KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

GREEN'S FUNCTIONS APPROACH TO INITIAL VALUE PROBLEMS

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Declaration

I hereby declare that this submission is my own work towards the award of the M.Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Abstract

One of the unique applications of Green's functions to solving ordinary differential equations arises with initial-value problems. A differential equation with auxiliary conditions specified only at the left end points is called an initial value problem. In this thesis a second order linear homogeneous and non- homogeneous differential equations with initial-value conditions are used, whiles the Green's functions approach was used to convert the differential equations too integral equations and then solved.

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CHAPTER 1

Introduction

1.1 Background of the study

Greens functions suffice as mathematical characterizations of significant physical concepts. These functions are of basal importance in many practical problems such as quantum field theory and electrodynamics. It's also used in the understanding of the theory of differential equations. Some authors call the Greens function an influence function(M.Rahman, 2007). The Greens function is a carefully worked-out function that was developed by George Greens in 1793 to 1841. Greens functions is also used to derive solutions to partial differential equation as well as linear differential equations of order two in diverse dimensions and also for both time-dependent and independent problems. (Mehdi Delkhosh, Mohammad Delkhosh, 2012). There are several ways in which homogeneous and non homogeneous linear differential equations of the second order with boundary conditions can be transformed into an integral equation. One arises from the differential equations with initial conditions imposed on them ,also known as initial value problems and those with boundary conditions imposed on them which is also known as the boundary value problems,whiles the others occur in the use of Mellin, Laplace and other transforms. In each case the integral obtained is an inversion or convolution integral .The method used in this work can be called the Greens functions approach to initial value problems. This approach was first used by Liouville who considered the unique equation $\frac{d^2u}{dx^2} + p^2u = q(y)u$. As an inhomogeneous differential equation which was then stated as

$$u(y) = u(0)cos\, p(y) + u(0)sin\, p(y) + \frac{1}{p}\int_0^y sin(y-t)q(t)u(t)dt$$

This integral equation was then used to study the asymptotic behavior of solutions for large p (J.Liouville, 1827).

Also ,Ikeda and Fubini later considered the problem by comparing the equation that has been stated with a parallel one whose solutions have been established. The transformation of differential equations into an integral equation is a probable procedure whenever the differential equation is not easily solved. However, the reverse is not true (Ikeda,1949,F.Fubini,1937).

1.2 Problem Statement

Finding solutions to problems of non -homogeneous and homogeneous differential equations of the second order with a definite source, and also with certain boundary conditions that must be satisfied by the differential equation, has been one of the fundamental problems of field theory . So this thesis seeks to employ the Greens functions approach to solving initial value problems to the non-homogeneous and the homogeneous differential equations of the second order

1.3 Objectives of the study

The core function of the study is to:

1. Construct Greens functions from second order non-homogeneous and homogeneous linear differential equations.

2. Use the integral equation with a Green's function as its kernel to find solutions to the homogeneous and non-homogeneous second order differential equations with initial value conditions.

1.4 Scope Of work

This research will utilize the use of academic papers and resources and will involve a study of the repository of works on Greens functions and differential equations and its associated topics. Well conventional and available textbooks and principal mathematical resources and software associated with this field of study will also be used as when it will be deemed necessary.

1.5 Purpose Of The Research And Its significance

To probe how the Green's functions technique can be use in finding solutions to initial value problems is the fundamental purpose of the study,thus by using the greens functions approach to transform a homogeneous and non-homogeneous linear differential equations of the second order into integral equations, by using a Green's function as a kernel in the integral equation and using it to find solutions to initial value problems and make it available to academia and mathematicians, especially those with keen interest in differential and integral equations thereby contributing to knowledge.

1.6 Organisation of the study

The thesis comprises of five chapters .

- Chapter one is an introduction dealing with background study, problem statement, objectives of the study, scope of work and purpose of research and significance.
- Chapter two is about the review of the related literature.
- Chapter three is about the derivation of Green's function from differential equations.

- Chapter four describes how the Green's function is used in deriving a solution to a second order non-homogeneous and homogeneous linear differential equation.
- Chapter five discusses the summary and recommendation

CHAPTER 2

Literature Review

This chapter is to review and discuss concepts and theories from the literature that are related to the Greens theorem approach to initial value problems. The literature review was discussed under the following themes.

- Theoretical framework
- Greens functions technique
- Green's Function integral representation

2.1 Theoretical framework

Greens function, a scientific function that was developed by George Greens. Greens functions can be used in finding solutions to ordinary and partial differential equations in different dimensions, especially time-independent and dependent problems. (Mehdi Delkhosh, Mohammad Delkhosh, 2014). Mathematically, the correlation that exist between a double integral across a level area which is confined and a line integral around a closed region is shown by the Greens theorem. In nuclear physics,the Green's theorem has many applications of which one is finding solutions to the flow in two-dimensions and integrals,which states that the addition of the total outflow around an encircling area is equivalent to the addition of the fluid out flows from a volume. In a two dimensional figure and also in a specific area survey, The Green's concept can be employed to define the specified area and the center of the two dimensional figure solving by integrating over the total distance around the plane figure.

Providing solutions to differential equations of the second order and beyond,which has a stated source and also when the differential equation is required to fulfill certain boundary condition is one of the fundamental problems those in field theories encounter (Kadanoff, 2018). One of the important applications of the Green's theorem to finding results to a homogeneous and a non-homogeneous linear differential equations arises from the ones with initial conditions imposed on them. A second order differential condition with auxiliary initial condition specified only at left endpoints is called an initial valued problem. There are several ways in which a differential equation with boundary value or initial value condition can be transformed into integral equations. One arises from differential equations with boundary value and those with initial value conditions, another occurs in the use of Laplace, Mellin or other transforms in both cases the integral obtained is an inversion or the convolution integral. Physical problems that are modeled involving differential equations are more often reduced to integral equations. The reason for converting a homogeneous and a non homogeneous differential equation to the required integral equation is that all settings specifying the boundary or the initial value problem for the non- homogeneous and homogeneous linear differential equations can be transformed into an equation with a single integral. It is for this reason that integral equation methods form a widely accepted tool to deal with many challenging aspects of mathematics as well us physics (Kanwal, 1971).

