to reorthogonalize them into basis that span Krylov subspace corresponding to the matrix A (Golub and Van Loan, 1996).

Definition(Krylov subspace): The $order-r$ Krylov subspace is generated by an $n \times n$ matrix A and a vector b of dimension n is the linear subspace spanned by the images of b under r powers of A. (Starting from $A^0 = I$), that is $K_r(A, b) = span(b, Ab, A^2b, ..., A^{r-1}b)$.

CHAPTER 2

The Inverse Eigenvalue Problem

In this chapter, we discuss some of the various forms of the inverse eigenvalue problems so far solved. As explained in chapter one, the IEP is concerned with the reconstruction of a matrix from prescribed spectral data.The spectral data involved may consist of the complete or only partial information of the eigenvalues or eigenvectors. Our main objective of the IEP is to construct a matrix that maintains a certain specific structure as well as that given spectral property as stated earlier. Depending on application, inverse eigenvalue problems may be described in several different forms. Translated into mathematics, it is often necessary in order that the inverse eigenvalue problem be meaningful to restrict the construction to special classes of matrices, especially to those with specific structures. Our solution to the IEP therefore satisfies two constrains: the spectral constraint referring to the prescribed spectral data and the structural constraint referring to the desirable structure. We mention that the entries of the matrix to be constructed usually represent physical parameters to be determined. So an IEP can generally be regarded as a parameter estimation problem. Each inverse eigenvalue problem carries it own characteristics.

We therefore discuss four forms of the inverse problem.

2.1 Inverse Eigenvalue Problem for the Quadratic Pencil

The IEP of the quadratic pencil arises from the active vibration control (AVC) of a dynamic system. When the eigenvalues of the system are the same as the external frequencies, there is an oscillations known as resonance. This resonance makes the system unstable and may cause dangerous vibrations in vibrating structures such as aircrafts and spacecrafts, buildings and bridges which can cause destruction that may lead to loss of human lives and properties. For example, the fall of Tacoma bridge in the USA, the collapse of Brought-on bridge in England and nobbling of the Millennium bridge over River Thames in London are believed to have been caused by resonance, (Datta and Sokolov, 2009). Active vibration control therefore means that we have to apply a control force in such a way that a few eigenvalues of the quadratic pencil which correspond to the external frequencies are eliminated from the structure while the remaining ones and their corresponding eigenvectors are preserved.

While the active vibration control controls the vibration in the structure, the Finite element model updating $(FEMU)$ updates the finite element model by using the few eigenvalues and their eigenvectors so that the system always attains its features such as symmetrical, orthogonal and positive definiteness, (Datta and Sarkissian,1999 and Datta and Sokolov, 2009). We expect that at any point in time the eigenvectors are orthogonal. Despite the importance of the AVC and the $FEMU$, they have some limitations. When the problem is large and sparse, the entire spectrum cannot be computed. Apart from this, the coefficient matrices of the dynamic system are either diagonal or tridiagonal but in our case, we are using full singular symmetric matrices.

The quadratic IEP is associated with a quadratic matrix pencil arising in a feedback control of a matrix second-order system, (Dong, Lin and Chu, 2009). Consider the following dynamic system which is associated with the quadratic pencil;

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t) \tag{2.1}$$

where M,D and K are $n \times n$ diagonal or tridiagonal nonsingular symmetric matrices; M a positive definite matrix denoted by $M > 0$, and $\ddot{x}(t)$ and $\dot{x}(t)$ denote second and first order derivatives respectively of time dependent vector $x(t)$. In vibration analysis,we consider the matrices M,K and D as the mass, stiffness and the damping matrices respectively. When we separate variables, the system gives rise to the quadratic eigenvalue problem for the pencil below:

$$p(\lambda) = \lambda^2 M + \lambda D + K \tag{2.2}$$

The characteristic polynomial $detP(\lambda) = 0$ of the above pencil has scalars λ which are the $2n$ roots of the polynomial. The above system has $2n$ eigenvalues and $2n$ corresponding eigenvectors. When we consider equation (2.1) as a vibrating system, then the eigenvalues of $p(\lambda)$ are related to the natural frequencies of the homogeneous system:

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0 \tag{2.3}$$

and the eigenvectors are referred to as the modes of the vibration of the system.

In order to avoid oscillations of the vibratory system modeled by equation (2.1), we introduce a control force $f = Bu(t)$, where B is an $n \times m$ matrix and $u(t)$ is a time dependent $m \times 1$ vector to equation (2.1). We choose $u(t)$ to be;

$$u(t) = F^T\dot{x}(t) + G^T x(t)$$

where F and G are constant matrices. The system (2.1) after substitution becomes;

$$M\ddot{x}(t) + (D - BF^T)\dot{x}(t) + (K - BG^T)x(t) = 0 \tag{2.4}$$

Mathematically, we choose the matrices F and G such that the eigenvalues of the associated closed-loop pencil becomes;

$$p_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + (K - BG^T) \tag{2.5}$$

can be altered as required in order to combat the effects of resonances or ensure and improve the stability of the system.

We choose a real control matrix B of order $n \times m$ $(n < m)$, and real feedback matrices F and G of order $n \times m$ such that the spectrum of the closed-loop pencil (2.5) is $\{\mu_1, \mu_2, ..., \mu_p; \lambda_{p+1}, ..., \lambda_{2n}\}$ and the eigenvector set $\{y_1, ..., y_p; x_{p+1}, ..., x_{2n}\}$, where $x_{p+1}, ..., x_{2n}$ are the eigenvectors of (2.2) corresponding to $\lambda_{p+1}, ..., \lambda_{2n}$. Direct Partial Modal Approach, as the name implies, the system is direct because the solution is obtained directly from a second-order setting without resulting to a first-order reformulation. It is Partial model because only part of the spectral data is needed for the solution. The solutions are obtained using only those small number of eigenvalues and the corresponding eigenvectors that are to be assigned and directly in terms of the coefficient matrices M, D and K. An algorithm for the Direct Partial Modal Approach which is based on the Single-Input case or the Multi-Input for the quadratic eigenvalue problem, (Datta and Sarkissian, 1999) is given here. We present the algorithm for the Single-Input case as follows; The inputs are,

1. The $n \times n$ matrices M,K and D; $M = M^T > 0$, $D = D_T$ and $K = K^T$
2. The $n \times 1$ control (input vector b)
3. The set $\{\mu_1, ..., \mu_p\}$, closed under complex conjugation.
4. The self-conjugate subset $\{\lambda_1, ..., \lambda_p\}$ of the open-loop spectrum

$$\{\lambda_1, ..., \lambda_p; \lambda_{p+1}, ..., \lambda_{2n}\}$$

   and the associated eigenvector set $\{x_1, ..., x_p\}$.

The feedback vectors f and g are chosen such that the spectrum of the closed-loop pencil (2.5) is $\{\mu_1, ..., \mu_p; \mu_{p+1}, ..., \mu_{2n}\}$. We use the following assumptions

i. The quadratic pencil is (partially) controlled with respect to the eigenvalues to be assigned $\mu_1, ..., \mu_p$.

ii. $\{\lambda_1, ..., \lambda_p\} \cap \{\lambda_{p+1}, ..., \lambda_{2n}\} = \emptyset$

Algorithm; Step 1. Form $\Lambda_1 = diag(\lambda_1..., \lambda_p)$ and $X_1 = (x_1, ..., x_p)$

Step 2. Solve for $y_1...., y_p$: $(\mu_j^2 M + \mu_j D + K)y_j = b$, $j = 1, 2, ..., p$

Form $Z_1 = \Lambda_1^{'} Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1$

where $Y_1 = (y_1, ..., y_p)$

and $\Lambda_1^{'} = diag(\mu_1, ..., \mu_p)$

Step 3. Solve for β: $Z_1 \beta = (1, 1, ..., 1)^T$

Step 4. Form

$f = MX_1\Lambda_1\beta$

$g = -KX_1\beta$

The algorithm for the Multi-Input case is similar to the Single-Input case.

The following theorem establishes the fact that the eigenvalues of the close loop pencil be orthogonal. (Datta and Sokolov,2009)

Theorem 2.1.1. *Theorem [Orthogonality of the Eigenvectors of the Quadratic Pencil] Let $P(\lambda) = \lambda^2 M + \lambda C + K$, where $M = M^T > 0, C = C^T$, and $K = K^T$. Assume that the eigenvalues $\lambda_1, ..., \lambda_{2n}$ are all distinct and different from zero. Let $\Lambda = diag(\lambda_1, ..., \lambda_{2n})$ be the eigenvalue matrix and $\phi = (\phi_1, ..., \phi_{2n})$ be the corresponding matrix eigenvectors. Then there exist diagonal matrices D_1, D_2 and D_3 such that;*

$$\Lambda\phi^T M\phi\Lambda - \phi^T K\phi = D_1 \tag{2.6}$$

$$\Lambda\phi^T C\phi\Lambda + \Lambda\phi^T K\phi + \phi^T K\phi\Lambda = D_2 \tag{2.7}$$

$$\Lambda\phi^T M\phi + \phi^T M\phi\Lambda + \phi^T C\phi = D_3 \tag{2.8}$$

and

$$D_1 = D_3\Lambda; D_2 = -D_1\Lambda; D_2 = -D_3\Lambda^2 \tag{2.9}$$

Furthermore, if $\{\lambda_1, ..., \lambda_k\}$ and $\{\lambda_{k+1}, ..., \lambda_{2n}\}$are disjoint, then

$$\Lambda_1 X_1^T M X_2 \Lambda_2 - X_1 K X_2 = 0 \tag{2.10}$$

where $\Lambda_1 = diag(\lambda_1, ..., \lambda_k$, $\Lambda_2 = diag(\lambda_{k+1}, ..., \lambda_{2n})$, X_1 and X_2 are the corresponding eigenvector matrices.

The proof of the above theorem is treated in the next chapter where the theorems related to some forms of the inverse eigenvalue problems treated are stated and proved.

The following is the algorithm for the quadratic inverse eigenvalue problem. The

Inputs are; $M = M^T > 0$, $K = K^T$, $\Sigma = diag(\mu_1, ..., \mu_k)$, Y_1 The following are the outputs; Updated stiffness matrix K_u and Y_2 such that $(Y^T MX_1)\Psi(Y^T MX_1)^T = Y^T M\Sigma^2 + Y^T KY$ is satisfied and $Y = col(Y_1, Y_2)$ is such that $Y^T MY$ is a diagonal matrix.

1. Compute Y_2 by solving $U_2^T M_2 Y_2 \Sigma^2 + U_2^T K_2 Y_2 - U_2^T(K_1 Y_1 + M_2 Y_1 \Sigma_1^2)$ and form the matrix

$$Y = \left((Y_1 Y_2)^T \right)$$

2. Orthogonalize matrix Y, by computing LDL decomposition of $Y^T MY = LDL^T$. Update the matrix Y by $Y \longleftarrow YL^{-T}$.

3. Compute Ψ by solving the following algebraic system of equation;

$(Y^T MX_1)\Psi(Y^T MX_1)^T = Y^T M\Sigma^2 + Y^T KY$

4. Update the stiffness matrix $K_u = K - MX_1 \Psi X_1^T M$

Finally, we present the inverse eigenvalue problem for the symmetric tridiagonal quadratic pencil with application to damped oscillatory systems, (Yitshak and Elhay, 1996). The problem is associated with the second-order differential equations of the form:

$$I\frac{d^2}{dt^2}x + C\frac{d}{dt}x + Kx = 0 \tag{2.11}$$

where C and K are n-square tridiagonal symmetric matrices, I is an identity matrix, and x is an n-vector depending on time t. This system may be solved by substituting $x = ve^{\lambda t}$, where v is a constant vector into (2.12), giving

$$Q(\lambda)v = 0, \tag{2.12}$$

where

$$Q(\lambda) = \lambda^2 I + \lambda C + K \tag{2.13}$$

The quadratic pencil $Q(\lambda)$ has $2n$ eigenvalues $\lambda_1, \lambda_2, .., \lambda_{2n}$ which are the roots of

$$det(Q(\lambda)) = 0 \tag{2.14}$$

The problem is that when given two sets of distinct numbers $\{\lambda_k\}_{k=1}^{2n}$ and $\{\mu_k\}_{k=1}^{2n-2}\}$, we are to determine tridiagonal symmetric matrices C and K which are such that the matrix $Q(\lambda) = \lambda^2 I + \lambda C + K$ satisfies $det(Q(\lambda))$ has zeros $\{\lambda_k\}_{k=1}^{2n}$ and the matrix $\hat{Q}(\lambda)$, is obtained by deleting the last row and column of $Q(\lambda)$, such that $det(\hat{Q}(\lambda))$ has zeros $\{\mu_k\}_{k=1}^{2n-2}$. (Yitshak and Sylvan , 1995), for the algorithm for determining the coefficients $\{\alpha_i, \gamma_i\}_{i=1}^{n}$and $\{\beta_i, \delta_i\}_{i=1}^{n-1}$ of the two symmetric tridiagonal matrices C and K.

2.2 Inverse eigenvalue problem for Jacobi matrices

A typical Jacobi matrix is a symmetric tridiagonal matrix. We show the construction of this type of matrix with two sets of eigenvalues which satisfy an interlacing property, (Boley and Golub, 1986). We have the eigenvalues of the main matrix given by $\{\lambda_i\}_1^n$ and the eigenvalues of the lower principal sub matrix $\{\mu_i\}_1^{n-1}$. The lower principal sub matrix has order $(n-1) \times (n-1)$ because it is obtained by deleting the first row and the last column of the main diagonal matrix. There have been considerable research interest in the IEP of the Jacobi and Periodic Jacobi matrices. See, for example, Xu, 1993, Ferguson, 1980, Andrea and Berry, 1992, Grey and Wilson and the references collected therein.

The following is one of the methods for constructing symmetric Jacobi matrices. Given a Jacobi matrix A which is a real symmetric tridiagonal matrix of the form:

$$A = \begin{pmatrix} a_1 & b_1 & ... & 0 \\ b_1 & a_2 & ... & 0 \\ 0 & ... & .... & b_{n-1} \\ 0 & ... & b_{n-1} & a_n \end{pmatrix}$$

with $b_i > 0$. We form arbitrary symmetric matrix $A = \begin{pmatrix} a_{11} & \hat{b}^t \\ \hat{b} & K \end{pmatrix}$ whose eigenvalues are $\{\lambda_i\}_1^n$ and whose lower principal sub matrix K has eigenvalues $\{\mu_i\}_1^{n-1}$. These eigenvalues are distinct and satisfy the following interlacing property $\lambda_i \geq \mu_i \geq \lambda_{i+1}$, $i = 1, ..., n-1$. We compute the first row eigenvectors of the matrix A using the relation

$$q_{1i^2} = \frac{\Pi_{j=1}^{n-1}(\mu_i - \lambda_i)}{\Pi_{j=1, j\neq i}^{n}(\lambda_j - \lambda_i)}.$$

All the eigenvectors are normalized to have norm 1. a_{11} is obtained from the relation $\Sigma_1^n \lambda_i - \Sigma_1^{n-1}\mu_i$. (Er-Xiong,2003) Lanczos algorithm is then applied to construct the tridiagonal matrix from the arbitrary symmetric tridiagonal matrix A.

