KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

Mathematical Modelling and Optimal Control of Rabies Transmission Dynamics

By

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(Bsc. Mathematics)

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Declaration

I hereby declare that this submission is my own work towards the award of the M.Phil. degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

I dedicate this work to the Almighty God, who made all things possible by His own understanding.

Abstract

In this thesis, we present an SEIRS model to describe the transmission dynamics of rabies virus in dogs and humans, using optimal control theory. We study the effect of pre and post-exposure prophylaxis on both compartments. Our analysis shows that the model is mathematically and epidemiologically meaningful, and well-posed. From the analysis, it shows that with an effective pre-exposure prophylaxis in the human population and the vector population, the rate of rabies transmission will be minimize, and additional control measure on the exposed dogs will minimize the spread of the rabies virus in both compartments, such measures could be post-exposure prophylaxis or culling of exposed dogs. Using the Routh-Hurwitz criterion, it shows that the disease-free equilibrium E_0, is locally asymptotically stable, if $R_0 < 1$, applying the Lyapunov function shows that the disease-free equilibrium is globally asymptotically stable, if $R_0 \leq 1$, and the endemic equilibrium is global asymptotically stable, if $R_0 > 1$. We also study the controllability of the control model, and then obtain an optimal cost-dependent and time-dependent effort, to minimize the spread of rabies virus in the exposed and infected classes. The simulation of our 8-differential equations using the forward-backward sweep scheme and the fourth order Range-Kutta numerical method, shows that applying pre-exposure prophylaxis (vaccination) and post-exposure prophylaxis (treatment) in both compartments have a considerable effect in reducing the number of infected dogs and humans with rabies than when a single control strategy is use.

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To the uncreated God, who is always with His creation, The Holy Spirit who will never leave nor forsake His children no matter what happens. In fact, this thesis was totally inspired by The Holy Spirit, and He has brought it to a fruitful end. My heartfelt thanks go to my supervisor, Dr. F. T. Oduro, for always being there for me and providing many relevant suggestions and worthy opinions on all the chapters of this thesis. I am grateful to him and may the good Lord bless him and enrich his life with good health. I would also like to acknowledge the lecturers of the Department of Mathematics, Knust, for bringing out the best in me. In addition, thanks go to Prof. S.K. Amponsah, Dr. Edward Prempeh, Dr. Richard Avuglah and Dr. Osei Frimpong, for their motivation, advice and invaluable contribution towards my study at Knust. My deepest gratitude goes to brother Joachim Asamoah, Sister Adwoa, Sister Freda and my parents for their love, moral support, patience and financial support, during my studies. I would like to express my gratitude to Frances Pairaudeau for reading through my work and my colleagues, especially, George Apaaboah, Patrick Anamuah Mensah, and Ibrahim Issaka for their encouragement. Lastly, I would like to extend my gratitude to all those who assisted me in diverse ways during the completion of this thesis.

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Chapter 1

Introduction

1.1 Background of the Study

Mathematical approach has been proved to be an important tool in epidemiology study (Murray et al., 1988). Utilizing this methodology on epidemiological study gives understanding of the epidemic features of the spreading law and control measure of pestilences (Murray et al., 1988). Generally, most human rabies cases are as a result of a dog bite (WHO, 2012). Rabies is an infection that mostly affect the brain of the infected animal or individual, through the saliva or tissues from the nervous system from an infected mammal to another mammal (WHO, 2012). Rabies is caused by a neurotropic virus (WHO, 2010).

Any warm-blooded mammal can get the rabies virus through a bit of infectious animal (WHO, 2010). This infection has turned into a worldwide problem, it is evaluated that rabies happens in more than 150 nations and regions (WHO, 2012). Raccoons, skunks, bats and foxes are the main animals that transmit the virus in the United States (WHO, 2010). Also this disease has been a major threat in Asia, Africa and Latin America where vaccination and medicines are sometimes not available (WHO, 2010). When the virus enters the human body or that of an animal, the infection (virus) moves rapidly along the neural pathways to the central nervous system. From there the virus continues to spread to other organs and causes injury by interrupting various nerves.

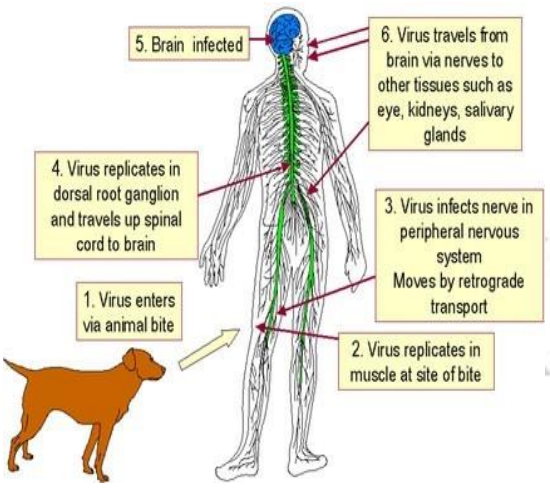
The symptoms of rabies are quite similar to those of the flu (Rupprecht et al., 2010). Treatment after exposure is known as post-exposure prophylaxis (PEP), and vaccination before exposure is known as pre-exposure prophylaxis. Due to movement of the carriers, the risk of not being exposed to the infection can never be guaranteed.

Optimal control has also been recently applied to an epidemic model for rabies in raccoons in researched performed by Artois et al. (1997); Allen et al. (2002). In their study, the control vector gives the rate of vaccination in the subpopulation that minimizes the exposed and infected classes over all sub-populations.

This accounts for the cost effectiveness of administering the vaccine. Rabies is estimated to have caused 55,000 human deaths in Africa and Asia (WHO, 2012). Individuals that are mostly at risk to the rabies infection live in rural territories, where human immunizations and immunoglobulin is not promptly accessible or open, but rather it is realized that individuals that get pre-exposure, immunization are at low risk of getting the rabies infection (Rupprecht et al., 2010). Hence, we want to study the effect of perexposure, post-exposure vaccines, and culling effect on the model, so as to minimize the infection in the expose and the infected classes, and also compare the existing model of rabies as proposed by Zhang et al. (2011), to the optimal control model.

1.1.1 Disease Transmission

Rabies transmission mostly happens due to the influence of an infected mammal biting a susceptible individual. Foxes, raccoons, coyote, skunks, dogs, bats, and cats are the mammals that mostly spread the rabies infection. The rabies infection has a short life range outside the vector, and stays alive in the cadaver of an infective warm-blooded animal for not more than 24 hours. Nonetheless, being exposed to the rabid animal does not necessarily imply that, the bitten warm blooded animal or individual will get to be contaminated. From research, it is speculated, that just 13% – 15% of exposed humans do contract the rabies virus after a bite from an infectious dog (WHO, 2010). Individuals that work very closely with wildlife, and veterinarians are at high risk to the rabies virus. Figure 1.1a demonstrates the transmission progression of the rabies infection in the human framework.



(a) Source: Department of infectious disease, Tehran University of Medical Science

(b) An example of a dog market in Ghana.

1.1.2 Symptoms

The advancement of the illness is moderately quick, and the normal time of rabies development from exposure stage to the brain is between 3 to 8 weeks in dogs, and 3 to 6 weeks in humans. At brain, it then moves to the salivary glands. The infected vector or host goes through one of the following stages with a different symptoms behaviour (WHO, 2010).

1.1.3 Prodromal Stage

In this stage, the virus usually lasts for 2–3 days in dogs, and in humans 5–6 days. When the infected mammal reaches this stage, the following symptoms are commonly shown; anxiety, uneasiness, isolation and fever. Most sociable animals may become nervous, whilst, hostile dogs may become friendly and submissive. The tainted puppy will as often as possible lick the site of the bite. The prodromal stage in cats mostly stay for 1–2 days. The cats usually develop fever spikes and flighty conduct behaviour than dogs (WHO, 2010).

1.1.4 Violent Stage

In this stage, the animal may become violent; cats are especially inclined to developing this stage. This malady stage in canines, for the most part, goes on for 1 to 7 days. The dog becomes restless and irritable and is hyper responsive to visual and auditory stimuli. As it gets to be more anxious, it can start to meander and turns out to be much more touchy and awful. The dog becomes ever more disoriented and then experiences seizures and could die (WHO, 2010).

1.1.5 The Paralytic (dumb) Stage

The dumb stage, for most dogs develop within 2 to 5 days after the primary signs emerge. The head and throat are the initially influenced, and the contaminated well-evolved animal may start to salivate as a consequence of their inability to swallow. Breathing gets to be more profound and the jaw may drop as a consequence of the stomach and facial muscles turning out to be progressively paralyzed. These infected mammals may make a gagging sound and many dog owners may think there is something in the dog's throat. The infected mammal will become weaker, finally develop lung failure and die (WHO, 2010).

1.1.6 Diagnosis

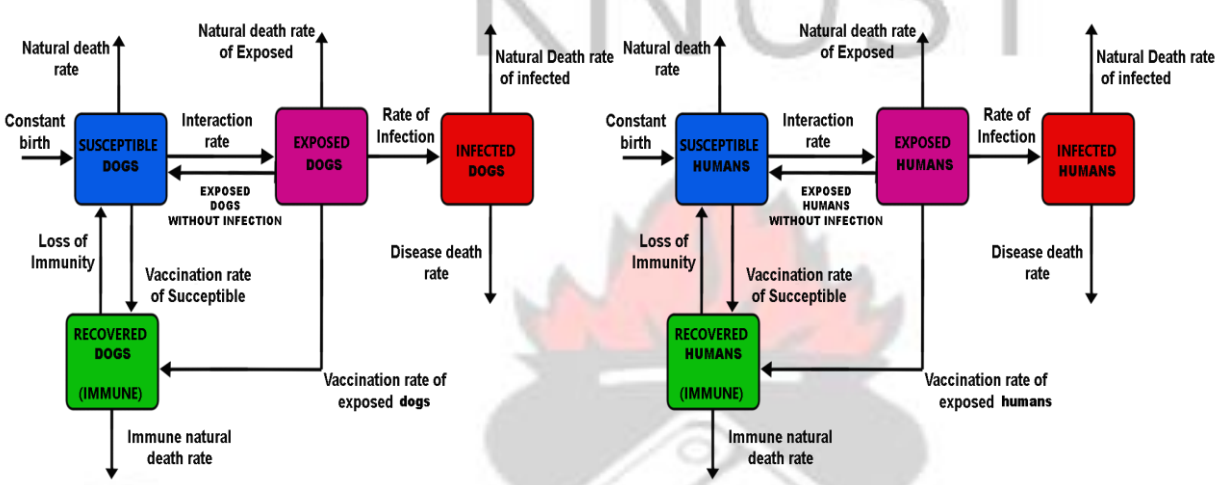
The most proficient approach to recognize rabies in mammals is to render the brain for a microscopic check (Rupprecht et al., 2010).

1.1.7 Treatment

Unfortunately, there is no cure when an individual gets into the paralytic stage and death is virtually assured. However, a very few people have survived a rabies infection after receiving intensive medical care. In dogs there is a very low survival rate (WHO, 2010).

1.1.8 Immunization

The best way to prevent the disease is through vaccination. A vaccinated individual stands small chance of contracting the rabies virus (WHO, 2010). Below is a model description of rabies transmission dynamics in dogs (vector) and humans (host) with vaccination effect.



(a) Dog (vector) sub-model. (b) Human (host) sub-model.

Figure 1.2: Model description of rabies transmission dynamics in Dogs and Humans

1.2 Statement of the Problem

Rabies potentially threatens over three (3.3) billion people in Africa and Asia (WHO, 2010). Individuals most at danger with the rabies infection live in provincial ranges where human antibodies and immunoglobulin are not promptly accessible. Albeit all age gatherings are vulnerable, rabies is generally common with children under 15 years. Overall 40% of post-exposure prophylaxis regimens are given to this class of age group 5 – 14 and the greater part are male (CDC, 2010). People could be exposed to the rabies infection, either by nature of their living arrangement or occupation. Also, needy individuals are at a higher risk, as the normal expense of rabies post-exposure prophylaxis after contact with a suspected rabid animal is US$ 40 in Africa and US$ 49 in Asia, where the normal everyday salary is about US$ 1–2 for each individual (WHO,

2010). But it is known that people that get pre-exposure vaccination are at low risk of getting the rabies virus (CDC, 2010). Hence, we want to study the effect of pre-exposure, post-exposure vaccines, and culling effect on the model, so as to minimize the infection in the expose and the infected classes, and also compare the existing model of rabies as proposed by Zhang et al. (2011), and optimal control to model, so that will know the time frame of controlling the disease in Africa and Asia.

1.3 Objectives of the Study

1. To develop an optimal control model for a vector-host transmission dynamics of rabies virus.
2. To determine the equilibrium points, basic reproduction number R_0, perform stability and sensitivity analysis of the optimal control model.
3. To determine controllability matrix of the model and solve an optimal control problem of the model.

1.4 Methodology

The diagram blow gives the entire idea, use in carrying out our task.

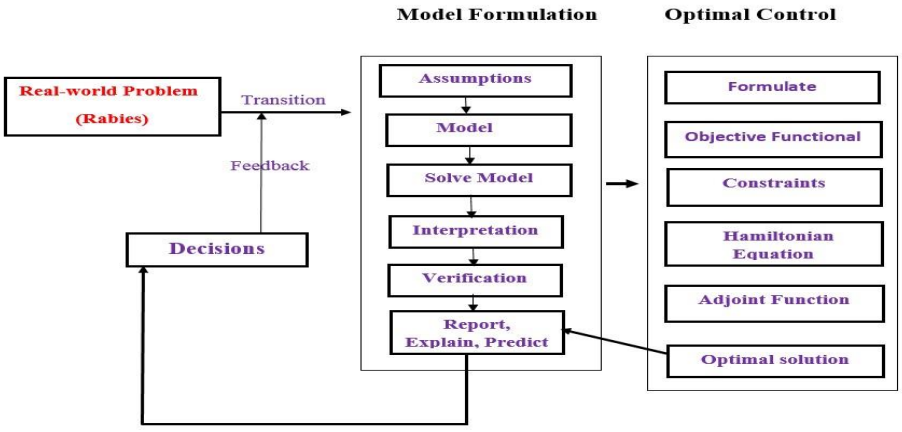


Figure 1.3: Steps taken to achieve our goal.

1.5 Justification of the Study

This thesis will contribute to the already exiting research knowledge of rabies in Asia and Africa. And will will also assist, decision makers in recognising the need to implement both pre and post-exposure vaccination program against the transmission of rabies. This will also create awareness of rabies, and promote the need to go for pre-exposure vaccination in other to prevent rabies infection, the optimal control strategies will provide the optimal cost and time dependent effort for controlling the disease.

1.6 Organization of the Thesis

Chapter 2 contains a review of related works and a brief explanation of mathematical epidemiology. Chapter 3 contains the model formulation, epidemiological and mathematical study, invariant region, equilibrium points, stability analysis, sensitivity analysis, controllability matrix and an optimal control problem. In Chapter 4 we present the simulation, sensitivity analysis, and discussion of results. Finally, Chapter 5 comprises of conclusion and recommendations.

Chapter 2

Literature Review

2.1 Overview

In this chapter, we will explain epidemiological modelling. We will also review related works which have been done by other researchers relating to rabies transmission between dogs and from dogs to humans.

2.2 Epidemiological Modelling

Epidemiology brings to bear the distribution of disease, that is, to answer who has, how much of what, where and when? Other functions of epidemiology are to identify the risk factors for the disease or causes, so as to find out why everyone does not have the same infection uniformly. We use epidemiology to build and test theories, plan, and measure the perception that infection has occurred, so as to lead to a cure, by developing control and prevention programs. Here, we focus on the modelling of rabies in dogs and the human population, and do not consider models of chronic diseases, such as cancer and heart diseases. In most cases, epidemiological modelling is divided into compartments per their epidemiological status. That is Susceptible (the ' S ' class), Exposed (the ' E ' class), Infectious (the 'I ' class) and Removed/Recover (the ' R ' class). These stages are dynamic, so we sometimes refer to epidemiological modelling as 'dynamic deterministic modelling', which refers to the movement between the individual compartments. In other words, susceptible individuals get exposed to the infection and then move to the infectious stage and then to the recovered/removed stage. Through vaccination or public education, the exposed or infected individual moves to the recovery stage, or the infections are eradicated from the susceptible population.

The movements between compartments are indicated by differential or difference mathematical equations. Although there are vaccines for many infectious diseases, such as rabies, these diseases still cause mortality and suffering in the world, especially in developing countries. Chronic ailments, for example, tumour and coronary illness have gotten more attention than infectious diseases in developed countries, but infectious diseases like rabies are still a more common cause of death in the world. There has been a recent re-emergence of rabies and other infectious diseases, such as Ebola, which has a very limited incubation period, hence this has led to a revived interest in infectious diseases in Africa and the world at large. The transmission interactions in a population are very complex. This makes it difficult to understand the huge scale

progression of illness spread without the formal structure of a mathematical model and flowchart (Hethcote, 2009).

Model formulation mechanism clarifies assumptions, variables, and parameters. Mathematical models also provides conceptual results such as, basic reproductive number

'R_0', thresholds 'θ,H_1' , contact rate number ' β ' and effective reproductive number 'E_{R0}'. Also mathematical models and computer simulations have become useful experimental devices for building and testing speculations, evaluating quantitative guesses, noting particular inquiries, deciding sensitivities to changes in parameter values, and estimating key parameters from data. Having a good knowledge of the transmission dynamics of infectious diseases such as rabies and Ebola in communities, regions and countries may lead to a better approach to minimize the transmission of these diseases. Mathematical models can be used in comparing, implementing, planning, measuring and optimizing various detection, prevention therapy and control programs.

Epidemiological displaying can likewise add to the configuration and examination of epidemiological overviews, foresee significant information that ought to be gathered, distinguish patterns, evaluate the vulnerability in estimates and make general gauges

(Nisbet et al., 1989) and (Hethcote et al., 1981). In the field of science, it is conceivable to lead trials to acquire data and test theories. Tests concerning the spread of irresistible maladies in human populaces are frequently outlandish, unscrupulous or costly to direct. For instance Hethcote (2009) states that getting information is now and again accessible from normally happening infection; nonetheless, the information is frequently fragmented because of under-reporting. The absence of dependable information makes the assessment of particular parameters exceptionally troublesome, with the goal that it might just be conceivable to gauge a scope of qualities for a few parameters. It is likewise a reality that not all inquiries can be addressed utilizing epidemiological modelling.

2.3 Review of Related Research Work

The investigation of rabies epidemiology and the study of this infection has been done by many researchers, however, the primary focus was on the spread of rabies in wild foxes, raccoons, bats, skunks and dogs. Moreover, there has been some study on rabies transmission in dogs in Ghana. The research done by Addo in 2012 in the Bongo District, in the Upper East Region of Ghana, was mostly on dog to dog transmission, without considering the transmission of rabies to the human population. He concluded that an increase in a rabies immunization program had a critical effect on the spread of rabies and a decrease in the immunization program increases the rate of transmission of rabies in dogs. The R_0 was found to be 1.3267 without immunization. This shows that the pervasiveness of rabies in dogs is viewed as an endemic. This is on the grounds that the transmission coefficient between dogs surpasses the death rate in dogs. Estimating the basic Reproductive ratio R_0 of rabies transmission with immunization, it was demonstrated that $R_0 = 0.3755 < 1$. This backs the outcome that an increase in immunization will diminish the rate of rabies transmissions in the community.

Research published by Aubert (1999), on the advancement of the expense of wildlife rabies in France incorporated various variables. They follow; immunization of domestic animals, the reinforcement of epidemiological reconnaissance system and the bolster gave to indicative research laboratories, the costs connected with outbreaks of rabies, the clinical perception of those mammals which had bitten humans and the preventive immunization and post-exposure treatment of people. A significant percentage of (72%) of the cost was the preventive immunization of local animals. In France, as in other European nations in which the red fox (Vulpes) is the species most affected, two primary procedures for controlling rabies were assessed in Aubert (1999) at the repository level to be specific: fox termination and the oral immunization of foxes. The consolidated costs and advantages of both systems were looked at and included either the expenses and of fox separation or the cost of oral immunization. The total yearly

costs of both techniques stayed practically identical until the fourth year, after which the oral immunization methodology turned out to be more cost effective. This estimate was made in 1988 and readjusted in 1993 and affirmed by ex-post investigation five years later. Accordingly it was presumed that fox termination brought about a transient diminishment in the event of the infection while oral immunization turned out to be equipped for wiping out rabies even in circumstances in which fox population were growing. Anderson et al. (1982) formulated a mathematical model based on each time step dynamics which were calculated independently in every cell. Later, Bohrer et al. (2002) published a paper on the viability of different rabies spatial immunization designs in a simulated host population.

Bohrer et al. (2002), inferred that the areas where the carrying capacity of the rabies host population varied over space, the spreading pattern of oral rabies immunization can considerable affect viability of the system and consequently its cost. The viability of a non-arbitrary spread of the immunization depends, to some extent, on the dispersal behaviour of the carriers. The outcomes likewise exhibit that, in a warm domain in a few high-density regions encompassed by populations with densities below the critical threshold for the spread of the disease, the rabies infection can persist. Nonetheless, eradication can be accomplished with a lower immunization level than generally acknowledge (50-70%) (Tischendorf et al., 1998). In reality, this pattern may present an additional advantage in that carriers in high density regions around human waste, may become non-territorial while carriers in the ‘ background ’ are expected to be regional. Subsequently, vaccination will have just minor effect on the population dynamics (Maher and Lott, 2000). Oral rabies vaccination (ORV) is as of now considered the best strategy for diminidhing rabies (Brochier et al., 1991) and (Fu, 1997).

This tells us that mathematical modelling has become one of the vital instrument in investigating the epidemiological conduct of irresistible infections, furthermore, gives valuable control measures. The study done by Smith et al. (1981) on a deterministic model comprising of three subclasses, Susceptible (S), Infectious (I) and Recovered (R)

additionally clarifies epidemiological features of rabies in fox populations in Europe. Levin et al. (2012), also presented a model for the immune responses to rabies virus in bats. Coyne et al. (1989), proposed a SEIR model, which was also used in a study predicting the local dynamics of rabies among raccoons in the United States. Childs et al. (2000), also researched on, rabies epidemics in raccoons with a seasonal birth pulse, using optimal control of a SEIRS model which describes the population dynamics. Hampson et al. (2007), also noted that rabies epidemic cycles have a period of 3 − 6 years in dog populations in Africa, so they built a susceptible, exposed, infectious and vaccinate model with an intervention response variable, which showed significant synchrony.

Carroll et al. (2011), used compartmental models to describe rabies epidemiology in dog populations and explored three control methods: vaccination, vaccination pulse fertility control, and culling. An ordinary differential equation model was used to characterize the transmission dynamics of rabies between humans and dogs by Wang and Lou (2008) and Yang and Lou (2009). There was also an extended work done by Zinsstag et al. (2009). They extended the existing models on rabies transmission between dogs to include dog-to-human transmission and concluded that human post exposure prophylaxis (PEP) with a dog-vaccination campaign was the more cost-effective in the long run. Optimal control has also been recently applied to an epidemic model for rabies in raccoons in researched performed by Artois et al. (1997) and Allen et al. (2002). The control vector gives the rate of vaccination in the subpopulation that minimizes the exposed, infected classes over all subpopulations. This accounts for the cost effectiveness of administering the vaccine. Ding et al. (2007), formulated an epidemic model for rabies in raccoons with discrete time and spatial features. Their goal was to analyze the strategies for optimal distribution of vaccine baits to minimize the spread of the disease and the cost of carrying out the control. Hong-tao et al. (2008) also established a mathematical model of rabies with similar controlling strategies in China. From their mathematical analysis and simulation, it indicates that a culling strategy is the most cost

effective, followed by vaccination. Zhang et al. (2011) compared three efficiency strategies for controlling rabies; vaccination, culling, culling and vaccination, and took vaccination as the most effective choice in controlling rabies transmission. Comparing Hong-tao et al. (2008) and Zhang et al. (2011), it is established that vaccination is the best control method of reducing rabies infections. The research done by Zhang et al. (2011) on rabies transmission in China, came out with a basic reproductive number $R_0 = 2$, for the transmission of rabies in China, and predicted that the number of human rabies is decreasing, but may reach another peak around 2030.

Zhang et al. (2011) also compared the effectiveness of immunization and culling of dogs. They showed that a decrease in the dog's birth rate and increase in the dog's immunization coverage rate are the most effective control method for reducing rabies cases in China, as opposed to large scale culling. A combined approach will be cheaper only when the per capita cost of vaccination is less than 20% of the per capita cost of culling.

For the past hundred years, Europe has been repeatedly subjected to rabies epidemics. But in Zhang et al. (2011) they did not consider pre-exposure vaccination of the human population and also used the same vaccination parameter for both exposed and susceptible class of the dog compartment, we will assume different parameter value for this model to see the effect of varying a parameter on the modified model, we will also incorporate the pre-exposure vaccination of the human population to see the effectiveness of vaccinating the susceptible humans. Below is the model description of Zhang et al. (2011).

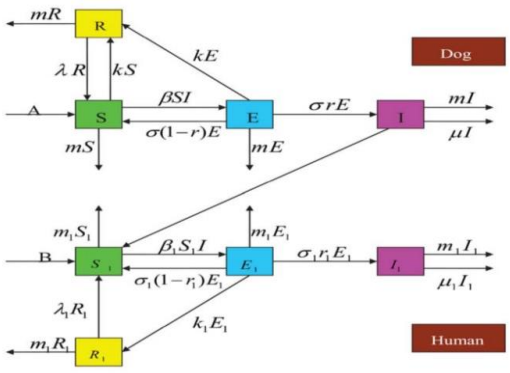


Figure 2.1: Zhang et al. (2011) model

There have likewise been some scientific models for dog's rabies that incorporate the controlling impact of human intervention Leung (2014) utilized a stochastic simulation model to research the effect of human intervention on the progression and persistence of rabies Leung (2014) investigated the impact of human intervention, such as culling infected dogs, is not merely equivalent to a reduction in the average infection period of the infection. Also research done by Castillo-Rodr´ıguez et al. (2015), on the costadequacy of a rabies immunization program in dogs to avert human rabies in Colombia. They utilized a Markov model with day by day cycles to plan mimicked progression of the transmission parameters inside the dog population and its transmission to people. The study considered two options: dogs immunization and no immunization by contrasting diverse immunization scope. Work done by Castillo-Rodr´ıguez et al. (2015), uncovered that with immunization, 58,591 canine rabies cases happened together with 4 human cases. With immunization at 68% scope and 75% of antibody adequacy, the evaluated number was decreased to 28,664 dog cases and 2 human case. Castillo-Rodr´ıguez et al. (2015), presumed that an expansion in immunization scope in dogs will diminish the occurrence of human rabies in Colombia. It is clear from the literature that to understand the transmission flow of rabies and investigate compelling control measures. Therefore, adjusting the current models gives extra understanding on how rabies could be controlled. In this study we will also apply

optimal control to know the time dependent factors and cost effect in controlling rabies in Africa and Asia.