In this study, the researcher adopted the integral equation method of conversion of differential equations by the method of Greens function technique as the theoretical framework to investigate the Green's theorem approach to initial value problems. Green's functions technique is well suited to handle a group of boundary value problems connected to a linear ordinary differential equation of some order greater than or equal to one. In certain circumstances, this technique can also be used to cast even nonlinear ordinary differential equations into the equivalent problems of solving integral or integro differential equation whose solutions can be derived by various methods. The Green's functions approach has many uses in diverse fields of study in physics.Thus the correlation function which is also known as the Green's function, can be applied in abstracting information from systems such as the relaxations times, the response functions and the density sates in classical differential equation to substantial many-body problems. Notwithstanding its strength and adaptability, it has been identified as a arduous and sometime a tedious method .

This method was first used by Liouville who considered the special equation

$$\frac{d^2u}{dy^2} + p^2u = q(y)u$$

as a non-homogeneous equation, which was then sated as

$$u(y) = u(0)cos\,p(y) + u(0)sin\,p(y) + \frac{1}{p}\int_0^x sin(y-t)q(t)u(t)dt$$

This integral equation was then uesd to study the asymptotic conduct of solution for large p . Ikeda, Fubini, and Tricomi later deliberated on the problem as a contrast of the given equation with a parallel one whose solutions are known. However the method of derivation using operators (effectively) the method used by Ikeda is certainly not the only one possible. The integral equation can also be derived by variation of parameters and in some cases Laplace transform methods. (Tricomi, 1957, F.Fubini, 1937, Ikeda, 1949).

(Marco Frasca, 2018), stated in his work titled A General Depiction for the Green's function for differential equations of the second order which are non linear that 'One of the essential unique methods when it comes to analysis of partial and linear differential equations and their attached systems is the famous influence function (Green's functions). If the influence functions of a linear differential equation, i.e, the required solution of the differential equation with the non-homogeneities substituted by the Dirac delta function is known, then its general solution is denoted in the form of the convolution of the influence function and the non-homogeneities. Using the superposition principle as a base, the Green's function method is believed to be applicable only to linear systems. However, an extension of the Green's function technique to differential equations are which nonlinear was developed by the author of this paper almost a decade ago and applied in derivation of important results in quantum field theory."

Green's functions(influence function) satisfy the special form of the differential equation form of interest usually referred to as the influence function or the Green's function form or singular form whose source term is a Dirac delta function. The influence function(Green's function) is a function belonging to the broader class of fundamental solutions of the original differential equation, this Green's functions not only satisfies the Green's functions form of differential equations but also a given set of boundary conditions (R.G.Harwood, 2014).

Hamid Safdari, Jones E. B (2015). In mathematics, the Green's functions can described as the impulse functions of a non-homogeneous differential equation specified on a particular area, which has a defined boundary or initial condition. . To derive the results of a non-homogeneous differential equation for a example the function $f(x)$ one can consider the principle of superposition and the Green's functions convolution to that arbitrary function ie $f(x)$ which has been define on an area.

2.2 Greens function Technique

Green's function was in existence since 1828 when George Green put out an article aimed at seeking a solution for the poison's equation $f = \Delta^2 u$ for the potential electric u defined in the interior of a boundary volume together with a given boundary condition at the surface of the volume. He presented the solution v of the potential equation

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} + \frac{\delta^2 v}{\delta z^2} = 0$$

in a region R in xyz-space with boundary surface S as

$$4\pi V = -\int v \frac{\delta^u}{\delta n} d\sigma$$

he presented the function u with the properties.

1. U = 0 on the surface S

2. U must fulfil the potential equation in R and

3. At a fixed undetermined point P in the inside, U infinite as $\frac{1}{r}$ where r is defined as the space of any point from P.

The function U was later named Green's function by Riemman.Green's accomplishment was shadowed with supplementary work by Gauss in the 1839 and Hilbert in the 1904. Hilbert used Green's functions to frame the sturm Liouville boundary value problem. (Teeravara, 1977). Also, the mathematical theory of electricity and magnesium was first propounded by George Green. This theory molded the basis of the works of other scientists such as James Clerk Maxwell William Thomson and others. Green's run similar to the great mathematician Gauss (potential theory), Green's function is connected to both ordinary and partial differential equations. (Mehdi Delkhosh, 2012).Green's functions serve as mathematical descriptions of essential physical concepts .These functions are of vital importance in many practical problems and in understanding of the theory of differential equations. (M.Rahman, 2007).

Moreover, Green's functions technique gave many uses in diverse fields in Physics,starting from quantum context many body problems to classical differential equations.Thus In the quantum frame work ,The correlation functions are seen as the Green's function, from which it is conceivable to obtain data from the systems that is been studied, for a example the relaxation times ,response functions and density states. Despite its uniqueness and flexibility, it is known as seen as complex and sometimes burdensome approach. Knowledge of Greens function of a certain medium can fundamentally simplify solving boundary value electromagnetic problems of the medium in question (Mariana M. Odashina & vernek, 2016).Green's functions perform a principal role in the discovery of results in integrals for boundary value problems in diverse computational physics. The concept of thermoelectricity which is a blend of the concept of elasticity and the conduction of heat is one of those fields (Sheremet, 2009).

Knowledge of the Greens function of diverse media seems quite restricted, due to inadequate knowledge of solution related to the corresponding differential operators (I. V. Lindell, 2001). Moreover, In other to obtain the Andreev bound states,the Josephson current and the local density states in a coherent manner in the concept of junctions of superconductors, the Greens function is used.It was shown that Green's functions can be constructed on the surface of topological insulator subsequent to Mcmillans's formalism where the spectrum energy of electrons response to a lined dispersion (Bo Lu, 2015). The Green's function as an integral kernel is useful in finding solutions to non-homogeneous and homogeneous differential equation with boundary conditions. it gives a fascinating physical implications when the operator of the convoluted differential is a Laplacian. For example, during an experiment in heat conduction the temperature caused by a concentrated energy sources is relative to the Green's function . (Shivani Dubey & Mishra, 2012). For problems which requires that a detailed analysis of wave guides and water waves, the function of Green is particularly well harmonized.the study of three dimensional wave by Kirchoff made him the principal figure in the building of Green's function for equations of waves (K.Ayedemir, 2013). Green's function provide an effective method for analyzing the local density of states conductance, and other transport-related properties of semiconductor systems (Rosen, 2009).
Also, (Tugce Akkaya & Horseen, 2016) in their work titled Green's functions

involving semi-infinite beams by means of boundary damping , employed the Green's function technique in finding solutions to the initial value problems for the fourth order partial differential equation ,where they considered a vibrating homogeneous square semi-infinite beam which has a classical boundary condition such as gliding clamped,pinned or with the ones without classical boundary condition such as the dampers, where Greens function was explicitly given using the Laplace transform in the framework of finding particular solution for semi-infinite beams that has a boundary damping. Greens function technique relies on defining a special function that incorporates the boundary conditions of the problem and which permits changing the differential equations into an integral equation which is sometimes easier to solve (H.Ameling, 1983).