Next we discuss the IEP for the periodic Jacobi matrix. This inverse problem normally arises in inverse scattering theory problems. The periodic Jacobi matrix is a tridiagonal matrix with real entries. The eigenvalues are therefore real and their corresponding eigenvectors are orthonormal. The eigenvalues of the main matrix and its leading principal sub matrix eigenvalues satisfy an interlacing property given by $\lambda_i \geq \mu_i \geq \lambda_{i+1}, ...$ $i = 1, ..., n-1$. This tridiagonal matrix is constructed with two sets of eigenvalues, the eigenvalues of the main matrix and the eigenvalues of the leading principal sub matrix and a set of scalars. The solution of the periodic Jacobi matrix is not unique and the number of solution is at most 2^{n-m-1}, where m is the number of common eigenvalues of the main matrix and the leading principal sub matrix. Throughout our discussions, we denote J_n and J_{n-1} by the main matrix and the leading principal sub matrix respectively (Xu and Jiang, 2006).

A periodic Jacobi matrix is any real, symmetric matrix of the form:

$$J_n = \begin{pmatrix} a_1 & b_1 & ... & b_n \\ b_1 & a_2 & b_2.. & 0 \\ 0 & b_2 & a_3..... & 0 \\ 0 & .... & a_{n-1} & b_{n-1} \\ b_n & 0...0 & b_{n-1} & a_n \end{pmatrix}$$

We form the following matrices from the matrix above;:

$$J^+ = \begin{pmatrix} a_1 & (b^+)^t \\ b^+ & K \end{pmatrix}$$

and

$$J^- = \begin{pmatrix} a_1 & (b^-)^t \\ b^{-1} & K \end{pmatrix},$$

where K is a Jacobi matrix given by J_{n-1}.
We denote the eigenvalues of J^+ by $\lambda_1^+ < \lambda_2^+ < ... < \lambda_n^+$,
those of J^- by $\lambda_1^- < \lambda_2^- < .... < \lambda_n^-$ which will be represented by the single scalar quantity $\beta = b_1 b_2 ....b_n$ and those of K by $\mu_1 < \mu_2 < .... < \mu_{n-1}$. The following theorems are necessary for the construction of the periodic Jacobi matrix.

We have seen that the eigenvalues of J_n and J_{n-1} satisfy an interlacing property which shows that they have common eigenvalues. The following theorem therefore provides the necessary and sufficient conditions for the two matrices to have common eigenvalues, (Xu and Jiang, 2006). Denote the first component of s_i as s_{1i} and the last one as $s_{n-1,i}$.

Theorem 2.2.1. . *For $j \in \{1, 2, ..., n-1\}$, μ_j is an eigenvalue of $J_n$if and only if*

$$b_n s_{1j} + b_{n-1} s_{n-1,j} = 0. \tag{2.15}$$

$b_n, n = 1, 2..., n$ are the eigenvalues of the matrix J_{n-1} which are represented by the single scalar quantity β and s_n are the components of the matrix.

Proof. .

Let $y^t = (b_n, 0, 0...., b_{n-1}) \in R^{n-1}$, then

$$J_n = \begin{pmatrix} J_{n-1} & y \\ y^t & a_n \end{pmatrix} \tag{2.16}$$

.

$$\begin{aligned} det(\lambda I - J_n) &= det \begin{pmatrix} (I & 0 & (\lambda I - J_{n-1} & -y \\ y^t(\lambda I - J_{n-1})^{-1} & I) & -y^t & \lambda - a_n) \end{pmatrix} \\ &= det \begin{pmatrix} \lambda I - J_{n-1} & -y \\ 0 & \lambda - a_n - y^t(\lambda I - J_{n-1})^{-1} y \end{pmatrix} \\ &= det(\lambda I - J_{n-1})(\lambda - a_n - y^t(\lambda I - J_{n-1})^{-1} y) \\ &= \Pi_{i=1}^{n-1}(\lambda - \mu_i)(\lambda - a_n - y^t(\lambda I - J_{n-1})^{-1} y) \end{aligned} \tag{2.17}$$

$(\lambda I - J_{n-1})^{-1}$ can be expressed as $(\lambda I - J_{n-1})^{-1} = \Sigma_{i=1}^{n-1} \frac{1}{\lambda - \mu_i} s_i s_i^t$, where μ_i are the eigenvalues of K and s_i are the components of the matrix J_{n-1}. s_i^t is the transpose of s_i.

Therefore,
$y^t(\lambda I - J_{n-1})^{-1} y = \Sigma_{i=1}^{n-1} \frac{(s_i^t y)^2}{\lambda - \mu_i} = \Sigma_{i=1}^{n-1} \frac{(b_n s_{1i} + b_{n-1} s_{n-1,i})^2}{\lambda - \mu_i}$
and

$$det(\lambda I - J_n) = \Pi_{j=}^{n-1}(\lambda - \mu_j)(\lambda - a_n - \Sigma_{i=1}^{n-1} \frac{(b_n s_{1i} + b_{n-1} s_{n-1,i})^2}{\lambda - \mu_i}). \tag{2.18}$$

From equation (2.18), we know

$$det(\mu_j I - J_n) = -\Pi_{i=1, i \neq j}^{n-1}(\mu_j - \mu_i) x (b_n s_{1j} + b_{n-1} s_{n-1,j})^2 \tag{2.19}$$

This implies that μ_j is an eigenvalue of J_n if and only if (2.15) is valid. Now on the other hand, if the two matrices J_n and J_{n-1} have no common eigenvalues, that is the eigenvalues are distinct, Then the following theorem holds. □

Theorem 2.2.2. *If*

$$b_n s_{1i} + b_{n-1} s_{n-1,j} \neq 0. j = 1, 2, ..., n-1. \tag{2.20}$$

then the eigenvalues of the matrix J_n are equal to n roots of the following equation:

$$F(\lambda) = \lambda - a_n - \Sigma_{i=1}^{n-1} \frac{(b_n s_{1i} + b_{n-1} s_{n-1,j})^2}{\lambda - \mu_i} = 0 \tag{2.21}$$

and μ_i strictly separate λ_i as follows:

$$\lambda_1 < \mu_1 < \lambda_2 < ... < \lambda_{n-1} < \mu_{n-1} < \lambda_n. \tag{2.22}$$

Proof. (Jiang, 2003).

Applying theorem (2.2), we can conclude that, under condition (2.20), for $i = 1, 2, .., n-1$, μ_1 are the eigenvalues of J_n. Combining this with (2.18), we know $det(\lambda I - J_n) = 0$, is equivalent to equation (2.23).

As $(b_n s_{1i} + b_{n-1} s_{n-1,j})^2 > 0, i = 1, 2, .., n-1$, for a sufficiently small positive number ϵ,

$F(\mu_i - \epsilon) > 0$, and $F(\mu_i + \epsilon) < 0, i = 1, 2, .., n-1$,

$F(-\infty) < 0$ and $F(+\infty) > 0$,

hence $2.22)$ holds. □

Suppose some of the eigenvalues of J_{n-1} are the eigenvalues of J_n. The following theorem therefore gives the relationship between the rest of the eigenvalues of J_n that do not belong to J_{n-1}.

Theorem 2.2.3. *Let $N = \{1, 2, .., n-1\}$. If there is a set $N_1 = \{i_1, i_2, ..i_m\} \subset N$ such that $b_n s_{1j} + b_{n-1} s_{n-1,j} = 0$, $j \in N_1$ and $b_n s_{1j} + b_{n-1} s_{n-1,j} \neq 0$, $j \in \frac{N}{N_1}$ then $\mu_1, \mu_2, ..., \mu_m$*

are the eigenvalues of J_n, and the rest of the eigenvalues of J_n are given as $n-m$ roots of the equation

$$F(\lambda) = \lambda - a_n - \Sigma_{i=1,i\notin N_i}^{n-1} \frac{(b_n s_{1i} + b_{n-1} s_{n-1,i})^2}{\lambda - \mu_i} = 0. \tag{2.23}$$

From the above theorem, we deduce that except for $\mu_{i1}, \mu_{i2}, .., \mu_{im}$, the rest of the eigenvalues of J_n are $n-m$ roots of $F(\lambda) = 0$. The necessary and sufficient conditions for an inverse eigenvalue problem for periodic Jacobi to be solvable are related to the following two theorems. If J_n and J_{n-1} have distinct eigenvalue then the periodic Jacobi problem is solvable provided the theorem below holds (Jiang, 2003).

Theorem 2.2.4. *If all the elements in the two sets $\lambda = \{\lambda_j\}_{j=1}^{n}$ and $\mu = \{\mu_j\}_{j=1}^{n-1}$ are distinct, then the periodic Jacobi inverse eigenvalue problem (PJI) is solvable if and only if*

$$\Pi_{i=1}^{n} |\mu_j - \lambda_i| \geq 4\beta(-1)^{n-j+1}, j = 1, 2, ..., n-1. \tag{2.24}$$

Furthermore, uniqueness of solution is not guaranteed and there are at most 2^{n-1} different solutions.

Finally, we state without proof, (see for example Jiang, 2003) for proof. The following theorem establishes the fact that if J_n and J_{n-1} have common eigenvalues then the inequality 2.26 holds.

Theorem 2.2.5. *If two sets $\lambda = \{\lambda_j\}_{j=1}^{n}$ and $\mu = \{\mu_j\}_{j=1}^{n-1}$ have common elements, and the number of common eigenvalues is m, then the periodic Jacobi inverse eigenvalue problem $(PJIEP)$ is solvable if and only if* (2.25) *is valid. Furthermore, if the problem $PJIEP$ is solvable, there are at most 2^{n-m-1} different solutions.*

The algorithm for constructing periodic Jacobi matrix follows. (See for example Boley and Golub, 1978).

Algorithm:

1. Two sets of eigenvalues $\{\lambda_i^+\}_{i=1}^{n}$, $\{\mu^i\}_{i=1}^{n-1}$ and the single scalar β

2. Compute the first row of Q, the eigenvectors of J^+ using

$$q_{1j}^2 = \frac{\Pi_{k=1}^{n-1}(\mu_k - \lambda_j^+)}{\Pi_{k=1,k\neq j}^{n}(\lambda_k^+ - \lambda_j^+)}$$

$j = 1, ..., n$

3. Compute b^+ and b^- using the equations below;

$$(b_k^+)^2 = -\frac{\Pi_{j=1}^{n}(\lambda_j^+ - \mu_k)}{\Pi_{j=1,j\neq k}^{n-1}(\mu_j - \mu_k)}$$

$$(b_k^-)^2 = -\frac{\Pi_{j=1}^{n}(\lambda_j^- - \mu_k)}{\Pi_{j=1,j\neq k}^{n-1}(\mu_j - \mu_k)}$$

$k = 1, ..., n-1.$

4. Compute the eigenvectors of K using, $P_{n-1} = \frac{b^+ - b^-}{2b_n}$

5. Compute the last row of Q, $z_n = r_n = [q_{n1}, ..., q_{nn}]$ using

$$q_{n,k} = -q_{1k}\Sigma_{j=1}^{n-1}\frac{P_{n-1,j}b_j^+}{(\mu_k - \lambda_k^+)}$$

6. Using the initial values z_1 and z_n and $\Lambda^+ = Q^t J^+ Q$, apply Lanczos algorithm to generate the tridiagonal matrix.

2.3 Parameterized Inverse Eigenvalue Problem $(PIEP)$

$PIEP$ is described as the process of adding or multiplying a vector X which contains parameters by an $n \times n$ square matrix A. We note that the parameter c is in the field F has the number of parameters m is not the same as the order n of the matrix.

Definition: Given an $n \times n$ square matrix A, we are to find the parameter $c = \{c_1, c_2, ..., c_n\} \in F^m$ where F is a field such that $(A(c)) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ where

$\{\lambda_1, \lambda_2, ..., \lambda n\} \subset F$ are scalars which are the eigenvalues of $A(c)$.

Three main kinds of $PIEP$ can be identified. These are Linear dependence on parameters($LiPIEP$), additive inverse eigenvalue problem($AIEP$) and the multiplicative inverse eigenvalue problem ($MIEP$). But we shall concentrate on $MIEP$ which we are applying in our work. The $PIEP$ format arise frequently in discrete modeling and factor analysis. By the definition, we see that the parameter c is in the field F where the number of parameters m is not the same as the order n of the matrix.

2.3.1 MIEP

The $MIEP$ is obtained from a process of pre-multiplying an $n \times n$ square symmetric matrix A by a vector X which contains the parameter $c = \{c_1, c_2, ..., c_n\}$ such that $XA = \Sigma_{i=1}^{n} c_i A_i$ where $c_i \in X$,(Oliveira, 1972). Some row(s) become linear combination of other row(s) after the multiplication. Some authors have treated the unsolvability of multiplicative IEP. See for example Sun, 1986. For the solvability to the MIEP, see for example (Silva, 1986, Hadeler, 1969, Oliveira, 1972, Shapiro, 1983 and J. Sun, 1986). There are also many numerical algorithms developed for computational purposes.

Example: Let $A = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ then

$XA = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} a+3b & 4a+b \\ 3 & 1 \end{pmatrix}$ The eigenvalues of the matrix therefore satisfy the system;

$\left\| \begin{matrix} \lambda_1 \lambda_2 = 1a \\ \lambda_1 + \lambda_2 = 1 + a + 3b \end{matrix} \right\|$. We therefore conclude that given any $\lambda \in R^2$, we can always find a pair (a, b) of real numbers that solves the $MIEP$. Indeed, the solution in this case is unique.

Even though there are many types of the $MIEP$, we will deal with the one which has $n \times n$ square symmetric matrices with real entries.

The $MIEP$ can arise from engineering application, (Chu and Golub, 2001, Yamamoto, 1990). For example, the vibration of particles on a string. Let us assume that four particles, each with mass m_i which are uniformly spaced with distance h and are vibrating vertically subject to a horizontal tension F. The equation of motion for such a system is a second-order differential equation of the form

$$\frac{d^2x}{dt^2} = -DAx, \tag{2.25}$$

where A is a tridiagonal real symmetric matrix, $x = [x_1, x_2, ..., x_n]^T$ and

$D = diag(d_1, d_2, ..., d_n)$ with $d_i = \frac{F}{m_i h}$.

To solve the second order differential equation (2.27), we consider the eigenvalue problem

$$DAx = \lambda x, \tag{2.26}$$

where λ is the square of the natural frequency of the system. The inverse problem then amounts to calculating the mass m_i, $i = 1, 2, ..., n$ so that the resulting system vibrates at a prescribed natural frequency. Generally, $PIEP$ can be solved using the Newton iterative method. We discuss a typical example below.

Let us consider the Newton's iterative method for solving $PIEP$. We concentrate exclusively on the Newton's method applied to Linearly dependent parameterized inverse eigenvalue problem ($LiPIEP$). We consider the case where all the matrices involved S_n are symmetric.

We choose to single out this method for consideration because, while Newton's iteration is typically regarded as the normal means to solve nonlinear differentiable equations, we demonstrate how iteration can be carried out by taking into account the matrix structure.

