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TRANSPORTATION PROBLEM WITH VOLUME DISCOUNT ON SHIPPING COST

BY:

HARUNA ISSAKA, MPHIL. MATHEMATICS

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DECLARATION

I, Haruna Issaka, hereby declare that this thesis, "***Transportation problem with volume discount on shipping cost***", consists entirely of my own work produced from research undertaken under supervision and that no part of it has been presented for another degree elsewhere, except for the permissible excepts/references from other sources, which have been duly acknowledged.

NAME: ……………………………. ……..……………… ……….…………….
(Name of candidate) Signature date

NAME: …………………................... ……………..... …………………. (Supervisor)
Signature date

NAME:………………………….. ……………… …………………. (Head of Department) Signature date

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Chapter 1

Introduction

1.1 Background information

When considering transportation, various considerations are apparent. This consideration includes port selection, inland movement, port to port carrier selection and delivery movement. In addition to these transportation concerns, distribution-related considerations must also be given attention to such as packing/packaging, transit insurance, terms of sale, import duties, handling/loading and method of financing. Nevertheless, even freight companies projecting large volume movements can encounter serious transportation problem in organizing for distribution.

Understanding these transportation problems especially that affects shipping costs is critical. Volume discount, more specifically, targets shipping costs and in minimizing the latter, volume discount must be acquired.

Network models and integer programs are well known variety of decision problems.

A very useful and widespread area of application is the management and efficient use of scarce resources to increase productivity.

These applications include operation operational problems such as the distributions of goods, production scheduling production and machine sequencing and planning problems such as capital budgeting facility allocation, portfolio selection, and design problems such as telecommunication and transportation network design.

The transportation problem which, is one of network integer programming problems is a problem that deals with distributing any commodity from any group of ‘sources’ to any group of destinations or ‘sinks’ in the most cost effective way with a given ‘supply’ and ‘demand’ constraints . Depending on the nature of the cost function, the transportation problem can be categorized into linear and nonlinear transportation problem.

In the linear transportation problem (ordinary transportation problem) the cost per unit commodity shipped from a given source to a given destination is constant, regardless of the amount shipped.

It is always supposed that the mileage (distance) from every source to every destination is fixed.

To solve such transportation problem we have the streamlined simplex algorithm which is very efficient. However, in actuality we can see at least two cases that the transportation problem fails to be linear.

First, the cost per unit commodity transported may not be fixed for volume discounts sometimes are available for large shipments. This would make the cost function either piecewise linear or just separable concave function. In this case the problem may be formulated as piecewise linear or concave programming problem with linear constraints.

Second, in special conditions such as transporting emergency materials when natural calamity occurs or transporting military during war time, where carrying network may be destroyed, mileage from some sources to some destination are no longer definite. So the choice of different measures of distance leads to nonlinear (quadratic, convex …) objective function.

In both the above cases solving the transportation problem is not as simple as that of the linear one.

In this work, solution procedures to the generalized transportation problem taking nonlinear cost function are investigated. In particular, the nonlinear transportation problem considered in this thesis is stated as follows;

We are given a set of n sources of commodity with known supply capacity and a set of m destinations with known demands.

The function of transportation cost, nonlinear, and differentiable for a unit of product from each source to each destination.

We are required to find the amount of product to be supplied from each source (may be market) to meet the demand of each destination in such a way as to minimize the total transportation cost.

Our approach to solve this problem is applying the existing general nonlinear programming algorithms to it making a suitable modification in order to use the special structure of the problem.

1.2 Problem Statement

The prices of commodities are determined by a number of factors; the prices of raw materials, labour, and transport. When price of raw materials increase, so does the price of the commodity. Transportation cost also affects the pricing system.

It is assumed that the cost of goods per unit shipped from a given source to a given destination is fixed regardless of the amount shipped.

But in actuality the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal cost of shipping one unit might follow a particular pattern.

This project therefore seeks to develop a mathematical model using optimization techniques to bridge the gap between demand and supply by discounting so as to minimize total transportation cost. The problem that will be addressed in this study centers on the transportation problems experienced by freight companies. Volumes of goods to be shipped incur costs hence acquiring volume discounts could effectively lead to reduced shipping costs. However, there are transportation problems that hinder the materialization of improved total output through reduced costs of shipping.
The key question to be answered is: How freight companies could improve their total output through effectively reducing shipping costs through volume discounts.

We will also provide algorithms and different solution procedures to the different cases that might arise.

1.3 Study Objectives

The main aim of this study is to design mathematical programme that would improve the total output of freight companies especially since they deal with shipping of goods by volume. Whether maximum profit will be realized with discounts on large volumes means to determine the best transportation route that would lead to low transportation cost and the effective transportation of these goods. In lieu of this, other study objectives are as follows:

(i) Develop a mathematical model using optimization techniques to bridge the gap between demand and supply.
(ii) Use the model developed to minimize cost and thereby maximize profit to enhance effective management.
(iii) Determine how freight companies could improve total output through addressing the transportation problems encountered
(iv) Determine how volume discount could advent the operation of freight companies

(v) Evaluate how freight companies could maximize their total output through limiting shipping cost.

1.4 Methodology

The research strategy that the study will utilize is the descriptive method. A descriptive research intends to present facts concerning the nature and status of a situation, as it exists at the time of the study (Creswell, 1994). It is also concerned with relationships and practices that exist, beliefs and processes that are ongoing, effects that are being felt, or trends that are developing (Best, 1970). In addition, such approach tries to describe present conditions, events or systems based on the impressions or reactions of the respondents of the research (Creswell, 1994).

In this study, primary and secondary research will be both incorporated. The reason for this is to be able to provide adequate discussion for the readers that will help them understand more about the issue and the different variables that involve with it. The primary data for the study will be represented by the survey results that will be acquired from the respondents. The secondary sources of data will come from published articles from books, journals and theses and related studies.

1.5 Justification

Until recently, heavy trucks could load up to any capacity without limit. These trucks normally exceed the average loading capacity of the truck. This was partially due to high transportation cost. Drivers and transport owners together with transport users had to find a way of compensating for the high cost of transport by increasing the truck load so as to maximize profit.

This had ripple effect on the state as a whole: increased road accidents, destruction of roads and longer time being spent on the road before getting destination. There is also the effect of increased cost of goods thereby increasing inflation. This drove the attention of the government to find a lasting solution to the problems. The government therefore went into agreement with transport owners to determine maximum loading capacity of trucks.

The purpose of this thesis is to find out whether giving discounts on transportation charges could minimize total transportation cost thereby increasing total revenue of both producers and retailers; and some of the problems aforementioned.

1.6 Transport Costs

1.6.1 Transport Costs and Rates

Transport systems face requirements to increase their capacity and to reduce the costs of movements. All users (e.g. individuals, enterprises, institutions, governments.) have to negotiate or bid for the transfer of goods, people, information and capital because supplies, distribution systems, tariffs, salaries, locations, marketing techniques as well as fuel costs are changing constantly. There are also costs involved in gathering information, negotiating, and enforcing contracts and transactions, which are often referred as the cost of doing business. Trade involves transactions costs that all agents attempt to reduce since transaction costs account for a growing share of the resources consumed by the economy.

Frequently, enterprises and individuals must take decisions about how to route passengers or freight through the transport system. This choice has been considerably expanded in the context of the production of lighter and high value consuming goods, such as electronics, and less bulky production techniques. It is not uncommon for transport costs to account for 10% of the total cost of a product. Thus, the choice of a transportation mode to route people and freight within origins and destinations becomes important and depends on a number of factors such as the nature of the goods, the available infrastructures, origins and destinations, technology, and particularly their respective distances. Jointly, they define transportation costs.

Transport costs are a monetary measure of what the transport provider must pay to produce transportation services. They come as fixed (infrastructure) and variable (operating) costs, depending on a variety of conditions related to geography, infrastructure, administrative barriers, energy, and on how passengers and freight are carried.

Three major components, related to transactions, shipments and the friction of distance, impact on transport costs.

Transport costs have significant impacts on the structure of economic activities as well as on international trade. Empirical evidence underlines that raising transport costs by 10% reduces trade volumes by more than 20%. In a competitive environment where transportation is a service that can be bided on, transport costs are influenced by the respective **rates** of transport companies, the portion of the transport costs charged to users.

Rates are the price of transportation services paid by their users. They are the negotiated monetary cost of moving a passenger or a unit of freight between a specific origin and destination. Rates are often visible to the consumers since transport providers must provide this information to secure transactions. They may not necessarily express the real transport costs.

The difference between costs and rates results in either a loss or a profit from the service provider. Considering the components of transport costs previously discussed, rate setting is a complex undertaking subject to constant change. For public transit, rates are often fixed and the result of a political decision where a share of the total costs is subsidized by the society. The goal is to provide an affordable mobility to the largest possible segment of the population even if this implies a recurring deficit, (public transit systems rarely make any profit). It is thus common for public transit systems to have rates that are lower than costs. For freight transportation and many forms of passenger transportation (e.g. air transportation) rates are subject to a competitive pressure. This means that the rate will be adjusted according to the demand and the supply. They either reflect costs directly involved with shipping (cost-of-service) or are determined by the value of the commodity (value-of-service). Since many actors involved in freight transportation are private, rates tend to vary, often significantly, but profitability is paramount.

1.6.2 Costs and Time Components

Among the most significant conditions affecting transport costs and thus transport rates are:

(i). Geography. Its impacts mainly involve distance and accessibility. Distance is commonly the most basic condition affecting transport costs.

The more it is difficult to trade space for a cost, the more the friction of distance is important. The friction of distance can be expressed in terms of length, time, economic costs or the amount of energy used. It varies greatly according to the type of transportation mode involved and the efficiency of specific transport routes. Landlocked countries tend to have higher transport costs, often twice as much, as they do not have direct access to maritime transportation.

(ii). Type of product. Many products require packaging, special handling, are bulky or perishable. Coal is obviously a commodity that is easier to transport than fruits or fresh flowers as it requires rudimentary storage facilities and can be transshipped using rudimentary equipment.

Insurance costs are also to be considered and are commonly a function of the value to weight ratio and the risk associated with the movement. As such, different economic sectors incur different transport costs as they each have their own transport intensity. For passengers, comfort and amenities must be provided, especially if long distance travel is involved.

(iii) Economies of scale. Another condition affecting transport costs is related to economies of scale or the possibilities to apply them as the larger the quantities transported, the lower the unit cost. Bulk commodities such as energy (coal, oil), minerals and grains are highly suitable to obtain lower unit transport costs if they are transported in large quantities.

(iv). Energy. Transport activities are large consumers of energy, especially oil. About 60% of all the global oil consumption is attributed to transport activities. Transport typically account for about 25% of all the energy consumption of an economy.

The costs of several energy intensive transport modes, such as air transport, are particularly susceptible to fluctuations in energy prices.

(v). Trade imbalances. Imbalances between imports and exports have impacts on transport costs. This is especially the case for container transportation since trade imbalances imply the repositioning of empty containers that have to be taken into account in the total transport costs. Consequently, if a trade balance is strongly negative (more imports than exports), transport costs for imports tend to be higher than for exports. Significant transport rate imbalances have emerged along major trade routes. The same condition applies at the national and local levels where freight flows are often unidirectional, implying empty movements.

(vi). Infrastructures. The efficiency and capacity of transport modes and terminals has a direct impact on transport costs. Poor infrastructures imply higher transport costs, delays and negative economic consequences. More developed transport systems tend to have lower transport costs since they are more reliable and can handle more movements.

(vii). Mode. Different modes are characterized by different transport costs, since each has its own capacity limitations and operational conditions. When two or more modes are directly competing for the same market, the outcome often results in lower transport costs. Containerized transportation permitted a significant reduction in freight transport rates around the world.

(viii). Competition and regulation: Concerns the complex competitive and regulatory environment in which transportation takes place. Transport services taking place over highly competitive segments tend to be of lower cost than on segments with limited competition (oligopoly or monopoly). International competition has favored concentration in many segments of the transport industry, namely maritime and air modes. Regulations, such as tariffs, cabotage laws, labor, security and safety impose additional transport costs.

The transport time component is also an important consideration as it is associated with the service factor of transportation. They include the transport time, the order time, the timing, the punctuality and the frequency. For instance, a maritime shipper may offer a container transport service between a number of North American and Pacific Asian ports. It may take 12 days to service two ports across the Pacific (transport time) and a port call is done every two days (frequency). In order to secure a slot on a ship, a freight forwarder must call at least five days in advance (order time). For a specific port terminal, a ship arrives at 8AM and leaves at 5PM (timing) with the average delay being two hours (punctuality).

1.6.3 Types of Transport Costs

Mobility tends to be influenced by transport costs. Empirical evidence for passenger vehicle use underlines the relationship between annual vehicle mileage and fuel costs, implying the higher fuel costs are, the lower the mileage. At the international level, doubling of transport costs can reduce trade flows by more than 80%. The more affordable mobility is, the more frequent the movements and the more likely they will take place over longer distances. A wide variety of transport costs can be considered.

(i). **Freight on board (FOB)**. Is a transport rate where the price of a good is the combination of the factory costs and the shipping costs from the factory to the consumer. In the case of FOB, the consumer pays for the freight transport costs. Consequently, the price of a commodity will vary according to transportation costs and distance.

(ii). **Costs-Insurance-Freight (CIF)**. Is a transport rate that considers the price of the good, insurance costs and transport costs.

It implies a uniform delivered price for all customers everywhere, with no spatially variable shipping price.