(Abushammala, 2014), Reported in her work title ''Iterative Methods for numerical solutions of Boundary value problems", That embedding Greens functions into well-established fixed points iterations, in Picards and krasnoselskii gives exact solutions very well and yields accurate results than Adomain Decomposition Method and Variation iteration method when one is looking for uniformly convergent iterations that relates to the wide class of nonlinear and linear boundary problems in the third order that arise in physical applications. It is well known that results of a stated boundary problem concur with the fixed points that are related to integral operators which have a related Green's function as its kernel . The Green's function as a results ,performs a very significant role in regards to the study of boundary value problems. (Alberto cabada, 2018) In addition, Greens functions method in the discrete problems is very effective in the determination (in solving convergence problem) of the rate of finite difference to the exact result of a differential equation as the discretization parameter approaches zero. (Al-Refai, 2017). (Steffem Borm, 2015), Stated in his work titled Approximation of Integral operators using the method of Green's quadrature and Nested cross approximation. In his work he introduced a firm algorithm that was used to build approximation of a data sparse for matrices that comes from the method of integral equations for elliptic partial differential equations frame work.In his novel algorithm , he adopted the Green's depiction formula together with the quadrature to achieve the first approximation of the function of the kernel and also the related nested cross approximation to achieve a more precise effective representation .

Abdelgabar Adam Hassan (2017). Stated that the differential equation series results produces an unbounded series which frequently converges gradually. it is perplexing for one to gain understanding into an over-all behavior of a solution. Instead of an unbounded series the Green's functions technique permits us to have a solution represented in the integral form .To achieve a filed caused by a distributed source, calculation is done on each of the effect of the basic portions of the source and then add integral to them completely.For example if r caused by a unit source r_0 whiles $G(r, r_0)$ is then considered as the field at the observers point, then r which is the field is caused by the distribution of $f(r_0)$ is the integral of $f(r_0)G(r, r_0)$ above the total domain of r_0 filled by the course.He later named the function$G$as the Green's function

In computing the Greens functions of a two or three-dimensional half space of Laplace and Helmholtz equations with impudence boundary conditions,the partial Fourier transform is used, and the Green's function obtained is used to arrive at a solution to the densely wave propagation problem of the densely perturbed impedance half spaced numerically by the method of integral equation together with the boundary element technique. (Hein, 2010).In find solutions to problems of automatic image annotation of a unique semi-supervised frame work of learning, a unique correlated multi-label Green's functions approach was used in graphing the annotated images, where the connections among the labels will be assimilated into the objective function will help in improving the performance considerably (Chris Ding & Huang, 2007).Green's function technique used in first –order differential equations varies from the normal claim in a differential equations of a second order, in fact, there should be a stated condition and the problem should not be considered a boundary value problem as usual, for example the Sturm – Liouville type equations. This inequity generates a vital asymmetry in deference to the variable which is independent, by permitting the casualty to be respected given that the the initial condition which has been chosen in deriving the Green's function completely. The application of the compacted form of the linear differential equation's general solution in the first order is permitted by constructing a straight relationship with the Green's functions approach (santos, 2010). (Snyder and Cruse, 1975),Proposed a method to modify the fundamental singular solution of a boundary integral equation, in a way in which a crack surface in a stress -free planar crack in an anisotropic elastic material will not be included in the integral path in their work titled determination of stress-free planar crack in anisotropic elastic material.In their attempt to achieve this they arrived at an improved singular solution called the influence function(Green's function) for stress-free planar crack in an anisotropic elastic material.it was realized that the Green's function derived provided extremely accurate results , particularly in during the calculation of the crack edges at the various stress. One of the disadvantages of the Green's function cradle was that, it demanded lot of mathematical effort in determining explicitly the known function for the class of crack problems that are limited. (Snyder and Cruse 1972), also stated that some authors have used a construction of a boundary integral which does not consider the use of Green's functions to find solutions to a class of plane elastic problems.In their method all that is required is to integrate over the surface of the crack to resolve the class of plane elasticity problems with homogeneous anisotropic materials with stress-free elastic problems with straight cracks at the center.A drawback to this approach is that integrating over the crack surface with stress singularity may be very complicated with any numerical integration technique that will be used.also,the formulation will not allow the crack to be modeled .In general modeling crack is not an easy task since the faces of the two cracks line on the same plane and on one another

(Qin and Mai, 2002), stated that the boundary element approach is usually suitable for stress concentration singularity problems. Where they used Greens function technique the hole and, inclusion problems and variational principles. Where the the continuity condition or the boundary condition of the crossing point at the middle of in matrix and inclusion is fulfilled consequently and priori, not intricate in the equation of the boundary element.

(Kazuhisa Tomita, 1963), Used the technique of two-time Green's function as an efficient way of probing the behavior of slackening in magnetic materials, where a stochastic hypothesis was introduced to let go the chain of higher order Green's function. As a result, the microscope constraint of relaxation in a localized spin was uttered through interaction constant and the spatial correction of spin deviations at a single in a symmetrical assembly".
By expressing the a boundary value problem solution in terms of Green's function then linearity is subjugated,which is computed first to give a cheap look up and speed operations. A collection of capacitance matrix algorithm based on the Sherman - Morrison wood bury formula is also used in finding solution to boundary value problems .Also reusing as well as serially updating previous capacitance of the inverses of matrix can be used in achieving sequential coherence. (James, 2001). properties of Green's functions related equation of hills attached to different two-point boundary value condition, can be applied to obtain the Green's function expression for anti-periodic, Dirichlet and,Neumann mixed problems as an amalgamation of the Greens function in relation to the ones that are periodic. This gives seemly results in the theory of spectrum and

also infers comparison of some results of the hills equation solution that has boundary value conditions that are in different dimensions. (Alberto Cabana & somoza, 2015).Linear embedding via Greens operators to spate a physical domain into smaller sub domains that are easily solved and then recombine the individual solutions by merging of the appropriate boundary conditions to obtain the full results of the domain the linearly embedding Greens operators facilitated the construction of equivalent source at the slab-background interfaces which accounted for radiation fields excited in the crystal. (Guerra, 2013).