The eigenvalues $\lambda = \lambda_{k}{}_{k=1}^{n}$ in this context are distinct and are arranged in ascending order. We consider an affine space of symmetric matrices together with isospectral surface $M(\Lambda)$ and a Lie transformation group $O(n)$. The isospectral surface contains special orthogonal matrices which are rotation matrices. Let the affine subspace be represented by,

$$\beta = \{B(c) \mid c \in R^n\} \tag{2.27}$$

and the isospectral surface

$$M(\wedge) = \{Q \wedge Q^T \mid Q \in O(n)\} \tag{2.28}$$

where Q is an orthogonal matrix. Q in this case is a rotation matrix given by Rodrigues rotation formula. When we use the fact that $Q(t)Q(t)^T = I$, then it follows that $Q(t)$ is a differentiable path which is embedded in $O(n)$ if and only if;

$$Q'(t) = K(t)Q(t), Q(t) \in O(n) \tag{2.29}$$

for some family of antisymmetric matrices $K(t)$. Q is an exponential function given by $Q = e^K$. When we translate this to the differentiable manifold $M(\wedge)$, then it follows that any tangent $S(X)$ to $M(\wedge)$, at a point $X \in M(\wedge)$ is a Lie algebra given by the Lie bracket;

$$S(X) = XK - KX \tag{2.30}$$

for some antisymmetric matrix $K \in R^{n \times n}$, where K is obtained from a rotation matrix Q in this case.

The Newton Method (Chu, 2005) here has a similar approach as the classical Newton Method for finding the roots of a one-variable differentiable function. Given a function $f(x)$ and its derivative $f'(x)$, we begin with an initial value of x_0 to obtain the first iterative value of x_1 using the fact that;

$$x_1 = x_0 - (f'(x_0))^{-1} f(x_0). \tag{2.31}$$

The process is repeated until a sufficiently accurate value is reached:

$$x_{n+1} = x_n - (f'(x_n))^{-1} f(x_n) \tag{2.32}$$

The new iterate x_{n+1} in the scheme above represents the $x - intercept$ of the line that is tangential to the graph of $f(x)$ at $(x_n, f(x_n))$. The point $(x_{n+1}, f(x_{n+1}))$ represents a natural "lift" of the intercept along the $y - axis$ to the graph of $f(x)$ from which the next tangent line begins.

Let us consider the isospectral surface $M(\wedge)$ as playing the role of the graph of $f(x)$ while the affine subspace β plays the role of the $x - axis$. We want to find the intersection of the isospectral surface $M(\wedge)$ and the affine subspace β which is given by:

$$B(c) = Q \wedge Q^T \tag{2.33}$$

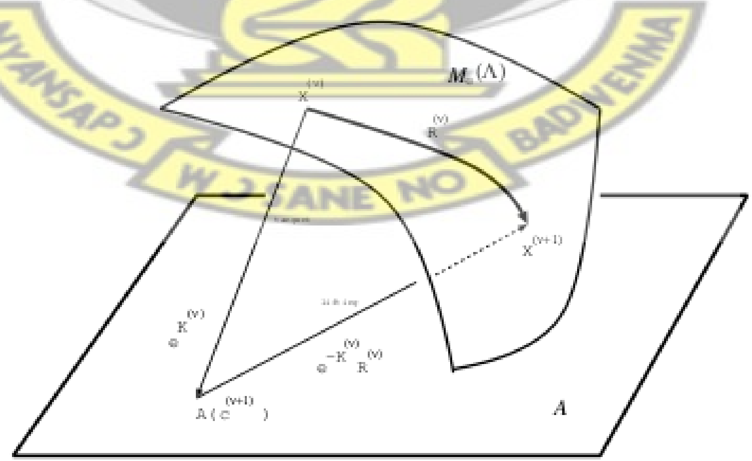
Given $X_n \in M(\wedge)$, there exists an orthogonal matrix $Q_n \in O_n$ such that;

$$Q_n^T X_n Q_n = \wedge \tag{2.34}$$

which is an inverse problem.

The matrix $X_n + KX_n - X_nK$ where K is an antisymmetric matrix represents a tangent vector to the surface of $M(\wedge)$ emanating from X_n. For one Newton iteration, we have to find $\beta - intercept$ $B(c_{n+1}) \in \beta$ which belongs to the affine subspace β and then "lift" up this point $B(c_{n+1}) \in \beta$ to the point $X_{n+1} \in M(\wedge)$.

See figure below



To find the $\beta - intercept$, we need to find an antisymmetric matrix K_n and a vector

c_{n+1} such that;

$$X_n + K_n X_n - X_n K_n = B(c_{n+1}) \tag{2.35}$$

The unknowns K_n and c_{n+1} in the above equation can be solved separately using equations $(2 \cdot 34)$ and $(2 \cdot 35)$ to obtain

$$\wedge + K_n^* \wedge - \wedge K_n^* = Q_n^T B(c_{n+1}) Q_n \tag{2.36}$$

where

$$K_n^* = Q_n^T K_n Q_n. \tag{2.37}$$

The above presentation of the inverse problem using Newton's iterative method is more theoretical and somehow cumbersome. We therefore present a more practicable and easy iterative method using a simpler algorithm in Chapter Five. The initial eigenvalue and the matrix for the iteration is given and therefore one can compute the rotation matrix and consequently, the skew symmetric matrix.

CHAPTER 3

Solvability and Computability of the Inverse Eigenvalue Problem

In this chapter, we discuss the theorems on the solvability of different types of IEP's and computability of the different methods we have discussed about the inverse eigenvalue problem (IEP). We consider the theorems related to the Inverse Eigenvalue problem of the Quadratic Pencil (IEQP), the Inverse Eigenvalue Problem of the Jacobi and Periodic Jacobi matrices and the Parameterized Inverse Eigenvalue problem (PIEP).

We first discuss theorems related to the quadratic inverse eigenvalue problem. This problem is as follows: Given

1. Real $n \times n$ matrices $M = M^T > 0$, $D = D^T$, $K = K^T$ of the quadratic pencil $p(\lambda) = \lambda^2 M + \lambda D + K$,

2. The self-conjugate subset $\{\lambda_1, ..., \lambda_p\}$, $p < n$ of the open-loop spectrum $\{\lambda_1, ..., \lambda_p; \lambda_{p+1}, ..., \lambda_{2n}\}$ and the corresponding eigenvector set $\{x_1, ..., x_p\}$.

3. The self-conjugate sets of numbers and vectors $\{\mu_1, ..., \mu_p\}$ and $\{y_1, ..., y_p\}$ such that $\mu_j = \bar{\mu_k}$ implies $y_i = \bar{y_k}$.

We are to find the control matrix B of order $n \times m$ $(m < n)$, and feedback matrices F and G of order $n \times m$ such that the spectrum of the closed-loop pencil $p_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + K - BG^T$ is $\{\mu_1, ..., \mu_p; \lambda_{p+1}, ..., \lambda_{2n}\}$ and the eigenvector set $\{y_1, ..., y_p; x_{p+1}, ..., x_{2n}\}$ where $x_{p+1}, ..., x_{2n}$ are the eigenvectors of $p(\lambda) = \lambda^2 M + \lambda D + K$ corresponding to $\lambda_{p+1}, , \lambda_{2n}$. We have to derive three orthogonality relations between the eigenvectors of a symmetric definite quadratic pencil. The results generalize the well-known results on orthogonality between the eigenvectors of a symmetric matrix and those of a symmetric definite linear pencil of the form $K - \lambda M$. The following two theorems are therefore important in solving problems related to the inverse problem of the quadratic pencil.

Theorem 3.0.1. *(Orthogonality of the Eigenvectors of Quadratic Pencil). Let* $p(\lambda) = \lambda^2 M + \lambda D + K$, *where* $M = M^T \succ 0$, $D = DT$, *and* $K = K^T$. *Let* X *and* $\Lambda = diag(\lambda_1, .., \lambda_{2n})$ *be, respectively, the eigenvector and eigenvalue matrix of the pencil* $p(\lambda) = \lambda^2 M + \lambda D + K$. *Assume that the eigenvalues* $\lambda_1, ..., \lambda_n$ *are all distinct and different from zero. Then there exist diagonal matrices* D_1,D_2 *and* D_3 *such that*

$$\Lambda X^T M X \Lambda - X^T K X = D_1 \tag{3.1}$$

$$\Lambda X^T D X \Lambda + \Lambda X^T K X + X^T K X \Lambda = D_2 \tag{3.2}$$

$$\Lambda X^T M X + X^T M X \Lambda + X^T D X = D3 \tag{3.3}$$

Furthermore

$$D_1 = D_3 \Lambda \tag{3.4}$$

$$D_2 = -D_1 \Lambda \tag{3.5}$$

$$D_2 = -D_3 \Lambda^2 \tag{3.6}$$

Proof. : (See for example, Biswa *et al*, 1996).

By definition, the pair (X, Λ) must satisfy the $n \times 2n$ system of equations (called the eigendecomposition of the pencil $p(\lambda) = (\lambda^2 M + \lambda D + K)$:

$$M X \Lambda^2 + D X \Lambda + K X = 0. \tag{3.7}$$

Isolating the terms in D, we have from above

$$-D X \Lambda = M X \Lambda^2 + K X.$$

Multiplying this on the left by ΛX^T gives

$$-\Lambda X^T DX\Lambda = \Lambda X^T MX\Lambda^2 + \Lambda X^T KX.$$

Taking the transpose gives

$$-\Lambda X^T DX\Lambda = \Lambda^2 X^T MX\Lambda + X^T KX\Lambda$$

When we subtract the latter from the former we have, on rearrangement,

$$\Lambda X^T MX\Lambda^2 - X^T KX\Lambda = \Lambda^2 X^T MX\Lambda - \Lambda X^T KX$$

or

$$(\Lambda X^T MX\Lambda - X^T KX)\Lambda = \Lambda(\Lambda X^T MX\Lambda - X^T KX).$$

Thus, the matrix $\Lambda X^T MX\Lambda - X^T KX$ which we denote by D_1, must be diagonal since it commutes with a diagonal matrix, the diagonal entries of which are distinct. We thus have the first orthogonality relation(3.1).

Similarly, isolating the term in M of the eigendecomposition equation, we get $-MX\Lambda^2 = DX\Lambda + KX$,
and multiplying this on the left by $\Lambda^2 X^T$ gives

$$-\Lambda^2 X^T MX\Lambda^2 = \Lambda^2 X^T DX\Lambda + \Lambda^2 X^T KX.$$

Taking transpose, we have

$$-\Lambda^2 X^T MX\Lambda^2 = \Lambda X^T DX\Lambda^2 + X^T KX\Lambda^2.$$

Subtracting the last equation from the previous one and adding $\Lambda X^T KX\Lambda$ to both sides gives, after some rearrangement,

$$\Lambda(\Lambda X^T DX\Lambda + \Lambda X^T KX + X^T KX\Lambda) + (\Lambda X^T DX\Lambda + \Lambda X^T KX + X^T KX\Lambda)\Lambda.$$

Again, this commutativity property implies, since Λ has distinct diagonal entries, that

$$\Lambda X^T DX\Lambda + \Lambda X^T KX + X^T KX\Lambda = D_2$$

is a diagonal matrix. This is the second orthogonality relation (3.2). The first and second orthogonality relations together easily imply the third orthogonality relation (3.3).

To prove (3.4) we multiply the last equation on the right by Λ giving

$\Lambda X^T MX\Lambda + X^T MX\Lambda^2 + X^T DX\Lambda = D_3\Lambda$,

which, using the eigendecomposition equation, becomes

$\Lambda X^T MX\Lambda + X^T(-KX) = D_3\Lambda$.

So, from the first orthogonality relation (3.1) we see that $D_1 = D_3\Lambda$

Next, using the eigendecomposition equation (3.7), we rewrite the second orthogonality relation (3.2) as

$$\begin{aligned} D_2 &= \Lambda X^T(DX\Lambda + KX) + X^T KX\Lambda \qquad (3.8) \\ &= \Lambda X^T(-MX\Lambda^2) + X^T KX\Lambda \\ &= (-\Lambda X^T MX\Lambda + X^T KX)\Lambda \end{aligned}$$

By the first orthogonality relation we then have $D_2 = -D_1\Lambda$

Finally, from $D_1 = D_3\Lambda$ and $D_2 = -D_1\Lambda$ we have $D_2 = -D_3\Lambda^2$.

Thus, using D_1, D_2 and D_3, we obtain the following results,

$$\begin{aligned} x_i^T(\lambda_i\lambda_j M - K)x_j &= 0 \qquad (3.9) \\ x_i^T(\lambda_i\lambda_j C + (\lambda_i + \lambda_j)K)x_j &= 0 \\ x_i^T((\lambda_i + \lambda_j)M + C)x_j &= 0, i \neq j \end{aligned}$$

We remind the reader that matrix and vector transposition here does not mean conjugate for complex quantities. □

The Direct modal approach as discussed in the previous chapter, is the process of finding a solution to the quadratic pencil problem using only a few eigenvalues of the characteristic polynomial ($p(\lambda)$) of the closed loop pencil. We remind our readers that Single-input and Multi-input are algorithms that were explained and used to compute the feedback matrices in the previous chapter.In order to solve the inverse eigenvalue problem of the quadratic pencil for both the Single- input and the Multi-input cases for problem

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t),$$

we apply the following theorems.

Theorem 3.0.2. *(Solution to the Single-Input Partial Eigenvalue Assignment Problem for a Quadratic Pencil). If* $\{\lambda_1, ..., \lambda_p\} \cap \{\lambda_{p+1}, ..., \lambda_{2n}\} = \emptyset$ *then*

(i). *For any arbitrary vector* β*, the feedback vectors* f *and* g *defined by*

$$f = MX_1\Lambda_1\beta \tag{3.10}$$

and

$$g = -KX_1\beta \tag{3.11}$$

are such that $2n - p$ *eigenvalues* $\lambda_{p+1}, ..., \lambda_{2n}$ *of the closed-loop pencil*

$p_c(\lambda) = \lambda^2 M + \lambda(D - bf^T) + K - bg^2$

are the same as these of the open-loop pencil $p(\lambda) = \lambda^2 M + \lambda D + K$

(ii). *Let* $y_1, ..., y_p$ *be the set of* p *vectors such that for each* $k = 1, 2, .., p$,

$\begin{pmatrix} y_k \\ 1 \end{pmatrix} \in null(\mu_k^2 M + \mu_k D + K, -b)$.

(Equivalently, the pencil $p(\lambda)$ *is partially controllable with respect to* $\mu_1, .., \mu_p$*).*

Define $Z_1 = \Lambda_1' Y_1^T MX_1\Lambda_1 - Y_1^T Kx_1$

where Z_1 is a single matrix.

The problem under discussion has a solution in the form (3.10)-(3.11) *if and only if the system of equations*

$Z_1\beta = (1, 1, ..., 1)^T$

has a solution.

The proof of the above theorem is related to Algorithm 3.2. (Datta and Sarkissian 1996).

Theorem 3.0.3. *(Solution to Multi-input Partial Eigenvalue Assignment Problem for a Quadratic Pencil)*

If $\{\lambda_1, ..., \lambda_p\} \cap \{\lambda_{p+1}, ..., \lambda_{2n}\} = \emptyset$. Then (i). For any arbitrary matrix Φ, the feedback matrices F and G defined by $F = MX_1\Lambda_1\Phi^T$ and $G = -KX_1\Phi^T$ are such that $2n-p$ eigenvalues $\lambda_{p+1}, ..., \lambda_{2n}$ of the closed-loop pencil $p_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + K - BG^T$ are the same as those of the open-loop pencil $p(\lambda) = \lambda^2 M + \lambda D + K$.

