KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

FAIR PRICING OF LIFE INSURANCE PARTICIPATING POLICIES WITH EMBEDDED SURRENDER OPTION UNDER STOCHASTIC INTEREST RATE MODEL

BY

ABDUL-RAHAMAN MUSTAPHA (BSc. Financial Mathematics)

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS,
KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF M.PHIL ACTUARIAL SCIENCE

JUNE 2016

DECLARATION

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university or elsewhere, except where due acknowledgment had been made in the text.

Abdul-Rahaman Mustapha	.... Signature	.... Date

Student (PG2533714)
Certified by:

Dr. F. T. Oduro	....	....
Supervisor	Signature	Date

Certified by:

Dr. R. K. Avuglah	....	....
Head of Department	Signature	Date

DEDICATION

I humbly dedicate this thesis to my dad and mum for the continuous support they accorded to me. It has been a journey which you have walked by my side despite the numerous challenges I encountered. Thanks a lot.

ABSTRACT

The study analysed life insurance endowment policy, paid by sequence of periodical premiums with guaranteed minimum return to the policyholder. The main objective of the research is to estimate the “fair price” of life insurance participating policy under stochastic interest rate model, in particular, the study considered both the case in which the premium is constant and the case in which it is periodically adjusted according to the performance of a special investment portfolio (reference portfolio). This premium is implicitly defined by an equation based on the actuarial principle of equivalence. Furthermore, the policy under examination is characterized by the presence of a surrender option, i.e., an American-style option that entitles policy owners to early terminate the contract and to receive a surrender cash value implied by a surrender charge. The study employed extended two-dimensional Cox- Ross and Rubinstein model to numerically estimate the “fair price” of the embedded surrender option.

ACKNOWLEDGMENT

I owe first a debt of gratitude to Almighty Allah whom by His Grace has seen me through this program successfully. I wish to register my deepest appreciation to my supervisor, Dr. F. T. Oduro for his careful reading through the manuscript, and the healthy and intellectual criticisms I received from him. Sir, you have really shaped my way of thinking. I am extremely grateful. My sincere appreciation continues to my dad, Afa Abdul-Rahaman Issahaku for the continuous support, encouragement and words of wisdom that kept me going even when the going got tough. I would also like to appreciate my mum, Mrs Afishetu Alhassan for her prayers and moral support throughout my research and studies. I wish to further extend my warmest gratitude to Mr and Mrs Alhassan Issahaku for their moral support and encouragement. To my brothers and sisters for believing in me, constant love, motivation, endless prayers and encouragement, without which this study would have not been successful. I am exceedingly grateful, and to all those who make it happen and make it all worthwhile. To them all I say thank you and may Allah richly bless you.

Contents

List of Tables

List of Figures

List of Acronyms

CIR	Cox Ingersoll and Ross
CRR	Cox Ross and Rubinstein
GDP	Gross Domestic Product
LIC	Life Insurance Companies
MCM	Monte Carlo Methods
NIC	National Insurance Commission
US	United States

Chapter 1

INTRODUCTION

1.1 Overview

This chapter is an introductory chapter that covers the background analysis of the major components of the study. It comprises the background of the study, problem statement, purpose, objectives, research questions, significance, limitations and how the search is organized.

1.2 Background of the Study

In life, losses are sometimes unavoidable. People may become ill and lose income or savings to pay off medical bills. Individuals or their relatives may die of illness or accidents. People's homes or other properties may suffer damage or theft. All these activities in the environment of man are subjected to the risk of loss from unforeseen events. To lessen this burden on humans, insurance companies have formed to provide products, with the common goal of pooling related risks together and offering a cushion to the unforeseen incidents. To expand the coverage of insurance policies, insurers have of late begun providing composite policies. Among them is the participating policies with interest rate guaranteed and surrender possibility. Bacinello (2001), defined participating policies as contracts in which policyholders are entitled to a share of excess profit if the realized interest rate during the insurance period is above the assumed interest rate. In many products, the guaranteed interest rate is fixed on a point-to-point basis (i.e. the guarantee is only applicable at expiration of the policy). In others, (common in the Ghanaian market), the so called cliquet-style (year-by-year) guarantee is used. Thus policy holder's account has to be credited each year, a certain rate of return (guarantee). The policy mechanisms allows the insured to receive dividends in addition to the amount assured implied by the guaranteed rate. Owing to the presence of the surrender possibilities and the guaranteed interest rate, the policy has some features that resembles that of American and the European style options. There exist numerous means through which the profit (bonus) allocation is achieved. The usual approach is to locate most of the premiums collected in a particular investment portfolio (reference portfolio), whenever a gain on the reference portfolio is above the guaranteed minimum interest rate of the period, an additional interest called bonus rate is credited to the policy.

Grosen and Jørgensen (2000), claimed that, interest rate guarantees, devidend sharing systems, and surrender options are common features of a standard participating life insurance policy issued in the United States(US), Europe, Japan and most developing countries of which Ghana is not left out. Each of these options contributes to the insurer's liability and present a value that constitutes a potential hazard to the company's solvency. Similar study by Briys and De Varenne (1997), revealed that, many life insurers disregard the importance of these features, thereby subjecting their companies to the risk of insolvency.

In Ghana, the National Insurance Commission (NIC) is the body mandated by law to regulate insurance activities in the country. The commission has objectives of; ensuring effective administration, supervision, regulation and the control of the insurance business. Competition in the industry has increased over the past decade as many new entrants have joined the insurance business. Despite the solid development in recent years, the market is still challenged by fair pricing specific life policies. This could partly be as a result of constraints on data availability and inadequate appropriate actuarial approaches (NIC, 2011). In view of the big market share of such polies in many countries including Ghana, the analysis of participating life insurance contracts with embedded surrender option is an important topic to consider. However, such policies according to Bauer et al. (2008), are hardly been analysed in the academic literature. Existing work on this area rather focused on life insurance policies where the growth of the insurer's obligation is directly connected to the performance of some reference portfolio, called unit-linked, equity-linked or variable products.

The efficient approach of pricing participating life insurance policies in the actuarial and financial literature is to model the interest rate by a stochastic process and price the contract according to the traditional actuarial equivalence principle. According to Bacinello (2001), the rules for computing premium(s) are fixed in any case, thus "fair pricing" is achievable by selecting the appropriate parameters that characterized the contract.

The study seek to price life insurance participating policies in the Ghanaian life insurance market introduced by Bacinello (2001), embedded a surrender option under stochastic interest rate model proposed by Liao et al. (2006). The focus is however, on the estimation of periodical premiums that makes the policy "fair" at issuance, which according to Bu¨hlmann (1985), is of much interest to the actuary.

1.3 Problem statement

The problem of this study is informed by the fact that, there exist limited research that has been made in the area of fair pricing life insurance participating policies in the Ghanaian insurance industry. Studies on pricing life insurance participating products in the country is very scanty. Product pricing plays a vital role in the development of an insurance policy. It saves as an indicator for product value and determine the rate at which premiums will be charge. The target market is very keen on the cost of acquiring insurance cover. If policies are overpriced, consumers will tend to avoid insurance perceiving it, as luxury and not necessity. On the contrary, if policies are under-priced, insurance companies will be expose to the substantial risk of insolvency, failure of business lines, and even bankruptcy of entire insurance enterprise (Obuba, 2014; Biener, 2013; Frank and Amankwah, 2011). As far as researchers are concerned, there is very little or no study, that has been published in the Ghanaian insurance market to inform stakeholders, especially the industry regulator, the National Insurance Commission (NIC) in relation to fair pricing of life insurance participating policies. Although competition in the industry as well as growth trend in the gross premium (both life and non-life policies) have improved in recent years (NIC, 2011, 2010). This does not support empirically that, insurance policies have been fairly priced in the country.

There is the need to investigate into the prospective methods for a more accurate (fair) pricing of life insurance policies, especially, the participating contracts with embedded surrender options in the Ghanaian insurance market.

1.4 Purpose of the study

For the above identified research problem, the main purpose of the study is to estimate the “fair price” of life insurance participating policies with embedded surrender option under stochastic interest rate model.

1.5 Research Objective

The objective of the study is to estimate the “fair price” of life insurance participating policy embedded a surrender option under stochastic interest rate model, paid by a sequence of periodical premiums.

1.6 Research Questions

For the above identified research problem and purpose, the study tried to find answers to the under listed questions.

1. What is the estimated "fair price" of life insurance participating policywith embedded surrender option paid by a sequence of constant periodical premiums?
2. What is the estimated "fair price" of life insurance participating policywith embedded surrender option paid by a sequence of periodically adjusted premiums?

1.7 Significance of the study

Life insurance industry plays a crucial role in the economic development of a nation in terms of long-term investment and contribution to the gross domestic product (GDP). The findings of the search is enormously important in various ways to various stakeholders in the Ghanaian insurance industry.

To insurance companies, the findings will assist them to have a better understanding of the importance of product pricing as a performance indicator in the life assurance business. This will also enable companies in the industry to assess the risk involved in developing life insurance products and the expected growth of the policies if priced fairly. Furthermore, findings of the study will help life insurers to appreciate the incidence of the different product divisions on the whole premium, and in case the need be, ascertain the likely variations in the process of designing the policy.

To policy makers, results of the research will provide an insight into "fair pricing" of life assurance participating policies, and assist to develop policies on product analysis and review. This will enable the regulatory body, the National Insurance Commission, to come up with guiding principles that will issue a directive to the insurance players in the industry on the requirements for commission of agents and brokers and the management expense incurred in life policy development.

To stakeholders such as investors and shareholders, successful completion of the search will offer valuable information that will enable them to provide constructive proposals to improve on the pricing policies of life assurance products in their respective companies.

1.8 Limitations of the Study

The major limitation of the research was its reliance on the Society of Actuaries life table which might not be applicable to fit the mortality trend in Ghana. Furthermore, time, resources and access also posed constraints to the study. Ideally, all life insurance companies that provide participating policies in the country should have been covered under the study but this was practically impossible due to time framed to complete the study coupled with financial difficulties. Again, academic calendar constraint within which the research ought to be carried out, not every life insurer that underwrite participating policies could be included in the study, though that is most appropriate for generalizing the result to the entire insurance business in the country. It was also not possible that, the researcher would have access to all life insurance policies in the country. As a result, the study was limited to, only the endowment life insurance policies. Therefore, a more extensive study of the different types of life insurance policies would be required to confirm the generality of the results.

1.9 Organization of the Study

The research will be structured into five chapters as follows: whereas Chapter One will present introduction that comprises the background of the study, problem statement, purpose of the study, research objective, research questions, significance of the study, limitations and organization of study. Chapter two is the literature review that covers the general review of the existing literature from related studies by researchers and professional bodies in the insurance industry in relation to fair pricing life policy. Detailed description of the methodology employed was carried out in to reflect the study type in Chapter three. Chapter four was dedicated to the analysis and discussion which composed of the estimates of the models and analytical techniques presented in methodology and Chapter five concludes the study with summary, conclusions and recommendations for further studies.

Chapter 2

LITERATURE REVIEW

2.1 Overview

The chapter presents a detail review of the existing literature and models on the study needed to uncover solution to the research questions. It summarizes findings from other studies that have been carried out by researchers and professional organizations in the insurance field in relation to fair pricing of life insurance participating policies.

2.2 General Review of Existing Literature

Contrast to a standard financial instrument such as put or call option, life insurance contracts are more complex products that incorporate features like mortality/survival, periodical premiums, guarantee interest rate and the right to surrender. Participating life insurance policies are contracts which incorporate an annual minimum guarantee interest rate and a bonus sharing schemes ascertained by the administrative assessment of the insurance company. They have several features such as surrender possibility, dividend options, interest rate guarantees, and so on, which posed liability to insurance companies and consequently composes the likely risks of the firm's solvency (Ballotta et al., 2006). Modeling and pricing participating policy that incorporate all these factors is complex and challenging. The ideal approach is to include the important features and at the same time keep the model tractable (Miltersen and Persson, 2003). As argued by Briys and De Varenne (1997), most life companies disregard the consequence of these factors and that exposes them to the risk of insolvency.

There exist abundant literature addressing the problems of valuation of participating policies, for instance in (Liao et al., 2006; Bacinello, 2003a, 2001; Tanskanen and Lukkarinen, 2003; Grosen and Jørgensen, 2000). In relation to pricing participating policies with guarantees and surrender option, a valuable contribution is made in (Bacinello, 2001, 2003a,b). Though many important contributions have been made in her literature, there exist several vital issues that have not been considered. Previous literature have also solely focused on "fair" valuation of equity-linked products. Thus policy premiums are determined in a manner that it equals the present value of benefits of the insured, which in a nut shell represents the insurer's expenses. Owing to the pioneering work by Brennan and Schwartz (1976) and Boyle and Schwartz (1977), much attention has been devoted to the valuation of equity-linked life insurance products with a minimum guarantee both in the financial and actuarial literature. As a consequence, participating life insurance policies have not been study considerable in a contingent-claim framework (Bacinello, 2003b).

Grosen and Jørgensen (2002), analyzed and priced participating policy with a guarantee minimum interest rate and argued that participating life insurance contracts should offer a low-risk, stable and nevertheless provide reasonable investment opportunities. Their model include the option to surrender and to receive the surrender value implied by a surrender charge. However, Grosen and Jørgensen ware unable to present a closed-form formulae of their bonus account owing to its path dependent nature hence the adoption of Monte Carlo Methods (MCM). Bacinello (2001), analysed participating life insurance contracts with a minimum interest rate guarantee and consider cases where by insurers recieved an upfront premium at inception, and in situations in which the premium is sequencially paid periodically in relation to the performance of a reference portfolio. She obtained a closed-form solution that makes the contract "fair" under Black and

Scholes (1973) and Merton (1973) framework. However, her analysis does not includes a surrender possibility. The inclusion of a surrender possibility in the policy infer that policy owners can sell back the policy to the issueing company prior to maturity. Jørgensen (2001), analysed an American-style contract with a guarantee interest rate using binomial lattice whereas Jensen et al. (2001), priced the embedded surrender option and the bonus policy by means of a finite difference approach. Bacinello (2003b), used Cox et al. (1979) model to determine the fair value of the the insurance company's policy that received an upfront single premium at inception of the contract and perform sensitivity analysis on the contractual parameters that characterized the contract.

Participating policies with embedded surrender option have been valued within the framework of stable risk-free interest rate. Ideally, the guarantee interest rate offered by the contract is more likely to change throughout the lifetime of the policy rather than been constant. Holders of participating policies with embedded surrender possibility might surrender their contract to take advantage of the higher yield in the financial market. As a result, Surrender options have become a major concern for life insurers especially during interest rate volatility. Owing to the long maturity nature of life insurance products, if the guarantee interest rate is not adequately fixed compared to other kinds of investments, policy holders may early terminate their existing policies in order to go in for the higher yields offered in the capital market (Bernard and Lemieux, 2008). As a consequence, this study considers the case of a stochastic interest rates.

