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NUMERICAL SOLUTION TO FRACTIONAL CATTANEO HEAT EQUATION IN A SEMI-INFINITE MEDIUM

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgment had been made in the text.

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Dedication

This thesis is dedicated to my late father, Mr. Cheyuo Martin Guo and my mother, Mrs Grace Cheyuo

Acknowledgment

First and foremost, I give thanks to the Almighty God for giving me the strength and knowledge to go through this studies. My sincere gratitude goes to my supervisor, Dr. Amoako-Yirenkyi Peter for his patience, guidance and the interest shown in my work which helped tremendously in bringing my research work to a successful completion. I am also grateful to members of Cheyuo's family especially Edmund C., Patience C., Gladys C. and Maurice C. for their support in diverse ways during my studies.

Abstract

In this study, a detailed review of the article published by Qi et al. (2013) on fractional Cattaneo heat equation in a semi-infinite medium has been made. In reviewing this article, two fractional Cattaneo heat equations modeling the heat flux and temperature distributions have been established and their exact solutions proofed in detail forms. Firstly, the solution of the fractional Cattaneo heat flux equation is established using Laplace transform. secondly, the exact solution of the fractional Cattaneo heat equation modeling temperature distribution is established in a series form through Fox-function using Laplace transform (and the inverse Laplace transform). In addition to the review, an implicit finite difference scheme has been used to solve the three c lasses of generalized fractional Cattaneo heat equations (GCE's) in a semi-infinite medium. Three numerical examples were provided using both the analytical solutions and finite difference solutions to demonstrate the effects of fractional derivatives of orders α and β on temperature distributions. Graphical representation of the solutions were presented using Matlab software. Finally, a comparison and discussion of the analytical and finite difference scheme solutions from the graphs of the various numerical examples have been made.

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Chapter 1

Introduction

The Fourier heat diffusion equation is an equation which is not only relevant in modeling heat diffusion in Physics, but also has wide spread application in several field of studies such as engineering, economics etc. Although the equation is mathematically correct, it predicts an infinite speed of heat transmission in a medium due to its parabolic nature. To make the law applicable to most physical processes, a relaxation time was introduced by Cattaneo and Vernotte to modify the Fourier law thus producing the Non-Fourier heat equation. This equation is called the Cattaneo-Vernotte heat equation. By its hyperbolic nature, this equation predict a finite speed of heat propagation. According to Turut and Guzel (2013), the Non-Fourier effects of heat conduction are of both fundamental interest and great potential value in practical engineering. The replacement of the integer orders of the derivatives of the generalized Cattaneo equation with fractional orders produces the generalized fractional Cattaneo heat equation. Quiet often, numerical approximation techniques are employed in solving many partial differential equations including the fractional Cattaneo heat equation.

1.1 Problem Statement

In solving problems using partial differential equations, the exact solutions or approximate solutions are often required. The exact or approximate solution derived for a particular partial differential equation usually depends to a large extend on the medium in which it is solved. The type of medium used influences the initial and boundary conditions to the problem. In terms of efficiency and accuracy, exact solutions of partial differential equations are often preferred to approximate solutions. However, establishing exact solutions of partial fractional differential equations such as the fractional Cattaneo heat equation can be stressful, time consuming and complex. Sometimes, special functions with complex properties are needed to establish the exact solution of a partial fractional differential equation. A typical example is the exact solution established by Qi et al. (2013) for fractional Cattaneo heat equation in a semi-infinite medium. In their work, an H-function which is a special function was used to establish the exact solution. Although Qi et al. (2013) established an exact solution for fractional Cattaneo heat equation in a semi-infinite medium, establishing and computing exact analytical solutions of such partial fractional differential equations is often a complex and difficult task. In view of such difficulties in establishing exact solutions, Turut and Guzel (2013) stated that many partial fractional differential equations are solved using numerical approximation techniques.

1.2 Objectives of Study

The objectives of this study includes:

(1) To establish in detail the exact solution of the generalized fractional Cattaneo heat equation in a semi-infinite provided by Qi et al. (2013)

(2) To solve the fractional Cattaneo heat equation in a semi-infinite medium using a numerical scheme.

(3) To compare the results of the analytical solution to the result of the numerical scheme method of solving the fractional Cattaneo heat equation in a semi infinite medium .

(4) To discuss the effects of fractional derivatives of order α and β on the temperature distribution.

1.3 Justification of Study

Considering the great potential of fractional models and their successful application in many fields, and the fact that the exact solutions are often difficult to be established or used for computations, a numerical scheme approximation of the partial differential equation often provides an easier way of approximating the solution. To the best of my knowledge, no researcher has solved the fractional Cattaneo heat equation in a semi-infinite medium with Neumann boundary conditions using a numerical scheme approximation technique. Hence, this study seeks to use one of the numerical schemes (implicit finite difference scheme) to solve the fractional Cattaneo heat equation in a semi-infinite medium with Neumann boundary conditions.

1.4 Methodology

In this work, a review of the article published by Qi et al. (2013) was made first. This was followed by using the implicit finite difference scheme to discretize the three classes of the generalized fractional Cattaneo heat equations(GCEs). Finally, graphical representations of the exact solutions and implicit finite difference solutions of the GCEs was made for the purpose of comparisons and discussions.

1.5 Outline of Study

Basically, this study is organized into five chapters. Chapter one contains the background to the problem, statement of problem and the justification of the study. In chapter two, related studies by other researchers which are relevant to this study will be cited. Furthermore, chapter three will provide a detailed review of the article published Qi et al. (2013). Firstly, the exact solution of the heat flux modeling equation will be established. Secondly, a detailed proof of the exact solutions of the fractional Cattaneo heat equation in a semi-infinite medium will be made. The final part of chapter three will use the implicit finite difference scheme to discretize the three classes of generalized Cattaneo heat equations (GCE's) to serve as numerical examples. Subsequently, chapter four will provide comparisons, analyses and discussion of the graphical representations of the analytical solutions and finite difference solutions of the fractional Cattaneo heat equation in a semi-infinite medium. Lastly, the conclusions and suggestions from this study will be made in chapter five.

Chapter 2

RELATED STUDIES

2.0.1 INTRODUCTION

This chapter presents a summary of related research works on fractional calculus and differential equations from other researchers. It cites recent works of other researchers that are related to this study. Research works on key functions of this study such as Mittag-Leffler function and Fox-function will be cited

2.0.2 Related Works from other Researchers

The Fourier heat diffusion law is an equation which establishes a linear relationship between the heat flux and the temperature gradient within a medium, i.e

$$q(x,t) = -D\partial_x u(x,t) \tag{2.1}$$

where $q(x,t)$ is the heat flux, $u(x,t)$ is the temperature function and D is diffusion constant. Due to the parabolic nature of the law, it predicts infinite speed of heat transmission. This makes the law not practically applicable to most physical phenomena. To make the Fourier heat conduction equation applicable to most physical phenomena, a relaxation of the flux is introduced. (shooda, 2009) stated the relaxed Cattaneo equation as:

$$\partial_t U(x,t) + \tau\partial_t^2 U(x,t) = D\partial_{xx} U(x,t) \tag{2.2}$$

where τ is the relaxation time, D is thermal diffusivity constant and $U(x,t)$ is temperature function. Such an equation is hyperbolic and it predicts a finite speed of heat transmission. This equation possess both diffusion and wave-like properties of heat transmission. Qi et al. (2013) gave the exact solution of (2.2) as:

$$G(x,t) = \sqrt{\frac{D}{\tau}} e^{-t/2\tau} I_0 \left(\frac{1}{2\tau} \sqrt{t^2 - \frac{\tau}{D} x^2} \right) u \left(t - x \sqrt{\frac{\tau}{D}} \right) \tag{2.3}$$

where $I_0(.)$ is the modified Bessel function of order zero and $u(.)$ denotes the unit step function. The replacement of the integer orders of (2.2) with fractional orders produces the generalized fractional Cattaneo heat equation:

$$\partial_t^{\beta} U(x,t) + \tau \partial_t^{\alpha} U(x,t) = D \partial_{xx} U(x,t) \tag{2.4}$$

(shooda, 2009) studied three Generalized Cattaneo Equations(GCEs) and concluded that GCEI and GCEIII models subdiffusion while GCEII models superdiffusion.

The GCEs include:

$$\partial_t^{\gamma} U(x,t) + \tau^{\gamma} \partial_t^{2\gamma} U(x,t) = D \partial_{xx} U(x,t) \quad (GCEI) \tag{2.5}$$

$$\partial_t^{2-\gamma} U(x,t) + \tau^{\gamma} \partial_t^{2} U(x,t) = D \partial_{xx} U(x,t) \quad (GCEII) \tag{2.6}$$

$$\partial_t^{\gamma} U(x,t) + \tau \partial_t^{1+\gamma} U(x,t) = D \partial_{xx} U(x,t) \quad (GCEIII) \tag{2.7}$$

Although the use of fractional calculus has been on the ascendancy in recent times, the concept of fractional calculus is not new. The concept has existed for several centuries. The study of fractional calculus began in 1695 when L'Hospital inquired in a letter to Leibniz what could happen if the order of a derivative is a fraction. Since 1695, fractional calculus has drawn the attention of famous mathematicians such as Euler, Laplace, Fourier, Abel, Liouville Rieman and Laurent. Rahimy (2010) stated three major definitions of fractional derivatives, namely:

Caputo, Rieman-Liouville and Grundwald-Letnikov fractional derivatives. He further added that, for zero initial conditions all the three definitions coincide. This allows a numerical solution of initial value problems for differential equations of non-integer order independent of the chosen definition of fractional derivative. Many researchers or engineers resort to the Caputo derivatives or use the Riemann-Liouville derivatives but avoid the problem of initial values of fractional derivative by treating only the cases with zero initial conditions. According to Ben and Cresson (2005), fractional differential equations associated with alpha-derivatives appear in many problems such the classical Schrodinger equation. Kilbas et al. (2006) in their monograph provides an extensive work on the properties of different kinds of fractional derivatives and integrals. In addition, Podlubny (1999) also provided a detailed account of the properties of fractional derivatives and some analytic solution methods of fractional derivatives.

To stress on the relevance and wide spread use of fractional differential equations in modeling, Turut and Guzel (2013), stated that fractional order partial differential equations are increasingly being used to model problems in fluid flow, finance, physical and biological processes and systems. They stated that fractional ordinary differential equations, fractional partial differential equations and fractional integral equations have received wide research and application. Gutierrez et al. (2010) further added that fractional order calculus and differential equations are tools used to better described many real systems as it is well suited in analyzing problems of fractal dimensions with long term memory and chaotic behavior. With these characteristics, engineers and almost all branches of science tend to apply it in solving problems. To demonstrate the practical uses of fractional calculus, Dizielinski et al. (2010) presented some practical application of fractional order system models namely: ultra capacitor fractional order modeling and fractional order beam heating modeling. In categorizing the media fractional partial differential equation may be best suited for, Dominik et al. (2011) pointed out that heat transfer in a solid(beam) can be described by an integer order partial differential equation while in a heterogeneous media, it can be described by sub-or hyper diffusion which often result in a fractional partial differential equation. Ting-Hui and Xiao-Yun (2011) also studied fractional heat conduction equation in spherical coordinate system. Ghazizadeh and Maerafat (2010) formulated a heat conduction constitutive equation using the recently introduced fractional Taylor formula by expanding the single-phase lag model. The equation has been shown to be capable of modeling Diffusion-to-Thermal wave behavior of heat propagation when the order of differentiation is changed. With courage, Emillia (2001) extended the study of fractional calculus into abstract mathematics by studying fractional calculus in Banach spaces.

According to Housbold et al. (2009), the Mittag-Leffler function, a fractional exponential function, arises naturally in the solution of fractional order differential equations or fractional order integral equations and especially in investigations of fractional generalizations of kinetic equation, random walks, super diffusive transport, Levy flights and complex systems. In recent decades, the interest in Mittag-Leffler function and Mittag-Leffler type functions is considerably on the rise among engineers and scientists due to its vast potential applications in several applied problems such as fluid flow, rheology, diffusive transport akin to diffusion, electric networks, probability, statistical distribution theory etc. (Mainardi and Pagnini, 2007) presented the fundamental solution of the fractional diffusion equation of distributed order based on its Mellin-Barnes representation. They also provided a series expansion to point out the distribution of time-scales related to the distribution of fractional orders. In a related study, Mainardi et al. (2005) presented the fundamental solution of the Cauchy problem for space-time fractional diffusion equation in terms of a special function(Fox H-function). The Fox-function, introduced by Charles Fox in 1961, is a special function of very general nature. It has been recognized to play a fundamental role in probability theory and fractional calculus as well as in their applications, including non-Gaussian stochastic processes, anomalous relaxation and diffusion.

Mostly, the solutions of fractional partial differential equations are achieved through numerical and approximations techniques since they do not often have an exact analytic solution. Over the years, different researchers have used different numerical methods to solve fractional differential equations. Ahmad et al. (2010) presented the Homotopy Analysis Method(HAM) to obtain symbolic approximate solution for linear and non-linear differential equations of fractional order. They stated that their results show that the Adomian Decomposition Method, Variational Iteration Method and Homotopy Perturbation Method are special cases of the Homotopy Analysis Method(HAM). Marek (2011) constructed a numerical scheme to solve two term sequential fractional differential equations with the orders of Caputo derivatives in the range(0,1). The proposed method is based on the existence and uniqueness theorem and the transformation of sequential fractional order differential equation into its equivalent fractional integral equations. Among the several methods of solving fractional differential equations, Beheshti et al. (2012) solved fractional differential equations using Jacobi polynomials. The method is based on expanding the derivative of the unknown solution in terms of Jacobi polynomials. Also, Saeedi (2012) presented an operational method known as the Haar wavelet method for approximating the solution of a non-linear fractional integro-differential equation of second kind. The technique of this method is based on reducing the main equation to system of algebraic equation by expanding the solution of the integro-differential equation as Haar wavelets with unknown coefficients. Furthermore, Diethelm and Neville (2002) discussed the existence, uniqueness and structural stability of solutions of nonlinear differential equations of fractional order. They investigated the dependency of the solution on the order of the differential equation and initial conditions. (Mariusz, 2009) presented numerical solution of Cattaneo-Vernotte equation using finite difference scheme. The theoretical models were verified experimentally. Xiao-Jun (2012a) from the geometric point of view explored the interpretation of local fractional derivative and integral equations. He investigated the Fourier law of heat conduction and heat conduction equation in fractal orthogonal system based on Cantor sets. Xiao-Jun (2012b) investigated local fractional Volterra/Fredholm integral equations, local fractional non-linear integral equations and local fractional singular equations. Manuel and Coito (2004) established a relation showing that the Grundwald-Letnikov and generalized Cauchy derivatives are equal. They presented an integral representation for both direct and reversed fractional differences. Miller and Stephen (2009) showed that polynomials and exponential functions can be deformed into their derivatives using μ- fractional derivative for $0 < \mu < 1$.

2.0.3 Concluding Remark:

In conclusion, this chapter has provided some useful relevant research related studies in fractional differential equations. The next chapter will make use of some of the functions mentioned in this chapter. Functions such as the H-function(Fox-function) and the generalized Mittag-Leffler function will be used extensively in establishing the exact solution of the generalized fractional Cattaneo heat equation in a semi-infinite medium with Neumann boundary conditions. The implicit finite difference scheme will also be used to solve the fractional Cattaneo heat equation in a semi-infinite medium with Neumann boundary conditions.

Chapter 3

Methodology

3.1 INTRODUCTION

This chapter discusses the modeling of the heat flux and temperature distribution in a semi-infinite medium with Neumann boundary conditions. It also provides a vivid proof of the exact solution of the heat flux and the temperature distribution function established by Qi et al. (2013). The short time and long time temperature distribution functions was established. Relevant theorems and definitions were applied in establishing the exact solutions. Implicit finite difference scheme was also applied to solve the three generalized Cattaneo heat equations(GCEs).