(Folkow & Krusten, 1996) .Stated in their report on the topic the Greens functions for Time Domain for the Timoshenko homogeneous beam, that apart from the Greens function approach being a unique tool to find solutions to direct problems, one major benefit of the Green's theorem approach is applying it to the inverse formulation. thus, the Green's theorem method can be widened to encompass inverse and the direct problems for a more complex structures, such as a non-homogeneous beam resting on layer of viscoelastic. A fairly accurate result is achieved without much effort, when the variation method is used to obtain the expression for line capacitance of transmissions line in an inhomogeneous isotropic/anisotropic media when is combined with the transverse transmissions line techniques and the Green's function method. (Guerra, 2013). In finding solutions to the unsteady flow problems in reservoirs that homogeneous the Green's functions method has been an important tool. The application of the method of Green's functions can be extended to problems in heterogeneous reservoirs that are diffusive to achieve a fundamental formula of which can be used to express the general pressure results to the diffusivity equation with non-homogeneous boundary and initial conditions in form of the Green's functions. Where appropriate technique were suggested for obtaining Green's functions that are more complex to find in heterogeneous than in non heterogeneous ones. (Deng, 1996). In situations where Green's function do not exist or is very difficult to find for investigative solutions, a generalized Green's function is constructed.

A generalized Green's function is constructed in situations where the Green's functions do not exist or are very difficult to find.A generalized Green's matrix systems for differential equations that have boundary conditions that are two points were developed by some authors.For example Brown in his attempt to find solutions to a linear stieltjes differential systems that have boundary conditions imposed on them, found the inverses and the generalized Green's functions to those equations. To establish the minimum norm least squares solutions for boundary value problems that are compatible, Locker used the investigation method to derive the generalized Green's functions for some nth order linear differential operators and linearly$K$ independent boundary values as a kernel of the Moore- Penrose inverses (Brown.R.C, 1974).
Green's matrix is usually important in the determination of the stability in tube flow problems. The Green's function was first used years ago in establishing the completeness in the axis symmetric case. There, the system was converted to one fourth-order differential equation. (Herron, 2013).
With a critical look at classical applied mathematics the Green's functions technique is one of the ancient methods and most straight forward approach finding solutions to boundary value problems in relation to linear elliptical partial differential equations. it is an established fact that a specific Greens function may be applied, in principle, and exist to build the solution for boundary value problems in a relation to an elliptical partial differential equations for any suitable domain. The only drawback of the Green's function technique is that is mostly difficult to derive explicitly the Greens functions for the geometries that are realistic. (J.Rizzo, 1997)

2.3 Green's Functions integral representation

The integral equation method was introduced in the year 1826 when Neils Abel arrived at the first integral equation while resolving a problem of mechanics. The problem involved determines the shape of the unknown path of descent of particles under the influence of gravity from a point, given the time of decent as a function of the vertical distance of fall. This was the beginning of an integral equation. Later George Green developed a new technique which is called the Green's functions technique to solve electrostatics. In certain circumstance, the Green's function technique was used to convert non-linear differential equations into equivalent problems of solving integral or differentiation equations into equivalent problems of integral equations whose solutions can be derived by various methods. (.I.Mushkhelish, 1946).

The concept of integral equations permits boundary value problems for partial differential equation to be changed to integral equations stating relevant importance, particularly, they give the basics of numerical solutions of difficult problems Integral equation methods revisited the works of Green during the early part of the 18th century and most of the current theory of integral equation introduced by Poincaré Fredholm and Hilbert, which were assigned to ensure the uniqueness and existence of solutions to equations. Because of the varieties in integral equations there are some distinguishing processes that helps one to identify a stated equation in a precise manner.In reference to integral equations of a single variable the justification system is established on the uniqueness(properties) of the integrals, where the unknown function is located in the equation ,as wells as the uniqueness of the kernel . integral equations are basically classified by their kind and type. The kind refers to where the unknown can be found or located as well as the nature of the function that is attached to the unknown, and the type refers to the nature of the integrals, being it definite or indefinite. (ortiz, 2004).

(R.G.Harwood, 2014) Examined the numerical evaluation of acoustic Greens functions where it was realized that all acoustic analogies, such as non-homogeneous linear differential equation of the second order solutions can be found by applying an integral technique , where the integral equation method will yield an integral equation displaying the acoustic field variable in terms of a volume integral and a source integral .where a particular configuration of the general caused the surface integral to vanish and the solution was expressed as the volume convolution between a specialized function kernel known as the kernel of the inhomogeneous term and the Green's function is also termed as the source term.

(Victor, 2014) also conducted a study on the new integral representation of the main thermo elastic Green's function centered on the introduction of solutions of the particular elliptical lame differential equation through the Green's function approach for the equation of Poisson .Where the resultant integral representation in the form of X-y coordinate allowed the prove of the theorem of the constructive formulas for the main thermo elastic Green's function in terms of the respective Greens function for the Poisson equations. The main advantage of this technique in contrast to the other convolution approaches for establishing the main thermo elastic Green's function is that the integral representation does not essentially require the establishment of the function of the unit force focused on the volume of elastic dilation and the calculation of a complicated integral volume of the product of the heat conduction function and the Green's function.

(Sheremet, 2009), Examined the integral representation in the dynamic uncoupled thermo-elasticity,where he employed the integral formula for the Greens function (also known as the influence function)and the general Green's function type formula for the integral to achieve the results in the cylindric form of coordinates. As a result, the influence function(The Green' function) type of the integral formula for displacement for the boundary and initial value problems of the detached dynamic thermo-elasticity for the wedge was achieved.