The concerpt of introducing stochastic interest rates into the modeling and pricing process of life insurance policy is not new. Miltersen and Persson (2003), adapted the General Health-Jarrow-Merton approach in a stochastic interest rates environment to model the minimum guaranteed rate of return. They used Vasicek (1977) and Cox et al. (1985) short rate models to derive pricing formulae for point-to-point and year-by-year style guarantee interest rates both on the short term interest rate process and the stock market return process.

Briys and De Varenne (1997), considered continuous time valuation case to modelled life insurance liability that explains both interest rate risk and the default risk. Their study employed instantaneous short rate by Vasicek (1977) and obtain a closed-form formulae of certain life insurance liability. Zaglauer and Bauer (2008), also considered the case of determining the risk-neutral value of with profit life insurance policy under stochastic interest rate environment. They used two asset market model and take into account the components of the insurance company's assets group directly by selecting sufficient volatilities and correlation among the interest rate process and the asset process . Their pricing model adapt the framework proposed by Bauer et al. (2008). They used Vasicek (1977) and Cox et al. (1985) short rates, to model the instantaneous risk-free interest rate to determine the risk-neutral price of the participating policy. Furthermore, Liao et al. (2006), priced participating contracts introduced by Bacinello (2001), embedded a surrender possibility in a stochastic interest rate environment. Their study proposed two-dimensional Cox et al. (1979) model capable of determining the "fair value" of the surrender possibility included in the policy. However, their study considered only a case of single upfront premium paid by the insured at the inception of the contract and did not account for situations in which the premium is paid periodically at the beginning of each contract year which is a common practice in the Ghanaian insurance market.

Ideally, the guarantee interest rate offered by the contract is more likely to change throughout the lifetime of the policy rather than being constant. In addition, policy premium is often paid by instalments, yet most authors have based their analysis on policies purchased by a single premium. An exception to this are (Calidonio-Aguilar and Xu, 2011; Bacinello, 2003b, 2001), that analysed policies where the premiums are paid periodically. The goal of the study is to seal this gap in literature by pricing participating policies with embedded surrender option under stochastic interest rate model paid by sequence of periodical premiums introduced by Liao et al. (2006), which in itself is an extension of Bacinello (2001).

2.3 History of Insurance

Insurance in some sense is as old as the story of mankind. This could be traced to the two types of economist in the human society: money economies (financial instruments, with markets e.t.c) and natural economist (involving barter trading without standardized financial instruments or money). The latter form is of more ancient type than the former, in that insurance, the category is seen as neighbors helping each other in the community. For instance, if one household got burnt down by fire, others in the neighborhood help built a new one. This kind of social aid continued should the same happened to others in the society. This form of insurance have lived to the present day in some areas where modern economy with money markets and standardized financial instruments is not widespread (Trenerry, 1926).

On the other hand, insurance in the modern money economy is seen as part of the financial cycle. The early means of transferring risk were first practiced by the Babylonians and the Chinese traders in the 2nd and 3rd millennia BC respectively. The Babylonians developed a scheme for the early Mediterranean sailing where merchants were recorded in codes of Hammurabi, c in 1750 BC. If a merchant obtained a loan to finance his shipment, in return, he would pay an additional amount to the lender for his guaranteed to cancel the loan in the event that the shipment got lost or stolen at sea. Likewise, the Chinese merchants travelling treacherous river rapids redistributes their wares over several vessels to reduce the loss that might result from the capsize of a single vessel (Vaughan, 1997).

According to Mehr and Cammack (1976), Achaemenian monarch were the first to officially insured their people by means of registering the process in a government attorney's office. With this arrangement, gift worth more than 10,000 Derricks (Achaemenian gold coin) were registered in special offices and presented to the monarch by heads of different ethnic groups. Other presents were also fairly assessed by the sister courts and then registered in the special offices. In that way, a person will receive help from the monarch and the court provided his/her gift is registered whenever he/she is in trouble. A decade later, dwellers of Rhodes devised the idea of general average where merchants paid proportionally divided premiums in order for their goods to be shipped together. The premiums collected would then use to reimburse any merchant whose goods were jettisoned as a result of storm or sinkage.

Greeks and Romans are recognized as originators of health and life insurance in 600 BC when they formed guilds called "benevolent societies" to care for the deceased family, as well as, pay funeral expenses of members in the event of death. Guilds in the middle ages aided the same goal. Likewise, Talmud transact in many features of insuring goods prior to the recognition of insurance in the 17th century. In England, individuals contributed amount of money to "friendly societies" which in turn could be use in cases of emergency (Mehr and Cammack, 1976). In the 14th century, insurance policies were seen as separate contracts believed to have originated from Genoa where insurance policies were backed by pledges of landed estate. These contracts allowed insurance to be separated from other form of investment, a separation of roles that first yield result in marine insurance, became more sophisticated in post-Renaissance Europe and specialized varieties developed. London's growing importance as a center for trade increased demand for marine insurance in the late 17th century. Edward Lloyd opened a coffee house in 1680 that subsequently became a popular haunt for merchants, ship captains and ship owners and eventually served as reliable source for latest shipping news. It subsequently became a meeting ground for persons willing to insure ships and cargos and those prepared to underwrite such ventures (Kingston, 2007).

Insurance known today can be traced to the great fire of London which devoured about 13,200 houses in 1666. Consequently, Nicholas Barbon established an insurance office in 1681 to insure building bricks and frames believed to be the first fire insurance company in England. Later in 1732, United States also had its first insurance company formed in Charles Town (present day Charleston), south Colombia, to underwrote fire insurance. Benjamin Frankling helped popularized and standardized the practice of insurance, particularly against fire in a form of perpetual insurance. Shortly in 1752, he founded the Philadelphia contributionship to insure houses against loss by fire (Poku, 2009).

2.4 Insurance and the Insurance Industry in Ghana

Bowers et al. (1986), defined insurance as a promise of compensation for specific potential future losses in exchange for a periodic payment. It therefore provides a means by which businesses or individuals are protected against financial losses resulting from specific perils. In order to receive reimbursement for accidental losses, one must purchase an insurance policy. In the policy document, the insurer states clearly the duration and conditions of agreement under which compensation will be provided for. The compensation to be awarded largely depends on the premium charged. This in turn based on the insured amount and the possibility of the insurance carrier having to pay (Akanlagm, 2011).

The insurance industry in Ghana composed of insurers and reinsurers, and insurance intermediaries. Whiles the insurers and the reinsurers are made of life, non-life and reinsurance companies, the insurance intermediaries composed of brokers, loss adjusters, reinsurance brokers and agents. More so, the insurance companies, on one hand, are seen as primary underwriters of the insurance policies and therefore assumed the risk covered by those policies. The insurance intermediaries on the other hand, market the insurance policies for the companies of which they are directly affiliated to. Others to, are independent and are free to sell policies from different insurance companies (NIC, 2011). Following the induction and passage of the new insurance law Act 724 of 2006, the National Insurance Commission (NIC) separates the composite insurance companies into its constituents as life and non-life companies. The industry as at June 2015, composed of 23 life companies, 26 non-life companies, 3 reinsurance companies, 69 broking companies, 1 loss adjuster, 1 reinsurance broker and about 6,000 insurance agents (National Insurance Commision).

2.4.1 General Insurance

General insurance also termed non-life insurance in Ghana, defined by Bowers et al. (1986), as an insurance policy not intended to provide life assurance. It provides policies to property owners, such as automobiles and homeowners, in a form of compensation dependent on the risk of loss from specified financial events in exchange for a predefined premium, over a relatively short period of time (usually monthly, quarterly, semiannual or yearly bases). There are 26 nonlife companies in Ghana as at June, 2015 registered to provide insurance in a form of: liability policies, engineering policies, fire/property insurance policies, motor insurance policies and others (National Insurance Commission).

2.4.2 Life Insurance

Life insurance is defined according to Bowers et al. (1986), as a contract between a policyholder and the insurer in which the insurer promised to pay a designated beneficiary the amount assured in exchange for a fixed or sequence of periodic premium(s), upon the death of the insured. It therefore provides insurance against the likelihood of death often on a guarantee bases over a longer period of time. However, it is possible for one to insure the risk of death over a relatively shorter period of time, such as a year, as it is usually done under general insurance (Angus, 2004). Life insurance policies are legal contracts with a policy document specifying the terms of which compensation will be provided for. Specific preclusions are often made to limit the insurer's liability. Common examples, are such relating to claims of fraud, suicide, riot, war, and others (Frank and Amankwah, 2011).

Life based policies are grouped into two major forms as:

Temporary policies - example term insurance, designed mainly to provide benefits usually of a lump sum in the event of the specified event.

Permanent policies (remains in force until policy matures) example whole life, universal life and variable life, the rationale is to facilitate the growth of capital either by single or periodic premiums. There are 23 life insurance companies licensed to underwrite life policies in Ghana as at June, 2015. Funeral, Keyman, Group life, Credit and Mortgage, whole life, Endowment and Term policies are some products of life policies found in the Ghanaian insurance market (National Insurance Commission).

2.5 Actuarial Product Pricing

Insurance provides a medium through which contingent future losses are exchange for fixed premium payments (Bowers et al., 1986). The underlying principle for the actuarial determination of premium is that, there need to be adequate on average to cover future loses. The equivalent principle is the consequent of this rationale, as a basis for pricing insurance products such that the present value of premiums equals the present value of the expected future losses (Young, 2004). Pricing both life and non-life insurance products originates from the equivalence principle, however, its application as observed by Biener (2013), requires divergent approaches in relation to different properties of risk in different line of business.

2.5.1 Pricing in Short Term

Estimating sufficient premium in a short term (non-life and some life) insurance products poses difficulty as both frequency and severity of the random loss are stochastic. Assumptions on the stochastic nature of the random loss and parameter settings are required to determine its distributional properties which formed the basis to apply the required premium principle (Young, 2004).

2.5.2 Pricing in Long Term

Contrary to short term pricing, the randomness of loss of a long term insurance (life) product relates to an individual's mortality table. Here, "the size of the loss but not the time of occurrence is typically known with certainty" (Bowers et al., 1986). This suggest the need for the inclusion of time value of money in terms of interest rates and mortality, into, the determination of the required insurance cover's premium (price). As showned by Bacinello (2001), expected losses for long term insurance products are dependent on the mortality, interest rates and a loading, for the variation in the actual mortality. This formed the basis for calculating most long term life insurance premiums of which assumptions are made (Wang, 1995).

2.5.3 Premium and Underwriting Life Policy

Underwriting is the signing and accepting liability, and guaranteeing payment in case loss or damage occurs (Bowers et al., 1986). It therefore involves measuring the risk exposure and determine the appropriate premium that needs to be charged to insure the risk (Young, 2004). Life insurance policy may be purchased by an upfront premium at inception or a series of periodical premiums while the contract is still in force, over an arranged period of time. Premium may be level or vary depending on the actuarial basis that is used. For instance, assumptions made about future movement of interest rates, mortality and other factors such as surrenders and cost of capital needed to support new business (Abdullah, 2011). Underwriters evaluate the risk and exposure of potential clients, and decide whether clients should accept and insure the risk, how much coverage they should receive or pay for. As part of the underwriting process for a life policy, medical report may be needed to examine the applicant's health status (other factors such as age and occupation may as well be considered). Whereas general insurance covers policyholders against specified loss of property, life insurance does not, the magnitude is basically determined by the holder depending on his/her needs and the available resources. However, the scope of the moral hazard should be clear to ensure that the amount insured is moderate in relation to the applicant's needs and resource (Angus, 2004).

2.6 Participating and Non-participating Policies

A policy that provides holders a share of profit from the excess earnings of the company if the realized interest rate during the insurance period is above the assumed interest rate is term as participating policy (Bacinello, 2001). This is so for the fact that, policy owners are entitled to participate in the surplus earnings of the company. An insurer incurs a loss if charged inadequate premium. A nonlife insurer kept premium up-to-date as it constantly monitor claim experience. For a life insurer, the possibility of doing this is far slashed as the experience takes long to work out (Dickson et al., 2013). As a result, realistic margins should be included in the premium so as to reduce, as low as possible, the possibility of turning out to be insufficient. This implies that premium would be determined assuming that:

1. Mortality would be worse than expected.
2. Less interest would be earned by the asset than is thought likely.
3. Management expenses would be more than expected.

Bacinello (2001), argued that, future course of first order technical basis (mortality, interest and expense) must be assumed to determine the premium for the policy. As a consequence, policy owners are likely to pay more than the company's needs to pay claims and that generates a considerable surplus over time. The generated surplus is return to policyholders in a form of dividends as a rewards for their heavily loaded premiums. This may be done in a form of: cash, by slashing their premiums or by enhancing their benefits, jointly known as bonus systems (Bauer et al., 2006). However, not all policies participates in the profit resulting in two separate classes of products transacted in the insurance market as participating (also term as with profit conversional products) and nonparticipating (also known as without profit conversional products). With profit conversional products mainly includes policies whose drive is savings (whole life and endowment), thereby include premiums with significant loadings aimed to generate investment surpluses, whiles without profit conversional products usually include policies whose purpose is pure insurance (term insurance) (Angus, 2004).

2.7 Features of Participating Policies

With profit policies gives a basic benefit, usually fixed at issuance of the contract. The insurer guaranteed that, the contract value will at least increase by a minimum guarantee each year (Bauer et al., 2006). The nature of these policies allows policy owners to have share in percentage wise of the company's earned profit commonly called participating rate (coefficient). The sharing mechanism is usually done as follows:

The insurance company allocate greater portion of the premiums received from participating policies in to a reference portfolio. Whenever a dividend on the reference portfolio is above the minimum guaranteed rate of that period, an additional interest called bonus rate is granted to the policy (Bacinello, 2003a). Owing to the uncertainty nature of the financial markets, reference portfolio performance might vary, and the likelihood of obtaining higher gains might equals that of heavy loses. However, loses are absorb by the insurance company since policy owners minimum interest rate is guaranteed (Calidonio-Aguilar and Xu, 2011).

2.8 Principles of Life Insurance Mathematics

According to Fischer (2004), the under listed assumptions are important for the theory of modern life insurance mathematics though stated in an informal way.

2.8.1 Independent Financial and Biometric Events

This is a common assumption that translates that biometric events such as age, sex and health status are independent of the financial market.

2.8.2 Complete Arbitrage-Free Financial Market

Though this assumption might seems unrealistic from the perspective of finance, however, it is reasonable from the view point of with profit conventional life insurance due to the fact that, life insurers does not typically develop pure financial products (Bowers et al., 1986). Hence one can argued that, financial products are either obtained from banks, traded on the market or can be replicated by self-financing strategy. However, it is obvious that claims dependent on technical events (death) cannot be hedged by financial instruments (Fischer, 2004).