(ii). Let $\{y_1, ..., y_p\}$ and $\{\gamma_1, ..., \gamma_p\}$ be the two sets of vectors chosen in such a way that $\mu_j = \bar{\mu_k}$ implies $\gamma_j = \bar{\gamma_k}$ and for each $k = 1, 2, .., p$, $\begin{pmatrix} y_k \\ \gamma_k \end{pmatrix} \in null(\mu_k^2 M + \mu_k D + K, -B)$ (equivalently, the pair $(p(\lambda), B)$ is partially controlled with respect to the modes $\mu_1, ..., \mu_p$) Define Z_1 and Y_1 as in Theorem (3.2). The problem under discussion (in the multi-input case) has a solution with F and G given by i, provided that Φ satisfies the linear system of equations: $\Phi Z_1^T = \Gamma$, where $\Gamma = (\gamma_1, .., \gamma_p)$.

Proof. :

Using the first orthogonality relation (3.1), it is easy to verify that

$$\begin{aligned} MX_2\Lambda_2^2 + (D - BF^T)X_2\Lambda_2 + (K - BG^T)X_2 &= (MX_2\Lambda_2^2 + DX_2\Lambda_2 KX_2) \\ &- B\Phi(\Lambda_1 X_1^T MX_2\Lambda_2 - X_1^T KX_2) \\ &= 0, \end{aligned}$$

which proves Part (i).

To prove (ii), we note using the closed-loop and the open loop pencil above, that

$$\begin{aligned} P_c(\mu_k)y_k &= (\mu_k^2 M + \mu_k(D - B\Sigma_{j=1}^{p}\phi_j\lambda_j x_j^T M) + (K + B\Sigma_{j=1}^{p}\phi_j x_j^T K))y_k \\ &= B\gamma_k - B(\Sigma_{j=1}^{p}\phi_j x_j^T(\mu_k\lambda_j M - K))y_k \\ &= B(\gamma_k - \Sigma_{j=1}^{p}\phi_j z_{kj}), \end{aligned}$$

where $\Phi = (\phi_1, ..., \phi_p)$ and $z'_{kj}s$ are the elements of the matrix Z_1. Then $P_c(\mu_k)y_k = 0$ for $k = 1, 2, .., p$ can be written in the form of the single matrix equation $\Phi Z_1^T = \Gamma$

We now show that the matrix F and G obtained this way are real matrices. Since, if $\gamma_1, .., \gamma_p$ are chosen in such a way that $\mu_j = \bar{\mu_k}$ implies $\gamma_j = \bar{\gamma_k}$, then this also implies $y_j = \bar{y_k}$ and, then, as in the proof of Theorem (3.2), there exist permutation matrices T and $T^{'}$ such that

$$\bar{X_1} = X_1 T, \bar{X_1\Lambda_1} = X_1\Lambda_1 T, \bar{\Gamma} = \Gamma T^{'}, \bar{Y_1} = Y_1 T^{'} and \bar{Y_1\Lambda_1'} = Y_1\Lambda_1' T^{'}.$$

Thus, conjugating $Z_1 = \Lambda_1^{'} Y_1^T M X_1 \Lambda_1 - Y_1^T K X_1$, gives $\bar{Z_1} = (T^{'})^T Z_1 T$ and, conjugating $\Phi Z_1^T = \Gamma$, we get

$$\bar{\Phi} T^T Z_1 T^{'} = \Gamma T^{'},$$

which implies that $\bar{\Phi} = \Phi T$. Therefore

$\bar{F} = M(X_1\Lambda_1 T)(T^T \Phi^T) = F$

and

$\bar{G} = -K(X_1 T)(T^T \Phi^T) = G$

showing that F and G are real matrices. □

The Quadratic Inverse Eigenvalue Problems, Active Vibration Control and Modal Updating, can be classified as Direct and Partial model. It is "direct" because the problem is solved directly in second-order setting, without transforming it to a standard first-order stage-space model. In this case, it avoids a possible ill-conditioned inversion of the mass matrix and the loss of some of the exploitable properties, very often offered by practical problems, such as definiteness, sparsity and bandness. It is "partial-model" because the problem is solved using only a few eigenvalues of $P(\lambda)$ that need to be reassigned and their corresponding eigenvectors. The "no spill-over" property in this case is established and confirmed in the proof of the theorem below and no model reduction is needed no matter how large the model may be, (Sarkissian, 2001).

Theorem 3.0.4. *Let the matrix B be of full rank. Let the scalars $\{\mu_1, ..., \mu_k\}$ and the eigenvalues of the pencil (M, C, K) be such that sets $\{\lambda_1, .., \lambda_k\}$, $\{\lambda_{k+1}, ..., \lambda_{2n}\}$ and $\{\mu_1, .., \mu_k\}$ are disjoint and each set is closed under complex conjugation. Let $Y = (y_1, .., y_k)$ be the matrix of left eigenvectors associated with eigenvalues $\{\lambda_1, .., \lambda_p\}$. Let the pair $(P(\lambda), B)$ be partially controllable with respect to $\{\lambda_1, .., \lambda_k\}$, i.e. $y_i^* B \neq 0, i = 1, .., k$. Let $\Gamma = (\gamma_1, .., \gamma_k)$ be a matrix such that $\gamma_j = \bar{\gamma}_j$, whenever $\mu_j = \bar{\mu}_i$. Set $\Lambda_1 = diag(\lambda_1, .., \lambda_k)$ and set $\Sigma = diag(\mu_1, .., \mu_k)$. Let Z be the unique nonsingular solution of the Sylvester equation*

$$\Lambda_1 Z - Z\Sigma = -Y^* B\Gamma,$$

Define the real feedback matrices by

$$F = \Phi Y^* M$$

, and

$$G = \Phi(\Lambda_1 Y^* M + Y^* C),$$

where Φ satisfies the linear system $\Phi Z = \Gamma$. Then matrices F and G are real and the closed-loop pencil $(M, C - BF^T, K - BG^T)$ will have $\{\mu_1, .., \mu_k, \lambda_{k+1}, .., \lambda_{2n}\}$ as its eigenvalues and the eigenvectors corresponding to the eigenvalues $\{\lambda_{k+1}, .., \lambda_{2n}\}$ will remain unchanged.

Proof. :

We omit the proof of the first part which is lengthy and can be found for example Brahma *et al* 2009. Let $\Lambda_2 = diag(\lambda_{k+1}, .., \lambda_{2n})$ and X_2 be the corresponding eigenvector matrix.

In matrix notations we then need to prove that

$$MX_2\Lambda_2^2 + (D - BF^T)X_2\Lambda_2 + (K - BG^T)X_2 = 0.$$

The result follows by substituting for F and G into the left-hand side of the equation and noting that (X_2, Λ_2) is a matrix eigenpair of $P(\lambda)$, that is:

$$MX_2\Lambda_2^2 + CX_2\Lambda_2 + KX_2 = 0$$

and the eigenpairs (Λ_1, X_1) and (Λ_2, X_2) satisfy the orthogonality relation $\Lambda_1 X_1^T M X_2 \Lambda_2 - X_1 K X - 2 = 0$. $\square$

Some recent results on Model Updating methods can be found in (Frishwell and Mottershead, 1995, Carvalho, Datta, Gupta and Lagadapati, 2007, Carvalho Datta, Lin, and Wang, 2006, Ewins, 2000, Friswell, Inman and Pilkey, 1998, Halevi and Bucher, 2003 and Kenigsbuch and Halevi, 1998)

We need the following theorems in order to discuss the Periodic Jacobi Inverse Eigenvalue Problem (PJI). First we state the following two lemmas which are fundamental to our approach to the Periodic Jacobi Inverse Eigenvalue Problem, (Ying-Hong and Jiang, 2006).

Lemma 3.0.1. *Let $\lambda_1 < \mu_1 < .... < \mu_{n-1} < \lambda_n$, then the following linear algebraic system*

$$\frac{x_1}{\lambda_i - \mu_1} + \frac{x_2}{\lambda_i - \mu_2} + .. + \frac{x_{n-1}}{\lambda_i - \mu_{n-1}} = \lambda_i - a_n, i = 1, 2, .., n-1, n. \qquad (3.12)$$

The term a_n on the right side is equal to $\Sigma_{i=1}^n \lambda_i - \Sigma_{i=1}^{n-1} \mu_i$, has a unique solution $x = (x_1, ..., x_{n-1})^T$ and

$$x_j = -\Pi_{i=1}^n (\lambda_i - \mu_j) \Pi_{i=1, i \neq j}^{n-1} (\mu_i - \mu_j)^{-1} > 0. \qquad (3.13)$$

We note that the linear system (3.12) is overdetermined with n equations and $n-1$

unknowns.

Lemma 3.0.2. *(Paige C C, 1971) For a Jacobi matrix J_{n-1}, we have*

$$s_{1j}s_{n-1,j} = \frac{b_1 b_2 ... b_{n-2}}{x'(\mu_j)}.j = 1, 2, .., n-1, \tag{3.14}$$

where

$$x'(\mu_j) = [det(\mu_j I - J_{n-1})] = \Pi_{i=1,i\neq j}^{n-1}(\mu_j - \mu_i) = (-1)^{n-j-1}\Pi_{i=1,i\neq j}^{n-1}|\mu_j - \mu_i|. \tag{3.15}$$

The following is the proof of Theorem 2.5 which was stated in Chapter Two.

Proof. : For a periodic Jacobi matrix J_n, if the eigenvalues of J_n and J_{n-1} are distinct, then the strict inequality (2.22) holds and by Theorem (2.3), its eigenvalues are the n roots of (2.21), that is,

$$\lambda_i - a_n - \Sigma_{k=1}^{n-1}\frac{x_k}{\lambda_i - \mu_k} = 0, i = 1, .., n. \tag{3.16}$$

where

$$x_j = (b_n s_{1j} + b_{n-1}s_{n-1,j})^2, j = 1, .., n-1 \tag{3.17}$$

By Lemma 3.1 , gives that the above equations (3.16) has a unique solution $x = (x_1, ..., x_{n-1})^T$, and

$$x_j = -\Pi_{i=1}^{n}(\lambda_i - \mu_j)\Pi_{i=1,i\neq j}^{n-1}(\mu_i - \mu_j)^{-1} > 0, j = 1, .., n-1. \tag{3.18}$$

By Lemma 3.2 and $\Pi_{i=1}^{n} b_i = \beta$, we get

$$s_{n-1,j} = \frac{\beta}{b_{n-1}b_n x'(\mu_j)s_{1j}}, j = 1, .., n-1 \tag{3.19}$$

Substituting (3.19) into (3.17) leads to

$$b_n^4[x'(\mu_j)]^2 s_{1j}^4 + [\frac{2\beta}{x'(\mu_j)} - x_j]b_n^2[x'(\mu_j)]^2 s_{1j}^2 + \beta^2 = 0.$$

Solving the above equation, we have

$$s_{1j}^2 = \frac{|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} \pm \sqrt{\Delta_j}}{2b_n^2|x^{'}(\mu_j|} \tag{3.20}$$

where

$$\begin{aligned} \Delta_j &= |x^{'}(\mu_j)x_j|^2 - 4\beta(-1)^{n-j-1}|x^{'}(\mu_j)x_j| \\ &= (\Pi_{i=1}^{n}|\mu_j - \lambda_i|)(\Pi_{i=1}^{n}|\mu_j - \lambda_i| - 4\beta(-1)^{n-j-1}). \end{aligned} \tag{3.21}$$

Since J_{n-1} is a Jacobi matrix, then by lemma 3.2, we have $s_{1j} > 0, j = 1, .., n-1$, and we get $\Delta_j \geq 0$

$$|x^{'}(\mu_j)\mu x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j} > 0.$$

or $\Delta_j \geq 0$

$$|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} - \sqrt{\Delta_j} > 0.$$

From the above inequalities, (2.24) follows. The necessary condition is proved. We are to show that (2.24) is also a sufficient condition for PJI. We first use the given data $\{\lambda_j\}_{j=1}^{n}, \{\mu_j\}_{j=1}^{n-1}$ and β to construct a periodic Jacobi matrix. We define

$$b_n = [\Sigma_{j=1}^{n-1}(\frac{|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} \pm \sqrt{\Delta_j}}{2|x^{'}(\mu_j)|})]^{\frac{1}{2}}, \tag{3.22}$$

and

$$s_{1j} = (\frac{|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} \pm \sqrt{\Delta_j}}{2b_n^2|x^{'}(\mu_j)|})^{\frac{1}{2}}, j = 1, .., n-1 \tag{3.23}$$

where Δ_j is defined in (3.21) and for each j, there are two choices for the sign of $\sqrt{\Delta_j}$. From (2.24), we see that

$$\Delta_j = \Pi_{i=1}^{n}|\mu_j - \lambda_i|^2 - 4\beta(-1)^{n-j-1}\Pi_{i=1}^{n}|\mu_j - \lambda_i| \geq 0$$

and

$$|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} - \sqrt{\Delta_j} > |x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} - \sqrt{\Delta_j + 4\beta^2} = 0.$$

Under condition (2.24), we know that $n-1$ terms of the sum of b_n and s_{1j}^2 are greater than 0 whenever the sign $\pm$ are chosen to be $+$ or $-$. Consequently, we observe that the signs $\pm$ in b_n can be chosen arbitrary, but we can choose the sign in s_{1j} the same way as the sign in b_n for the same value of j. When the sign $\pm$ in 3.23 is chosen to be positive, we denote s_{1j} by s_{1j}^+ and we denote s_{1j} by s_{1j}^- when the sign is negative. If we have a choice for each sign of $\sqrt{\Delta_j}$, then we can obtain a b_n from (3.22), and let $g = (g_1, ..., g_{n-1})^T$ be a $(n-1) \times 1$ vector whose jth component g_j is equal to s_{1j}^+ or s_{1j}^- determined by b_n. It has been established that by $\mu_1, .., \mu_{n-1}$ and g, we can construct a matrix J_{n-1} in a unique manner, (Boley and Golub 1987, Parlett B. 1980).We then compute

$$b_{n-1} = \frac{\beta}{b_1 b_2 ... b_{n-2} b_n} \tag{3.24}$$

We have

$$a_n = \Sigma_{i=1}^{n}\lambda_i - \Sigma_{i=1}^{n-1}\mu_i \tag{3.25}$$

This completes the reconstruction of the matrix J_n.