Chapter 3

Methodology

3.1 Introduction

In this chapter, we will focus on an SEIRS model, and optimal control strategies for the transmission of rabies to support the research knowledge of rabies transmission in Africa and Asia. We will consider the following; the feasible region, equilibrium points, herd immunity, stability analysis, sensitivity analysis, controllability, optimal control strategies and an optimality system of the model.

3.2 Model Formulation and Assumption

The vector (dog) to host (human) SEIRS model transmission dynamics are formulated by dividing the population into compartments based on the idea provided by Zhang et al. (2011). The nature and time of transmission from one compartment to another was also considered. We assumed that the total population size of dogs at any time t is $N_d(t)$ and the total human population size at any time t is $N_H(t)$. We assume that the dog is the main carrier of the rabies infection into the human population. We also assume that the dogs mix homogeneously with the human population. Due to the territorial nature of dogs, they sometimes engage in a fight with new dogs who enters their territory. This makes it easier for an infectious dog to transmit the rabies virus to a susceptible dog. We assume that the human population gets infected only when they come in contact with an infectious dog. We assume that the susceptible dog population is increased by recruitment at a rate A, and B is the recruitment rate of humans. We denote the transmission rate of rabies in the

dog (vector) compartment as β_d. Applying vaccination to the susceptible dogs, the transmission dynamics becomes $(1 - \nu)\beta_d S_d I_d$, where ν is the pre-exposure prophylaxis (vaccination). This implies that, with pre-exposure prophylaxis (vaccination) on the model, the transmission rate decreases at a rate $(1 - \nu)$. We also denoted the transmission dynamics from the dog compartment into the human compartment at a rate β_{dH}. Similarly applying vaccination on the susceptible humans the transmission dynamics becomes $(1 - \nu_H)\beta_{dH} S_H I_d$, where ν_H is the pre-exposure prophylaxis (vaccination). The exposed dogs and humans are treated at the rate ρ and ρ_H respectively. The post-exposure prophylaxis (treatment), ρ_d and ρ_H, decreases the progression of the rabies virus, at the exposed state to the infectious state at a rate $(1-\rho)$, and $(1-\rho_H)$ respectively. The loss of immunity for both compartments is represented by α and α_H respectively, $\delta\varepsilon$ represents the rate of dogs without clinical rabies and moves back to the susceptible compartment.

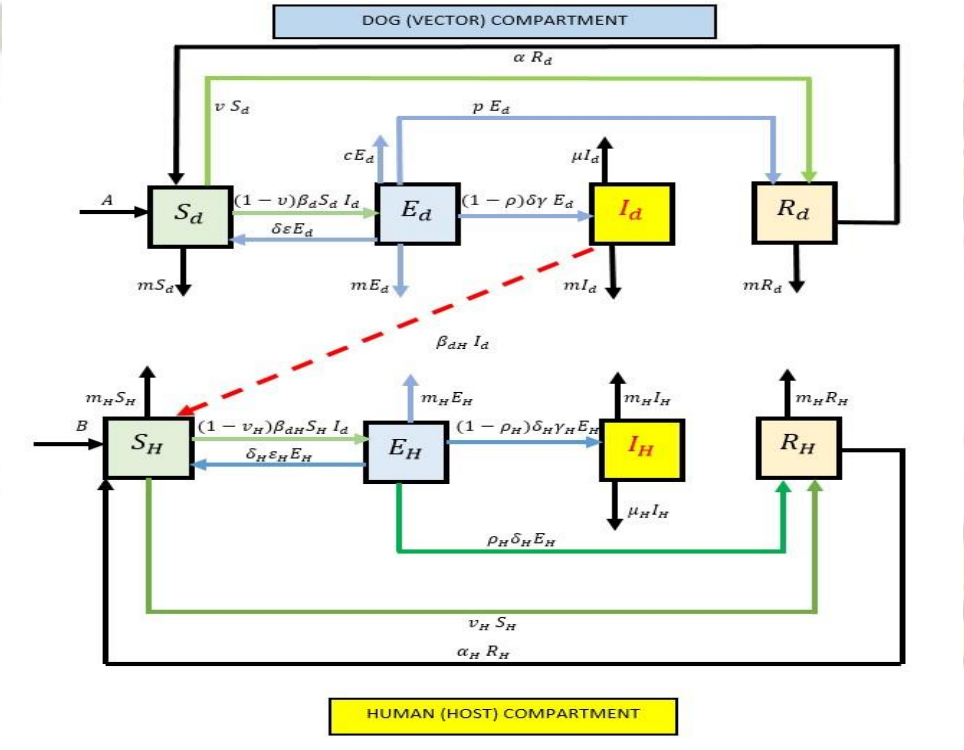


Figure 3.1: Optimal control model of rabies transmission dynamics

The red dash arrow line from the infectious dogs population I_d, to the susceptible human population S_H, indicates the rate of contact of an infectious dog to a susceptible individual. The exposed humans without clinical rabies that moves back to the susceptible population is denoted as $\delta_H\varepsilon_H$. The natural death rate of dogs is m, and m_H denote the mortality rate of humans (natural death rate), $\beta_{dH}S_HI_d$ describes the transmission of rabies from infectious dogs to susceptible humans, μ_d represents the death rate associated with rabies infection in dogs, μ_H, represents the disease induce death in humans. The fraction of dogs that die due to culling is cE_d. The total susceptible, exposed, infectious, and recovered (temporary immunity) in the vector compartment at any time t is denoted by $S_d(t)$, $E_d(t)$, $I_d(t)$ and $R_d(t)$ respectively. Similarly, $S_H(t)$, $E_H(t)$, $I_H(t)$ and $R_H(t)$ represents the susceptible, exposed, infected and recovered (temporary immunity) in human compartment at any time t.

From the above assumptions and transmission flowchart,lead to the following system of ordinary differential equations which describe the dynamics of the rabies disease in both compartments.

$$\left.\begin{aligned}
\frac{dS_d}{dt} &= A-(1-\nu)\beta_dS_dI_d-(m+\nu)S_d+\delta\varepsilon E_d+\alpha R_d,\\
\frac{dE_d}{dt} &= (1-\nu)\beta_dS_dI_d-((1-\rho)\delta\gamma+m+\rho+\delta\varepsilon+c)E_d,\\
\frac{dI_d}{dt} &= (1-\rho)\delta\gamma E_d-(m+\mu)I_d,\\
\frac{dR_d}{dt} &= \nu S_d+\rho E_d-(m+\alpha)R_d,\\
\frac{dS_H}{dt} &= B-(1-\nu_H)\beta_{dH}S_HI_d-(m_H+\nu_H)S_H+\delta_H\varepsilon_HE_H+\alpha_HR_H,\\
\frac{dE_H}{dt} &= (1-\nu_H)\beta_{dH}S_HI_d-((1-\rho_H)\delta_H\gamma_H+m_H+\rho_H+\delta_H\varepsilon_H)E_H,\\
\frac{dI_H}{dt} &= (1-\rho_H)\delta_H\gamma_HE_H-(m_H+\mu_H)I_H,\\
\frac{dR_H}{dt} &= \nu_HS_H+\rho_HE_H-(m_H+\alpha_H)R_H,
\end{aligned}\right\} \quad (3.1)$$

with $S_d(0) > 0$, $E_d(0) \geq 0$, $I_d(0) \geq 0$, $R_d(0) \geq 0$,

and $S_H(0) > 0,\ E_H(0) \geq 0,\ I_H(0) > 0,$ and $R_H(0) > 0.$

3.3 Feasible and Non-Negative Solutions

In this section, we will show that the model system (3.1) is mathematically well-posed and epidemiologically meaningful.

3.3.1 Invariant Region

Lemma 1

The solution set $\{S_d, E_d, I_d, R_d, S_H, E_H, I_H, R_H\} \in R_+^8$ of model system(3.1) is contained in the feasible region Ω.

Proof.

Suppose $\{S_d, E_d, I_d, R_d, S_H, E_H, I_H, R_H\} \in R_+^8$ for all $t > 0$. We want to show that the region Ω is positively invariant, so that it becomes sufficient to look at the dynamics of the model system (3.1). Therefore, given that

$$N_d(t) = S_d(t) + E_d(t) + I_d(t) + R_d(t),$$

and

$$N_H(t) = S_H(t) + E_H(t) + I_H(t) + R_H(t),$$

where, $N_d(t)$, is the total population of dogs at any time (t) and $N_H(t)$ is total population of humans at any time (t).

Therefore

$$\frac{dN_d}{dt} = \frac{dS_d}{dt} + \frac{dE_d}{dt} + \frac{dI_d}{dt} + \frac{dR_d}{dt}, \text{ and}$$
$$\frac{dN_H}{dt} = \frac{dS_H}{dt} + \frac{dE_H}{dt} + \frac{dI_H}{dt} + \frac{dR_H}{dt}.$$

Hence

$$\frac{dN_d}{dt} = A - (S_d + E_d + I_d + R_d)\, m - \mu I_d - cE_d,$$

this yields

$$\frac{dN_d}{dt} = A - mN_d - \mu I_d - cE_d, \tag{3.2}$$

similarly

$$\frac{dN_H}{dt} = B - m_H N_H - \mu_H I_H \quad . \tag{3.3}$$

Now assuming that there is no disease induced death rate and culling effect, then ($\mu_d = \mu_H = c = 0$), hence equation (3.2) and (3.3) becomes

$$\frac{dN_d}{dt} = A - mN_d, \tag{3.4}$$

$$\frac{dN_H}{dt} = B - m_H N_H \quad . \tag{3.5}$$

Suppose,

$$\frac{dN_d}{dt} \leq 0 \text{ and } \frac{dN_H}{dt} \leq 0,$$

$$N_d \leq \frac{A}{m} \text{ and } N_H \leq \frac{B}{m_H}.$$

Now imposing the theorem proposed by Birkhoff and Rota (1989) on differential inequality result in $0 \leq N_d \leq \frac{A}{m}$ and $0 \leq N_H \leq \frac{B}{m_H}$. Therefore equation (3.4) and equation (3.5) becomes

$$\frac{dN_d}{dt} \leq A - mN_d. \tag{3.6}$$

$$\frac{dN_H}{dt} \leq B - m_H N_H \quad . \tag{3.7}$$

Solving equation (3.6) and equation (3.7) using the integrating factor (*I.F*) method, we have

$$\frac{dy}{dt} + p(t)y = Q,$$

$$I.F = e^{\int p(t)dt}.$$

Gives the following deductions

$$e^{mt}\frac{dN_d}{dt} + mN_d e^{mt} \leq Ae^{mt}$$

$$\int dN_d e^{mt} \leq \int Ae^{mt}\, dt,$$

$$N_d = \frac{A}{m} + K_1 e^{-mt},$$

$$\frac{A}{m} \leq N_d - K_1 e^{-mt}$$, where $K_1 e^{-mt} \leq 0$.

This gives

$$A - mN_d \geq K_1 e^{-mt},$$

where K_1 is the constant of integration.

Hence the feasible solution of the dogs population in the system model (3.1) is in the region:

$$\Omega_d = \left\{ (S_d, E_d, I_d, R_d \in R_+^4, N_d \leq \frac{A}{m} \right\}.$$

Similarly the humans population follows suit and from equation (3.7) this implies that

$$B - m_H N_H \geq K_2 e^{-(m_H)t}.$$

Where K_2 is the constant of integration. Therefore, the feasible solution of the human population of system model (3.1) is in the region

$$\Omega_H = \left\{ (S_H, E_H, I_H, R_H) \in R_+^4, N_H \leq \frac{B}{m_H} \right\}.$$

Therefore the feasible solutions are contained in Ω. Thus $\Omega = \Omega_d \times \Omega_H$. From the standard comparison theorem used on differential inequality by Lakshmikantham and Kaul (1994) it is implied that

$$N_d(t) \leq N_d(0)e^{-mt} + \frac{A}{m}\left(1 - e^{-mt}\right),$$

and

$$N_H(t) \leq N_H(0)e^{-(m_H)t} + \frac{B}{m_H}\left(1 - e^{-m_H t}\right).$$

Hence, the vector population size $N_d(t) \rightarrow \frac{A}{m} \text{ as } t \rightarrow \infty$.

Similarly for the host population size $N_H(t) \rightarrow \frac{B}{m_H} \text{ as } t \rightarrow \infty$. This means that the infected state ($E_d, I_d, E_d I_d$) of the two populations tends to zero as time goes to infinity. This means region Ω is pulling (attracting) all the solutions in R^8_+. This gives the feasible solution set of model system (3.1) as

$$\Omega = \left\{ \begin{pmatrix} S_d \geq 0 \\ E_d \geq 0 \\ I_d \geq 0 \\ I_d \geq 0 \\ R_d \geq 0 \\ S_H \geq 0 \\ E_H \geq 0 \\ I_H \geq 0 \\ R_H \geq 0 \\ N_d \leq \frac{A}{m} \\ N_H \leq \frac{B}{m_H} \end{pmatrix} \right\} \in R^8_+. \qquad (3.8)$$

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Therefore, equation (3.8) is the feasible set of the model system (3.1). Hence, the model system (3.1) is mathematically well-posed, biologically and epidemiologically correct. Furthermore, the usual existence, uniqueness and continuation results hold for the model system (3.1).

3.3.2 Non-Negative Solutions

We need to verify whether or not the system's solutions are non-negative, and also show that the region Ω is positively invariant for the model system (3.1).

Theorem 3.3.1 *Let the initial conditions be* $\{(S_d(0),S_H(0)) > (E_d(0),E_H(0),I_d(0),I_H(0),R_d(0),R_H(0)) \geq 0\} \in \Omega$. *Then the solution set* $\{S_d,E_d,I_d,R_d,S_H,E_H,I_H,R_H\}(t)$ *of the model system (3.1) is non-negative for all* $t > 0$.

Proof.

Considering the susceptible classes of model system (3.1) we have

$$\frac{dS_d}{dt} = A - (1-\nu)\beta_d S_d I_d - (m+\nu)S_d + \delta\varepsilon E_d + \alpha R_d$$

$$\frac{dS_H}{dt} = B - (1-\nu_H)\beta_{dH} S_H I_d - (m_H+\nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H,$$

with no new births and from the initial conditions $(R_d(0),I_d(0),E_d(0) \geq 0)$ we have

$$\frac{dS_d}{dt} \geq -(m+\nu)S_d,$$
$$\frac{dS_d}{S_d} \geq -(m+\nu)dt,$$
$$\int \frac{dS_d}{S_d} \geq \int -(m+\nu)dt,$$
$$ln|S_d| \geq -(m+\nu)t + K,$$
$$S_d \geq Ke^{-(m+\nu)t},$$

$S_d \geq S_d(0)e^{-(m+\nu)t} \geq 0.$

Similarly

$$\frac{dS_H}{dt} \geq -(m_H+\nu_H)S_H,$$
$$S_H \geq S_H(0)e^{-(m_H+\nu_H)t} \geq 0.$$

Next considering the exposed class of both populations of model system (3.1) we have the following:

$$\frac{dE_d}{dt} = (1-\nu)\beta_d S_d I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d, \quad (3.9)$$
$$\frac{dE_H}{dt} = (1-\nu_H)\beta_{dH} S_H I_d - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H \quad . \quad (3.10)$$

From equation (3.9) we have

$$\frac{dE_d}{dt} \geq ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d,$$
$$\int \frac{dE_d}{E_d} \geq \int -((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)\,dt,$$
$$\ln|E_d| \geq -((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)t + K,$$
$$E_d(t) \geq Ke^{-((1-\rho)\delta\gamma+m+\rho+\delta\varepsilon+c)t},$$

$E_d(t) \geq E_d(0)e^{-((1-\rho)\delta\gamma+m+\rho+\delta\varepsilon+c)t} \geq 0.$ Similarly

equation (3.10) gives

$$\frac{dE_H}{dt} \geq -((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H,$$

$$\int \frac{dE_H}{E_H} \geq \int -((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)dt,$$

$$\ln|E_H| \geq -((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)t + K,$$

$$E_H(t) \geq Ke^{(-(1-\rho+H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)t},$$

$E_H(t) \geq E_H(0)e^{(-(1-\rho^H)\delta^H\gamma^H + m^H + \rho^H + \delta^H\varepsilon^H)t} \geq 0.$

Now considering the infected classes of the model system (3.1) gives

$$\frac{dI_d}{dt} = (1-\rho)\delta\gamma E_d - (m+\mu)I_d, \tag{3.11}$$

$$\frac{dI_H}{dt} = (1-\rho)\delta_H\gamma_H E_H - (m_H+\mu_H)I_H \quad . \tag{3.12}$$

From equation (3.11) we have

$$\frac{dI_d}{dt} \geq (1-\rho)\delta\gamma E_d - (m+\mu)I_d \geq -(m+\mu)I_d$$

$$\frac{dI_d}{dt} \geq -(m+\mu)I_d,$$

$$\int \frac{dI_d}{I_d} \geq \int -(m+\mu)\,dt,$$

$$\ln|I_d| \geq -(m+\mu)t + K,$$

$$e^{\ln|I_d|} \geq e^{-(m+\mu)t+K},$$

$I_d \geq Ke^{-(m+\mu)t},$

$I_d(t) \geq I_d(0)e^{-(m+\mu)t} \geq 0.$

Equation (3.12) becomes

$$\int \frac{dI_H}{I_H} \geq \int -(m_H+\mu_H)\,dt,$$

$$\ln|I_H| \geq -(m_H+\mu_H)t + K,$$

$$e^{\ln|I_H|} \geq e^{-(m_H+\mu_H)t+K},$$

$$I_H \geq Ke^{-(m^H+\mu^H)t},$$

$$I_H(t) \geq I_H(0)e^{-(m_H+\mu_H)t} \geq 0.$$

Finally, we consider the recovery classes of the model system (3.1) which gives

$$\begin{aligned}
\frac{dR_d}{dt} &= \nu S_d + \rho E_d - (m+\alpha)R_d, \\
\frac{dR_H}{dt} &= \nu_H S_H + \rho_H E_H - (m_H+\alpha_H)R_H \\
\frac{dR_d}{dt} &\geq -(m+\alpha)R_d, \\
\int \frac{dR_d}{R_d} &\geq \int -(m+\alpha)dt,
\end{aligned}$$

,

$$\ln|R_d| \geq -(m+\alpha)t + K,$$

$$R_d \geq Ke^{-(m+\alpha)t},$$

$$R_d(t) \geq R_d(0)e^{-(m+\alpha)t} \geq 0.$$

Similarly, we have

$$\begin{aligned}
\frac{dR_H}{dt} &\geq -(m_H+\alpha_H)R_H, \\
\int \frac{dR_H}{R_H} &\geq \int -(m_H+\alpha_H)dt, \\
\ln|R_H| &\geq -(m_H+\alpha_H)t + K, \\
R_H &\geq Ke^{-(m_H+\alpha_H)t}, \\
R_H(t) &\geq R_H(0)e^{-(m_H+\alpha_H)t} \geq 0.
\end{aligned}$$

From the above demonstration, it shows that Ω is positively invariant. The right hand of equation (3.4) and equation (3.5) are both bounded above by $A - mN_d$ and $B - m_HN_H$ respectively. This follows that $\frac{dNd}{dt} < 0$, if $N_d(t) > \frac{A}{m}$ and $\frac{dN_H}{dt} < 0$, if $N_H(t) > \frac{B}{m_H}$. Hence using the standard comparison theorem (Zhang, 1988). It shows that;

$$N_d(t) \leq \frac{A}{m}\left(1 - e^{-mt}\right) + N_d(0)e^{-mt},$$

and

$$N_H(t) \leq \frac{B}{m_H}\left(1 - e^{-m_H t}\right) + N_H(0)e^{-m_H t}.$$

Particularly, if $N_d(0) < \frac{A}{m}$, then $N_d(t) \leq \frac{A}{m}$ and if $N_H(0) < \frac{B}{m_H}$, then $N_H(t) \leq \frac{B}{m_H}$, therefore Ω is positively invariant.

If $N_d(0) > \frac{A}{m}$ and $N_H(0) > \frac{B}{m_H}$ this means that either the solution enters Ω in finite time or $N_d(t)$ approaches $\frac{A}{m}$ and $N_H(t)$ approaches $\frac{B}{m_H}$ asymptotically and the exposed and infected variables E_d,I_d,E_H and I_H approaches zero as t approaches ∞, thus $\lim_{t\to\infty} \frac{dS_d}{dt} \to 0$ and $\lim_{t\to\infty} \frac{dS_H}{dt} \to 0$.

This implies that when proper control measures are taken, that is vaccinating all dogs against rabies even in a dense population of dogs, they will reduce rabies associated disease and mortality in our society as time increases.

3.4 Steady Analysis

This section, takes care of the disease-free equilibrium, the basic reproduction number, the endemic equilibrium, the stability analysis and the sensitivity analysis of the optimal control model. At the equilibrium point, we set the right hand side of model (3.1) to zero. Therefore model system (3.1) becomes

□ (3.13)

$$A - (1-\nu)\beta_d S_d I_d - (m+\nu)S_d + \delta\varepsilon E_d + \alpha R_d = 0,$$
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$$(1-\nu)\beta_d S_d I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d = 0, \delta\gamma E_d -$$

$$(m+\mu)I_d = 0, \nu S_d + \rho E_d - (m+\alpha)R_d = 0,$$

$$(1 - \nu_H)\beta_{dH} S_H I_d - ((1 - \rho_H)\delta_H\gamma_H + m_H(1 -$$

$$\rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H)I_H = 0, \nu_H S_H$$

$$B - (1+-\rho\nu_{HH}E)_H\beta_{dH} - S(m_H I_{Hd} + - \alpha(m_{HH})R + {}_H\nu = 0_H)S._H + \delta + {}_H\varepsilon\rho_{HH}E + {}_H\delta + {}_H\varepsilon\alpha_{HH})ER_{HH} =$$
$$0 = 0,,$$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

3.4.1 Disease-Free Equilibrium E_0

Suppose there is no infection of rabies in both populations, then ($E_d = 0, I_d = 0, E_H = 0, I_H = 0$).

Incorporating this into equation (3.13) gives

$$A \quad -(m+\nu)S_d + \alpha R_d = 0, \quad (3.14)$$

$$\nu S_d - (m+\alpha)R_d = 0, \quad (3.15)$$

$$B \quad -(m_H + \nu_H)S_H + \alpha_H R_H = 0, \quad (3.16)$$

$$\nu_H S_H - (m_H + \alpha_H)R_H = 0. \quad (3.17)$$

From equation (3.15), making R_d the subject gives

$$R_d = \frac{\nu}{m+\alpha} S_d .$$

Substituting R_d into equation (3.14) yields.

$$S_d = \frac{A(m+\alpha)}{m^2 + m\alpha + m\nu + \nu\alpha - \nu\alpha},$$

$$\therefore \quad S_d^* = \frac{A(m+\alpha)}{m(m+\alpha+\nu)}.$$

Plugging S_d^* into R_d gives

$$R_d^* = \frac{A\nu}{m\,(m+\alpha+\nu)}.$$

Similarly making R_H the subject from equation (3.17) results in

$$R_H = \frac{\nu_H}{m_H+\alpha_H} S_H.$$

Substituting R_H into equation (3.16) yields

$$S_H = \frac{B\,(m_H+\alpha_H)}{m_H^2 + m_H\alpha_H + m_H\nu_H + \nu_H\alpha_H - \nu_H\alpha_H}.$$

$$\therefore \quad S_H^* = \frac{B\,(m_H+\alpha_H)}{m_H\,(m_H+\alpha_H+\nu_H)}.$$

Plugging S_H^* into R_H gives

$$R_H^* = \frac{B\nu_H}{m_H\,(m_H+\alpha_H+\nu_H)}.$$

Hence, the disease-free equilibrium $E_0 = (S_d^0, E_d^0, I_d^0, R_d^0, S_H^0, E_H^0, I_H^0, R_H^0)$ is given as

$$\mathcal{E}_0 = \left(\frac{A\,(m+\alpha)}{m\,(m+\alpha+\nu)}, 0, 0, \frac{A\nu}{m\,(m+\alpha+\nu)}, \frac{B\,(m_H+\alpha_H)}{m_H\,(m_H+\alpha_H+\nu_H)}, 0, 0, \frac{B\nu_H}{m_H\,(m_H+\alpha_H+\nu_H)}\right).$$

3.4.2 Basic Reproduction Number R_0

To evaluate the stability of the disease-free equilibrium, we used the computation of the basic reproductive number R_0 and next generation matrix operator $G = FV^{-1}$, $J = F - V$ to ascertain R_0, as described by Diekmann et al. (1990). They defined the basic reproduction number R_0, as the number of secondary infections that an infectious individual would produce in the entire susceptible compartment during the duration of the spread of the infection. Considering that everyone is susceptible to the disease then we can rearrange the model system (3.1), in the order of $E_d, I_d, S_d, R_d, E_H, I_H, S_H, R_H$, respectively, this yields

$f_1 = (1-\nu)\beta_d S_d I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d$, $f_2 = (1-\rho)\delta\gamma E_d - (m+\mu)I_d$,

$f_3 = (1-\nu_H)\beta_{dH} S_H I_d - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H$,

$f_4 = (1-\rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H)I_H$, $f_5 = A - (1-\nu)\beta_d S_d I_d - (m+\nu)S_d + \delta\varepsilon E_d + \alpha R_d$, (3.18)

$f_6 = \nu S_d + \rho E_d - (m+\alpha)R_d$,

$f_7 = \nu B_H - S_H(1-\rho\nu_{HH}E)_H\beta_{dH} - S(m_H I_{Hd} - + \alpha(m_{HH})R + {}_H,\nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H$,

where, $f_1 = \frac{dE_d}{dt}, f_2 = \frac{dI_d}{dt}, f_3 = \frac{dE_H}{dt}, f_4 = \frac{dI_H}{dt}, f_5 = \frac{dS_H}{dt}, f_6 = \frac{dR_d}{dt}, f_7 = \frac{dS_d}{dt}, f_8 = \frac{dR_H}{dt}$.