The average shipping price is built into the price of a good. The CIF cost structure can be expanded to include several rate zones, such as one for local, another for the nation and another for exports.

(iii). **Terminal costs**. Costs that is related to the loading, transshipment and unloading. Two major terminal costs can be considered; loading and unloading at the origin and destination, which are unavoidable, and intermediate (transshipment) costs that can be avoided.

(iv). **Line-haul costs**. Costs that are a function of the distance over which a unit of freight or passenger is carried. Weight is also a cost function when freight is involved. They include labor and fuel and commonly exclude transshipment costs.

(v). **Capital costs**. Costs applying to the physical assets of transportation mainly infrastructures, terminals and vehicles. They include the purchase or major enhancement of fixed assets, which can often be a one-time event. Since physical assets tend to depreciate over time, capital investments are required on a regular basis for maintenance.

Transport providers make a variety of decisions based on their cost structure, a function of all the above types of transport costs. The role of transport companies has sensibly increased in the general context of the global commercial geography. However, the nature of this role is changing as a result of a general reduction of transport costs but growing infrastructure costs, mainly due to greater flows and competition for land. Each transport sector must consider variations in the importance of different transport costs. While operating costs are high for air transport, terminal costs are significant for maritime transport. Several indexes, such as the Baltic Dry Index, have been developed to convey a pricing mechanism useful for planning and decision-making.

Technological changes and their associated decline in transport costs have weakened the links transport modes and their terminals. There is less emphasis on heavy industries and more importance given to manufacturing and transport services (e.g. warehousing and distribution). Indeed, new functions are being grafted to transport activities that are henceforward facilitating logistics and manufacturing processes. Relations between terminal operators and carriers have thus become crucial notably in containerized traffic.

They are needed to overcome the physical and time constraints of transshipment, notably at ports.

1.6.4 TRANSPORTATION COST ANALYSIS

A typical application of the transportation problem is to determine an optimal plan for shipping goods from various sources to various destinations given supply and demand constraints in order to minimize total shipping cost. It is assumed that the cost of goods per unit shipped from a given source to a given destination is fixed regardless of the amount shipped. However, in actuality the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal cost of shipping one unit might follow a particular pattern.

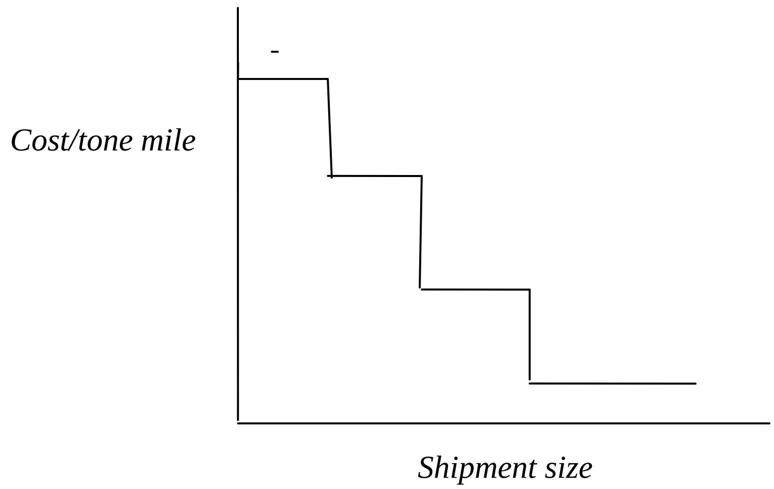
A transportation service incurs a number of costs: labor, fuel, maintenance etc. this cost can be divided into two: those cost that vary with services or volumes called variable cost and those that do not vary with services called fixed cost.

If a long enough period of time and greater volumes are considered then the cost is said to be variable. For the purposes of transportation pricing, it is useful to consider costs that are constant over the normal operating volume of the carrier as fixed and all other costs are treated as variables.

1.6.5 Cost characteristics by mode

(i) **Rail**: the rail road has the characteristics of high fixed cost and relatively low variable cost. Loading unloading billing etc. all contribute to high variable cost.

Increased per shipment volume and its effect on reducing fixed cost result in some substantial economies. I.e. lower per unit cost for increased per shipment volume. Variable costs vary proportionally by distance and volume.



Generalized railed road cost structure based on shipment size.

(i) **High way**: Motor carriers show contrasting cost characteristics with rail. Their fixed costs are the lowest compared with variable cost. Variable cost tend to be high because highway construction and maintenance cost are charged to the users in the form of fuel taxes, tolls and weight-mile taxes.

(ii). **Water**: the major capital investment that a water carrier makes is in transport equipment and to some extent terminal facility. Water ways and harbors are publicly owned and operated. Little of their costs are charged back to water carriers. The predominantly fixed costs in a water carrier's budget are associated with terminal operations.

Fixed costs include the harbor fees as the carrier enters the seaport, and the cost for loading and unloading cargo. These typically high fixed costs are somewhat offset by very low variable costs. Variable costs include costs associated with operating the transport equipment.

(iii) **Air:** air terminals and their air spaces are generally not owned by the airline companies. Airlines purchase airport services as needed in the form of fuel, storage, space rental and landing fees. if we include ground handling, and pickup and delivery in the case of air freight operations, these costs are the terminal costs for the air transportation.

Airlines own their own equipment which when depreciated over its economic life becomes an annual fixed expense. In the short run airline variable expenses are influenced more by distance than by shipment size.

Volume has indirectly influenced variable cost as greater demand for air transportation services has brought larger aircraft that have lower operating cost per available ton-mile.

1.6.6 Volume Related Rate

The economies of the transportation industry indicate that costs of services are related to size of the shipment. Rate structures in general reflect these economies as shipments in constantly high volumes are transported at lower rates than smaller shipments. Volume is reflected in the rate structure in several ways.

First, rates may be quoted directly on the quantity shipped. If the shipment is small, and results in very low revenue for the carrier, the shipment will be assessed either a minimum charge or an any–quantity rate.

Larger shipments that result in charges greater than the minimum charge but which is less than a full vehicle load quantity are charge at a less than-vehicle load rate that varies with the particular volume. Larger shipment sizes that equal or exceed the designated vehicle load quantity are charged the vehicle-load rate.

The system of freight classification permits some allowance for volume. High volumes can be considered justifications for quoting a shipper special rates on particular commodities. These special rates are considered deviations from the regular rate that apply to products shipped in lesser volumes.

The thesis is organized as follows: In the first chapter, we give a brief introduction to the transportation problem as well as other works that have been carried out in this respect and in the second chapter; the linear transportation problem will be looked at.

In the third chapter, where the main objective of the thesis lies, we discuss different algorithms to the nonlinear transportation problem.

Finally, in the fourth chapter, application to real life situation and some practical matters associated with solving the problem are discussed. Discussion and conclusion are also given under this chapter.

Chapter 2

Literature Review

2.1. *Linear and nonlinear transportation review*

The transportation problem was formalized by the French mathematician Gaspard Monge in (1781). Major advances were made in the field during World War II by the Soviet/Russian mathematician and economist Leonid Kantorovich. Consequently, the problem as it is now stated is sometimes known as the Monge-Kantorovich transportation problem as reported by www.historyofmathematics.com.

Many scientific disciplines have contributed toward analyzing problems associated with the transportation problem, including operation research, economics, engineering, Geographic information science and geography. It is explored extensively in the mathematical programming and engineering literatures.

Sometimes referred to as the facility location and allocation problem, the transportation optimization problem can be modeled as a large-scale mixed integer linear programming problem.

The problem with the production capacity of each source fixed with constant unit transportation cost was originally formulated by Hitchcock (1941) and was subsequently dealt with independently by Koopmans during Second World War

Analytical solution to this problem has been given by several authors. Stringer and Haley have developed a method of solution using mathematical analogue.

In (1988) Denardo, Eric, Rothblum, Uriel, Swersey, Arthur J. developed a problem which uses supplied item travel time averages to determine the 'cost' of satisfying the demand at a particular location. Items that arrive first receive the greatest weight, and decreasing weights are given to each succeeding item. An equivalent transportation problem is used for problems with a known demand.

If the demand is stochastic a transportation problem whose aim is to minimize the sum of a linear function is used. The function is linearized by substituting the product and a linear term for the convex function.

May be the first algorithm to find an optimal solution for the uncapacitated transportation problem was that of Efroymson and Ray (1964). They assumed that each of the unit production cost functions has a fixed charge form.

But they remark that their branch-and – bound method can be extended to the case in which each of these functions is concave and consists of several linear segments. And each unit transportation cost function is linear.

George Dantzig (1951) adapted the simplex method to solve the transportation problem formulated earlier by Hitchcock and Koopmans.

Abraham Charnes and William Cooper (1954) derived an intuitive presentation of Dantzig's procedure called the stepping-stone method which follows the basic logic of the simplex method but avoids the use of the tableau and the pivot operations required to get the inverse of the basis.

Algorithms for the capacitated case have been presented by Davis Ray, Ellwein, Gray and Marks (1966). In all of these the cost functions are assumed to be linear and the production cost is linear where ever the production is and zero where not. Ellwein's technique allows the easy incorporation of configuration constraints that restrict the allowable combinations of open plants and generalization of the production.

Vidal has given a graphical solution based on successive approximations when the production costs vary with the volume of resource produced.

Two dynamic programming algorithms are developed for a problem involving the minimization of the number of needed workstations for an assembly line when processing times are normally distributed, independent random variables.

The algorithms are based on the works of Held (1963) and Kao (1976), and results indicate that they are more efficient than the alternative dynamic programming approach suggested by Henig (1986).

A study was conducted to develop a model based on the initial solution and Goal Programming to obtain the values of the decision variables. Findings by Parra, M. Arenas, Terol, A. Bilbao, Uria, M.V. Rodriguez (1999) revealed the development of a decision vector which allows the approximation of the goal functions with fuzzy coefficients to the fuzzy solutions specifically obtained for such programs. It was also revealed that the new fuzzy solution in the set of objectives verifies that each component is a triangular fuzzy number.

Frank Sharp.et.al developed an algorithm for reaching an optimal solution to the production-transportation problem for the convex case.

The algorithm utilizes the decomposition approach it iterates between a linear programming transportation problem which allocates previously set plant production quantities to various markets and a routine which optimally sets plant production quantities to equate total marginal production cost, including a shadow price representing a relative location cost determined from the transportation problem.

The TRIMAP linear programming package for solving three-objective transportation problems was evaluated by Climaco, Joao, Antunes, Henggeler, Alves, and Maria. TRIMAP was designed (1993) to help in decision making by removing non dominated solutions through the use of weight space decomposition, introduction of constraints on the objective function space and constraints on the weight space. The package also supports objective function value constraints for automatic translation into the weight space.

Williams (1964) applied the decomposition principle of Dantzing and Wolf to the solution of the Hitchcock transportation problem and to several generalization of it. In this generalizations, the case in which the costs are piecewise linear convex functions is included.

He decomposed the problem and reduced to a strictly linear program. In addition, he argued that the two problems are the same by a theorem that he called the reduction theorem.

The algorithm given by him, to solve the problem, is a variation of the simplex method with “generalized pricing operation”.

It ignores the integer solution property of the transportation problem so that some problems of not strictly transportation type, and for which the integer solution property may not hold be solved.

Shetty (1959) also formulated an algorithm to solve transportation problems taking nonlinear costs. He considered the case when a convex production cost is included at each supply center besides the linear transportation cost.

Feldman (1983) assessed, the concavity of the cost curve brought about by economics of scale leads to multiple-optima, and thus problems like these are not susceptible to conventional mathematical techniques. The power of the simplex method in solving linear programs is based on the general theorem which states that the number of variables – including slack variables, whose values are positive in an optimal solution, is at most equal to the number of constraints in the problem.

For this reason, nearsighted computational techniques are used to examine the corners of the feasible region (basic solution). Unfortunately, these myopic computational and optimality testing techniques can be employed only when the problem involves a convex feasible region and increasing marginal cost (the case on which we are also will be focusing).

Martins, Lucia, Craveirinha (2005) presented a study of a bi-dimensional dynamic routing model for telecommunications network. The model uses heuristic methods to solve instability problems. The routing methods through heuristics are compared with the discrete-event simulation in the dynamic routing system.

The branch and bound algorithm approach is based on using a convex approximation to the concave cost functions. It is equivalent to the solution of a finite sequence of transportation problems.

The algorithm was developed as a particular case of the simplified algorithm for minimizing separable concave functions over linear polyhedral as Falk and Soland.

Piece-wise linear over approximation is also the other approach to solve the nonlinear concave transportation problem.

Richard Soland (1971) presented a branch and bound algorithm to solve concave separable transportation problem which is called it “the simplified algorithm” in comparison with similar algorithm given by Falk and himself in 1969.

The algorithm reduces the problem to a sequence of linear transportation problem with the same constraint set as the original problem.

Gay (1989) presented two algorithms to solve mixed integer second-order cone programming problems: a branch-and-cut method and an outer approximation based branch-and-bound approach.

Different techniques were presented for the generation of linear and convex quadratic cuts and investigate their impact on the branch-and-cut procedure. The presented outer approximation based branch-and-bound algorithm is an extension of the well-known outer approximation based branch-and-bound approach for continuous differentiable problems to sub-differentiable constraint functions.

Convergence can be guaranteed, since the sub-gradients, that satisfy the KKT conditions, can be identified using the dual solution of the occurring second order cone problems. Computational results for test problems and real world applications are given.