Next, we are to show that the reconstruction matrix J_n is a solution to the PJI. It is therefore sufficient to prove that $\{\lambda_j\}_{j=1}^{n}$ are the eigenvalues of J_n. We can also as well assume that g_i be s_{1j}^{+}. From (3.19), (3.22) and (3.23),

$$
\begin{aligned}
(b_n s_{1j} + b_{n-1}s_{n-1,j})^2 &= (b_n s_{1j})^2 + 2b_n b_{n-1} s_{1j} s_{n-1,j} + (b_{n-1}s_{n-1,j})^2 \\
&= b_n^2 s_{1j}^2 + \frac{2\beta}{x'(\mu_j)} + \frac{\beta^2}{[x'(\mu_j)]^2 b_n^2 s_{1j}^2} \\
&= \frac{[x'(\mu_j)]^2 b_n^4 s_{1j}^4 + 2\beta x'(\mu_j) b_n^2 s_{1j}^2 + \beta^2}{[x'(\mu_j)]^2 b_n^2 s_{1j}^2} \\
&= \frac{[|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j}]^2}{2|x'(\mu_j)|[|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j}} \\
&+ \frac{4\beta(-1)^{n-j-1}[|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j}] + 4\beta^2}{2|x'(\mu_j)|[|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j}]} \\
&= \frac{|x'(\mu_j)x_j|^2 + \Delta_j + 2|x'(\mu_j)x_j|\sqrt{\Delta_j}}{2|x'(\mu_j)|[|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j}]} \\
&= \frac{2|x'(\mu_j)x_j|^2 - 4\beta(-1)^{n-j-1}|x'(\mu_j)x_j| + 2|x'(\mu_j)x_j|\sqrt{\Delta_j}}{2|x'(\mu_j)|[|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j}]} \\
&= x_j
\end{aligned}
$$

We therefore have,

$$\lambda_i - a_n - \Sigma_{k=1}^{n-1} \frac{(b_n s_{1k} + b_{n-1}s_{n-1,k})^2}{\lambda_i - \mu_k} = 0, i = 1, 2, ..., n$$

which in agreement with Theorem 2.3, and we know that

$$det(\lambda_i I - J_n) = 0, i = 1, 2, .., n,$$

we therefore conclude that the reconstruction matrix J_n is a solution to PJI.

We note that different choices of the signs of $\sqrt{\Delta_j}$ in b_n will consequently, give rise to different vectors g and thus different J_{n-1}. As the choices of the signs in b_n are more than 2^{n-1}, the number of the constructed periodic Jacobi matrices J_n is at most 2^{n-1}. $\square$

Theorem 2.6 covers the case that the matrices J_n and J_{n-1} have common eigenvalues.

Proof. We prove Theorem 2.6. For simplicity, suppose $m = 1$, that is J_n and J_{n-1} have common eigenvalues $\lambda_p = \mu_p$. By theorem 2.2, we have

$$\begin{aligned} x_p &= (b_n s_{1p} + b_{n-1} s_{n-1,p})^2 \\ &= 0 \end{aligned} \tag{3.26}$$

Combining this with (3.19),

$$s_{1p}^2 = \frac{(-1)^{n-p}\beta}{b_n^2 |X'(\mu_p)|} \tag{3.27}$$

and as $s_{1p}^2 > 0$, it follows that

$$(-1)^{n-p-1} < 0 \tag{3.28}$$

In fact, (3.32) is equivalent to

$$0 = \Pi_{i=1}^{n} |\mu_p - \lambda_i| \geq 4\beta(-1)^{n-p-1}. \tag{3.29}$$

Consider equation (3.16), for $i = p$, since $x_p = 0$, the equation can be rewritten as

$$\Sigma_{k=1}^{p-1} \frac{x_k}{\lambda_i - \mu_k} + \Sigma_{k=p+1}^{n-1} \frac{x_k}{\lambda_i - \mu_k} = \lambda_i - a_n, i = 1, 2, .., p-1, p+1, ..., n. \tag{3.30}$$

This system is a special case of equation 3.12 with $n-1$ equations and $n-2$ unknowns. The system has a unique solution in accordance with lemma 3.1,

$$x = (x_1, .., x_{p-1}, x_{p+1}, .., x_{n-1})^T,$$

$$x_j = -\Pi_{i=1, i\neq p}^{n} (\lambda_i - \mu_j) \Pi_{i=1, i\neq p,j}^{n-1} (\mu_i - \mu_j)^{-1} > 0,$$

$j = 1, 2, .., p-1, p+1, .., n-1$. For $j \neq p$, we observe that, $\lambda_p - \mu_j = \mu_p - \mu_j \neq 0$ yields the following:

$$x_j = -\Pi_{i=1}^{n} (\lambda_i - \mu_j) \Pi_{i=1, i\neq j}^{n-1} (\mu_i - \mu_j)^{-1} > 0, j = 1, 2, .., p-1, p+1, .., n-1 \tag{3.31}$$

By Theorem 2.4, we have, $x_j = (b_n s_{1j} + b_{n-1}s_{n-1,j})^2, j = 1, .., p-1, p+1, .., n-1$. As discussed in theorem 2.5 and combining (3.29), we see that (2.24) holds. The necessity condition is therefore completed. We use the data given to reconstruct a periodic Jacobi matrix. Computing b_n from the expression below

$$b_n = [\Sigma_{j=1, j\neq p}^{n-1}(\frac{|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} \pm \sqrt{\Delta_j}}{2|x'(\mu_j)|}) + \frac{\beta}{|x'(\mu_p)|}]^{\frac{1}{2}}. \quad (3.32)$$

Let $g = (g_1, ..., g_{n-1})^T$, where for $j = p$,

$$g_p = s_{1p} = (\frac{\beta}{b_n^2|x'(\mu_p)|})^{\frac{1}{2}} \quad (3.33)$$

for $j = 1, 2, .., p-1, p+1, .., n-1$

$$g_j = s_{1j} = (\frac{|x'(\mu_j)x_j| - 2\beta(-1)^{n-j-1} \pm \sqrt{\Delta_j}}{2b_n^2|x'(\mu_j)|})^{\frac{1}{2}}. \quad (3.34)$$

When we consider equation (2.24), we can easily see that $n-2$ terms of b_n and s_{1j} are always greater than 0 whenever the signs $\pm$ are chosen to be $+$ or $-$. If $j \neq p$, for the same value of j, the sign in the definition of b_n and g_j should be chosen the same. Analogous to the construction in theorem 2.5, the matrix J_n can be determined completely.

We now show that the matrix J_n is a solution to PJI. As μ_p is an eigenvalue common to the matrices J_n and J_{n-1}, then from equation (2.24), we have $(-1)^{n-p-1} \leq 0$. From (3.19) and (3.33) we have,

$$\begin{aligned} b_n s_{1p} + b_{n-1}s_{n-1,p} &= b_n(\frac{\beta}{b_n^2|x'(\mu_p)|})^{\frac{1}{2}} + \frac{\beta}{b_n x'(\mu_j)s_{1p}} \\ &= (\frac{\beta}{|x'(\mu_p)|})^{\frac{1}{2}} + (-1)^{n-p-1}(\frac{\beta}{|x'(\mu_p)|})^{\frac{1}{2}} \\ &= 0 \end{aligned} \quad (3.35)$$

Notwithstanding, if $j \neq p$ and supposing g_j is the same as s_{1j}^+ then from (3.19), (3.32) and

(3.34), we have

$$
\begin{aligned}
(b_n s_{1j} + b_{n-1}s_{n-1,j})^2 &= b_n^2 s_{1j}^2 + \frac{2\beta}{x^{'}(\mu_j)} + \frac{\beta^2}{[x^{'}(\mu_j)]^2 b_n^2 s_{1j}^2} \\
&= \frac{2|x^{'}(\mu_j)x_j|^2 - 4\beta(-1)^{n-j-1}|x^{'}(\mu_j)x_j| + 2|x^{'}(\mu_j)|\sqrt{\Delta_j}}{2|x^{'}(\mu_j)|[|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} + \sqrt{\Delta_j}]} \\
&= x_j \\
&= -\Pi_{i=1,i\neq p}^{n}(\lambda_i - \mu_j)(\lambda_p - \mu_j)\Pi_{i=1,i\neq p,j}^{n-1}(\mu_i - \mu_j)^{-1}(\mu_p - \mu_j)^{-1} \\
&= -\Pi_{i=1,i\neq p}^{n}(\lambda_i - \mu_j)\Pi_{i=1,i\neq p,j}^{n-1}(\mu_i - \mu_j)^{-1}.
\end{aligned}
$$

Therefore, for $i = 1, 2, .., p-1, p+1, .., n$,

$$
\lambda_i - a_n - \Sigma_{k=1}^{p-1}\frac{(b_n s_{1k} + b_{n-1}s_{n-1,k})^2}{\lambda_i - \mu_k} - \Sigma_{k=p+1}^{n-1}\frac{(b_n s_{1k} + b_{n-1}s_{n-1,k})^2}{\lambda_i - \mu_k} = 0,
$$

then

$$
det(\lambda_i I - J_n) = 0, i = 1, .., p-1, p+1, .., n.
$$

We can therefore conclude that the matrix J_n is a solution to the problem PJI. The choices of the signs in b_n are no more than 2^{n-2}, so there are at most 2^{n-2} different solutions. $\square$

We want to summarize the above discussions by assuming that $\mu_j j_i \in \{1, 2, .., n-1\}$ are elements of the sets λ and μ and letting q be a subscript set of μ_j. If the solution to PJI exists, the following procedure can be used to construct the solution. Compute

$$
b_n = [\Sigma_{j=1,j\neq q}^{n-1}(\frac{|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} \pm \sqrt{\Delta_j}}{2|x^{'}(\mu_j)|}) + \Sigma_{j=1,j\neq q}^{n-1}\frac{\beta}{|x^{'}(\mu_j)|}]^{\frac{1}{2}}. \tag{3.36}
$$

where $|X^{'}(\mu_j)x_j| = \Pi_{i=1}^{n}|\mu_j - \lambda_i|, |x^{'}(\mu_j)| = \mu\Pi_{i=1,i\neq q}^{n-1}|\mu_j - \mu_i|$. We can pick a $(n-1)\times 1$ vector $g = (g_j)$, where

$$
g_j = s_{1j} = (\frac{\beta}{b_n^2|x^{'}(\mu_j)|})^{\frac{1}{2}}, j \in q. \tag{3.37}
$$

$$
g_j = s_{1j} = (\frac{|x^{'}(\mu_j)x_j| - 2\beta(-1)^{n-j-1} \pm \sqrt{\Delta_j}}{2b_n^2|x^{'}(\mu_j)|})^{\frac{1}{2}}, j \notin q. \tag{3.38}
$$

Finally, we state some theorems and their proofs of the solvability and computability of the $(PIEP)$. Let $A_n \in H_n, n = 1, 2, .., n$ be the set of all $n \times n$ Hermitian matrices. We are to find a sufficient condition under which the following problem is solvable. Let $A = [a_{ij}] \in H_n$ be given, and $\lambda = (\lambda_1, .., \lambda_n)$ be a given vector in R^n. We are to find $c = (c_1, .., c_n) \in R^n$ such that the matrix

$$A(c) = A + \Sigma_{i=1}^{n} c_i A_i \tag{3.39}$$

has eigenvalues $\lambda_1, .., \lambda_n$.

The following are the definitions and terms we are going to use in the lemmas and theorems below.

$$\begin{aligned}
a &= (a_1, a_2, .., a_n) = (a_{11}, a_{22}, .., a_{nn}), \\
A^{(0)} &= A - diag(a_1, a_2, ..., a_n) \\
A_t^{(0)} &= A_t - diag(a_{11}^{(t)}, a_{22}^{(t)}, ..., a_{nn}^{(t)}, \\
\tilde{A} &= A^{(0)} - \Sigma_{t=1}^{n} a_t A_t^{(0)}.
\end{aligned}$$

For $1 \leq k < j \leq n$ denote

$$\begin{aligned}
\overline{\lambda}_{k,j} &= \lambda_{max}(\tilde{A}[k, k+1, .., j]), \\
\underline{\lambda}_{k,j} &= \lambda_{min}(\tilde{A}[k, k+!, .., j]), \\
\overline{\lambda}_{k,j}^{(t)} &= \lambda_{max}(A_t^{(0)}[k, k+1, .., j]), \\
\underline{\lambda}_{k,j}^{(t)} &= \lambda_{min}(A_t^{(0)}[k, k+1, .., j]), \\
\overline{\omega}_k^{(0)} &= \overline{\lambda}_{1,k+1}^{(t)} - \underline{\lambda}_{k,n}^{(t)}, \\
\underline{\omega}_k^{(t)} &= \underline{\lambda}_{1,k+1}^{(t)} - \overline{\lambda}_{k,n}^{(t)}, \\
r_{k,j} &= g(\tilde{A}[k, k+1, .., j]), g_{k,j}^{(t)} = g(A_t^{(0)}[k, k+1, .., j]), \\
r_k^{(t)} &= g_{1,k+1}^{(t)} + g_{k,n}^{(t)}.
\end{aligned}$$

Since $tr\tilde{A} = trA_t^{(0)}[k, k+1, .., j] = 0$, then

$$\underline{\lambda}_{k,j} \leq 0 \leq \overline{\lambda}_{k,j}$$

$$\underline{\lambda}_{k,j}^{t} \leq 0 \leq \overline{\lambda}_{k,j}^{(t)},$$

$$\underline{\omega}_k^{(t)} \leq 0 \leq \overline{\omega}_k^{(t)}(t = 1, 2, ..n)$$

Before we state any theorem for the solvability of the above problem, we need the following Lemmas provided in (Horn and Johnson, 1985).

Lemma 3.0.3. *(Cauchy-Poincare). Let $A \in H_n$. Then*

$\lambda_k(A) \leq \lambda_k(A[i_1, .., i_m]) \leq \lambda_{n-m+k}(A)$

for each $k = 1, .., m$, where $m \leq n$. As a result, we have

$\lambda_{min}(A[k, k+1, .., n]) \leq \lambda_k(A) \leq \lambda_{max}(A[1, 2, .., k])$

for each $k = 1, 2, .., n$.

Lemma 3.0.4. *(Weyl). Let $A, B \in H_n$. Then*

$$\lambda_k(A) + \lambda_{min}(B) \leq \lambda_k(A+B)\lambda \leq \lambda_k(A) + \lambda_{max}(B) \tag{3.40}$$

for each $k = 1, 2, .., n$.

For real vectors u and v, "$u > v$" means that u majorizes v.

Lemma 3.0.5. *(Schur). Let $A = (a_{ij}) \in H_n$. Then*

$$(a_{11}, a_{22}, .., a_{nn}) > (\lambda_1(A), \lambda_2(A), .., \lambda_n(A)). \tag{3.41}$$

Lemma 3.0.6. *. Let $\{d_i\}_{i=1}^n, \{y_i\}_{i=1}^n$ be real numbers and $\Sigma_{i=1}^k y_i \leq \Sigma_{i=1}^k d_i (k = 1, 2, .., n)$. Then for $v_1 \geq, .., \geq v_n \geq 0$ one has the inequalities*

$$\Sigma_{i=1}^k y_i v_i \leq \Sigma_{i=1}^k d_i v_i, k = 1, 2, .., n. \tag{3.42}$$

Proof. : Writing the sums using summation by parts, we have

$$\begin{aligned} \Sigma_{i=1}^{k} y_i v_i &= \Sigma_{j=1}^{k-1} \Sigma_{i=1}^{j} y_i (v_j - v_{j+1}) + v_k \Sigma_{i=1}^{k} y_i \\ &\leq \Sigma_{j=1}^{k-1} (v_j - v_{j+1}) \Sigma_{i=1}^{j} d_i + v_k \Sigma_{i=1}^{k} d_i. \end{aligned}$$

The inequality (3.42) can be established easily. (See for example,Horn and Johnson 1991)

Similarly, if

$$\Sigma_{i=1}^{k} d_i \geq \Sigma_{i=1}^{k} y_i (k = 1, 2, .., n),$$

then for $v_1 \leq v_2 \leq, .., \leq v_n \leq 0$, we have

$\Sigma_{i=1}^{k} d_i v_i \leq \Sigma_{i=1}^{k} y_i v_i$, $k = 1, .., n$.