Taken the Jacobian Matrix of f_1, f_2, f_3, andf_4 with respect to, E_d, I_d, E_H, andI_H, respectively, yields

$$J = \begin{bmatrix} -((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c) & (1-\nu)\beta_d S_d & 0 & 0 \\ (1-\rho)\delta\gamma & -(m+\mu) & 0 & 0 \\ 0 & (1-\nu_H)\beta_{dH} S_H & -((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H) & 0 \\ 0 & 0 & (1-\rho_H)\delta_H\gamma_H & -(m_H+\mu_H) \end{bmatrix}.$$

Using the idea that $J = F - V$ gives

$$F(\mathcal{E}_0) = \begin{bmatrix} 0 & \frac{(1-\nu)\beta_d A(m+\alpha)}{m(m+\nu+\alpha)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{(1-\nu_H)\beta_{dh}(m_H+\alpha_H)B}{m_H(m_H+\nu_H+\alpha_H)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.19)$$

The element in matrix F, constitutes the new infection that will arise from E_d and E_H, while that of matrix V, constitutes the new transfer of infections from one compartment to another. Now, V evaluated at E_0, from the matrix J, is given as

$$V(E_0) = \begin{bmatrix} ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c) & 0 & 0 & 0 \\ -(1-\rho)\delta\gamma & (m+\mu) & 0 & 0 \\ 0 & 0 & ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H) & 0 \\ 0 & 0 & -(1-\rho_H)\delta_H\gamma_H & (m_H+\mu_H) \end{bmatrix} \quad (3.20)$$

From the Next Generation Matrix Approach, we can divide matrix V, into a 2 × 2 submatrices, and then find its corresponding inverses. Therefore the inverse of equation (3.20) is given as

$$V^{-1}(\mathcal{E}_0) = \begin{bmatrix} \frac{1}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)} & 0 & 0 & 0 \\ \frac{(1-\rho)\delta\gamma}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\mu)} & \frac{1}{(m+\mu)} & 0 & 0 \\ 0 & 0 & \frac{1}{((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)} & 0 \\ 0 & 0 & \frac{(1-\rho_H)\delta_H\gamma_H}{((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)(m_H+\mu_H)} & \frac{1}{(m_H+\mu_H)} \end{bmatrix} \quad (3.21)$$

Therefore, $G = FV^{-1}$ is given as

$$G = \begin{bmatrix} \frac{(1-\nu)\beta_d A(m+\alpha)}{m(m+\nu+\alpha)} & 0 & 0 \\ \frac{(1-\rho)(1-\nu)\delta\gamma\beta_d A(m+\alpha)}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\mu)m(m+\nu+\alpha)} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{(1-\rho)\delta\gamma(1-\nu_H)\beta_{dH}B(m_H+\alpha_H)}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\mu)m_H(m_H+\nu_H+\alpha_H)} & \frac{(1-\nu_H)\beta_{dH}(m_H+\alpha_H)B}{(m+\nu)m_H(m_H+\nu_H+\alpha_H)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.22)$$

Let $a = \frac{(1-\rho)(1-\nu)\delta\gamma\beta_d A(m+\alpha)}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\mu)m(m+\nu+\alpha)}$, $b = \frac{(1-\nu)\beta_d A(m+\alpha)}{m(m+\nu+\alpha)}$,

$$c = \frac{(1-\rho)(1-\nu_H)\delta\gamma\beta_{dH}(m_H+\alpha_H)}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)\,(m+\mu)\,m_H\,(m_H+\nu_H+\alpha_H)} \text{ and } d = \frac{(1-\nu_H)\beta_{dH}(m_H+\alpha_H)B}{(m+\nu)m_H(m_H+\nu_H+\alpha_H)}.$$

$$\Rightarrow$$

$$G = \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.23}$$

Finding the matrix determinant of equation (3.23) and denoting it by D, gives the expression, $D = |G - I\lambda|$ where, I is the identity matrix of a 4×4 matrix, thus

$$D = \begin{vmatrix} a-\lambda & b & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ c & d & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0 \tag{3.24}$$

This gives a characteristic equation of the form $\lambda^3(a - \lambda) = 0$, solving the characteristic polynomial, we have the following eigenvalues of equation (3.24) as $\lambda_i = [0,0,0,a]$. The basic reproduction number R_0, is the spectral radius (largest eigenvalue) $\rho(FV^{-1})$, also defined as the dominant eigenvalue of FV^{-1}.

Therefore,

$$\mathcal{R}_0 = \frac{(1-\rho)(1-\nu)\delta\gamma\beta_d A(m+\alpha)}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)\,(m+\mu)\,m\,(m+\nu+\alpha)}. \tag{3.25}$$

Remark

The R_0 above contains the secondary infection produced by the infective compartment of dogs (in the presence of vaccination and treatment). When $R_0 < 1$, the infection gradually dies off with or without pre-exposure vaccine and post-exposure treatment. But when $R_0 > 1$, the infection stays in the environment at a longer period, this therefore impose a high spread of the virus on the susceptible individuals and dogs.

3.4.3 Stability Analysis of E_0

In this section, we shall consider the model system (3.1) and determine whether the model system (3.1) is stable or unstable at the disease-free equilibrium point. Therefore, by linearizing equation (3.1) at E_0, and subtracting λ along the main diagonal, we have

$$\mathbb{J} = \begin{bmatrix} b_1 - \lambda & b_7 & a_1 & \alpha & 0 & 0 & 0 & 0 \\ 0 & & a_2 - \lambda & a_3 & 0 & 0 & 0 & 0 \\ 0 & & b_{10} & b_2 - \lambda & 0 & 0 & 0 & 0 \\ \nu & \rho & & & & & & \\ 0 & 0 & & & & & & \\ 0 & 0 & 0 & b_3 - \lambda & 0 & 0 & 0 & 0 \\ & & a_4 & 0 & b_4 - \lambda & b_5 & 0 & \alpha_H \\ & & a_5 & 0 & 0 & a_6 - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_9 & b_6 - \lambda & 0 \\ 0 & 0 & 0 & 0 & \nu_H & \rho_H & 0 & b_8 - \lambda \end{bmatrix}, \quad (3.26)$$

where,

$$a_1 = \frac{-(1-\nu)\beta_d(m+\alpha)A}{m(m+\nu+\alpha)},\ a_2 = -\left((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c\right),\ a_3 = \frac{(1-\nu)\beta_d(m+\alpha)A}{m(m+\nu+\alpha)}$$

$$a_4 = \frac{-(1-\nu_H)\beta_{dH}(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H)},\ a_5 = \frac{(1-\nu_H)\beta_{dH}(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H)}$$

$$a_6 = -((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H),$$

$$b_1 = -(m+\nu),\ b_2 = -(m+\mu),\ b_3 = -(m+\alpha),\ b_4 = -(\nu_H + m_H),\ b_5 = \delta_H\varepsilon_H,$$

$$b_6 = -\left(m_H + \mu_H\right),\ b_7 = \delta\varepsilon,\ b_8 = -(m_H+\alpha_H),\ b_9 = (1-\rho_H)\delta_H\gamma_H,\ b_{10} = (1-\rho)\delta\gamma,$$

Simplifying matrix $J(E_0)$, we have

$$\begin{aligned}&(b_8-\lambda)[(b_6-\lambda)(a_6-\lambda)(b_4-\lambda)\nu(1-\rho)\delta\gamma a_3\\&+(b_6-\lambda)(a_6-\lambda)(b_4-\lambda)\nu(b_2-\lambda)(a_2-\lambda)\alpha\\&-(b_6-\lambda)(a_6-\lambda)(b_4-\lambda)(b_3-\lambda)(1-\rho)\delta\gamma(b_1-\lambda)a_3\\&+(b_6-\lambda)(a_6-\lambda)(b_4-\lambda)(b_3-\lambda)(b_2-\lambda)(b_1-\lambda)(a_2-\lambda)]=0,\end{aligned}$$

this implies that

$$\begin{aligned}&(b_6-\lambda)(a_6-\lambda)(b_4-\lambda)(b_8-\lambda)\left[\lambda^4+(b_2b_1-b_2-a_2-b_1-b_3)\lambda^3\right.\\&+(b_3b_2+b_1b_3+b_3a_2+b_1a_2+(1-\rho)\delta\gamma a_3+b_2a_2-\nu\alpha)\lambda^2\\&+(a_2\nu\alpha+b_2\nu\alpha-(1-\rho)\delta\gamma a_3b_3-b_1(1-\rho)\delta\gamma a_3-b_3b_2a_2-b_1b_3a_2-b_2b_1a_2)\lambda\\&\left.+(b_1b_2b_3a_2+(1-\rho)\delta\gamma a_3b_3b_1-\nu\delta a_3\alpha-\nu\alpha b_2a_2)\right]=0.\end{aligned}$$

$$(b_6-\lambda)(a_6-\lambda)(b_4-\lambda)(b_8-\lambda)\left[\lambda^4+a_{11}\lambda^3+a_{12}\lambda^2+a_{13}\lambda+a_{14}\right]=0, \tag{3.27}$$

where

$a_{11}=(-b_2-a_2-b_1-b_3)$, $a_{12}=\nu\alpha+a_2b_3+a_2b_1+b_2b_3+b_2b_1+b_3b_1+b_2a_2-(1-\rho)\delta\gamma a_3$, $a_{13}=-a_2\nu\alpha-b_2\nu\alpha+a_3(1-\rho)\delta\gamma b_2+a_3(1-\rho)\delta\gamma b_1-a_2b_2b_3-b_2a_2b_1-a_2b_3b_1-b_2b_3b_1$, $a_{14}=(b_1b_2b_3a_2+(1-\rho)\delta\gamma a_3b_3b_1+(1-\rho)\delta\gamma a_3\nu+\nu\alpha b_2a_2)$.

From equation (3.27) we can extract four characteristic factors that are negative, thus

$$\lambda_1=b_6,\ \lambda_2=a_6,\ \lambda_3=b_4,\ \lambda_4=b_8.$$

where $a_6=-((1-\rho_H)\delta_H\gamma_H+m_H+\rho_H+\delta_H+\delta_H\varepsilon_H)$, $b_6=-(m_H+\mu_H)$, $b_4=-(\nu_H+m_H)$, $b_8=-(m_H+\alpha_H)$. The other four characteristic factors can be obtained using the Routh-Hurwitz criterion. Routh-Hurwitz stability criterion is a test to ascertain the nature of the eigenvalues. If the roots of the polynomial are all positive, then the polynomial has a negative real part (Pielou, 1969; May, 1973).

3.4.4 Routh-Hurwitz Criterion

Theorem 3.4.1 *Given the polynomial:*

$$p(\lambda) = \lambda_k + a_1\lambda_{k-1} + a_2\lambda_{k-2} + \ldots + a_{k-1}\lambda + a_k = 0,$$

where, the coefficients a_i are real constants, i = 1,2,...,k.

Proof:

We define the k Hurwitz matrices using the coefficients a_i of the characteristic polynomial

$$H_1 = \begin{bmatrix} a_{11} \end{bmatrix}, H_2 = \begin{bmatrix} a_{11} & 1 \\ a_{13} & a_{12} \end{bmatrix}, H_3 = \begin{bmatrix} a_{11} & 1 & 0 \\ a_{13} & a_{12} & a_{11} \\ a_{15} & a_{14} & a_{13} \end{bmatrix}, H_j = \begin{bmatrix} a_{11} & 1 & 0 & 0 & \ldots & 0 \\ a_{13} & a_{12} & a_{11} & 1 & \ldots & 0 \\ a_{15} & a_{14} & a_{13} & a_{12} & \ldots & \\ & & & & & \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \ldots & 0 \\ & & & & & a_{1n} \end{bmatrix}$$

$$H_k = \begin{bmatrix} a_{11} & 1 & 0 & 0 & \ldots & 0 \\ a_{13} & a_{12} & a_{11} & 1 & \ldots & 0 \\ a_{15} & a_{14} & a_{13} & a_{12} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{1k} \end{bmatrix}$$

0

where the (n,m) term in the matrix H_j is:

$$a_{2n-m} \text{ for } 0 < 2n - m < k,$$

$$1 \text{ for } 2n = m,$$

$$0 \text{ for } 2n < m, \text{ or } 2n > k + m.$$

Now suppose all the eigenvalues have negative real parts, then the disease-free equilibrium point is stable, if and only if, the determinants of all Hurwitz matrices are positive. Thus $\det H_j > 0 (j = 1,2,3,4,...,k)$. (Pielou, 1969; May, 1973). The Routh-Hurwitz criteria for $k = 3,4,5$ is given as

$$k = 3 : a_{11} > 0, a_{13} > 0, a_{11}a_{12} > a_{13}, k = 4 : a_{11} > 0, a_{13} > 0, a_{14}, a_{11}a_{12}a_{13} > a_{213} + a_{211}a_{14}, k = 5 : a_{1i} > 0, i = 1,2,3,4,5, a_{11}a_{12}a_{13} > a^2{}_{11}a_{14}.$$

From equation (3.27) we consider

$$\lambda_4 + a_{11}\lambda_3 + a_{12}\lambda_2 + a_{13}\lambda + a_{14} = 0.$$

We then verify whether the remaining four eigenvalues satisfy the Routh-Hurwitz criterion for polynomial of order four.

Hence, simplifying the co-efficient of the above characteristic polynomial we have

$$a_{11} = ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c) + (m+\mu) + (m+\alpha) + (m+\nu),$$

$$a_{12} = \nu\alpha + ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)\left[(m+\alpha) + (m+\nu)\right] + (m+\mu)\left[(m+\alpha) + (m+\nu)\right]$$
$$+ (m+\alpha)(m+\nu) + ((1-\rho)\delta\gamma + m + \rho + c)(m+\mu)(1-\mathcal{R}_0),$$

$$a_{13} = ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)\nu\alpha + (m+\mu)\nu\alpha + (m+\mu)(m+\alpha)(m+\nu)$$
$$+ ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\alpha)(m+\mu)\left[1 - \frac{\mathcal{R}_0(m+\mu)}{m+\alpha}\right]$$
$$+ ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\alpha)(m+\nu)\left[1 - \frac{\mathcal{R}_0}{m+\alpha}\right],$$

$$a_{14} = \nu\alpha(m+\nu)((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c) + \frac{(1-\rho)\delta\gamma(1-\nu)\beta_d(m+\alpha)A\nu}{m(m+m+\nu+\alpha)}$$
$$+ (m+\nu)(m+\mu)(m+\alpha)((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(1-\mathcal{R}_0)$$

Therefore, from the Routh-Hurwitz criterion of order four, it implies that the following conditions: $a_{11} > 0$, $a_{12} > 0$, $a_{13} > 0$, $a_{14} > 0$ and $a_{11}a_{12}a_{13} > a_{13}^2 + a_{11}^2 a_{14}$, is satisfied, if $R_0 < 1$, which implies that the disease-free equilibrium E_0 is locally asymptotically stable (Martcheva, 2015).

3.4.5 Global Stability of E_0

Theorem 3.4.2 *The disease-free equilibrium* E_0 *of model (3.1) is globally asymptotically stable if* $R_0 \leq 1$ *and unstable if* $R_0 > 1$.

Proof

We will prove the above Theorem 3.4.2 by forming a Lyapunov function V. We will consider the *SEIRS* model on the space of the eight state variables.

Let V be a Lyapunov function with positive constants, K_1, K_2, K_3, K_4 such that

$$\mathcal{V} = \left(S_d - S_d^0 - S_d^0 \ln\frac{S_d}{S_d^0}\right) + \mathcal{K}_1 E_d + \mathcal{K}_2 I_d + \left(R_d - R_d^0 - R_d^0 \ln\frac{R_d}{R_d^0}\right)$$
$$+ \left(S_H - S_H^0 - S_H^0 \ln\frac{S_H}{S_H^0}\right) + \mathcal{K}_3 E_H + \mathcal{K}_4 I_H + \left(R_H - R_H^0 - R_H^0 \ln\frac{R_H}{R_H^0}\right). \tag{3.28}$$

Taken the derivative of the Lyapunov function with respect to time we obtain

$$\frac{d\mathcal{V}}{dt} = \left(1 - \frac{S_d^0}{S_d}\right)\frac{dS_d}{dt} + \mathcal{K}_1\frac{dE_d}{dt} + \mathcal{K}_2\frac{dI_d}{dt} + \left(1 - \frac{R_d^0}{R_d}\right)\frac{dR_d}{dt} + \left(1 - \frac{S_H^0}{S_H}\right)\frac{dS_H}{dt} + \mathcal{K}_3\frac{dE_H}{dt} + \mathcal{K}_4\frac{dI_H}{dt} + \left(1 - \frac{R_H^0}{R_H}\right)\frac{dR_H}{dt} \tag{3.29}$$

Plugging equation (3.1) into equation (3.29) we have

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &= \left(1 - \frac{S_d^0}{S_d}\right)[A - (1-\nu)\beta_d S_d I_d - (m+\nu)S_d + \delta\varepsilon E_d + \alpha R_d] \\ &+ \mathcal{K}_1[(1-\nu)\beta_d S_d I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d] \\ &+ \mathcal{K}_2[(1-\rho)\delta\gamma E_d - (m+\mu)I_d] + \left(1 - \frac{R_d^0}{R_d}\right)[\nu S_d + \rho E_d - (m+\alpha)R_d] \\ &+ \left(1 - \frac{S_H^0}{S_H}\right)[B - (1-\nu_H)\beta_{dH} S_H I_d - (m_H + \nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H] \\ &+ \mathcal{K}_3[(1-\nu_H)\beta_{dH} S_H I_d - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H] \\ &+ \mathcal{K}_4[(1-\rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H)I_H] \\ &+ \left(1 - \frac{R_H^0}{R_H}\right)[\nu_H S_H + \rho_H E_H - (m_H + \alpha_H)R_H],\end{aligned} \tag{3.30}$$

Setting,

$$S_d \leq S_d^0 = \frac{A(m+\alpha)}{m(m+\alpha+nu)},\ R_d \leq R_d^0 = \frac{A\nu}{m(m+\alpha+\nu)}, S_H \leq S_H^0 = \frac{B(m_H+\alpha_H)}{m_H(m_H+\alpha_H+nu_H)},$$
$$R_H \leq R_H^0 = \frac{B\nu_H}{m_H(m_H+\alpha+H+\nu_H)},$$

this yields

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &\leq \mathcal{K}_1\left[\frac{(1-\nu)\beta_d A(m+\alpha)}{m(m+\alpha+\nu)}I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d\right] \\ &+ \mathcal{K}_2[(1-\rho)\delta\gamma E_d - (m+\mu)I_d] \\ &+ \mathcal{K}_3\left[\frac{(1-\nu_H)\beta_{dH}B(mH+\alpha_H)}{m_H(m_H+\alpha_H+\nu_H)}I_d - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H\right] \\ &+ \mathcal{K}_4[(1-\rho_H)\delta_H\gamma_H E_H - (m_H+\mu_H)I_H],\end{aligned} \tag{3.31}$$

This implies that

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &\leq \left[\frac{\mathcal{K}_1(1-\nu)\beta_d A(m+\alpha)}{m(m+\alpha+\nu)} - \mathcal{K}_2(m+\mu) + \frac{\mathcal{K}_3(1-\nu_H)\beta_{dH}B(m_H+\alpha_H}{m_H(m_H+\alpha_H+\nu_H})\right]I_d \\ &+ [\mathcal{K}_2(1-\rho)\delta\gamma - \mathcal{K}_1((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)]E_d \\ &+ [\mathcal{K}_4(1-\rho_H)\delta_H\gamma_H - \mathcal{K}_3((1-\rho_H)\delta_H\gamma_H + m_H + \delta_H\varepsilon_H]E_H - \mathcal{K}_4(m_H+\mu_H).\end{aligned} \tag{3.32}$$

Equating the coefficient of I_d, E_d, I_H, and E_H in equation (3.32) to zero gives

$K_4 = K_3 = 0$, $K_2 = ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)$, and$K_1 = (1-\rho)\delta\gamma$,

we obtain

$$\frac{d\mathcal{V}}{dt} \leq ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\mu)(\mathcal{R}_0 - 1)I_d,$$

$$\leq 0, \quad \text{if } R_0 \leq 1.$$

Additionally $\frac{d\mathcal{V}}{dt} = 0$ if and only if $I_d = 0$. Therefore, plugging $E_d = I_d = E_H = I_H = 0$ into the equations (3.1). It shows that $S_d(t) \longrightarrow \frac{A(m+\alpha)}{m(m+\alpha+\nu)}$, $R_d(t) \longrightarrow \frac{A\nu}{m(m+\alpha+\nu)}$, $S_H(t) \longrightarrow \frac{B(m_H+\alpha_H)}{m_H(m_H+\alpha_H+\nu_H)}$ and $R_H(t) \longrightarrow \frac{B\nu_H}{m_H(m_H+\alpha_H+\nu_H)}$ as $t \longrightarrow \infty$. Hence, the largest compact invariant set in $\left\{(S_d, E_d, I_d, R_d, S_H, E_H, I_H, R_H) \in \Omega : \frac{d\mathcal{V}}{dt} \leq 0\right\}$, is the singleton set $\{E_0\}$. Therefore, from the La Salle's invariance principle, we conclude that E_0 is globally asymptotically stable in Ω if $R_0 \leq 1$ (Yusuf and Benyah, 2012; Dorothy et al., 2011).

3.4.6 Endemic Equilibrium Points E_1

In this section we shall consider the presence of rabies virus in both populations. That is if $E_d \neq 0, I_d \neq 0, E_H \neq 0$ and $I_H \neq 0$, then we say that the model system (3.1) has an endemic equilibrium point. We denote the endemic equilibrium point as $E_1 = S_d^*, E_d^*, I_d^*, R_d^*, S_H^*, E_H^*, I_H^*, R_H^* \neq 0$. Equating the state variables in equation (3.1) to zero, we obtain

$$A - (1-\nu)\beta_d S_d I_d - (m+\nu)S_d + \delta\varepsilon E_d + \alpha R_d = 0, \tag{3.33}$$

$$(1-\nu)\beta_d S_d I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d = 0, \tag{3.34}$$

$$(1-\rho)\delta\gamma E_d - (m+\mu)I_d = 0, \tag{3.35}$$

$$\nu S_d + \rho E_d - (m+\alpha)R_d = 0, \tag{3.36}$$

$$B - (1-\nu_H)\beta_{dH} S_H I_d - (m_H + \nu_H)S_H + \delta_H \varepsilon_H E_H + \alpha_H R_H = 0, \tag{3.37}$$

$$(1-\nu_H)\beta_{dH} S_H I_d - ((1-\rho_H)\delta_H \gamma_H + m_H + \rho_H + \delta_H \varepsilon_H)E_H = 0, \tag{3.38}$$

$$(1-\rho_H)\delta_H\gamma_H E_H - (m_H+\mu_H)I_H = 0, \tag{3.39}$$

$$\nu_H S_H + \rho_H E_H - (m_H+\alpha_H)R_H = 0. \tag{3.40}$$

From the above equations, we can ascertain the following endemic equilibrium points as follows;

Making I_d the subject in (3.35) we have

$$I_d = \frac{(1-\rho)\delta\gamma}{(m+\mu)}E_d$$ intoequation(3.34)gives,

$$(1-\nu)\beta_d S_d \frac{(1-\rho)\delta\gamma}{(m+\mu)}E_d - (1-\rho)\delta\gamma E_d - (m+\rho+c)E_d - \delta\varepsilon E_d = 0,$$

$$S_d = \frac{(m+\mu)\left((1-\rho)\delta\gamma + (m+\rho+\delta\varepsilon+c)\right)}{(1-\nu)\beta_d(1-\rho)\delta\gamma}. \tag{3.41}$$

Substituting $E_d = \frac{(m+\mu)}{(1-\rho)\delta\gamma}I_d$ and equation (3.41) into equation (3.36) gives

$$(m+\alpha)R_d = \frac{\nu(m+\mu)((1-\rho)\delta\gamma + m+\rho+\delta\varepsilon+c) + (1-\nu)\beta_d\rho(m+\mu)I_d}{(1-\nu)\beta_d(1-\rho)\delta\gamma},$$

$$R_d = \frac{\nu(m+\mu)((1-\rho)\delta\gamma + m+\rho+\delta\varepsilon+c) + (1-\nu)\beta_d\rho(m+\mu)I_d}{(1-\nu)\beta_d(1-\rho)\delta\gamma(m+\alpha)}. \tag{3.42}$$

Also substituting

$E_d = \frac{(m+\mu)}{(1-\rho)\delta\gamma}I_d$, equation (3.41) and equation (3.42), into equation (3.33) yields

$$\begin{aligned} &A - (1-\nu)\beta_d \frac{(m+\mu)((1-\rho)\delta\gamma + m+\rho+\delta\varepsilon+c)}{\beta_d(1-\rho)(1-\nu)\delta\gamma}I_d \\ &+ \alpha\frac{\nu(m+\mu)((1-\rho)\delta\gamma + m+\rho+\delta\varepsilon+c) + (1-\nu)\beta_d\rho(m+\mu)I_d}{(1-\nu)\beta_d(1-\rho)\delta\gamma(m+\alpha)} \\ &- (m+\nu)\frac{(m+\mu)((1-\rho)\delta\gamma + m+\rho+\delta\varepsilon+c)}{(1-\nu)\beta_d(1-\rho)\delta\gamma} + \delta\varepsilon\frac{(m+\mu)}{(1-\rho)\delta\gamma}I_d = 0. \end{aligned}$$

Multiplying through by $(1-\nu)\beta_d(1-\rho)\delta\gamma(m+\alpha)$ and making I_d the subject gives

$$I_d = \frac{\left[(1-\rho)\delta\gamma + m+\rho+\delta\gamma\right](m+\mu)m(m+\nu+\alpha)(\mathcal{R}_0-1)}{(m+\alpha)(1-\nu)\beta_d\left[(1-\rho)\delta\gamma + m+c\right] + m(1-\nu)\beta_d\rho},$$

From equation (3.39) and (3.40) we have

$$I_H = \frac{(1-\rho_H)\delta_H\gamma_H}{m_H+\mu_H}E_H \quad , \tag{3.43}$$

$$R_H = \frac{\nu_H S_H + \rho_H E_H}{m_H+\alpha_H}. \tag{3.44}$$

Furthermore, plugging equation (3.44) into (3.37) gives

$$S_H[(1-\nu_H)(m_H+\alpha_H)\beta_{dH}I_d + (m_H+\nu_H)(\alpha_H+m_H) - \alpha_H\nu_H] = B(m_H+\alpha_H) + [\delta_H\varepsilon_H+\alpha_H\rho_H]E_H,$$

therefore

$$S_H = \frac{B(m_H+\alpha_H)+[\delta_H\varepsilon_H+\alpha_H\rho_H]\,E_H}{[(1-\nu_H)(m_H+\alpha_H)\beta_{dH}I_d + m_H(m_H+\alpha_H+\nu_H)]}. \tag{3.45}$$

Replacing equation (3.45) into equation (3.44) yields

$$R_H = \frac{B\nu_H(m_H+\nu_H)+[(\nu_H\delta_H\varepsilon_H+\nu_H\alpha_H\rho_H)+\rho_H(1-\nu_H)(m_H+\alpha_H)\beta_{dH}I_d+\rho_H m_H(m_H+\alpha_H+\nu_H)]\,E_H}{[(1-\nu_H)(m_H+\alpha_H)^2\beta_{dH}I_d+(m_H+\alpha_H)m_H(m_H+\alpha_H+\nu_H)]}. \tag{3.46}$$

Finally, solving E_H we have

$$E_H = \frac{(1-\nu_H)B(m_H+\alpha_H)\beta_{dH}I_d}{(m_H+\alpha_H)\,[(1-\nu_H)\beta_{dH}I_d((1-\rho_H)\delta_H\gamma_H+m_H+\rho_H)+(m_H+\nu_H)((1-\rho_H)\delta_H\gamma_H+m_H+\rho_H+\delta_H\varepsilon_H)]-(1-\nu_H)\beta_{dH}I_d\alpha_H\rho_H}. \tag{3.47}$$

Hence, the endemic equilibrium is given as

$$
\begin{aligned}
S_d^* &= \frac{A(m+\alpha)}{m(m+\nu+\alpha)\mathcal{R}_0}, \\
E_d^* &= \frac{(m+\mu)}{(1-\rho)\delta\gamma} I_d^*, \\
I_d^* &= \frac{[(1-\rho)\delta\gamma + m + \rho + \delta\gamma]\,(m+\mu)m(m+\nu+\alpha)(\mathcal{R}_0-1)}{(m+\alpha)(1-\nu)\beta_d\,[(1-\rho)\delta\gamma + m + c] + m(1-\nu)\beta_d\rho}, \\
R_d^* &= \frac{A\nu(1-\nu)\beta_d(1-\rho)\delta\gamma(m+\alpha) + (1-\nu)\beta_d\rho(m+\mu)I_d^*}{m(m+\nu+\alpha)\mathcal{R}_0(1-\nu)\beta_d(1-\rho)\delta\gamma(m+\alpha)}, \\
S_H^* &= \frac{B(m_H+\alpha_H) + [\delta_H\varepsilon_H + \alpha_H\rho_H]\,E_H^*}{[(1-\nu_H)(m_H+\alpha_H)\beta_{dH}I_d^* + m_H(m_H+\alpha_H+\nu_H)]}, \\
E_H^* &= \frac{(1-\nu_H)B(m_H+\alpha_H)\beta_{dH}I_d^*}{(m_H+\alpha_H)\,[(1-\nu_H)\beta_{dH}I_d^*((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H) + (m_H+\nu_H)((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)] - (1-\nu_H)\beta_{dH}I_d^*\alpha_H\rho_H} \\
I_H^* &= \frac{(1-\rho_H)\delta_H\gamma_H}{m_H+\mu_H} E_H^*, \\
R_H^* &= \frac{B\nu_H(m_H+\nu_H) + [(\nu_H\delta_H\varepsilon_H + \nu_H\alpha_H\rho_H) + \rho_H(1-\nu_H)(m_H+\alpha_H)\beta_{dH}I_d^* + \rho_H m_H(m_H+\alpha_H+\nu_H)]\,E_H^*}{[(1-\nu_H)(m_H+\alpha_H)^2\beta_{dH}I_d^* + (m_H+\alpha_H)m_H(m_H+\alpha_H+\nu_H)]}.
\end{aligned}
$$

We note that, if $R_0 = 1$, then we will obtain the disease-free equilibrium, if $R_0 > 1$, then there exist a unique endemic equilibrium, if $R_0 < 1$, the there exist two endemic equilibrium.