The optimal design of a Water Distribution Network (WDN) is a real-world optimization problem known and studied since the 1970s. Several approaches have been proposed, for example, heuristics, metaheuristics, Mixed Integer Linear Programming models and global optimization methods. Despite this interest, it is still an open problem, since it is very hard to find good solutions for even medium sized instances. Work on non-convex Mixed Integer Non-Linear Programming (MINLP) model that was presented by Bragalli, Lee, Lodi and Toth which accurately approximates the problem, and was solved with an ad-hoc modified branch-and-bound for minimization linear programming problems.

For the general class of minimization linear programming problems where relaxing the integrality on integer variables yields a non-convex problem, a commonly used solution method is Branch-and-Bound (BB). Two crucial components of a BB algorithm are: a convex relaxation, often an LP relaxation, to obtain lower bounds; and branching rules for partitioning the solution set.

Caputo et al (1991) presented a methodology for optimally planning long-haul road transport activities through proper aggregation of customer orders in separate full-truckload or less-than-truckload shipment in order to minimize total transportation cost. They have demonstrated that evolutionary computation technique may be effective in tactical planning of transportation activities. The model shows that substantial savings on overall transportation cost may be achieved adopting the methodology in a real life scenario.

2.2. Economic optimum size order and price discount (EOQ) review

Based on the simple EOQ model, a model for determining the optimal stock replenishment strategy for temporary price reduction can be derived. Barman and Tersine (1995) extended the logic to a composite EOQ model that can be segmented into a family of hybrid models with broader operational flexibility. The composite EOQ provides malleability and flexibility to changing operational requirements by desegregating complexity. The resourcefulness of an expert system with its attendant economics is approached.

Lu and Qui (1994) derived the worst-case performance of a power-of-two policy in an all-unit quantity discount model with one price break point, which extends the sensitivity analysis for the classical EOQ model. The model showed that the worst-case performance will depend on the discount rate α and is within 7.66 % of optimality when $\alpha < 7.51$ % and approximately within 100 α % of optimality when $\alpha \geq 7.51$ %.

Bastian (1992) developed a dynamic lot-size problem under discounting which allows a speculative motive for holding inventory. He derived a procedure that determined the first lot-size decision in a rolling horizon environment, using forecast data of the minimum possible number of future periods.

Chao (1992) generalized Bather's EOQ model with discounting. The model considers two cases: one with demand backlogging and the other, without backlogging. Important implications of discounting are investigated useful insights and formulas are provided.

Martin (1993) provides an alternative perspective on the quantity discount-pricing problem. He considered the multiple price breaks excluding the buyer's operating parameter from consideration, with the exception of price dependent demand.

Johnson (1975) described a graphical approach to price-break analysis. He argued that in any stock control situation the main problem is how many units of an item should be procured at any one time, the "economic batch quantity; (EBQ)" and having determined EBQ, the problem of quantity discount arises.

Would it be profitable to order in quantities other (usually more) than the EBQ in order to take advantage of price discount. This approach provided:

(a). a rationale foundation for a very many real life purchasing decisions;

(b) a tool which is readily understood by the clever man in the business.

It suggested "Given a discount structure for a product group, find the most economical purchase quantities for all members of that group".

Fitzpatric and Roy (1997) incorporated quantity and freight discounts in inventory decision making when demand is considered to be dependent upon price (rather being constant). An algorithm was developed to determine the optimal lot-size and selling price for a class of demand functions, including constant price-elasticity and linear demand. A numerical example was illustrated to develop the model and computer program to implement the model.

Followill (1990) studied managerial decision to accept a quantity discount, if total, per period inventory and acquisition costs are reduced. They developed an EOQ model within wealth maximization framework, when volume discounts were available. They established that the traditional method of analyzing volume discount opportunities may invoke wealth decreasing decisions.

Wang (1993), in this article, investigated the managerial insights related in using the all- unit quantity discount policies under various conditions.

The models developed here were general treatments that deal with four major issues:

a. One or multiple buyers;

b. Constant or price elastic demand;

c. The relationship between the supplier's production schedule or ordering policy and buyer's ordering sizes;

d. The supplier either purchasing or manufacturing the item.

The two main objectives of the developed models were the supplier's profit improvement or the supplier's increased profit share analysis. Algorithms were developed to find optimal decision policies.

The present analysis provided the supplier with both optimal all-unit quantity discount policy and the optimal production or ordering strategy. The concept was illustrated with numerical examples.

Benton (1991) considered quantity discount procedures under conditions of multiple items, resource limitations and multiple suppliers.

He offered an efficient heuristic programming procedure for evaluating alternative discount schedules. The article provided encouraging findings for the managers. Gaither and Park (1991) developed optimal ordering policies for a group of inventory items when a supplier offers a one time discount for the group by minimizing total inventory costs within a firm's ware house space or funds availability constraints.

Carlson and Miltenburg (1993) examined order quantities for families of items when regular and special discount schedules are available, and the objective is to minimize the present value of the relevant cash flows. Order quantities can be large when special discounts are available, traditional cost minimization models which ignore the effects of the timing of cash flows are less accurate than the discounted cash flow models. They extended the earlier research work on families of items, special quantity discounts and discounted cash flows. Order cost, invoice cost, physical inventory carrying cost and financial carrying cost were identified. The amount and the timing of the cash out flow for each cost were described. Regular as well as two special types of discounts was also considered. Regular discounts were schedules of discounts and break point order quantities that were always available. The special discounts are an opportunity to order before a price increase and a ‘sale’ or special one-time price reduction.

Sorger (1994) derived a condition that must be satisfied by a cost function $h: X \to X$ in order to be the optimal function of a strictly concave deterministic dynamic programming problem which defined on the state space X and which has a given discount factor ζ. This condition was used to show that there was no such DPP on a one-dimensional state space that generates optimal solutions that are periodic with minimal period three unless the discount rate exceeds 82%. This bound held uniformly for all strictly concave problems and all period three cycles.

In many practical situations, co-ordination between replenishment orders for a family of items can be cost saving. A well-known class of strategies for the case where cost savings are due to reduced joint ordering costs is the class of can-order strategies.

However, these strategies, which are simple to implement do not take price discount possibilities into consideration. Duyn, Van and Heuts (1994) proposed a method to incorporate discounts in the framework of can-order strategies. A continuous review multi-item inventory system was considered with independent compound Poisson demand processes for each of the individual items.

The supplier as a percentage of the total dollar value offers discounts whenever this value exceeds a given threshold. They developed a simple heuristic to evaluate these discount opportunities taking the can-order strategy as a basic rule. The performance of the can-order strategy with discount evaluation was compared with that of another class of discount evaluation rules as proposed by Miltenburg and Silver.

Crainic and Laporte (1997) extensively review the optimization models for freight transportation. A main distinction can be established between strategic-tactical and operational models that respectively consider a national or an international multimodal network, such as in the Service Network Design Problem (SNDP) (see Crainic (2000)), and the unimodal distribution management models that are variants of the Vehicle Routing Problem (VRP) (see Toth and Vigo (2002)).

Macharis and Bontekoning (2004) present a freight logistics literature review focused on intermodal transportation. They propose a classification based on two criteria: the type of operator and the length of the problem's time horizon. Four types of operators are distinguished: drayage operators, terminal managers, network planners, and intermodal operators. The time horizon criterion results in the classical differentiation of strategic, tactical, and operational levels. In this classification matrix of twelve categories, the M++TP would correspond to operational problems faced by an intermodal operator, since the problem can be stated as the selection of routes and of services in a multimodal network. This problem category, according to Macharis and Bontekoning (2004) and to our own updated survey (see Section 2.3), is one of the least studied.

Bontekoning et al. (2004) reviewed the intermodal literature related to the rail-truck combination. This paper, like the previous one, highlights the need for more research on operational problems faced by intermodal operators, like the optimal route selection. Container based transportation is the key enabler of intermodalism because of various advantages like higher productivity during the transfers, less product damage, etc.

Consequently, Crainic and Kim (2006) focused their recent intermodal logistics literature review on the container related aspects of the transportation industry.

In particular, empty container repositioning and container terminal management problems are thoroughly discussed.

Kim and Pardalos (1999) introduced a dynamic slope scaling algorithm to heuristically solve the Fixed Charge Network Flow Problem (FCNFP). Kim and Pardalos (2000) applied similar algorithms to the Concave Piecewise Linear Network flow Problem (CPLNFP).

In fact, through an arc separation procedure, the CPLNFP can be transformed into an FCNFP on an extended graph. A more refined algorithm variant, which employs a trust interval technique, was also presented in the same paper. The dynamic slope scaling concept was exploited by Crainic et al. (2004) to solve the multicommodity version of the FCNFP. The authors propose a heuristic algorithm that combines slope scaling, Lagrangean relaxation, intensification and diversification mechanisms as in metaheuristics.

Croxton (2003a) prove that three textbook mixed-integer linear programming formulations of a generic minimization problem with separable non-convex piecewise linear costs are equivalent. Their LP relaxations approximate the piecewise linear cost function with its lower convex envelope.

Independently, Keha (2004, 2006) derived a similar result. Croxton (2007) present valid inequalities based upon variable disaggregation for network flow problems with piecewise linear costs. Croxton (2003) study an application, the merge-in-transit problem, where the above mentioned technique shows its efficacy.

Crowther (1964) established the relationship between the prices on orders of different sizes after the seller has achieved a few fundamental decisions regarding the manner in which he wants a smooth and efficient business.

There are two general types of quantity discount schedules offered by supplies: the all-units discounts and the incremental discount. Purchasing large quantity in all-units discount schedule results in a lower unit price for the entire lot; whereas, in incremental discount schedule, lower unit facility is available only to units purchased above a specified quantity. The quantity at which price change is called price-break quantity. Gorham ((1970) calculated break-even demand volumes to determine quantity discount desirability.

CHAPTER 3

Methodology

Introduction

A typical application of the transportation problem is to determine an optimal plan for shipping goods from various sources to various destinations given supply and demand constraints in order to minimize total shipping cost. It is assumed that the cost of goods per unit shipped from a given source to a given destination is fixed regardless of the amount shipped. But in actuality the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal cost of shipping one unit might follow a particular pattern. When volume discounts are offered, the objective function or the constraint functions assume a nonlinear form. We therefore use the nonlinear method of solution to solve such a problem.

LINEAR TRANSPORTATION PROBLEM

3.1.1. Transportation Model Problem

Transportation is an example of network optimization problem. It deals with the efficient distribution (transportation) of product (goods) and services from several supply locations (sources) with limited supply, to several demand locations (destinations) with a specified demand with the objective of minimizing total distribution cost; a typical example of which this project represents (in *analogy*).

This objective is achieved under the following constraints;

1. Each demand point receives its requirement.
2. Distributions from supply points do not exceed its available capacity. This goal is achieved contingent on availability and requirements constraints.

3.1.2 Model Formulation

The formulation of the transportation model employs double – subscripted variables of the form x_{ij}. Thus, the general formulation of the transportation problem with *m* sources and *n* destinations, with the following defined notations;

i= index for origins (supply points), $i = 1, 2, 3. . . .m$.

j= index for destinations (demand point), $j = 1, 2, 3 \ldots n$.

x_{ij} = number of units transported from origin *i* to destination *j*.

C_{ij} = per unit cost of transporting from origin *i* to destination *j*.

S_i = supply or capacity in units at origin ***i***.

d_j = demand in units at destination ***j***.

is given by

Minimize

$$\sum_{j=1}^{n} . \sum_{i=1}^{m} C_{ij} x_{ij}$$

$$subject\ to \qquad \sum_{i=1}^{n} s_i = S \qquad i = 1,2,3 \ldots \ldots ..$$

$$\sum_{j=1}^{m} d_j = d \qquad j = 1,2,3 \ldots \ldots$$

At instances where total supply from all the sources equals total demand at all destinations, the transportation model is expressed as

$$\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j$$

Under such circumstances, the transportation problem is said to be **balanced**.

3.1.3 Problem Variation

Variation of the basic transportation problem may involve one or more of the following situations

I. Total supply not equal to total demand.

II. Maximization of objective function rather than minimization

III. Unacceptable routes.

These situations can be easily accommodated with some modifications in the linear programming model. Modifications to the above situations are shown next.

(i). Total Supply Not Equal To Total Demand

These conditions arise in most realistic cases when total supply is less than or exceeds total demand. Either of these situations occasions an unbalanced transportation problem.

$$\sum_{i=1}^{n} S_i < \sum_{j=1}^{m} d_j$$

a. **Total demand exceeds total supply**

$$\sum_{j=1}^{m} d_j > \sum_{i=1}^{n} s_i$$

Under this situation the linear programming model of the transportation problem will not have a feasible solution. The modification of this situation for a feasible solution requires an addition of a ***dummy origin*** with a supply equal to the difference between the total demand and the total supply to the network representation to modify it. To reflect this modification in the transportation tableau, a ***dummy row*** is added to all the units' demands for which supply is not available.

Thus an imaginary (fictitious) supply point with an amount available = total demand – total supply is added to balance supply and demand. A zero per-unit cost is, however, assigned to each cell of the ***dummy row***.

The ***dummy cells*** in the transportation tableau are analogous to slack variable, which have zero C_{ij} values in the objective function and so does not affect the initial solution.

b. Total Supply Exceeds Total Demand

$$ie. \quad \sum_{i=1}^{n} s_i > \sum_{j=1}^{m} d_j$$

When total supply exceeds total demand, the excess will appear as a slack in the linear programming solution. Slack for any origin is the unused supply or the amount not transported from the origin to demand points.

To construct a balanced model, we create a "fictitious" demand point with an amount equal to the excess supply. Thus a ***dummy column*** is added to the tableau.