If $\Sigma_{i=k}^{n} d_i \leq \Sigma_{i=k}^{n} y_i (k = 1, 2, .., n)$, then for $0 \leq v_i \leq ... \leq v_n$, we have

$\Sigma_{i=k}^{n} d_i v_i \leq \Sigma_{i=k}^{n} y_i v_i$, $k = 1, 2, .., n$. □

Lemma 3.0.7. *Let $\{b^{[t]}\}_{t=1}^{n}$ be the increasing arrangement of the real numbers $\{b_t\}_{t=}^{n}$. Then one has the inequalities*

$\Sigma_{t=1}^{m} c_t b^{[n-t+1]} \leq \Sigma_{t=1}^{m} c_t b_t \leq \Sigma_{t=1}^{m} c_t b^{[t]}$,

$\Sigma_{t=m+1}^{n} c_t b_t \leq \Sigma_{t=m+1}^{n} c_t b^{[t]}$

for real numbers

$$c_1 \leq .. \leq c_m < 0 \leq c_{m+1} \leq .. \leq c_n$$

.

Proof. :

In reference to the first inequality in this lemma, we set $d_i = b_i$ and $y_i = b^{[i]}(i = 1, 2, .., n)$, while $v_i = c_i$ for $i > m$ and $v_i = 0$ for $i \leq m$. We then get the second inequality. The proof of the first inequality is similar. □

Lemma 3.0.8. *Let $A = X + B$, where $X = diag(x_1, .., x_n)$ with $x_1 \leq ... \leq x_n$ and*

$B = b_{ij} \in H_n$ *with* $b_{ii} = 0 (i = 1, 2, .., n)$. *Then*

$$\begin{aligned} \lambda_{k+1}(A) - \lambda_k(A) &\leq (x_{k+1} - x_k) + \lambda_{max}(B[1, 2, .., k+1]) \\ &- \lambda_{min}(B[k, k+1, .., n]) \end{aligned}$$

Proof. :

We have to note that $A[k, k+1, .., k+m] = diag(x_k, x_{k+1}, .., x_{k+m}) + B[k, k+1, .., k+m]$. Lemmas (3.7) and (3.8) imply that

$$\begin{aligned} \lambda_{k+1}(A) &\leq \lambda_{max}(A[1, 2, .., k+1]) \\ &\leq x_{k+1} + \lambda_{max}(B[1, 2, .., k+1]) \end{aligned}$$

and

$$\begin{aligned} \lambda_k(A) &\geq \lambda_{min}(A[k, k+1, .., n]) \\ &\geq x_k + \lambda_{min}(B[k, k+1, .., n]) \end{aligned}$$

We see that the above lemma is easily established. □

Theorem 3.0.5. *Let* $a_{ii}^{(t)} = \delta_{it}(i, t = 1, .., n)$ *and* $\lambda_1 \leq \lambda_2 \leq .. \leq \lambda_s < 0 \leq ... \leq \lambda_{s+1} \leq .. \leq \lambda_n$. *Suppose*

$\lambda_{k+1} - \lambda_k \geq \bar{\lambda}_{1,k+1} - \underline{\lambda}_{k,n} + \Sigma_{t=1}^{s} \lambda_t \underline{\omega}_k^{[t]} + \Sigma_{t=s+1}^{n} \lambda_t \bar{\omega}_k^{[t]}$

for each $k = 1, 2, .., n-1$. *Then problem* (3.39) *is solvable.*

Proof. :(Adapted from Luoluo, 1995)

We assume that the assumptions in the statement of the above theorem hold. For the vectors λ and a define

$$\begin{aligned} D(\lambda, a) &= \{x = (x_1, .., x_n) \in R^n : x + a \\ &> \lambda, x_1 + a_1 \leq x_2 + a_2 \leq .. \leq x_n + a_n\} \end{aligned}$$

We can verify that $D(\lambda, a)$ is a nonempty, bounded, convex and closed set in R^n. For $x \in D(\lambda, a)$ define $A(x) = A + \Sigma_{t=1}^{n} x_t A_t$. Let $\lambda_i(x)$ be the ith smallest eigenvalue of $A(x)$. Let us assume that j_x is the index satisfying $1 \leq j_x \leq n$ and

$$\begin{aligned} x_1 + a_1 &\leq x_2 + a_2 \leq ... \leq x_{j_x} + a_{j_x} < 0 \\ &\leq x_{j_x+1} + a_{j_x+1} \leq ... \leq x_n + a_n \end{aligned}$$

Since $A(x) = diag(x_1 + a_1, x_2 + a_2, .., x_n + a_n) + \tilde{A} + \Sigma_{t=1}^{n}(x_t + a_t)A_t^{(0)}$, then by Lemma (3.8) we have

$$\begin{aligned} \lambda_{k+1}(x) - \lambda_k(x) &- (x_{k+1} + a_{k+1}) + (x_k + a_k) \\ &\leq \lambda_{max}((\tilde{A} + \Sigma_{t=1}^{n}(x_t + a_t)A_t^{(0)})[1, .., k+1]) \\ &- \lambda_{min}((\tilde{A} + \Sigma_{t=1}^{n}(x_t + a_t)A_t^{(0)})[k, k+1, .., n]) \end{aligned}$$

Applying Lemma (3.4), we have

$$\begin{aligned} &\lambda_{max}((\tilde{A} + \Sigma_{t=1}^{n}(x_t + a_t)A_t^{(0)})[1, .., k+1]) && (3.43) \\ &\qquad \leq \overline{\lambda}_{1,k+1} + \Sigma_{t=1}^{n}(x_t + a_t)\underline{\lambda}_{1,k+1}^{(t)} \\ &\qquad + \Sigma_{t=j_x+1}^{n}(x_t + a_t)\overline{\lambda}_{1,k+1}^{(t)} \end{aligned}$$

and

$$\begin{aligned} &\lambda_{min}((\tilde{A} + \Sigma_{t=1}^{n}(x_t + a_t)A_t^{(0)})[k, k+1, .., n]) && (3.44) \\ &\qquad \geq \underline{\lambda}_{k,n} + \Sigma_{t=1}^{j_x}(x_t + a_t)\overline{\lambda}_{k,n}^{(t)} + \\ &\Sigma_{t=j_x+1}^{n}(x_t + a_t)\underline{\lambda}_{k,n}^{t}. && (3.45) \end{aligned}$$

Combining the two inequalities above, we have

$$\lambda_{k+1}(x) - \lambda_k(x) - (x_{k+1} + a_{k+1}) + (x_k + a_k) \tag{3.46}$$

$$\leq \overline{\lambda}_{1,k+1} - \underline{\lambda}_{k,n}$$

$$+\Sigma_{t=1}^{j_x}(x_t + a_t)(\underline{\lambda}_{1,k+1}^{(t)} - \overline{\lambda}_{k,n}^{t} \tag{3.47}$$

$$+ \Sigma_{t=j_x+1}^{n}(x_t + a_t)(\overline{\lambda}_{1,k+1}^{(t)} - \underline{\lambda}_{k,n}^{(t)}.$$

The following is the implication of the inequalities in lemma (3.6) and lemma (3.7)

$$\Sigma_{t=j_x+1}^{n}(x_t + a_t)(\overline{\lambda}_{1,k+1}^{(t)} - \underline{\lambda}_{k,n}^{(t)}) \tag{3.48}$$

$$= \Sigma_{t=j_x+1}^{n}(x_t + a_t)\overline{\omega}_k^{(t)}$$

$$\leq \Sigma_{t=j_x+1}^{n}(x_t + a_t)\overline{\omega}_k^{(t)}$$

$$\leq \Sigma_{t=j_x+1}^{n}\lambda_t\overline{\omega}_k^{(t)}$$

$$\leq \Sigma_{t=s+1}^{n}\lambda_t\overline{\omega}_k^{(t)}.$$

We have to note that the last inequality holds for either $j_x \geq s$ or $j_x < s$. Similarly, we note again that the inequalities in lemmas (3.6) and (3.7) above imply that

$$\Sigma_{t=1}^{J_x}(x_t + a_t)(\underline{\lambda}_{1,k+1}^{(t)} - \overline{\lambda}_{k,n}^{(t)}) \tag{3.49}$$

$$\leq \Sigma_{t=1}^{j_x}(x_t + a_t)\underline{\omega}_k^{(t)}$$

$$\leq \Sigma_{t=1}^{j_x}\lambda_t\underline{\omega}_k^{(t)}$$

$$\leq \Sigma_{t=1}^{s}\lambda_t\underline{\omega}_k^{(t)}.$$

We therefore conclude from the inequalities above that there hold for $k = 1, 2, .., n-$

1 the inequalities

$$\lambda_{k+1}(x) - \lambda_k(x) - (x_{k+1} + a_{k+1}) + (x_k + a_k)$$
$$\leq \overline{\lambda}_{1,k+1} - \underline{\lambda}_{k,n} + \Sigma_{t=1}^{s}\lambda_t\underline{\omega}_k^{(t)}$$
$$+ \Sigma_{t=s+1}^{n}\lambda_t\overline{\omega}_k^{(t)}$$

We define the continuous map $f : D(\lambda, a) \to R^n, f(x) = \lambda + x - \lambda(x)$, where $\lambda(x) = (\lambda_1(x), \lambda_2(x), .., \lambda_n(x))$. By the assumption of the above Theorem (3.5) and the above inequality, we can verify that for each $k = 1, 2, .., n-1$

$$f_k(x) + a_k \leq f_{k+1}(x) + a_{k+1}, \tag{3.50}$$

where $f_k(x)$ is the kth component of $f(x)$. In addition to this, since $\{x_t + a_t\}_{t=1}^{n}$ are the diagonal entries of $A(x)$, then by Lemma (3.5), we have $x + a > \lambda(x)$ and therefore

$$f(x) + a = x + a - \lambda(x) + \lambda > \lambda. \tag{3.51}$$

The two equations above mean that $f(x) \in D(\lambda, a)$. By Brouwer's fixed-point theorem we know that there exists a vector $c = (c_1, c_2, .., c_n \in D(\lambda, a)$ such that $f(c) = c$, that is $\lambda(c) = \lambda$. In other words, the matrix $A(c) = A + \Sigma_{t=1}^{n} c_t A_t$ has eigenvalues $\lambda_1, \lambda_2, .., \lambda_n$ and hence problem (3.39) is solvable. □

Theorem 3.0.6. *Let* $a_{ii}^{(t)} = \delta_{it}(i, t = 1, 2, .., n)$ *and* $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_s < 0 \leq ... \leq \lambda_{s+1} \leq ... \leq \lambda_n$. *Suppose*

$$\lambda_{k+1} - \lambda_k \tag{3.52}$$
$$\geq r_{1,k+1} + r_{k,n} + \Sigma_{t=1}^{s}|\lambda_t|r_k^{[n-t+1]}$$
$$+ \Sigma_{t=s+1}^{n}\lambda_t r_k^{[t]}$$

for each $k = 1, 2, .., n-1$. *The problem* (3.39) *is solvable.*

Proof. : (Luoluo, 1995).

The proof of this theorem is similar to the above proved theorem. We only have to replace two inequalities in theorem (3.5) by

$$
\begin{aligned}
\lambda_{k+1}(x)-\lambda_k(x)-(x_{k+1}+a_{k+1})+(x_k+a_k) &\leq \\
\lambda_{max}((\tilde{A}+\Sigma_{t=1}^{n}(x_t+a_t)A_t^{(0)})[1,..,k+1]) & \qquad (3.53) \\
&- \lambda_{min}((\tilde{A}+\Sigma_{t=1}^{n}(x_t+a_t)A_t^{(0)})[k,..,n]) \\
&\leq r_{1,k+1}+r_{k,n}-\Sigma_{t=1}^{j_x}(x_t+a_t)(g_{1,k+1}^{(t)}+g_{k,n}^{(t)}) \\
&+ \Sigma_{t=j_x+1}^{n}(x_t+a_t)(g_{1,k+1}^{(t)}+g_{k,n}^{(t)}) \\
&\leq r_{1,k+1}+r_{k,n}-\Sigma_{t=1}^{j_x}(x_t+a_t)r_k^{[n-t+1]} \\
&+ \Sigma_{t=j_x+1}^{n}(x_t+a_t)r_k^{(t)} \\
&\leq r_{1,k+1}+r_{k,n}-\Sigma_{t=1}^{j_x}\lambda_t r_k^{[n-t+1]} \\
&+ \Sigma_{t=j_x+1}^{n}\lambda_t r_k^{(t)} \qquad (3.54) \\
&\leq r_{1,k+1}+r_{k,n}-\Sigma_{t=1}^{s}\lambda_t r_k^{[n-t+1]} \\
&+ \Sigma_{t=s+1}^{n}\lambda_t r_k^{(t)} \qquad (3.55) \\
&= r_{1,k+1}+r_{k,n}+\Sigma_{t=1}^{s}|\lambda_t| r_k^{[n-t+1]} \\
&+ \Sigma_{t=s+1}^{n}\lambda_t r_k^{(t)}, \qquad (3.56)
\end{aligned}
$$

the second inequality comes from the Gerschgorin disc theorem, the third comes from lemma (3.7), and the fourth comes from lemma (3.6). The details are omitted. □

CHAPTER 4

IEP for a class of Singular Hermitian Matrices

In this chapter, we consider the solution of the IEP for a class of Hermitian matrices. We restrict ourselves to singular symmetric matrices. The IEP has been more famous by Mack Kac's 1966 article "Can I hear the shape of a drum?" Literally it is difficult, if not impossible to determine the shape of a drum from its vibration modes. The problem becomes solvable only when certain parameters are prescribed. This includes prescribing that the matrix is tridiagonal. This case has been extensively investigated already. The case for singular symmetric matrices, however, have received little to no attention. We therefore discuss the inverse eigenvalue problem for singular Hermitian matrices. We will consider full singular symmetric matrices. Hermitian matrices are endowed with real eigenvalues. This makes the problem more tractable. We therefore want to construct singular $n \times n$ symmetric matrices of this form from their eigenvalues. We therefore develop and prove an algorithm using matrix invariants for solving direct eigenvalue problem for singular symmetric matrices.

For an $n \times n$ symmetric matrix, the following properties hold:

1. The matrices are singular and symmetric
2. The eigenvalues are real and the number of distinct values is the same as the rank.
3. The matrices concerned have arbitrary non zero elements for $r > 1$.

Let M and N denote the subset of square matrices where $M \in Hn$, Hn is the set of Hermitian matrices. Our M in this case is the set of some singular symmetric matrices with real entries. N denotes a class of matrices which contains parameters. Given a matrix $A \in M$, scalar $\{\lambda_1, \lambda_2, ..., \lambda_n\} \in F$ and a class of matrices N, we are to find some parameters $X \in N$ such that $\sigma(XA) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. The IEP under the outlined conditions for 'large' matrices is generally unsolvable. As such we begin with the 2×2 singular symmetric matrices and extend to the highest possible $n \times n$ singular symmetric matrix.