3.4.7 Global Stability of Endemic Equilibrium E_1

Theorem 3.4.3 *The endemic equilibrium* E_1 *of model (3.1) is globally asymptotically stable whenever* $R_0 > 1$.

Proof

Suppose $R_0 > 1$, then the existence of the endemic equilibrium point is assured. Using the common quadratic Lyapunov function

$$V(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \frac{c_i}{2}(x_i - x_i^*)^2,$$

as illustrated in De Leo´n (2009), we consider a Lyapunov function with the following candidates

$$\mathcal{V}(S_d,\, E_d,\, I_d,\, R_d,\, S_H,\, E_H,\, I_H,\, R_H) = \frac{1}{2}\left[(S_d - S_d^*) + (E_d - E_d^*) + (I_d - I_d^*) + (R_d - R_d^*)\right]^2 + \frac{1}{2}\left[(S_H - S_H^*) + (E_H - E_H^*) + (I_H - I_H^*) + (R_H - R_H^*)\right]^2 \tag{3.48}$$

Now, differentiating equation (3.48) along the solution curve of equation (3.1) gives

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &= \left[(S_d - S_d^*) + (E_d - E_d^*) + (I_d - I_d^*) + (R_d - R_d^*)\right]\frac{d(S_d + E_d + I_d + R_d)}{dt} \\ &+ \left[(S_H - S_H^*) + (E_H - E_H^*) + (I_H - I_H^*) + (R_H - R_H^*)\right]\frac{d(S_H + E_H + I_H + R_H)}{dt}, \\ &= \left[(S_d - S_d^*) + (E_d - E_d^*) + (I_d - I_d^*) + (R_d - R_d^*)\right](A - m(S_d + E_d + I_d + R_d) - cE_d - \mu I_d) \\ &+ \left[(S_H - S_H^*) + (E_H - E_H^*) + (I_H - I_H^*) + (R_H - R_H^*)\right](B - m(S_H + E_H + I_H + R_H) - \mu_H I_H).\end{aligned}$$

Setting

$$A = m(S_{d*} + E_{d*} + I_{d*} + R_{d*}) + cE_{d*} + \mu I_{d*}$$

$$B = m_H(S_{H*} + E_{H*} + I_{H*} + R_{H*}) + \mu_H I_{H*}$$

Therefore, we have

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &= \left[(S_d - S_d^*) + (E_d - E_d^*) + (I_d - I_d^*) + (R_d - R_d^*)\right](m(S_d^* + E_d^* + I_d^* + R_d^*) + cE_d^* + \mu I_d^* \\ &- m(S_d + E_d + I_d + R_d) - cE_d - \mu I_d) + \left[(S_H - S_H^*) + (E_H - E_H^*) + (I_H - I_H^*)\right. \\ &\left.+ (R_H - R_H^*)\right](m_H(S_H^* + E_H^* + I_H^* + R_H^*) + \mu_H I_H^* - m(S_H + E_H + I_H + R_H) - \mu_H I_H). \\ \frac{d\mathcal{V}}{dt} &= \left[(S_d - S_d^*) + (E_d - E_d^*) + (I_d - I_d^*) + (R_d - R_d^*)\right](-m(S_d - S_d^*) - m(E_d - E_d^*) - m(I_d - I_d^*) \\ &- m(R_d - R_d^*) - c(E_d - E_d^*) - \mu(I_d - I_d^*)] + \left[(S_H - S_H^*) + (E_H - E_H^*) + (I_H - I_H^*)\right. \\ &\left.+ (R_H - R_H^*)\right](-m_H(S_H - S_H^*) - m_H(E_H - E_H^*) - m_H(I_H - I_H^*) - m_H(R_H - R_H^*) - \mu_H(I_H - I_H^*)]\end{aligned}$$

This implies

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &= -m(S_d - S_d^*)^2 - (c+m)(E_d - E_d^*)^2 - (m+\mu)(I_d - I_d^*)^2 - (2m+c)(S_d - S_d^*)(E_d - E_d^*) \\ &- m(R_d - R_d^*)^2 - (2m+\mu)(S_d - S_d^*)(I_d - I_d^*) - (2m+\mu+c)(E_d - E_d^*)(I_d - I_d^*) \\ &- 2m(R_d - R_d^*)(I_d - I_d^*) - (2m+\mu+c)(R_d - R_d^*)(I_d - I_d^*) - m_H(S_H - S_H^*)^2 \\ &- m_H(E_H - E_H^*)^2 - (m_H - \mu_H)(I_H - I_H^*)^2 - 2m_H(S_H - S_H^*)(E_H - E_H^*) \\ &- (2m_H - \mu_H)(S_H - S_H^*)(I_H - I_H^*) - (2m_H + \mu_H)(E_H - E_H^*)(I_H - I_H^*) \\ &- m_H\left[(I_H - I_H^*)(R_H - R_H^*) + (S_H - S_H^*)(R_H - R_H^*)\right]\end{aligned}$$

This shows that $\frac{d\mathcal{V}}{dt}$ is negative, and $\frac{d\mathcal{V}}{dt} = 0$, if and only if $S_d = S_d^*,\, E_d = E_d^*,\, I_d =$

I_d^*, $R_d = R_d^*$, $S_H = S_H^*$, $E_H = E_H^*$, $I_H = I_H^*$, $R_H = R_H^*$. It follows that every solution of the model (3.1) with the initial conditions, approaches E_1 as $t \longrightarrow \infty$, hence, the largest compact invariant set in $\left\{(S_d, E_d, I_d, R_d, S_H, E_H, I_H, R_H) \in \Omega : \frac{d\mathcal{V}}{dt} \leq 0\right\}$, is the singleton set $\{E_1\}$. Therefore, from the Lasalle's invariant principle LaSalle (1976), it implies that the endemic equilibrium E_1, is globally asymptotically stable in Ω whenever $R_0 > 1$.

3.5 Herd Immunity Threshold H_1

In this section, we consider the percentage of dogs that need to be immune in order to control the rabies transmission using the equation proposed by Diekmann and Heesterbeek. (1990). That is

$$H_1 = 1 - \frac{1}{\mathcal{R}_0}.$$

Therefore, our herd immunity is given as

$$H_1 = 1 - \frac{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)(m+\mu)m(m+\nu+\alpha)}{(1-\nu)\beta_d A(m+\alpha)(1-\rho)\delta\gamma}.$$

As the vaccination rate increases, the herd immunity threshold H_1 also increases. Whenever the herd immunity threshold decreases there is a potential decrease in the vaccination rate in the susceptible group. (Johnson and McQuarrie, 2009).

3.6 Optimal Control Problem

In this section, we shall consider the controllability matrix of model system (3.1) and then set an objective functional for the model system (3.1), using the Pontryagin's maximization principle. Finally we will set an optimal system and its corresponding Hamiltonian equation for the model system (3.1).

3.6.1 Controllability Matrix

To check whether the model system (3.1) is controllable, we used the Kalmann's controllability rank condition, with the linearised system

$$\begin{cases} \dfrac{dx}{dt} := f(x,u) = Ax(t) + Bu(t) \\ x \in \mathbb{R}^n, u \in U \in \mathbb{R}^m. \end{cases} \quad (3.49) \qquad \frac{dx}{dt} = \left(\frac{dS_d}{dt}, .., \frac{R_H}{dt} \right), \quad \text{where,.}$$

Its tangent linear system around the equilibrium point ($\bar{x}$,$\bar{u}$) is given by

$$\frac{d\xi}{dt} = A\xi + Bu, \quad (3.50)$$

where $A = \dfrac{\partial f}{\partial x_i}(\bar{x}, \bar{u})$ and $B = \dfrac{\partial f}{\partial u_i}(\bar{x}, \bar{u})$ are respectively $n \times n$ and $n \times m$ matrices. Therefore, the controllability matrix for this eighth-order system model (3.1) is given as

$$C = [B : AB : A^2B : A^3B : A^3B : A^4B : A^5B : A^6B : A^7B.] \quad (3.51)$$

Using the Jacobian matrix of model system (3.1) at the disease-free equilibrium point E_0, we have

$$\frac{d\xi}{dt} = \begin{pmatrix} -m & a_{12} & a_{13} & \alpha & 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{53} & 0 & a_{55} & a_{56} & 0 & a_{58} \\ 0 & 0 & a_{63} & 0 & 0 & a_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{76} & a_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88} \end{pmatrix} \xi + \begin{pmatrix} a_{13} \\ 0 \\ 0 \\ -a_{13} \\ a_{53} \\ 0 \\ 0 \\ -a_{53} \end{pmatrix} u \quad (3.52)$$

where,

$$\delta\gamma,\ a_{33} = -(m+\mu),\ a_{44} = -(m+\alpha),\ a_{53} = \frac{-\beta_{dH}(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H)},\ a_{56} = -m_H,\ a_{57} = \frac{\beta_H(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H},\ a_{58} = \alpha_H,\ a_{63} = \delta_H\varepsilon_H,\ a_{66} = -(\delta_H\gamma_H + m_H + \delta_H\varepsilon_H),\ a_{76} = \delta_H\gamma_H,\ a_{77} = -(m_H+\mu_H),\ a_{88} = -(m_H+\alpha_H)$$

$$a_{12} = \delta\varepsilon,\ a_{13} = \frac{-\beta(m+\alpha)A}{m(m+\nu+\alpha)},\ a_{22} = -(\delta\gamma+m+\delta\varepsilon+c),\ a_{23} = \frac{\beta(m+\alpha)A}{m(m+\nu+\alpha)},\ a_{32} =$$

After some algebraic manipulation we have

$$AB = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ -a_{22} & -a_{66} & -a_{32} & -a_{76} & -a_{53}a_{88} & -a_{13}a_{44} & -a_{53}a_{882} & -a_{13}a_{442} \end{bmatrix}, A_2B = \begin{bmatrix} A_{19} & A_{23} & A_{21} & A_{24} & A_{22} & A_{25} & A_{33} & A_{39} \\ -a_{13}a_{443} & -a_{53}a_{883} & -a_{13}a_{443} & -a_{53}a_{883} & -a_{13}a_{443} & -a_{53}a_{883} & -a_{13}a_{443} & -a_{53}a_{883} \end{bmatrix}$$

$$A_3B = \begin{bmatrix} A_{26} & A_{29} & A_{27} & A_{31} & A_{28} & A_{32} & A_{46} \\ -a_{13}a_{444} & -a_{53}a_{884} & -a_{13}a_{444} & -a_{53}a_{884} & -a_{13}a_{444} & -a_{53}a_{884} & -a_{13}a_{444} \end{bmatrix}, A_4B = \begin{bmatrix} \end{bmatrix}$$

$$\begin{vmatrix} & & & & & \\ & A_{34} & & & A_{47} & \\ & & & & & \\ & & & & & \\ & A_{35} & & A_{42} & A_{48} & \\ & & & & & \\ & & & & & \end{vmatrix}$$

$A_5B = -a_{13}a_{445}$, $A_6B = -a_{13}a_{446}$, $A^7B = -a_{13}a_{447}$.

$$\begin{vmatrix} & A_{36} & & & A_{43} & & & A_{49} & \\ & & & & & & & & \\ & & & & & & & & \\ & A_{37} & & & A_{44} & & & A_{51} & \\ & & & & & & & & \\ A_{38} & & & A_{45} & & & A_{52} & & \\ & & & & & & & & \\ & & & & & & & & \\ -a_{53}a_{885} & & & -a_{53}a_{886} & & & -a_{53}a_{887} & & \end{vmatrix}$$

Hence, equation (3.54) becomes

$$C = \begin{vmatrix} & & & & & & & \\ -a_{13} & A_{11} & A_{13} & A_{19} & A_{26} & A_{33} & A_{39} & A_{46} \\ & & & & & & & \\ 0 & -a_{22} & A_{14} & A_{21} & A_{27} & A_{34} & A_{41} & A_{47} \\ & & & & & & & \\ & & & & & & & \\ 0 & -a_{32} & A_{15} & A_{22} & A_{28} & A_{35} & A_{42} & A_{48} \\ & & & & & & & \\ & & & & & & & \\ -a_{13} & 2a_{44} & 2a_{442} & 2a_{443} & 2a_{444} & 2a_{445} & 2a_{446} & 2a_{447} \\ & & & & & & & \\ & & & & & & & \\ a_{53} & A_{12} & A_{16} & A_{23} & A_{29} & A_{36} & A_{43} & A_{49} \\ & & & & & & & \\ & & & & & & & \\ 0 & -a_{66} & A_{17} & A_{24} & A_{31} & A_{37} & A_{44} & A_{51} \end{vmatrix}$$

$$\begin{bmatrix} 0 & -a_{76} & A_{18} & A_{25} & A_{32} & A_{38} & A_{45} & A_{52} \\ -a_{53} & -a_{53}a_{88} & -a_{53}a_{882} & -a_{53}a_{883} & -a_{53}a_{884} & -a_{53}a_{885} & -a_{53}a_{886} & -a_{53}a_{887} \end{bmatrix} \tag{3.53}$$

Using MATLAB (14) we obtained the rank of matrix (C) to be n=8, Therefore, the model system (1) is controllable.

The definition of the controllability parameters is given in Appendix *A*.

3.7 Objective Functional

Given that $y(t) \in Y \in \mathrm{R}^n$ is a state variable of model system (3.1) and $u(t) \in U \in \mathrm{R}^n$ are the control variables at any time (t) with $t_{(0)} \leq t \leq t_{(f)}$, then an optimal control problem consists of finding a piecewise continuous control $u(t)$ and its corresponding state $y(t)$.

This optimizes the cost functional $J[y(t),u(t)]$ using Pontryagin's maximum principle (Sharomi and Malik, 2015).

Theorem 3.7.1 *If* $u^*(t)$ *and* $y^*(t)$ *are optimal for a given problem. Then* $\min_u J\,[y(t),u(t)]$ *where;* $J\,[y(t),u(t)] = \min_u \int_{t_0}^{t_f} f\,(t, y_1(t), y_2(t), y_3(t), u_1(t), u_2(t), u_3(t))\,dt.$

Subject to

$$\left.\begin{aligned} \frac{dy_1}{dt} &= g_1\,(t, y_1(t), y_2(t), y_3(t), u_1(t), u_2(t), u_3(t))\,, \\ \frac{dy_2}{dt} &= g_2\,(t, y_1(t), y_2(t), y_3(t), u_1(t), u_2(t), u_3(t))\,, \\ \frac{dy_3}{dt} &= g_3\,(t, y_1(t), y_2(t), y_3(t), u_1(t), u_2(t), u_3(t))\,. \end{aligned}\right\} \tag{3.54}$$

Then there exists a piecewise differential adjoint variable $\lambda(t)$*, such that the Hamiltonian equation (H) given as*

$$H\,(t,y(t),u(t),\lambda(t)) = f\,(t,y(t),u(t)) + g\,(t,y(t),u(t)), \tag{3.55}$$

with the necessary conditions

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial y_1}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y_2}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial y_3},$$

$$\lambda_1(t_f) = 0, \lambda_2(t_f) = 0, \lambda_3(t_f) = 0,$$

This implies that

$$\lambda'(t) = -\frac{\partial H\,(t, y^*(t), u^*, \lambda(t))}{\partial y},$$

with controls

$$\frac{\partial H}{\partial u_1} = 0, \frac{\partial H}{\partial u_2} = 0, \frac{\partial H}{\partial u_3} = 0,$$

we then set

$$\frac{dy_1}{dt} = \frac{\partial H}{\partial \lambda_1}, \frac{dy_2}{dt} = \frac{\partial H}{\partial \lambda_2}, \frac{dy_3}{dt} = \frac{\partial H}{\partial \lambda_3}.$$

Sufficient Condition

Theorem 3.7.2 *The condition of maximum principle of (3.54) are sufficient for global minimization of* $J\,[y(t),u(t)]$ *if the minimized Hamiltonian function H defined in (3.55) is convex in the variable* (y) *for all* (t) *in the time interval* $[t_0,t_f]$ *for a given* λ *. Hence, if* $y(t)$ *represents the class of individuals to be vaccinated and* $u(t) \in U$ *represents the controls, then the control set* (u) *is given as:* $U = \{u(t) : [t_0,t_f] \to [0,K]$ *lebesgue measurable* $\}$.

Consider the following interventions made in model system (3.1) thus

1. $u_1 = v$: Is the control effort aimed at reducing the infection of susceptible dogs (pre-exposed vaccination).

2. $u_2 = \rho$: Is the control effort aimed at treating the exposed dogs (post-exposed vaccination).

3. $u_3 = \nu_H$: Is the control effort aimed at reducing the infection of susceptible humans (Pre-exposure vaccination).

4. $u_4 = \rho_H$: Is the control effort aimed at treating the exposed humans (post-exposed vaccination)

Our main goal is to seek optimal controls ν^*, ρ^*, ν_H^* and ρ_H^* that minimize the objective functional:

$$J = \min \int_{t_0}^{t_f} \left[A_1 E_d + A_2 E_H + A_3 I_d + A_4 I_H + \frac{B1}{2}\nu^2 + \frac{B2}{2}\rho^2 + \frac{B3}{2}\nu_H^2 + \frac{B4}{2}\rho_H^2 \right] dt. \tag{3.56}$$

Therefore, equation (3.56) is subject to

$$\begin{aligned}
\frac{dS_d}{dt} &= A - (1-\nu)\beta_d S_d I_d - (m+\nu)S_d + \delta\varepsilon E_d + \alpha R_d, \\
\frac{dE_d}{dt} &= (1-\nu)\beta_d S_d I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d, \\
\frac{dI_d}{dt} &= (1-\rho)\delta\gamma E_d - (m+\mu)I_d, \\
\frac{dR_d}{dt} &= \nu S_d + \rho E_d - (m+\alpha)R_d, \\
\frac{dS_H}{dt} &= B - (1-\nu_H)\beta_{dH} S_H I_d - (m_H + \nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H, \\
\frac{dE_H}{dt} &= (1-\nu_H)\beta_{dH} S_H I_d - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H \\
\frac{dI_H}{dt} &= (1-\rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H)I_H, \\
\frac{dR_H}{dt} &= \nu_H S_H + \rho_H E_H - (m_H + \alpha_H)R_H, \\
S_d \geq 0, E_d \geq 0, I_d \geq 0, R_d \geq 0, S_H \geq 0, E_H \geq 0, I_H \geq 0, R_H \geq 0. &,
\end{aligned}$$

From equation (3.56) the quantities A_1 and A_2 denote the weight constants of the exposed classes and A_3 and A_4 are the weight of the infected classes. $B1, B2, B3, B4$ are the weight constants for dogs control and personal protection (Prevention of infected dog-human contact). $B1\nu^2, B2\rho^2, B3\nu_H^2, B4\rho_H^2$ describe the cost associated with rabies control and

prevention. The square of the control variables shows the severity of the side effects of the vaccination and treatment. Employing, the Pontryagin's maximum principle, we form the Hamiltonian equation as

$$\begin{aligned}
H = {} & A_1E_d + A_2E_H + A_3I_d + A_4I_H + \frac{B1}{2}\nu^2 + \frac{B2}{2}\rho^2 + \frac{B3}{2}\nu_H^2 + \frac{B4}{2}\rho_H^2 \\
& + \lambda_1\left[A - (1-\nu)\beta_d S_d I_d - (m+\nu)S_d + \delta\varepsilon E_d + \alpha R_d\right], \\
& + \lambda_2\left[(1-\nu)\beta_d S_d I_d - ((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)E_d\right], \\
& + \lambda_3\left[(1-\rho)\delta\gamma E_d - (m+\mu)I_d\right], \\
& + \lambda_4\left[\nu S_d + \rho E_d - (m+\alpha)R_d\right], \\
& + \lambda_5\left[B - (1-\nu_H)\beta_{dH} S_H I_d - (m_H+\nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H\right], \\
& + \lambda_6\left[(1-\nu_H)\beta_{dH} S_H I_d - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H\right], \\
& + \lambda_7\left[(1-\rho_H)\delta_H\gamma_H E_H - (m_H+\mu_H)I_H\right], \\
& + \lambda_8\left[\nu_H S_H + \rho_H E_H - (m_H+\alpha_H)R_H\right].
\end{aligned}$$

Considering the existence of adjoint functions $\lambda_i, i = 1,2,\ldots 8$ satisfying

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial S_d}, \\
&= -\left[-\lambda_1(1-\nu^*)\beta_d I_d - \lambda_1(m+\nu^*) + \lambda_2(1-\nu^*)\beta_d I_d + \lambda_4\nu^*\right], \\
&= \lambda_1\left((1-\nu^*)\beta_d I_d + m + \nu^*\right) - \lambda_2(1-\nu^*)\beta_d I_d - \lambda_4\nu^*, \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial E_d}, \\
&= -\left[A_1 - \lambda_1\delta\varepsilon - \lambda_2((1-\rho^*)\delta\gamma + m + \rho^* + \delta\varepsilon + c) + \lambda_3(1-\rho^*)\delta\gamma + \lambda_4\rho^*\right],
\end{aligned}$$

$$= \lambda_2((1-\rho^*)\delta\gamma + m + \rho^* + \delta\varepsilon + c) - \lambda_1\delta\varepsilon - \lambda_3(1-\rho^*)\delta\gamma - \lambda_4\rho^* - A_1,$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I_d},$$

$$= -\left[A_3 - \lambda_1(1-\nu^*)\beta_d S_d + \lambda_2(1-\nu^*)\beta_d S_d - \lambda_3(m+\mu) - \lambda_5(1-\nu_H^*)\beta_{dH} S_H + \lambda_6(1-\nu_H^*)\beta_{dH} S_H\right],$$

$$= \lambda_3(m+\mu) + \lambda_1(1-\nu^*)\beta_d S_d + \lambda_5(1-\nu_H^*)\beta_{dH} S_H - \lambda_2(1-\nu^*)\beta_d S_d - \lambda_6(1-\nu_H^*)\beta_{dH} S_H - A_3,$$

$$\frac{d\lambda_4}{dt} = \frac{\partial H}{\partial R_d},$$

$$= -\left[\lambda_1\alpha - \lambda_4(m+\alpha)\right],$$

$$= \lambda_4(m+\alpha) - \lambda_1\alpha,$$

$$\frac{d\lambda_5}{dt} = \frac{\partial H}{\partial S_H},$$

$$= -\left[-\lambda_5(1-\nu_H^*)\beta_{dH} I_d - \lambda_5(m_H + \nu_H^*) + \lambda_6(1-\nu_H^*)\beta_{dH} I_d + \lambda_8\nu_H^*\right],$$

$$= \lambda_5\left((1-\nu_H^*)\beta_{dH} I_d + m_H + \nu_H^*\right) - \lambda_6(1-\nu_H^*)\beta_{dH} I_d - \lambda_8\nu_H^*,$$

$$\frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial E_H},$$

$$= -\left[A_2 - \lambda_5(\delta_H\varepsilon_H) - \lambda_6((1-\rho_H^*)\delta_H\gamma_H + m_H + \rho_H^* + \delta_H\varepsilon_H) + \lambda_7(1-\rho_H^*)\delta_H\gamma_H + \lambda_8\rho_H\right]$$

$$= \lambda_6((1-\rho_H^*)\delta_H\gamma_H + m_H + \rho_H^* + \delta_H\varepsilon_H) - \lambda_5\delta_H\varepsilon_H - \lambda_7(1-\rho_H^*)\delta_H\gamma_H - \lambda_8\rho_H^* - A_2,$$

$$\frac{d\lambda_7}{dt} = \frac{\partial H}{\partial I_H},$$

$$= -\left[A_4 - \lambda_7(m_H + \mu_H)\right],$$

$$= \lambda_7(m_H + \mu_H) - A_4,$$

$$\frac{d\lambda_8}{dt} = \frac{\partial H}{\partial R_H},$$

$$= -\left[\lambda_5\alpha_H - \lambda_8(m_H + \alpha_H)\right],$$

$$= \lambda_8(m_H + \alpha_H) - \lambda_5\alpha_H,$$

with transversality condition $\lambda_i(t_f) = 0$ for $i = 1...8$ for the control set u_i hence

$$\frac{\partial H}{\partial u_i} = 0 \text{ where } i = 1,2,3,4$$

$$\frac{\partial H}{\partial \nu} = B1\nu - \lambda_1 S_d + \lambda_4 S_d + \lambda_1 \beta_d S_d I_d - \lambda_2 \beta_d S_d I_d,$$

$$\frac{\partial H}{\partial \nu}|_{\nu=\nu^*} := B1\nu^* - \lambda_1 S_d + \lambda_4 S_d + \lambda_1 \beta_d S_d I_d - \lambda_2 \beta_d S_d I_d = 0,$$

$$\nu^* = \frac{(\lambda_1 S_d^* - \lambda_4 S_d^*) + (\lambda_2 - \lambda_1)\beta_d I_d^* S_d^*}{B1},$$

$$\frac{\partial H}{\partial \rho} = B2\rho - \lambda_2 E_d + \lambda_2 \delta\gamma E_d - \lambda_3 \delta\gamma E_d$$

$$\frac{\partial H}{\partial \rho}|_{\rho=\rho^*} := B2\rho^* - \lambda_1 E_d + \lambda_4 E_d + \lambda_2 E_d \delta\gamma E_d - \lambda_3 \delta\gamma E_d = 0,$$

$$\rho^* = \frac{(\lambda_2 E_d^* - \lambda_4 E_d^*) + (\lambda_3 - \lambda_2)\delta\gamma E_d^*}{B2},$$

$$\frac{\partial H}{\partial \nu_H} = B3\nu_H - \lambda_5 S_H + \lambda_8 S_H + \lambda_5 \beta_{dH} S_H I_d - \lambda_6 \beta_{dH} S_H I_d,$$

$$\frac{\partial H}{\partial \nu_H}|_{\nu_H=\nu_H^*} := B3\nu^* - \lambda_5 S_H + \lambda_8 S_H + \lambda_5 \beta_{dH} S_H I_d - \lambda_6 \beta_{dH} S_H I_d = 0,$$

$$\nu_H^* = \frac{(\lambda_5 S_H^* - \lambda_8 S_H^*) + (\lambda_6 - \lambda_5)\beta_{dH} S_H^* I_d^*}{B3},$$

$$\frac{\partial H}{\partial \rho_H} = B4\rho_H - \lambda_6 E_H + \lambda_8 E_H + \lambda_6 \delta_H \gamma_H E_H - \lambda_7 \delta_H \gamma_H E_H,$$

$$\frac{\partial H}{\partial \rho_H}|_{\rho_H=\rho_H^*} := B4\rho^* - \lambda_6 E_H + \lambda_8 E_H + \lambda_6 \delta_H \gamma_H E_H - \lambda_7 \delta_H \gamma_H E_H = 0$$

$$\rho_H^* = \frac{(\lambda_6 E_H^* - \lambda_8 E_H^*) + (\lambda_7 - \lambda_6)\delta_H \gamma_H E_H^*}{B4}. \quad ,$$