3.2 MODELING THE HEAT FLUX

The heat flux which is the amount of heat energy flowing per unit area in a region of space is modeled classically by the Fourier law(equation 2.1). The infinite speed of heat transmission predicted by the Fourier law does not make it applicable to real situations. Hence, the constitutive heat flux model proposed by Cattaneo and Vernotte is one of the most widely used(Qi et al. (2013)):

$$q(x,t) + \tau\partial_t q(x,t) = -\lambda\partial_x T(x,t) \tag{3.1}$$

where $q(x,t)$, $T(x,t)$, τ and λ are heat flux vector, the temperature, relaxation time and the thermal conductivity respectively. τ has the dimension $\lceil\tau\rceil = s^{\alpha-\beta}$

let $q_v(t) = -\partial_x T(x,t)$, thus (3.1) becomes

$q(x,t) + \tau\partial_t q(x,t) = \lambda q_v(t)$

where $q_v(t)$ is the temperature gradient with respect to space.

Let $q(x,t) = \partial_t^0 q(x,t)$ in (3.1). Replacing the integer order derivatives with fractional order derivatives $\beta - 1$ and $\alpha - 1$ respectively produces the fractional differential heat flux equation

$$\partial_t^{\beta-1} q(x,t) + \tau \partial_t^{\alpha-1} q(x,t) = \lambda q_v(t) \tag{3.2}$$

3.2.1 Fractional Derivatives

A fractional derivative is a derivative whose order is a fraction. There are three kinds of fractional derivatives, namely: Caputo derivatives, Grundwald Letnikov derivatives and Riemann Liouville derivatives. For easier treatment of the boundary and initial conditions in modeling of physical phenomenons, the Caputo derivatives are usually preferred to other types of fractional derivatives because their initial conditions are stated in integer order derivatives and they are easy to solve. Three major kinds of fractional derivatives are defined below.

Caputo derivative(Qi et al. (2013)):

$$\partial_t^{\gamma} f(x,t) \overset{def}{=} \begin{cases} \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{f'(x,t)}{(t-t')^{\gamma}} dt', & 0 < \gamma < 1 \\ \frac{1}{\Gamma(-\gamma)} \int_0^t \frac{f(x,t')}{(t-t')^{1+\gamma}} dt', & \gamma < 0 \end{cases} \tag{3.3}$$

Riemann Liouville derivative(Rahimy (2010)):

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \quad (n-1 \leq \alpha < n)$$

Grundwald-Letnikov derivative(Rahimy (2010)):

$${}_aD_t^{\alpha} f(t) = \lim_{h \longrightarrow 0} h^{-\alpha} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{\alpha}{j} f(t-jh)$$

where n is an integer, γ and α are fractional orders of the derivatives above

3.2.2 Solving Fractional Derivatives(Caputo Derivatives)

All derivatives considered in this study are treated as Caputo derivatives. Laplace transform and the Laplace Convolution theorem shall be employed to solve the fractional derivatives in the semi-infinite medium considered in this work.

Laplace Transform: The Laplace transform of a function h(t) is given by

$$\overline{h}(s) = \int_0^\infty h(t)e^{-st}dt \tag{3.4}$$

Laplace transform of Caputo derivative

For a derivative of integer order n , the Laplace transform is given by

$L\left\{\frac{d^n y}{dx^n}\right\} = s^n y(s) - s^{n-1}y^{n-1}(0) - s^{n-2}y^{n-2}(0) - \cdots - sy'(0) - y(0)$

$= s^n y(s) - \sum_{k=0}^{n-1} s^{n-k-1}y^k(0), \quad n-1 < \gamma < n$

Given that n is a fraction denoted by γ, then the Laplace transform of a fractional derivative is given by:

$$L\left\{\frac{d^\gamma y}{dt^\gamma}\right\} = s^\gamma Y(s) - \sum_{k=0}^{\gamma-1} s^{\gamma-k-1}y^k(0+) \tag{3.5}$$

Convolution Theorem: The Laplace convolution theorem is used to express the inverse Laplace transform of a product of two transformed functions. For two functions in the Laplace domain F(s) and G(s), the inverse Laplace transform of their product is given by:

$$L^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Theorem 3.2.1 *: If $f(t)$ and $g(t)$ are causal functions, then*

$$(f * g)(t) = \int_0^t f(t-t')g(t')dt'$$

3.2.3 The solution of the Fractional Heat flux equation

To solve for the heat flux, the Laplace transform of a fractional derivative and theorem (3.2.1) are used. For easy representation $q(x,t)$ and $T(x,t)$ are represented by q and T respectively.

$L\{{}_0^c\partial_t^{\beta-1}q\} + \tau L\{{}_0^c\partial_t^{\alpha-1}q\} = -\lambda L\{\partial_x T\}$

$s^{\beta-1}q(s) - \sum_{k=0}^{1} s^{\beta-k-1}q^k(0_+) + \tau s^{\alpha-1}q(s) - \tau\sum_{k=0}^{1} s^{\alpha-k-1}q^k(0_+)$

$= -\lambda\partial_x T$

$$(s^{\beta-1} + \tau s^{\alpha-1})q(s) = c - \lambda gradT(s) \tag{3.6}$$

With zero initial conditions:

$c = (\sum_{k=0}^{1} s^{\beta-k-1}q^k(0_+) + \tau\sum_{k=0}^{1} s^{\alpha-k-1}q^k(0_+)) = 0$

$$q(s) = -\frac{\lambda gradT(s)}{(s^{\beta-1} + \tau s^{\alpha-1})} = -\frac{\lambda}{\tau}\frac{s^{\alpha-1}}{(\frac{s^{\beta-\alpha}}{\tau} + 1)}[gradT(s)] \tag{3.7}$$

To solve equation (3.7), let

$$G(s) = \frac{s^{\alpha-1}}{(\frac{s^{\beta-\alpha}}{\tau} + 1)} = \sum_{k=0}^{\infty}(-\frac{1}{\tau})^k s^{-(\alpha-1)-(\alpha-\beta)k}, \quad F(s) = gradT(s) \tag{3.8}$$

Hence, the heat flux, $q(t) = L^{-1}\{G(s)F(s)\}$

$$L^{-1}\{G(s)F(s)\} = -\frac{\lambda}{\tau}\int_0^t g(t-t')f(t')dt'$$

$$g(t') = L^{-1}(\sum_{k=0}^{\infty}(-\frac{1}{\tau})^k s^{-(\alpha-1)-(\alpha-\beta)k})$$

$$= L^{-1}(\sum_{k=0}^{\infty} -\frac{1}{\tau^k}s^{-p})$$

where $p = (\alpha-1) + (\alpha-\beta)k$

using the Laplace inverse transform formula,

$L^{-1}\{s^{-p}\} = \frac{t^{p-1}}{\Gamma(p)}$

$$g(t') = L^{-1}(\sum_{k=0}^{\infty}(-\frac{1}{\tau})^k s^{-p}) = \sum_{k=0}^{\infty}(-\frac{1}{\tau})^k \frac{t'^{p-1}}{\Gamma(p)} \tag{3.9}$$

$$= \sum_{k=0}^{\infty}(-\frac{1}{\tau})^k \frac{t'^{(\alpha-1)+(\alpha-\beta)k-1}}{\Gamma((\alpha-\beta)k+(\alpha-1))} \tag{3.10}$$

$$g(t') = t'^{(\alpha-1)-1}\sum_{k=0}^{\infty}\frac{(-\frac{t'^{(\alpha-\beta)}}{\tau})^k}{\Gamma[(\alpha-\beta)k+(\alpha-1)]} \tag{3.11}$$

let, $v = (\alpha - 1)$ and $\mu = (\alpha - \beta)$

$$g(t') = t'^{\alpha-2}E_{\alpha-\beta,\,\alpha-1}(-\frac{t'^{\alpha-\beta}}{\tau}) \tag{3.12}$$

where, $E_{\mu,\,v}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+v)}$ is the Mittag-Leffler function

$$f(t') = L^{-1}[gradT(s)] = gradT(t')$$

Given that:

$f(t') = gradT(t'), \quad g(t') = t'^{\alpha-2}E_{\alpha-\beta,\alpha-1}\left(-\frac{t'^{\alpha-\beta}}{\tau}\right)$

The heat flux,

$$q(t) = -\frac{\lambda}{\tau}\int_0^t g(t-t')f(t')dt'$$

$$q(t) = -\frac{\lambda}{\tau}\int_0^t (t-t')^{\alpha-2}E_{\alpha-\beta,\,\alpha-1}\left(-\frac{(t-t')^{\alpha-\beta}}{\tau}\right)[gradT(t')]dt' \tag{3.13}$$

$$q(t) = -\frac{\lambda}{\tau}\int_0^t (t-t')^{\alpha-2}E_{\alpha-\beta,\,\alpha-1}\left(-\frac{(t-t')^{\alpha-\beta}}{\tau}\right)[\partial_x T(t')]dt' \tag{3.14}$$

Hence, equation(3.13) (Qi et al. (2013) equation(5)) is the solution of the fractional heat flux equation in a semi- infinite medium

3.3 MODELING THE TEMPERATURE DISTRIBUTION IN THE SEMI-INFINITE MEDIUM

In modeling the temperature distribution in a semi-infinite medium, the fractional heat flux equation(3.2), the energy conservation law and the divergence theorem are used.

Energy conservation law:

$$\rho c \partial_t T(x,t) = \text{-div q(x,t)} \tag{3.15}$$

Applying divergence theorem to (3.2) above produces

$\partial_x \cdot (\partial_t^{\beta-1} q(x,t) + \tau \partial_t^{\alpha-1} q(x,t) = -\lambda \partial_x T(x,t)), \quad (0 < \beta \leq \alpha \leq 2)$

From properties of fractional order derivative, i.e

$\frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\beta q}{\partial t^\beta}) = \frac{\partial^{\alpha+\beta} q}{\partial t^{\alpha+\beta}} = \partial_t^{\alpha+\beta} q$ and $\frac{\partial}{\partial x}(\frac{\partial}{\partial y}) = \partial_x(\partial_y) = \partial_y(\partial_x)$

$$\partial_t^{\beta-1}(\partial_x q(x,t)) + \tau \partial_t^{\alpha-1}(\partial_x q(x,t)) = -\lambda \partial_x(\partial_x T(x,t)) \tag{3.16}$$

substitute (3.15) into (3.16)

$\partial_t^{\beta-1}(\partial_t^1 T(x,t)) + \tau \partial_t^{\alpha-1}(\partial_t^1 T(x,t)) = \dfrac{\lambda}{\rho c} \partial_{xx} T(x,t)$

$$\partial_t^\beta T(x,t) + \tau \partial_t^\alpha T(x,t) = D \partial_{xx} T(x,t), \quad D = \frac{\lambda}{\rho c} \tag{3.17}$$

$$\partial_t^\beta T(x,t) + \tau \partial_t^\alpha T(x,t) = D \Delta T(x,t), \quad \Delta = \partial_{xx} \tag{3.18}$$

Hence, the partial differential equation modeling the temperature distribution in the semi-infinite medium is given by:

$$\partial_t^\beta T(x,t) + \tau \partial_t^\alpha T(x,t) = D \partial_{xx} T(x,t), \quad 0 \leq x < \infty, \quad t > 0 \tag{3.19}$$

3.3.1 Assumptions of the model

In modeling the heat transmission in a semi-infinite medium in this study, the following assumptions are employed:

- the temperature distributions occurs in a semi-infinite medium($0 \leq x < \infty$), which is initially at uniform temperature.

- the boundary surface temperature gradient is given by a time-dependent function.

- heat can only enter or leave the body through the surface at x=0 and that any thermal process begins at time t=0. Hence, (3.19) becomes a one dimensional fractional Cattaneo heat equation.
  $\partial_t^\beta T + \tau \partial_t^\alpha T = D\partial_{xx} T, \quad 0 \leq x < \infty, \quad t > 0$

- the temperature and the time derivative of the temperature are initially zero throughout the medium.

- temperature far from the surface will be neglected.

The above assumptions leads to the following initial and Neumann boundary conditions:

$$T(x,0) = \frac{\partial T(x,0)}{\partial t} = 0, \quad 0 \leq x < \infty \tag{3.20}$$

$$T(\infty,t) = 0, \quad t \geq 0 \tag{3.21}$$

$$-\lambda \frac{\partial T(0,t)}{\partial x} = \frac{\partial^{\beta-1} q(0,t)}{\partial t^{\beta-1}} + \tau \frac{\partial^{\alpha-1} q(0,t)}{\partial t^{\alpha-1}} \tag{3.22}$$

Let $\frac{\partial T(0,t)}{\partial x} = -q_w(t)$ and (3.22) can be written as

$$q_w(t) = \frac{1}{\lambda} \left(\frac{\partial^{\beta-1} q(0,t)}{\partial t^{\beta-1}} + \tau \frac{\partial^{\alpha-1} q(0,t)}{\partial t^{\alpha-1}} \right) \tag{3.23}$$

The initial-boundary value problem (3.22) can be solve by using the discretization method of solving the inverse Laplace transform(Qi et al. (2013))

3.3.2 Establishing the exact Solution of the fractional Cattaneo heat equation in terms of temperature

Since the temperature distribution modeling equation occurs in a semi-infinite medium ($0 \leq x < \infty$), the transform (3.4) is a good choice in solving it. The solution in the Laplace domain shall be converted to a Fox-function via Taylor series expansion. This makes it easier to solve using the inverse Laplace transform of the H-function(Fox-function).

Solving for the exact solution

Applying the transform (3.4) to (3.19) yields

$$s^{\beta}\overline{T}(x,s) - \sum_{k=0}^{1} s^{\beta-1}\overline{T}^{(0)}(x,0) + \tau s^{\alpha}\overline{T}(x,s) - \tau\sum_{k=0}^{1} s^{\alpha-1}\overline{T}^{(0)}(x,0)$$

$$= D\frac{\partial^2\overline{T}(x,s)}{\partial x^2} - D\frac{\partial\overline{T}(x,0)}{\partial x} - \overline{T}(x,0)$$

$$(s^{\beta} + \tau s^{\alpha})\bar{T}(x,s) - c_0 = D\frac{\partial^2\overline{T}(x,s)}{\partial x^2} - \overline{T}_x(x,0) - \overline{T}(x,0) \tag{3.24}$$

$$c_0 = \sum_{k=0}^{1} s^{\beta-1}\overline{T}^{(0)}(x,0) + \tau\sum_{k=0}^{1} s^{\alpha-1}\overline{T}^{(0)}(x,0)$$

From initial conditions:, $(c_0, \quad \overline{T}_x(x,0), \quad \overline{T}(x,0)) = 0$

$$\frac{\partial^2\bar{T}(x,t)}{\partial x^2} = \frac{s^{\beta} + \tau s^{\alpha}}{D}\bar{T}(x,t) \tag{3.25}$$

Equation(3.25) is a homogeneous second order differential equation with a general solution of the form:

$$\bar{T}(x,s) = c_1 e^{m_1 x} + c_2 e^{m_2 x} \tag{3.26}$$

(3.25) is of the form:

$$\frac{\partial^2 \bar{T}}{\partial x^2} - A\bar{T} = 0, \tag{3.27}$$

where, $A = \frac{s^{\beta}+\tau s^{\alpha}}{D}$.