(ii). Maximization Objective

The objective function in some transportation problem is maximized rather than minimized. This is done when the objective is to find a solution that maximizes profit or revenue.

The profit or revenue per-unit values are used as coefficients in the objective function under this circumstance, and maximization rather than a minimization linear programming model is got, while the constraints remain unaffected by this change.

(iii). Unacceptable Routes

Special situations do arise in transportation problems where it may not be possible to establish a route from every origin to every destination.

This solution makes some routes unacceptable. This is handled by dropping the corresponding arc from the network and removing the corresponding variable from the linear programming formulation.

3.1.4 FINDING INITIAL FEASIBLE SOLUTION TO TRASPORTATION PROBLEM

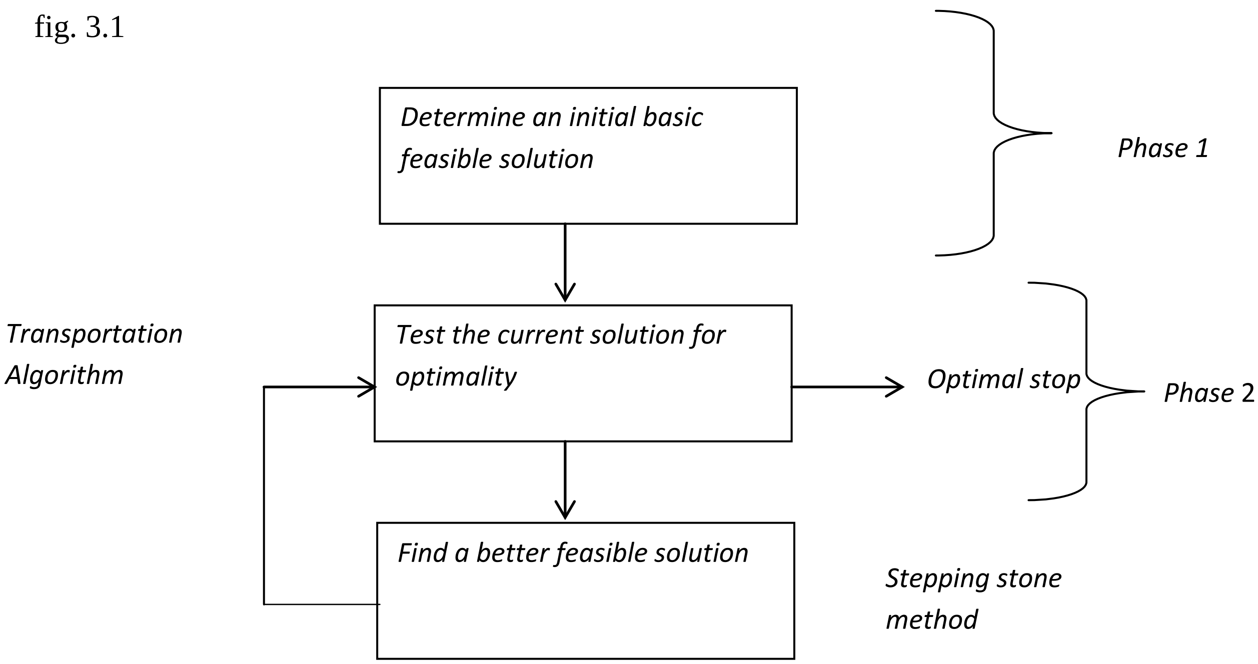
The general formulation of the transportation problem reveals that **m** supply constraints and **n** demand constraints translate into *m + n* total constraints. In the transportation problem however, one of the constraints is redundant resulting in the fact that if, in a balance condition,

$$\sum_{i=1}^{n} S_i \geq \sum_{j=1}^{m} d_j$$

m + n constraints are met then m + n equations will also be met. Only *m + n – 1* independent equations, thus, exist and so the initial solution will have only *m+ n -1* basic variables.

The flow chart below illustrates the various phases leading to the optional solution of a transportation problem

fig. 3.1



3.15 Transportation Tableau

The transportation tableau is a unique tabular representation of the transportation problem. The tableau has **m x n** cells**,** where **m** is the number of supplies (sources) and **n** the number of destinations (demand). The demand at each destination is entered in the bottom row, while the supply from each source is listed in the right-hand column.

The lower right hand corner represents the quantity of total demand and total supply.

The x_{ij} variable gives the number of units transported from source i to destination j (which is to be solved for) while the unit cost for the transportation from i to j, denoted by C_{ij}, is recorded in a small box in the upper – right – hand corner of each cell. Below is the form of the general transportation tableau.

Table 2.1 Transportation Tableau

To (i) *From (j)*	*DESTINATIONS*							
S O U R C E S		1	2	....	j	....	n	*Supply*
	1	C_{11} x_{11}	C_{12} x_{12}	....	C_{1j} x_{1j}	....	C_{1n} x_{1n}	S_1
	2	C_{21} x_{21}	C_{22} x_{22}	......	C_{2j} x_{2j}	....	C_{2n} x_{2n}	S_2
	: : :	: : :	: : :	......	: : :	....	: : :	: : :s
	i	C_{i1} x_{i1}	C_{i2} x_{i2}	......	C_{ij} x_{ij}	....	C_{in} x_{in}	Si
	: : :	: : :	: : :	......	: : :	.....	: : :	: : :
	m	C_{m1} x_{m1}	C_{m2} x_{m2}	......	C_{mj} x_{mj}	....	C_{mn} x_{mn}	S_m
Demand		d_1	d_2	......	d_j	....	d_n	$\Sigma S_i = \Sigma d_j$

3.1.6 Methods for Finding Initial Basic Feasible Solutions

The first phase of the solving a transportation problem for optimal solution involves finding the initial basic feasible solution. An initial feasible solution is a set of arc flows that satisfies each demand requirement without supplying more from any origin node than the supply available. Heuristic, a common – sense procedure for quickly finding a solution to a problem is a producer most employed to find an initial feasible solution to a transportation problem. This project examines three of the more popular heuristics for developing an initial solution to transportation problem.

i. The Northwest corner method
ii. The Least Cost Method
iii. The Vogel's Approximation Method

(i). The Northwest Corner Method

This method is the simplest of the three methods used to develop an initial basic feasible solution. This notwithstanding, it is the least likely to give a good "low cost" initial solution because it ignores the relative magnitude of the costs C_{ij} in making allocations The procedure of this method is as follows.

1. Start at the northwest corner (upper-left-hand corner) cell of the tableau and allocate as much as possible to$x_{\mathbf{11}}$ without violating the supply or demand constraints (i.e. $x_{\mathbf{11}}$ is equal to the minimum of the values of S_i or d_j .)

2. This will exhaust the supply at source **i** and or the demand for destination **j**. As a result, no more units can be allocated to the exhausted row or column, and it is eliminated. Next, allocate as much as possible to the adjacent cell in the row or column that has not been eliminated. If both row and column are exhausted move diagonally to the next cell.

3. Continue the process in the same manner until all supply has been exhausted and demand requirements have been met. Following is an example to illustrate the use of the Northwest Corner Method of finding an initial basic feasible solution to transportation problems.

(ii). Least – Cost Method

The Least– Cost Method tries to reflect the objective of cost minimization by systematically allocating to cells according to the magnitude of their unit costs.

Following is the general procedure for the Least –Cost Method.

1. Select the $\boldsymbol{x_{ij}}$ variable (cell) with the minimum C_{ij} unit transportation cost C_{ij} and allocate as much as possible thus, for minimum C_{ij}.

x_{ij} = minimum ($S_{i,}$,d_j). This wills exhaust either row i or column j**.**

2. From the remaining cells that are feasible (i.e. have not been filled or their row or column eliminated), select the minimum C_{ij} value and allocate as much a possible

3. Continue the process until all supply and demand requirements are satisfied

4. In case of ties between the min C_{ij} values select between the tied cells arbitrarily and apply the procedure.

(iii). VOGEL'S APPROXIMATION METHOD

The Vogel's Approximation Method (VAM) is by far the best method (better than the Northwest Corner Method and the Last-Cost Method) of developing an initial basic feasible solution to transportation problems. In many cases the initial solution obtained by the VAM will be optimal.

It consists of making allocations in a manner that will minimize the penalty (regret or opportunity cost) for selecting the wrong cell for an allocation. The procedure for the use of the VAM is as follows;

1. Calculate the penalty cost for each row and column. The penalty costs for each row *i* are computed by subtracting the smallest C_{ij} values in the row from the next smallest C_{ij} values in the same row.
2. Column penalty costs are similarly obtained, by subtracting the smallest c_{ij} value in each column from the next smallest column C_{ij} value. These costs are the penalty for mot selecting the minimum cell cost.

3. Select the row or column with the greatest penalty cost (breaking any ties arbitrarily) and allocate as much as possible to the cell with the minimum C_{ij} value in the selected row or column, that is for minimum C_{ij}, x_{ij} = minimum (S_i, d_j). This will avoid the greatest penalties.
4. Adjust the supply and demand requirements to reflect the allocations already made. Eliminate any rows and columns in which supply and demand have been exhausted.
5. If all supply and demand requirements have not been satisfied, go to the first step and recalculate new penalty costs. If all row and column values have been satisfied the initial solution has been obtained.

3.1.7 Optimality-Test Algorithm for Transportation Problems

These are methods of determining the optimal solutions for transportation problems following the determination of the initial basic feasible solution. Two methods,

(1) The stepping stone method
(2) The Modified Distribution Method shall be the focus of this project.

(1). The Stepping Stone Method: This optimality test begins, once an initial basic feasible solution is obtained for the transportation problem, by determining if the total transportation cost can be further reduced by entering a nonbasic variable (i.e. allocating units to an empty cell) into the solution. Thus each empty cell is evaluated to determine if the cost of shifting a unit to that cell from a cell containing a positive unit will decrease. A closed loop of occupied cells is used to evaluate each nonbasic valuable. An initial basic feasible solution is considered optimal if the total transportation cost cannot be lowered/ decreased by reallocating units between cells.

The following three steps are involved in the stepping-stone method

1. Determine an initial feasible solution by using any of the afore-discussed initial feasible solution determination methods
2. Compute a cell evaluator for each empty cell, determined by computing the next cost of shifting one unit from a cell containing a positive unit to the empty cells. The sign of cell evaluators are then checked for optimality
3. If a cell evaluator fails the sign test, if the solution is not optimal, determine a new lower total cost solution, accomplished by shifting the maximum amount to that empty cell so that the supply or demand constraints are not violated.

(2). The Modified Distribution Method (MODI)

The modified distribution method of solution is a variation of the steeping-stone method based on the dual formulation. The difference between the two is that with the MODI, unlike the stepping-stone method, it is not necessary to determine all closed paths for nonbasic variable. The C^*_{ij} values are instead determined simultaneously and the closed path is identified only for the entering nonbasic variable. In the MODI method, a value u_i is defined for each row (i) and a value v_j is defined for each column (j) in the transportation tableau. For each basic variable, (occupied cell), x_{ij} the following relationship exists.

$C_{ij} = u_i + v_j$, where C_{ij} is the unit cost of transportation.

The steps employed in the MODI method are;

1. Determine u_i values for each row and v_j value for each column by using the relationship $C_{ij} = u_i + v_j$ for all basic variables beginning with an assignment of zero to u_1.
2. Compute the net cost change C^*_{ij}, for each nonbasic variable using the formula $C^*_{ij} = C_{ij} - u_i - v_j$.
3. If a negative C^*_{ij} value exists, the solutions is not optimal. Select the x_{ij} variable with the greatest negative C^*_{ij} value as the entering nonbasic variable.

4. Allocate units to the entering C^*_{ij} value as the entering nonbasic variable, x_{ij}, according to the stepping-stone procedure. Return to step 1.

The Non Linear programming problem

Preliminaries

3.2. CONVEX SETS

Definition: a line segment joining the points x_1 and x_2 in $\mathbf{R}^n$ is the set $[x_1, x_2] = \{x \in \mathbf{R}^n : x = \lambda x_1 + (1 - \lambda)x_2\}$, $0 \leq \lambda \leq 1$.

A point on the line segment for which $0 < \lambda < 1$, is called an interior point of the line segment.

Definition 3.1.1: A subset S of $\mathbf{R}^n$ is said to be convex if for any two elements x_1, x_2 in S the line segment $[x_1, x_2]$ is contained in S . Thus x_1 and x_2 in S imply $\lambda x_1 + (1 - \lambda)x_2\} \in S$ for all $0 \leq \lambda \leq 1$ if S is **convex**.

1. **Extreme Points.**

Definition 3.2.2 Let P be non-empty convex set in E^n. A vector $x \in$ P is called an extreme point of P if $x = \lambda x_1 + (1 - \lambda)x_2$ with x_1 and x_2 elements of P and $\lambda \in (0,1)$ then $x = x_1 = x_2$.

The following are basic theorems concerning extreme points; and for their proofs one can refer to any analysis book.

Theorem 3.2.1 Let $P = \{x: Ax = b, x \geq 0\}$, where A is mXn matrix of rank m, and b is an m vector. A point x is an extreme point of P if and only if A can be decomposed in to [B, N] such that $\boldsymbol{x} = \binom{\boldsymbol{XB}}{\boldsymbol{XN}} = \binom{\boldsymbol{B}}{\boldsymbol{0}}$; where B, is an mXm invertible matrix satisfying $B^{-ib} \geq 0$. Any such solution is called a basic feasible solution (BFS) for P.

Corollary: The number of extreme points of P is finite.