2.8.3 Independent Biometric States of Persons

This is a normal assumption of a conventional life insurance. Nevertheless, one could even overlook the likelihood of epidemic diseases, and the principle still seems applicable in the modern world. Even the well-known argument that couples bear some dependences, especially when they have contract with the same company or joint contract on two lifes, falls out of place, in that couples themselves will usually be independent (Fischer, 2004).

2.8.4 Similar Biometric persons are the same

For the sake of fairness, two or more persons with the same biometric events should be required to charge the same premium under the same policy (Bowers et al., 1986). Also, any cause of action an insurer does as a result of two persons with the same type of policy is considered identical, as much as their future biometric growth is identically identified from the stochastic perspective (Fischer, 2004).

2.8.5 Arbitrage-Free Pricing

An important assumption from the theory of finance for rational pricing technique is the arbitrage-free opportunity, that is, nonexistence of "riskless" gains (Hull, 2006). Thus, it should not be possible for one to beat the pricing market of life insurance simply by buying and selling a policy in an prevailing or hypothetical reinsurance market (Fischer, 2004).

2.8.6 Large type of Similar Persons

Regarding the law of Large Numbers as applied in conventional life insurance mathematics, an implied hypothesis is usually made concerning the number of individuals under a policy in a particular company. Thus insurers should be able to handle successfully volumes of such persons even if they all subscribe to the same policy (Fischer, 2004).

2.8.7 Fair Minimum Prices Allows for Hedging such that Mean Balance Converges to Zero Almost surely

The independent biometric state is very similar to the principle of expectation of a conventional life policy. Financial markets in the conventional situation, presumed to be deterministic and states that, the present value at time zero of an upfront net premium of a policy is the expectation of the sum of its benefits (Fischer, 2004). The link among these principles is the Law of Large Numbers. Premiums are calculated in a manner that for an increasing number of policies (as a result of independent persons), the mean final balance per policy (mean balance) turns to zero nearly surely. Similar to the conventional case, it is usually demanded that the "minimum fair price of any policy should at least covers the price of a pure financial hedging strategy" such that the mean balance per policy converges to zero for increasing number of clients (Fischer, 2004).

2.8.8 Equivalence Principle

Under a rational pricing principle, the principle of equivalence requires that insurer's premiums should be determined in such a way that it's present value equals that of the future benefits of the insured (Bowers et al., 1986). The idea could translates as, the insurers liability in some way can be hedged working forward with the premiums (Fischer, 2004).

2.9 Risk-neutral Valuation

To be able to price life policies, it is necessary to apply techniques and theories from modern financial mathematics, most importantly, the theory of riskneutral valuation (Fischer, 2004). Let $(\Phi, G, \mathbf{P})$ be a probability space with state space Φ describing the likely occurrences in the financial market with a natural filtration $G = \{G_t\}_{t\in[0,T]}$ containing the available information of the financial market. If financial securities in the market are adapted to the filtration G described by a stochastic process, specifically the existence of the num´eraire process $\mathbf{B} = (B_t)_{t\in[0,T]}$ and asset development $(S_t)_{t\in[0,T]}$. Further assume the existence of a $\mathbf{Q}$ measure equivalent to $\mathbf{P}$ so that discounted securities $(\mathbf{S_t}/\mathbf{B_t})$ are $\mathbf{Q}$ local martingales. Zaglauer and Bauer (2008), maintained that, the existence of the equivalent martingale measure implies the no arbitrage opportunity of the financial market. Contrast to a financial derivative, life insurance does not only depends on the changes in the financial market, but also, on the biometric progress of individuals (Zaglauer and Bauer, 2008; Bauer et al., 2006; Fischer, 2004; Bacinello, 2001).

If $(\Theta, H, L, \mathbf{H})$ denotes the stochastic base where Θ describe the state space of the biometric growth of the insured, $\mathbf{H} = (H_t)_{t\in[0,T]}$ is the filtration and L as the associate probability measure. Pricing life insurance products requires the construction of a filtered probability space to model simultaneously the biometric events and the financial market. Assuming principles (2.8.1) to (2.8.5) makes life insurance company asymptotically risk-neutral in relation to the biometric events and premium determination (Fischer, 2004).

Axiom 2.9.1. *Suppose a combined filtered probability space is given by:* $(\Omega, F, Q, \mathbf{F}, \mathbf{P}) = (\Phi, G, \mathbf{Q}, \mathbf{G})^{N}(\Psi, H, L, \mathbf{H})$ *defining the effect of the financial market and the biometric events.*

Where $F_t = G_t\, ^{N}H_t$ denotes the information of the insurance market and $Q^{N}L$ identifies the insurance cover. As argued by Zaglauer and Bauer (2008), Q is again a martingale measure on the combined filtered probability space $(\Omega, F, Q, \mathbf{F}, \mathbf{P})$.

If X is an insurance claim (benefit) contingent on the financial market and the biometric progress of the insured. The risk-neutral pricing (premium) relation put forward by Bingham and Kiesel (2013), is:

$$\pi(X_t) = B(0,t)E_Q\,[X_t/F_t] \qquad (2.9.1)$$

Chapter 3

RESEARCH METHODOLOGY

3.1 Overview

The chapter discusses the methodology developed to achieve the study objective. The search is an experimental exploratory study designed to estimate the "fair price" of life insurance participating policies with embedded surrender option paid by a series of regular premiums under stochastic interest rate model.

3.2 The Policy Structure

With profit conventional life policies are contracts that have minimum guarantee interest rate of and a bonus distribution mechanism determined by the management decision of the insurance company (Zaglauer and Bauer, 2008). An attempt is made to give a description of a standard endowment participating policy issued in the Ghanaian insurance market.

3.3 The Policy Premium

Consider a standard endowment participating life insurance policy issued in the Ghanaian insurance market at time ($t = 0$) and matures at time ($t = T$) . Under the policy arrangement, the beneficiary receive a specified amount of money if the insured dies within the contractual period or survives the maturity date. Assumed in the event of death during the t-th year of the contract ($t = 1,2,...,T - 1$) benefit is paid at the end of the year of death otherwise paid at maturity T. Again, if x represents the age at issue of the policy at time t, b_0 as the initial amount insured, payable if death occurs within the first year of the policy and b_t as the benefit due at time t ($t = 2,3,...,T$). If the policy is paid by a series of periodical premiums due at the beginning of each policy year if the insured is still alive. The initial premium P_0 paid at the time of issuance of the contract is giving by:

$$P_0 = b_1 P_{x:\overline{T|}} = b_1 \frac{A_{x:\overline{T|}}}{\ddot{a}_{x:\overline{T|}}}$$

$$= b_1 \frac{\sum_{t=1}^{T-1} (1+i)^{-t} \ {}_{t-1/}q_x + (1+i)^{-T} \ {}_{(T-1)}p_x}{\sum_{t=0}^{T-1} (1+i)^{-t} \ {}_t p_x} \quad (3.3.1)$$

Where $i > 0$, is the annual technical interest rate, ${}_{t-1/}q_x$ represents the probability of the insured dying within the $t-th$ policy year and ${}_t p_x$ is the probability of the insured surviving to time t. The probabilities introduced are dependent on the age, x of the insured and are typically obtained from an appropriate risk-neutral mortality table implied on the technical interest rate. Traditionally, (3.3.1) translates that, the expected value of the initial benefit b_1 discounted from the current time with the technical interest rate to the time of random payment (= $b_1 A_{x:T\rceil}$) equals the expected value of the sequence of the regular premiums P_0 of the current time also discounted with the same rate (= $P_0 \ddot{a}_{x:T\rceil}$). Hence P_0 makes the policy "fair" at issuance on the grounds of first order technical bases.

3.4 The policy Reserve

The benefit reserve at time t , of a policy issued at age x , that is still in force t years latter is defined according to Bowers et al. (1986), as the excess actuarial present value at age $x + 1$ of future premiums including any premium payable at age $x + 1$. This excess represents a liability to the insurer and are usually calculated at the end of each policy year. If P_0 from (3.3.1) is charged by the insurer, the technical rate could be seen as the return granted to the policy's reserve at the onset of each policy year. As a result, the benefit is annually adjusted, thereby resulting the dependence of the benefits on the performance of the special investment portfolio (Bacinello, 2003a). However, the adjustment is done in a manner that, the policy remains fair on the grounds of first order technical bases in relation to the remaining contract policy duration.

Denote by P_t, the premium at time $t(t = 1,2,...,T-1)$ if the insured is alive and the contract is still in force and V_t^- (V_t^+) as the insurers policy's reserve at time t, shortly prior to the payment of the periodical premiums P_t and (shortly after) an adjustment respectively. Given b_t and P_{t-1}, V_t^- is estimated as:

$$V_{t-} = b_t A_{x+t:T-t|} - P_{t-1}\ddot{a}_{x+t:T-t|}$$

$$= b_t \sum_{h} (1+i)_{-t}\, {}_{h-1/}q_{x+t} + (1+i)_{-(T-t)}\, {}_{(T-t)}p_{x+t}$$

$$- P_{t-1} \sum (1+i)^{-h}\, {}_h p_x, \qquad t = 1,2,...,T-1 \qquad (3.4.1)$$

Where ${}_{h-1/}q_{x+t}$ is the probability of the insured dying within $(t+h)$-th year of the contract (between, $t + h - 1$ and $t + h$) conditioned on the event of survivorship of the insured at time, t and ${}_hp_{t+x}$ is the probability that the insured survives to time $(t + h)$ conditioned on the same occurrence.

3.4.1 The Reserve, the Benefit and the Premium Adjustment Rates

Following the computation of V_t^- from (3.4.1), V_t^+ is directly modified by an adjustment rate δ_t such that:

$$V_t^+ = V_t^-(1+\delta_t)\ , \quad t = 1,2,\ldots,T-1 \qquad (3.4.2)$$

Where δ_t is defined as:

$$\delta_t = max\left\{\frac{\eta\, g_t - i_g}{1+i_g}\ ,\ 0\right\}\ , \quad t = 1,2,\ldots T-1 \qquad (3.4.3)$$

Here, η is the participating coefficient between 0 and 1, g_t represents the return on the investment portfolio during the t-th policy year and i_g is the (minimum) guaranteed interest rate. Following the adjustment of the mathematical reserve, if P_0 is defined by (3.3.1), as argued by Bacinello (2003a), the return credited to the insured during the t-th contract year implied on the guaranteed rate riches its maximum between i_g and $\eta\, g_t$ given by:

$$max\{i_g, \eta\, g_t\} = (1 + i_g)\,(1 + \delta_t) - 1\,, \qquad t = 1,2,...,T - 1 \qquad (3.4.4)$$

Now, assume α_t and γ_t as the benefit and the premium adjustment rates respectively. It is usual in practice that, the benefit and the periodical premiums of a participating policy be adjusted in the same measure as the bonus rate δ_t, credited to the benefit reserve. Hence the benefit and the periodical premiums respectively becomes:

$$b_{t+1} = b_1\,(1 + \alpha_t)\,, \quad t = 1,2,...,T - 1 \qquad (3.4.5)$$

and

$$P_t = P_{t-1}\,(1 + \gamma_t)\,, \quad t = 1,2,...,T - 1 \qquad (3.4.6)$$

With regards to the residual contract period, the policy is fairly priced on the basis of first order technical bases as:

$$V_{t+} = b_{t+1}\, A_{x+t:\overline{T-t|}} - P_t\, \ddot{a}_{x+t:\overline{T-t|}}\,, \qquad t = 1,2,...,T - 1 \qquad (3.4.7)$$

Analyzing (3.4.1), (3.4.2), (3.4.5) and (3.4.6) revealed that, (3.4.7) imposes constraint among the adjustment rates such that α_t becomes the appropriate average of the residual rates. Thus

$$\alpha_t = \omega_t\, \delta_t\,(1 - \omega_t)\,, \qquad t = 1,2,...,T - 1 \qquad (3.4.8)$$

Where

$$\omega_t = \frac{R_t^+}{b_t\, A_{x+t:\overline{T-t|}}}$$

At this stage, the study assumes two extreme cases that seems common to most life insurance participating policies, that is, identical adjustment rates and constant periodical premiums.

Identical Adjustment Rates

In what follows, the study assumed that, life insurers set the same adjustment rates i.e. $\alpha_t = \gamma_t = \delta_t$ at any time t such that the policy reserve, the benefit and the regular premiums are all modified using the same measure (Bacinello, 2003a). For a given b_1, one can ascertain an initial premium P_0, of which the contract is fairly priced at inception. In that instance the adjusted benefit and the premium from (3.4.5) and (3.4.6) respectively, becomes

$$b_{t+1} = b_1 (1 + \delta_t) , \qquad t = 1,2,...,T - 1 \qquad (3.4.9)$$

and

$$P_t = P_{t-1} (1 + \delta_t) , \qquad t = 1,2,...,T - 1 \qquad (3.4.10)$$

where δ_t is obtained from (3.4.3).

Iteratively, (3.4.9) and (3.4.10) can be express as:

$$b_t = b_1 \prod_{k=1}^{t-1} (1 + \delta_k) , \qquad t = 1,2,...,T \qquad (3.4.11)$$

and

$$P_t = P_0 \prod_{k=1}^{t-1} (1 + \delta_k) , \qquad t = 1,2,...,T - 1 \qquad (3.4.12)$$

Constant Periodical Premiums

In order to uphold the idea of constant periodical premiums, it is apparent to assume $\gamma_t = 0$ so that P_t will be constant at any given time t. Suppose P denote the constant periodical premium, analysing (3.4.1), (3.4.5) and (3.4.8) imply that:

$$b_{t+1} \quad = \quad b_t(1 + \omega_t \delta_t) = b_t(1 + \delta_t) - b_t(1 - \omega_t)$$

$$= b_t(1+\delta_t) - \frac{P\,\delta_t}{P_{x+t:\overline{T-t}|}} \tag{3.4.13}$$

Where

$$P_{x+t:\overline{T-t}|} = \frac{A_{x+t:\overline{T-t}|}}{\ddot{a}_{x+t:\overline{T-t}|}}$$

Equation (3.4.13) depicts that, the benefit adjustment rate is determined by the pair $(x+t, T-t)$. However, it is prudent to assume in practice that, the adjustment rate depends only on the time, t and maturity, T and not on the policy holder's age. (Bacinello, 2003a). Therefore, (3.4.13) has to be approximated by replacing, P with premium obtained from (3.3.1) for policies belonging to the same portfolio as:

$$b_{t+1} = b_t(1+\delta_t) - b_1\delta_t\left(1-\frac{t}{T}\right) \tag{3.4.14}$$

Applying a conversion that:

$$\prod_{h=1}^{t-1}(1+\delta_t) = 1$$

Relation

(3.4.14) becomes:

$$b_t = b_1\left\{\prod_{k=1}^{t-1}(1+\delta_k) - \sum_{k=1}^{t-1}\left[\delta_k(1-\frac{k}{T})\prod_{h=k+1}^{t-1}(1+\delta_h)\right]\right\}, \quad t = 2,3,\ldots,T \tag{3.4.15}$$

3.5 Surrender Condition

A surrender option defined by Bacinello (2003b), is a type of American put option that empowers policy owners to resell their policy to the insurer at a surrender value. The inclusion of this possibility in the policy infers that the insured can sell back the policy to the issuing company before maturity. Thus policy owners has the right to early terminate their contracts, so as to receive the surrender value implied by a surrender charge. Assume that surrender decisions are made at the commencing of the policy year shortly following the declaration of the renewal benefit for the impending policy anniversary , and prior to the payment of the regular premiums. If S_t denote the surrender value at any time t defined by:

$$S_t = \begin{cases} 0 \,, & t = 1, 2 \\ b_{t+1}\, \nu\, A_{x+t:\overline{T-t|}} \,, & t = 3, 4, \ldots, T-1 \end{cases} \tag{3.5.1}$$

Where ν is a surrender parameter and

$$A_{x+t:\overline{T-t|}} = \sum_{t=3}^{T-1} (1+i_g)^{-(h-t)}\ {}_{h-1/}q_{x+t} + (1+i_g)^{-(T-t)}\ {}_{(T-t)}p_{x+t}$$

The equation is consistent with most life insurance policies sold in the Ghanaian insurance market. Thus no benefit is paid back to the policy owner unless at least three periodical premiums are collected.