The characteristic differential equation of (3.27) is

$m^2 - A = 0, \Longrightarrow m = \pm\sqrt{A}$

$$\Longrightarrow m_1 = +\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}}, \quad m_2 = -\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}}$$

$$\bar{T}(x,s) = c_1 e^{+x\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}}} + c_2 e^{-x\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}}} \tag{3.28}$$

$$\overline{T}(x,s) = c_1 e^{+x\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}}} \tag{3.29}$$

Considering the boundary conditions:

$\frac{\partial\overline{T}(0,s)}{\partial x} = -\overline{q}_w(s)$ and $\overline{T}(\infty,s) = 0$

equation (3.29) doesn't satisfy the boundary condition, $\overline{T}(\infty,s) = 0$

Hence, the solution of the temperature distribution is:

$$\overline{T}(x,s) = c_2 e^{-x\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}}} \tag{3.30}$$

Determining the constant of the solution: c_2

Differentiating (3.30) with respect to x produces

$$\frac{\partial\overline{T}(x,s)}{\partial x} = (-\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}})c_2 e^{-x\sqrt{\frac{s^{\beta}+\tau s^{\alpha}}{D}}}$$

Using the model assumptions, $\frac{\partial \overline{T}(0,s)}{\partial x} = -\overline{q}_w(s)$ and $\overline{T}(x,0) = 0$

$$\Longrightarrow -q_w = (-\sqrt{\frac{s^\beta + \tau s^\alpha}{D}})c_2 e^{-(0)\sqrt{\frac{s^\beta+\tau s^\alpha}{D}}}$$

$$\Rightarrow c_2 = \overline{q}_w(s)\sqrt{\frac{D}{s^\beta + \tau s^\alpha}} \tag{3.31}$$

Thus, the solution of the fractional Cattaneo heat equation in the Laplace domain is:

$$\overline{T}(x,s) = \overline{q}_w(s)\frac{\sqrt{D}}{\sqrt{s^\beta + \tau s^\alpha}}e^{-\frac{x}{\sqrt{D}}\sqrt{s^\beta+\tau s^\alpha}} \text{(Qi et al. (2013))} \tag{3.32}$$

Using the convolution theorem, the inverse Laplace transform of the temperature distribution function is given by:

$$T(x,t) = \int_0^t G(x,t')q_w(t-t')dt' \tag{3.33}$$

3.4 EXAMINING THE INFLUENCE OF THE FRACTIONAL DERIVATIVES OF ORDERS α and β ON THE TEMPERATURE DISTRIBUTION

This section examines the influence of the fractional Cattaneo derivatives of orders α and β on the temperature distribution. The values of α and β lies within the interval: $0 < \beta \leq \alpha \leq 2$. The α- order will be used to examine the temperature distribution at the boundary and within the medium for a short time. The long time effect on the temperature distribution within the medium and the boundary will be examined using the fractional order parameter β.

3.4.1 The Influence of fractional derivative of order α on temperature distribution

For a large value of α, the inverse Laplace transform of s^{α} or $s^{\beta-\alpha}$ in the Laplace domain will lead to a short time for temperature distribution within the medium or at the boundary. To examine the effects of the α-order derivative on temperature distribution, let the Laplace transform of $G(x,t)$, be $\overline{G}(x,s)$

From (3.32),

$$\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{s^{\beta}+\tau s^{\alpha}}} e^{-\frac{x}{\sqrt{D}}\sqrt{s^{\beta}+\tau s^{\alpha}}}$$

$$\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{\tau s^{\alpha}}\sqrt{(1+\tau^{-1}s^{\beta-\alpha})}} e^{-\frac{x}{\sqrt{D}}\sqrt{\tau s^{\alpha}}\sqrt{(1+\tau^{-1}s^{\beta-\alpha})}}$$

For $\tau \neq 0$,

$$\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{\tau s^{\alpha}}} \frac{e^{-r\sqrt{z}}}{\sqrt{z}} \tag{3.34}$$

Where

$$z = 1+\tau^{-1}s^{\beta-\alpha}, \quad r = \frac{x\sqrt{\tau s^{\alpha}}}{\sqrt{D}} \tag{3.35}$$

let

$$g(z) = e^{-r\sqrt{z}}/\sqrt{z} \tag{3.36}$$

3.4.2 Representing the Solution in terms of Taylor series

Establishing the solution in Taylor series makes it easier to convert it into a Fox-function. When the solution is written in Taylor series, the derivative component of the Taylor series formula can be transformed into a Fox-function(H-function). In the Fox-function domain, it is easier to take the inverse Laplace transform of the solution of a fractional partial differential equation.

Taylor series representation

Taylor series expansion of $g(z)$ about a point z=1 is given by

$$g(z) = \sum_{k=0}^{\infty} \frac{g^k(1)}{k!}(z-1)^k \tag{3.37}$$

3.4.3 Fox-Function(H-function)

The H-function is a generalized function. In its special cases, it can be used to represent almost every named mathematical function and continuous statistical distribution. The Laplace and Fourier Transforms (and their inverses) and the derivatives of H-function are also expressed in terms of the H-function. The H-function is firmly rooted in gamma functions, integral transform theory, complex analysis and statistical distribution theory. The H-function is often represented in terms of Mellin-Barnes inversion integral. The H-function can be solved using the residue theorem. Examples of statistical distributions that can be represented in terms of the H-function include: exponential, Rayleigh, Chi-Square, Weibull and beta functions. More information about the H-function(Fox-function) can be obtained from the monographs of Kilbas et al. (2006), Podlubny (1999), Mathai et al. (2009) and Carl (1992). The H-function is usually represented by the notation below.

Notation of H-Function(Fox-function):

$$H(x) = H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_q,A_q)}\right] = H_{p,q}^{m,n}[z|_{(b_1,B_1).....(b_q,B_q)}^{(a_1,A_1).....(a_p,A_p)}] \tag{3.38}$$

Mellin transform inversion integral(Carl (1992))

$$= \frac{1}{2\pi i}\int_{L_1} \frac{\{\prod_{j=1}^{m}\Gamma(b_j+B_js)\}\{\prod_{j=1}^{n}\Gamma(1-a_j-A_js)\}}{\{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\}\{\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\}} z^{-s} ds \tag{3.39}$$

General Mellin-Barnes integral:

$$= \frac{1}{2\pi i}\int_{L_2} \frac{\{\prod_{j=1}^{m}\Gamma(b_j-B_js)\}\{\prod_{j=1}^{n}\Gamma(1-a_j+A_js)\}}{\{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\}\{\prod_{j=n+1}^{p}\Gamma(a_j-A_js)\}} z^{s} ds \tag{3.40}$$

where, $i = (-1)^{\frac{1}{2}}, z \neq 0$ and $z^{-s} = exp(-s\{In|z| + iarg(z)\}$, $z^s = exp(s\{In|z| + iarg(z)\})$. $In\,|z|$ is the natural logarithm of, $|z|$ and $arg(z)$ is not the principal value. $m, n, p, q \in N_0$ with $0 \leq n \leq p, \quad 1 \leq m \leq q, \quad A_i, B_j \in R_+, \quad z, a_i, b_j \in R$ or $C, \quad i = 1 \cdots p, \quad j = 1 \cdots q, \quad L_1$, is a suitable contour separating the poles $\zeta_{jv} = -(\frac{b_j+v}{B_j}), \quad j = 1 \cdots q, v = [0, 1, 2 \cdots]$, of the gamma function, $\Gamma(b_j + B_j S)$, to the left of L_1 from the pole $w_{\lambda k} = \frac{(1-a_\lambda+k)}{A_\lambda}$ of the gamma function, $\Gamma(1 - a_\lambda - A_\lambda s)$, which lie to the right of L_1. When the parameters A_j, B_j in notation (3.38) above reduces to one, Meijer G-function is produced below:

$G(x) = G^{m,n}_{p,q}(z) = G^{m,n}_{p,q}\left[z|^{(a_q,1)}_{(b_q,1)}\right]$

Expressing the solution in terms of Fox H-function

The inverse Laplace transform of exponential solutions of fractional nature are not usually achieved by simply looking into a standard inverse Laplace transform table. However, if the solution can be expressed in terms of a derivative, the opportunity exist to get the inverse Laplace transform. Hence, the Fox-function provides this opportunity to obtain the inverse Laplace transform of a fractional derivative or solution.

using the identity (1.125)(Mathai et al. (2009)), the solution (i.e 3.34) can be transformed into an H-function(Fox-function)

$H^{1,0}_{0,1}\left[z|_{(b,B)}\right] = B^{-1} z^{\frac{b}{B}} exp\left(-z^{\frac{1}{B}}\right)$

From the solution: $g(z) = e^{-r\sqrt{z}}/\sqrt{z}$

Let $u = r\sqrt{z}$

$g(z) = ru^{-1} exp(-u) = r\left(1^{-1} u^{\frac{-1}{1}}\right) exp\left(-u^{\frac{1}{1}}\right)$

$$= rH^{1,0}_{0,1}\left[u|^{-}_{(-1,1)}\right] \tag{3.41}$$

Using the derivative definition of the H-function (Equation(1.69)) of (Mathai et al. (2009))

$$\left(\frac{d}{dz}\right)^n \left\{z^{\rho-1} H_{p,q}^{m,n}\left[az^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = z^{\rho-n-1} H_{p+1,q+1}^{m,n+1}\left[az\Big|_{(b_q,B_q),(1-\rho+n,\sigma)}^{(1-\rho,\sigma),(a_p,A_p)}\right] \quad (3.42)$$

$$g^k(z) = r\left(\frac{d}{dz}\right)^k \left\{z^0 H_{0,1}^{1,0}\left[rz^{\frac{1}{2}}\Big|_{(-1,1)}^{-}\right]\right\} \quad (3.43)$$

$g^k(z) = z^{-k} r H_{1,2}^{1,1}\left[(r^2 z)^{\frac{1}{2}}\Big|_{(-1,1),(k,1/2)}^{(0,1/2)}\right]$
$g^k(z) = r H_{1,2}^{1,1}\left[(r^2 z)^{\frac{1}{2}}\Big|_{(-1,1),(k,1/2)}^{(0,1/2)}\right]$

$$g^k(1) = r H_{1,2}^{1,1}\left[r\Big|_{(-1,1),(k,1/2)}^{(0,1/2)}\right] \quad (3.44)$$

substitute(3.44) into(3.37)

$$g(z) = \sum_{k=0}^{\infty} \frac{(z-1)^k}{k!} r H_{1,2}^{1,1}\left[r\Big|_{(-1,1),(k,1/2)}^{(0,1/2)}\right] \text{ (Qi et al. (2013))} \quad (3.45)$$

$$g(z) = \sum_{k=0}^{\infty} \frac{(z-1)^k}{k!} \left(\frac{1}{2\pi i}\int_L \Theta(s)(r)^{-s} ds\right)$$

$\theta(s) = \dfrac{\Gamma(s-1)\Gamma(1-\frac{1}{2}s)}{\Gamma(1-k-\frac{1}{2}s)}$

Modification of the solution

The Fox-function has a some properties which makes it possible to modify the solution.

Using property 1.4(of Mathai et al. (2009)), the last part of (3.45) is modified as follows:

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_q,A_q)}\right] = \kappa H_{p,q}^{m,n}\left[z^{\kappa}\Big|_{(b_q,\kappa B_q)}^{(a_q,\kappa A_q)}\right] \quad (3.46)$$

$$rH_{1,2}^{1,1}\left[r|_{(-1,1),(k,1/2)}^{(-k,1/2)}\right] = 2rH_{1,2}^{1,1}\left[r^2|_{(-1,2),(k,1)}^{(0,1)}\right]$$

Using property 1.6,(Mathai et al. (2009))

$$H_{p+1,q+1}^{m,n+1}\left[z|_{(b_1,B_1).....(b_q,B_q),(r,\gamma)}^{(0,\gamma),(a_1,A_1).....(a_p,A_p)}\right] = (-1)^r H_{p+1,q+1}^{m+1,n}\left[z|_{(r,\gamma),(b_1,B_1).....(b_q,B_q)}^{(a_1,A_1).....(a_p,A_p),(0,\gamma)}\right] \quad (3.47)$$

$$2rH_{1,2}^{1,1}\left[r^2|_{(-1,2),(k,1)}^{(0,1)}\right] = (-1)^k 2rH_{1,2}^{2,0}\left[r^2|_{(k,1),(-1,2)}^{(0,1)}\right] \quad (3.48)$$

Using property 1.5,(Mathai et al. (2009))

$$z^{\sigma}H_{p,q}^{m,n}\left[z|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z|_{(b_q+\sigma B_q,B_q)}^{(a_q+\sigma A_p,A_p)}\right] \quad (3.49)$$

$$(-1)^k 2rH_{1,2}^{2,0}\left[r^2|_{(k,1),(-1,2)}^{(0,1)}\right] = (-1)^k 2(r^2)^{\frac{1}{2}}H_{1,2}^{2,0}\left[r^2|_{(k+1/2,1),(0,2)}^{(1/2,1)}\right]$$

$$= 2(-1)^k H_{1,2}^{2,0}\left[r^2|_{(k+1/2,1),(0,2)}^{(1/2,1)}\right] \quad (3.50)$$

using the Mellin inversion integral

$$2(-1)^k H_{1,2}^{2,0}\left[r^2|_{(k+1/2,1),(0,2)}^{(1/2,1)}\right] = 2(-1)^k\left(\frac{1}{2\pi i}\int_L \frac{\Gamma(k+\frac{1}{2}+s)\Gamma(2s)}{\Gamma(\frac{1}{2}+s)}(r^2)^{-s}ds\right)$$

Using gamma duplication rule,

$$\frac{\Gamma(2s)}{\Gamma(\frac{1}{2}+s)} = \frac{\Gamma(s)2^{2(s)-1}}{\sqrt{\pi}} \quad (3.51)$$

$$= 2(-1)^k\left(\tfrac{1}{2\pi i}\textstyle\int_L \Gamma(s)\Gamma(k+1/2+s)\tfrac{2^{2(s)-1}}{\sqrt{\pi}}(r^2)^{-s}ds\right)$$

$$= \frac{(-1)^k}{\sqrt{\pi}}\left(\frac{1}{2\pi i}\int_L \Gamma(s)\Gamma(k+1/2+s)\left(\frac{r^2}{4}\right)^{-s}ds\right)$$

$$= \frac{(-1)^k}{\sqrt{\pi}}H_{0,2}^{2,0}\left[\frac{r^2}{4}|_{(0,1),(k+1/2,1)}^{-}\right] \quad (3.52)$$

Substituting (3.52) into (3.45)

$$g(z) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{(z-1)^k}{k!} H_{0,2}^{2,0} \left[\frac{r^2}{4} \Big|_{(0,1),(k+1/2,1)}^{-} \right] \tag{3.53}$$

But $\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{\tau s^\alpha}} g(z)$, hence

$$\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{\tau s^\alpha}} \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{(z-1)^k}{k!} H_{0,2}^{2,0} \left[\frac{r^2}{4} \Big|_{(0,1),(k+1/2,1)}^{-} \right] \tag{3.54}$$

Substituting, $z - 1 = \tau^{-1} s^{\beta-\alpha}$ and $r = \frac{x\sqrt{\tau s^\alpha}}{\sqrt{D}}$, into (3.54) yields

$$\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{\tau s^\alpha}} \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{(\tau^{-1} s^{\beta-\alpha})^k}{k!} H_{0,2}^{2,0} \left[\frac{x^2 \tau s^\alpha}{4D} \Big|_{(0,1),(k+1/2,1)}^{-} \right] \tag{3.55}$$

Inverse Laplace transform of(3.55)

Using the inverse Laplace transform of the H-function((2.21) in (Mathai et al. (2009) p.51)

$$L^{-1}\left\{ s^{-\rho} H_{p,q}^{m,n} \left[a s^{\sigma} \Big|_{(b_q,B_q)}^{(a_p,A_p)} \right]; t \right\} = t^{\rho-1} H_{p+1,q}^{m,n} \left[a t^{-\sigma} \Big|_{(b_q,B_q)}^{(a_p,A_p),(\rho,\sigma)} \right] \tag{3.56}$$

let $\rho = k(\alpha - \beta) + \alpha/2$ in (3.55)

The new solution of the temperature Modeling equation:

Let the inverse Laplace transform of $\overline{G}(x,s)$ be $G(x,t)$. Thus,

$$G(x,t) = L^{-1}\left\{ \frac{\sqrt{D}}{\sqrt{\pi\tau}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{s^{-\rho}}{\tau^k} H_{0,2}^{2,0} \left[\frac{x^2 \tau s^\alpha}{4D} \Big|_{(0,1),(k+1/2,1)}^{-} \right] \right\}$$

$$G(x,t) = \frac{\sqrt{D}}{\sqrt{\pi\tau}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{k(\alpha-\beta)+\alpha/2-1}}{\tau^k} H_{1,2}^{2,0} \left[\frac{\tau x^2}{4Dt^\alpha} \Big|_{(0,1),(k+1/2,1)}^{(k(\alpha-\beta)+\alpha/2,\alpha)} \right] \tag{3.57}$$

(Qi et al. (2013)) established (3.57) as the new exact solution of the generalized fractional Cattaneo heat equation in a semi-infinite medium.