4.1 Preliminaries

In what follows, we denote the $n \times n$ singular symmetric matrix (a_{ij}) such that $a_{ij} = a_{ji}, i, j = 1, ..., n$ by A_n. Without loss of generality, singularity is achieved by multiplying the first row by prescribed scalars. We denote a singular symmetric matrix of rank r by $A_{(n,r)}$. In this case we write $R_i = kR_{i+1}$ to denote that the ith row is k times the first row, where $k \in \mathbb{R}$. Finally, we denote the spectrum of A_n by $\Lambda_n = \{\lambda_1, ..., \lambda_n\}$. If the rank of A is r, then we assume $\lambda_i \neq 0$ for $i = 1, ..., r$, but $\lambda_i = 0, i = r+1, ..., n$.

Lemma 4.1.1. *Let A be a non-traceless, symmetric matrix of rank r with non-vanishing elements. Then there exits an isomorphism between the elements of A and its distinct non-zero eigenvalues if and only if $r = 1$.*

Corollary: The inverse eigenvalue problem has a unique solution for singular symmetric matrices of rank 1 with prescribed linear dependence relation.

4.1.1 Specific Case 1

(n=2, r=1): We begin by considering $A_{(2,1)}$. By definition, $A_{(2,1)}$ is of the form:

$$\begin{aligned} A_{(2,1)} &= \begin{pmatrix} a_{11} & ka_{11} \\ ka_{11} & k^2 a_{11} \end{pmatrix} \\ &= a_{11} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}. \end{aligned}$$

Let $\Lambda_2 = \{\lambda_1, \lambda_2\}$. Since $A_{(2,1)}$ is singular of rank 1, it follows that $\lambda_2 = 0$. We have: $Tr(A_{(2,1)}) = \lambda = a_{11}(1 + k^2)$. Therefore $a_{11} = \frac{\lambda}{1+k^2}$.

Hence:

$$A_{(2,1)} = \frac{\lambda}{1+k^2}\begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}.$$

Thus $A_{(2,1)}$ has been reconstructed for given λ and prescribed scalar k.

We see from this formula that for any given λ and parameter k, we can generate any 2×2 singular symmetric matrix of rank one. For example if $k = 2, \lambda = 5$, we have

$$A_{(2,1)} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

4.1.2 Extension to Hermitian matrices

We now extend the above to Hermitian matrices of order 2×2. A is Hermitian implies that, $a_{21} = \bar{a_{12}}$. Linear dependence of rows is given by $a_{21} = ka_{11}$ and $a_{22} = ka_{12}$, so that $a_{21} = \bar{a_{12}} = \bar{k}a_{11}$. Then $a_{22} = k(\bar{k}a_{11}) = |k|^2 a_{11}$. (Note that the diagonal elements of Hermitian matrix are real.) We now rewrite the matrix as;

$$A_{(2,1)} = \begin{pmatrix} a_{11} & \bar{k}a_{11} \\ ka_{11} & |k|^2 a_{11} \end{pmatrix}$$

$$= a_{11}\begin{pmatrix} 1 & \bar{k} \\ k & |k|^2 \end{pmatrix}$$

Thus $TrA = \lambda = a_{11}(1 + |k|^2) \Longrightarrow a_{11} = \frac{\lambda}{1+|k|^2}$. From this, we see that any 2×2 Hermitian matrix which has a parameter with the same value as the modulus k satisfies the above formula. Example: Let $k = 1 + i, \lambda = 5$ and $\bar{k} = 1 - i$. We have $a_{11} = \frac{5}{3}$ and

$$A_{(2,1)} = \frac{5}{3}\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}$$

4.2 New Results

We generalize the method above in the following two theorems, first for an $n \times n$ singular symmetric matrix of rank 1 and then of rank r, where $1 \leq r < n$.

Theorem 4.2.1. *Given the spectrum and the row multipliers $k_i, i = 1, ..., n-1$, the inverse eigenvalue problem for a $n \times n$ singular symmetric matrix of rank 1 is solvable.*

Proof. Given the spectrum $\Lambda_n = \{\lambda_1, \lambda_2, ..., \lambda_n\}$, since rank $A_n = 1$, it follows from our notation above that$\lambda_1 \neq 0$ and $\lambda_i = 0, i = 2, ..., n$. Let $k_i, i = 1, ..., k_{n-1}$ be the row multiples. Letting

$$A_{(n,1)} = \begin{pmatrix} a_{11} & a_{11}k_1 & a_{11}k_1k_2 & \cdots & a_{11}k_1k_2\cdots k_{n-1} \\ a_{11}k_1 & a_{11}k_1^2 & a_{11}k_1^2k_2 & \cdots & a_{11}k_1^2k_2\cdots k_{n-1} \\ a_{11}k_1k_2 & a_{11}k_1^2k_2 & a_{11}k_1^2k_2^2k_3 & \cdots & a_{11}k_1^2k_2^2k_3\cdots k_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{11}k_1k_2\cdots k_{i-1} & a_{11}k_1^2k_2\cdots k_{i-1} & a_{11}k_1^2k_2^2k_3\cdots k_{i-1} & \cdots & a_{11}k_1^2k_2^2k_3^2\cdots k_{n-1}^2 \end{pmatrix}$$

Then $Tr(A) = \lambda = a_{11}(1 + k_1^2 + k_1^2k_2^2 + k_1^2k_2^2k_3^2 + \cdots + k_1^2k_2^2\cdots k_{n-1}^2)$. Hence

$$a_{11} = \frac{\lambda}{1 + k_1^2 + k_1^2k_2^2 + k_1^2k_2^2k_3^2 + \cdots + k_1^2k_2^2\cdots k_{n-1}^2}.$$

The result follows by induction on n. $\square$

We state the following theorem for the general case where A_n has rank r:

Theorem 4.2.2. *The inverse eigenvalue problem for an $n \times n$ singular symmetric matrix of rank r is solvable provided that $n - r$ arbitrary parameters are prescribed.*

Proof. Let $A_{(n,r)} = [a_{ij}]$, where $n \geq 2$. It is obvious that,

$$a_{ij} = a_{11} \left\{ \begin{array}{cc} \prod_{s=0}^{j-1} k_s, & i < j \\ (\prod_{s=0}^{i-1} k_s)^2, & i = j \\ \prod_{s=0}^{i-1} k_s, & i > j \end{array} \right\}$$

where

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + k_1^2 k_2^2 k_3^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-r}^2}$$

and

$$\begin{aligned} a_{ij} &= \lambda_2, \quad \text{where} \quad i = j = n - r + 2 \\ \vdots &= \vdots \\ a_{nn} &= \lambda_r \end{aligned} \tag{4.1}$$

$\lambda_1, \lambda_2 ...... \lambda_r \in \mathbb{R}$ such that $R_i = k_{i-1} R_1$ where $k_i \in R, i = 2, ...n - r + 1$ and R_i is the ith row of A.

$Tr(A) = 1 + \sum_{s=1}^{n} (\prod_{s=1}^{i-1} k_s)^2$. We see that for $r = 1$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + k_1^2 k_2^2 k_3^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-1}^2}.$$

$r = 2$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + k_1^2 k_2^2 k_3^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-2}^2}.$$

$r = 3$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + k_1^2 k_2^2 k_3^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-3}^2}.$$

$r = 4$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + k_1^2 k_2^2 k_3^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-4}^2}.$$

The result follows for any rank $1 < r < n$. $\square$

4.3 Numerical Examples

In this section, we illustrate the results above with small matrices of size $3 \leq n \leq 5$ of rank $1 \leq r \leq 4$. We begin with the singular 3×3 symmetric matrix of rank 1, $A_{(3,1)}$, which is of the form:

$$A_{(3,1)} = a_{11} \begin{pmatrix} 1 & k_1 & k_1 k_2 \\ k_1 & k_1^2 & k_1^2 k_2 \\ k_1 k_2 & k_1^2 k_2 & k_1^2 k_2^2 \end{pmatrix}$$

$a_{11} = \frac{\lambda}{1+k_1^2+k_1^2 k_2^2}$

For $k_1 = -1, k_2 = 2, \lambda = 6$, we have;

$$A_{(3,1)} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$$

We extend the above result to Hermitian matrix of order 3×3. By symmetry, we have

$a_{12} = \bar{a_{21}}, a_{13} = \bar{a_{31}}, a_{23} = \bar{a_{32}}$. Row dependency is of the form; $k_1 R_1 = R_2, k_2 R_2 = R_3 \Longrightarrow k_1 k_2 R_1 = R_3$. The off diagonal elements are;

$a_{12} = \bar{a_{21}} = \bar{k_1} a_{11}$

$a_{13} = \bar{a_{31}} = \bar{k_1}\bar{k_2} a_{11}$

$a_{23} = \bar{a_{32}} = \bar{k_1}\bar{k_2}\bar{a_{12}} = \bar{k_1}\bar{k_2} k_1 a_{11} = |k_1|^2 \bar{k_2} a_{11}$

The diagonal elements are;

$a_{22} = k_1 a_{12} = k_1(\bar{k_1}) = |k_1|^2 a_{11}$

$a_{33} = k_1 k_2 a_{13} = k_1 k_2 (\bar{k_1}\bar{k_2}) a_{11} = |k_1|^2 |k_2|^2 a_{11}$.

Thus the general Hermitian 3×3 matrix is presented below;

$$A_{(3,1)} = a_{11} \begin{pmatrix} 1 & \bar{k_1} & \bar{k_1}\bar{k_2} \\ k_1 & |k_1|^2 & |k_1|^2 \bar{k_2} \\ k_1 k_2 & |k_1|^2 k_2 & |k_1|^2 |k_2|^2 \end{pmatrix}$$

The trace $TrA = \lambda = (1 + |k_1|^2 + |k_1|^2|k_2|^2)a_{11} \Rightarrow a_{11} = \frac{\lambda}{1+|k_1|^2|k_2|^2}$.

Any parameter which has the same value as the modulus of k_1 and k_2, generates the same 3×3 Hermitian matrix.

Numerical Example: For $\lambda = 3, k_1 = 2i, \bar{k_1} = -2i, k_2 = 2+i, \bar{k_2} = 2-i$, we have $a_{11} = \frac{3}{10}$ and

$$A_{(3,1)} = \frac{3}{10}\begin{pmatrix} 1 & -2i & -(2+4i) \\ 2i & 4 & 8-4i \\ -2+4i & 8+4i & 20 \end{pmatrix}$$

$\underline{\mathbf{n = 4, r = 1}}$

Given k_1, k_2, and k_3, we obtain the following singular symmetric matrix:

$$A_{(4,1)} = a_{11}\begin{pmatrix} 1 & k_1 & k_1k_2 & k_1k_2k_3 \\ k_1 & k_1^2 & k_1^2k_2 & k_1^2k_2k_3 \\ k_1k_2 & k_1^2k_2 & k_1^2k_2^2 & k_1^2k_2^2k_3 \\ k_1k_2k_3 & k_1^2k_2k_3 & k_1^2k_2^2k_3 & k_1^2k_2^2k_3^2 \end{pmatrix}$$

In this case, $a_{11} = \frac{\lambda}{1+k_1^2+k_1^2k_2^2+k_1^2k_2^2k_3^2}$. When $\lambda = 2$, $k_1 = 3$, $k_2 = 2$ and $k_3 = 4$, we obtain the following 4×4 singular symmetric matrix of rank one:

$$A_{(4,1)} = \begin{pmatrix} \frac{1}{311} & \frac{3}{311} & \frac{6}{311} & \frac{24}{311} \\ \frac{3}{311} & \frac{9}{311} & \frac{18}{311} & \frac{72}{311} \\ \frac{6}{311} & \frac{18}{311} & \frac{36}{311} & \frac{144}{311} \\ \frac{24}{311} & \frac{72}{311} & \frac{144}{311} & \frac{576}{311} \end{pmatrix}$$

Finally, we present singular Hermitian matrix of order 4×4. By symmetry, we have: $a_{12} = \bar{a}_{21}, a_{13} = \bar{a}_{31}, a_{23} = \bar{a}_{32}, a_{14} = \bar{a}_{41}, a_{24} = \bar{a}_{42}$ and $a_{34} = \bar{a}_{43}$. Row dependence is given by the expressions;

$$\begin{aligned} k_1R_1 &= R_2 \\ k_2R_2 &= R_3 \qquad (4.2) \\ k_3R_3 &= R_4 \end{aligned}$$

$\Rightarrow k_1k_2R_1 = R_3, k_1k_2k_3R_1 = R_4$. The off-diagonal elements are;

$$\begin{aligned} a_{12} &= \bar{a}_{21} = \bar{k}_1 a_{11} \\ a_{13} &= \bar{a}_{31} = \bar{k}_1\bar{k}_2 a_{11} \\ a_{23} &= \bar{a}_{32} = |k_1|^2\bar{k}_2 \qquad (4.3) \\ a_{14} &= \bar{a}_{41} = \bar{k}_1\bar{k}_2\bar{k}_3 a_{11} \qquad (4.4) \\ a_{24} &= \bar{a}_{42} = |k_1|^2\bar{k}_2\bar{k}_3 a_{11} \qquad (4.5) \\ a_{34} &= \bar{a}_{43} = |k_1|^2|k_2|^2\bar{k}_3 a_{11} \end{aligned}$$

The following are the diagonal elements;

$$\begin{aligned} a_{22} &= k_1a_{12} &&= |k_1|^2 a_{11} \\ a_{33} &= k_1k_2a_{13} &&= |k_1|^2|k_2|^2 a_{11} \\ a_{44} &= k_1k_2k_3a_{14} = k_1k_2k_3(\bar{k}_1\bar{k}_2\bar{k}_3)a_{11} &&= |k_1|^2|k_2|^2|k_3|^2 a_{11} \end{aligned}$$

We obtain the following singular Hermitian matrix;

$$A_{(4,1)} = a_{11}\begin{pmatrix} 1 & \bar{k}_1 & \bar{k}_1\bar{k}_2 & \bar{k}_1\bar{k}_2\bar{k}_3 \\ k_1 & |k_1|^2 & |k_1|^2\bar{k}_2 & |k_1|^2\bar{k}_2\bar{k}_3 \\ k_1k_2 & |k_1|^2k_2 & |k_1|^2|k_2|^2 & |k_1|^2|k_2|^2\bar{k}_3 \\ k_1k_2k_3 & |k_1|^2k_2k_3 & |k_1|^2|k_2|^2k_3 & |k_1|^2|k_2|^2|k_3|^2 \end{pmatrix}$$

The trace, $TrA = \lambda = a_{11}(1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2)$. Rewriting the above equation, $a_{11} = \frac{\lambda}{1+|k_1|^2+|k_1|^2|k_2|^2+|k_1|^2|k_2|^2|k_3|^2}$.

Lemma 4.3.1. *Entries of singular Hermitian matrix $A = [a_{ij}]$ of $rank1$ with nonzero eigenvalue $\lambda \in R$ and which is such that $R_i = k_{i-1}R_{i-1}$ where $k_i \in C, i = 2, ...n$ and the R_i is the ith row of A can be generated from; $a_{11} = \frac{\lambda}{1+|k_1|^2+|k_1|^2|k_2|^2+....+|k_1|^2...|k_{n-1}|^2}$ and $a_{ij} = a_{11}(\Pi_{s=0}^{i-1} ks)$*

Singular Hermitian matrix of $rankr \geq 2$ is left out for future research work.