CHAPTER 3

Derivation of Greens functions from Differential Equations

The chapter presents the derivation of the Greens function from a linear differeential eqaution of order two, which has been stated in the form

$$L[u] = f \tag{3.1}$$

The differential operator is represented by the L in the above equation and the required solution will be stated as

$$u = L^{-1}[f] \tag{3.2}$$

Since L has been defined as the differential operator, then the inverse of a differential operator is an integral operator, which we will be seeking to write equation (3.2) in the form

$$u = \int G(x, \tau) r(\tau) d\tau \tag{3.3}$$

$G(x, \tau)$ which is a function will be termed as the kernel of the integral operator and is named as the Green's function (influence function) for the differential equation order two. The thesis is restricted to Green's function for a linear differential equations of order two. The researcher began his investigation by examining the solutions of a second non-homogeneous and homogeneous second order linear differential equation using the variation of parameters method. It could be identified that problems of Green's function for initial value , properties of Green's function and the derivation of the Green's function is also conceivable by using the Sturm –liouville concept,this has lead to the several depiction of the Green's functions (M. Rahman, 2007) .

3.1 Initial Value Green's Function

Lets Consider the non-homogeneous linear differential equation.

$$y'' + p(x)y' + q(x)y = r(x) \quad a < x < b \tag{3.4}$$

Where $P(x), q(x), r(x)$ are continuously differentiable on (a, b). The required solution of the above linear differential equation of the second order is found as addition of the required homogeneous equation

$$y'' + p(x)y' + q(x)y = 0, \tag{3.5}$$

and the particular solution the non-homogeneous equations. There are several approaches in deriving the particular solutions to a non-homogeneous differential equation, such as the undetermined co-efficient method .However, a more precise method which is first seen in a first course in differential equation is the variation of parameters technique. While it is sufficient to derive the method for the general differential equation above the researcher considered the equations that are in the Sturm Liouville or the self adjoint form. (P. Kanwal,1971). Therefore the above equation was rewritten as

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}(x)\right) + q(x)y(x) = r(x). \tag{3.6}$$

Assume two independent solutions that are linear has been deteremined for the eqaution of the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}(x)\right) + q(x)y(x) = 0. \tag{3.7}$$

The general solution is presented as $y(x) = u_1(x)y_1(x)+u_2(x)y_2(x)$. To determine the particular solution the the non-homogeneous equation we consider varying the parameters u_1 and u_2 of the solution of the homogeneous problem by making them functions of the independent variables .Thus we are seeking to obtain a particular solution to the non-homogeneous equation which will be stated in the form $y_p = u_1y_1 + u_2y_2$ in order for this to be a solution it must be established that it fulfills the differential equation ,we first compute the derivatives $y_p(x)$. The first derivative

$$y_p'(x) = u_1'(x)y_1(x) + u_1(x)y_1'(x) + u_2'(x)y_2(x) + u_2(x)y_2'(x) \tag{3.8}$$

Without lossing its generality we set

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \tag{3.9}$$

Now

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) \tag{3.10}$$

Taking the second derivative of the remaining terms

$$y_p''(x) = u_1'(x)y_1'(x) + u_1(x)y_1''(x) + u_2'(x)y_2'(x) + u_2(x)y_2''(x) \tag{3.11}$$

We expand the terms of the derivatives in equation (3.6)

$$p(x)y_p''(x) + p'(x)y_p'(x) + q(x)y_p(x) = r(x) \tag{3.12}$$

Putting in the expressions for y_p $y_p'(x)$ and $y_p''(x)$ we obtain

$$p(x)[u_1'(x)y_1'(x) + u_1(x)y_1''(x) + u_2'(x)y_2'(x) + u_2(x)y_2''(x)] + p'(x)[u_1(x)y_1'(x) + u_2(x)y_2'(x)]$$
$$+ q(x)[u_1(x)y_1(x) + u_2(x)y_2(x)] = r(x) \tag{3.13}$$

Rearranging the terms, it was found that

$$u_1(x)[p(x)y_1''(x) + p'(x)y_1'(x) + q(x)y_1(x)] + u_2[p(x)y_2''(x) + p'(x)y_2'(x) \\ + q(x)y_2(x)] + p(x)[u_1'(x)y_1'(x) + u_2'(x)y_2'(x)] = r(x) \tag{3.14}$$

$$p(x)y_1''(x) + p'(x)y_1'(x) + q(x)y_1(x) = 0 \tag{3.15}$$

Since it satisfy the homogeneous part of the equation (3.5) moreover u_1 and u_2 are solutions to the homogeneous eqaution.
Now we obtain

$$p(x)[u_1'(x)y_1'(x) + u_2'(x)y_2'(x)] = r(x). \tag{3.16}$$

Dividing both sides by $p(x)$, we obtain

$$u_1'y_1'(x) + u_2'y_2'(x) = \frac{r(x)}{p(x)}. \tag{3.17}$$

The aim is to determine $u_1(x)$ and $u_2(x)$. In this analysis, it can be determined that the derivatives of these functions fulfill a linear system of equations.

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = 0 \tag{3.18}$$

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = \frac{r(x)}{p(x)} \tag{3.19}$$

Putting the systems of equations in (3.18)and (3.19) into the matrix form we obtain,

$$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{r(x)}{p(x)} \end{bmatrix}$$

solving by applying the Cramer's rule to the matrix eqaution we obtain,

$$u_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ \frac{r(x)}{p(x)} & y_2'(x) \end{vmatrix}}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}$$

$$u_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \frac{r(x)}{p(x)} \end{vmatrix}}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}$$

The system is easily solved to give,

$$\begin{aligned} u_1'(x) &= \frac{-r(x)y_2(x)}{p(x)[y_1(x)y_2'(x) - y_1'(x)y_2(x)]} \\ u_2'(x) &= \frac{r(x)y_1(x)}{p(x)[y_1(x)y_2'(x) - y_1'(x)y_2(x)]}. \end{aligned} \tag{3.20}$$

But $y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is the Wronskian then the results can be presented in another form by representing the Wronskian by ω, rewriting the solution in (3.20) we obtain,

$$\begin{aligned} u_1'(x) &= \frac{-r(x)y_2(x)}{p(x)\omega(x)} \\ u_2'(x) &= \frac{r(x)y_1(x)}{p(x)\omega(x)}. \end{aligned} \tag{3.21}$$