Reducing $G(x,t)$ to Lower order H-function

Representing $G(x,t)$ in Mellin-Barnes inverse integral provides a means of eliminating some pairs of $\Gamma(.)$ functions. This is achieved using Stirling's approximation formula.

$$G(x,t) = \frac{\sqrt{D}}{\sqrt{\pi\tau}} t^{\frac{\alpha}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{\tau} t^{(\alpha-\beta)})^k}{k!} \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(k+1/2+s)}{\Gamma(k(\alpha-\beta)+\frac{\alpha}{2}+\alpha s)} \left(\frac{\tau x^2}{4Dt^\alpha}\right)^{-s} ds$$

Using Stirling's approximation:

$$\lim_{\frac{1}{2}\to\infty} \frac{\Gamma(k+\frac{1}{2}+s)}{\Gamma(\frac{1}{2})} = \frac{\sqrt{2\pi}\left(\frac{1}{2}\right)^{\frac{1}{2}+k+s-\frac{1}{2}} e^{-\frac{1}{2}}}{\sqrt{2\pi}\left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{2}} e^{-\frac{1}{2}}}$$

$$= \left(\frac{1}{2}\right)^{k+s} \tag{3.58}$$

$$= \frac{\sqrt{D}}{\sqrt{\tau}} t^{\frac{\alpha}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2\tau} t^{(\alpha-\beta)})^k}{k!} \frac{1}{2\pi i} \int_L \frac{\Gamma(s)}{\Gamma(k(\alpha-\beta)+\frac{\alpha}{2}+\alpha s)} \left(\frac{\tau x^2}{2Dt^\alpha}\right)^{-s} ds$$

$$= \frac{\sqrt{D}}{\sqrt{\tau}} t^{\frac{\alpha}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2\tau} t^{(\alpha-\beta)})^k}{k!} H_{1,1}^{1,0}\left[\frac{\tau x^2}{2Dt^\alpha}\Big|_{(0,1)}^{(k(\alpha-\beta)+\frac{\alpha}{2},\alpha)}\right] \tag{3.59}$$

3.4.4 Evaluating Fox-Function through Series Expansion

The H-function can be evaluated as a series expansion using the residue theorem. If a complex function f has singularity at the point z_0, then f has a Laurent series representation. That is

$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$

Which converges for all z near z_0 and valid within the open disk of radius R, $0 < |z-z_0| < R$

Residue:

The co-efficient a_{-1}, of $\frac{1}{(z-z_0)}$ in the Laurent series above is called the residue of the function f at the isolated singularity z_0. The notation $a_{-1} = Res(f(z), z_0)$ denotes the residue at z_0

Residue theorems

Theorem 3.4.1 *: If f has a simple pole at $z = z_0$, then*

$$Res(f(z), z_0) = \lim_{z \longrightarrow z_0} (z - z_0) f(z)$$

Theorem 3.4.2 *: If the function f has a pole of order, n at $z = z_0$, then*

$$Res(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \longrightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

Applying theorem(3.4.1), equation(3.59) can be evaluated as a series expansion at the pole of the gamma function $\Gamma(s)$.

Residue at the poles of $\Gamma(s)$ is given by

$$= \frac{\sqrt{D}}{\sqrt{\tau}} t^{\frac{\alpha}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2\tau} t^{(\alpha-\beta)})^k}{k!} \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} \frac{1}{\Gamma(k(\alpha-\beta) + \frac{\alpha}{2} - \alpha v)} \left(\frac{\tau x^2}{2Dt^\alpha} \right)^v \tag{3.60}$$

$$G(x,t) = \frac{\sqrt{D}}{\sqrt{\tau}} t^{\frac{\alpha}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2\tau} t^{(\alpha-\beta)})^k}{k!} \frac{\left(1 - \frac{\tau x^2}{2Dt^\alpha}\right)^{(k(\alpha-\beta)+\frac{\alpha}{2}-1)}}{\Gamma(k(\alpha-\beta)+\frac{\alpha}{2})} \tag{3.61}$$

If $\alpha = 2$

$$= \frac{\sqrt{D}}{\sqrt{\tau}} t^{\frac{\alpha}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2\tau} z^{(\alpha-\beta)})^k}{k!\Gamma(k(\alpha-\beta)+\frac{\alpha}{2})}, \tag{3.62}$$

where $z = te^{-\frac{\tau x^2}{2Dt^\alpha}}$

$$G(x,t) = \frac{\sqrt{D}}{\sqrt{\tau}} t^{\frac{\alpha}{2}-1} \varepsilon_{\alpha-\beta,\frac{\alpha}{2}} \left(-\frac{1}{2\tau} z \right) \tag{3.63}$$

(3.63) is a Fox-Wright function.

Using convolution theorem (3.2.1), and using integration by parts, the temperature function is obtained. i.e

$T(x,t) = \int_0^t G(x,t')q_w(t-t')dt'$,

Hence, the equation for temperature distribution within a short period of time inside the medium is given by:

$$T(x,t) = \frac{\sqrt{D}}{\sqrt{\tau}} t^{\frac{\alpha}{2}} \varepsilon_{\alpha-\beta,\frac{\alpha}{2}+1}\left(-\frac{1}{2\tau}z\right) q_w \tag{3.64}$$

3.4.5 The influence of the fractional derivative of order β on temperature distribution

Since β is less than or equal to α (i.e $\beta \leq \alpha$), the inverse Laplace transform of s^β or $s^{\alpha-\beta}$ in the solution will result in large time for the temperature distribution. To solve the solution in terms of s^β let the Laplace transform of $G(x,t)$, be $\overline{G_1}(x,s)$

From (3.32),

$$\overline{G_1}(x,s) = \frac{\sqrt{D}}{\sqrt{s^\beta + \tau s^\alpha}} e^{-\frac{x}{\sqrt{D}}\sqrt{s^\beta+\tau s^\alpha}}$$

$$\overline{G_1}(x,s) = \frac{\sqrt{D}}{\sqrt{s^\beta}\sqrt{(1+\tau s^{\alpha-\beta})}} e^{-\frac{x}{\sqrt{D}}\sqrt{s^\beta}\sqrt{(1+\tau s^{\alpha-\beta})}}$$

For $\tau \neq 0$,

$$\overline{G_1}(x,s) = \frac{\sqrt{D}}{\sqrt{s^\beta}} \frac{e^{-r_1\sqrt{z_1}}}{\sqrt{z_1}} \tag{3.65}$$

Where

$$z_1 = 1 + \tau s^{\alpha-\beta}, \quad r_1 = \frac{x\sqrt{s^\beta}}{\sqrt{D}} \tag{3.66}$$

Expressing the solution in terms of Taylor series and Fox-function

From (3.65), let

$$g_1(z_1) = \frac{e^{-r_1\sqrt{z_1}}}{\sqrt{z_1}} \tag{3.67}$$

The Taylor series representation of (3.67) about the point $z_1 = 1$ can be written as:

$$g_1(z_1) = \sum_{k=0}^{\infty} \frac{g_1^k(1)}{k!}(z_1 - 1)^k \tag{3.68}$$

The exponential part of (3.67) can be expressed as a Fox-function using the identity (1.125)(Mathai et al. (2009)), (equation 3.67) as follows:

$H_{0,1}^{1,0}\left[z|_{(b,B)}\right] = B^{-1} z^{\frac{b}{B}} exp\left(-z^{\frac{1}{B}}\right)$

From the solution: $g_1(z_1) = e^{-r_1\sqrt{z_1}}/\sqrt{z_1}$

Let $u_1 = r_1\sqrt{z_1}$

$g_1(z_1) = r_1 u_1^{-1} exp(-u_1) = r_1\left(1^{-1} u_1^{\frac{-1}{1}}\right) exp\left(-u_1^{\frac{1}{1}}\right)$

$$= r_1 H_{0,1}^{1,0}\left[u_1|_{(-1,1)}^{-}\right] \tag{3.69}$$

Using the derivative definition of the H-function (Equation(1.69)) of (Mathai et al. (2009))

$$\left(\frac{d}{dz}\right)^n \left\{z^{\rho-1} H_{p,q}^{m,n}\left[az^{\sigma}|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = z^{\rho-n-1} H_{p+1,q+1}^{m,n+1}\left[az|_{(b_q,B_q),(1-\rho+n,\sigma)}^{(1-\rho,\sigma),(a_p,A_p)}\right]$$

$$g_1^k(z_1) = r_1\left(\frac{d}{dz_1}\right)^k \left\{z^0 H_{0,1}^{1,0}\left[r_1 z_1^{\frac{1}{2}}|_{(-1,1)}^{-}\right]\right\} \tag{3.70}$$

$g_1^k(z_1) = z_1^{-k} r_1 H_{1,2}^{1,1}\left[(r_1^2 z_1)^{\frac{1}{2}}|_{(-1,1),(k,1/2)}^{(0,1/2)}\right]$

$g_1^k(z_1) = r_1 H_{1,2}^{1,1}\left[(r_1^2 z_1)^{\frac{1}{2}}|_{(-1,1),(k,1/2)}^{(0,1/2)}\right]$

For $z_1 = 1$,

$$g_1^k(1) = r_1 H_{1,2}^{1,1}\left[r_1|_{(-1,1),(k,1/2)}^{(0,1/2)}\right] \tag{3.71}$$

substituting (3.71) into (3.68) yields

$$g_1(z_1) = \sum_{k=0}^{\infty} \frac{(z_1-1)^k}{k!} r_1 H_{1,2}^{1,1}\left[r_1|_{(-1,1),(k,1/2)}^{(0,1/2)}\right] \tag{3.72}$$

$$g_1(z_1) = \sum_{k=0}^{\infty} \frac{(z_1-1)^k}{k!} \left(\frac{1}{2\pi i} \int_L \Theta(s)(r_1)^{-s} ds \right)$$

$$\theta(s) = \frac{\Gamma(s-1)\Gamma(1-\frac{1}{2}s)}{\Gamma(1-k-\frac{1}{2}s)}$$

Modifying the solution

Similarly, using property 1.4 (of Mathai et al. (2009)), the last part of (3.72) is modified as follows:

$$H_{p,q}^{m,n}\left[z|_{(b_q,B_q)}^{(a_q,A_q)}\right] = \kappa H_{p,q}^{m,n}\left[z^{\kappa}|_{(b_q,\kappa B_q)}^{(a_q,\kappa A_q)}\right] \tag{3.73}$$

$$r_1 H_{1,2}^{1,1}\left[r_1|_{(-1,1),(k,1/2)}^{(-k,1/2)}\right] = 2r_1 H_{1,2}^{1,1}\left[r_1^2|_{(-1,2),(k,1)}^{(0,1)}\right]$$

Using property 1.6, (Mathai et al. (2009))

$$H_{p+1,q+1}^{m,n+1}\left[z|_{(b_1,B_1).....(b_q,B_q),(r,\gamma)}^{(0,\gamma),(a_1,A_1).....(a_p,A_p)}\right] = (-1)^r H_{p+1,q+1}^{m+1,n}\left[z|_{(r,\gamma),(b_1,B_1).....(b_q,B_q)}^{(a_1,A_1).....(a_p,A_p),(0,\gamma)}\right] \tag{3.74}$$

$$2r_1 H_{1,2}^{1,1}\left[r_1^2|_{(-1,2),(k,1)}^{(0,1)}\right] = (-1)^k 2r_1 H_{1,2}^{2,0}\left[r_1^2|_{(k,1),(-1,2)}^{(0,1)}\right] \tag{3.75}$$

using property 1.5, (Mathai et al. (2009))

$$z^{\sigma} H_{p,q}^{m,n}\left[z|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z|_{(b_q+\sigma B_q,B_q)}^{(a_q+\sigma A_p,A_p)}\right]$$

$$(-1)^k 2r_1 H_{1,2}^{2,0}\left[r_1^2|_{(k,1),(-1,2)}^{(0,1)}\right] = (-1)^k 2(r_1^2)^{\frac{1}{2}} H_{1,2}^{2,0}\left[r_1^2|_{(k+1/2,1),(0,2)}^{(1/2,1)}\right]$$

$$= 2(-1)^k H_{1,2}^{2,0}\left[r_1^2|_{(k+1/2,1),(0,2)}^{(1/2,1)}\right] \tag{3.76}$$

using the Mellin inversion integral

$$2(-1)^k H_{1,2}^{2,0}\left[r_1^2|_{(k+1/2,1),(0,2)}^{(1/2,1)}\right] = 2(-1)^k \left(\frac{1}{2\pi i} \int_L \frac{\Gamma(k+\frac{1}{2}+s)\Gamma(2s)}{\Gamma(\frac{1}{2}+s)} (r_1^2)^{-s} ds \right) \tag{3.77}$$

Using gamma duplication rule,

$$\frac{\Gamma(2s)}{\Gamma(\frac{1}{2}+s)} = \frac{\Gamma(s)2^{2(s)-1}}{\sqrt{\pi}}$$

equation(3.77) is can be reduced to:

$= 2(-1)^k \left(\frac{1}{2\pi i} \int_L \Gamma(s)\Gamma(k+1/2+s) \frac{2^{2(s)-1}}{\sqrt{\pi}} (r_1^2)^{-s} ds \right)$

$$= \frac{(-1)^k}{\sqrt{\pi}} \left(\frac{1}{2\pi i} \int_L \Gamma(s)\Gamma(k+1/2+s) \left(\frac{r_1^2}{4}\right)^{-s} ds \right)$$

$$= \frac{(-1)^k}{\sqrt{\pi}} H_{0,2}^{2,0} \left[\frac{r_1^2}{4} |_{(0,1),(k+1/2,1)}^{-} \right] \tag{3.78}$$

Substituting (3.78) into (3.72)

$$g_1(z_1) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{(z_1-1)^k}{k!} H_{0,2}^{2,0} \left[\frac{r_1^2}{4} |_{(0,1),(k+1/2,1)}^{-} \right] \tag{3.79}$$

But $\overline{G_1}(x,s) = \frac{\sqrt{D}}{\sqrt{s^\beta}} g_1(z_1)$, hence

$$\overline{G_1}(x,s) = \frac{\sqrt{D}}{\sqrt{s^\beta}} \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{(z_1-1)^k}{k!} H_{0,2}^{2,0} \left[\frac{r_1^2}{4} |_{(0,1),(k+1/2,1)}^{-} \right] \tag{3.80}$$

Substituting, $z_1 - 1 = \tau s^{\alpha-\beta}$ and $r_1 = \frac{x\sqrt{s^\beta}}{\sqrt{D}}$, into (3.80) yields

$$\overline{G_1}(x,s) = \frac{\sqrt{D}}{\sqrt{s^\beta}} \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{(\tau s^{\alpha-\beta})^k}{k} H_{0,2}^{2,0} \left[\frac{x^2 s^\beta}{4D} |_{(0,1),(k+1/2,1)}^{-} \right] \tag{3.81}$$

Inverse Laplace transform of(3.81)

Using the inverse Laplace transform of the H-function((2.21) in Mathai et al. (2009), p.51)

$$L^{-1} \left\{ s^{-\rho} H_{p,q}^{m,n} \left[as^\sigma |_{(b_q,B_q)}^{(a_p,A_p)} \right] ; t \right\} = t^{\rho-1} H_{p+1,q}^{m,n} \left[at^{-\sigma} |_{(b_q,B_q)}^{(a_p,A_p),(\rho,\sigma)} \right] \tag{3.82}$$

let $\rho_1 = k(\beta - \alpha) + \beta/2$ in (3.81)

Representing the inverse Laplace transform of $\overline{G_1}(x,s)$ as $G_1(x,t)$ implies

$$G_1(x,t) = L^{-1} \left\{ \frac{\sqrt{D}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k!} s^{-\rho_1} H_{0,2}^{2,0} \left[\frac{x^2 s^\beta}{4D} |_{(0,1),(k+1/2,1)}^{-} \right] \right\}$$

$$G_1(x,t) = \frac{\sqrt{D}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k!} t^{k(\beta-\alpha)+\beta/2-1} H_{1,2}^{2,0} \left[\frac{x^2}{4Dt^\beta} \Big|_{(0,1),(k+1/2,1)}^{(k(\beta-\alpha)+\beta/2,\beta)} \right] \tag{3.83}$$

Reducing $G_1(x,t)$ to a lower order Fox-function

Expressing (3.83) in Mellin-Barnes integral will provide a means of reducing the equation to a lower order Fox-function,

$$G_1(x,t) = \frac{\sqrt{D}}{\sqrt{\pi}} t^{\frac{\beta}{2}-1} \sum_{k=0}^{\infty} \frac{(-\tau t^{(\beta-\alpha)})^k}{k!} \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(k+1/2+s)}{\Gamma(k(\beta-\alpha)+\frac{\beta}{2}+\beta s)} \left(\frac{x^2}{4Dt^\beta} \right)^{-s} ds \tag{3.84}$$