Now, using an appropriate variation argument and taking the bounds into account, the optimal control strategies are given as $\nu^*, \rho^*, \nu_H^*, \rho_H^*$.

$$\nu^* = \min\left\{\max\left(0, \frac{(\lambda_1 - \lambda_4)S_d^* + (\lambda_2 - \lambda_1)\beta_d I_d^* S_d^*}{B1}\right), \nu_{\max}\right\},$$

$$\rho^* = \min\left\{\max\left(0, \frac{(\lambda_2 - \lambda_4)E_d^* + (\lambda_3 - \lambda_2)\delta\gamma E_d^*}{B2}\right), \rho_{\max}\right\},$$

$$\nu_H^* = \min\left\{\max\left(0, \frac{(\lambda_5 - \lambda_8)S_H^* + (\lambda_6 - \lambda_5)\beta_{dH} S_H^* I_d^*}{B3}\right), \nu_{H\max}\right\},$$

$$\rho_H^* = \min\left\{\max\left(0, \frac{(\lambda_6 - \lambda_8)E_H^* + (\lambda_7 - \lambda_6)\delta_H \gamma_H E_H^*}{B4}\right), \rho_{H\max}\right\}.$$

3.8 Optimality System

Substituting the representation of the optimal vaccination controls, we have the optimality system as

$$\begin{aligned}
\frac{dS_d}{dt} &= A - \left(1 - \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_d^* + (\lambda_2-\lambda_1)\beta_d I_d^* S_d^*}{B_1}\right), \nu_{\max}\right\}\right)\beta_d S_d I_d - mS_d \\
&\quad - \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_d^* + (\lambda_2-\lambda_1)\beta_d I_d^* S_d^*}{B_1}\right), \nu_{\max}\right\} S_d \\
&\quad + \delta\varepsilon E_d + \alpha R_d, \\
\frac{dE_d}{dt} &= (1 - \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_d^* + (\lambda_2-\lambda_1)\beta_d I_d^* S_d^*}{B_1}\right), \nu_{\max}\right\})\beta_d S_d I_d \\
&\quad - \left(\left(1 - \min\left\{\max\left(0, \frac{(\lambda_2-\lambda_4)E_d^* + (\lambda_3-\lambda_2)\delta\gamma E_d^*}{B2}\right), \rho_{\max}\right\}\right)\delta\gamma + m + \delta\varepsilon + c\right) E_d \\
&\quad - \min\left\{\max\left(0, \frac{(\lambda_2-\lambda_4)E_d^* + (\lambda_3-\lambda_2)\delta\gamma E_d^*}{B2}\right), \rho_{\max}\right\} E_d, \\
\frac{dI_d}{dt} &= \delta\gamma E_d - (m+\mu)I_d, \\
\frac{dR_d}{dt} &= \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_d^* + (\lambda_2-\lambda_1)\beta_d I_d^* S_d^*}{B_1}\right), \nu_{\max}\right\} S_d - (m+\alpha)R_d \\
&\quad + \min\left\{\max\left(0, \frac{(\lambda_2-\lambda_4)E_d^* + (\lambda_3-\lambda_2)\delta\gamma E_d^*}{B2}\right), \rho_{\max}\right\} E_d, \\
\frac{dS_H}{dt} &= B - \left(1 - \min\left\{\max\left(0, \frac{(\lambda_5-\lambda_8)S_H^* + (\lambda_6-\lambda_5)\beta_{dH} S_H^* I_d^*}{B3}\right), \nu_{H\max}\right\}\right)\beta_{dH} S_H I_d - m_H S_H \\
&\quad - \min\left\{\max\left(0, \frac{(\lambda_5-\lambda_8)S_H^* + (\lambda_6-\lambda_5)\beta_{dH} S_H^* I_d^*}{B3}\right), \nu_{H\max}\right\} S_H \\
&\quad + \delta_H\varepsilon_H E_H + \alpha_H R_H, \\
\frac{dE_H}{dt} &= \left(1 - \min\left\{\max\left(0, \frac{(\lambda_6-\lambda_8)E_H^* + (\lambda_7-\lambda_6)\delta_H\gamma_H E_H^*}{B4}\right), \rho_{H\max}\right\}\right)\beta_{dH} S_H I_d \\
&\quad - (\delta_H\gamma_H + m_H + \delta_H\varepsilon_H)E_H \\
&\quad - \min\left\{\max\left(0, \frac{(\lambda_6-\lambda_8)E_H^* + (\lambda_7-\lambda_6)\delta_H\gamma_H E_H^*}{B4}\right), \rho_{H\max}\right\} E_H \\
\frac{dI_H}{dt} &= \delta_H\gamma_H E_H - (m_H+\mu_H)I_H \\
\frac{dR_H}{dt} &= \min\left\{\max\left(0, \frac{(\lambda_5-\lambda_8)S_H^* + (\lambda_6-\lambda_5)\beta_{dH} S_H^* I_d^*}{B3}\right), \nu_{H\max}\right\} S_H - (m_H+\alpha_H)R_H \\
&\quad + \min\left\{\max\left(0, \frac{(\lambda_6-\lambda_8)E_H^* + (\lambda_7-\lambda_6)\delta_H\gamma_H E_H^*}{B4}\right), \rho_{H\max}\right\} E_H,
\end{aligned}$$

$\frac{1}{dt}, \frac{2}{dt}, \frac{3}{dt}, \frac{4}{dt}, \frac{5}{dt}, \frac{6}{dt}, \frac{7}{dt}, \frac{8}{dt}$ $d\lambda$ $d\lambda$ $d\lambda$ $d\lambda$ $d\lambda$ $d\lambda$ $d\lambda$ $d\lambda$, with, $\lambda_i(t_f) = 0, i = 1,2,3,4,5,6,7,8.$

Chapter 4

Numerical Analysis and Discussion

4.1 Introduction

In this chapter, we will use the analysis in chapter three, to estimate the basic reproductive number without vaccination $\mathcal{R}_0^*$, basic reproductive number with vaccination R_0, the herd immunity H_1, the disease equilibrium point and the stability analysis of the model system (3.1) numerically. We will use Matlab to simulate the sensitivity of the basic reproduction number, with pre and post exposure prophylaxis, we will also compare the existing model of (Zhang et al., 2011) with the optimal control model, and then we will simulate optimality system using the backward swap method.

4.2 Parameter Values

WHO (2010) reports that the incubation period of rabies is 1 − 3 months. Therefore we assumed the median value of 2 months for the incubation period of rabies. We assume loss of immunity to be $\alpha = \alpha_H = 1$. The probability of the clinical outcome of the exposed is 30% -70%. (WHO, 2010) and we considered that it is 40%. Hence, the rate at which the exposed dogs and humans move to the infected classes is $\gamma = 0.4$ and $\gamma_H = 0.4$ respectively. The rate of vaccination is the product of efficiency of the coverage rate of rabies vaccine (Zhang et al., 2011). Hence, the efficiency of rabies vaccine is about 90%

(WHO, 2010). However, in Africa the rates of vaccination and treatment is below 40% (WHO, 2010). We obtained the transmission rates β_d, β_{dH} by comparing the recent trend of rabies cases in Africa and Asia, so we used the transmission rate estimated by

Zhang et al. (2011). Since getting data for the studies was very difficult, we assumed a constant natural death rate $m = 0.056$, for dogs. All parameter as measured in years, the details of these parameters are shown below.

Parameter	Description	Standard Value	Source
A	Birth/Recruitment rate of Dogs	$3 \times 10^6 y^{-1}$	Zhang et al. (2011)
α	Loss of immunity in dogs	$1y^{-1}$	Zhang et al. (2011)

c	Death rate of dogs due to external factors	$0.3y^{-1}$	Assumed
m	Natural death rate of dogs	$0.056y^{-1}$	Zhang et al. (2011)
μ	Disease induced mortality in dogs	$1y^{-1}$	Zhang et al. (2011)
ν	Pre-exposure prophylaxis for dogs	$0.25y^{-1}$	GVMA 2014
ρ	Post-exposure prophylaxis for dogs	$0.2y^{-1}$	Zhang et al. (2011)
β_d	Transmission rate in dogs	$1.58 \times 10^{-7}y^{-1}$	Zhang et al. (2011)
γ	Latency period in dogs	$\frac{37}{6}y^{-1}$	Zhang et al. (2011)
$\delta\varepsilon$	Rate of no clinical rabies	$0.4y^{-1}$	Zhang et al. (2011)
B	Birth/Recruitment rate (humans)	$0.0314y^{-1}$	GDP 2014
β_{dH}	Transmission rate (dog - humans)	$2.29 \times 10^{-12}y^{-1}$	Zhang et al. (2011)
α_H	Loss of immunity (humans)	$1y^{-1}$	Zhang et al. (2011)
m_H	Natural death rate (humans)	$0.0074y^{-1}$	GDP 2014
μ_H	Disease induced mortality (humans)	$1y^{-1}$	Zhang et al. (2011)
ν_H	Pre-exposure prophylaxis for humans	$0.54y^{-1}$	GVMA 2014
ρ_H	Post-exposure prophylaxis for humans	$0.1y^{-1}$	Zhang et al. (2011)
γ_H	Latency rate (humans)	$\frac{1}{6}y^{-1}$	Zhang et al. (2011)
$\gamma_H\varepsilon_H$	Rate of no clinical rabies (humans)	$2.4y^{-1}$	Zhang et al. (2011)

Table 4.1: Parameter values

4.2.1 Basic Reproduction Number without controls $\mathcal{R}_0^*$

In this subsection, we consider model system (3.1) without control measures. We denote the basic reproduction number without control as $\mathcal{R}_0^*$. Therefore, R_0 is given as:

$$\mathcal{R}_0^* = \frac{\beta_d A \delta \gamma}{(\delta\gamma + m + \delta\varepsilon + c) \times (m + \mu)\, m},$$

$$\therefore \quad \mathcal{R}_0^* = \frac{2.4 \times (3 \times 10^6) \times (1.58 \times 10^{-7})}{(2.4 + 0.056 + 3.6 + 0.3)\,(0.056 + 1)\,(0.056)},$$

$$\mathcal{R}_0^* = 3.027$$

This indicates that without vaccination, and assuming the disease occurs, the infection will spread faster and may cause more deaths in the human population, since $\mathcal{R}_0^* = 3.027 > 1$. Hence, the presence of one infectious dog could produce more than one secondary infections in susceptible dogs or humans.

4.2.2 Herd Immunity Threshold

We seek to know the number of humans or dogs that should be vaccinated during the outbreak of the disease.

$$H_1 = 1 - \frac{1}{\mathcal{R}_{0,}^*}$$

$$H_1 = 1 - \frac{1}{3.027,}$$

$$H_1 = 0.66.$$

This shows that about 66%, of susceptible humans and dogs should be vaccinated during an outbreak of rabies, in order to control the disease. That is if the proportion of immune humans and dogs goes beyond 66%, due to mass vaccination, the disease could be eradicated.

4.3 Equilibrium points and stability analysis

From chapter three we obtained the disease-free equilibrium point as:

$$\mathcal{E}_0 = \left(\frac{A\,(m+\alpha)}{m\,(m+\alpha+\nu)},\, 0,\, 0,\, \frac{A\nu}{m\,(m+\alpha+\nu)},\, \frac{B\,(m_H+\alpha_H)}{m_H\,(m_H+\alpha_H+\nu_H)},\, 0,\, 0,\, \frac{B\nu_H}{m_H\,(m_H+\alpha_H+\nu_H)} \right),$$

incorporating the parameter values gives:

$$\mathcal{E}_0 = \left(4.3317 \times 10^7,\, 0,\, 0,\, 1.025 \times 10^7,\, 2.7625,\, 0,\, 0,\, 1.4808\right).$$

4.3.1 Stability analysis of disease-free equilibrium point

The Jacobian matrix of the rabies model at the disease-free equilibrium point is given as:

$$\begin{bmatrix} b_1 - \lambda & b_7 & \alpha & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J(E_0) = \begin{bmatrix} a_2 - \lambda & a_1 & 0 & 0 & 0 & 0 & 0 \\ \delta\gamma & a_3 & 0 & 0 & 0 & 0 & 0 \\ \rho & b_2 - \lambda & b_3 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 - \lambda & b_5 & 0 & 0 \\ 0 & a_4 & 0 & 0 & a_6 - \lambda & 0 & 0 \\ 0 & a_5 & 0 & \nu_H & \delta_H\gamma_H\rho_H & b_6 - \lambda & \alpha_H \\ 0 & 0 & 0 & 0 & 0 & 0 & b_8 - \lambda \\ 0 & 0 & \nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where, $b_1 = -(m+\nu),\ b_7 = \delta\varepsilon,\ a_1 = \frac{-\beta(m+\alpha)A}{m(m+\nu+\alpha)},\ a_2 = -(\delta\gamma + m\rho + \delta\varepsilon + c),\ a_3 = \frac{\beta(m+\alpha)A}{m(m+\nu+\alpha)},\ b_2 = -(m+\mu),\ b_3 = -(m+\alpha),\ a_4 = \frac{-\beta_{dH}(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H}},\ b_4 = -(\nu_H + m_H),\ b_5 = \frac{\beta_{dH}(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H)},\ a_5 = \delta_H\varepsilon_H,\ a_6 = -(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)$ and $b_8 = -(m_H + \alpha_H)$

Substituting the parameter values gives: $b_1 = -0.306$, $b_7 = 3.6$, $a_1 = -6.8440$, $a_2 = -6.556$, $a_3 = 6.8440$, $b_2 = -1.056$, $b_3 = -1.056$, $a_4 = -2.7625$, $b_4 = -0.5474$, $b_5 = 2.7625$, $a_5 = 2.4$, $a_6 = -6.1074$, $b_6 = -1.0074$ and $b_8 = -1.0074$.

$$J(E_0) = \begin{bmatrix} -0.306 & 3.6 & -6.8440 & 1 & 0 & 0 & 0 \\ 0 & -6.556 & 6.8440 & 0 & 0 & 0 & 0 \\ & 3.6 & -1.056 & 0 & 0 & 0 & 0 \\ & 0.2 & 0 & -1.056 & 0 & 0 & 0 \\ 0 & 0 & -2.7625 & 0 & -0.5474 & 2.7625 & 0 \\ & 0 & 2.4 & 0 & 0 & -6.1074 & 1 \\ 0.25 & 0 & 0 & 0 & 0 & 3.6 & -1.0074 \\ & 0 & 0 & 0 & 0.54 & 0.1 & 0 \end{bmatrix}$$

0

0

0

0

−1.0074

Now, finding the eigenvalues, using $\det|J(E_0) - \lambda I|$ gives:

$$(-1.0074-\lambda)(-6.1074-\lambda)(-0.5474-\lambda)(-1.0074-\lambda)\left[\lambda^4 a_{11}\lambda^3 + a_{12}\lambda^2 + a_{13}\lambda + a_{14}\right] = 0 \quad (4.1)$$

,

where; $a_{11} = (b_2b_1 - b_2 - a_2 - b_1 - b_3)$, $a_{12} = (b_3b_2 + b_1b_3 + b_3a_2 + b_1a_2 + \delta\gamma a_3 + b_2a_2 - \nu\alpha)$, $a_{13} = (a_2\nu\alpha + b_2\nu\alpha - \delta\gamma a_3b_3 - b_1\delta\gamma a_3 - b_3b_2a_2 - b_1b_3a_2 - b_2b_1a_2)$ and $a_{14} = (b_1b_2b_3a_2 + \delta\gamma a_3b_3b_1 - \nu\delta a_3\alpha - \nu\alpha b_2a_2)$.

Therefore the first four eigenvalues are given as $\lambda_1 = -1.0074$, $\lambda_2 = -6.1074$, $\lambda_3 = -0.5474$ and $\lambda_4 = -1.0074$. Using the Routh-Hurwitz criterion, as described in Chapter three, gives the following:

$a_{11} = 9.2971$, $a_{12} = 33.3325$, $a_{13} = 43.2022$ and $a_{14} = 2.3083$.

From the Routh-Hurwitz criterion the eigenvalues of the matrix system, have negative real parts if and only if the following conditions are satisfied under the polynomial of degree 4.

Conditions:

$$a_{11} > 0,\ a_{12} > 0,\ a_{13} > 0,\ a_{14} > 0,\ a_{11}a_{12}a_{13} > a_{13}^2 + a_{11}^2a_{14}.$$

Substituting the estimated parameter values into the above conditions yields

$$a_{11} = 9.2971 > 0,\ a_{12} = 33.3325,\ a_{13} = 43.2022 > 0,\ a_{14} = 2.3083 > 0,$$

$$a_{11}a_{12}a_{13} = 13387.97 > a_{13}^2 + a_{11}^2 a_{14} = 2065.95.$$

From this analysis, it clearly shows that the disease-free equilibrium E_0 is locally asymptotically stable.

4.3.2 Basic reproductive number R_0 with controls

$$\mathcal{R}_0 = \frac{(1-\nu)(1-\rho)\delta\gamma\beta_d A(m+\alpha)}{((1-\rho)\delta\gamma + m + \rho + \delta\varepsilon + c)\,(m+\mu)\,m\,(m+\nu+\alpha)}.$$

Therefore,

$$\mathcal{R}_0 = \frac{(1-0.25)(1-0.2)\times 2.4\times(3\times 10^6)\times(1.58\times 10^{-7})(0.056+1)}{((1-0.2)\times 2.4 + 0.056 + 3.6 + 0.3 + 0.2)\,(0.056+1)\,(0.056+0.25+1)\times 0.056}.$$

$$R_0 = 1.536.$$

Therefore, applying controls to the model, shows that R_0 can be brought down with effective controls. Therefore, this shows that to minimize the spread of rabies, we need effect control measures, thus more of pre-exposure vaccination and post-exposure vaccination (post-exposure treatment).

4.3.3 Numerical simulations

In this section, we consider the graphical interpretation of the model by using the parameter values in Table 4.2. We will use MATLAB and the fourth order Runge-Kutta method for our simulations. We will also consider the sensitivity analysis of the basic reproduction number. We will simulate our optimality solution using the forward-backward sweep scheme.

In Figure 4.1, we look at the effect of the initial conditions of $S_d(0)$,$E_d(0)$,$I_d(0)$,$R_d(0)$, on the infected humans. We found out that an increase in the initial conditions of susceptible dogs could increase the number of infected humans, as indicated by the red line in Figure 4.1(A). The number of exposed, infected and recovered has the same influence on the number of infected human, as shown in Figure 4.1(B), 4.1(C), and 4.1(D) respectively. The plot below suggest that the best way to control the infection in the human population is to reduce the recruitment rate in the dog population, whenever the transmission starts, or increase vaccination in the susceptible dog class.

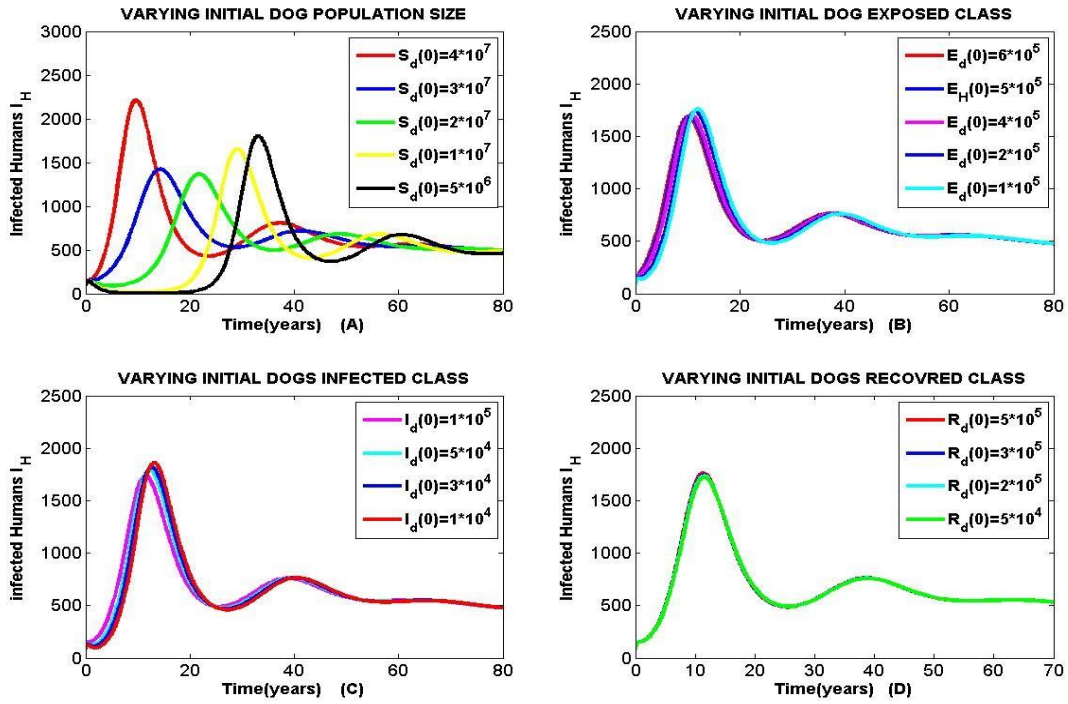


Figure 4.1: Varying the initial condition of $S_d(0)$,$E_d(0)$, $I_d(0)$ and $R_d(0)$ respectively, on the infected humans

From Figure 4.2, the first sub-plot shows that as the human population grows, with a continuous growth in the dog population there is a likelihood of a higher rate of rabies cases in the human compartment, as indicated by the red line. Figure 4.2 also indicates that as the population of humans reduces, the infection could also be reduced as indicated by the black line in Figure 4.2 (A). In sub-plot B in Figure 4.2, it shows that varying the

exposed class of the human population with a constant transmission rate from the infectious dogs has little effect on the spread of the disease in the human population. Varying the number of recovered humans maintains the same likelihood of becoming infected again at a constant transmission rate from the dog's population, since the recovered person does not build a permanent immunity. Hence, Figure 4.2(C), also suggest that the best way to control the disease in the human population is to increase vaccination coverage in the susceptible dogs.

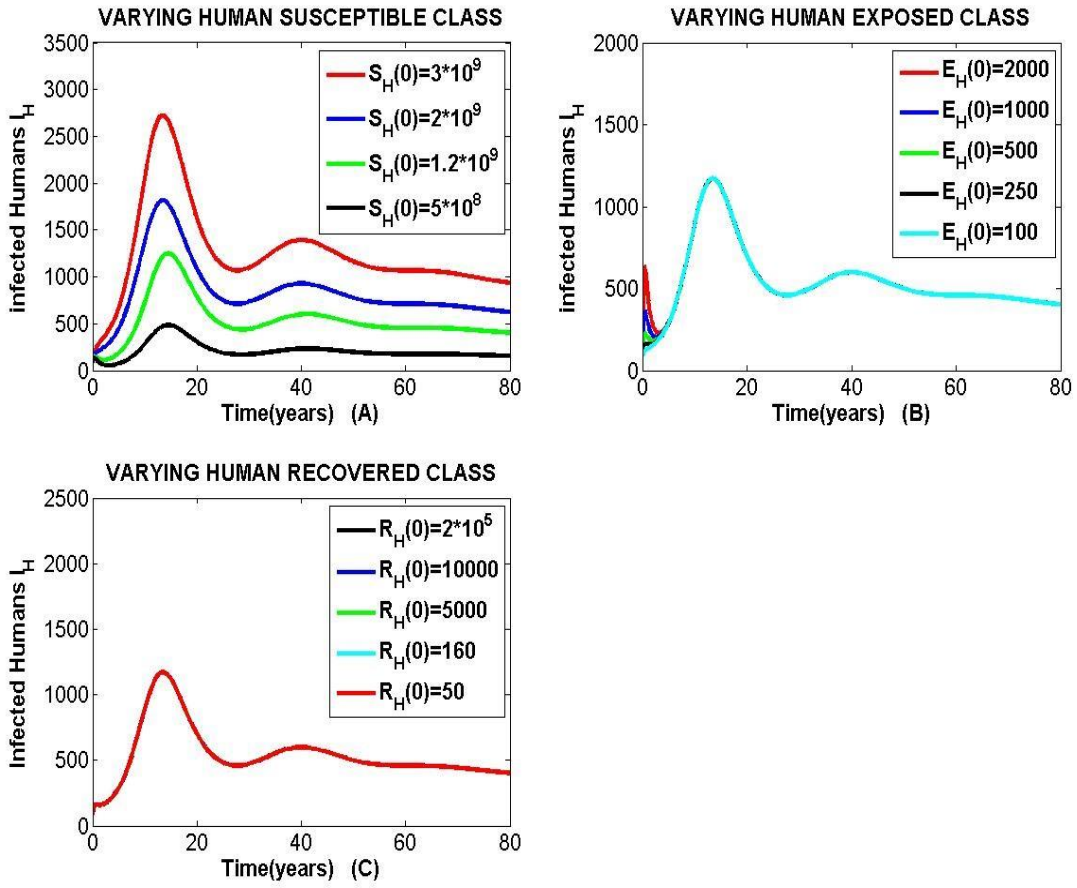


Figure 4.2: Varying the initial conditions of $S_H(0)$,$E_H(0)$ and $R_H(0)$ respectively

4.3.4 Sensitivity Analysis (S.I)

In this section, we want to determine the parameters that contribute most to the transmission dynamics of the rabies infection.