Integrating the equations in (3.21) we obtain,

$$u_1'(x) = \int \frac{-r(x)y_2(x)}{p(x)\omega(x)} dx$$

and

$$u_2'(x) = \int \frac{r(x)y_1(x)}{p(x)\omega(x)} dx$$

then the particular solution can be written us,

$$y_p = y_1(x) \int \frac{-r(x)y_2(x)}{p(x)\omega(x)} dx + y_2(x) \int \frac{r(x)y_1(x)}{p(x)\omega(x)} dx$$

. Let x_0 represent any point on the interval (a,b) and the indefinite integral be replaced by definite integral with reverence to τ which is a dummy variable from x_1 to x then the required solution can be expressed as

$$y_p = y_1(x)\int_{x_1}^{x}\frac{-r(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}d\tau + y_2(x)\int_{x_0}^{x}\frac{r(\tau)y_1(\tau)}{p(\tau)\omega(\tau)}d\tau. \tag{3.22}$$

Considering equation (3.6) with an initial condition

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}(x)\right) + q(x)y(x) = r(x), \quad y(0) = 0\,,\, y'(0) = 0$$

Then it can be assumed that the particular solution also satisfy the initial condition, such that $y_p(0) = 0$, $y_p'(0) = 0$. Therefore we need to focus only on solving for the particular solution that satisfy homogeneous initial conditions. From (3.22),

$$y_p = y_1(x)\int_{x_1}^{x}\frac{-r(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}d\tau + y_2(x)\int_{x_0}^{x}\frac{r(\tau)y_1(\tau)}{p(\tau)\omega(\tau)}d\tau. \tag{3.23}$$

Seeking values for x_0 and x_1 which fulfills the homogeneous initial conditions $y_p = 0$, $y_p' = 0$. Considering $y_p(0) = 0$ we obtain,

$$y_p = y_1(0)\int_{x_1}^{0}\frac{-r(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}d\tau + y_2(0)\int_{x_0}^{0}\frac{r(\tau)y_1(\tau)}{p(\tau)\omega(\tau)}d\tau. \tag{3.24}$$

Because $y_1(x)$ and $y_2(x)$ are the homogeneous differential equation solutions ,then we can put $y_1 = 0$ and $y_2 \neq 0$ then we obtain,

$$y_p = y_2(0)\int_{x_0}^{0}\frac{r(\tau)y_1(\tau)}{p(\tau)\omega(\tau)}d\tau$$

. Also, considering $y_p'(0) = 0$. We differentiate equation (3.24) to obtain,

$$y_p' = y_1'(x)\int_{x_0}^{x}\frac{-r(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}d\tau + y_2'(x)\int_{0}^{x}\frac{r(\tau)y_1(\tau)}{p(\tau)\omega(\tau)}d\tau. \tag{3.25}$$

Since differentiating the integral will cancel out we evaluate the result at $x_0 = 0$ then we obtain,

$$y'_p = y'_1(0)\int_{x_0}^{0} \frac{-r(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}d\tau$$

. Assume that $y' \neq 0$, then we can put $x_0 = 0$ then we obtain,

$$\begin{aligned} y_p =& y_1(x)\int_0^x \frac{-r(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}d\tau + y_2(x)\int_0^x \frac{r(\tau)y_1(\tau)}{p(\tau)\omega(\tau)}d\tau \\ =& \int_0^x \left[\frac{-y_2(\tau)y_1(x)}{p(\tau)\omega(\tau)} + \frac{y_1(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}\right] r(\tau)d\tau. \end{aligned} \tag{3.26}$$

Let

$$G(x,\tau) = \frac{-r(\tau)y_2(\tau)y_1(x)}{p(\tau)\omega(\tau)} + \frac{r(\tau)y_1(\tau)y_2(\tau)}{p(\tau)\omega(\tau)}$$

then equation (3.26) becomes,

$$y_p = \int_0^x G(x,\tau)r(\tau)d\tau$$

where $G(x,\tau)$ is initial value Greens function.

3.1.1 Properties of Green's functions

1. It fulfils the homogeneous part of the second order linear differential equation, thus $\frac{\delta}{\delta x}(p(x)\frac{\delta G(x,\tau)}{\delta x}) + q(x)G(x,\tau) = 0$, $x \neq \tau$ such that $x < \tau$ then $G(x,\tau)$ proportionate to $y_1(x)$. Since $y_1(x)$ is a solution of the homogeneous equation. Similarly $G(x,\tau)$ for $x > \tau$. $G(x,\tau)$ is proportionate to $y_2(x)$ since $y_2(x)$ is a solution to linear differential equation of the second order.

2. Boundary conditions for $x = a$, since $G(x,\tau)$, is relative to $y_1(x)$. implying that whatever condition $y_1(x)$ fulfills $G(x,\tau)$ and $x = b$ will also fulfill.

3. Symmetrical or Reciprocity: The function $G(x,\tau)$,is symmetric in its arguments, hence $G(x,\tau) = G(\tau,x)$.

4. The function $G(x,\tau)$ is continuous at $x=\tau$ Thus,

$$G(\tau^+,\tau) = G(\tau^-,\tau)$$

$$G(\tau^-,x) = \lim_{x\to\tau^-} G(x,\tau), x<\tau$$

,

$$G(\tau^+,x) = \lim_{x\to\tau^+} G(x,\tau), x>\tau$$

. Setting $x=\tau$ we obtain,

$$\frac{y_1(\tau)y_2(\tau)}{p\omega} = \frac{y_1(\tau)y_2(\tau)}{p\omega}$$

.