Theorem 3.2.2 (Existence of extreme points)

Let $P = \{x: Ax = b, x \geq 0\}$ be non empty: where A is an mXn matrix of rank m and b is an m – vector. Then P has at least one extreme point.

2. **Extreme Direction**

Definition 3.2.3 Let P be a non empty polyhedral set in E^n. A non zero vector d in E^n is called direction or a recession direction of P if $x + \lambda d \in P$ for each $x \in P$ and all $\lambda \geq 0$.

It follows that; d is a direction of P if and only if

$Ad = 0, \; d \geq 0$

Theorem 3.2.3 Characterizations of Extreme Directions

Let $p = \{x : Ax = b, x \geq 0\} \neq \emptyset$, where A is an mXn matrix of rank m, and b is an m vector. A vector D is an extreme direction of P if and only if A can be decomposed into [B,N] such that $B^{-1} a_j \leq 0$ for some column a_j of N and D is a positive multiple of D $=\begin{pmatrix} -Baj \\ ej \end{pmatrix}$ where e_j is an n-m vector of zeros except for position j which is 1.

<u>Solution procedures to the Nonlinear Transportation problem (NTP)</u>

In this section, we consider a transportation problem with nonlinear cost function. We try to find different solution procedures depending on the nature of the objective function.

Before going to the different special cases, let's formulate the KKT condition and general algorithm for the problem.

Given a differentiable function

$C : \mathbb{R}^{nm} \longrightarrow \mathbb{R}$

We consider a nonlinear transportation problem (NTP),

$$\text{Min } \; c(x)$$

$$\textit{subject to} \;\; Ax = 0,$$

$$x \geq 0$$

Where

$$x = \begin{pmatrix} x_{11} \\ . \\ . \\ x_{ij} \\ . \\ . \\ x_{nm} \end{pmatrix} \qquad b = \begin{pmatrix} s_1 \\ . \\ s_n \\ . \\ d_1 \\ . \\ d_m \end{pmatrix} \qquad A = \begin{pmatrix} 1\ 1\ 1\ 1 \ldots\ldots\ldots\ldots.. \\ \ldots 1\ 1\ 1 \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ \ldots\ldots\ldots\ldots 1\ 1\ 1 \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ \ldots\ldots\ldots 1\ 1\ 1\ 1\ \ldots\ . \\ 1\ 1\ 1\ \ldots\ldots\ldots\ldots 1\ 1 \end{pmatrix}$$

The KKT Optimality Condition for the NTP

The transportation table is given as:-

Table 2.2

$\frac{\partial c(\bar{x})}{\partial x_{11}}$	. . .	. . .	. . .	$\frac{\partial c(\bar{x})}{\partial x_{1m}}$	S_1	u_1
. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	$\frac{\partial c(\bar{x})}{\partial x_{ij}}$	. . .	. . .	s_i	u_i
$\frac{\partial c(\bar{x})}{\partial x_{n1}}$	. . .	. . .	. . .	$\frac{\partial c(\bar{x})}{\partial x_{nm}}$	s_n	u_n
d_1	. . .	. . .	. . .	d_m		
v_1	. . .	v_j	. . .	v_m		

where $\bar{x}$ is the current basic solution.

The Lagrange function for the NTP is formulated as

$$Z(x, \lambda, \omega) = C(x) + \omega(b - Ax) - \lambda x$$

Where λ and ω are Lagrange multipliers and $\lambda \epsilon \mathbb{R}_+^{nm} U\{0\}$ $\omega \in \mathbb{R}^{n+m}$

The optimal point $\bar{x}$ should satisfy the KKT conditions;

$$\nabla z = \nabla C(\bar{x}) - \omega^T A - \lambda = 0$$

$$\lambda \bar{x} = 0$$

$$\lambda \geq 0$$

$$\bar{x} \geq 0$$

Specifically for each cell (i, j,) we have

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial c(x)}{\partial x_{ij}} - (u, v)\left(e_i, e_{n+j}\right) - \lambda_k = 0 \qquad (3.1)$$

$$\lambda_{ij} x_{ij} = 0$$

$$x_{ij} \geq 0$$

$$\lambda_k \geq 0$$

Where $k = 1 \ . \ . \ . \ nm$ and $w = (u, v) = (u_1, u_2, \cdots, u_n, \ v_1, \cdots, v_m)$, $e_k \in \mathbb{R}^{m+n}$

Is a vector of zeros except at position k which is 1.

From the conditions (3.1) and $\lambda_k \geq 0$, we get,

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial c(\bar{x})}{\partial x_{ij}} - \left(u_i + v_j\right) \geq 0 \qquad (3.2)$$

$$x_{ij} \frac{\partial z}{\partial x_{ij}} = \frac{\partial c(\bar{x})}{\partial x_{ij}} - \left(u_i + v_j\right) \geq 0 \qquad (3.3)$$

$$x_{ij} \geq 0$$

General Solution Procedure for the NTP

Initialization

Find an initial basic feasible solution $\bar{\boldsymbol{x}}$

Iteration

Step 1 If $\bar{\boldsymbol{x}}$ is KKT point, stop. Otherwise go to the next step.

Step 2 Find the new feasible solution that improves the cost function and go to step 1.

3.1 Transportation Problem with Concave Cost Functions

For large shipments, volume discount may be available sometimes. In this case the cost function of the transportation problem generally takes concave structure for it is separable and the marginal cost (cost per unit commodity shipped) decreases with increase of the amount of shipment; and increasing, because of the total cost increase per addition of unit commodity shipped.

The discount may be either directly related to the unit commodity.

Or have the same rate for some amount.

Case 1:

If the discount is directly related to the unit commodity the resulting cost function will be continues and have continues first partial derivatives.

The graph of C_{ij} (x_{ij}) looks like,

Fig 3.2

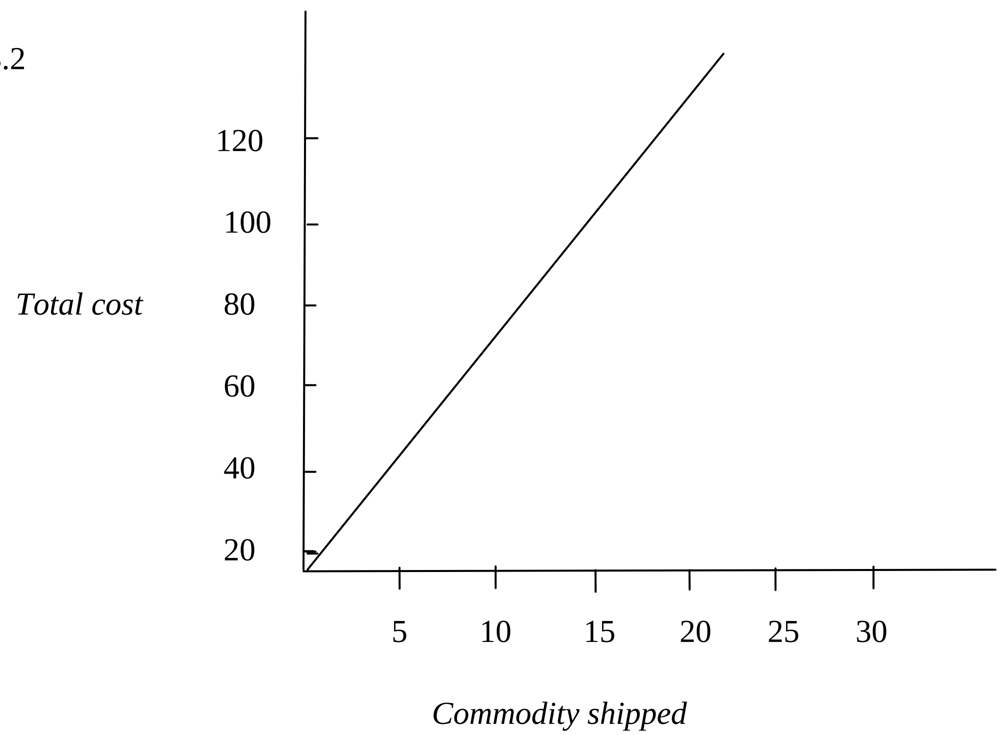


Figure 3.1*: **Transportation problem with continuous volume discount***

Nonlinear programming formulation of such a problem is given by

$$\sum_{i=1}^{n} c_{ij}\, x_{ij} = d_j$$

$$subject\ to$$

$$\sum_{I=1}^{m} x_{ij} = s_i$$

$$\sum_{J=1}^{n} x_{ij} = d_j$$

Where

$$C_{ij} : \mathbb{R} \longrightarrow \mathbb{R}$$

Now before going to look for an optimal solution let's state an important theorem:

Theorem 3.2.4

Let f be concave and continues function and **P** be a non empty compact polyhedral set. Then the optimal solution to the problem

min$f(x)$, x Є**P** exists and can be found at an extreme point of P.

<u>Proof</u>

Let E = (x_1, x_2,,x_k. , x_n) be the set of extreme points of P,

and $\boldsymbol{x}_k$ Є E such that f(x_k) = min {f($\boldsymbol{x}_i$) : i= 1, . . .,n}. Now since P is compact and f is continuous, f attains its minimum in P, call it $\bar{x}$, If $\bar{x}$ is extreme point, we are done. Otherwise, we have that,

$$\bar{x} = \sum_{i=1}^{n} \lambda_i x_i$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i > 0$$

Where $(x_{1,}\, x_{2\ldots\ldots}\, x_n)$ are extreme points of P.

Then by concavity of f it follows that,

$f(\bar{x}) = f(\sum_{i=1}^{n} \lambda_i x_i) \geq \sum_{i=1}^{n} \lambda_i f(x_i) \geq f(x_k) \sum_{i=1}^{n} \lambda_i$

$\Rightarrow f(\bar{x}) \geq f(x_k)$ $\quad (since\ for\ each\ i = 1, \ldots \ldots \ldots;\ f(x_k) \leq f(x_i)\ and\ \sum_{i=1}^{n} \lambda_i = 1).$

Since $\bar{x}$ is minimize, in addition we have,

$$f(\bar{x}) \leq f(x_k)$$

It then follows that

$$f(\bar{x}) = f(x_k).$$

Solution Procedure

Because of the above theorem, it suffices to consider only the extreme points to find the minimum; the following is the procedure

Let $\bar{x}$ be the basic solution we have in the current iteration, i.e. n+m-1.

Next let's decompose our x to ($\bar{x}_B$, $\bar{x}_N$) where $\bar{x}_B$ and $\bar{x}_N$ are the basic and nonbasic variables respectively. Since $\boldsymbol{x_B} > \mathbf{0}$, the complementary slackness condition given in equation (3.3) giver us m + n – 1 equations;

$$\frac{\partial z}{\partial x_{Bij}} = \frac{\partial c\,(\bar{x})}{\partial x_{Bij}} - \left(u_i + v_j\right) = 0 \tag{3.4}$$

From the above relation we can determine the values of u_i and v_j by assigning one of u'_is the value zero for we have m + n variables, u_i and v_j.

Then we calculate $\partial z/\partial x_{ij}$ for the non basic variables x_{ij}. Since all x_{ij} are zero at the extreme, the complementary slackness condition is satisfied.

Therefore if equation (3.2) is satisfied for all non basic variable x_{ij}, $\bar{x}$ is a KKT point.

Otherwise, if

$$\frac{\partial z}{\partial x_{ij}} - \left(u_i + v_j\right) < 0$$

We will move to look for better basic solution such that all the constraints (feasibility conditions) are satisfied. We do this by using the same procedure as the transportation simplex algorithm.

3.1.1 The Transportation Concave Simplex Algorithm (TCS)

Initialization

Find the initial basic feasible solution using some rule like west corner rule.

Iteration

Step 1: Determine the values of u_i and v_i from the equation,

$$\frac{\partial c\,(\bar{x})}{\partial x_{Bij}} - \left(u_i + v_j\right) = 0$$

Where x_{Bij} are the basic variables.

Step 2

If

$$\frac{\partial c(x_{ij})}{\partial x_{ij}} - (u_i + v_j) = 0$$

For all x_{ij} non-basic, stop, x is KKT point. Else go to step 3.

Step 3

Calculate

$$\frac{\partial z}{\partial x_{rl}} = min\,\{\frac{\partial c(x_{ij})}{\partial x_{ij}} - (u_i + v_j)\}$$

x_{rl} will enter the basis.

Allocate $x_{rl} = \theta$ where θ is found as in the linear transportation case.

Adjust the allocations so that the constraints are satisfied.

Determine the leaving variable say x_{Brk}, where x_{Brk} is the basic variable which comes to zero first while making the adjustment.

Then find the new basic variables and go to step 1.

Finite Convergence of the Algorithm

The feasible set of our problem is a non empty polyhedral set. And by definition, a polyhedral set P is a set bounded with a finite number of hyper planes from which it follows that it possesses finite number of extreme points.

In each step of the algorithm, we jump from one extreme point to another looking for a better feasible solution implying that the algorithm will terminate after a finite iteration. In addition since for all i and j, $0 \leq x_{ij} \leq \max\ \{s_i, d_j\}$, P is bounded that guarantees the existence of minimum value.

Example: The following example illustrates the algorithm.