3.6 The Model Dynamics

Consider life insurance endowment participating contract with embedded surrender option issued in the Ghanaian insurance market, the insured at maturity receive a guaranteed benefit in addition to a participating bonus in return for the periodical premiums paid. In a risk neutral world, the insurer is a subject of mortality and financial risks. Assumed independence between both risks, the value of an insurance claim is formulated as a discounted expected value in relation to the risk neutral mortality and the financial elements. As outlined by Bernard et al. (2005), the dynamic process of a zero coupon bond in the risk neutral world with maturity τ and a reference portfolio is given by:

$$\frac{dB(t,\tau)}{B(t,\tau)} = r_t\, dt + \sigma_0(t\,,\,\tau)\, dZ_1(t) \tag{3.6.1}$$

$$\frac{dS(t)}{S(t)} = r_t\, dt + \sigma_1\, dZ(t) \tag{3.6.2}$$

Where $Z(t)$ and $Z_1(t)$ are standard Brownian motions under the risk neutral measure Q and r_t is the instantaneous risk-free interest rate. If ρ denote the correlation factor between the two Brownian movements such that $dZ(t)dZ_1(t) = \rho dt$. Now consider another Brownian motion $Z_2(t)$ independent from $Z_1(t)$ such that $dZ_1(t)dZ_2(t) = 0$. Therefore the Brownian motion $Z(t)$ is formulated as:

$$dZ(t) = \rho dZ_1(t) + \sqrt{1-\rho^2}\, dZ_2(t) \tag{3.6.3}$$

Equation (3.6.3) decorrelates the interest rate risk from the reference portfolio risk. Therefore the reference portfolio dynamics from (3.6.2) is rewritten as:

$$\frac{dS(t)}{S(t)} = r_t\, d_t + \sigma_1 \rho\, dZ_1(t) + \sigma_1 \sqrt{1-\rho^2}\, dZ_2(t) \tag{3.6.4}$$

Axiom 3.6.1. *Suppose Q^T as the forward risk-neutral equivalent probability measure with num´eraire process defined through its Radon-Nikodym derivative*

$$\frac{dQ^T}{dQ} = e^{-\int_0^t \sigma(u,t)\, dZ_1 - \left(\frac{1}{2}\right)\int_0^t \sigma^2(u,t)\, du}$$

Using Girsanov's theorem, the existence of such measure is guaranteed, therefore, $Z_1^T(t)$ defined by $dZ_1^T(t) = dZ_1(t) + \sigma(u,t)du$ is a Q^T Brownian motion. Furthermore, building the process, Z_2^T in such a way that Z_1^T and Z_2^T are uncorrelated and applying Itˆ*o*s formula to (3.6.1) and (3.6.4), the dynamics of the reference portfolio and the zero coupon bond under the transformed Q^T measure are respectively formulated as:

$$\frac{S(t)}{B(0,t)} = \frac{S(0)}{B(0,t)} \exp \left\{ \begin{array}{c} \int_0^t \sigma_1 \rho - \sigma_0 (u,t)\, dZ_1^T(u) + \int_0^t \sigma_1 \sqrt{1-\rho^2}\, dZ_2^T(u) \\ -\frac{1}{2}\int_0^t \left((\sigma_1 \rho - \sigma_0 (u,t))^2 + \sigma_1^2 (1-\rho^2) \right) du \end{array} \right\} \tag{3.6.5}$$

and

$$B(t-1,t) = \frac{B(0,t)}{B(0,t-1)} \exp \left\{ \begin{array}{c} \int_0^{t-1} \sigma (u,t-1) + \sigma (u,t)\, dZ_1^T(u) \\ -\frac{1}{2}\int_0^{t-1} \left((\sigma (u,t-1) + \sigma(u,t))^2 \right) du \end{array} \right\} \tag{3.6.6}$$

Owing to the lengthy maturity nature of most life insurance contracts and the constraints imposed on the estimation viability, the search assumed $\sigma(u,t) = \sigma(t - u)$ where σ is constant as proposed by (Liao et al., 2006; Nielsen and Sandmann, 1995). The underlying forces of the relative price, $R(t)$ in consecutive years $(t - 1, t)$ of the reference portfolio and the zero coupon bond under the transformed Q^T measure are therefore reduced to:

$$R(t) = \frac{1}{B(0,t)} \exp\left\{ \begin{array}{c} \int_{t-1}^{t} \sigma_1 \rho - \sigma_0 (t-u)\, dZ_1^T(u) + \int_{t-1}^{t} \sigma_1 \sqrt{1-\rho^2}\, dZ_2^T(u) \\ -\frac{1}{2} \int_{t-1}^{t} \left((\sigma_1 \rho - \sigma_0 (t-u))^2 + \sigma_1^2 (1-\rho^2) \right) du \end{array} \right\} \quad (3.6.7)$$

Respectively

$$B(t-1, t) = \frac{B(0,t)}{B(0,t-1)} \exp\left\{ \frac{1}{2} \sigma^2 (t-1) + \sigma Z_1^T (t-1) \right\} \quad (3.6.8)$$

The surrender possibility in the contract makes the pricing process complex as a result of the issue of optimizing the stopping time (Liao et al., 2006). The CoxRoss-Rubinstein (CRR) model presents an effective means of managing pathdependency option such as the surrender option. Hence, the section that follows elaborates more on an extended two-dimensional CRR model for contract pricing that is paid by sequence of periodical premiums, be it constant or adjusted.

3.7 Fair Policy Pricing Using Extended TwoDimensional CRR Model

Constant interest rate within the CRR modeling framework is a modest and proficient means of pricing options in which early exercise may represent the optimal choice. However, for a given stochastic investment portfolio and a zero coupon bond, it is essential to ascertain an extended two-dimensional CRR model for a two-dimensional lognormal distribution of the relative price of the investment portfolio and that of the zero coupon bond to proficiently estimate the fair price of the participating policy with a surrender possibility that is paid by a series of periodical premiums.

3.7.1 The Extended Two-Dimensional CRR Model

In order to approximate a two-dimensional lognormal density of the investment portfolio and the zero coupon bond, an extended two-dimensional CRR is formulated with the origin of the tree representing the bond price $B(0,1)$ and a detailed tree formulation structure given by:

State 1: Given an origin of bond price $B(t-1,t)$, construct a node of relative price $R(t)$ minus the periodical premium paid at time t.

State 2: For the constructed nodes of $R(t)$ in state 1, create nodes of bond prices $B(t,t+1)$.

As proposed by (Liao et al., 2006; Ho and Lee, 1986), the joint distribution of the relative price $R(t)$ and the bond price $B(t,t+1)$ conditioned on the filtered information F_{t-1} is only dependent on $B(t-1,t)$. This implies that, the extended two-dimensional CRR model has a Markov property as the relative price $R(t)$ and the future bond price $B(t,t+1)$ development are only dependent on the immediate past bond price $B(t-1,t)$ (Liao et al., 2006). The dynamics of the relative price $R(t)$ conditioned on the bond price $B(t-1,t)$ under the forward Q^T measure is calculated as:

$$R(t) = \frac{1}{B(t-1,t)} \exp\left\{ \begin{array}{c} \int_{t-1}^{t} \sigma_1 \rho - \sigma_0 (t-u)\, dZ_1^T(u) + \int_{t-1}^{t} \sigma_1 \sqrt{1-\rho^2}\, dZ_2^T(u) \\ -\frac{1}{2}\int_{t-1}^{t} \left((\sigma_1 \rho - \sigma_0 (t-u))^2 + \sigma_1^2 (1-\rho^2) \right) du \end{array} \right\}$$

$$\stackrel{d}{=} \frac{1}{B(t-1,t)} \exp\left\{ \sigma_L \tilde{Z}(1) - \frac{1}{2}\sigma_L^2 \right\} \qquad (3.7.1)$$

Where $\tilde{Z}(1)$ is the new Brownian motion and

$$\sigma_L^2 = \int_{t-1}^{t} \left((\sigma_1 \rho - \sigma(t-u))^2 + \sigma_1^2 (1-\rho) \right) du$$

3.8 The Pricing Framework

The section device the "fair pricing" of the contract and the recursive backward algorism based on the CRR model. The policy described above is a typical example of a contingent claim since it is affected by both mortality and financial risks. Assume independent between these risks and that financial and insurance markets are perfectly competitive, frictionless and free of arbitrage opportunities. Also, assume policyholders are rational and non-satiated, and to share the same information. As a result, surrender decision can only be made following the comparison of the surrender value and the continuation value, such that policyholder's surrender their contract, if and only if, the surrender value is more than the continuation value otherwise keeps the policy until the end of the coming year. The financial risk that affects the policy under study comes from the stochastic evolution of the rates of return on the investment portfolio and the zero coupon bond. In such an instance, assume the investment portfolio is carefully diversified and divided in to components such that any yield realized is immediately reinvested and spreed among all components (Bacinello, 2003a). As a result, the reinvested yield affects only the unit component price and not the overall number of components involve when a withdrawal or new investments are done. This suggest that the reference portfolio price is entirely ascertained by the development of its relative unit component price. If $R(\tau)$ denote the relative price at time ($\tau > 0$), then

$$g_t = \frac{R(t)}{R(t-1)} - 1 \quad , \qquad t = 1, 2, \ldots, T - 1 \qquad (3.8.1)$$

To describe the stochastic evolution of the reference portfolio and the zero coupon bond, the study proposed an extended two-dimensional CRR model. The CRR model is seen as an exact model for which exact values for both European and American style contracts can be valued as an approximation of Black and Scholes (1973), and Merton (1973), models to which it is asymptotically converges (Bacinello, 2003a). To fairly priced the participating policy with the embedded surrender option under a stochastic interest rate model, the study split every contract year to form N-equal sub-period such that $\nabla = 1/N$. Following binomial evolution as proposed by Cox et al. (1979), the relative price, at each period has two possible values, a good one denoted by U_R and a bad one represented by D_R with mathematical relations as:

$$\sqrt{\ } U_R = e^{r_{R(t)} \nabla}$$

and

$$D_R = \frac{1}{U_R}$$

Wth corresponding probabilities given respectively by:

$$P_U = \frac{1 - D_R}{U_{RP} - D_R} \tag{3.8.2}$$

and

$$1 - P_U = \frac{U_R - 1}{U_R - D_R} \tag{3.8.3}$$

In order to avoid any arbitrage opportunity, a volatility parameter is fixed for a given drift term such that $U_R > e^{r\nabla} > D_R$. This result a strictly non negative number between 0 and (1) for both P_U and $1-P_U$. More so, the above assumptions infer that $g_t, t = 1,2,...,T-1$ is independent and identically distributed and can assume $N+1$ possible values as:

$$\psi_i = \frac{1}{B(t-1\,,t)} U_R^{N-1} D_R^{N} - 1 \quad , \qquad i = 0,1,2,\ldots,N \tag{3.8.4}$$

With corresponding risk-neutral probability as:

$$P_{R(t)}^{(i)} = \begin{pmatrix} N \\ i \end{pmatrix} P_R^{N-i} (1 - P_R)^i \quad , \qquad i = 0,1,2,\ldots,N \tag{3.8.5}$$

Furthermore δ_t is also independent and identically distributed and can assume $n + 1$ likely values given by:

$$\Psi_i = \frac{\eta\,\psi_i - i_g}{1 + i_g} \quad , \qquad i = 0, 1, 2, \ldots, n - 1 \tag{3.8.6}$$

Where

$$n = \left\lfloor \frac{N}{2} + 1 - \frac{\ln(1 + i_g/\eta)}{2\,\ln(U_R)} \right\rfloor \tag{3.8.7}$$

With bzc represents the integer part of a real number z, indicating the minimum number of downs such that a call option on the rate of return of a policy in a specified time with exercise price i_g/η expires out of the money.

The zero coupon bond price, $B(t,t+1)$ under the transformed measure Q^T is given by:

$$B(t\,,t+1) = \tfrac{B(0\,,t+1)}{B(0\,,t)} \exp\left\{\sigma Z_1^T(t) + \tfrac{1}{2}\sigma^2\, t\right\} \tag{3.8.8}$$

Now dividing (3.8.8) by (3.6.8) result:

$$B(t\,,t+1) = B(t-1\,,t)\tfrac{B(0\,,t-1)B(0\,,t+1)}{B(0\,,t)^2}\, e^{-\sigma^2(t-1)} \exp\left\{\int_{t-1}^{t} \sigma\, dZ_1^T - \tfrac{1}{2}\sigma^2\right\} \tag{3.8.9}$$

Applying log transformation to (3.6.7) and (3.8.9) such that $\log B(t,t+1)$ and $\log R(t)$ follow Gaussian distribution. Employing linear regression, the distribution of $\log B(t,t+1)$ conditioned on $\log R(t)$ has bivariate Gaussian distribution with conditional mean μ_t and volatility σ_t having equivalent distribution as:

$$\log B(t\,,t+1) \overset{d}{=} \mu_t + \sigma_t\,\varepsilon \overset{d}{=} \log\kappa_t + r_{R(t)} - \frac{1}{2}\sigma_t^2 + \sigma_t \tilde{Z}_2(2) \tag{3.8.10}$$

Where κ_t is the initial value of the CRR model dependent on the bond price $B(t-1,t)$, $r_{R(t)}$ is the drift term dependent on both bond price $B(t-1,t)$ and the relative price $R(t)$, $\varepsilon \sim N(0,1)$ and $\tilde{Z}_2(2)$ is the new Brownian motion.