Definition 1 *The normalized forward-sensitivity index of* R_0 *to any parameter, say* ρ*, as given in Martcheva (2015) can be defined as:*

$$\Gamma^{\rho}_{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \rho}\frac{\rho}{\mathcal{R}_0}. \tag{4.2}$$

Therefore from equation (3.25), the sensitivity indices of R_0 with respect to the parameter values in Table 4.2 are

$$, \begin{cases} \Gamma^{\beta_d}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial \beta_d}\dfrac{\beta_d}{\mathcal{R}_0} = 1, \\ \Gamma^{A}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial A}\dfrac{A}{\mathcal{R}_0} = 1, \\ \Gamma^{\mu}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial \mu}\dfrac{\mu}{\mathcal{R}_0} = \dfrac{-\mu}{(m+\mu)} = -0.95, \\ \Gamma^{\delta\varepsilon}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial \delta\varepsilon}\dfrac{\delta\varepsilon}{\mathcal{R}_0} = \dfrac{\delta\varepsilon}{((1-\rho)\delta\gamma - \delta\varepsilon - c - m - \rho)} = -1.61 \\ \Gamma^{c}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial c}\dfrac{c}{\mathcal{R}_0} = \dfrac{c}{((1-\rho)\delta\gamma - \delta\varepsilon - c - m - \rho)} = -0.45, \\ \Gamma^{\alpha}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial \alpha}\dfrac{\alpha}{\mathcal{R}_0} = 0.28, \\ \Gamma^{m}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial m}\dfrac{m}{\mathcal{R}_0} = -1.64, \\ \Gamma^{\delta\gamma}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial \delta\gamma}\dfrac{\delta\gamma}{\mathcal{R}_0} = 1.33, \\ \Gamma^{\rho}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial \rho}\dfrac{\rho}{\mathcal{R}_0} = -0.05, \\ \Gamma^{\nu}_{\mathcal{R}_0} = \dfrac{\partial \mathcal{R}_0}{\partial \nu}\dfrac{\nu}{\mathcal{R}_0} = -0.52. \end{cases} \tag{4.3}$$

Below is the signs of the sensitivity indices in R_0.

Parameter	Description	Sensitivity index
β_d	Transmission rate of dogs	+ve
ρ	Post-exposure vaccination (treatment)	-ve
ν	Pre-exposure vaccination	-ve
A	Birth/Recruitment rate of dogs	+ve
m	Natural death rate of dogs	-ve
$\delta\gamma$	Rate at which expose dogs becomes infective (Infective rate)	+ve
α	Loss of immunity	+ve
$\delta\varepsilon$	Rate of no clinical rabies	-ve

μ	Disease induce death rate	-ve

Table 4.2: Sensitivity indices of R_0 to the parameters in equation (3.25).

The positive signs in Table (4.2), promotes the propagation of the disease. Therefore an addition or reduction in the values of β_d, α, $\delta\gamma$ and A, will have an increase or decrease in the spread of the disease. For example $\Gamma_{\beta R_{d0}} = 1$ indicates that increasing or reducing the transmission rate by 10%, may increase or reduce the number of secondary infection by 10%. The negative sign in Table 4.2, will have a reduction in the basic reproduction number, R_0, when the values of those parameters are increased, and a reduction in the values of ρ, ν, μ, m, and $\delta\varepsilon$, will lead to an increase in the number of secondary infections. Therefore, this show that in the presence of vaccination and treatment in model (3.1) decreases the spreading rate of the disease. Hence, the sensitivity analysis gives a wonderful insight into the control of the rabies transmission dynamics. Additionally, the sensitivity indices could help public/private health authorities in focusing on the best method for preventing and controlling the disease in the course of any outbreak of the disease.

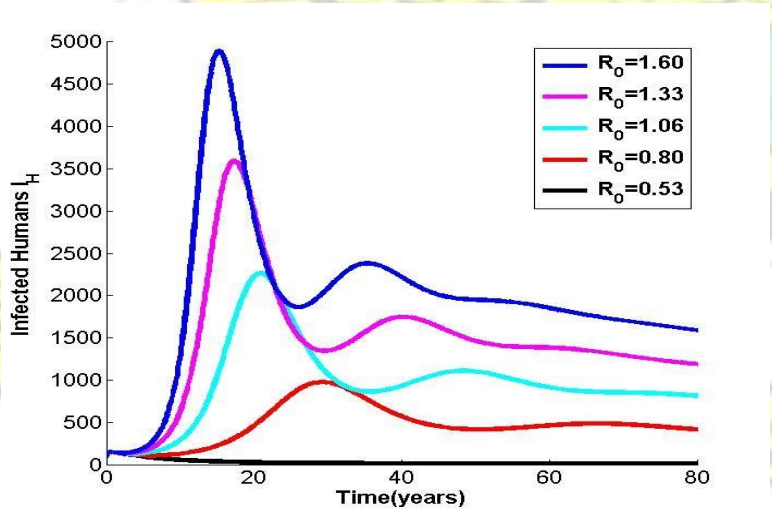
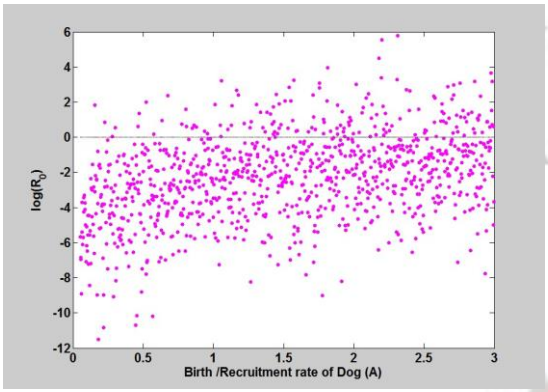
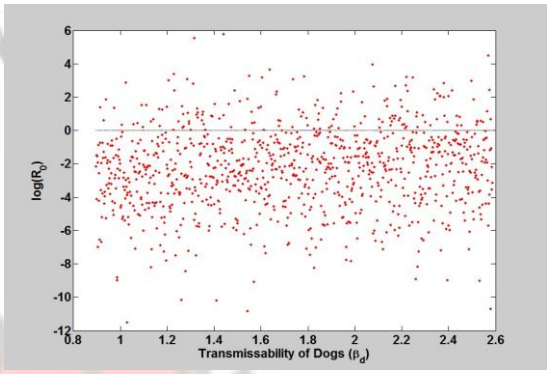


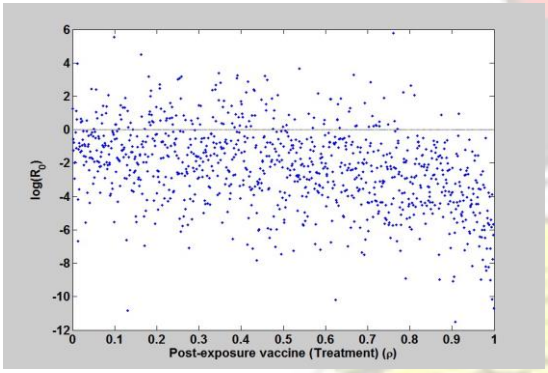
Figure 4.3: Varying the value of R_0

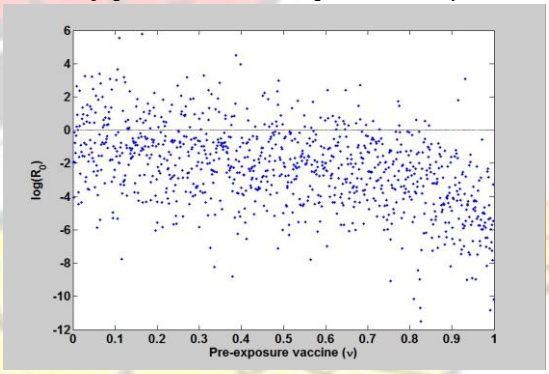
From Figure 4.3, it shows that as R_0 increases, the disease remains in the community (country) for a longer period, and as R_0 decreases, the epidemic (disease) gets eradicated in the community (country) with time, as indicated by the black line in Figure 4.3.



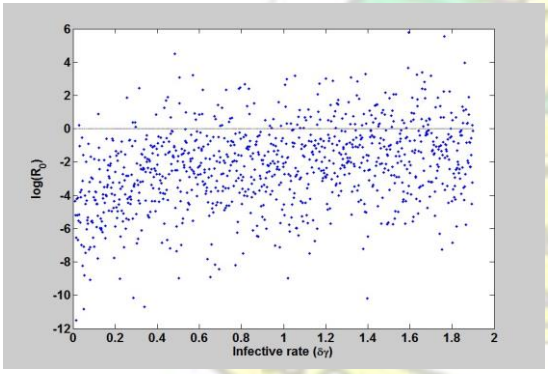


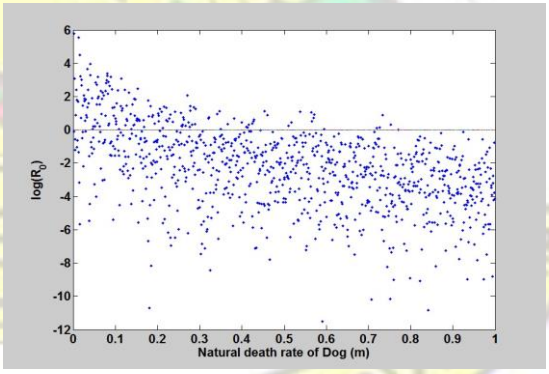
(a) Sensitivity plot of R_0 to the parameter A. (b) Sensitivity plot of R_0 to the parameter β_d





(c) Sensitivity plot of R_0 to the parameter ρ. (d) Sensitivity plot of R_0 to the parameter ν.



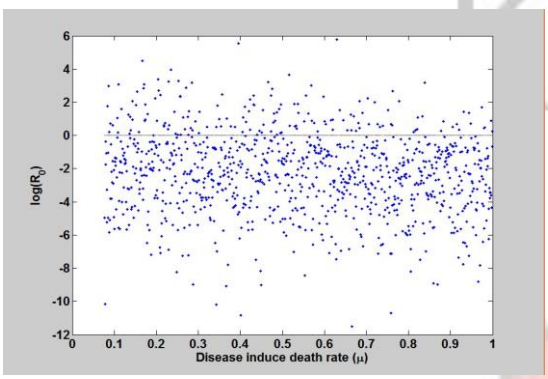


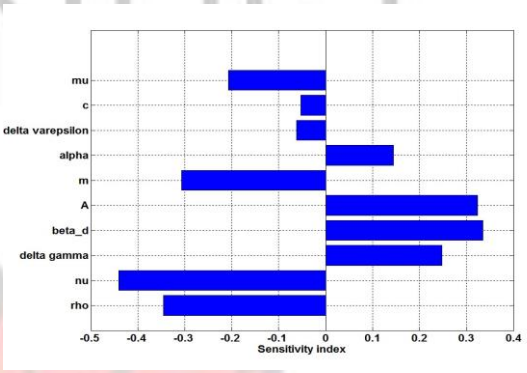
(e) Sensitivity plot of R_0 to the parameter $\delta\gamma$. (f) Sensitivity plot of R_0 to the parameter m.

Figure 4.4: Scatter/Sensitivity plots

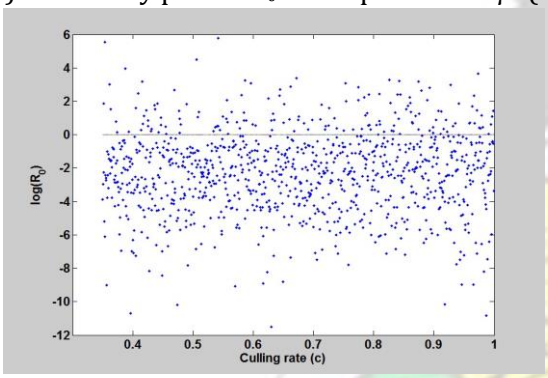
Figure 4.4a, 4.4b, 4.4e shows that β, A and $\delta\gamma$, have a positive relation with the basic reproduction number. Therefore to control the spread of the rabies virus, there is a need

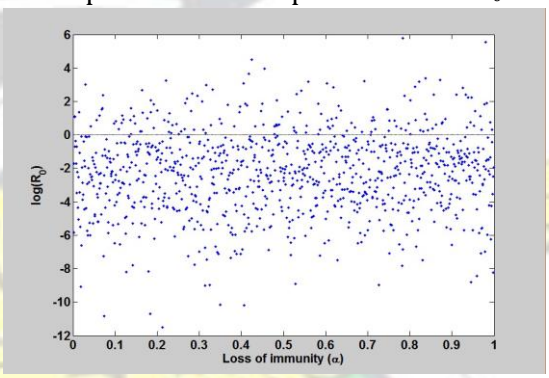
of reducing the rate of those parameters. Figure 4.4c and 4.4d shows that preexposure prophylaxis and post-exposure prophylaxis helps in controlling the spreading rate of rabies in the susceptible dogs and humans. Figure 4.4f shows that, the natural death rate m has negative relation in the spread of the rabies virus.



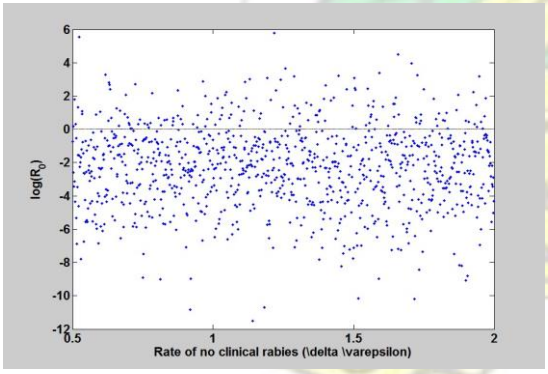


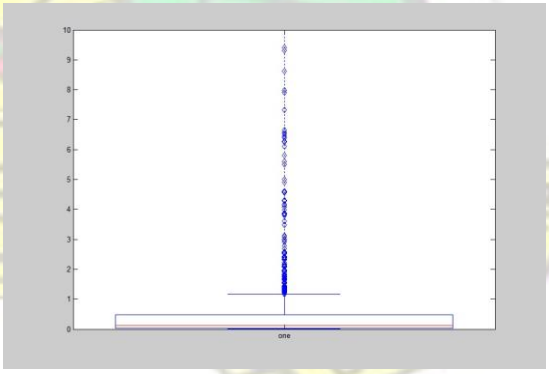
(a) Sensitivity plot of R_0 to the parameter μ. (b) Tornado plots for the ten parameters in R_0





(c) Sensitivity plot of R_0 to the parameter c. (d) Sensitivity plot of R_0 to the parameter α.





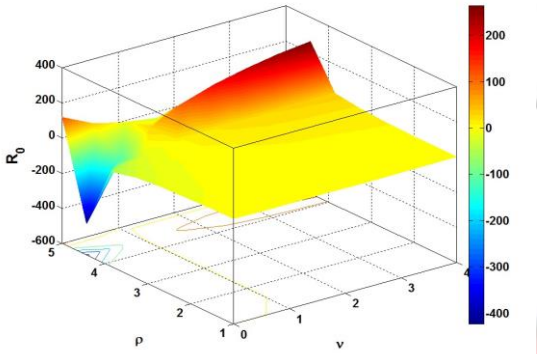
(e) Sensitivity plot of R_0 to the parameter $\delta\varepsilon$. (f) Box plot for the ten parameters in R_0.

Figure 4.5: Sensitivity Analysis plots

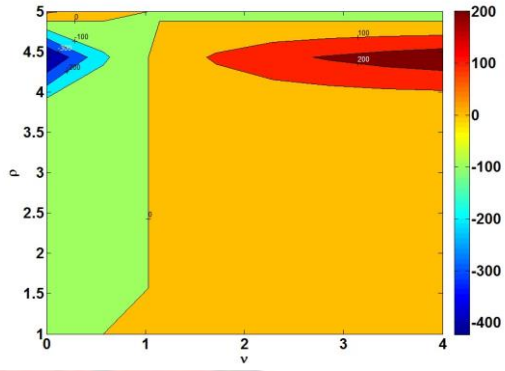
Figure 4.5a, 4.5c, 4.5d, 4.5e shows that μ, c, α, $\delta\gamma$, have a minimal influence in the rate at which the disease is spread. Hence, the culling rate introduce into the model, is not an

effective way of controlling the disease. Figure 4.5b shows that the most influential parameter in spreading the infection is β_d followed by A, and ν, ρ are the most influential parameters in controlling the rabies virus.

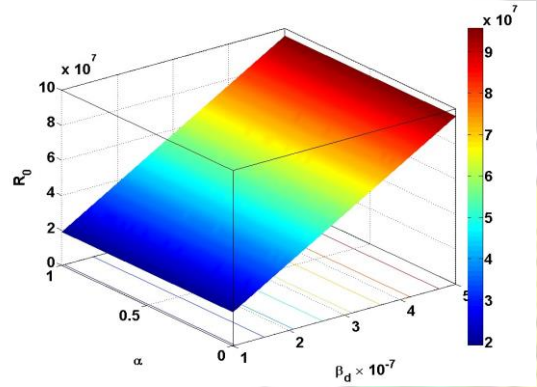
Figure 4.6 and 4.7 shows the topological effect of some parameters in R_0.



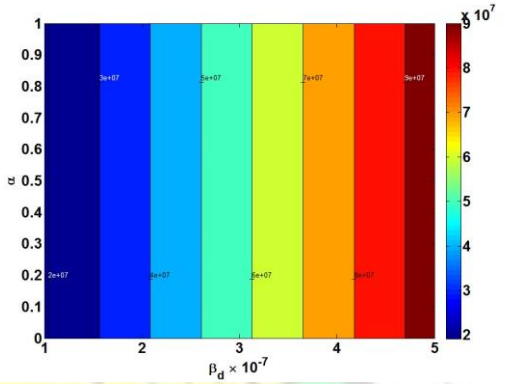
(a) *The 3D plot of* R_0 *to* ν *and* ρ.



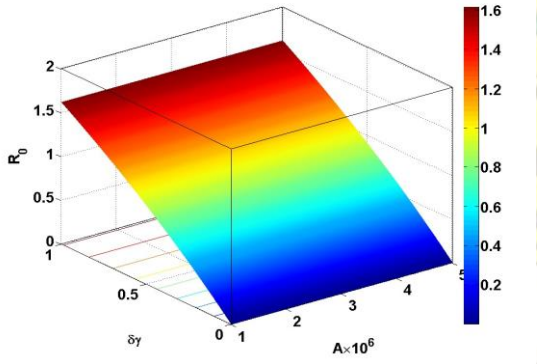
(b) *The contour plot of* ν *and* ρ *to* R_0.



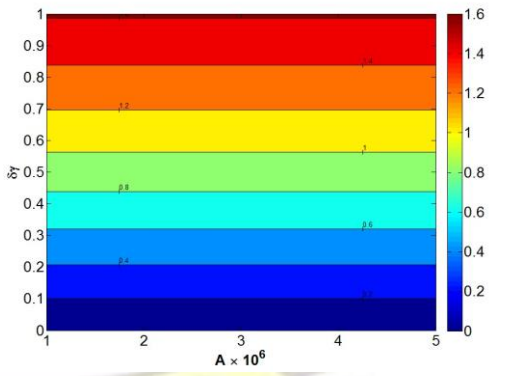
(c) *The 3D plot of* R_0 *to* α *and* β_d



(d) *The contour plot of* α *and* β_d *to* R_0.

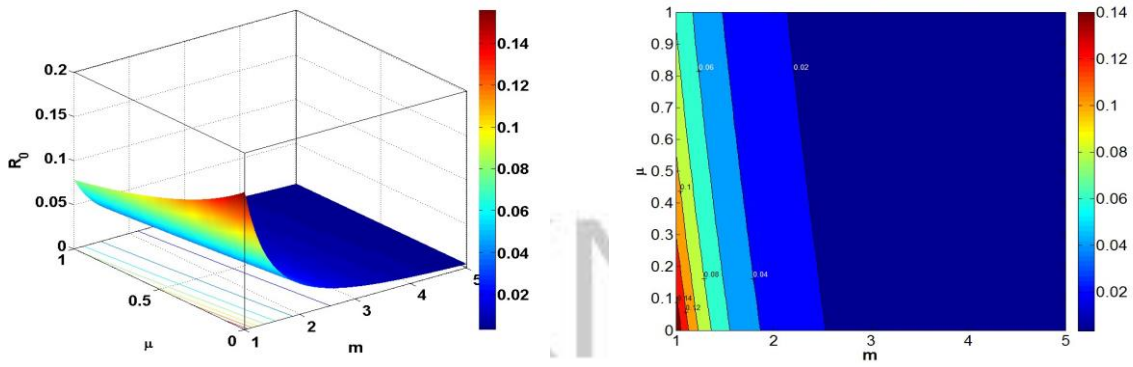


(e) *The 3D plot of* R_0 *to* $\delta\gamma$ *and* A



(f) *The contour plot of* $\delta\gamma$ *and* A *to* R_0.

Figure 4.6: The topological relation of some parameter in R_0.



(a) *The 3D plot of* R_0 *to* μ *and m.* (b) *The contour plot of* μ *and m to* R_0.

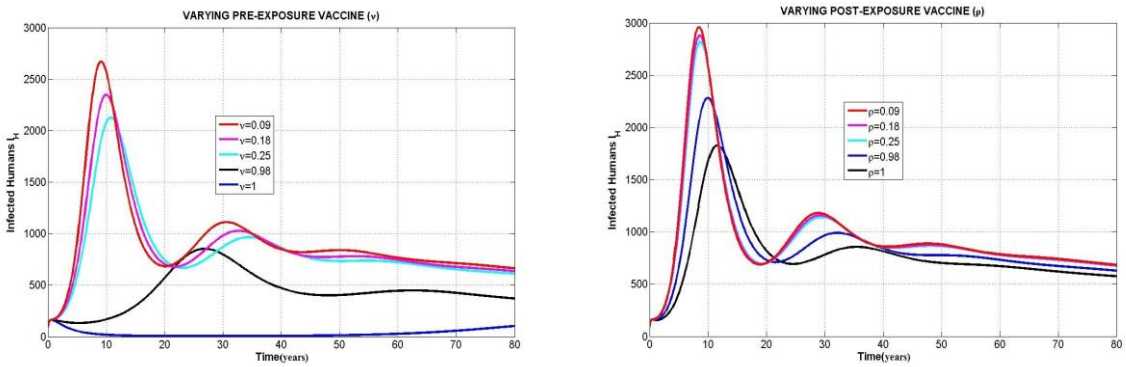
Figure 4.7: The topological relation of some parameter in R_0.

4.3.5 The Simulation Effect of Vaccination, Recruitment rate and Culling rate on the model

To find a good control method for the transmission of the disease, we performed some sensitivity analysis on the basic reproductive number as demonstrated in Figure 4.3. From the analysis, it shows that R_0 is really the threshold for the establishment of the disease in the susceptible classes. In this section we establish, with graphical results, the effect of vaccination on susceptible dogs (ν), vaccination on exposed dogs (ρ), and an equal rate of both vaccines (ν and ρ) on the model. Figure 4.8a shows the effect of pre-exposure vaccine on the model. Figure 4.8b shows the effect of post-exposure vaccine on the model. In Figure 4.9a we consider an equal vaccination rate on the model. Figure 4.9b shows the effect of the Recruitment rate of dogs on the model. Lastly, Figure 4.10a and 4.10b shows the effect of culling rate on the model and pre-exposure vaccination of humans on the model.

Figure 4.8a and Figure 4.8b shows that concentrating only on post-exposure vaccination of the exposed class of the dogs or humans, without pre-exposure vaccination on the susceptible classes, has a minimal effect in controlling the transmission rate of rabies.

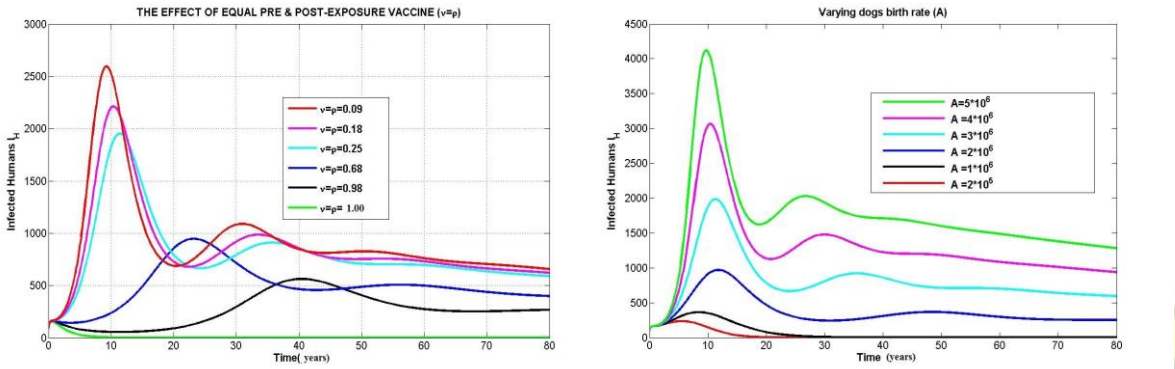
In Figure 4.9a below, we consider that there is an equal rate of vaccination in dog compartment. This shows that, if the disease starts and we implement the same rate of



(a) Varying the value of ν (b) Varying the value of ρ

Figure 4.8: The effect of pre-exposure and post-exposure vaccine of dogs on the model

vaccination at the same time, the disease will decline faster and can be eradicated, as indicated by the green line $\nu = \rho = 1.00$. Figure 4.9b shows that as recruitment rate decreases, the rate of transmission equally decreases and the rate of infection in the human subclass is reduced, as indicated by the red and the black lines in Figure 4.9b.