Using Stirling's approximation:

$$\lim_{\frac{1}{2}\to\infty} \frac{\Gamma(k+\frac{1}{2}+s)}{\Gamma(\frac{1}{2})} = \frac{\sqrt{2\pi}\left(\frac{1}{2}\right)^{\frac{1}{2}+k+s-\frac{1}{2}} e^{-\frac{1}{2}}}{\sqrt{2\pi}\left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{2}} e^{-\frac{1}{2}}}$$

$$= \left(\frac{1}{2}\right)^{k+s} \tag{3.85}$$

substituting (3.85) into (3.84) produces

$$= \sqrt{D} t^{\frac{\beta}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}\tau t^{(\beta-\alpha)})^k}{k!} \frac{1}{2\pi i} \int_L \frac{\Gamma(s)}{\Gamma(k(\beta-\alpha)+\frac{\beta}{2}+\beta s)} \left(\frac{x^2}{2Dt^\beta} \right)^{-s} ds$$

$$= \sqrt{D} t^{\frac{\beta}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}t^{(\beta-\alpha)})^k}{k!} H_{1,1}^{1,0} \left[\frac{x^2}{2Dt^\beta} \Big|_{(0,1)}^{(k(\beta-\alpha)+\frac{\beta}{2},\beta)} \right] \tag{3.86}$$

solving (3.86) as a series expansion at the pole of $\Gamma(s)$.

$$= \sqrt{D} t^{\frac{\beta}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}\tau t^{(\beta-\alpha)})^k}{k!} \frac{\left(1-\frac{x^2}{2Dt^\beta}\right)^{(k(\beta-\alpha)+\frac{\beta}{2}-1)}}{\Gamma(k(\beta-\alpha)+\frac{\beta}{2})} \tag{3.87}$$

$$= \sqrt{D} t^{\frac{\beta}{2}-1} \left(1-\frac{x^2}{2Dt^\beta}\right)^{\frac{\beta}{2}-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}\tau\phi^{(\beta-\alpha)})^k}{k!\Gamma(k(|\beta-\alpha|)+\frac{\beta}{2})} \tag{3.88}$$

$$G_1(x,t) = \sqrt{D} t^{\frac{\beta}{2}-1} \left(1 - \frac{x^2}{2Dt^\beta}\right)^{\frac{\beta}{2}-1} E_{\beta-\alpha,\frac{\beta}{2}} \left(-\frac{1}{2}\tau\phi\right) \tag{3.89}$$

Where $\phi \simeq te^{-\frac{x^2}{2Dt^\beta}}$

Solving the convolution,

$$T_1(x,t) = \int_0^t G_1(x,t') q_w(t-t') dt' \tag{3.90}$$

produces the long time temperature distribution function.

The temperature distribution for long time inside the medium is given by:

$$T_1(x,t) = \sqrt{D} t^{\frac{\beta}{2}} \left(1 - \frac{x^2}{2Dt^\beta}\right)^{\frac{\beta}{2}-1} E_{\beta-\alpha,\frac{\beta}{2}+1} \left(-\frac{1}{2}\tau\phi\right) \tag{3.91}$$

3.4.6 Examining Temperature Distribution at the Boundary

While equation(3.64) and (3.91) can be used to study the temperature distributions inside the medium, it is important to also study the temperature distribution at the boundary. This section examines the boundary surface temperature for short time and long time based on the fractional derivatives of orders α and β. Asymptotic expressions of the boundary surface temperatures are established below:

From (3.32), the boundary temperature is given by

$$T(0,t) = \int_0^t G(0,t') q_w(t-t') dt' \tag{3.92}$$

(Qi et al. (2013)) established the boundary temperature as:

$$\overline{T}(0,s) = \frac{q_w \sqrt{D}}{s\sqrt{s^\beta + \tau s^\alpha}} \tag{3.93}$$

Let the Laplace transform of $G(0,t)$, be $\overline{G}(x,s)$

Effects of fractional derivative of order α on boundary temperature

For short time boundary temperature, the α -parameter is considered since it is larger than β. In the Laplace domain

$$\overline{T}(0,s) = \overline{q}_w(s)\frac{\sqrt{D}}{\sqrt{\tau s^{\alpha}}}\frac{1}{s\sqrt{\tau^{-1}s^{\beta-\alpha}+1}}$$

$$\overline{T}(0,s) = \overline{q}_w(s)\frac{\sqrt{D}}{s\sqrt{\tau s^{\alpha}}}\left(\tau^{-1}s^{\beta-\alpha}+1\right)^{-\frac{1}{2}}$$

Let

$$\overline{G}1(0,s) = \frac{\sqrt{D}}{s\sqrt{\tau s^{\alpha}}}\left(\tau^{-1}s^{\beta-\alpha}+1\right)^{-\frac{1}{2}}$$

$L^{-1}\{\overline{G}1(0,s)\} = L^{-1}\{\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{1}{2}\right)_k\tau^{-k}s^{-[k(\alpha-\beta)+\frac{\alpha}{2}+1]}}{k!};t\}$

$$= t^{\frac{\alpha}{2}}\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{1}{2}\right)_k\tau^{-k}t^{k(\alpha-\beta)}}{k!\Gamma(k(\alpha-\beta)+\frac{\alpha}{2}+1)} \tag{3.94}$$

$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$, $\quad (a)_0 = 1$, where $(a)_k$ is the Pochammar symbol

$$G1(0,t') = \sqrt{\frac{D}{\tau}}t^{\frac{\alpha}{2}}E^{\frac{1}{2}}_{(\alpha-\beta),\frac{\alpha}{2}+1}(-\frac{1}{\tau}t^{\alpha-\beta}) \tag{3.95}$$

$$G1(0,t) \propto t^{\frac{\alpha}{2}}\left(1-\frac{t^{(\alpha-\beta)}}{\tau}+\frac{t^{2(\alpha-\beta)}}{\tau^2}\cdots\right)$$

For a short time, the boundary surface temperature is given by:

$T(0,t) = G1(0,t)q_w$

$$T(0,t) = \sqrt{\frac{D}{\tau}}t^{\frac{\alpha}{2}}E^{\frac{1}{2}}_{(\alpha-\beta),\frac{\alpha}{2}+1}(-\frac{1}{\tau}t^{\alpha-\beta})q_w \tag{3.96}$$

Effects of fractional derivative of order β on boundary temperature distribution

To examine the temperature distribution for a long period of time at the boundary, the the β-parameter is considered since it has small value compared to α. In the Laplace domain,

$$\overline{T}(0,s) = \overline{q}_w(s)\frac{\sqrt{D}}{s\sqrt{s^{\beta}}}\frac{1}{s\sqrt{\tau s^{\alpha-\beta}+1}}$$

$$\overline{G}2(0,s) = \sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{1}{2}\right)_k \tau^k s^{-[k(\beta-\alpha)+\frac{\beta}{2}+1]}}{k!}$$

The inverse Laplace transform of $\overline{G}2(0,s)$ is given by:

$$G2(0,t) = L^{-1}\{\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{1}{2}\right)_k \tau^k s^{-[k(\beta-\alpha)+\frac{\beta}{2}+1]}}{k!}\}$$

$$= t^{\frac{\beta}{2}}\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{1}{2}\right)_k \tau^k t^{k(\beta-\alpha)}}{k!\Gamma(k(\beta-\alpha)+\frac{\beta}{2}+1)} \tag{3.97}$$

$$= \sqrt{D}t^{\frac{\beta}{2}}E^{\frac{1}{2}}_{(\beta-\alpha),\frac{\beta}{2}+1}(-\tau t^{\beta-\alpha}) \tag{3.98}$$

To examine a long period of temperature distribution at the boundary, the equation below is considered:

$$T(0,t) = \sqrt{D}t^{\frac{\beta}{2}}E^{\frac{1}{2}}_{(|\beta-\alpha|),\frac{\beta}{2}+1}(-\tau t^{\beta-\alpha})q_w \tag{3.99}$$

$\overline{G}2(x,s)$ can also be express as an infinite series in the Laplace domain as:
$\overline{G}2(0,s) = s^{-(\frac{\beta}{2}+1)}\left(1-\frac{1}{2}\tau s^{-1(\beta-\alpha)}+\frac{1}{4}\tau s^{-2(\beta-\alpha)}\cdots\right)$

$$L^{-1}\{\overline{G}2(0,s)\} = L^{-1}\{s^{-(\frac{\beta}{2}+1)}\left(1-\frac{1}{2}\tau s^{-1(\beta-\alpha)}+\frac{1}{4}\tau^2 s^{-2(\beta-\alpha)}\cdots\right)\} \tag{3.100}$$

$$G2(0,t) \propto t^{\frac{\beta}{2}}\left(1-\tau t^{(\beta-\alpha)}+\tau^2 t^{2(\beta-\alpha)}\cdots\right) \tag{3.101}$$

3.4.7 The generalized fractional Cattaneo heat equation and its special cases

This section looks at how the generalized fractional Cattaneo heat equation can be transformed into other forms of diffusion equations under special conditions. The classical solution of (3.1) of integer orders

$\alpha = 2$,$\beta = 1$ is given by (2.3),i.e

$$G(x,t) = \sqrt{\frac{D}{\tau}} e^{-t/2\tau} I_0 \left(\frac{1}{2\tau} \sqrt{t^2 - \frac{\tau}{D} x^2} \right) u \left(t - x\sqrt{\frac{\tau}{D}} \right)$$

The new solution established for the fractional Cattaneo

The new solution (Qi et al. (2013)) ,i.e(3.57) is given below:

$$G(x,t) = \frac{\sqrt{D}}{\sqrt{\pi\tau}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{k(\alpha-\beta)+\alpha/2-1}}{\tau^k} H_{1,2}^{2,0} \left[\frac{\tau x^2}{4Dt^\alpha} |_{(0,1),(k+1/2,1)}^{(k(\alpha-\beta)+\alpha/2,\alpha)} \right]$$

Special cases of the fractional Cattaneo model(3.19)

(i) when $\tau = 0$,

(3.19) becomes the fractional wave- diffusion (Qi et al. (2013)), i.e

$$\frac{\partial^\beta U(x,t)}{\partial t^\beta} = D\frac{\partial^2 U(x,t)}{\partial x^2} \tag{3.102}$$

The above equation has its solution as:

$$\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{s^\beta}} exp\left(-\frac{x}{\sqrt{D}} \sqrt{s^\beta} \right)$$

Proof: The Laplace transform of equation(3.102) is given below.

$$L\{\frac{\partial^\beta U(x,t)}{\partial t^\beta}\} = L\{D\frac{\partial^2 U(x,t)}{\partial x^2}\}$$

$$S^\beta \overline{U}(x,s) + \sum_{k=0}^{1} S^{\beta-1} \overline{U}^{(0)}(x,s) = D\frac{\partial^2 \overline{U}(x,t)}{\partial x^2} \tag{3.103}$$

Considering zero initial conditions

$$S^{\beta}\overline{U}(x,s) = D\frac{\partial^2\overline{U}(x,t)}{\partial x^2} \tag{3.104}$$

solving (3.104) in the Laplace domain produces the solution

$$\overline{G}(x,s) = \frac{\sqrt{D}}{\sqrt{s^{\beta}}} exp(-\frac{x}{\sqrt{D}}\sqrt{s^{\beta}}) \tag{3.105}$$

$$= \frac{\sqrt{D}}{\sqrt{s^{\beta}}} H^{1,0}_{0,1}\left[\frac{x}{\sqrt{D}}\sqrt{s^{\beta}}|_{(0,1)}\right] \tag{3.106}$$

Using the inverse Laplace transform for the Fox H-function (3.106) is obtained as:

$$= \sqrt{D}t^{\frac{\beta}{2}-1} H^{1,0}_{0,1}\left[\frac{x}{\sqrt{Dt^{\beta}}}|^{(\frac{\beta}{2},\frac{\beta}{2})}_{(0,1)}\right]$$

$$= t^{\frac{\beta}{2}-1} W\left(-\frac{x}{\sqrt{Dt^{\beta}}}; -\frac{\beta}{2}, \frac{\beta}{2}\right) \tag{3.107}$$

where:

$W(z;\mu,v) = \sum_{k=0}^{\infty} \frac{z^{\mu}}{k!\Gamma(\mu k+v)}$ is the Wright function(Qi et al. (2013))

(ii) if $\tau = 0$ and $\beta = 1$,

(3.19) becomes the classical heat equation, i.e

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^2 U(x,t)}{\partial x^2}$$

which has the solution

$$G(x,t) = \sqrt{\frac{D}{\pi t}} exp\left(-\frac{x^2}{\sqrt{4Dt}}\right) \tag{3.108}$$

3.5 USING A NUMERICAL SCHEME TO STUDY THE FRACTIONAL CATTANEO HEAT EQUATION IN A SEMI-INFINITE MEDIUM

The implicit finite difference scheme is used in solving the numerical examples in this study. The implicit finite difference scheme is a scheme that evaluates a derivative at future time step. In all numerical examples in this chapter : Surface temperature gradient, $q_w, = 1.0$, relaxation time, $\tau = 0.1$, Diffusivity constant, $D = 1.0$

3.5.1 Discretizing a fractional derivative

The Grundwald- Letnikov definition of a fractional derivative is the bases of finite difference schemes for fractional derivatives. (Mariusz (2009)) stated that a Caputo derivative can be converted to a Grundwald- Letnikov derivative using the relation

$$^{C}\partial_t^{\beta} U(x,t) =^{GL} \partial_t^{\beta} \left(U(x,t) - U(x,t)|_{t=0} - t\frac{\partial U(x,t)}{\partial t}|_{t=0} \right) \tag{3.109}$$

where C stands for Caputo derivative and GL means Grundwald-Letnikov derivative

$$U(x,t)|_{t=0} = p_0, \quad \frac{\partial U(x,t)}{\partial t}|_{t=0} = p_1 \tag{3.110}$$

$$^{GL}\tfrac{\partial^{\beta} U(x,t)}{\partial t^{\beta}} = \lim_{\delta t \longrightarrow \infty} \tfrac{1}{(\delta t)^{\beta}} \textstyle\sum_{k=0}^{\frac{t}{\delta t}} (-1)^k \binom{\beta}{k} \left(U(x,t) - p_0 - k\delta t p_1\right)$$

$$\simeq \frac{1}{(\delta t)^{\beta}} \sum_{k=0}^{j} (-1)^k \binom{\beta}{k} \left(U_i^{j-k} - p_0 - k\delta t p_1 \right) \tag{3.111}$$

The Grundwald-Letnikov weight:,

$$w_k^{\beta} = (-1)^k \binom{\beta}{k} = (-1)^k \frac{\Gamma(\beta+1)}{\Gamma(k+1)\Gamma(\beta-k+1)} \tag{3.112}$$

$$w_0^\beta = 1, \quad w_k^\beta = (1 - (\beta+1)/k)\, w_{k-1}^\beta \tag{3.113}$$

In order to discretize a fractional differential equation, two homogeneous grids are defined below.

spatial: $0 = x_0 < x_1 < x_2 \cdots, x_n = a$, and temporal: $0 = t_0 < t_1 < t_2 \cdots t_m = T$.

To discretize space and time, let

$\delta x = \frac{a}{n+1}, \quad \delta t = \frac{T}{m}, \quad x = i\delta x, \quad t = j\delta t, \qquad 0 < j < m, \quad 0 < i < n+1.$

The temperature at any time at any point in the medium is represented by

$U(x,t) = U_i^j, \quad 0 < x_i < a, \quad 0 < t_j < T$

3.5.2 Discretization for Example 1a

The generalized Cattaneo heat equation (4.2) is used, i.e.

$$\partial_t^{2-\gamma} U(x,t) + \tau^\gamma \partial_t^2 U(x,t) = D\partial_{xx} U(x,t) \qquad GCEII,$$

$$\partial_t^\beta U(x,t) + \tau^\gamma \partial_t^\alpha U(x,t) = D\partial_{xx} U(x,t) \quad 0 \le x < \infty, \quad t > 0 \tag{3.19}$$

comparing the fractional orders of GCEII with the generalized factional Cattaneo heat equation

$\beta = 2 - \gamma, \quad \alpha = 2$

The exact solution of the above equation is:

$$U(x,t) = e^{-\frac{x^2}{2Dt^\beta}} t^{\beta/2} \sum_{k=0}^{\infty} \frac{\left(-\tau^\beta t e^{-\frac{x^2}{2Dt^\beta}}\right)^k}{k!(k\beta + \frac{\beta}{2} + 1)} q_w$$

The initial and boundary conditions include:

$U(x,0) = 0$, $U(x,\infty)$=0, $U_x(0,t) = -q_w f(t)$, $U_t(x,0) = 0$.