Consider the transportation problem

$$\sum_{i=1}^{3} . \sum_{j=1}^{3} C_{ij}\, x_{ij} = d_j$$

$$x_{11} + x_{12} + x_{13} = 150$$

$$x_{21} + x_{22} + x_{23} = 175$$

$$x_{31} + x_{32} + x_{33} = 275$$

$$x_{11} + x_{21} + x_{31} = 200$$

$$x_{12} + x_{22} + x_{32} = 100$$

$$x_{13} + x_{23} + x_{33} = 600$$

$$C_{11}(x_{11}) = 6x_{11} - 0.01{x^2}_{11} \qquad C_{23}(x_{23}) = 11x_{23} - 0.015{x^2}_{23}$$

$$C_{12}(x_{12}) = 7x_{12} - 0.01{x^2}_{12} \qquad C_{31}(x_{31}) = 4x_{31} - 0.02{x^2}_{31}$$

$$C_{13}(x_{13}) = 10x_{13} - 0.02{x^2}_{13} \qquad C_{32}(x_{32}) = 5x_{32} - 0.02{x^2}_{32}$$

$$C_{21}(x_{21}) = 7x_{21} - 0.03{x^2}_{21} \qquad C_{33}(x_{33}) = 12x_{33} - 0.015{x^2}_{33}$$

$$C_{22}(x_{22}) = 11x_{22} - 0.01{x^2}_{22}$$

Using the West Corner rule we get the initial basic solution.

$$\bar{x} = (x_{B11},\ x_{12},\ x_{13},\ x_{B21},\ x_{B22},\ x_{B23},\ x_{31},\ x_{32},\ x_{B33})$$

$$= (150, 0, 0, 50, 100, 25, 0, 0, 275)$$

The partial derivatives at $\bar{x}$ are given as:

$$\frac{\partial f\,(\bar{x})}{\partial x_{11}} = 3, \qquad \frac{\partial f\,(\bar{x})}{\partial x_{21}} = 4, \qquad \frac{\partial f\,(\bar{x})}{\partial x_{22}} = 9, \qquad \frac{\partial f\,(\bar{x})}{\partial x_{23}} = 10.25, \qquad \frac{\partial f\,(\bar{x})}{\partial x_{33}} = 3.75$$

Now we find,

$$\frac{\partial z}{\partial x_{Bij}} = \frac{\partial f\,(\bar{x})}{\partial x_{Bij}} - \left(u_i + v_j\right) = 0$$

ie

$$\frac{\partial f(x)}{\partial x_{Bij}} = \left(u_i + v_j\right).$$

Hence

$$u_1 + v_1 = 3,\ \ u_2 + v_1 = 4,\ \ u_2 + v_2 = 9,\ \ u_2 + v_3 = 10.25, \qquad u_3 + v_3 = 3.75$$

Letting$u_1 = 0$, from the above equations,we have

$$u_0 = 0, \quad u_2 = 1, \quad u_3 = -5.5$$

$$v_1 = 3,\ v_2 = 8 \quad v_3 = 9.25$$

Then the reduced costs for the non basic variables is

$$\frac{\partial z}{\partial x_{12}} = \frac{\partial f\,(\bar{x})}{\partial x_{12}} - (u_1 + v_2) = 5 \qquad \frac{\partial z}{\partial x_{13}} = \frac{\partial f\,(\bar{x})}{\partial x_{13}} - (u_1 + v_3) = 0.75$$

$$\frac{\partial z}{\partial x_{31}} = \frac{\partial f\,(\bar{x})}{\partial x_{31}} - (u_3 + v_1) = 6.5 \qquad \frac{\partial z}{\partial x_{32}} = \frac{\partial f\,(\bar{x})}{\partial x_{32}} - (u_3 + v_2) = 1.25$$

Since all are non-negative, $\bar{x}$ is **kkt** point and optimal solution to the problem.

Case 2:

In the case when the volume discount is fixed for some amount of commodity, rather than varying with unit amount shipped, the transportation cost function will be piecewise linear concave yet increasing.

fig. 3.3

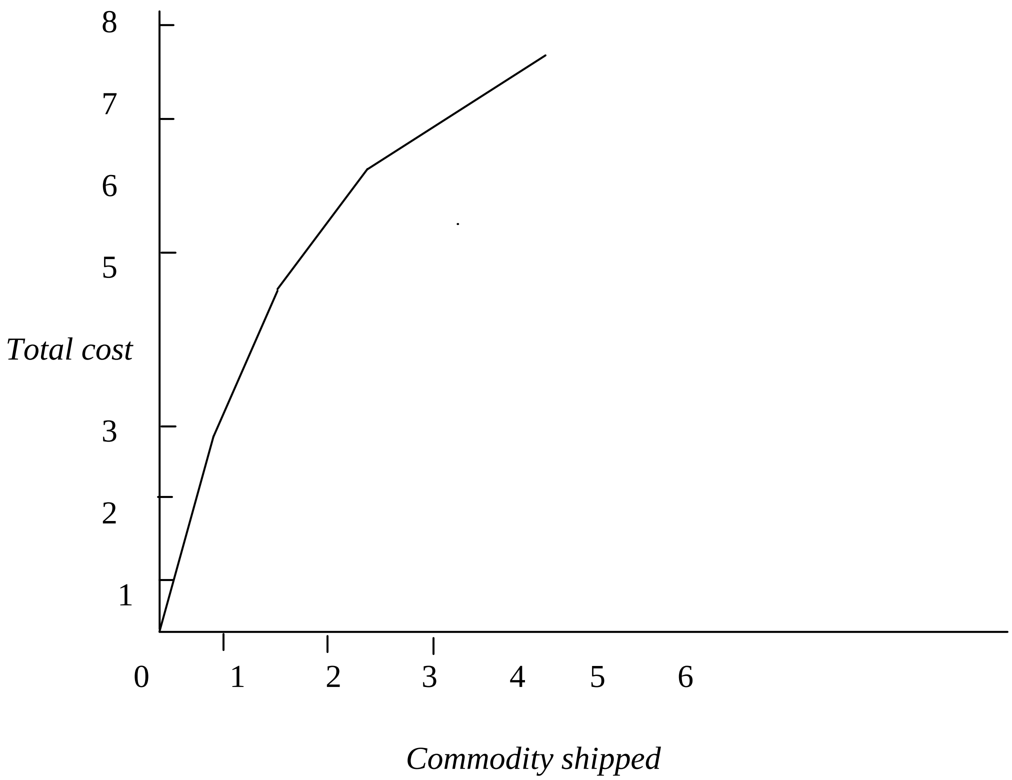


Figure 3.2**:** ***Transportation problem with piecewise linear concave cost***

To avoid complication, assuming that to each combination of source and destination, the interval in which the marginal cost (cost per unit commodity) changes is the same, the cost of shipping x_{ij} units from source i to destination j is given by C_{ij} (x_{ij}), then the nonlinear programming formulation of the problem is given by

$$minimize \qquad \sum_{i=1}^{n} . \sum_{j=1}^{m} C_{ij}\, x_{ij}$$

$$subject\ to \qquad \sum_{I=1}^{m} x_{ij} = s_i$$

$$\sum_{j=1}^{n} x_{ij} = d_j$$

$$i = 1,2, \dots , n \quad j = 1,2,3 \dots , m$$

Where,

$$C_{ij}\, x_{ij} = \begin{cases} C^{0}{}_{ij}\, x_{ij} & 0 \le x_{ij} \le a_1 \\ C^{1}{}_{ij}\, x_{ij} & a_1 \le x_{ij} \le a_2 \\ \dots \dots \dots \dots \dots \dots \dots .. \dots \dots \dots \dots \dots \dots . \\ C^{l}{}_{ij}\, x_{ij} & a_l \le x_{ij} \le a_{l+1} \\ \dots \dots \dots \dots \dots \dots \dots . \dots \dots \dots \dots \dots \dots .. \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ C^{k-1}{}_{ij}\, x_{ij} & a_{k-1} \le x_{ij} \le a_k \\ C^{k}{}_{ij}\, x_{ij} & a_k \le x_{ij} \le b = max(s_i,\ d_j) \end{cases}$$

and

1. $(0, a_1, \dots . a_l, \dots\ a_{k-1},\ a_k,\ b)$ is the partition of the interval *(0, b)* in to $k + 1$ sub intervals

2. Each $C^{l}{}_{ij}$ is linear in the sub interval (a_l, a_{l+1})

To solve this problem, as we can see from the structure of the cost function, it’s impossible to directly apply the algorithm of the previous section for non-differentiability of the total cost function.

But, since the function, also, has a simple structure and differentiability fails at discrete points, it can be easily approximated using differentiable functions like Chebyshev trigonometric or Legendre polynomials.

We choose to approximate it by the so called shifted Legendre polynomials.

These set of Legendre polynomials say (p_0, p_1, . . . p_r) are orthogonal in (0,1) with respect to weight function w(x) = 1, where the inner product on C(0,1) is defined by

$$< f, g > = \int_0^1 f(x)g(x)dx, \textit{ for all } f, g \in C\ (0,1)$$

Where *C (0,1)* is the space of continuous functions on (0,1).

The first three terms are,

$p_0(x) = 1$

$p_1(x) = x$

$p_2(x) = 3x^2 - 1$

And the others can be obtained from

$$p_r(x) = \frac{1}{2^r!}\frac{d^r[(x^2-1)^r]}{dx^r}$$

Then, the space spanned by $(p_0, p_1, \dots \dots \dots p_r)$ is a subspace of *C(0, 1).* Hence, given any *f(x)* $\in$ *C (0,1),* we can find a unique least square approximation of *f* in the subspace. Note that every element of the subspace spanned $(p_0, p_1, \dots \dots \dots p_r)$ is at least twice differentiable.

The least square approximation of any function f(x) with r of these polynomials in (0,1) is given by,

$$f(x) = a_o p_o(x) + a_1 p_1(x) \dots \dots \dots . a_i p_i(x) \dots \dots a_r p_r(x)$$

where

$$a_i = \frac{\int_0^1 p_i f(x)dx}{\int_0^1 [p_i(x)]^2 dx} \qquad i = 0, 1, 2, \ldots \ldots . r$$

To approximate our functions $C_{ij}(x_{ij})$, in the same manner, we define a one to one correspondence between (*0*, *b*) to (*0*,*1*) by

$$g: (0,\ b) \to (0,\ 1)$$

$$g(x_{ij}) = \frac{1}{b} x_{ij}$$

That is, we substitute x_{ij} by $(\frac{1}{b} x_{ij})$so that its domain will be (0,1).

Then we have,

$$x_{ij} \to C^*{}_{ij}(x_{ij}) = C_{ij}(\frac{1}{b} x_{ij}) = \begin{cases} C^0{}_{ij}(\frac{1}{b} x_{ij}) & 0 \le x_{ij} \le \frac{a_i}{b} \\ C^1{}_{ij}(\frac{1}{b} x_{ij}) & \frac{a_i}{b} \le x_{ij} \le \frac{a_2}{b} \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \ldots \ldots \ldots \ldots \ldots \ldots . & \\ C^l{}_{ij}(\frac{1}{b} x_{ij}) & \frac{a_l}{b} \le x_{ij} \le \frac{a_{l+1}}{b} \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \ldots \ldots \ldots \ldots \ldots \ldots . . & \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \\ C^{k-1}{}_{ij}(\frac{1}{b} x_{ij}) & \frac{a_{k-1}}{b} \le x_{ij} \le \frac{a_k}{b} \\ C^k{}_{ij}(\frac{1}{b} x_{ij}) & \frac{a_k}{b} \le x_{ij} \le 1 = max(s_i,\ d_j) \end{cases}$$

Now, after approximating $C^*{}_{ij}(x_{ij})$ by the shifted Legendre polynomials on (0,1), assume we have found it's best approximation at $C^\wedge{}_{ij}(x_{ij})$.

Then, substituting back the x_{ij} in $C^\wedge{}_{ij}$ by $\boldsymbol{b}x_{ij}$ gives us the approximation to

$C_{ij}(x_{ij})$ over (0,b). Therefore the best approximation of $C_{ij}(x_{ij})$ over (0,b) will be

$$C_{ij}(\overline{x}_{ij}) = C^\wedge{}_{ij}(bx_{ij}).$$

This has continuous derivatives.

Consequently, we solve the problem

$$minimize \qquad \sum_{i=1}^{n} . \sum_{j=1}^{m} C_{ij} x_{ij} = \sum_{l=0}^{2} . \sum_{i=1}^{n} . \sum_{j=1}^{m} a_l p_l(x_{ij})$$

$$s.t. \qquad \sum_{j=1}^{m} x_{ij} = \frac{s_i}{b}$$

$$\sum_{i=1}^{n} x_{ij} = \frac{d_j}{b}$$

$$i = 1, 2, \dots \dots \dots \dots \dots .., n \qquad j = 1, 2, \dots \dots \dots \dots \dots \dots m,$$

We then use the same procedure as above to solve the problem.

3.2. **Convex Transportation Problem:** This case may arise when the objective function is composed of not only the unit transportation cost but also of production cost related to each commodity. Or in the case when the distance from each source to each destination is not fixed,

The problem can be formulated as

$$minimize \qquad C(x)$$

$$subject\ to \qquad Ax = b$$

$$x \geq 0$$

Where C (x) is convex, continuous and has continuous first order partial derivatives.

The Convex Simplex solution procedure for Transportation problem

In the case when the cost function is convex, the minimum point may not be attained necessarily at an extreme; it may be found before reaching a boundary of the feasible set.

What precisely happens is that there may be non basic variable with positive allocation while none of the basis is driven to zero.

To solve this problem, we use the idea of the convex simplex algorithm of Zangwill (32) which was originally designed to take care of convex and pseudo convex problem with linear constraints. Actually the original procedure is used to look for a local optimal solution for any other linearly constrained programming problem.