3.9 Fair Pricing of the Contract Components

The section intend to estimate the initial premium P_0 that makes the policy "fair" at issuance on the basis of the assumption of arbitrage-free. Thus P_0 translates that the insurer's liability equals the sequence of regular premiums if and only if their present values are equivalent (Bacinello, 2003a). The study considered the case where by the premium is constant and the case by which it is annually adjusted. In both instances, the policy is split in to three separate components as:

1. Basic contract (without profit and without surrender) equivalent to a zerocoupon bond.
2. Bonus option (with profit but without surrender).
3. Surrender option (whole contract).

These imply that, the basic contract and the bonus option policy's components are of the European-style option while the surrendable participating policy is of the American type (Liao et al., 2006). There exist a closed-form formulae for each premium of the European-style. It is intuitive that, the basic contract's premium for both constant and periodical adjustment cases are the same and stays constant as through out the contract period. However, the non-surrendable and the surrendable participating policies are both adjusted at the rate δ_t. Hence the decomposition only applies to the initial premium P_0 and the incidence of the different policy units on the whole premium varies stochastically as time goes by (Bacinello, 2003a).

3.9.1 Pricing the Basic Contract

The insurer's liability of the basic contract is the deterministic benefit b_1 due at random time of death or maturity. The market value of the basic contract at time zero, is its expected value with regard to the risk-neutral mortality of the benefit discounted from the random payment time to time zero implied by the risk-free interest rate given by:

$$b_1\, A_{x:\overline{T}|} = b_1 \left[\sum_{t=0}^{T} B(0\,,t)\,{}_{t-1/}q_x + B(0\,,t)\,{}_{T}p_x \right] \tag{3.9.1}$$

More so, the sequence of constant periodical premium, P^{BC} due at beginning of each policy year has its time zero market value given as:

$$P^{BC}\,\ddot{a}_{x:\overline{T}|} = P^{BC} \sum_{t=0}^{T} B(0\,,t)\,{}_{t}p_x \tag{3.9.2}$$

Therefore, the periodical premium which equal these two is:

$$P^{BC} = b_1\, \frac{A_{x:\overline{T}|}}{\ddot{a}_{x:\overline{T}|}} = b_1\, P_{x:\overline{T}|} \tag{3.9.3}$$

3.9.2 Fair Pricing of the Bonus Option

Under the non-surrendable participating policy, the insurer's liability represents the stochastic benefit due at the insured random death or at maturity. From (2.9.1), the fair price of the insurer's liability is computed as:

$$\pi(b_t) = \sum_{t=0}^{T} B(0,t) E_Q\,[b_t]\,{}_{t-1/}q_x + B(0,T) E_Q\,[b_T]\,{}_{T}p_x \tag{3.9.4}$$

The time zero value of the sequence of periodical premiums are also determined in the same manner since both are adjusted by the same measure . The study identified a distinction between the adjusted and the constant periodical premiums.

Identical Adjustment Rate

Assume the benefit b_t is defined by equation (3.4.11) and examining the stochastic dependence of δ_k, for $k = 1,2,...,T-1$, one obtained from equation (2.9.1) as:

$t-1$

$$\pi(b_t) = b_1 \, [B(0,t)] \prod_{k=0} E_Q \, (1 + \delta_k) \qquad (3.9.5)$$

Also, taking in to account the identically distributed nature of δ_k, equation (3.9.5) becomes:

$$\pi(b_t) = b_1 \, B(0,t)(1 + \Psi)^{t-1} \qquad (3.9.6)$$

Where

$$\Psi = E_Q\left[\delta_k\right] = \sum \Psi \, P_{R(t)}^{(i)}$$

Therefore $\pi(b_t)$ from equation (3.9.4) is rewritten as:

$$\begin{aligned} \pi(b_t) &= \frac{b_1}{1+\Psi}\left[\sum_{t=1}^{T} B(0\,,t)\,{}_{t-1/}q_x + B(0\,,t)_T p_x\right](1+\Psi)^t \\ &= \frac{b_1}{1+\Psi}\, A_{x:\overline{T}|}\,(1+\Psi)^t \end{aligned} \qquad (3.9.7)$$

The regular premiums have similar form as the benefit since both are adjusted by an equivalent rate. If P_0^{BO} denote the initial premium with a market value of $\pi(P_0^{BO}) = P_0^{BO}$, then from equation (3.4.12) one obtained

$$P_t^{BO} = P_0^{BO} \prod_{k=1}^{t} (1 + \delta_t) \qquad (3.9.8)$$

Also,

$$\begin{aligned} \pi(P_t^{BO}) &= E_Q\left[B(0\,,t)\,P_0^{BO}\right] \\ &= P_0^{BO}\,B(0\,,t)\,(1+\Psi)^t \end{aligned}$$

Hence, the fair price at time zero of the sequence of regular premiums, P_t^{BO} is estimated as:

$$\sum_{t=0}^{T-1} \pi(P_t^{BO}) \, {}_t p_x = P_0^{BO} \sum_{t=0}^{T-1} B(0,t) \, {}_t p_x \, (1 + \Psi)$$

$$= P_0^{BO}\, \ddot{a}_{x:\overline{T}|}\, (1+\Psi)^t \tag{3.9.9}$$

Relation (3.9.9) is equal to the fair price of the insurance company's liability, if and only if:

$$P_0^{BO} = \frac{b_1\, A_{x:\overline{T}|}\, (1+\Psi)^t}{(1+\Psi)\, \ddot{a}_{x:\overline{T}|}\, (1+\Psi)^t}$$

$$= \frac{b_1}{1+\Psi}\, P_{x:\overline{T}|} \tag{3.9.10}$$

Constant Periodical Premiums

Consider the constant periodical benefit defined by equation (3.4.15) and again the *i.i.d* nature of δ_k, for $k = 1,2,...,T-1$, from (2.9.1), the insurer's liability of the benefit is given by:

$$\pi(b_t) = b_1\, B(0\,,t) \left[(1+\Psi)^{t-1} - \sum_{k=1}^{t-1} \Psi(1+\Psi)^{t-k-1} \left(1 - \frac{k}{T}\right) \right]$$

$$= b_1 \left(1 - \frac{1}{\Psi T}\right) B(0\,,t) + \frac{b_1}{\Psi T}\, B(0\,,t) - \frac{b_1}{T}\, tB(0\,,t) \tag{3.9.11}$$

Consequently, (3.9.4) is rewritten in the constant periodical premium case as:

$$\pi(b_t) = b_1 \left(1 - \frac{1}{\Psi T}\right) A_{x:\overline{T}|} + \frac{b_1}{\Psi T}\, A_{x:\overline{T}|} - \frac{b_1}{T} (IA)_{x:\overline{T}|} \tag{3.9.12}$$

Where

$$(IA)_{x:\overline{T}|} = \sum_{t=0}^{T} tB(0\,,t)\; {}_{t-1/}q_x + TB(0\,,t)\; {}_T p_x$$

If P^{BO} denote the constant periodical premium due at the onset of every policy anniversary if the insured is alive. As it is in the basic contract, the time zero market value of the sequence of regular premiums is given by $P^{BO}\, \ddot{a}_{x:\overline{T}|}$. Therefore, the fair price of the bonus option in the policy become:

$$P^{BO} = P^{BC} \left(1 - \frac{1}{\Psi T}\right) b_1 \left[\frac{A_{x:\overline{T}|} - \Psi\, (IA)_{x:\overline{T}|}}{\Psi\, T\, \ddot{a}_{x:\overline{T}|}} \right] \tag{3.9.13}$$

Where P^{BC} is the fair premium of the basic contract.

3.9.3 Pricing the Whole Contract

The section seeks to ascertain an initial premium P_0 that makes the policy fair at issuance on the grounds of arbitrage-free opportunity. The insurer's liability under the whole contract is the stochastic benefit, b_t due at a random death time of the insured or at maturity, if only policy owners does not surrender their contract. Likewise, policy owner's liability is presented by the regular premiums (adjusted or constant) due at the commencement of every policy anniversary until either death, surrender or at maturity. For any stated initial premium P_0 , and the regular premiums P_t , the whole contract premium is given by the difference among the insurer's liability and the policy owner's liability, and the continuation premium. If C_t and W_t for $t = 1,2,...,T - 1$ denotes respectively the continuation and the whole contract values at the onset of (t+1)-th contract anniversary. For any specified time t , the benefit is b_{t+1}, required at time (t+1) , in the event of the insured dying within times, (t and t+1), else the whole policy value W_{t+1} is paid . Employing equation (2.9.1) and the independence between financial and mortality risk, the continuation value C_t at a specific time t on the event of survivalship of the insured is determined as:

$$C_t = B(t,t+1)\,[q_{x+t}\,b_{t+1} + p_{x+t}\,E_Q\,(W_{t+1}/\mathrm{F}_t)] - P_t \quad , \quad t = 1,2,...,T-1 \qquad (3.9.14)$$

Special case is at time $T-1$, if the contract is still in force, policy owners received b_T, at the end of the T-th year irrespective of either the insured died within the T-th year or survive the policy. Hence the policy value at maturity equals b_T. Consequently, equation (3.9.14) becomes:

$$C_{T-1} = b_T\,B\,(T-1,T) - P_{T-1} \qquad (3.9.15)$$

The value of the whole policy is then determine as the greatest of the continuation value and the surrender value since policy owners are rational and non-satiated.

$$W_t = max\{C_t, S_t\} \quad (3.9.16)$$

Now, if P_t, b_t, and W_t are calculated at time t, with the assumption of a given P_0, then, are all functions of P_0. Specifically, if

$$C_0 = f(P_0) \quad (3.9.17)$$

A contract is "fair", if and only if, the time zero market value of the insurer's liability equals that of the policy owner's liability. Hence the whole policy is "fairly priced", if:

$$f(P_0) = 0 \quad (3.9.18)$$

3.10 Existence and Uniqueness of P_0

The section introduced a proposition which presents the existence and uniqueness of the initial premium that makes the policy fair at issuance.

Axiom 3.10.1. *Suppose that the function is continuous over the interval* $[0,+\infty[$ *, such that* $f(0) > 0$. *Because* $b_1 > 0$ *and* $\lim_{P_0 \to +\infty} f(P_0) = -\infty$ *, the existence of this premium is assured.*

Proposition 3.10.1. *The function f defined by (3.9.17) is strictly monotonic in relation to* P_0.

Proof. Since b_t and C_t are independent of P_0, either in the constant premium case or in the regular adjustment case. In those instances, the regular premium P_t is strictly increasing and so does P_0, therefore C_{T-1} is strictly decreasing alongside P_0 (3.9.15). Assume now that, C_t is strictly decreasing with respect to P_0. It stand to argue from (3.9.15) that, W_t is weakly decreasing at any time t, and using backward induction $C_0 = f(P_0)$ is strictly decreasing with P_0 .□

Chapter 4

ANALYSIS AND DISCUSSION

4.1 Overview

The chapter presents numerical analysis that makes the initial premium of the policy and its components "fairly priced" at issuance, both in the constant case and in the case in which the premium is annually adjusted at a rate δ_t, according to the performance of a special investment portfolio(reference portfolio). More so, reasonable values were assigned to the parameters of the model so as to fairly close to reality and subsequently, discusses the effects of the various model parameters on premium.

4.2 Parameter Choice

In this section, model parameters are selected in a manner to model the actual conduct of a normal life insurer in Ghana fairly close to reality. Currently, the minimum interest rate which Ghana life insurers have to guarantee policy owners is fixed around 20%. However, guaranteed interest rate have to be credited for the entire life of the policy, and since the guaranteed interest rate has varied over the years, life insurers reference portfolio contains policies of higher minimum guaranteed interest rate up to 25%. The research assumed 24.6% guaranteed interest which was the bank of Ghana Treasury bill rate. Furthermore, a minimum of 50% participation level on book value earnings have to be credited to the policy's reserve.

The mortality table for the study was extracted from the Society of Actuaries (SOA) web site. Also, the study fixed the term to maturity $T = 10$, initial benefit $b_1 = 1000$, $N = 250$ and the spot rate of the zero coupon bond is 26%. The choice of N is informed by the day-to-day change in the component price of the relative reference portfolio price since there exist approximately 250 trading days in a year. Furthermore, the study also computed the premium defined by (3.3.1) for static analysis.

4.3 Computational Basics

The study denote by:

P_0 - The premium defined by relation (3.3.1)

S_t - The initial premium of the surrender option at time t

P_{BC} - The initial premium of the basic contract

P_{cBO} - The initial premium of the non-surrendable participating policy in the constant case

P_a^{BO} - The initial premium of the non-surrendable participating policy in the adjustable case

P_c^{WC} - The initial premium of the whole policy in the constant case

P_a^{WC} - The initial premium of the whole policy in the adjustable case

U_c - The initial premium of the bonus option in the constant case

U_a - The initial premium of the bonus option in the adjustable case

4.4 Numerical Results

The "fair price" of the whole contract is estimated as the summation of the individual components of the policy as:

Premium of the whole contract = Premium of the basic contract + Premium of non-surrendable participating option + Premium of the surrender option.

Whiles the premium of the non-surrendable participating policy is determined as:

Premium of non-surrendable participating policy = Premium of the basic contract + Premium of the bonus option.

With the above parameter values, the study obtained the initial premiums of the various components that makes the policy fair at inception as:

$P_0 = 29.09$, $S = 0.32$, $P^{BC} = 27.20$, $P_c^{BO} = 27.48$, $U_c = 0.28$,

$P_c^{WC} = 27.80$, $P_a^{BO} = 28.99$, $U_a = 1.89$, $P_a^{WC} = 29.31$

The estimated values revealed that, the premium of the non-surrendable participating policy is smaller in the constant premium case than in the periodical adjustment case. Also, the premium of the bonus option in the surrendable participating policy looks cheap in the constant premium case than in the periodical adjustment case. Thus it's about 1.03% and 6.95% respectively of the total premium for the constant and for the periodical adjustment cases. Furthermore, the premium defined by (3.3.1) is below the whole policy premium in the periodical adjustment case ($P_0 < P_a^{WC}$) and at the same time, above the premium in the constant case ($P_0 > P_c^{WC}$) . Detailed results of the study are tabulated in tables

1 to 4. Table1 shows the result obtained when the insured age (x) varies from 40 to 60. Table 2 present those obtained when g_t varies from 20% to 30% with a step of 1%. Table 3 report the result of a varying i_g from 20% to 30% with a step of 0.5%. In table 4 η varies from 15% to 100% with a step of 5%.