The differential equation is discretize below using implicit finite difference scheme

$\frac{1}{(\delta t)^\beta} \sum_{k=0}^{j+1} w_k^\beta \left(U_i^{j+1-k} - p_0 - k\delta t p_1\right) + \frac{\tau}{(\delta t)^2}\left(U_i^{j+1} - 2U_i^j + U_i^{j-1}\right)$
$= \frac{D}{(\delta x)^2}\left(U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}\right)$

$$c_0 \sum_{k=0}^{j+1} w_k^\beta \left(U_i^{j+1-k} - p_0 - k\delta t p_1\right) + \left(U_i^{j+1} - 2U_i^j + U_i^{j-1}\right) = r_0 \left(U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}\right)$$

$$-m_0 U_i^{j+1} + r_0 \left(U_{i+1}^{j+1} + U_{i-1}^{j+1}\right) = 2U_i^j - U_i^{j-1} - c_0 \sum_{k=1}^{j+1} w_k^\beta \left(U_i^{j+1-k} - p_0 - k\delta t p_1\right) \tag{3.114}$$

where, $m_0 = c_0 w_0^\beta + 1 + 2r_0, \quad p_0 = 0, \quad p_1 = 0$

Writing out the above implicit finite scheme line by line produces the following:

For j = 0:

$$-m_0 U_i^1 + r_0 \left(U_{i+1}^1 + U_{i-1}^1\right) = 2U_i^0 - U_i^{-1} - c_0 w_1^\beta U_i^0$$

For j = 1:

$$-m_0 U_i^2 + r_0 \left(U_{i+1}^2 + U_{i-1}^2\right) = 2U_i^1 - U_i^0 - \left(c_0 w_1^\beta U_i^1 + c_0 w_2^\beta U_i^0\right)$$

For j = 2:

$$-m_0 U_i^3 + r_0 \left(U_{i+1}^3 + U_{i-1}^3\right) = 2U_i^2 - U_i^1 - \left(c_0 w_1^\beta U_i^2 + c_0 w_2^\beta U_i^1 + c_0 w_3^\beta U_i^0\right)$$

:

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For j = n-1:

$$-m_0 U_i^n + r_0 \left(U_{i+1}^n + U_{i-1}^n\right) = 2U_i^{n-1} - U_i^{n-2} - c_0 \left(w_1^\beta U_i^{n-1} + w_2^\beta U_i^{n-2} + \cdots + w_{n-1}^\beta U_i^1 + w_n^\beta U_i^0\right)$$

From the above, a tridiagonal matrix equation is established below.

$A.U^{j+1} = b(i)$

$$A = \begin{pmatrix} -m_0 & 2r_0 & 0 & 0\cdots\cdots 0 & \\ r_0 & -m_0 & r_0 & 0\cdots\cdots 0 & \\ 0 & r_0 & -m_0 & r_0\cdots\cdots 0 & \\ \vdots & \vdots & \ddots & \ddots\vdots & \\ 0 & 0 & r_0 & -m_0 & r_0 \\ 0 & 0 & 0 & r_0 & -m_0 \end{pmatrix}, \quad U^{j+1} = \begin{bmatrix} U_1^{j+1} \\ U_2^{j+1} \\ \vdots \\ U_{n+1}^{j+1} \end{bmatrix} \qquad (3.115)$$

where

$$b(i) = -2U_i^j + U_i^{j-1} + c_0 \sum_{k=1}^{j+1} w_k^{\beta} U_i^{j+1-k} - 2\delta t q_w f(t)$$

$c_0 = \frac{(\delta t)^{(2-\beta)}}{\tau}, \quad r_0 = \frac{D(\delta t)^2}{\tau(\delta x)^2}, \quad p_0 = 0, \quad p_1 = 0$

The 'ghost points': U_i^{-1} and U_{-1}^j

can be replaced in the implicit finite scheme by applying the central difference scheme to the Neumann boundary conditions:

$$\frac{U_i^1 - U_i^{-1}}{2\delta t} = 0, \quad \frac{U_1^j - U_{-1}^j}{2\delta x} = -q_w f(t)$$

From the above, $U_i^{-1} = U_i^1$ and $U_{-1}^j = U_1^j + 2\delta x q_w f(t)$

The short time heat flux(surface temperature gradient) for example 1a is defined below

$$q(x,t) = e^{-x}\sqrt{\frac{D}{\tau}}\, t^{\frac{\alpha}{2}} E^{\frac{1}{2}}_{(\alpha-\beta),\frac{\alpha}{2}+1}\left(-\frac{1}{\tau}t^{\alpha-\beta}\right) \qquad (3.116)$$

U^{j+1} is a vector of unknown values of U. In computation at each time step, the value of U always depend on the previous values of U. This produces the memory effect that is usually associated with fractional derivatives.

3.5.3 Example 1c: Discretization of Boundary surface Temperature,T(0,t)

Using (4.2), the finite difference scheme for the boundary surface temperature is presented below. From example 1a, the GCEII (4.2) was discretized as:

$$c_0 \sum_{k=0}^{j+1} w_k^\beta \left(U_i^{j+1-k} - p_0 - k\delta t p_1 \right) + \left(U_i^{j+1} - 2U_i^j + U_i^{j-1} \right) = r_0 \left(U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1} \right)$$

At the boundary, x_0

For $i = 0$

$c_0 \sum_{k=0}^{j+1} w_k^\beta \left(U_0^{j+1-k} - p_0 - k\delta t p_1 \right) + \left(U_0^{j+1} - 2U_0^j + U_0^{j-1} \right) = r_0 \left(U_1^{j+1} - 2U_0^{j+1} + U_{-1}^{j+1} \right)$

Using the Neumann boundary condition, $\frac{U_1^{j+1} - U_{-1}^{j+1}}{2\delta x} = -q_w f(t)$

$\Longrightarrow U_{-1}^{j+1} = U_1^{j+1} + 2\delta x q_w f(t)$

and initial conditions ($p_0 = 0 \quad p_1 = 0$), the boundary temperature at the future time step, U_1^{j+1} is given by:

$$U_1^{j+1} = \frac{1}{2r_0} \left(m_0 U_0^{j+1} - 2U_0^j + U_0^{j-1} + c_0 \sum_{k=1}^{j+1} w_k^\beta U_0^{j+1-k} - 2\delta x r_0 q_w f(t) \right) \quad (3.117)$$

where, $m_0 = (c_0 w_0^\beta + 1 + 2r_0)$

A few lines of the scheme for matlab implementation are written below:

For j = 0:

$$U_1^1 = \frac{1}{2r_0} \left(m_0 U_0^1 - 2U_0^0 + U_0^{-1} + c_0 w_1^\beta U_0^0 - 2\delta x r_0 q_w f(t) \right)$$

For j = 1:

$$U_1^2 = \frac{1}{2r_0} \left(m_0 U_0^2 - 2U_0^1 + U_0^0 + c_0 \left(w_1^\beta U_0^1 + w_2^\beta U_0^0 \right) - 2\delta x r_0 q_w f(t) \right)$$

For j = 2:

$$U_1^3 = \frac{1}{2r_0}\left(m_0U_0^3 - 2U_0^2 + U_0^1 + c_0\left(w_1^\beta U_0^2 + w_2^\beta U_0^1 + w_3^\beta U_0^0\right) - 2\delta x r_0 q_w f(t)\right)$$

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For j = n-1

$$U_1^n = \frac{1}{2r_0}\left(m_0U_0^n - 2U_0^{n-1} + U_0^{n-2}\right)$$

$$+\frac{1}{2r_0}\left(c_0\left(w_1^\beta U_0^{n-1} + w_2^\beta U_0^{n-2} + w_3^\beta U_0^{n-3} + \cdots + w_{n-1}^\beta U_0^1 + w_n^\beta U_0^0\right)\right) - \delta x q_w f(t)$$

3.5.4 Discretization for Example 2.1a: Temperature distribution within the medium

The exact solution of GCEI established from this study is:

$$U(x,t) = e^{-\frac{x^2}{2Dt^\beta}} t^{\beta/2} \sum_{k=0}^{\infty} \frac{\left(-\tau^{\beta/2} t e^{-\frac{x^2}{2Dt^\beta}}\right)^k}{k!(k\beta + \frac{\beta}{2} + 1)} q_w$$

The initial and boundary conditions for this example will include:

$U(x,0) = 0$, $U(x,\infty)$=0, $U_x(0,t) = -q_w f(t)$, $U_t(x,0) = 0$, $q_w f(t) = u(t) - u(t - t_s)$.

Applying the operator $\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}$, GCEI (4.1) is discretized below.

$$\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\left(\frac{\partial^\beta U(x,t)}{\partial t} + \tau\frac{\partial^{2\beta} U(x,t)}{\partial t^\beta}\right) = D\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\frac{\partial^2 U(x,t)}{\partial x^2}$$

$$\left(U_i^{j+1} - U_i^j\right) + c_{10}\sum_{k=0}^{j+1} w_k^{1+\beta}\left(U_i^{j+1-k} - p_0 - k\delta t p_1\right)$$

$$= r_{10} \sum_{k=0}^{j+1} w_k^{1-\beta} \left(U_{i+1}^{j+1-k} - 2U_i^{j+1-k} + U_{i-1}^{j+1-k} \right) \tag{3.118}$$

where,

$c_{10} = \tau(\delta t)^\beta, \quad r_{10} = \frac{D(\delta t)^\beta}{(\delta x)^2} \quad p_0 = 0, \quad p_1 = 0, \quad m_{10} = c_{10} w_0^{1+\beta} + 1 + 2w_0^{1-\beta} r_{10}$

Re-writing the above equation yields

$$-m_{10} U_i^{j+1} + r_{10} \left(U_{i+1}^{j+1} + U_{i-1}^{j+1} \right) = -U_i^j + c_{10} \sum_{k=1}^{j+1} w_k^{1+\beta} \left(U_i^{j+1-k} - p_0 - k\delta t p_1 \right)$$

$$-r_{10} \sum_{k=1}^{j+1} w_k^{1-\beta} \left(U_{i+1}^{j+1-k} - 2U_i^{j+1-k} + U_{i-1}^{j+1-k} \right)$$

Below are few lines of the scheme for this example

For j = 0

$$-m_{10} U_i^1 + r_{10} \left(U_{i+1}^1 + U_{i-1}^1 \right) = -U_i^0 + c_{10} w_1^{1+\beta} \left(U_i^0 \right) - r_{10} w_1^{1-\beta} \left(U_{i+1}^0 - 2U_i^0 + U_{i-1}^0 \right)$$

For j = 1

$$-m_{10} U_i^2 + r_{10} \left(U_{i+1}^2 + U_{i-1}^2 \right) = -U_i^1 + c_{10} \left(w_1^{1+\beta} U_i^1 + w_2^{1+\beta} U_i^0 \right)$$

$$-r_{10} \left(w_1^{1-\beta} \left(U_{i+1}^1 - 2U_i^1 + U_{i-1}^1 \right) + w_2^{1-\beta} \left(U_{i+1}^0 - 2U_i^0 + U_{i-1}^0 \right) \right)$$

For j = 2

$$-m_{10} U_i^3 + r_{10} \left(U_{i+1}^3 + U_{i-1}^3 \right) = -U_i^2 + c_{10} \left(w_1^{1+\beta} U_i^2 + w_2^{1+\beta} U_i^1 + w_3^{1+\beta} U_i^0 \right)$$

$$-r_{10} \left(w_1^{1-\beta} \left(U_{i+1}^2 - 2U_i^2 + U_{i-1}^2 \right) + w_2^{1-\beta} \left(U_{i+1}^1 - 2U_i^1 + U_{i-1}^1 \right) + w_3^{1-\beta} \left(U_{i+1}^0 - 2U_i^0 + U_{i-1}^0 \right) \right)$$

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For j = n-1

$$-m_{10}U_i^n + r_{10}\left(U_{i+1}^n + U_{i-1}^n\right) =$$

$$-U_i^{n-1} + c_{10}\left(w_1^{1+\beta}U_i^{n-1} + \cdots + w_{n-1}^{1+\beta}U_i^1 + w_n^{1+\beta}U_i^0\right) - r_{10}\left(w_1^{1-\beta}\left(U_{i+1}^{n-1} - 2U_i^{n-1} + U_{i-1}^{n-1}\right)\right)$$

$$- \cdots - r_{10}\left(w_{n-1}^{1-\beta}\left(U_{i+1}^1 - 2U_i^1 + U_{i-1}^1\right) + w_n^{1-\beta}\left(U_{i+1}^0 - 2U_i^0 + U_{i-1}^0\right)\right)$$

From the above a tridiagonal matrix is established below

$A_0.U^{j+1} = B(i)$

Where,

$$A_0 = \begin{pmatrix} -m_{10} & 2r_{10} & 0 & 0\cdots\cdots 0 & \\ r_{10} & -m_{10} & r_{10} & 0\cdots\cdots 0 & \\ 0 & r_{10} & -m_{10} & r_{10}\cdots\cdots 0 & \\ : & : & \ddots & \ddots : & \\ 0 & 0 & r_{10} & -m_{10} & r_{10} \\ 0 & 0 & 0 & r_{10} & -m_{10} \end{pmatrix}, \quad U^{j+1} = \begin{bmatrix} U_1^{j+1} \\ U_2^{j+1} \\ : \\ U_{n+1}^{j+1} \end{bmatrix} \tag{3.119}$$

$$B(i) = -U_i^j + c_{10}\sum_{k=1}^{j+1} w_k^{1+\beta}U_i^{j+1-k} - r_{10}\sum_{k=1}^{j+1} w_k^{1-\beta}\left(U_{i+1}^{j+1-k} - 2U_i^{j+1} + U_{i-1}^{j+1}\right)$$

$$U^{j+1} = A_0 \backslash B(i)$$

3.5.5 Discretization for Example 3

The GCEIII (4.3) is used to examine the temperature distribution.