We use the special structure of transportation problem in the procedure so as to make it efficient for our particular problem.

The method reduces to the ordinary transportation simplex algorithm whenever the objective function is linear, to the method of Beal when it is quadratic and to the above concave simplex procedure when the function is concave.

We partition the variable, $x = (x_{11},\ x_{12} \dots\dots x_{nm})\ to\ (x_B, x_N)$. Where x_B is ***n + m – 1*** component vector of basic variables and $x_N$is *nm – (n + m -1)* component vector of non-basic variables, corresponding to the *(n + m – 1) X(n + m – 1)* basis sub matrix and *(n +m – 1)X(nm – (n + m – 1)* non basic sub matrix of A.

Suppose we have the initial basic feasible solution $\bar{x}^0$.

In the procedure what we do is to find a mechanism in which non optimal basic solution x at a given iteration is improved until it satisfies the KKT conditions which are also sufficient conditions for convex transportation problem,

i.e, until for each cell we have;

$$x_{ij}\left(\frac{\partial f(\bar{x})}{\partial x_{ij}} - (u_i + v_j)\right) = 0 \qquad \text{and} \qquad \frac{\partial f(\bar{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0$$

Since we have each basic variable $x_{Bij} > 0$, the above complementary

Slackness condition implies that for each basic cell,

we must have

$$\frac{\partial f(\overline{x})}{\partial x_{Bij}} - (u_i + v_j) = 0$$

x_{Bij} basic variable.

Since we have ***n* + *m* – *1*** of such equations, by letting $u_1 = 0$ we obtain all the values of u_i and v_i as we have done exactly for the concave and linear cases.

Now for a non basic cell, at a feasible iterate point x; we may have:

(i) $\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) > 0$ $\qquad x_{st}\frac{\partial f(\overline{x})}{\partial x_{st}} - (u_i + v_j) > 0$

(ii) $\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) < 0$ $\qquad x_{st}\frac{\partial f(\overline{x})}{\partial x_{st}} - (u_i + v_j) < 0$

(iii) $\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0$ $\qquad x_{st}\frac{\partial f(\overline{x})}{\partial x_{st}} - (u_i + v_j) > 0$

From the KKT conditions given earlier, the last case occurs when x is optimal.

But if the solution x falls on either of the other three, it must be improved as follows.

Let $IJ = (ij : x_{ij}\ is\ non - basic\ variable)$ and suppose that we are in the k^{th}iteration.

We first begin by computing;

$$\frac{\partial z}{\partial x_{rl}} = min\left\{\frac{\partial f(\overline{x}^k)}{\partial x_{ij}} - (u_i + v_j)\right\} \qquad ij \in IJ$$

$$x_{st}\frac{\partial z}{\partial x_{st}} = max\left\{x_{ij}\left[\frac{\partial f(\overline{x}^k)}{\partial x_{ij}} - (u_i + v_j)\right]\right\} \qquad ij \in IJ$$

Here we do not want to improve (decrease) a positive valued non-basic variable x_{ij} unless its partial derivative is positive.

Therefore we only focus on the positive values of the product

$$x_{ij}\frac{\partial z}{\partial x_{ij}}$$

Now the variable to be selected is as below,

Case 1

$$for \qquad \frac{\partial z}{\partial x_{rl}} \geq 0 \quad and \quad x_{st}\frac{\partial z}{\partial x_{st}} \geq 0$$

Decrease x_{st} by the value θ using the transportation table as in the linear and concave cases.

Let $y^k = (y^k{}_{11}, y^k{}_{12}, \dots \dots \dots . y^k{}_{nm})$ be the value of $x^k = (x^k{}_{11}, x^k{}_{12}, \dots \dots \dots . x^k{}_{nm})$ after making the necessary adjustment by adding and subtracting θ in the loop containing x_{st} so that all the constraints are satisfied.

By doing so, either x_{st} itself or a basic variable say x_{Bst} will be driven to zero.

Now y^k may not be the next iterate point; since the function is convex, a better point could be found before reaching y^k to check this, we solve problem;

$$f(x^{k+1}) = \ min\{\,f(\lambda x^k + (1-\lambda)y^k : 0 \leq \lambda \leq 1\} \qquad 3.5$$

$and\ get\ (\bar{x}^{k+1}) = \ \{\,\bar{\lambda}\bar{x}^k + (1-\bar{\lambda}\,)y^k\}\,where\ \bar{\lambda}$ is the optimal solution of equation (3.5)

Before the next iteration,

If $\bar{x}^{k+1} = \lambda x^k$ and if a basic variable became zero during adjustment made, we change the basis.

$if\ x^{k+1} \neq y^k or\ x^{k+1} \equiv y^k and x_{st}$ is driven to zero, we don't change the basic by substituting the leaving basic variable by x_{st}.

Case 2

$$for \quad \frac{\partial z}{\partial x_{rl}} < 0 \;\; and \;\; x_{st}\frac{\partial z}{\partial x_{st}} \leq 0$$

In this case the value of x_{rl} should be increased by θ and then we find y^k, where θ and y^k are defined as in the case 1.

Note that: as we increase the value of x_{rl} one of the basic variables, say, x_{Bt} will be driven to zero, and this is the exit criteria of the linear and concave transportation simplex algorithm and y^k would have been the next iterate point of the procedure.

But now after solving for $\bar{x}^{k+1}$ from 3.5, before going to the next iteration, we will have the following possibilities.

$if \; x^{k+1} = y^k$ we change the former basis, substitute x_{Bt} by x_{rl}.

$if \; x^{k+1} \neq y^k$ we do not change the basis. All the basic variables outside of the loop will remain unchanged.

Case 3

$$for \qquad \frac{\partial z}{\partial x_{rl}} < 0 \;\; and \;\; x_{st}\frac{\partial z}{\partial x_{st}} > 0$$

In this case either we decrease x_{st} as in the case 1 or increase x_{rl} according to case 2.

3.2.1 **The Transportation Convex Simplex Algorithm**

Now we write the formal algorithm for solving the convex transportation problem.

Initialization

Final the initial basic feasible solution

Iteration

Step 1: Determine all u_{i} and v_{j} from

$$\frac{\partial f\left(x^{k}\right)}{\partial x_{Bij}} - u_i - v_j = 0$$

For each basic cell

Step 2: For each non basic cell, calculate;

$$\frac{\partial z}{\partial x_{rl}} = min.\left\{\frac{\partial f\left(x^k\right)}{\partial x_{ij}} - u_i - v_j\right\}$$

$$x_{st}\frac{\partial z}{\partial x_{rl}} = \max\left\{x_{ij}\left(\frac{\partial f\left(x^k\right)}{\partial x_{ij}} - u_i - v_j\right)\right\}$$

$$If \qquad \frac{\partial z}{\partial x_{rl}} \geq 0 \ \ and \ \ x_{st}\frac{\partial z}{\partial x_{st}} = 0$$

Step 3

Determine the non-basic variable to change.

Decrease x_{rl} according to case if $\frac{\partial z}{\partial x_{rl}} < 0 \ \ and \ \ x_{st}\frac{\partial z}{\partial x_{st}} > 0$.

Increase x_{rl} according to case 2 if $\frac{\partial z}{\partial x_{rl}} < 0 \ \ and \ \ x_{st}\frac{\partial z}{\partial x_{st}} \leq 0$.

Either increase x_{rl} or decrease x_{st} if $\frac{\partial z}{\partial x_{rl}} < 0 \ \ and \ \ x_{st}\frac{\partial z}{\partial x_{st}} < 0$.

Step 4:

Find the values of y^k, by means of θ, and $\overline{x}^{k+1}$, from 3.5

If $y^k = \overline{x}^{k+1}$and a basic variable is driven to zero, change the basic otherwise do not change the basis.

$\overline{x}^k = x^{k+}$ go to step 1.

An Example.

This example is used by Zangwill to illustrate the Convex Simplex Algorithm.

$\min z = f(x) = x_{11} + 2x_{12} + x^2{}_{12} + x^2{}_{13} + 3x_{22} + 2x^2{}_{23} + e(x_{11} + x_{21})$

$\quad subject\ to$

$$x_{11} + x_{12} + x_{13} = 2 = s_1$$

$$x_{21} + x_{22} + x_{23} = 2 = s_2$$

$$x_{11} + x_{21} = 1 = d_1$$

$$x_{12} + x_{22} = 2 = d_2$$

$$x_{13} + x_{23} = 2 = d_3$$

$$x_{ij} \geq 0$$

Using the west corner rule, we found

$\bar{x}^1 = x_{B11} + x_{B12} + x_{13} + x_{21} + x_{B22} + x_{B23} = (1, 2, 0, 0, 0, 2)$

$\quad is\ initial\ basic\ feasible\ solution.$

Then

$$\frac{\partial f(x^1)}{\partial x_{11}} = 1 \qquad \frac{\partial f(x^1)}{\partial x_{12}} = 2 \qquad \frac{\partial f(x^1)}{\partial x_{13}} = 0$$

$$\frac{\partial f(x^1)}{\partial x_{21}} = 1 \qquad \frac{\partial f(x^1)}{\partial x_{22}} = 3 \qquad \frac{\partial f(x^1)}{\partial x_{23}} = 8$$

$the\ reduced\ cost$

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial f(x^1)}{\partial x_{ij}} - u_i - v_j$$

are calculated and we get the values

$$\frac{\partial z}{\partial x_{11}} = 0 \quad \frac{\partial z}{\partial x_{12}} = 0 \quad \frac{\partial z}{\partial x_{13}} = -7$$

$$\frac{\partial z}{\partial x_{21}} = -1 \quad \frac{\partial z}{\partial x_{22}} = 0 \quad \frac{\partial z}{\partial x_{23}} = 0$$

$$\frac{\partial z}{x_{13}} = \ min\ \left\{ \frac{\partial f\,(\bar{x}^1)}{\partial x_{ij}} - u_i - v_j \right\} = -7 < 0$$

$$\frac{\partial z}{\partial x_{rl}} = \max\ \left\{ x_{ij} \left(\frac{\partial \bar{f}\,(x^1)}{\partial x_{ij}} - u_i - v_j \right) \right\} = 0$$

Therefore x_{13} *must be increased by* θ *where*

$$\theta = \min(x_{12}\ \ x_{23}) = 2$$

Thereby making the value of

$y^1 = (1, 0, 2, 0, 2, 0)$

Now to find the value of $\bar{x}^2$ we solve

$f(x^2) = \ min\,\{ f(\lambda\bar{x}^k + (1-\lambda)\bar{y}^1\} : \quad 0 \le \lambda \le 1$

This gives us

$$\lambda = \frac{5}{12}\ and\ x^2 = (x_{B11}, x_{B12}, x_{13}, x_{21}, x_{B22}, x_{B23})$$

$$= (1\ \ \tfrac{5}{6}\ \ \tfrac{7}{6}\ \ 0\ \ \tfrac{7}{6}\ \ \tfrac{5}{6})$$

Where $\lambda = \frac{\sum x_B}{no.of\ cells}$ and $x^2 = \lambda x^1 + (1-\lambda)\, y^1$

Bases are not changed.

Calculating the reduced costs, we get,

$$\frac{\partial z}{\partial x_{rl}} = \min\left(\frac{\partial f(x^2)}{\partial x_{ij}} - u_i - v_j\right) = -1 < 0$$

$$x_{ij}\ \frac{\partial z}{\partial x_{st}} = max\left\{x_{ij}\left(\frac{\partial f(\overline{x}^2)}{\partial x_{ij}} - u_i - v_j\right)\right\} = 0$$

Therefore we increase x_{21} by θ calculated as before.

This time x_{21} becomes basic while x_{11} leaves the basis

Then we get $y^2 = (0, \frac{11}{6}, \frac{7}{6}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6})$

Continuing the process, we find $x^3 = y^2$

The reduced costs are

$$\frac{\partial z}{\partial x_{11}} = 1 \quad \frac{\partial z}{\partial x_{12}} = 0 \quad \frac{\partial z}{\partial x_{13}} = 0$$

$$\frac{\partial z}{\partial x_{11}} = 0 \quad \frac{\partial z}{\partial x_{12}} = 0 \quad \frac{\partial z}{\partial x_{13}} = 0$$

And all

$$x_{ij}\ \frac{\partial z}{\partial x_{ij}} = \max\left\{x_{ij}\left(\frac{\partial f(\overline{x}^k)}{\partial x_{ij}} - u_i - v_j\right)\right\} = 0$$

Therefore

$\overline{x_3} = (0, \frac{11}{6}, \frac{7}{6}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6})$

is the optimal solution.

CHAPTER 4

DATA ANALYSIS

Introduction

In this chapter we examine a practical application of the above solution procedures. Emphasis will be on the concave transportation problem. We will examine the data obtained from the GALCO Company limited; manipulate the data to suit our transportation problem.

DATA AND ANALYSIS

The GALCO Ghana limited is a manufacturing company located in Kumasi. They produce pomade (obaatannku), body powder, machine oil, candle wax etc. these products are supplied to the 10 regional capitals of the country. For the purpose of this project only 4 of these demand points will be considered; Accra sunyani tamale and bolga. The estimated supply capacities of the three products, the demand requirements at the four sites (regions) and the transportation cost per box of each product are given below.