4.5 Policy components and the Insured Age

As anticipated, the premium of the basic contract and that of the one obtained from relation (3.3.1) are both increasing with the insured age. As a result, the premium of the non-surrendable participating contract increases both in the constant premium case and in the periodical adjustment case. More so, the premium of the surrender option also increases with the insured age (x), and so does the

premium of the whole contract since $P_c^{WC} = P_c^{BC} + U_c + S$ in the constant case and, $P_a^{WC} = P_a^{BC} + U_a + S$ in the periodical adjustment case. Furthermore, the incidence of the premium of the bonus option on the whole policy's premium decreases from 1.04% to 1.02% in the constant case, and 6.14% to 6.07% in the

Table 4.1: The Policy Components and the Insured Age

				Constant Premiums			Adjustable Premiums		
x	P_0	P_{BC}	S	P_c^{BO}	U_c	P_c^{WC}	P_a^{BO}	U_a	P_a^{WC}
40	26.67	24.76	0.150	25.02	0.260	25.17	26.39	1.630	26.54
41	26.82	24.92	0.160	25.18	0.260	25.34	26.56	1.640	26.72
42	26.99	25.09	0.170	25.35	0.260	25.52	26.74	1.650	26.91
43	27.18	25.27	0.190	25.54	0.270	25.73	26.94	1.670	27.13
44	27.38	25.48	0.200	25.75	0.270	25.95	27.16	1.680	27.36
45	27.61	25.70	0.220	25.97	0.270	26.19	27.40	1.700	27.62
46	27.85	25.90	0.230	26.22	0.270	26.45	27.66	1.710	27.89
47	28.12	26.22	0.250	26.49	0.270	26.74	27.95	1.730	28.20
48	28.42	26.52	0.270	26.79	0.270	27.06	28.26	1.740	28.51
49	28.74	26.84	0.290	27.12	0.280	27.41	28.61	1.770	28.90
50	29.09	27.20	0.320	27.48	0.280	27.80	28.99	1.790	29.31
51	29.48	27.59	0.350	27.87	0.280	28.22	29.40	1.810	29.75
52	29.91	28.01	0.370	28.30	0.290	28.67	29.86	1.850	30.23
53	30.38	28.48	0.410	28.78	0.300	29.19	30.36	1.880	30.77
54	30.89	28.99	0.440	29.30	0.310	29.74	30.90	1.910	31.34

55	31.45	29.56	0.480	29.86	0.310	30.34	31.50	1.940	31.98
56	32.07	30.17	0.520	30.49	0.320	31.01	32.16	1.990	32.68
57	32.75	30.85	0.570	31.17	0.320	31.74	32.88	2.030	33.45
58	33.49	31.59	0.620	31.92	0.330	32.54	33.67	2.090	33.45
59	34.30	32.40	0.680	32.74	0.340	33.42	34.54	2.140	32.22
60	35.19	33.29	0.730	33.64	0.350	34.37	35.49	2.200	36.22

periodical adjustment case. Likewise, the incidence of the premium of the surrender option on the whole policy increases from 0.60% to 2.12% in the constant case and from 0.57% to 2.02% in the periodical adjustment case. Lastly, the premium, P_c^{WC} is observed to be smaller than the premium P_0, whereas P_a^{WC} seems to be very close to P_0 for ages 40 to 44, and thereafter greater than the premium P_0.

4.6 The Influence of Age of the Insured on Premium

The graph below demonstrate the influence of the insured age on premium, both in the constant case and in the case in which the premium is periodically adjusted according to the performance of the reference portfolio, with special references made to the premium computed by insurance companies. Figure 4.1 revealed

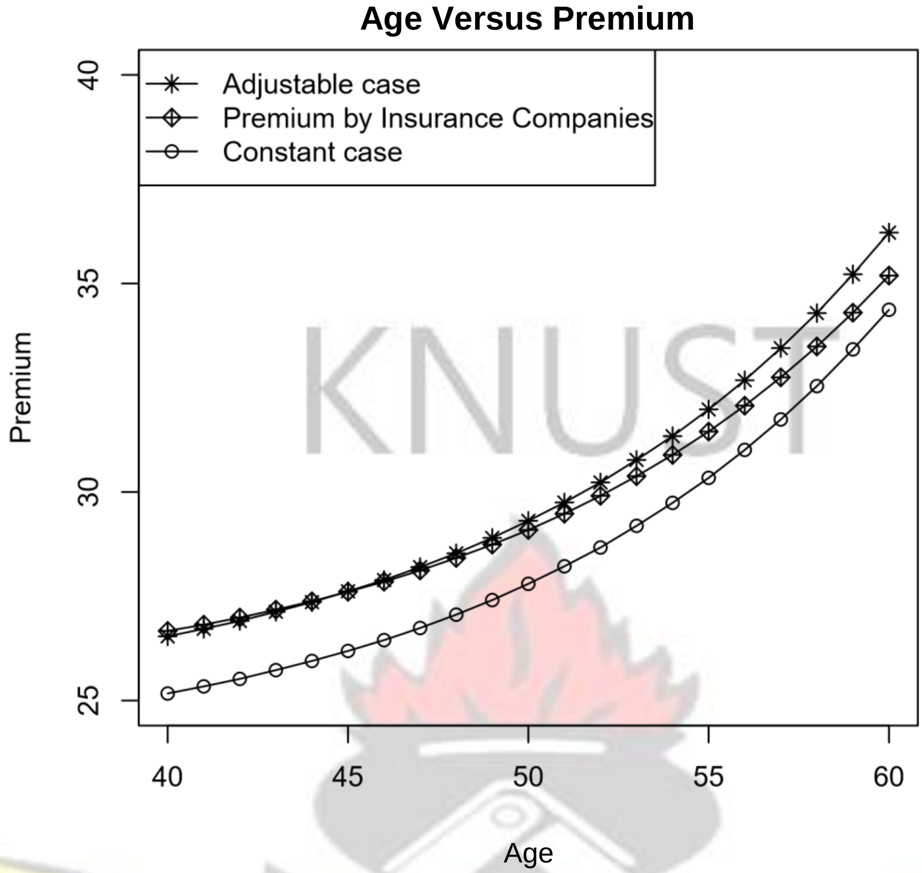

Figure 4.1: The Influence of Age of the Insured on Premium

that, the insured age has a great impact on the premium, at least for the range of values examined. For all things being equal, age is an incredible factor that influences the premium paid by the insured. Thus the younger the age of the insured the smaller the premium paid and vice versa. This is evident from figure 4.1 as an appreciable much premium is charged for older persons in all the different cases of premium for the same life cover. The reason every year inches up the premium is that, every policy anniversary put the insured closer to his/her life expectancy, and thus become expensive to insure. More so, younger persons are less likely to pass on within the term of the policy. The premium computed by relation (3.3.1) lies between the constant premiums and the periodical adjusted premiums and increases from about 1.5% to 8.9% for every year of age as compared to 1.7% to 9.5% and 1.8% to 10.0% respectively in the constant premium case and in the periodical adjustment case. To be able to hold life insurance premiums steady, rather than raising premiums every policy anniversary, insurers spread the premiums over the life time of the policy and average them into a single payment.

4.7 The Policy components and the Rate of Return

Table 4.2: The Policy Components and the Rate of Return

P_0 = 29.09								
			Constant Premiums			Adjustable Premiums		
g_t	P_{BC}	S	P_c^{BO}	U_c	P_c^{WC}	P_a^{BO}	U_a	P_a^{WC}
0.200	36.52	0.270	37.04	0.520	37.31	40.27	3.750	40.54
0.210	34.74	0.280	35.21	0.470	35.49	38.08	3.340	38.36
0.220	33.05	0.280	33.48	0.430	33.76	36.06	3.010	36.34
0.230	31.46	0.290	31.85	0.390	32.14	34.10	2.640	34.39
0.240	29.96	0.300	30.31	0.350	30.61	32.29	2.330	32.59
0.250	28.54	.0310	28.86	0.320	29.17	30.59	2.050	30.90
0.260	27.20	0.320	27.48	0.280	27.80	28.99	1.790	29.31
0.270	25.93	0.330	26.19	0.260	26.52	27.48	1.550	27.81
0.280	24.73	0.330	24.96	0.230	25.29	26.07	1.340	26.40
0.290	23.60	0.340	23.80	0.200	24.14	24.74	1.140	25.08
0.300	22.54	0.350	22.72	0.180	23.07	23.50	0.960	23.85

As anticipated, all the components presented in table 4.2 above are very sensitive to the rate of return of the reference portfolio. The premium of the basic contract is clearly decreasing with raising rate of return, and so is the premium of the non-surrendable participating policy as well as whole policy despite an increasing trend in the premium of the walk away option. A study conducted by Bacinello (2003a), on fair valuation of guaranteed life insurance participating contract embedding a surrender option produced similar results. This is due to the fact that policyholders are entitled to participate in the profit sharing of their heavily loaded premiums. The bonus option cost-low in the constant premium case and its incidence on the whole premium decreases from 1.39% to 0.78%.

Whereas the premium of the bonus option in the periodical adjustment case appears expensive with its incidence on the total premium decreasing from about

9.25% to 4.03%. Finally, there exist a value of g_t between 25% and 25.5% for which $P_c^{WC} = P_0$ in the constant premium case, and between 26% and 26.5% in the periodical adjustment case for which $P_a^{WC} = P_0$.

4.8 The Influence of the Rate of Return on Premium

In any life insurance policy, the rate of return forms an integral element both to the insurer and the policyholder. It is in the interest of both parties to get a favorable return for their policy.

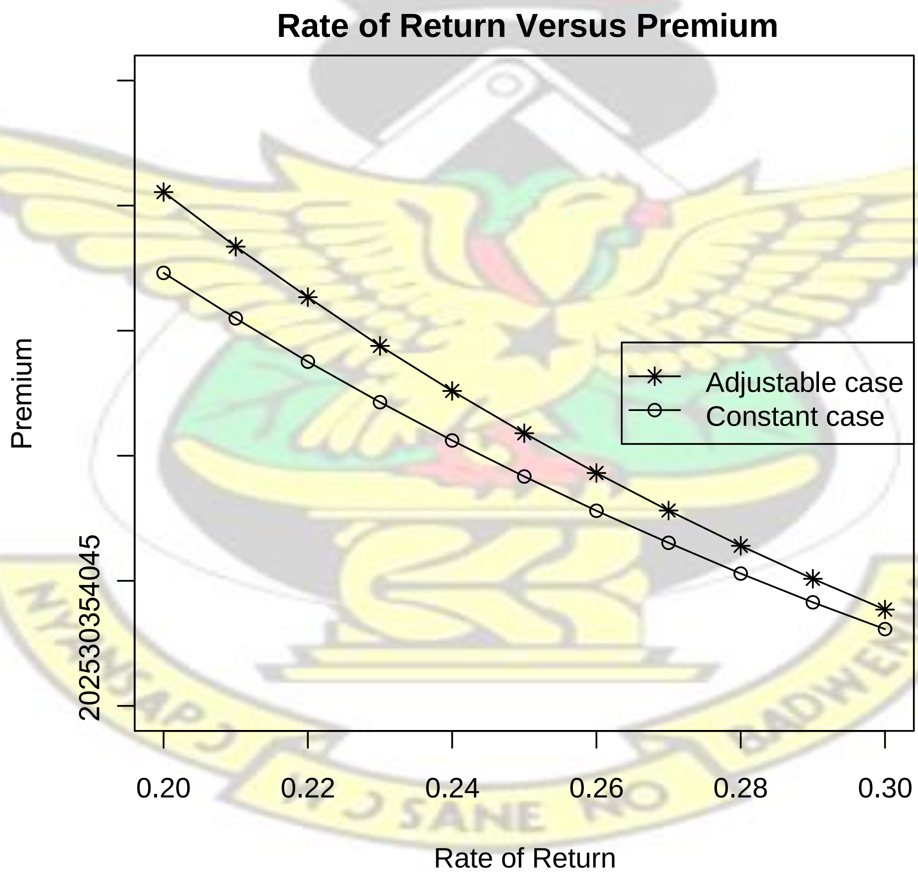

Figure 4.2: The Influence of Rate of Return on Premium

The insurer's portfolio consist of contracts with different levels of rates of return that leads to different levels of premium, since for instance a policy with 20% rate of return does not required the same premium as one with 25% rate of return. Figure 4.2 above shows how rate of return influences premium. It is observed from the figure that, the premiums both in the constant and in the periodical adjustment cases decreases, for an increasing rate of return for the same insurance cover. For a fixed term to maturity of an insurance policy, as well as, age of the insured, higher rate of return implies that more profit are being made by the insurance companies on their investment. This in effect result lesser premiums being charged by the insurer due the participating effect of the policy.

4.9 The Policy Components and the Guaranteed Interest Rate

As observed in table 4.3 below, the guarantee interest rate, has little impact on the premium of the non-surrendable participating policy as well as the surrendable participating contract, at least in the range of values considered. With increasing guaranteed interest rate, the premium of the bonus option also increases both in the constant and in the periodical adjustment cases. However, the premium of the basic contract, the non-surrendable participating policy as well as the premium of the whole contract decreases with increasing guaranteed interest rate. This seems surprising, but it is obvious, as i_g raises, the probability that, policy owners finding more profitable investment after surrendering their policy decreases. Also, there exist a level of i_g that makes the premium computed by relation (3.3.1) equals that of the whole policy's premium. That is between 25% and 25.5% in the constant premium case ($P_0 = P_c^{WC}$) and between 26% and 26.5% in the

periodical adjustment case ($P_0 = P_a^{WC}$).

Table 4.3: The Policy Components and the Guaranteed Interest Rate

$P_0 = 29.09$

			Constant Premiums			Adjustable Premiums		
i_g	P_{BC}	S	P_c^{BO}	U_c	P_c^{WC}	P_a^{BO}	U_a	P_a^{WC}
0.200	36.52	0.270	36.79	0.270	37.06	37.49	0.970	37.76
0.205	35.61	0.270	35.89	0.280	36.16	36.71	1.100	36.98
0.210	34.74	0.280	35.02	0.280	35.30	35.95	1.210	36.23
0.215	33.88	0.280	34.17	0.290	34.45	35.21	1.330	35.49
0.220	33.05	0.280	33.34	0.290	33.62	34.50	1.450	34.78
0.225	32.24	0.290	32.54	0.300	32.83	33.79	1.550	34.08
0.230	31.46	0.290	31.76	0.300	32.05	33.10	1.640	33.39
0.235	30.70	0.300	31.00	0.300	31.30	32.43	1.730	32.73
0.240	29.96	0.300	30.26	0.300	30.56	31.78	1.820	32.08
0.245	29.24	0.310	29.54	0.300	29.85	31.14	1.900	31.45
0.250	28.54	0.310	28.85	0.310	29.16	30.51	1.970	30.82
0.255	27.86	0.310	28.17	0.310	28.48	29.91	2.050	30.22
0.260	27.20	0.320	27.51	0.310	27.83	29.31	2.110	29.63
0.265	26.55	0.320	26.87	0.310	27.19	28.73	2.180	29.05
0.270	25.93	0.330	26.24	0.320	26.57	28.17	2.240	28.50
0.275	25.32	0.330	25.64	0.320	25.97	27.62	2.300	27.95
0.280	24.73	0.330	25.05	0.320	25.38	27.08	2.350	27.41
0.285	24.16	0.340	24.48	0.320	24.82	26.56	2.400	26.40
0.290	23.60	0.340	23.92	0.320	24.26	26.06	2.460	26.40
0.295	23.06	0.350	23.38	0.320	23.73	25.55	2.490	25.90
0.300	22.53	0.350	22.85	0.320	23.20	25.05	2.530	25.41

Moreover, the incidence of the surrender option on the total premium increases from 0.73% to 1.51% in the constant premium case and from 0.72% to 1.38% in the periodical adjustment case, whiles that of the bonus option on the whole policy's premium increases from 0.73% to 1.38% in the constant premium case and from 1.36% to 2.57% in the periodical adjustment case.