5. Jump discontinuity: The derivatives of $G(x,\tau)$ with reverence to τ is discontinuous at $x=\tau$ this can be written as follows

$$\frac{\delta G(\tau^+,\tau)}{\delta x} - \frac{\delta G(\tau^-,\tau)}{\delta x} = \frac{1}{p(\tau)}$$

. By computing the derivatives ,

$$-\frac{1}{p(\omega)}y_1(\tau)y_2'(\tau) + \frac{1}{p(\omega)}y_1'(\tau)y_2(\tau)$$

$$= \frac{-y_1'(\tau)y_2(\tau) - y_1(\tau)y_2'(\tau)}{p(\tau)(y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau))}$$

CHAPTER 4

Application of Green's functions to linear Differential Equation of the second order

This chapter deals with using the Green's functions theorem to find solutions to a linear non-homogeneous and homogeneous differential equations of the second order when an initial condition has been imposed on it .This will be achieved by constructing the Green's functions that are appropriate for a differential eqaution of order two which can be stated in the form

$$L[u] = f$$

and then seek how to write or transform it to the form

$$u = \int G(x, \tau)r(\tau)d\tau$$

. Where L is the differential operator and f will also represent the known function, where u is the an unknown function. To obtain a solution to a differential equations in this form, one of the methods is to find the inverse operator L^{-1} , which will be an integral operator with $G(x, \tau)$ as its kernel, because L has already been defined as a differential operator. Then the eqaution will be written in the form,

$$u = L^{-1}f = \int G(x, \tau)r(\tau)d\tau. \tag{4.1}$$

Hence , the kernel of the integral operator has been termed as the Green's functions for the differential operator. This integral theorem is then used to show that the initial value problem of a differential equation can be solved in the form of appropriately defined Green's function. Meaning, the results of the differential equation can be stated when the Green's function can be determined (.P.Kanwal, 1971).

The Green's functions approach is a unique technique for deriving the solutions for boundary and initial value problem problems (Sheremet, 2009).

Second Order linear non-homogenous Differential Eqautions

Example 1. (Kanwal, 1971)

Solve the Differential Equation

$$y'' + 4y = r(x), \tag{4.2}$$

where,

$$r(x) = 1, \quad y(\pi) = 0, \quad y'(\pi) = 1$$

.

Solution

Assume the fundamental solution to the homogeneous part of the problem is

$$y'' + 4y = 0, \tag{4.3}$$

where

$$y = e^{\lambda x}, \quad y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

substituting into the homogeneous part of the equation we obtain the auxiliary equation

$$e^{\lambda x}(\lambda^2 + 4) = 0.$$

Solving

$$(\lambda^2 + 4) = 0,$$

we obtain

$$\lambda = \pm 2i.$$

Hence obtain

$$e^{0x}(cos2x + sin2x).$$

since we obtain two complex roots.

Then

$$y_1(x) = cos2x \quad , \quad y_2(x) = sin2x$$

Then the corresponding solution to the homogeneous equation is of the form

$$y_h(x) = Acos2x + bsin2x.$$

For the particular solution

$$y_p(x) = \int_{x_0}^{x} G(x,\tau)r(\tau)d\tau,$$

we first compute the Greens functions $G(x,\tau)$ and Wronskian (w)

$$G(x,\tau) = \frac{y_1(\tau)y_2(\tau) - y_1(x)y_2(\tau)}{p(\tau)w(\tau)}.$$

$$w = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = \begin{bmatrix} cos2x & sin2x \\ -2sin2x & 2cos2x \end{bmatrix} = 2cos^2 2x + 2sin^2 2x = 2.$$

Comparing the differential equation(4.2) to the equation of the form

$$\frac{d}{dx}(p(x)\frac{dy}{dx}(x)) + q(x)y(x) = r(x), \tag{4.4}$$

then $P(x) = 1$ then the Greens function is written us

$$G(x,\tau) = \frac{y_1(\tau)y_2(x) - y_1(x)y_2(\tau)}{p(\tau)w(\tau)} = \frac{cos2\tau sin2x - sin2\tau cos2x}{2(1)}$$

$$= \frac{cos2\tau sin2x - sin2\tau cos2x}{2} = \frac{sin[2(x-\tau)]}{2}.$$

Hence

$$G(x,\tau) = \frac{sin[2(x-\tau)]}{2}.$$

Then the integral

$$y_p(x) = \int_{x_0}^{x} G(x,\tau)r(\tau)d\tau =] \int_{\pi}^{x} \left[\frac{sin[2(x-\tau)]}{2}\right] d\tau.$$

Solving the integral using substitution

$$u = 2(x-\tau) \quad , \quad \frac{-1}{2}du = dr$$

$$\frac{1}{2}\int \frac{-1}{2} sinudu = \frac{1}{2}\left[-\frac{1}{2}(cos(2(x-\tau))) = \frac{1}{2}(1-cos2x)\right]$$

then,

$$y_p(x) = \int_{x_0}^{x} G(x,\tau)r(\tau)d\tau = \frac{1}{2}(1-cos2x),$$

Substituting the initial conditions into y_h yield

$$y_h(\pi) = Acos2\pi + Bsin2\pi$$

$$A(-1) + B(0) = 0.$$

Hence $A = 0$,

substituting the second initial condition into y_h yielded

$$y_h'(\pi) = -2Acos2\pi + 2Bsin2\pi,$$

$$= -2A(0) + 2B(-1) = 1.$$

Solving we obtain $B = -\frac{1}{2}$, then correspond general solution to the initial value problem is written as

$$y = y_h(x) + y_p(x),$$

$$y = \frac{1}{2}sin2x + \frac{1}{4} - \frac{1}{4}cos2x.$$

Example 2 (Kanwal, 1971)

Solve the differential Equation

$$y'' - 3y' + 2y = 4x, \tag{4.5}$$

Where $r(x) = 4x, \quad y(0) = 0, \quad y'(0) = 0$ then we assume that the fundamental solution to the homogeneous part of the differential equation is

$$y = e^{\lambda x} \quad , \quad y' = \lambda e^{\lambda x} \quad y'' = \lambda^2 e^{\lambda x}$$

substituting into equation will yield

$$e^{\lambda x}(\lambda'' + 3\lambda + 2) = 0.$$

Considering the auxiliary equation

$$\lambda'' - 3\lambda + 2 = 0$$

We obtain $(\lambda - 1)(\lambda - 2) = 0$ then $\lambda_1 = 1$, $\lambda_2 = 2$ and this will give the corresponding linearly two independent solutions

$$y_1 = e^x \quad , \quad y_2 = e^{2x}$$

that satisfies the homogeneous differential

$$y'' - 3y' + 2y = 0.$$

Hence the corresponding solution to the homogeneous equation is stated as

$$y_h(x) = k_1 e^x + k_2 e^{2x},$$

The particular solution is obtained as,

$$y_p(x) = \int_{x_0}^{x} G(x,\tau)r(\tau)d\tau,$$

we compute

$$w = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} = 2e^{3x} - e^{3x} = e^3x.$$