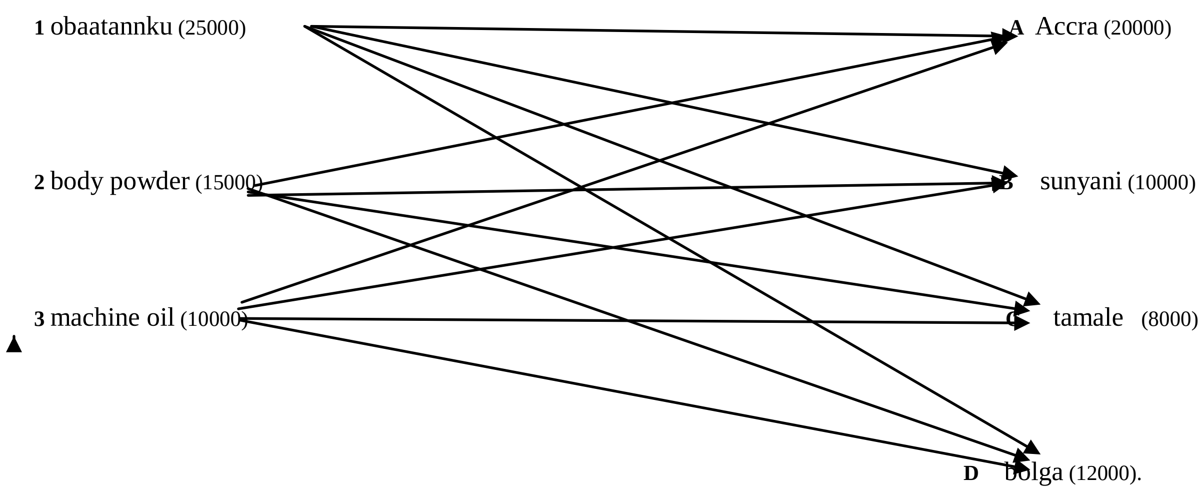
	Accra	Tamale	Sunyani	Bolga	Supply
1. Body powder	15	10	4	20	15000
2. Obaatannku	7	6	8	3	25000
3. Machine oil	1	9	5	3	10000

Demand	20000	10000	8000	12000	**50000**

The problem is to determine how many boxes of each product to be transported from the source to each destination on a monthly basis in order to minimize the total transportation cost.

A diagram of the different transportation routes with supply and demand figures is shown below.

<u>*Supply*</u> <u>*demand*</u>



Forming the transportation tableau.

To form transportation tableau, let

i= product to be shipped.

j = destination of each product.

s_i =the capacity of source node *i*,

d_j =the demand of destination *j*,

x_{ij}= the total capacity from source *i* to destination *j*

C_{ij} = the per unit cost of transporting commodity from *i* to destination *j*.

p_{m} = percentage discount allowed for transporting from *i* to destination *j*.

If we suppose that discount is given on each box transported from *i* to *j* then the non linear transportation problem can be formulated as:

$$min \quad \sum_{i=1}^{3} . \sum_{j=1}^{4} C_{ij}\, x_{ij}$$

$$x_{11} + x_{12} + x_{13} + x_{14} = 15$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 25$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 10$$

$$x_{11} + x_{21} + x_{31} = 20$$

$$x_{12} + x_{22} + x_{32} = 10$$

$$x_{13} + x_{23} + x_{33} = 8$$

$$x_{14} + x_{24} + x_{34} = 12$$

Where

$$C_{11}x_{11} = 15x_{11} - p_{11}x^2{}_{11}C_{23}x_{23} = 8x_{23} - p_{23}x^2{}_{23}$$

$$C_{12}x_{12} = 10x_{12} - p_{12}x^2{}_{12}C_{24}x_{24} = 3x_{24} - p_{24}x^2{}_{24}$$

$$C_{13}x_{13} = 4x_{13} - p_{13}x^2{}_{13}C_{31}x_{31} = x_{31} - p_{31}x^2{}_{31}$$

$$C_{14}x_{14} = 20x_{14} - p_{14}x^2{}_{14}C_{32}x_{32} = 9x_{32} - p_{32}x^2{}_{32}$$

$$C_{21}x_{21} = 7x_{21} - p_{21}x^2{}_{21}C_{33}x_{33} = 5x_{33} - p_{33}x^2{}_{33}$$

$C_{22}x_{22} = 6x_{22} - p_{22}{x^2}_{22} C_{34}x_{34} = 2x_{34} - p_{34}{x^2}_{34}$ If we allow the following discounts on each transported product i from the source to each of the destinations,

$(p_{11}, p_{12}, p_{13}, p_{14}, p_{21}, p_{22}, p_{23}, p_{24}, p_{31}, p_{32}, p_{33}, p_{14})$

= (0.02, 0.01, 0.04, 0.07, 0.01, 0.04, 0.03, 0.02, 0.005, 0.03, 0.015, and 0.01).

$then\ the\ cost\ function\ C_{ij}\ can\ be\ expressed\ as$

$$C_{11}x_{11} = 15x_{11} - 0.02{x^2}_{11} \qquad C_{23}x_{23} = 8x_{23} - 0.03{x^2}_{23}$$

$$C_{12}x_{12} = 10x_{12} - 0.01{x^2}_{12} \qquad C_{24}x_{24} = 3x_{24} - 0.02{x^2}_{24}$$

$$C_{13}x_{13} = 4x_{13} - 0.04{x^2}_{13} \qquad C_{31}x_{31} = x_{31} - 0.005{x^2}_{31}$$

$$C_{14}x_{14} = 20x_{14} - 0.07{x^2}_{14} \qquad C_{32}x_{32} = 9x_{32} - 0.03{x^2}_{32}$$

$$C_{21}x_{21} = 7x_{21} - 0.01{x^2}_{21} \qquad C_{33}x_{33} = 5x_{33} - 0.015{x^2}_{33}$$

$$C_{22}x_{22} = 6x_{22} - 0.04{x^2}_{22} \qquad C_{34}x_{34} = 2x_{34} - 0.01{x^2}_{34}$$

We develop the tableau as below,

x^4	Accra	Sunyani	Tamale	Bolga	S_1	u_1
1	15	10	4	20	15000	u_1
2	7	6	8	3	25000	u_2
3	1	9	5	2	10000	u_3
d_1	20000	10000	8000	12000		
v_i	v_1	v_2	v_3	v_3		

Total supply = 50000

Total demand = 50000

Hence the tableau is balanced.

Using the West Corner rule we get the initial basic solution.

The solution tableau is as shown below,

	Accra	sunyani	tamale	Bolga	S_1	u_1
1	15 **15**	10	4	20	15000	u_1
2	7 **5**	6 **10**	8 **8**	3 **2**	25000	u_2
3	1	9	5	2 **10**	10000	u_3
d_1	20000	10000	8000	12000		
v_{i}	v_1	v_2	v_3	v_3		

$\overline{x} = (x_{B11}, x_{12}, x_{13}, x_{14}, x_{B21}, x_{B22} x_{B23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{B34})$

$= (15, 0, 0, 0, 5, 10, 8, 2, 0, 0, 0, 10), in\ thousands.$

$With\ total\ transportation\ cost\ of$

$1500 * 15 + 5000 * 5 + 10000 * 6 + 8000 * 8 + 2000 * 3 + 10000 * 2$

$= ₵\,400000$

The partial derivatives at x are given as:

$$\frac{\partial f(x)}{\partial x_{11}} = 14.4 \qquad \frac{\partial f(x)}{\partial x_{12}} = 10 \qquad \frac{\partial f(x)}{\partial x_{13}} = 4 \qquad \frac{\partial f(x)}{\partial x_{21}} = 20$$

$$\frac{\partial f(x)}{\partial x_{21}} = 6.9 \qquad \frac{\partial f(x)}{\partial x_{22}} = 5.2 \qquad \frac{\partial f(x)}{\partial x_{23}} = 7.52 \qquad \frac{\partial f(x)}{\partial x_{24}} = 2.92$$

$$\frac{\partial f(x)}{\partial x_{31}} = 1 \qquad \frac{\partial f(x)}{\partial x_{32}} = 9 \qquad \frac{\partial f(x)}{\partial x_{33}} = 5 \qquad \frac{\partial f(x)}{\partial x_{34}} = 1.8$$

Now we find,

$$\frac{\partial z}{\partial x_{Bij}} = \frac{\partial f(\bar{x})}{\partial x_{Bij}} - u_i - v_j = 0$$

$$ie \quad \frac{\partial f(x)}{\partial x_{Bij}} = u_i + v_j$$

hence,

$$u_1 + v_1 = 14.4, \qquad u_1 + v_2 = 10, \qquad u_2 + v_2 = 5.2$$

$$u_1 + v_1 = 14.4 \qquad u_1 + v_2 = 10 \qquad u_2 + v_2 = 5.2$$

Letting $u_1 = 0$, from the equations;

We have

$$u_0 = 0, \qquad u_2 = -7.5, \qquad u_3 = -6.62$$

$$v_1 = 14.4, \quad v_2 = 12.7, v_3 = 15.02, v_4 = 8.42$$

Then the reduced costs for the non-basic variables is

$$\frac{\partial z}{\partial x_{12}} = \frac{\partial f(\bar{x})}{\partial x_{12}} - u_1 - v_2 = 2.3 \qquad \frac{\partial z}{\partial x_{13}} = \frac{\partial f(\bar{x})}{\partial x_{13}} - u_1 - v_3 = -11.02$$

$$\frac{\partial z}{\partial x_{14}} = \frac{\partial f(\bar{x})}{\partial x_{14}} - u_1 - v_4 = 11.58 \qquad \frac{\partial z}{\partial x_{31}} = \frac{\partial f(\bar{x})}{\partial x_{31}} - u_3 - v_1 = -6.78$$

$$\frac{\partial z}{\partial x_{32}} = \frac{\partial f(\bar{x})}{\partial x_{32}} - u_3 - v_2 = 2.92 \qquad \frac{\partial z}{\partial x_{33}} = \frac{\partial f(\bar{x})}{\partial x_{33}} - u_3 - v_3 = -3.4$$

$$\underline{\partial z} = \min \left\{ \underline{\partial z} \right\} = -11.02$$

∂x_{13} ∂x_{ij}

The presence of negative values for the reduced cost signifies non optimality; hence we readjust.

Therefore x_{13} should enter the basis since it is the most negative reduced cost; after adjusting the values x_{23} left the basic.

Continuing in the same manner, after three iterations (excluding the first)

The reduced costs for the non-basic ones at a basic feasible point

$$\bar{x}^4 = (x_{11}, x_{B12}, x_{B13}, x_{14}, x_{B21}, x_{B22}, x_{23}, x_{B24}, x_{31}, x_{B32}, x_{34})$$

$$= (0, 7, 8, 0, 10, 3, 0, 12, 0, 10, 0, 0)$$

Will be;

$$\frac{\partial z}{\partial x_{11}} = \frac{\partial f(\bar{x})}{\partial x_{11}} - u_1 - v_1 = 4.1 \qquad \frac{\partial z}{\partial x_{14}} = \frac{\partial f(\bar{x})}{\partial x_{14}} - u_1 - v_4 = 13.38$$

$$\frac{\partial z}{\partial x_{23}} = \frac{\partial f(\bar{x})}{\partial x_{23}} - u_2 - v_3 = 8.74 \qquad \frac{\partial z}{\partial x_{32}} = \frac{\partial f(\bar{x})}{\partial x_{32}} - u_3 - v_2 = 9.14$$

$$\frac{\partial z}{\partial x_{33}} = \frac{\partial f(x)}{\partial x_{33}} - u_3 - v_3 = 11.64 \qquad \frac{\partial z}{\partial x_{34}} = \frac{\partial f(x)}{\partial x_{34}} - u_3 - 4 = 5.38$$

All are non-negative, implying that x^4 is a KKT point.

In fact, the optimal solution to our problem.

Hence the following allocation should be made:

7000 boxes of obaatan nku should be supplied to Sunyani, 8000 of same product be supplied to Tamale. Allocate 10000 boxes of body powder to Accra, 3000 boxes to sunyani and 12000 boxes to Bolga. Finally allocate 10000 boxes of machine oil to Accra.

$$\boldsymbol{Total\ cost} = 10000\,x\,7 + 8000\,x\,4 + 10000\,x\,7 + 3000\,x\,6 + 12000\,x\,3 + 10000\,x\,9$$

$$= ¢\mathbf{316,000}$$

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

Sometimes, there may be different ways to model a particular problem, but choosing the best approach minimizes the complexity of the problem and time to solve.

Since any programming problem with constraint matrix structure the same as the transportation problem, it can be regarded as a transportation type problem regardless of its physical meaning and because of its simple structure, modeling such problems as transportation problem requires much less effort to solve than modeling it differently.

In this paper the nonlinear transportation problem is considered as a nonlinear programming problem and algorithms to solve this particular problem are given. The first algorithm is similar to that of the transportation simplex algorithm except for the nonlinearity assumption. The second algorithm is dependent on the simplex algorithm of Zangwill that we modified to use the special property of the coefficient matrix of the transportation problem so that we may take shortcuts to make problem solving simple.

For the purposes of even distribution of each product, reasonable reallocation could be made to unassigned destinations.

However, the algorithms are not compared to any other previous algorithms therefore in the future further work should be done to:

1. *Measure the efficiency of the algorithm.*
2. *Check how near the solution of the approximated problem of the piecewise nonlinear*

$transportation\ problem\ is\ to\ the\ optimal\ solution\ of\ the\ original\ problem.$

3. $To\ implement\ the\ algorithm\ to\ complex\ real\ life\ problems.$

We then conclude that given discounts on cost of transportation could lead to increased productivity of producers. This as a result of the fact that wholesalers and retailers, will have to pay less on transport for buying in large quantities; subsequently, consumers will buy at lower cost comparatively.

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