4.10 The Influence of Guaranteed Interest Rate on Premium

The insurer's reference portfolio consist of contracts of several interest guaranteed. Policies with different guaranteed interest rate requires different premiums, as for example, a policy with guaranteed of interest rate of 20% requires different premium from a contract of 25% interest rate guaranteed.

Figure 4.3 below illustrates how guaranteed interest rate influences premium. The respective policy's

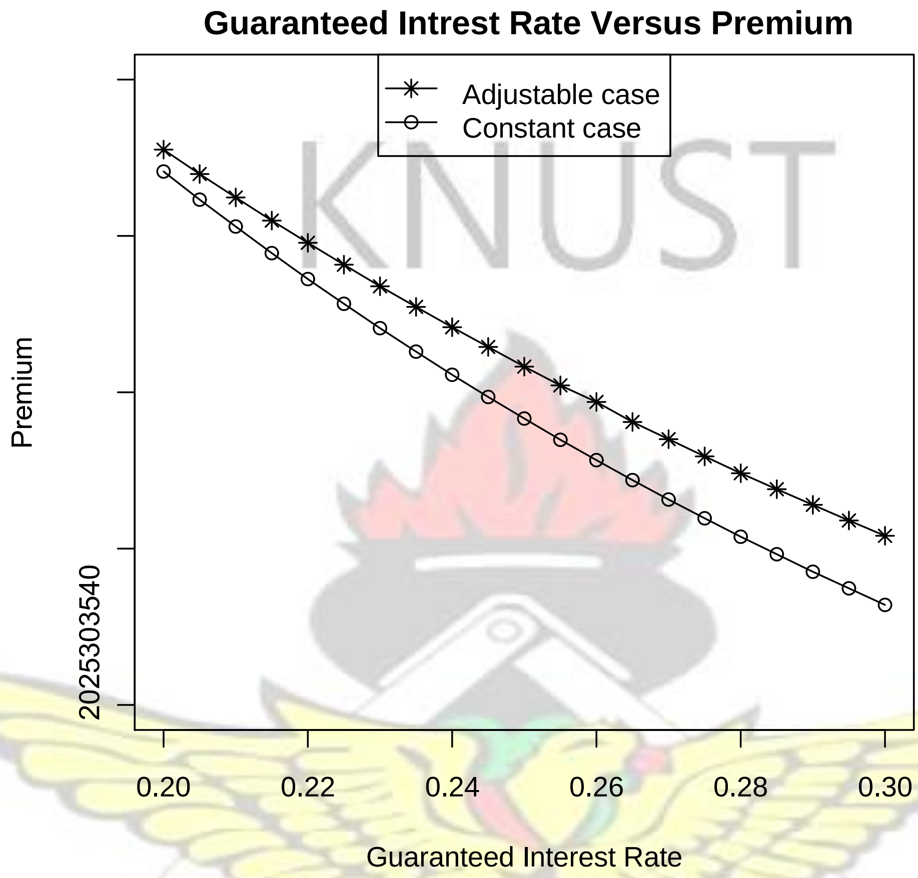

Figure 4.3: The Influence of Guaranteed Interest Rate on Premium

premiums have being estimated for the different levels of guaranteed interest rate both in the constant case and in the periodical adjustment case. It is observed from the above figure that, as the interest rate guaranteed increases, premium required by those policies decreases both in the constant and the periodical adjustment cases. Similar search by Zaglauer and Bauer (2008), on "risk neutral valuation of participating life insurance policy under stochastic interest environment" revealed the same trend. This seems surprising, but it obvious as, the possibility of policy holders surrendering their policies and finding more profitable investment decreases as guaranteed interest rate increases. Furthermore, premiums of insurance policies decrease with increasing guaranteed interest rate, since for potential clients, the choice of patronizing the money market becomes less lucrative than buying insurance cover for a changing minimum interest rate guaranteed. The above figure also revealed that, the impact of guaranteed interest rate on premium is higher in the periodical adjustment case than in the constant case. Finally, it is possible to find a "fair policy premium" both in the constant and the periodical adjustment cases. That is, a premium equals the premium computed by relation (3.3.1).

4.11 The Policy Components and the Participating Co-efficient

As far as the participating co-efficient is concerned, the findings reported in table 4.4 below revealed a very strong influence of the participating co-efficient on the premium of the surrendable participating policy as well as the bonus option both

Table 4.4: The Policy Components and the Participating Co-efficient

P_0=29.09, P^{BC} = 27.20, S = 0.32						
	Constant Premiums			Adjustable Premiums		
η	P_c^{BO}	U_c	P_c^{WC}	P_a^{BO}	U_a	P_a^{WC}
0.150	27.34	0.140	27.66	27.46	0.260	27.78
0.200	27.41	0.210	27.73	27.76	0.560	28.08
0.250	27.55	0.350	27.89	28.06	0.860	28.38
0.300	27.71	0.510	28.08	28.36	1.160	28.86
0.350	27.90	0.700	28.22	28.67	1.470	28.99
0.400	28.08	0.880	28.40	28.99	1.790	29.31
0.450	28.26	1.060	28.58	29.32	2.120	29.64
0.500	28.44	1.240	28.76	29.65	2.450	29.97
0.550	28.62	1.420	28.94	29.99	2.790	30.31
0.600	28.81	1.610	29.13	30.34	3.140	30.66
0.650	29.00	1.800	29.32	30.70	3.500	31.02
0.700	29.20	2.000	29.52	31.07	3.870	31.39
0.750	29.40	2.200	29.72	31.44	4.240	31.76
0.800	29.61	2.410	29.93	31.82	4.620	32.14
0.850	29.87	2.670	30.19	32.32	5.120	32.64
0.900	30.03	2.830	30.35	32.62	5.420	32.94
0.950	30.25	3.050	30.57	33.03	5.830	33.35
1.000	30.48	3.280	30.80	33.46	6.260	33.78

in the constant case and in the periodical adjustment case. An increasing trend is reported for all the values presented in the table, even though the trend in the non-surrendable participating policy in the periodical adjustment case beats that of the constant case. Furthermore, the participating co-efficient has no or little influence on the premiums of the basic contract and that lives its value same for different levels of the participating co-efficient. Also, the premiums of the bonus option in the periodical adjustment case is almost twice fold that of the premiums in the constant case, at least in the range of values reported for the different levels of η . The incidence of the bonus policy on the whole premium in the constant premium case increases from 0.51% to 9.80% whiles from 0.94% to 18.53% in the periodical premium case. Finally, there exist a value of η for which the whole premium equals P_0. That is between 55% and 60% in the constant premium case, and between 40% and 45% in the periodical adjustment case.

4.12 The Influence of Participating Co-efficient on Premium

Contracts with different levels of participating co-efficient presents different liabilities to the insurer, for instance a policy with η = 55% posed greater risk to the insurer than a policy with a participating co-efficient of 45%. Figure 4.4 below illustrate how participating level influences premium. All things being equal, policies with high participating level requires high premium than those with less participating co-efficient. This is depicted in the graph below, as raising participating level turns to increase the premium required both in the constant case and in the periodical adjustment case. Similar searched by (Zaglauer and Bauer, 2008; Liao et al., 2006; Bacinello, 2003a) revealed the same trend. This is due to the fact that, if a participating co-efficient is high enough, cost of insurance turned to be high and the likelihood of the insurer failing to fulfill its promised on its own increases.

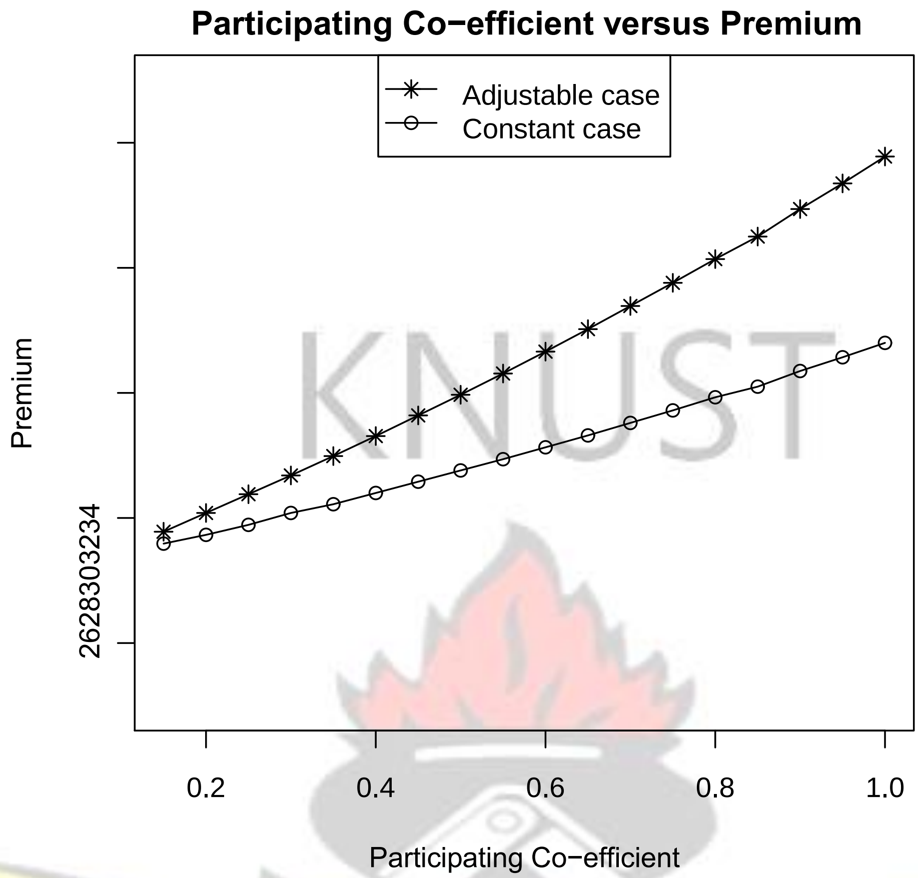

Figure 4.4: The Influence of Participating Co-efficient on Premium

Moreover, the value of the bonus option increases since there exist likelihood that a capital shot will be required. More specifically, if the participating level is increased, insurance becomes expensive, too. This result from the fact that, as life insurers raises the portion of benefit credited to the policy's reserve, the non-surrendable participating policy becomes more valuable. As reference portfolio remains high, policyholders have high funds on revenue for which they can participate.

Chapter 5

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The thesis presents a pricing model for life insurance participating policy, which particularly, permits for the stochastic evolution of interest rates. To emphasis on the basic properties, only a specific type of life insurance policy is considered, namely the endowment policy, paid by series of regular premiums. Moreover, the study presents two premium determination schemes for the insurance policy, the constant premium case and the periodical adjustment case of which the benefit and the regular premiums are periodical adjusted in relation to the performance of a reference portfolio. The premium determination mechanism and the adjustment technique are done in a manner that a minimum interest rate is guaranteed to policy owners. More so, a special bonus is periodically granted to the mathematical reserve of the policy. These introduced in to the contract some features of embedded options, of the European and the American style options that can be valued in a contingent-claim framework once an independent assumption is made between financial and mortality risks.

The life insurance policy is priced and analyzed employing techniques from modern financial mathematics, that requires the fulfillment the conditions for the risk-neutral valuation. Specially, a given underlying security and the equivalent martingale measure. Using the extended two-dimensional Cox-Ross and Rubinstein model for the evolution of the reference portfolio and the zero coupon bond, and exploiting the martingale approach, a simple closed-form formula that depicts “fair” policy is obtained, i.e., policies valued consistently with the normal norms in the financial markets and, particularly, the arbitrage-free opportunity. This relation links together the contractual parameters (i.e., the minimum interest rate guaranteed and a “participation” coefficient) with the market interest rate and the riskiness of the reference portfolio.

Besides calculating the policy's premium, the influence of the most important parameters on the premium ware also investigated. However, the influence of the other parameters of the model is not less interesting, their behavior was examined concisely, since their influence has been studied in detail by (Zaglauer and Bauer, 2008; Liao et al., 2006; Bacinello, 2003a) and most of their findings still valid under stochastic interest rate model. It turned out that due to the additional source of uncertainty in the model, for a comparable parameter choice, the premium computed by relation (3.3.1) always exceeds the premium in the constant case and below the premium in the periodical adjustment case. However, the premiums does not differ tremendously for a realistic parameter choice. With rising guaranteed interest rate, rate of return on the investment portfolio, the premium of the whole policy decreases both in the constant and the periodical adjustment cases whiles an increase in the participating co-efficient and age of the insured leads to an increase in the total premium both in the constant case and in the periodical adjustment case.

5.2 Conclusion

Undoubtedly the pricing model presented is simple and captures all the features of a participating life insurance endowment policy. The incorporation of stochastic interest rates makes the model more practical, since market rates most likely to change through out the lifetime of the insurance cover. Nevertheless, it is very problematic to select a suitable model and, within a stated interest rate model, to calibrate the parameters effectively. For example, the correlation between reference portfolio returns and money market returns can be altered significantly over time.

The model presented can be useful to an insurance company for computing premiums as well as determining the participation co-efficient, once all the residual parameters are given. However, the model is based on the assumption of a constant participating level, even though insurance companies typically reserve themselves the right to determine year by year the value of such parameter. Therefore, if there is a change in the market rate or volatility of the reference portfolio, the model can be used with the new parameters in order to update the participating coefficient, if and only if, the market rate remains above the guaranteed interest rate.

5.3 Recommendations

Empirical studies revealed that, the distribution of the lognormal returns of the reference portfolio might not be the same as the assumed normal distribution. Therefore, a more extensive and an in-depth study is required to assess the distributional properties of the insurer's reference portfolio. An area to consider for further research could be other processes such as stochastic volatility and/or pariticipating co-efficient to model the reference portfolio dynamics.