$\partial_t^\gamma U(x,t) + \tau\partial_t^{1+\gamma}U(x,t) = D\partial_{xx}U(x,t) \quad (GCEIII)$

$\alpha = 1+\gamma$, where $\beta = \gamma, \beta \in [0,1]$

The initial and boundary conditions include:

$U(x,0) = 0$, $U(x,\infty)$=0, $U_x(0,t) = -q_w f(t)$, $U_t(x,0) = 0$,

$q_w f(t) = exp(-\mu t) - exp(-\upsilon t)$

3.5.6 Example 3.1: Discretization of boundary temperature gradient,f(t)

Apply the differential operator, $\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}$ to GCEIII(4.3) produces:

$$\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\left(\partial_t^{\beta}U(x,t)+\tau\partial_t^{1+\beta}U(x,t)=D\partial_{xx}U(x,t)\right) \tag{3.120}$$

$$\frac{\partial U(x,t)}{\partial t}+\frac{\partial^2 U(x,t)}{\partial t^2}=\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\left(\frac{\partial^2 U(x,t)}{\partial x^2}\right) \tag{3.121}$$

$$\frac{1}{\delta t}\left(U_i^{j+1}-U_i^j\right)+\frac{\tau}{(\delta t)^2}\left(U_i^{j+1}-2U_i^j+U_i^{j-1}\right)$$

$$=(\delta t)^{-(1-\beta)}\frac{D}{(\delta x)^2}\sum_{k=0}^{t/\delta t}w_k^{1-\beta}\left(U_{i+1}^{j+1-k}-2U_i^{j+1-k}+U_{i-1}^{j+1-k}\right)$$

$$\frac{(\delta t)}{\tau}\left(U_i^{j+1}-U_i^j\right)+\left(U_i^{j+1}-2U_i^j+U_i^{j-1}\right)=\frac{D(\delta t)^{1+\beta}}{\tau(\delta x)^2}\sum_{k=0}^{t/\delta t}w_k^{1-\beta}\left(U_{i+1}^{j+1-k}-2U_i^{j+1-k}+U_{i-1}^{j+1-k}\right)$$

$$k_0\left(U_i^{j+1}-U_i^j\right)+\left(U_i^{j+1}-2U_i^j+U_i^{j-1}\right)=r_{20}\sum_{k=0}^{j+1}w_k^{1-\beta}\left(U_{i+1}^{j+1-k}-2U_i^{j+1-k}+U_{i-1}^{j+1-k}\right)$$

Where,

$$k_0=\frac{(\delta t)}{\tau},\quad r_{20}=\frac{D(\delta t)^{1+\beta}}{\tau(\delta x)^2}$$

At the boundary, x_0

$$k_0\left(U_0^{j+1}-U_0^j\right)+\left(U_0^{j+1}-2U_0^j+U_0^{j-1}\right)=r_{20}\sum_{k=0}^{j+1}w_k^{1-\beta}\left(U_1^{j+1-k}-2U_0^{j+1-k}+U_{-1}^{j+1-k}\right)$$

Using the Neumann boundary condition, $\frac{U_1^{j+1}-U_{-1}^{j+1}}{2\delta x}=-q_w f(t)$

$$U_1^{j+1}=1/(2r_{20}w_0^{1-\beta})\left((k_0+1+2r_{20}w_0^{1-\beta})U_0^{j+1}-(k_0+2)U_0^j+U_0^{j-1}-2r_{20}\delta x q_w(t)\right)$$

$$-1/2\sum_{k=1}^{j+1}w_k^{1-\beta}\left(U_1^{j+1-k}-2U_0^{j+1-k}+U_{-1}^{j+1-k}\right) \tag{3.122}$$

Example 3.2: Discretization of boundary surface Temperature,T(0,t)

With the boundary surface temperature, (3.122) above is used. The boundary temperature for this example is given by

$$U(0,t) = \sqrt{D}t^{\frac{\beta}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(-\tau t^{\beta-\alpha}\right)^k}{k!\left(k(|\beta-\alpha|)+\frac{\beta}{2}+1\right)}$$

$$\begin{aligned} U_1^{j+1} &= 1/(2r_{20}w_0^{1-\beta})\left(m_{20}U_0^{j+1} - (k_0+2)U_0^j + U_0^{j-1} - 2r_{20}\delta x q_w(t)\right) \\ &\quad -1/2\sum_{k=1}^{j+1} w_k^{1-\beta}\left(U_1^{j+1-k} - 2U_0^{j+1-k} + U_{-1}^{j+1-k}\right) \end{aligned} \tag{3.123}$$

where, $m_{20} = (k_0 + 1 + 2r_{20}w_0^{1-\beta})$

A few lines of the finite difference scheme for this example are written below.

For j =0

$$U_1^1 = 1/(2r_{20}w_0^{1-\beta})\left(m_{20}U_0^1 - (k_0+2)U_0^0 + U_0^{-1} - 2r_{20}\delta x q_w(t)\right) - 1/2 w_1^{1-\beta}\left(U_1^0 - 2U_0^0 + U_{-1}^0\right)$$

For j =1

$$U_1^2 = 1/(2r_{20}w_0^{1-\beta})\left(m_{20}U_0^2 - (k_0+2)U_0^1 + U_0^0 - 2r_{20}\delta x q_w(t)\right)$$

$$-1/2\left(w_1^{1-\beta}\left(U_1^1 - 2U_0^1 + U_{-1}^1\right) + w_2^{1-\beta}\left(U_1^0 - 2U_0^0 + U_{-1}^0\right)\right)$$

For j =2

$$U_1^3 = 1/(2r_{20}w_0^{1-\beta})\left(m_{20}U_0^3 - (k_0+2)U_0^2 + U_0^1 - 2r_{20}\delta x q_w(t)\right)$$

$$-1/2\left(w_1^{1-\beta}\left(U_1^2 - 2U_0^2 + U_{-1}^2\right) + w_2^{1-\beta}\left(U_1^1 - 2U_0^1 + U_{-1}^1\right) + w_3^{1-\beta}\left(U_1^0 - 2U_0^0 + U_{-1}^0\right)\right)$$

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For j = n-1

$$U_1^n = 1/(2r_{20}w_0^{1-\beta})\left(m_{20}U_0^n - (k_0+2)U_0^{n-1} + U_0^{n-2} - 2r_{20}\delta x q_w(t)\right)$$

$$-1/2\left(w_1^{1-\beta}\left(U_1^{n-1} - 2U_0^2 + U_{-1}^{n-1}\right) + w_2^{1-\beta}\left(U_1^{n-2} - 2U_0^{n-2} + U_{-1}^{n-2}\right)\right)$$

$$-\cdots - 1/2\left(w_{n-1}^{1-\beta}\left(U_1^1 - 2U_0^1 + U_{-1}^1\right) + w_n^{1-\beta}\left(U_1^0 - 2U_0^0 + U_{-1}^0\right)\right)$$

Example 3.3: Discretization scheme for temperature distribution inside the medium,T(x,t)

Applying the differential operator $\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}$ to GCEIII produces

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial^2 U(x,t)}{\partial t^2} = \frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\left(\frac{\partial^2 U(x,t)}{\partial x^2}\right) \tag{3.124}$$

Discretizing the differential equation above produces:

$$k_0\left(U_i^{j+1} - U_i^j\right) + \left(U_i^{j+1} - 2U_i^j + U_i^{j-1}\right) = r_{20}\sum_{k=0}^{j+1} w_k^{1-\beta}\left(U_{i+1}^{j+1-k} - 2U_i^{j+1-k} + U_{i-1}^{j+1-k}\right)$$

$$k_0\left(U_i^{j+1} - U_i^j\right) + \left(U_i^{j+1} - 2U_i^j + U_i^{j-1}\right) =$$

$$r_{20}w_0^{1-\beta}\left(\left(U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}\right)\right) + r_{20}\sum_{k=1}^{j+1} w_k^{1-\beta}\left(U_{i+1}^{j+1-k} - 2U_i^{j+1-k} + U_{i-1}^{j+1-k}\right) \tag{3.125}$$

Re-arranging the above equation produces

$$-m_{20}U_i^{j+1} + r_{20}w_0^{1-\beta}\left(U_{i+1}^{j+1} + U_{i-1}^{j+1}\right) =$$

$$U_i^{j-1} - (k_0+2)U_i^j - r_{20}\sum_{k=1}^{j+1} w_k^{1-\beta}\left(U_{i+1}^{j+1-k} - 2U_i^{j+1-k} + U_{i-1}^{j+1-k}\right)$$

where, $m_{20} = k_0 + 1 + 2r_{20}w_0^{1-\beta}$,

For the matlab implementation of this example, here are few lines of the scheme.

For j = 0

$$-m_{20}U_i^1+r_{20}w_0^{1-\beta}\left(U_{i+1}^1+U_{i-1}^1\right)=U_i^{-1}-(k_0+2)U_i^0-r_{20}w_1^{1-\beta}\left(U_{i+1}^0-2U_i^0+U_{i-1}^0\right)$$

For j = 1

$$-m_{20}U_i^2+r_{20}w_0^{1-\beta}\left(U_{i+1}^2+U_{i-1}^2\right)=U_i^0-(k_0+2)U_i^1-r_{20}w_1^{1-\beta}\left(U_{i+1}^1-2U_i^1+U_{i-1}^1\right)$$

$$-r_{20}w_2^{1-\beta}\left(U_{i+1}^0-2U_i^0+U_{i-1}^0\right)$$

:

:

For j = n-1

$$-m_{20}U_i^n+r_{20}w_0^{1-\beta}\left(U_{i+1}^n+U_{i-1}^n\right)=U_i^{n-2}-(k_0+2)U_i^{n-1}-r_{20}w_1^{1-\beta}\left(U_{i+1}^{n-1}-2U_i^{n-1}+U_{i-1}^{n-1}\right)$$

$$-r_{20}w_2^{1-\beta}\left(U_{i+1}^{n-2}-2U_i^{n-2}+U_{i-1}^{n-2}\right)-\cdots-r_{20}w_{n-1}^{1-\beta}\left(U_{i+1}^1-2U_i^1+U_{i-1}^1\right)$$

$$-r_{20}w_n^{1-\beta}\left(U_{i+1}^0-2U_i^0+U_{i-1}^0\right)$$

A tridiagonal matrix system is established below:

$A_{20}.U^{j+1}=d(i)$

where,

$$A_{20}=\begin{pmatrix} -m_{20} & 2r_{20} & 0 & 0\cdots\cdots 0 & \\ r_{20} & -m_{20} & r_{20} & 0\cdots\cdots 0 & \\ 0 & r_{20} & -m_{20} & r_{20}\cdots\cdots 0 & \\ \vdots & \vdots & \ddots & \ddots\ \vdots & \\ 0 & 0 & r_{20} & -m_{20} & r_{20} \\ 0 & 0 & 0 & r_{20} & -m_{20} \end{pmatrix},\quad U^{j+1}=\begin{bmatrix} U_1^{j+1} \\ U_2^{j+1} \\ \vdots \\ U_{n+1}^{j+1} \end{bmatrix} \tag{3.126}$$

$d(i)=U_i^{j-1}-(k_0+2)U_i^j-r_{20}\sum_{k=1}^{j+1}w_k^{1-\beta}\left(U_{i+1}^{j+1-k}-2U_i^{j+1-k}+U_{i-1}^{j+1-k}\right)$

3.5.7 Conclusion

In this chapter, the exact solutions of the fractional Cattaneo heat equation in a semi-infinite medium have been established properly . It has also Provided the discretization schemes for the three generalized fractional Cattaneo heat equations(GCEs). The next chapter will make use of the various discretized examples in this chapter to provide graphical solutions of the three GCEs as examples.

Chapter 4

ANALYSIS AND DISCUSSION OF RESULTS

4.0.8 INTRODUCTION

In this chapter, graphical results of the analytical solutions and numerical scheme solutions are presented, compared and discussed. The implicit finite difference schemes for all the numerical examples in chapter 3 are implemented using Matlab software. The three Generalized Cattaneo heat equations (GCE's) discretized in chapter 3 are used to plot the finite difference graphs in this chapter. The steps used in solving the generalized Cattaneo heat equation(3.19) are followed in establishing the analytical solutions of each of these three generalized Cattaneo heat equations.

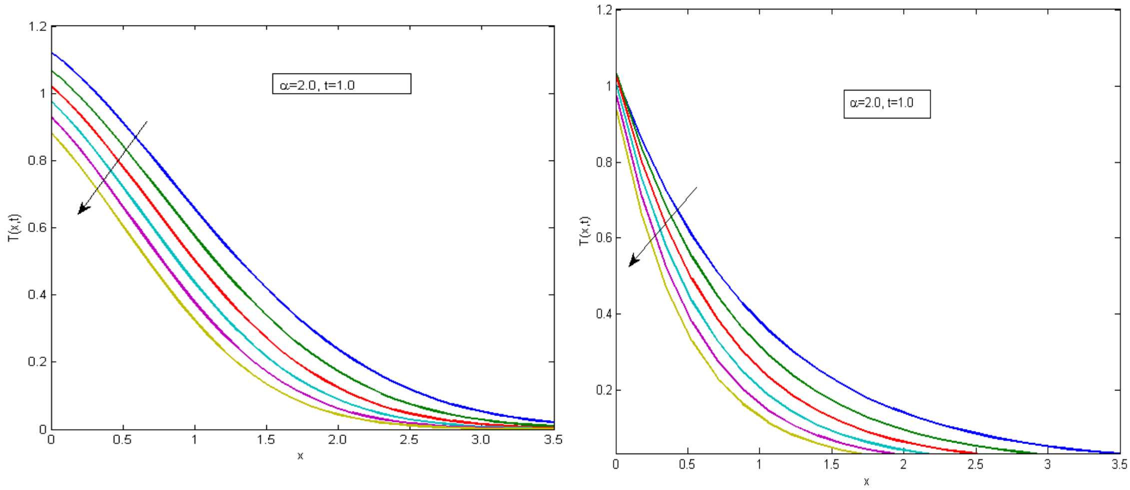
$$\partial_t^{\gamma} U(x,t) + \tau^{\gamma} \partial_t^{2\gamma} U(x,t) = D\partial_{xx} U(x,t) \quad (GCEI) \tag{4.1}$$

$$\partial_t^{2-\gamma} U(x,t) + \tau^{\gamma} \partial_t^{2} U(x,t) = D\partial_{xx} U(x,t) \quad (GCEII) \tag{4.2}$$

$$\partial_t^{\gamma} U(x,t) + \tau \partial_t^{1+\gamma} U(x,t) = D\partial_{xx} U(x,t) \quad (GCEIII) \tag{4.3}$$

4.0.9 Comparison of results of the analytical solution and finite difference solution of Example 1(GCEII)

The effects of the α-order derivative on temperature distribution for a short period of time is shown in this example. Figure 4.1(a) and (b) respectively show the analytical and finite difference graphs for temperature distribution profile inside the semi-infinite medium for different values of fractional orders β at a fixed time.



(a) analytical graph (b) finite difference graph

Figure 4.1: Example 1a :Graphs of $T(x,t)$ versus x. The arrows in the graphs (a) and (b) indicate increasing order of, β=[1.0, 1.2, 1.4, 1.6, 1.8, 2.0]

Though the temperature values decreases in both the finite difference and analytical graphs as β increase, the temperature values in the finite difference graph(i.e 4.1(b)) are slightly lower than the temperature values in the analytical graph (4.1(a)). According to Qi et al. (2013) the temperature values of GCEII falls between those of the wave equation, ($\beta = 2$),and the Cattaneo equation ($\beta = 1$). The propagation speeds of thermal disturbances corresponding to the Cattaneo equation and the wave equation are, $c_t = \sqrt{D/\tau}$ and, $c_w = \sqrt{D/(1+\tau)}$, respectively(Qi et al. (2013)). When the GCEII (4.2) is used, the speed for heat

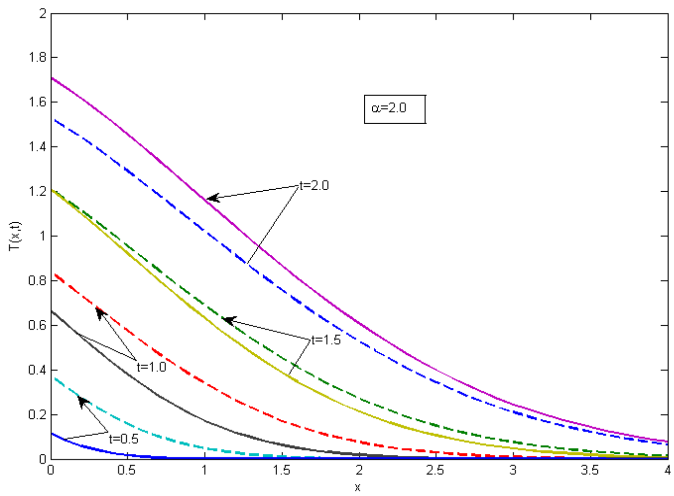
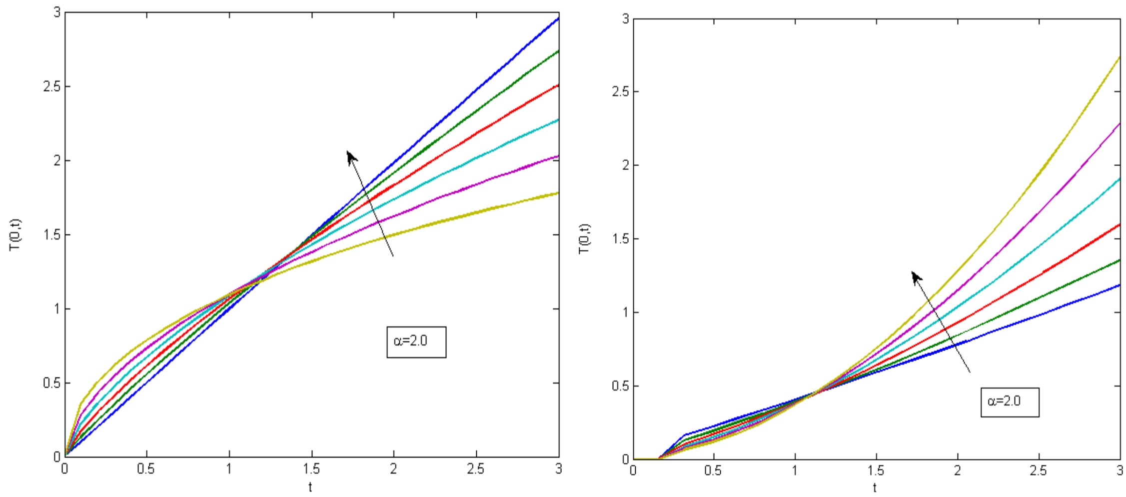