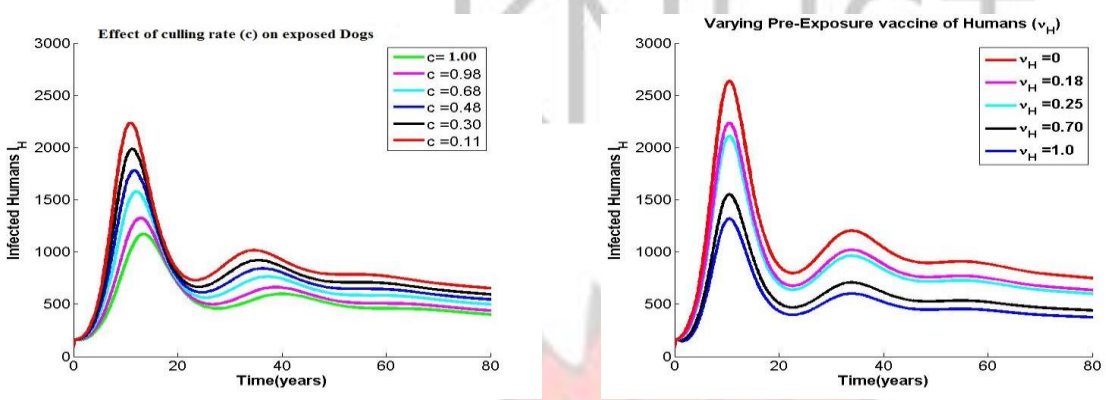


(a) Varying the value of $\nu = \rho$ (b) Varying the birth/recruitment rate of dogs

Figure 4.9: The combined effect of pre-exposure, post-exposure vaccine and recruitment rate of dogs on the model

Figure 4.10a shows that culling rate plays a little role in curbing the rate at which exposed dogs gets infected, Figure 4.10a also indicates that as culling rate increases on the exposed dogs, it reduces the spread rabies in the human population. Figure 4.10b shows the

importance of including pre-exposure vaccination on the human compartment, it indicates that as pre-exposure vaccination increase in the susceptible humans, many individuals builds temporary immunity against the disease, which prolong the rate of getting rabies infection when bitten by an infectious dog.



(a) Effect of culling on exposed dogs (b) The effect of pre-exposure vaccination

Figure 4.10: The effect of culling rate c of exposed dogs and pre-exposure vaccination ρ_H on the model

4.4 Optimal control effect on the model.

In this section we used the forward-backward sweep scheme; starting with an initial guess for the optimal controls v, ρ, v_H and ρ_H. The state system is solved in forward time, whilst the solution to the states together with the initial guess for the controls, are used to solve the co-state system backwards in time. Subsequently, we determined control strategies; v,ρ,v_H and ρ_H, as given in objective functional. We began an iteration of the model until convergence was achieved. The results of the simulation of the control strategies are displayed below. We consider equal weights of ($A_1 = 1, A_2 = 1, A_3 = 1, A_4 = 1$) for both exposed and infected classes. We varied the cost associated with the objective functional, which indicate that; with low cost of vaccination, the rate at which individual will seek for treatment of their exposed dogs will increase, and this could result in low transmission of rabies in an heterogeneous population. We consider the various cost of

pre-exposure prophylaxis and post-exposure prophylaxis to be ($B1 = 1, B2 = 4, B3 = 1, B4 = 4$). We observe that for optimality to be achieved, we either use more of the control with lesser weight of cost, or more of the control with the lesser weight of both exposed and infected classes. The results shows that applying more of post-exposure vaccine does not appreciably bring down the number of infected individuals, as compared to the case where we applied more of pre-exposure vaccine (treatment) on the susceptible classes. However, the peak attained in the normal simulation in the previous figures seems to be significantly different from the case when the four controls were used. We found that, with minimal cost of pre-exposure and post-exposure vaccination, the optimal time in controlling the infection will be the first eight (8) years, as shown in Figure 4.11.

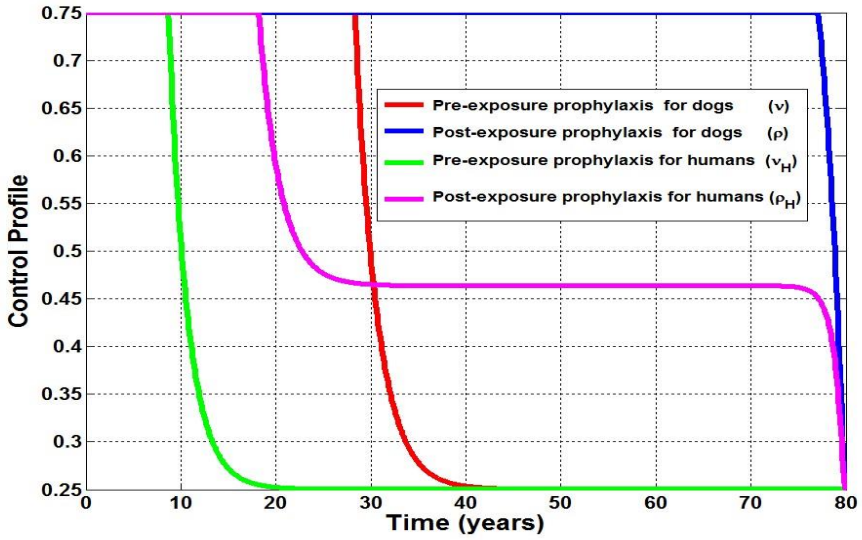
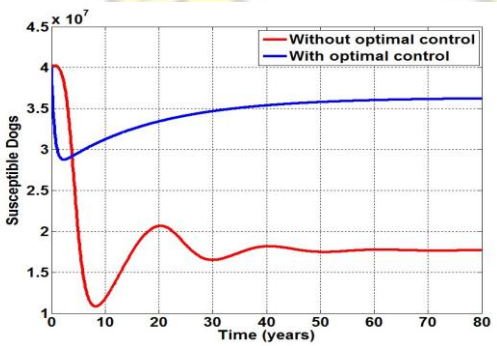
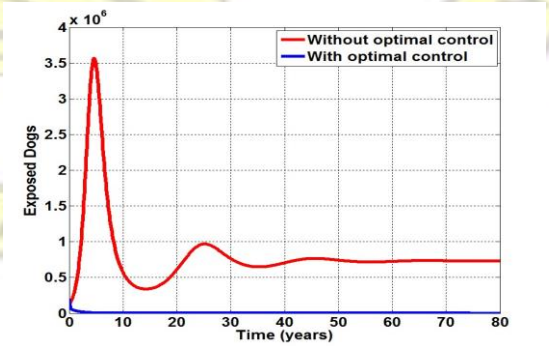


Figure 4.11: The simulation effect of the controls

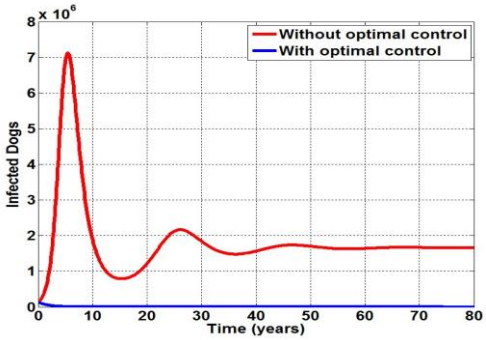


(a) *Susceptible dogs with and without control.*

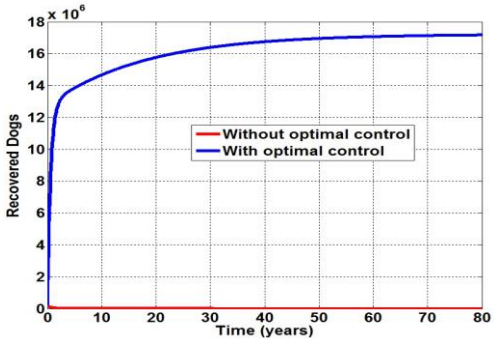


(b) *Exposed dogs with and without control.*

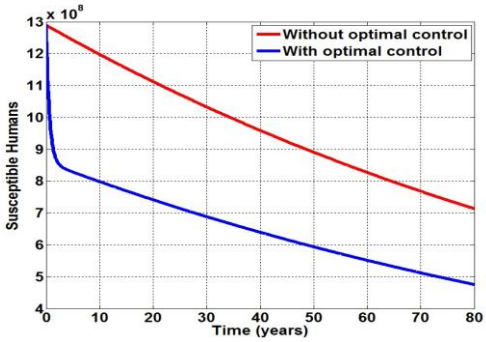
Figure 4.12: The trajectories of the model with and without optimal control on individual compartments



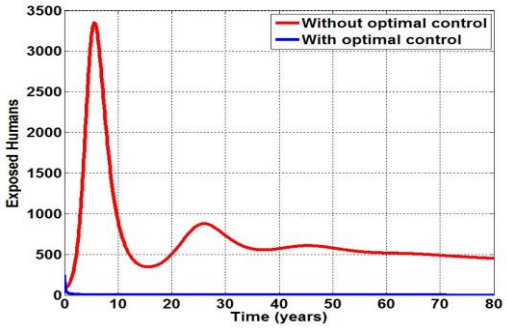
(a) *Infected dogs with and without control.*



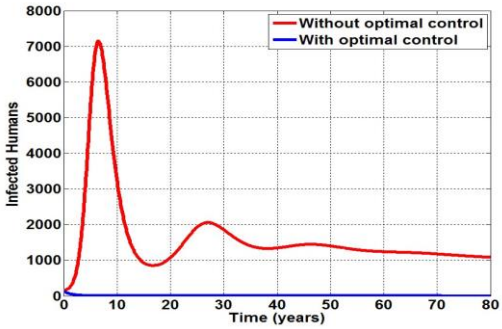
(b) *Recovered dogs with & without control.*



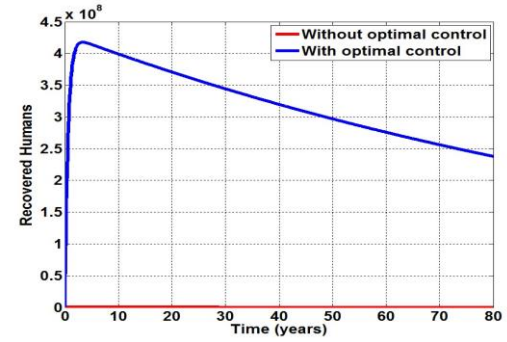
(c) *Susceptible humans with and without control.*



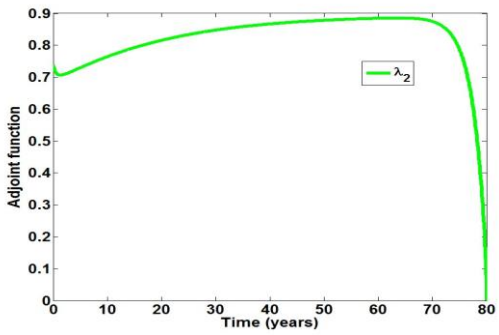
(d) *Exposed humans with and without control.*



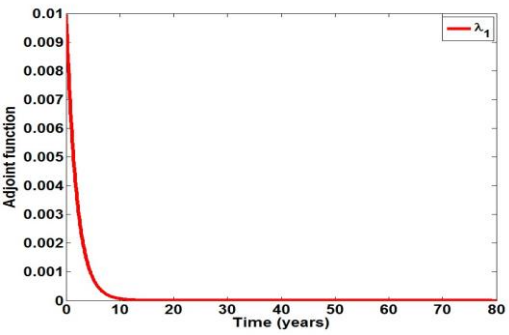
(e) *Infected humans with and without control.*



(f) *Recovered humans with and without control.*

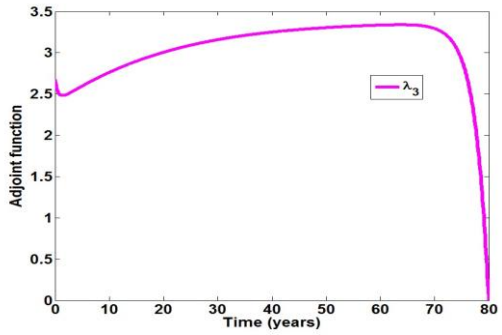


(g) *The marginal cost of* λ_1

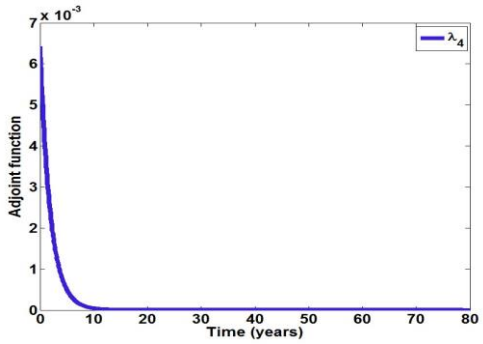


(h) *The marginal cost of* λ_2

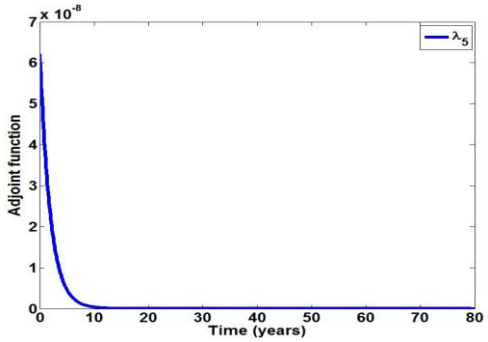
Figure 4.13: The trajectories of the model with and without optimal control on individual compartments with adjoint function



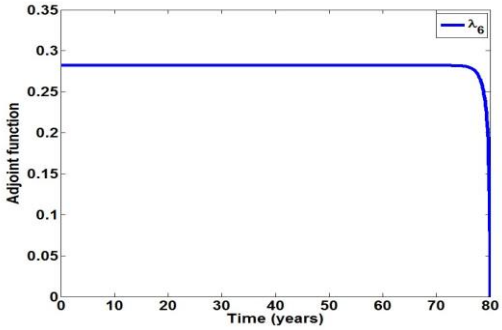
(a) *The marginal cost of* λ_3



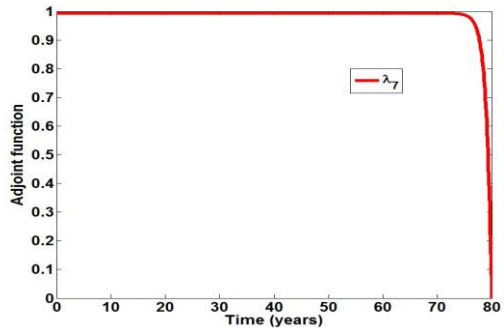
(b) *The marginal cost of* λ_4



(c) *The marginal cost of* λ_5



(d) *The marginal cost of* λ_6



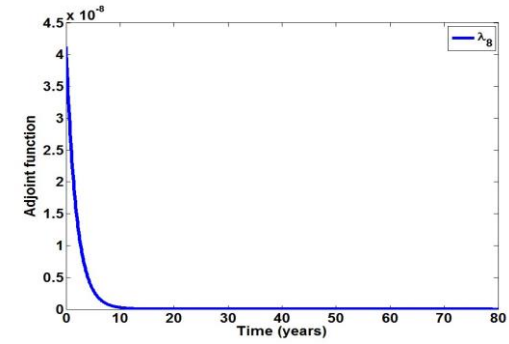
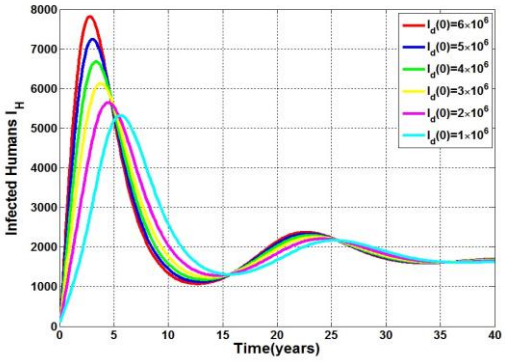
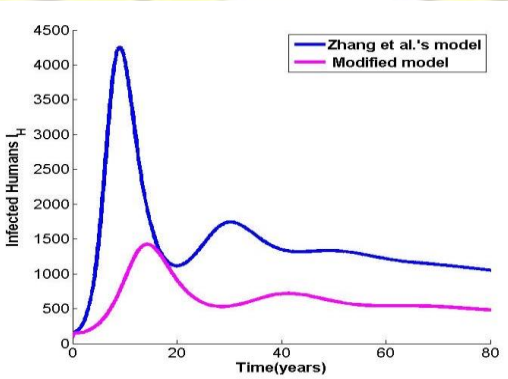
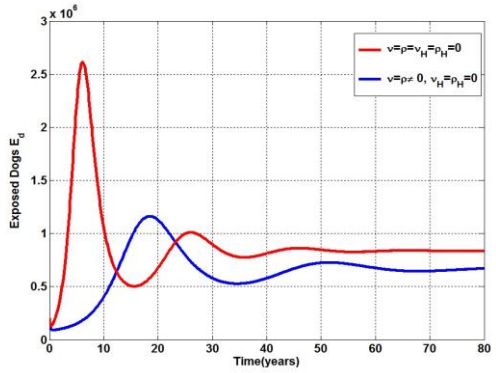


Figure 4.14: The simulation effect of the adjoint function



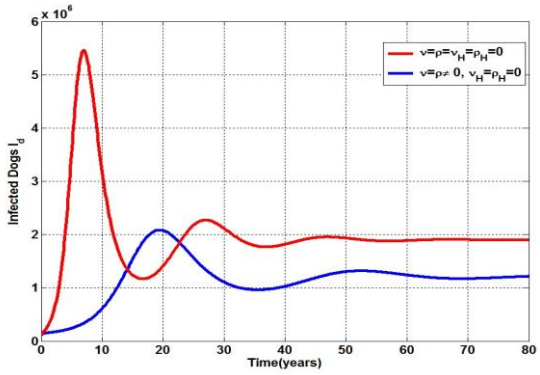


(a) The effect of varying the initial infected dog population size $Id(0)$ humans I_H

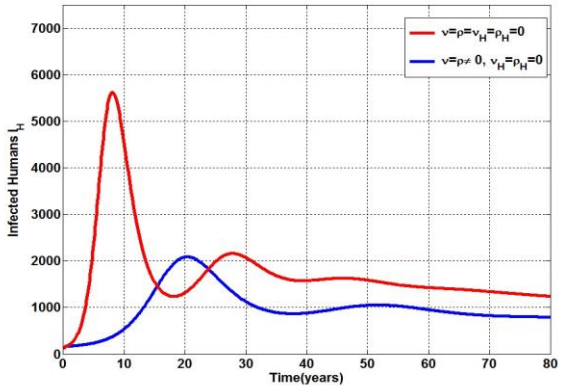


(a) A plot of $\nu = \rho$ 6= 0, and $\nu_H = \rho_H = 0$

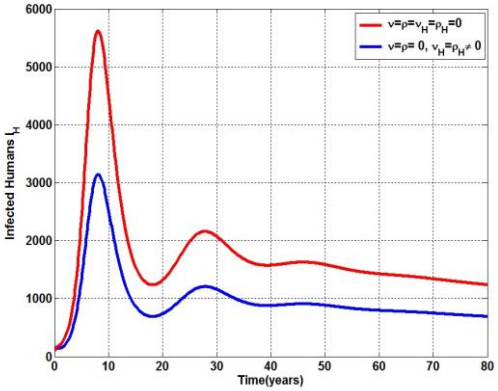
(b) Comparing Zhang et al.(2011) with our current model on the number of infected



(b) A plot of $\nu = \rho$ = 06, and $\nu_H = \rho_H = 0$



(a) A plot of $\nu = \rho$ 6= 0, and $\nu_H = \rho_H = 0$



(b) A plot of $\nu = \rho = 0$, and $\nu_H = \rho_H$ 6= 0

4.5 Discussion

The numerical simulations of the resulting optimality system shows that, during the case where it is more expensive to vaccinate than treatment, more resources should be invested in treating affected individuals until the disease prevalence begins to fall. This option, however, does not reduce the number of individuals expose to the disease quickly enough, thus resulting in an overall increase in the infected population. On the other hand, if it is more expensive to treat than to vaccinate, then more susceptible dogs should be vaccinated, so as to lower the rate at which new born dogs get infected. Nevertheless, in the case where both measures are equally expensive, the simulation shows that, the optimal way to drive the epidemic towards eradication within the specified period is to

use more of the vaccination control, thus pre-exposure prophylaxis in both compartments, and also reducing the number of new born dogs into the susceptible population.

Figure 4.12b and Figure 4.13a shows that, there is a proportional decrease in the number of exposed and infected dogs when the control measures are applied, similarly, Figure 4.13d shows a significant decrease in the number of exposed and infected humans when the control measures are applied. Figure 4.13b and Figure 4.13f shows that there is a proportional increase in the number of recovered dogs and humans when the control measures are applied. The Tornado plot in 4.5b, shows that, a high number of birth/recruitment, and transmission rate in the dog compartment, during an outbreak of rabies, will have a major impact on the spread of the disease in the human population. Similarly 4.5b, shows that pre-exposure vaccine plays a major role in controlling the disease in both compartments than post-exposure vaccine (treatment).

Chapter 5

Conclusion

5.1 Overview

In this chapter, we draw conclusion from the study and suggest some recommendations to researchers and stakeholders.

5.2 Conclusion

In this thesis, we studied an optimal control model of rabies transmission dynamics in dogs and humans. This gave us more insight to the dynamics of rabies transmission from dogs to the human population. We obtained the equilibrium points of the model and stability analysis, The stability analysis shows that the disease free equilibrium is locally and globally asymptotically stable, The endemic equilibrium shows a global stability if $R_0 > 1$. We obtained the controllability matrix, which indicates that the model is controllable, we also obtained an optimal control solutions for the model which indicates that, with mass vaccination in both compartments the optimal time for controlling the disease could be 8 years. From the simulation, it shows that vaccinating the susceptible humans and applying post-exposure prophylaxis to the exposed dogs has a major impact in controlling the spread of rabies infection. Also, the simulation of the resulting optimality system indicates that, during periods where it is more expensive to use pre-exposure prophylaxis than post-exposure prophylaxis, more attention should be given to the infected dogs or individuals through treatment, so as to bring the disease propagation down. This method, however, does not minimize the number of expose individuals faster enough, as compared to the period where more susceptible dogs and humans are given pre-exposure prophylaxis. However, the case where both controls are equally expensive showed that the optimal way to curb the epidemic towards eradication within the specified time is to

use more of pre-exposure vaccination in both compartment and less of the post-exposure vaccination in the dogs compartment.

5.3 Recommendation

Rabies transmission has been one of the biggest challenges facing Africa and Asia (WHO, 2010). To eradicate or control rabies transmission dynamics, we suggest the following:

1. Administering both pre and post-exposure prophylaxis at a lower cost in both compartments will help to eradicate rabies transmission.

2. From our simulation in Figure 4.9a, it shows that, if the rate of post-exposure vaccination could be increased to 80–90%, this could go a long way to eradicating rabies in Africa and Asia, as compare to the current rate of post-exposure vaccine of 40% as reported in WHO (2012).

3. Although dogs are the primary source of rabies, rabies can affect other animals too and it is wise to vaccinate all animals against rabies, particularly livestock.

4. Vaccinate your family's animals against rabies to protect them and help protect you and your family too.

5.3.1 Further Study

Further work can be done on rabies, by studying the transmission dynamics between dogs and cats (vectors), then from the two vector to the human population.