We now consider the IEP for $n \times n$ singular symmetric matrices of rank 2. $A_{(3,2)}$ is of the form:

$$A_{(3,2)} = \begin{pmatrix} a_{11} & ka_{11} & a_{13} \\ ka_{11} & k^2a_{11} & ka_{13} \\ a_{13} & ka_{13} & a_{33} \end{pmatrix}$$

Here, $Tr(A) = \lambda_1 + \lambda_2 = a_{11}(1 + k^2) + a_{33}$ and $\lambda_1\lambda_2 = a_{11}(1 + k^2)a_{33}$. Hence $a_{33} = \frac{\lambda_1\lambda_2}{a_{11}(1+k^2)}$. Thus

$$a_{11}^2(1 + k^2)^2 - a_{11}(1 + k^2)(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 0.$$

which yields $a_{11} = \frac{\lambda_1}{1+k^2}$ and $\lambda_2 = a_{33}$. Therefore a_{13} becomes a free variable. When $\lambda_1 = 2, \lambda_2 = 3, k = 4$ and $a_{13} = 5$, for example, we obtain the following singular symmetric matrix:

$$A_{(3,2)} = \begin{pmatrix} \frac{2}{17} & \frac{8}{17} & 5 \\ \frac{8}{17} & \frac{32}{17} & 20 \\ 5 & 20 & 3 \end{pmatrix}.$$

In general, the solution of the IEP for $A_{n,r}$ leads to the solution of an rth degree polynomial equation in a_{11} of the form:

$$\begin{aligned} 0 &= a_{11}^r(1+k_1^2+...+k_{n-r}^2) - (\sum_{i=1}^{r}\lambda_i)(1+k_1^2+...+k_{n-r}^2)a_{11}^{r-1} \\ &+ \sum_{k=1}^{r}(\Pi_{i=k}^{k+1}\lambda_i)(1+k_1^2+...+k_{n-r}^2)a_{11}^{r-2} \\ &- \sum_{k=1}^{r}(\Pi_{i=k}^{k+2}\lambda_i)(1+k_1^2+...+k_{n-r}^2)a_{11}^{r-3} + ... - (\Pi_{i=1}^{r}\lambda_i) \end{aligned} \tag{4.6}$$

To solve the case for $n = 4$ and $r = 2$, we deduce from the general polynomial equation above that the following quadratic in a_{11} holds:

$$a_{11}^2(1+k_1^2+k_1^2k_2^2)^2 - (\lambda_1+\lambda_2(1+k_1^2+k_1^2k_2^2)a_{11} + \lambda_1\lambda_2 = 0.$$

This yields: $a_{11} = \frac{\lambda_1}{1+k_1^2+k_1^2k_2^2}$, $a_{44} = \lambda_2$ and a_{14} becomes a free variable. For $\lambda_1 = 6, \lambda_2 = 5, k_1 = 4, k_2 = 3$ and $a_{14} = 2$, we obtain a singular symmetric matrix below:

$$A_{(4,2)} = \begin{pmatrix} \frac{6}{161} & \frac{24}{161} & \frac{72}{161} & 2 \\ \frac{24}{161} & \frac{96}{161} & \frac{288}{161} & 8 \\ \frac{72}{161} & \frac{288}{161} & \frac{864}{161} & 24 \\ 2 & 8 & 24 & 5 \end{pmatrix}$$

Similarly, for $A_{(5,2)}$, we obtain the following quadratic in a_{11}:

$$a_{11}^2(1+k_1^2+k_1^2k_2^2+k_1^2k_2^2k_3^2)^2 - (\lambda_1+\lambda_2)(1+k_1^2+k_1^2k_2^2+k_1^2k_2^2k_3^2)a_{11} + \lambda_1\lambda_2 = 0.$$

The solution gives: $a_{11} = \frac{\lambda_1}{1+k_1^2+k_1^2k_2^2+k_1^2k_2^2k_3^2}$ and $\lambda_2 = a_{55}$ where a_{15} is a free variable. When $\lambda_1 = 2, \lambda_2 = 5, k_1 = 3, k_2 = 5, k_3 = 7$ and $a_{15} = 4$, we obtain a singular symmetric matrix below:

$$A_{(5,2)} = \begin{pmatrix} \frac{2}{11260} & \frac{6}{11260} & \frac{30}{11260} & \frac{210}{11260} & 4 \\ \frac{6}{11260} & \frac{18}{11260} & \frac{90}{11260} & \frac{630}{11260} & 12 \\ \frac{30}{11260} & \frac{90}{11260} & \frac{450}{11260} & \frac{3150}{11260} & 60 \\ \frac{210}{11260} & \frac{630}{11260} & \frac{3150}{11260} & \frac{22050}{11260} & 420 \\ 4 & 12 & 60 & 420 & 5 \end{pmatrix}$$

By the same method, $A_{(4,3)}$ leads to the following cubic equation:

$$a_{11}^3(1+k^2)^3 - (\lambda_1+\lambda_2+\lambda_3)a_{11}^2(1+k^2)^2 + a_{11}(1+k^2)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3 = 0.$$

Solving the above cubic equation we obtain the following roots. $\lambda_1 = a_{11}(1+k^2) \Rightarrow a_{11} = \frac{\lambda_1}{1+k^2}$, $\lambda_2 = a_{33}$ and $\lambda_3 = a_{44}$, where a_{13}, a_{14} and a_{34} are free variables.

Finally, we want to consider 5×5 singular symmetric matrix of rank 4. Using equation (2). we obtain the following quartic equation in a_{11} where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the nonzero members of the spectrum.

$$\begin{aligned} a_{11}^4(1+k^2)^4 - (\lambda_1+\lambda_2+\lambda_3+\lambda_4)a_{11}^3(1+k^2)^3) &\quad + \\ (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3\lambda_2\lambda_4+\lambda_3\lambda_4)a_{11}^2(1+k^2)^2 &\quad - \\ (\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4)a_{11}(1+k^2)+\lambda_1\lambda_2\lambda_3\lambda_4 &= 0. \end{aligned}$$

Factoring the above quartic equation, we obtain the following results: $a_{11} = \frac{\lambda_1}{1+k^2}$, $\lambda_2 = a_{33}, \lambda_3 = a_{44}$ and $\lambda_4 = a_{55}$. The free variables are $a_{13}, a_{14}, a_{15}, a_{35}, a_{34}$ and a_{45}.

As an example, if we let $\lambda_2 = 4, \lambda_3 = 7, \lambda_4 = 11, k = 2, a_{13} = 8, a_{14} = -4, a_{15} =$

$5, a_{35} = 14, a_{34} = 6$ and $a_{45} = 17$

$$A = \begin{pmatrix} 1.3 & 2.7 & 8 & -4 & 5 \\ 2.6 & 5.4 & 16 & -8 & 10 \\ 8 & 16 & 4 & 6 & 14 \\ -4 & -8 & 6 & 7 & 17 \\ 5 & 10 & 14 & 17 & 12 \end{pmatrix}$$

CHAPTER 5

Numerical analytic interpretation of the IEP for Hermitian Matrices using fibre bundles with structure group $SO(n)$

In this chapter we give numerical analytic interpretation of the IEP using fibre bundles with structural group $SO(n)$. The previous work on IEP using the idea of tangent bundles was more theoretical. The initial matrix for the iteration was not known and therefore the special orthogonal matrix Q could not be determined and consequently the skew symmetric matrix K. Our work is more practicable and provide information for the initial matrix which in our case is a singular symmetric matrix. Given the initial matrix, we could obtain the matrices Q and K for the iterative process. Finally, we expect the iteration to converge to a matrix which is a nonsingular symmetric matrix such that its eigenvalues are in the neighbourhood of the eigenvalues of Λ, a diagonal matrix.

We consider the general linear group $GL(n, R)$ which is a Lie group. The general linear group is a group of all invertible and square matrices whose subgroup is the orthogonal group $O(n, R)$. Our Lie group under consideration is a group of matrices.

We will deal with two subspaces, the affine subspace which contains symmetric matrices and the isospectral surface $O(n, R)$ whose normal subgroup is $SO(n, R)$. The $S0(n, R)$ is a special orthogonal group. The normal subgroups we will deal with are $SO(2)$ and $SO(3)$ whose corresponding Lie algebras are $so(2)$ and $so(3)$ respectively, which are real $n \times n$ skew symmetric matrices with null trace. Let us denote the isospectral surface by $M(\Lambda)$ which contains the same spectrum.We know from Proposition 1.3 that a fibre from the affine subspace to a point $X \in M(\Lambda)$ which is a differentiable manifold is of the form $T(X) = XK - KX$. This is the Lie bracket of the Lie algebra $so(n)$.

Let

$$M(\Lambda) = \{Q\Lambda Q^t | Q \in SO(n)\}$$

where Λ contains the set of distinct eigenvalues. This implies $detQ = 1$ and $QQ^t = I_n$. If $Q(t)$ is a one-parameter subgroup of $SO(n)$ parameterized by t, then differentiating QQ^t with respect to t gives $Q^{'}(0) + Q^{'}(0)^t = 0$ and so the Lie algebra $so(n)$ consists of all skew-symmetric $n \times n$ matrix. We have

$$Q\Lambda Q^t = X \Rightarrow \Lambda = Q^t X Q$$

which is an inverse problem. We know that by Lemma 1.1, Q is a rotation matrix given by Rodrigues rotation formula below,

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

But by Lemma 1.1, we have

$$Q = e^K = I + K + \frac{K}{2!}...., = \Sigma_{p\geq 0}\frac{K^p}{p!}.$$

Ignoring highest powers of K, we have $Q = I + K$ and we obtain

$$K = Q - I = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

where θ is a real number between $[0, 2\pi]$.

Our function is therefore given by $f(x) = Q\Lambda Q^t - X = 0$ and the derivative is the Lie bracket $XK - KX$. We therefore formulate a Direct iterative method instead of the Newton Ralphson method. At the isospectral surface, $M(\Lambda)$, we have

$$Q(A^i)\Lambda Q^t(A^i) = X^{i+1}...(1), i = 1, 2...$$

where A^i is a singular symmetric matrix, Q the normalized eigenvectors of the matrix A^i and Λ a diagonal matrix which is similar to the matrix X^{i+1} and therefore, they have the same eigenvalues. At the tangent space of the Lie group which is the Lie Algebra, we obtain the following equation,

$$X^{i+1} + X^{i+1}K - KX^{i+1} = A^{i+1}...(2), i = 1, 2...$$

where K is a skew symmetric matrix which is given by, $K = \frac{1}{2}(Q - Q^t)$ or $K = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$, $0 \leq \theta \leq 2\pi$. At this point, the iteration continues with A^{i+1} being the initial matrix until the iteration converges at where the eigenvalues of A^{i+1} are in the same neighbourhood of the eigenvalues of Λ. At the convergence stage, the skew symmetric matrix is zero.

5.1 Numerical Example

In this section, we illustrate the results above with small matrix of order 2×2. We begin with an initial 2×2 singular symmetric matrix of the form;

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

with the eigenvalues as, $\lambda_1 = 5, \lambda_2 = 0$. The normalized eigenvectors are the column vectors of the matrix

$$Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

and

$$Q^t = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Let $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

Step 1: We use the equation; $Q^1(A^i)\Lambda Q^{(1)t}(A^i) = X^i$, to obtain;

$$\begin{aligned} X^1 &= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -0.2 & 1.6 \\ 1.6 & 2.2 \end{pmatrix} \end{aligned} \tag{5.1}$$

Step 2:

The following equation is used;

$$X^1 + X^1K - KX^1 = A^1$$

where $K = \frac{1}{2}(Q - Q^t) = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ to obtain the following results;

$$A^1 = \begin{pmatrix} -0.2 & 1.6 \\ 1.6 & 2.2 \end{pmatrix} + [\begin{pmatrix} -0.2 & 1.6 \\ 1.6 & 2.2 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} - \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -0.2 & 1.6 \\ 1.6 & 2.2 \end{pmatrix}] \tag{5.2}$$

$$= \begin{pmatrix} 2.6622 & 3.7466 \\ 3.7466 & -0.6622 \end{pmatrix} \tag{5.3}$$

The normalized eigenvectors of the matrix A^1 are the column space vectors of

$$Q^2 = \begin{pmatrix} 0.83831 & 0.54519 \\ 0.54519 & -0.83831 \end{pmatrix}.$$

This is used for the next iteration.

Step 3:We solve for X^2:

$$
\begin{aligned}
Q^2\Lambda Q^{(2)t} &= \begin{pmatrix} 0.83831 & 0.54519 \\ 0.54519 & -0.83831 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0.83831 & 0.54519 \\ 0.54519 & -0.83831 \end{pmatrix} \\
&= \begin{pmatrix} 1.8111 & 1.8282 \\ 1.8282 & 0.18893 \end{pmatrix} \qquad (5.4)
\end{aligned}
$$

We conclude from the above results that since the nonsingular matrix X^2 has the eigenvalues $\lambda_1 = 3.001, \lambda_2 = -1.0$, it is similar to the matrix Λ. We have therefore used a singular symmetric matrix to obtain a nonsingular symmetric matrix. We state the following lemma to justify our claim:

Lemma 5.1.1. *A nonsingular symmetric matrix can be generated using a singular symmetric matrix as initial matrix the following recursive equations are used;*

$$Q^i(A^i)\Lambda Q^{(i)t}(A^i) = X^i, i = 1, 2, ...$$

and

$$X^i + X^iK^i - K^iX^i = A^{i+1}.$$

We provide in the appendix, a program that generates any dense $n \times n$ singular symmetric matrix of rank 1 for given row multipliers. The program could be easily modified for rank $1 < r < n$.

5.2 Appendix

The following is the program to generate any $n \times n$ singular symmetric matrix of $rank1$

```
function ()
%UNTITLED2 Summary of this function goes here
%   Detailed explanation goes here
```

```
lambda=input('Enter a positive number as trace of the
      generalized matrix')
k = input('1.Enter m positive numbers that characterize the
      generalided matrix')
m = length(k);h(1)=1;ss=[];
for i=1:m
    h(i+1)=h(i)*k(i);
end
h;kk=1;
for i=m:-1:1
    kk=kk*(k(i)^2)+1;
end
kk;
format short eng
a=lambda/kk
ss=a*h;
mm(1,:)=ss;
for i=2:(m+1);
 mm(i,:)=k(i-1)*mm((i-1),:);
end
 mm
 %fprintf('The required singular matrix of rank 1 is:\n %d \n',mm)
tr=trace(mm)
sing=det(mm)
```

CHAPTER 6

Conclusion and Future Work

We have developed algorithms to generate singular symmetric matrices with an extension to singular Hermitian matrices of rank 1 when the eigenvalues and some parameters are given. In our presentation, numerical examples are provided. We have also developed direct iterative method to generate non singular symmetric matrices of orders two and three when the eigenvalues are provided. Singular symmetric matrix is used as an initial iterative matrix to generate a non singular symmetric matrix.

We shall consider the following for future work:

1. Consider singular Hermitian matrices of $rank \geq 2$.

2. Use singular symmetric matrix to generate non singular symmetric matrix of order greater than three using direct iterative method.

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