In order to obtain a more applicable model, it would be interesting to determine hedging policies for the insurance policy and for the embedded policies. Additionally, One could include more complex insurance policies such as, whole life insurances.

By and large, the thesis modeled the "fair price" of a participating life insurance policy under the influence of a stochastic interest rates. It gives insights into the interactions of the diverse factors that influences the policy's premium and helps to understand the risks that come along with the insurer's liabilities.

REFERENCES

Abdullah, M. S. (2011). Assessing the Management of Life Insurance Funds: A case of SIC Life Company Limited. Thesis.

Akanlagm, J. A. (2011). Promoting Customer Satisfaction in the Insurance Industry in Ghana: A Case Study of selected Insurance Companies in the Tamale Metropolis. Thesis.

Angus, S. M. (2004). Life insurance. In Teugels, J. L. and Sundt, B., editors, *Encyclopedia Of Actuarial Science*, volume 2, pages 979–986. John Wiley & Sons, Ltd, Chichester.

Bacinello, A. R. (2001). Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. *Astin Bulletin*, 31(02):275–297.

Bacinello, A. R. (2003a). Fair valuation of a guaranteed life insurance participating contract embedding a surrender option. *Journal of risk and insurance*, 70(3):461–487.

Bacinello, A. R. (2003b). Pricing guaranteed life insurance participating policies with annual premiums and surrender option. *North American Actuarial Journal*, 7(3):1–17.

Ballotta, L., Haberman, S., and Wang, N. (2006). Guarantees in With-Profit and Unitized With-Profit Life Insurance Contracts: Fair Valuation Problem in Presence of the Default Option. *Journal of Risk and Insurance*, 73(1):97–121.

Bauer, D., Kiesel, R., Kling, A., and Ru\s s, J. (2006). Risk-neutral valuation of participating life insurance contracts. *Insurance: Mathematics and Economics*, 39(2):171–183.

Bauer, D., Kling, A., and Russ, J. (2008). A universal pricing framework for guaranteed minimum benefits in variable annuities. *Astin Bulletin*, 38(02):621–651.

Bernard, C., Le Courtois, O., and Quittard-Pinon, F. (2005). Market value of life insurance contracts under stochastic interest rates and default risk. *Insurance: Mathematics and Economics*, 36(3):499–516.

Bernard, C. and Lemieux, C. (2008). Fast simulation of equity-linked life insurance contracts with a surrender option. In *Proceedings of the 40th Conference on Winter Simulation*, pages 444–452. Winter Simulation Conference.

Bu¨hlmann, H. (1985). Premium calculation from top down. *Astin Bulletin*, 15(02):89–101.

Biener, C. (2013). Pricing in microinsurance markets. *World Development*, 41:132–144.

Bingham, N. H. and Kiesel, R. (2013). *Risk-neutral valuation: Pricing and hedging of financial derivatives*. Springer Science & Business Media.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *The journal of political economy*, pages 637–654.

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A., and Nesbitt, C. J. (1986). *Actuarial mathematics*, volume 2. Society of Actuaries Itasca, Ill.

Boyle, P. P. and Schwartz, E. S. (1977). Equilibrium prices of guarantees under equity-linked contracts. *Journal of Risk and Insurance*, pages 639–660.

Brennan, M. J. and Schwartz, E. S. (1976). The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics*, 3(3):195–213.

Briys, E. and De Varenne, F. (1997). On the risk of insurance liabilities: debunking some common pitfalls. *Journal of Risk and Insurance*, pages 673–694.

Calidonio-Aguilar, P. R. and Xu, C. (2011). Design of life insurance participating policies with variable guarantees and annual premiums. *International Journal of Innovative Computing Information and Control*, 7(8):4741–4753.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

Cox, J. C., Ross, S. A., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of financial Economics*, 7(3):229–263.

Dickson, D. C., Hardy, M., Hardy, M. R., and Waters, H. R. (2013). *Actuarial mathematics for life contingent risks*. Cambridge University Press.

Fischer, T. (2004). *Valuation and risk management in life insurance*. PhD thesis, TU Darmstadt.

Frank, D. K. and Amankwah, T. (2011). An Analysis and Assessment of Customer Satisfaction with Service Quality in Insurance Industry in Ghana. Thesis, Lulea University of Tecnology, Sweden.

Grosen, A. and Jørgensen, P. L. (2000). Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender options, and bonus policies. *Insurance: Mathematics and Economics*, 26(1):37–57.

Grosen, A. and Jørgensen, P. L. (2002). Life insurance liabilities at market value: an analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. *Journal of risk and insurance*, 69(1):63–91.

Ho, T. S. and Lee, S.-B. (1986). Term structure movements and pricing interest rate contingent claims. *Journal of Finance*, pages 1011–1029.

Hull, J. C. (2006). *Options, futures, and other derivatives*. Pearson Education India.

Jensen, B., Jørgensen, P. L., and Grosen, A. (2001). A finite difference approach to the valuation of path dependent life insurance liabilities. *The Geneva Papers on Risk and Insurance Theory*, 26(1):57–84.

Jørgensen, P. L. (2001). Life Insurance Contracts with Embedded Options: Valuation, Risk Management, and Regulation. *Risk Management, and Regulation (December 13, 2012). Journal of Risk Finance*, 3(1):19–30.

Kingston, C. (2007). "Marine Insurance in Britain and America, 1720–1844: A Comparative Institutional Analysis,". *Journal of Economic History*, Vol. 67(Issue 2):pp 379–409.

Liao, S.-L., Chang, C.-K., and Lin, S.-K. (2006). Fair Valuation of Participating Policies in Stochastic Interest Rate Models: Two-dimensional Cox-RossRubinstein Approaches.

Mehr, R. I. and Cammack, E. (1976). *Principles of insurance*. Irwin series in insurance and security. R D Irwin, 6th edition edition.

Merton, R. C. (1973). Theory of rational option pricing, The Bell Journal of Economics and Management Science 4 (1): 141–183. *URL: http://www. jstor. org/stable/3003143*.

Miltersen, K. R. and Persson, S.-A. (2003). Guaranteed investment contracts: distributed and undistributed excess return. *Scandinavian Actuarial Journal*, 2003(4):257–279.

NIC (2010). National Insrurance Commision Annual Report. Technical report, Ghana.

NIC (2011). National Insrurance Commision Annual Report. Technical report, Ghana.

Nielsen, J. A. and Sandmann, K. (1995). Equity-linked life insurance: A model with stochastic interest rates. *Insurance: Mathematics and Economics*, 16(3):225–253.

Obuba, C. W. (2014). The effects of product pricing on the growth of microinsurance by insurance underwrtters in Kenya. Thesis, University of Nairobi, Kenya.

Poku, S. (2009). Exploring the implications of seperating Life Insurance from General Insurance Business. Thesis.

Tanskanen, A. J. and Lukkarinen, J. (2003). Fair valuation of path-dependent participating life insurance contracts. *Insurance: Mathematics and Economics*, 33(3):595–609.

Trenerry, C. F. (1926). *The origin and early history of insurance: including the contract of bottomry*. The Lawbook Exchange, Ltd.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of financial economics*, 5(2):177–188.

Vaughan, E. J. (1997). *Risk Management. New York: John Willey and Sons*. Inc.

Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, 17(1):43– 54.

Young, V. R. (2004). Premium principles. *Encyclopedia of Actuarial Science*.

Zaglauer, K. and Bauer, D. (2008). Risk-neutral valuation of participating life insurance contracts in a stochastic interest rate environment. *Insurance: Mathematics and Economics*, 43(1):29–40.

APPENDICES

Appendix A: Markovian Property

Let $L_t(X)$ represent the conditional distribution of the random variable X given information set $\mathcal{F}_t = \sigma\left\{\left(Z_1^T(u), Z_2^T(u)\right), t \geq u\right\}$ from (3.7.2) and (3.8.9), the conditional distribution of $(\log R(t), \log B(t,t+1))$ given information set F_{t-1} is:

$$L_{t-1}\left(\begin{array}{c}\log R(t)\\ \log B(t,t+1)\end{array}\right)$$

$$= N_2 \left[\begin{pmatrix} -\log B(t-1\,,t) - \frac{1}{2}\sigma_R^2 \\ \log B(t-1\,,t)\frac{B(t-1\,,0)\,B(0\,,t+1)}{B(0\,,t)^2} - \sigma^2(t-\frac{1}{2}) \end{pmatrix} \begin{pmatrix} \sigma_R^2 & \sigma\left(\sigma_1 - \frac{1}{2\sigma}\right) \\ \sigma\left(\sigma_1 - \frac{1}{2}\right) & \sigma^2 \end{pmatrix} \right]$$

Where,

$$Cov\left(\log R(t)\,, \log B(t\,,t+1)\right) = \int_{t-1}^{t} \sigma\left(\sigma_1 - \sigma\left(t-u\right)\right)\, du = \sigma\left(\sigma_1 - \frac{1}{2}\,\sigma\right)$$

Particularly, the conditional distribution of ($\log R(t)$,$\log B(t,t+1)$) given information set F_{t-1} depends on the bond price $B(t,t+1)$ which is F_{t-1} measurable. As a result, the dynamics of the bond price $B(t,t+1)$ and the relative price $R(t)$ are Markovian stochastic processes.

Appendix B: CRR model Parameters

From appendix A, the conditional distribution of the bond price $B(t,t+1)$ given information set F_{t-1} (or bond price $B(t-1,t)$) and the relative price $R(t)$ is:

$$\mathcal{L}_{t-1} \log\left(\log B(t\,,t+1)/R(t)\right) = N(\mu_t\,, \sigma_t^2) = \log \alpha_t + r_{R(t)} - \frac{1}{2}\,\sigma_t^2 + \sigma_t\,\tilde{Z}_2(1)$$

Where,

$$\mu_t = E\,\left(\log B(t\,,t+1)\right) + \frac{Cov\left(\log R(t)\,, \log B(t\,,t+1)\right)}{Var\left(\log R(t)\right)}\left[\log R(t) - E\,\left(\log R(t)\right)\right]$$

$$r_{R(t)} = -\sigma\,\left(t - \frac{1}{2}\right) + \frac{\sigma\,\left(\sigma_1 - \frac{1}{2}\,\sigma\right)}{\sigma_R}\left(\log R(t) + \log B(t-1\,,t) + \frac{1}{2}\,\sigma_R\right) + \frac{1}{2}\,\sigma_t^2$$

$$\alpha_t = B(t-1\,,t)\,\frac{B(0\,,t+1)\,B(0\,,t)}{B(0\,,t)^2}$$

$\sigma_t^2 = \sigma^2\,(1 - \rho^2)$

$$\rho = \frac{\sigma_1 - \frac{1}{2}\,\sigma}{\sigma_R}$$

Appendix C: SOA 2008 Life Table

Age (x)	l_x	p_x	e_x
0	10000000	0.9949901	71.3469199

1	9949901	0.9949648	70.7061635
2	9899801	0.9949394	70.0639824
3	9849702	0.9949136	69.4203550
4	9799602	0.9948876	68.7752590
5	9749503	0.9990991	68.1286718
6	9740720	0.9990983	67.1901020
7	9731937	0.9990975	66.2507405
8	9723154	0.9990967	65.3105853
9	9714371	0.9990959	64.3696342
10	9705588	0.9991375	63.4278850
11	9697217	0.9991367	62.4826410
12	9688845	0.9991360	61.5366275
13	9680474	0.9991352	60.5898427
14	9672102	0.9991345	59.6422844
15	9663731	0.9990495	58.6939507
16	9654545	0.9990486	57.7497950
17	9645359	0.9990476	56.8047932
18	9636174	0.9990467	55.8589431
19	9626988	0.9990458	54.9122421
20	9617802	0.9989700	53.9646879

21	9607896	0.9989383	53.0203269
22	9597695	0.9989033	52.0766801
23	9587169	0.9988650	51.1338564
24	9576288	0.9988230	50.1919569
25	9565017	0.9987770	49.2511010

2695533190.998726548.3114087

2795411530.998671247.3730110

2895284750.998610546.4360426

2995152350.998544045.5006562

3095013810.998471144.5670008

3194868540.998391143.6352453

3294715910.998303542.7055613

3394555220.998207341.7781364

3494385710.998102040.8531670

3594206570.997986439.9308520

3694016880.997859739.0114171

3793815660.997720838.0950905

3893601840.997568737.1821133

3993374270.997401736.2727328

4093131660.997218835.3672241

4192872640.997018234.4658626

4292595710.996798333.5689412

4392299250.996557332.6767627

4491981490.996292931.7896481

4591640510.996003430.9079324

4691274260.995685930.0319546

4790880490.995337829.1620780

4890456790.994956528.2986733

4990000570.994538327.4421215

5089509010.994080126.5928265

5188979130.993577925.7511895

5288407700.993027524.9176343

5387791280.992424424.0925917

5487126210.991763723.2765003

5586408610.991039622.4698053

5685634350.990246121.6729652

5784799080.989377020.8864437

5883898260.988424920.1107025

5982927130.987381819.3462118

6081880740.986239618.5934455

6180754030.984988517.8528681

6279541790.983618717.1249510

6378238790.982118816.4101524

6476839790.980476915.7089274

6575339640.978679715.0217208

6673733380.976713014.3489646

6772016350.974560913.6910758

6870184320.972206813.0484557

6968233670.969632012.4214822

7066161550.966816711.8105118

7163966090.963739211.2158752

7261646630.960375910.6378735

7359203940.956701710.0767805

7456640510.95268939.5328344

7553960810.94830909.0062360

7651171520.94352918.4971537

7748281820.93831598.0057142

7845303600.93263297.5320030

7942251630.92644127.0760636

8039143650.91969916.6378984

8136000380.91236316.2174674

8232845420.90438675.8146850

8329704960.89572045.4294236

84	2660734	0.8863141	5.0615157
85	2358246	0.8761130	4.7107473
86	2066090	0.8650635	4.3768718
87	1787299	0.8531074	4.0595961
88	1524758	0.8401878	3.7585991
89	1281083	0.8262470	3.4735228
90	1058491	0.8112265	3.2039762
91	858676	0.7950694	2.9495456
92	682707	0.7777260	2.7097964
93	530959	0.7591396	2.4842558
94	403072	0.7392749	2.2724625
95	297981	0.7180894	2.0739074
96	213977	0.6955514	1.8880908
97	148832	0.6716634	1.7145238
98	99965	0.6463962	1.5526534
99	64617	0.6197905	1.4020149
100	40049	0.5918999	1.2620790
101	23705	0.5627083	1.1322506
102	13339	0.5323488	1.0121448
103	7101	0.5010562	0.9012815
104	3558	0.4688027	0.7987634
105	1668	0.4358513	0.7038369
106	727	0.4016506	0.6148556
107	292	0.3698630	0.5308219
108	108	0.3333333	0.4351852
109	36	0.3055556	0.3055556