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Appendix A

Controllability matrix parameters

Where,

$A_{11} = 2\alpha+m-a_{12}$ $A_{12} = 2a_{58}-a_{56}-a_{55}$ $A_{13} = 2\alpha a_{44}+ma_{12}-m_2-a_{12}a_{22}-a_{13}a_{32}-2\alpha m$, $A_{14} = -a_{222}$

$-a_{23}a_{32}$, $A_{15} = -a_{22}a_{32}-a_{32}a_{33}$, $A_{16} = 2a_{55}a_{58}-a_{32}a_{53}-a_{55}a_{56}-a_{552}-a_{56}a_{66}+2a_{58}a_{88}$,

$A_{17} = -a_{662}-a_{32}a_{63}$, $A_{18} = -a_{66}a_{76}-a_{76}a_{77}$, $A_{19} = -(a_{22}a_{32}+a_{32}a_{33})a_{13}-$

$2m(\alpha a_{44}-\alpha m)+m(a_{12}a_{22}-ma_{12}+a_{13}a_{32})-(a_{222}+a_{23}a_{32})a_{12}+2\alpha a_{442}+m_3$,

$A_{21} = -(a_{22}a_{32}+a_{32}a_{33})a_{23}-(a_{222}+a_{23}a_{32})a_{22}$, $A_{22} = -(a_{22}a_{32}+a_{32}a_{33})a_{33}-$

$(a_{222}+a_{23}a_{32})a_{32}$,

$A_{23} = 2a_{58}a_{882}-a_{56}a_{662}-a_{553}-(a_{22}a_{32}+a_{32}a_{33})a_{53}-(a_{55}a_{56}+a_{56}a_{66})a_{55}+2(a_{55}a_{58}+a_{58}a_{88})a_{55}-a_{32}$

$a_{53}a_{55}-a_{32}a_{56}a_{63}-a_{663}-a_{32}a_{63}a_{66}-(a_{22}a_{32}+a_{32}a_{33})a_{63}$,

$A_{24} = -a_{663}-a_{32}a_{63}a_{66}-(a_{22}a_{32}+a_{32}a_{33})a_{63}$, $A_{25} = -a_{662}a_{76}-(a_{66}a_{76}+a_{76}a_{77})a_{77}-$

$a_{32}a_{63}a_{76}$,

$A_{26} = 2(\alpha a_{44}-\alpha m)a_{442}-(a_{22}a_{32}+a_{32}a_{33})(a_{12}a_{23}-ma_{13}+a_{13}a_{33})-(a_{222}+a_{23}a_{32})$

$(a_{12}a_{22}-ma_{12}+a_{13}a_{32})+2m_2$

$(\alpha a_{44}-\alpha m)-m_4-m_2(a_{12}a_{22}-ma_{12}+a_{13}a_{32})$,

$A_{27} = -(a_{22}a_{23}+a_{23}a_{33})(a_{22}a_{32}+a_{32}a_{33})-(a_{222}+a_{23}a_{32})_2$,

$A_{28} = -(a_{22}a_{32}+a_{32}a_{33})(a_{222}+a_{23}a_{32})-(a_{22}a_{32}+a_{32}a_{33})(a_{332}+a_{23}a_{32})$,

$A_{29} = 2(a_{55}a_{58}+a_{58}a_{88})a_{552}-(a_{22}a_{32}+a_{32}a_{33})(a_{33}a_{53}+a_{53}a_{55}+a_{56}a_{63})$

$-(a_{55}a_{56}+a_{56}a_{66})a_{552}-(a_{55}a_{56}+a_{56}a_{66})a_{662}-a_{554}+2(a_{55}a_{58}+a_{58}a_{88})a_{882}-a_{32}a_{53}a_{552}-$

$(a_{55}a_{56}+a_{56}a_{66})a_{32}a_{63}$

$-(a_{222}+a_{23}a_{32})a_{32}a_{53}$,

$A_{31} = -a_{664}-(a_{22}a_{32}+a_{32}a_{33})(a_{33}a_{63}+a_{63}a_{66})-a_{32}a_{63}a_{662}-(a_{222}+a_{23}a_{32})a_{32}a_{63}$,

$A_{32} = -(a_{66}\,a_{76} + a_{76}\,a_{77})\,a_{662}$

$-\,(a_{66}\,a_{76} + a_{76}\,a_{77})\,a_{772} - (a_{22}\,a_{32} + a_{32}\,a_{33})\,a_{63}\,a_{76}$

$-\,(a_{66}\,a_{76} + a_{76}\,a_{77})\,a_{32}\,a_{63},$

$$\begin{aligned}A_{33} = {} & -2\,m\,\left((\alpha\,a_{44} - \alpha\,m)\,{a_{44}}^2 + m^2\,(\alpha\,a_{44} - \alpha\,m)\right) - \left((a_{22}\,a_{32} + a_{32}\,a_{33})\right. \\ & \left.\left({a_{22}}^2 + a_{23}\,a_{32}\right) + (a_{22}\,a_{32} + a_{32}\,a_{33})\left({a_{33}}^2 + a_{23}\,a_{32}\right)\right)\,a_{13} \\ & + m\,\left((a_{22}\,a_{32} + a_{32}\,a_{33})\,(a_{12}\,a_{23} - m\,a_{13} + a_{13}\,a_{33})\right. \\ & \left. + \left({a_{22}}^2 + a_{23}\,a_{32}\right)\,(a_{12}\,a_{22} - m\,a_{12} + a_{13}\,a_{32}) + m^2\,(a_{12}\,a_{22} - m\,a_{12} + a_{13}\,a_{32})\right) \\ & + 2\,\alpha\,{a_{44}}^4 + m^5 - \Big((a_{22}\,a_{23} + a_{23}\,a_{33}) \\ & (a_{22}\,a_{32} + a_{32}\,a_{33}) + \left({a_{22}}^2 + a_{23}\,a_{32}\right)^2\Big)\,a_{12},\end{aligned}$$

$$\begin{aligned}A_{34} = {} & -\left((a_{22}\,a_{32} + a_{32}\,a_{33})\,\left({a_{22}}^2 + a_{23}\,a_{32}\right) + (a_{22}\,a_{32} + a_{32}\,a_{33})\,\left({a_{33}}^2 + a_{23}\,a_{32}\right)\right)\,a_{23} \\ & - \Big((a_{22}\,a_{23} + a_{23}\,a_{33})\,(a_{22}\,a_{32} + a_{32}\,a_{33}) + \left({a_{22}}^2 + a_{23}\,a_{32}\right)^2\Big)\,a_{22},\end{aligned}$$

$$\begin{aligned}A_{35} = {} & -\left((a_{22}\,a_{32} + a_{32}\,a_{33})\,\left({a_{22}}^2 + a_{23}\,a_{32}\right) + (a_{22}\,a_{32} + a_{32}\,a_{33})\,\left({a_{33}}^2 + a_{23}\,a_{32}\right)\right)\,a_{33} \\ & - \Big((a_{22}\,a_{23} + a_{23}\,a_{33})\,(a_{22}\,a_{32} + a_{32}\,a_{33}) + \left({a_{22}}^2 + a_{23}\,a_{32}\right)^2\Big)\,a_{32},\end{aligned}$$

$$\begin{aligned}A_{36} = {} & 2\,a_{58}\,{a_{88}}^4 - \left((a_{22}\,a_{32} + a_{32}\,a_{33})\,\left({a_{22}}^2 + a_{23}\,a_{32}\right) + (a_{22}\,a_{32} + a_{32}\,a_{33})\,\left({a_{33}}^2 + a_{23}\,a_{32}\right)\right)\,a_{53} \\ & - a_{55}\,\left((a_{22}\,a_{32} + a_{32}\,a_{33})\,(a_{33}\,a_{53} + a_{53}\,a_{55} + a_{56}\,a_{63}) + a_{32}\,a_{53}\,{a_{55}}^2\right. \\ & \left. + (a_{55}\,a_{56} + a_{56}\,a_{66})\,a_{32}\,a_{63} + \left({a_{22}}^2 + a_{23}\,a_{32}\right)\,a_{32}\,a_{53}\right) - a_{56}\,{a_{66}}^4 - {a_{55}}^5 \\ & - a_{56}\,\left((a_{22}\,a_{32} + a_{32}\,a_{33})\,(a_{33}\,a_{63} + a_{63}\,a_{66}) + a_{32}\,a_{63}\,{a_{66}}^2 + \left({a_{22}}^2 + a_{23}\,a_{32}\right)\,a_{32}\,a_{63}\right) \\ & - \left((a_{55}\,a_{56} + a_{56}\,a_{66})\,{a_{55}}^2 + (a_{55}\,a_{56} + a_{56}\,a_{66})\,{a_{66}}^2\right)\,a_{55} \\ & + 2\,\left((a_{55}\,a_{58} + a_{58}\,a_{88})\,{a_{55}}^2 + (a_{55}\,a_{58} + a_{58}\,a_{88})\,{a_{88}}^2\right)\,a_{55},\end{aligned}$$

$$
\begin{aligned}
A_{37} = & -{a_{66}}^5 - \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\right. \\
& \left.\left({a_{22}}^2 + a_{23}\, a_{32}\right) + \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{33}}^2 + a_{23}\, a_{32}\right)\right)\, a_{63} \\
& - a_{66}\, \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left(a_{33}\, a_{63} + a_{63}\, a_{66}\right) + a_{32}\, a_{63}\right. \\
& \left.{a_{66}}^2 + \left({a_{22}}^2 + a_{23}\, a_{32}\right)\, a_{32}\, a_{63}\right), \\
& A_{38} \\
& = -{a_{66}}^4\, a_{76} - a_{76}\, \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left(a_{33}\, a_{63} + a_{63}\, a_{66}\right) + a_{32}\, a_{63}\right. \\
& \left.{a_{66}}^2 + \left({a_{22}}^2 + a_{23}\, a_{32}\right)\, a_{32}\, a_{63}\right) \\
& - \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, a_{63}\, a_{76} + \left(a_{66}\, a_{76} + a_{76}\, a_{77}\right)\, a_{32}\, a_{63}\right)\, a_{77} - \left(\left(a_{66}\, a_{76} + a_{76}\right.\right. \\
& \left.\left.a_{77}\right)\, {a_{66}}^2 + \left(a_{66}\, a_{76} + a_{76}\, a_{77}\right)\, {a_{77}}^2\right)\, a_{77}, \\
& A_{39} \\
& = 2\, \left(\alpha\, a_{44} - \alpha\, m\right) \\
& {a_{44}}^4 - \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{22}}^2 + a_{23}\, a_{32}\right) + \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{33}}^2 + a_{23}\, a_{32}\right)\right)\, \left(a_{12}\, a_{23}\right. \\
& \left. - m\, a_{13} + a_{13}\, a_{33}\right) + 2\, m^2\, \left(\left(\alpha\, a_{44} - \alpha\right.\right. \\
& \left.\left.m\right)\, {a_{44}}^2 + m^2\, \left(\alpha\, a_{44} - \alpha\, m\right)\right) - \left(\left(a_{22}\, a_{23} + a_{23}\, a_{33}\right)\, \left(a_{22}\, a_{32}\right.\right. \\
& \left.\left. + a_{32}\, a_{33}\right) + \left({a_{22}}^2 + a_{23}\, a_{32}\right)^2\right)\, \left(a_{12}\, a_{22} - m\, a_{12} + a_{13}\, a_{32}\right) - m^6 - m^2 \\
& \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left(a_{12}\, a_{23} - m\, a_{13} + a_{13}\, a_{33}\right)\right. \\
& \left. + \left({a_{22}}^2 + a_{23}\, a_{32}\right)\, \left(a_{12}\, a_{22} - m\, a_{12} + a_{13}\, a_{32}\right) + m^2\, \left(a_{12}\, a_{22} - m\, a_{12} + a_{13}\, a_{32}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
A_{41} = & -\left({a_{22}}^2 + a_{23}\, a_{32}\right)\, \left(\left(a_{22}\, a_{23} + a_{23}\, a_{33}\right)\, \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right) + \left({a_{22}}^2 + a_{23}\, a_{32}\right)^2\right) \\
& - \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{22}}^2 + a_{23}\, a_{32}\right) + \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{33}}^2 + a_{23}\, a_{32}\right)\right)\, \left(a_{22}\, a_{23}\right. \\
& \left. + a_{23}\, a_{33}\right),
\end{aligned}
$$

$$
\begin{aligned}
A_{42} = & -\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left(\left(a_{22}\, a_{23} + a_{23}\, a_{33}\right)\, \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right) + \left({a_{22}}^2 + a_{23}\, a_{32}\right)^2\right) \\
& - \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{22}}^2 + a_{23}\, a_{32}\right) + \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{33}}^2 + a_{23}\, a_{32}\right)\right)\, \left({a_{33}}^2\right. \\
& \left. + a_{23}\, a_{32}\right),
\end{aligned}
$$

$$
\begin{aligned}
A_{43} = & 2\, \left(\left(a_{55}\, a_{58} + a_{58}\, a_{88}\right)\, {a_{55}}^2 + \left(a_{55}\, a_{58} + a_{58}\, a_{88}\right)\, {a_{88}}^2\right)\, {a_{55}}^2 \\
& - \left(\left(a_{55}\, a_{56} + a_{56}\, a_{66}\right)\, {a_{55}}^2 + \left(a_{55}\, a_{56} + a_{56}\, a_{66}\right)\, {a_{66}}^2\right)\, {a_{55}}^2 - {a_{55}}^6 \\
& - \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{22}}^2 + a_{23}\, a_{32}\right) + \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left({a_{33}}^2 + a_{23}\, a_{32}\right)\right)\, \left(a_{33}\, a_{53}\right. \\
& \left. + a_{53}\, a_{55} + a_{56}\, a_{63}\right) - {a_{55}}^2\, \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left(a_{33}\, a_{53} + a_{53}\, a_{55} + a_{56}\, a_{63}\right) + a_{32}\, a_{53}\, {a_{55}}^2\right. \\
& \left. + \left(a_{55}\, a_{56} + a_{56}\, a_{66}\right)\, a_{32}\, a_{63} + \left({a_{22}}^2 + a_{23}\, a_{32}\right)\, a_{32}\, a_{53}\right) - \left(a_{55}\, a_{56} + a_{56}\, a_{66}\right)\, {a_{66}}^4 \\
& + 2\, \left(a_{55}\, a_{58} + a_{58}\, a_{88}\right)\, {a_{88}}^4 - \left(a_{55}\, a_{56} + a_{56}\, a_{66}\right)\, \left(\left(a_{22}\, a_{32} + a_{32}\, a_{33}\right)\, \left(a_{33}\, a_{63} + a_{63}\, a_{66}\right)\right. \\
& \left. + a_{32}\, a_{63}\, {a_{66}}^2 + \left({a_{22}}^2 + a_{23}\, a_{32}\right)\, a_{32}\, a_{63}\right) \\
& - \left(\left(a_{22}\, a_{23} + a_{23}\, a_{33}\right)\, \left(a_{22}\, a_{32} + a_{32}\, a_{33}\right) + \left({a_{22}}^2 + a_{23}\, a_{32}\right)^2\right)\, a_{32}\, a_{53},
\end{aligned}
$$

$$
\begin{aligned}
A_{44} = & -{a_{66}}^{6} - \left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left({a_{22}}^{2} + a_{23}\,a_{32}\right)\right. \\
& \left. + \left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left({a_{33}}^{2} + a_{23}\,a_{32}\right)\right)\,\left(a_{33}\,a_{63} + a_{63}\,a_{66}\right) \\
& - {a_{66}}^{2}\,\left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left(a_{33}\,a_{63} + a_{63}\,a_{66}\right) + a_{32}\,a_{63}\,{a_{66}}^{2} + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)\,a_{32}\,a_{63}\right) \\
& - \left(\left(a_{22}\,a_{23} + a_{23}\,a_{33}\right)\,\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right) + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)^{2}\right)\,a_{32}\,a_{63},
\end{aligned}
$$

$$
\begin{aligned}
A_{45} = & -\left(\left(a_{66}\,a_{76} + a_{76}\,a_{77}\right)\,{a_{66}}^{2} + \left(a_{66}\,a_{76} + a_{76}\,a_{77}\right)\,{a_{77}}^{2}\right)\,{a_{77}}^{2} - \left(a_{66}\,a_{76} + a_{76}\,a_{77}\right)\,{a_{66}}^{4} \\
& - \left(a_{66}\,a_{76} + a_{76}\,a_{77}\right)\,\left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left(a_{33}\,a_{63} + a_{63}\,a_{66}\right) + a_{32}\,a_{63}\,{a_{66}}^{2}\right. \\
& \left. + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)\,a_{32}\,a_{63}\right) \\
& - \left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,a_{63}\,a_{76} + \left(a_{66}\,a_{76} + a_{76}\,a_{77}\right)\,a_{32}\,a_{63}\right)\,{a_{77}}^{2} - \left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\right. \\
& \left.\left({a_{22}}^{2} + a_{23}\,a_{32}\right) + \left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left({a_{33}}^{2} + a_{23}\,a_{32}\right)\right)\,a_{63}\,a_{76},
\end{aligned}
$$

$$
\begin{aligned}
A_{46} = & -\left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left({a_{22}}^{2} + a_{23}\,a_{32}\right)\right. \\
& \left. + \left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left({a_{33}}^{2} + a_{23}\,a_{32}\right)\right)\,\left(\left(a_{22}\,a_{23} + a_{23}\,a_{33}\right)\,a_{12}\right. \\
& \left. - m\,\left(a_{12}\,a_{23} - m\,a_{13} + a_{13}\,a_{33}\right) + \left({a_{33}}^{2} + a_{23}\,a_{32}\right)\,a_{13}\right) \\
& - 2\,m^{3}\,\left(\left(\alpha\,a_{44} - \alpha\,m\right)\,{a_{44}}^{2} + m^{2}\,\left(\alpha\,a_{44} - \alpha\,m\right)\right) \\
& - \left(\left(a_{22}\,a_{23} + a_{23}\,a_{33}\right)\,\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right) + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)^{2}\right)\,\left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,a_{13}\right. \\
& \left. - m\,\left(a_{12}\,a_{22} - m\,a_{12} + a_{13}\,a_{32}\right) + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)\,a_{12}\right) \\
& + m^{7} - 2\,\left(m\,\left(\alpha\,a_{44} - \alpha\,m\right) - \alpha\,{a_{44}}^{2}\right)\,{a_{44}}^{4} \\
& + m^{3}\,\left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left(a_{12}\,a_{23} - m\,a_{13} + a_{13}\,a_{33}\right)\right. \\
& \left. + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)\,\left(a_{12}\,a_{22} - m\,a_{12} + a_{13}\,a_{32}\right) + m^{2}\,\left(a_{12}\,a_{22} - m\,a_{12} + a_{13}\,a_{32}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
A_{47} = & -\left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,a_{23} + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)\,a_{22}\right)\,\left(\left(a_{22}\,a_{23} + a_{23}\,a_{33}\right)\,\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\right. \\
& \left. + \left({a_{22}}^{2} + a_{23}\,a_{32}\right)^{2}\right) - \left(\left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left({a_{22}}^{2} + a_{23}\,a_{32}\right)\right. \\
& \left. + \left(a_{22}\,a_{32} + a_{32}\,a_{33}\right)\,\left({a_{33}}^{2} + a_{23}\,a_{32}\right)\right)\,\left(\left(a_{22}\,a_{23} + a_{23}\,a_{33}\right)\,a_{22} + \left({a_{33}}^{2} + a_{23}\,a_{32}\right)\,a_{23}\right),
\end{aligned}
$$

$$A_{48} = -\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;a_{33}+\left({a_{22}}^2+a_{23}\,a_{32}\right)\;a_{32}\right)\left((a_{22}\,a_{23}+a_{23}\,a_{33})\;(a_{22}\,a_{32}+a_{32}\,a_{33})\right.$$
$$\left.+\left({a_{22}}^2+a_{23}\,a_{32}\right)^2\right)-\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{22}}^2+a_{23}\,a_{32}\right)\right.$$
$$\left.+(a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{33}}^2+a_{23}\,a_{32}\right)\right)\left((a_{22}\,a_{23}+a_{23}\,a_{33})\;a_{32}+\left({a_{33}}^2+a_{23}\,a_{32}\right)\;a_{33}\right),$$

$$A_{49} = 2\left((a_{55}\,a_{58}+a_{58}\,a_{88})\;{a_{55}}^2+(a_{55}\,a_{58}+a_{58}\,a_{88})\;{a_{88}}^2\right){a_{55}}^3$$
$$-\left((a_{55}\,a_{56}+a_{56}\,a_{66})\;{a_{55}}^2+(a_{55}\,a_{56}+a_{56}\,a_{66})\;{a_{66}}^2\right){a_{55}}^3-{a_{55}}^7$$
$$-{a_{66}}^4\left(a_{56}\,{a_{66}}^2+(a_{55}\,a_{56}+a_{56}\,a_{66})\;a_{55}\right)+2\,{a_{88}}^4\left(a_{58}\,{a_{88}}^2+(a_{55}\,a_{58}+a_{58}\,a_{88})\;a_{55}\right)$$
$$-\left(a_{56}\,{a_{66}}^2+(a_{55}\,a_{56}+a_{56}\,a_{66})\;a_{55}\right)\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;(a_{33}\,a_{63}+a_{63}\,a_{66})+a_{32}\,a_{63}\,{a_{66}}^2\right.$$
$$\left.+\left({a_{22}}^2+a_{23}\,a_{32}\right)\;a_{32}\,a_{63}\right)-{a_{55}}^3\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;(a_{33}\,a_{53}+a_{53}\,a_{55}+a_{56}\,a_{63})\right.$$
$$\left.+a_{32}\,a_{53}\,{a_{55}}^2+(a_{55}\,a_{56}+a_{56}\,a_{66})\;a_{32}\,a_{63}+\left({a_{22}}^2+a_{23}\,a_{32}\right)\;a_{32}\,a_{53}\right)$$
$$-\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{22}}^2+a_{23}\,a_{32}\right)\right.$$
$$\left.+(a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{33}}^2+a_{23}\,a_{32}\right)\right)\left(a_{55}\;(a_{33}\,a_{53}+a_{53}\,a_{55}+a_{56}\,a_{63})\right.$$
$$\left.+(a_{33}\,a_{63}+a_{63}\,a_{66})\;a_{56}+\left({a_{33}}^2+a_{23}\,a_{32}\right)\;a_{53}\right)-\left((a_{22}\,a_{23}+a_{23}\,a_{33})\;(a_{22}\,a_{32}+a_{32}\,a_{33})\right.$$
$$\left.+\left({a_{22}}^2+a_{23}\,a_{32}\right)^2\right)\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;a_{53}+a_{32}\,a_{53}\,a_{55}+a_{32}\,a_{56}\,a_{63}\right),$$

$$A_{51} = -{a_{66}}^7-\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;a_{63}+a_{32}\,a_{63}\,a_{66}\right)\left((a_{22}\,a_{23}+a_{23}\,a_{33})\;(a_{22}\,a_{32}+a_{32}\,a_{33})\right.$$
$$\left.+\left({a_{22}}^2+a_{23}\,a_{32}\right)^2\right)-\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{22}}^2+a_{23}\,a_{32}\right)\right.$$
$$\left.+(a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{33}}^2+a_{23}\,a_{32}\right)\right)\left((a_{33}\,a_{63}+a_{63}\,a_{66})\;a_{66}+\left({a_{33}}^2+a_{23}\,a_{32}\right)\;a_{63}\right)$$
$$-{a_{66}}^3\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;(a_{33}\,a_{63}+a_{63}\,a_{66})+a_{32}\,a_{63}\,{a_{66}}^2+\left({a_{22}}^2+a_{23}\,a_{32}\right)\;a_{32}\,a_{63}\right),$$

$$A_{52} = -\left((a_{66}\,a_{76}+a_{76}\,a_{77})\;{a_{66}}^2+(a_{66}\,a_{76}+a_{76}\,a_{77})\;{a_{77}}^2\right){a_{77}}^3$$
$$-{a_{66}}^4\left({a_{66}}^2\,a_{76}+(a_{66}\,a_{76}+a_{76}\,a_{77})\;a_{77}\right)$$
$$-\left({a_{66}}^2\,a_{76}+(a_{66}\,a_{76}+a_{76}\,a_{77})\;a_{77}\right)\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;(a_{33}\,a_{63}+a_{63}\,a_{66})\right.$$
$$\left.+a_{32}\,a_{63}\,{a_{66}}^2+\left({a_{22}}^2+a_{23}\,a_{32}\right)\;a_{32}\,a_{63}\right)$$
$$-\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{22}}^2+a_{23}\,a_{32}\right)+(a_{22}\,a_{32}+a_{32}\,a_{33})\;\left({a_{33}}^2\right.\right.$$
$$\left.\left.+a_{23}\,a_{32}\right)\right)\left((a_{33}\,a_{63}+a_{63}\,a_{66})\;a_{76}+a_{63}\,a_{76}\,a_{77}\right)-\left((a_{22}\,a_{32}+a_{32}\,a_{33})\;a_{63}\,a_{76}\right.$$
$$\left.+(a_{66}\,a_{76}+a_{76}\,a_{77})\;a_{32}\,a_{63}\right)\;{a_{77}}^3$$
$$-\left((a_{22}\,a_{23}+a_{23}\,a_{33})\;(a_{22}\,a_{32}+a_{32}\,a_{33})+\left({a_{22}}^2+a_{23}\,a_{32}\right)^2\right)\;a_{32}\,a_{63}\,a_{76}$$

M file codes for rabies model transmission

All matlab codes for optimal simulation can be requested through my email: topeljoshua@gmail.com

```
function dydt = topel(t,y) dydt
= zeros(size(y));
A=3*10^6;B=0.0314;
B1=1.58*10^-7;B2= 2.29*10^-
12; r=1;r1=1; v=0.25;
v˙H=0.54; k=0.2; k˙H=0.1;
m=0.056;m1=0.0074;
g=0.4;g1=0.4; u=6;u1=6;
d=1;d1=1; c=0.3;
Sd=y(1);
Ed=y(2);
Id=y(3);
Rd=y(4);
SH=y(5);
EH=y(6);
IH=y(7); RH=y(8); dydt(1) = A-B1*Sd*Id+r*Rd-
(m+v)*Sd+u*(1-g)*Ed;

        dydt(2) = B1*Sd*Id-(u*g+m+k+u*(1-g)+c)*Ed;

        dydt(3) = u*g*Ed-(m+d)*Id;

        dydt(4) = v*Sd+k*Ed-(m+r)*Rd;

        dydt(5) = B-B2*SH*Id-(m1+v˙H)*SH+u1*(1-g1)*EH+r1*RH;
```

```
dydt(6) = B2*SH*Id-(m1+k˙H+(u1*g1)+u1*(1-g1))*EH;
dydt(7) = u1*g1*EH-(m1+d1)*IH;


dydt(8) = v˙H*SH+k˙H*EH-(m1+r1)*RH;
```

Basic Reproduction Number R_0 **** note the statement is not part of the code?? R0=

(B1*A*(m+r)/(m*((m+k+(u*(1-g))+c+(u*g))*(m+r+v)*(m+d)))) end

Appendix B

codes for varying the susceptible ,exposure , infectious and Recovered classes


```
set(figure(3),'position',[528 81 1320 897]);
subplot(221)
[t,y] = ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 3*10^9 250 89 2*10^5]);
plot(t,y(:,7),'LineWidth',3,'Color','r') axis([0
80 0 5000])
set(gca,'fontsize',16)
hold on
[t,y] = ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 2*10^9 250 89 2*10^5]);
plot(t,y(:,7),'LineWidth',3,'Color','b')
[t,y]= ode45('topel',[0 80],[3*10^7 2.10^5 1*10^5 2*10^5 1.29*10^9 250 89 2*10^5]);
plot(t,y(:,7),'LineWidth',3,'Color','g')
[t,y] = ode45('topel',[0 80],[3*10^7 2.10^5 1*10^5 2*10^5 5*10^8 250 89 2*10^5]);
plot(t,y(:,7),'LineWidth',3,'Color','k')
legend('S¨H(0)=3*10^9','S¨H(0)=2*10^9','S¨H(0)=1.2*10^9','S¨H(0)=5*10^8')
xlabel('Time(year)');ylabel('infected Humans I¨H');
title('VARYING HUMAN SUSCEPTIBLE CLASS');
set(gca,'fontsize',16)
subplot(222)
[t,y] = ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 2000 89 2*10^5]);
plot(t,y(:,7),'LineWidth',3,'Color','r') axis([0
80 0 3000])
set(gca,'fontsize',16)
hold on
[t,y]= ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 1000 89 2*10^5]);
```

plot(t,y(:,7),'LineWidth',3,'Color','b')

[*t,y*]= ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 500 89 2*10^5]);

plot(t,y(:,7),'LineWidth',3,'Color','g')

[*t,y*] = ode45('topel',[0 80],[3.*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 250 89 2*10^5]);

plot(t,y(:,7),'LineWidth',3,'Color','k')

[*t,y*] = ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 100 89 2*10^5]);

plot(t,y(:,7),'LineWidth',3,'Color','c')

legend('**E˙H(0)=2000**','**E˙H(0)=1000**','**E˙H(0)=500**','**E˙H(0)=250**','**E˙H(0)=100**')

xlabel('**Time(year)**');ylabel('**infected Humans I˙H**');

title('**VARYING HUMAN EXPOSED CLASS**');

set(gca,'fontsize',16)

subplot(223)

[*t,y*]= ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 250 89 2*10^5]);

plot(t,y(:,7),'LineWidth',3,'Color','k') axis([0

80 0 2500])

set(gca,'fontsize',16)

hold on

[*t,y*] = ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 250 89 10000]);

plot(t,y(:,7),'LineWidth',3,'Color','b')

[*t,y*]= ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 250 89 5000]);

plot(t,y(:,7),'LineWidth',3,'Color','g')

[*t,y*] = ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 250 89 160]);

plot(t,y(:,7),'LineWidth',3,'Color','c')

[*t,y*] = ode45('topel',[0 80],[3*10^7 2*10^5 1*10^5 2*10^5 1.29*10^9 250 89 50]);

plot(t,y(:,7),'LineWidth',3,'Color','r')

legend('**R˙H(0)=2*10^5**','**R˙H(0)=10000**','**R˙H(0)=5000**','**R˙H(0)=160**', '**R˙H(0)=50**')
xlabel('**Time(year)**');ylabel('**Infected Humans I˙H**');

title('**VARYING HUMAN RECOVERED CLASS**');