Figure 4.2: Example 1b: Analytical graph showing $T(x,t)$ versus x at different times, t for $\beta = 1.2$ (dashed lines) and $\beta = 1.8$ (continuous line)

propagation is between , c_t and c_w. The thermal speed attains lower values as the values of β increases. This is consistent with the intermediate processes between Cattaneo equation and the wave equation (Qi et al. (2013)). Figure 4.2 shows the analytical graph for temperature profile for a range of time t as a function of variable x for $\beta = 1.2$ and $\beta = 1.8$ inside the medium. The temperature values inside the medium decreases with increasing β, for small time t, whereas increasing β, raises the temperature for large t. Figure 4.3(a) and (b) respectively show the temperature profile at the boundary using the analytical solution and finite difference method. From both the analytic graph and finite difference graph



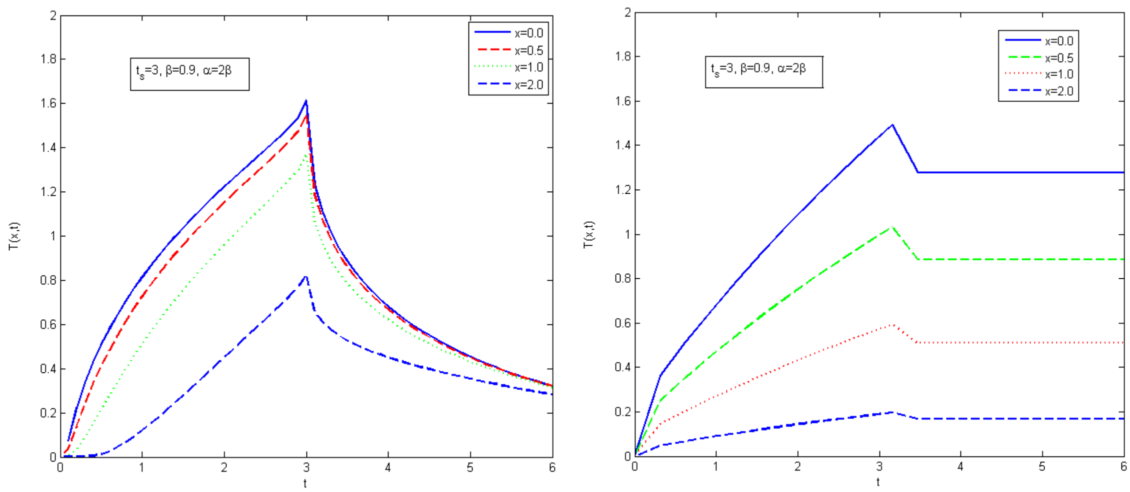
(a) analytical graph (b) finite difference graph

Figure 4.3: Example 1c: Graph of $T(0, t)$ versus t. The arrows in the graphs (a) and (b) indicate increasing order of, β=[1.0 ,1.2 ,1.4, 1.6 ,1.8 ,2.0]

(figure 4.3), the boundary temperature at the early heating period increase with decreasing β while the boundary surface temperature decreases with increasing β at the later stage of the heating process at the boundary. Comparing the results from this example, the temperature values from the finite difference graphs tend to approach the temperature values corresponding to that of the exact solution as depicted in the analytical graphs.

4.0.10 Comparison of finite difference and analytic solutions of example 2(GCEI)

Figure 4.4(a) and (b) respectively show the analytic and finite difference graphs of the temporal variation of the temperature distribution at different locations inside the medium for, $\beta = 0.9$. From the graphs (4.4(a) and (b)), it can be seen



(a) analytical graph (b) finite difference graph

Figure 4.4: Example 2a: Graph of $T(x, t)$ versus t.

that the temperature rises sharply in the early heating period because of internal energy gains from the source. As the heating continues, the temperature rise become gradual due to the enhancement of heat transfer from the surface region to the medium. The the heating process starts at time $t = 0$ and ends at $t = 3$. When $t > 3$, the temperature reduces rapidly and then decays gradually with time as shown in both the analytic graph and finite difference graph (i.e 4.4(a) and (b)) for different locations inside the medium. Comparing figure 4.4(a) and (b) during the cooling process, the temperature decay in the analytical graph (i.e fig.4.4(a)) is sharper than that in the finite difference graph (fig. 4.4(b)). By comparison, both the analytical solution and finite difference (i.e fig.4.4(a) and (b)) graphs produced similar trends of temperature distribution during the heating and the cooling periods.

Figure 4.5 shows the heating and cooling process at the boundary of the medium

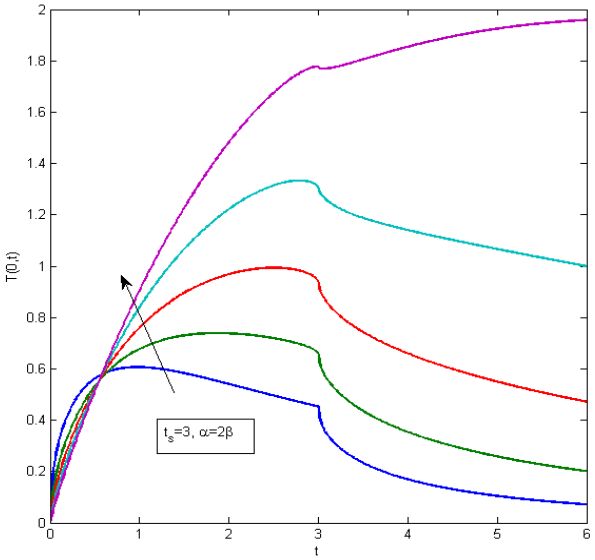
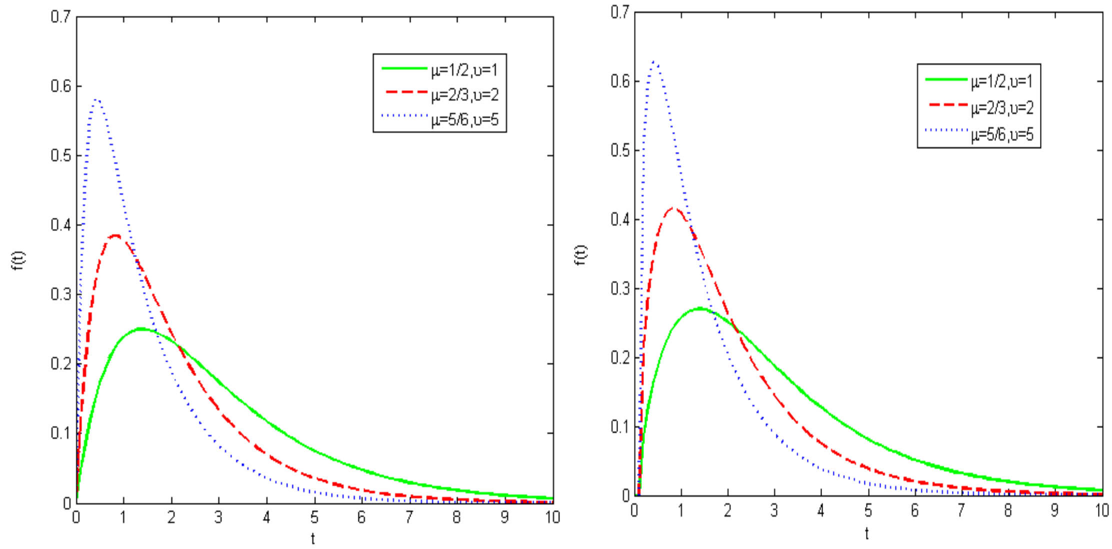


Figure 4.5: Example 2b(analytical graph): $T(0,t)$ versus x at different times, t. The arrow in the graph shows increasing order of $\beta = 0.5, 0.6, 0.7, 0.8, 0.9$

using GCEI. The heating process at the boundary for different values of β in figure 4.5 (example 2b) is similar to the heating process of figure 4.3 (example 1c). Both figure 4.5 and figure 4.3 showed similar temperature rise at the boundary from time $t = 0$ to time $t = 3s$. For a long time period of temperature distribution at different locations inside the medium, the β-order derivative is demonstrated in this example.

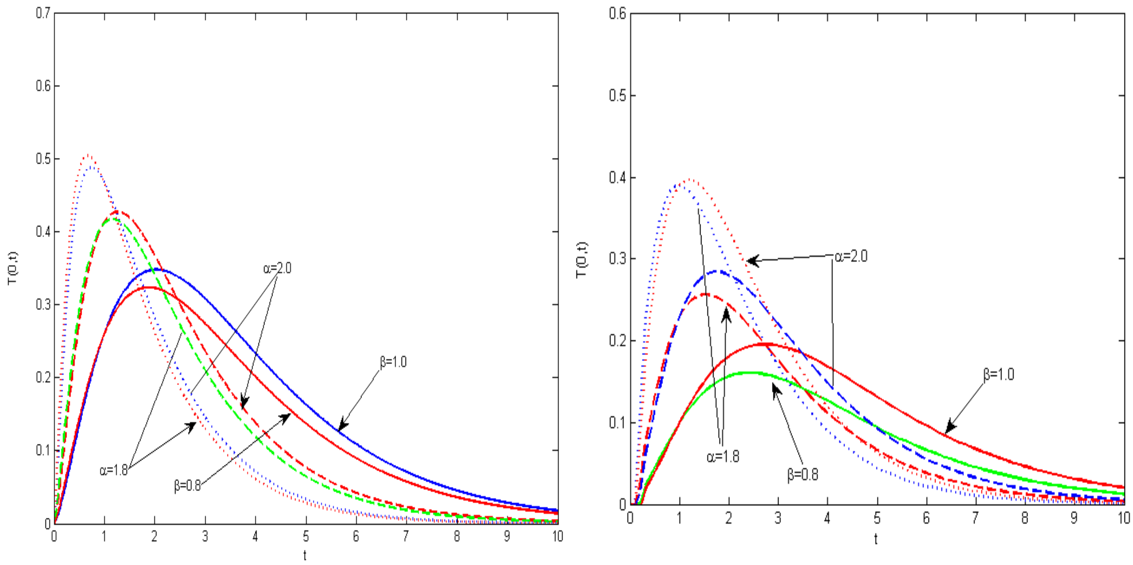
4.0.11 Comparison of finite difference solution to the analytic solution for example 3(GCEIII)

Figure 4.6 (a) and (b) respectively show the analytic and finite difference graphs for the temporal variation of exponential function $f(t)$ and their corresponding temperature rise at the boundary surface. The finite difference graph(4.6(b)) temperature values are almost the same as that of the analytic graph(4.6(a)). Figure 4.7 (a) and (b) also show analytic and finite difference graphs of the boundary surface temperature for the classical Cattaneo model ($\beta = 1$) and the fractional Cattaneo model ($\beta = 0.8$). In the early heating period the rise of boundary temperature for classical Cattaneo model ($\beta = 1$) and the fractional



(a) analytical graph (b) finite difference graph

Figure 4.6: Example 3a: Graph of $f(t)$ versus t.



(a) analytical graph (b) finite difference graph

Figure 4.7: Example 3b: Graph of $T(0,t)$ versus t for $\mu = 1/2$, $\upsilon = 1$ (continuous line), $\mu = 2/3$, $\upsilon = 2$ (dashed line) and $\mu = 5/6$, $\upsilon = 5$ (dotted line).

Cattaneo model ($\beta = 0.8$), is almost the same. However, the boundary temperature reaches higher values for the classical Cattaneo model than the fractional Cattaneo model in both the analytic and finite difference graphs. It is also noted that the temporal variation of temperature (4.7 (a) and (b)) does not follow exactly the temporal variation of $f(t)$ (i.e fig. 4.6 (a) and fig. 4.6(b)) because of the energy transfer from the surface region to the medium. Figure 4.8(a) and

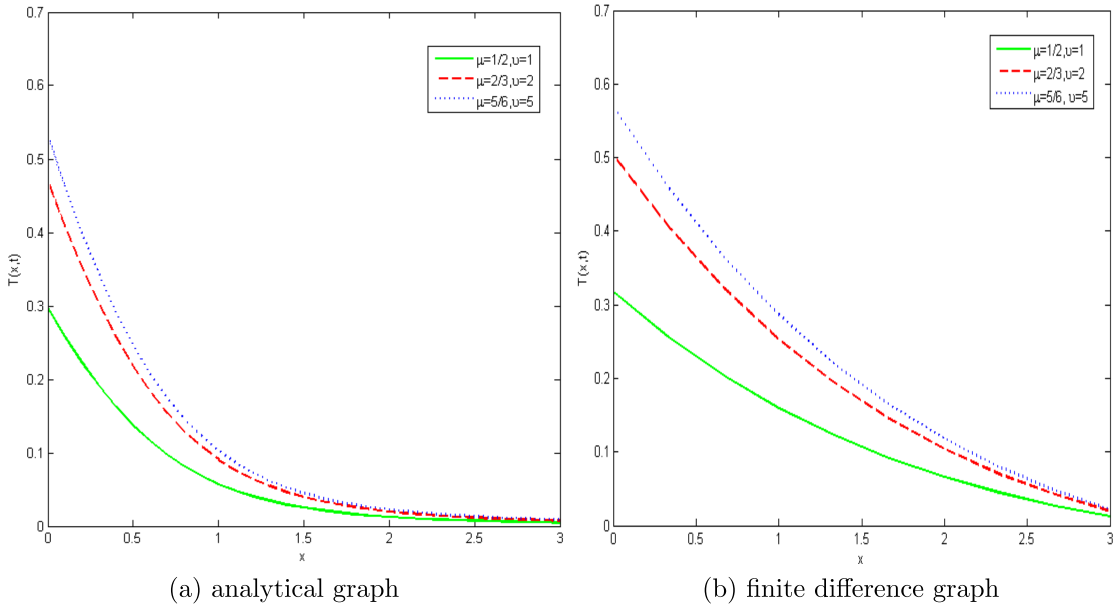


Figure 4.8: Example 3c: Graph of $T(x,t)$ versus x for three exponential pulses at time $t = 1$ when $\alpha = 1.8$ and $\beta = 0.8$

(b) respectively show the analytic and finite difference graphs of the temperature distribution inside the medium at a constant time $t = 1$. The boundary gradient distribution with time has a significant influence on the temperature distribution inside the medium(4.8(a) and (b)). By comparison from all the graphs in this example, the finite difference approach and the analytical solution produced almost similar results.

Chapter 5

Conclusions

1. A detailed proof of the exact solutions (in Qi et al. (2013)) of the fractional Cattaneo heat equation in a semi-infinite medium have been established.

2. A comparison between the analytic and Implicit finite difference solution of the fractional Cattaneo heat equation in a semi-infinite medium using graphical representations have been made in this work.

3. The implicit finite difference method of solving the fractional Cattaneo heat equation in the semi-infinite medium produced results which are very close to the temperature values obtained using the exact solutions established by (Qi et al. (2013)).

4. The numerical examples of the fractional cattaneo heat equation using the finite difference scheme showed similar trend of influence of the fractional derivatives of orders, α and β, on the temperature distribution just as in the case of the analytical graphs.

5.1 RECOMMENDATIONS

1. Since this study did not examine the stability of the exact solutions or numerical scheme used for the fractional Cattaneo heat equations in this study, further research work on stability analysis of the exact solutions of the fractional Cattaneo heat equation in a semi-infinite medium is needed.

2. For easier but reliable results, engineers and scientists can use the implicit finite difference scheme since it gives very good approximations to the temperature values of the exact solutions of the fractional Cattaneo heat equa-

tion and saves the user the complexity and time consuming nature of using special functions such as the H-function to establish exact solutions.

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