Hence the Green's function is stated as

$$G(x,\tau) = \frac{e^{\tau} \cdot e^{2x} - e^{x} \cdot e^{2\tau}}{e^{3\tau}} = e^{-2\tau} \cdot e^{2x} - e^{x} \cdot e^{-\tau},$$

by the above formula we obtain

$$4e^{2x}\left[\frac{1}{4}(-2y-1)e^{-2\tau}\right]_x^0.$$

Evaluating it at the limits we obtain

$$4e^{2x}\left[\frac{1}{4}(-2y-1)e^{-2\tau}\right]_x^0 = 2x - 1 + e^{2x}.$$

Also solving the second part

$$4e^{2x}\int_0^x \tau e^{-2\tau}d\tau = 4e^{x}[e^{\tau}(-y-1)]_x^0$$

evaluating it at the limits

$$4e^{x}[e^{\tau}(-y-1)]_x^0 = -4x - 4 + 4e^{x}.$$

Hence the solution to

$$4e^{2x}\int_0^x \tau e^{-2\tau}d\tau - 4e^{x}\int_0^x \tau e^{-1\tau}d\tau,$$

is

$$y_p(x) = 2x + 3 + e^{2x} - 4e^{x}$$

Initial condition substituted in

$$y_h(x) = k_1e^{x} + k_2e^{2x}.$$

yield

$$k_1 + k_2 = 0 \cdots\cdots (4.6)$$

Substituting the initial conditions into

$$y_h(x) = k_1e^{x} + 2k_2e^{2x}.$$

Yield

$$k_1 + k_2 = 0 \cdots\cdots (4.7)$$

Solving equations (4.6) and (4.7) simultaneously it was obtained that

$$k_1 = -1 \quad , \quad k_2 = 1$$

Thus, the general solution is stated as

$$y_h(x) + y_p(x) = -e^{x} + e^{2x} + 2x + 3 + e^{2x} - 4e^{x}.$$

Example 3 (Rahman, 2007)

Solve the second order linear non homogeneous ordinary differential equations

$$y'' + y = secx$$

Comparing it with the general form of the equation in (3.0.3)

$$p(x) = 1, \quad r(x) = secx, \quad y(0) = 0, \quad y'(0) = 1$$

Then it can be assumed that

$$y = e^{\lambda x}, \quad y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

substituting it into the ordinary differentiation equation the auxiliary equation is obtained as

$$\lambda^2 + 1 = 0.$$

Finding solutions to the auxiliary equation we obtain

$$\lambda = \pm i,$$

then

$$y_1 = cosx \ , \ y_2 = sinx.$$

so the corresponding solution to the homogeneous equation

$$y'' + y = 0,$$

can be stated as

$$y_h(x) = c_1 cosx + c_2 sinx$$

For the Particular solution we first compute the Green's functions $G(x, \tau)$ and the Wronskian ω

$$\omega = \begin{bmatrix} cosx & sinx \\ -sinx & cosx \end{bmatrix} = cos^2x - sin^2x$$

$$G(x, \tau) = \frac{-r(\tau)y_2(\tau)y_1(x)}{p(\tau)\omega(\tau)} + \frac{r(\tau)y_1(\tau)y_2(x)}{p(\tau)\omega(\tau)} = cos(\tau)sin(x) - sin(\tau)cosx$$

Initial condition substituted in y_h yield

$$0 = Acos(0) + Bsin(0)$$

$$A = 0$$

And

$$y_h'(x) = -Asinx + Bcosx.$$

Substituting in the initial conditions

$$-Asin(0) + Bcos(0) = 1.$$

$$B = 1.$$

The particular solution is obtained as

$$y_p(x) = \int_0^x G(x,\tau)r(\tau)d\tau = \int_0^x [cos(\tau)sinx - sin(\tau)cosx]sec(\tau)d\tau.$$

Solving

$$\int_0^x [cos(\tau)sinx - sin(\tau)cosx]sec(\tau)d\tau$$

$$\int_0^x [cos(\tau)sinx - sin(\tau)cosx]\frac{1}{cos(\tau)}d\tau.$$

Computing the boundaries

$$\int_0^x [cos(\tau)sin(x) - sin(\tau)cos(x)]sec(\tau)d\tau = xsin(x) + \frac{1}{2}cos(x)ln|cos^2x| - c$$

$$y_p(x) = \int_0^x [cos(\tau)sin(x) - sin(\tau)cos(x)]sec(\tau)d\tau. = xsin(x) + \frac{1}{2}cos(x)ln|cos^2x|.$$

Hence

$$y_p(x) = xsin(x) + \frac{1}{2}cos(x)ln|cos^2x|.$$

The general solution

$$y = y_h(x) + y_p(x).$$

$$y = sinx + xsinx + \frac{1}{2}cos(x)ln|cos^2x|.$$

Homogeneous Differential Equations

Example 4 (Rahman, 2007)

Find solution to the second order linear differential Equation

$$y'' + y' + 6y = 0 \quad y(0) = 4, \quad y'(0) = -6 \quad r(x) = 0$$

Let

$$y = e^{\lambda x}, \quad y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}.$$

Substituting y, y', y'' into the differential eqaution gives the auxiliary equation

$$\lambda^2 + \lambda - 6 = 0.$$

finding solution to the auxiliary equation, we obtain

$$(\lambda + 3)(\lambda - 2) = 0 \quad \lambda = -3, \; \lambda = 2.$$

Then

$$y_1(x) = e^{-3x} \quad y_2(x) = e^{2x}.$$

So, the corresponding solution to the homogeneous equation is obtain as

$$y_h(x) = k_1 e^{-3x} + k_2 e^{2x}$$

For the particular solution we must solve for the wronskian and the Green's functions.

$$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x). \end{bmatrix} = \begin{bmatrix} e^{-3} & e^{2} \\ -3e^{-3} & 2e^{2x} \end{bmatrix}$$

$$\omega = \begin{bmatrix} e^{-3} & e^{2} \\ -3e^{-3} & 2e^{2x} \end{bmatrix} = 2e^{-x} + 3e^{-x} = 5e^{-x}$$

For the Green's functions

$$G(x,\tau) = \frac{e^{-3\tau} \cdot e^{2x} - e^{-3x} \cdot e^{2\tau}}{5e^{-3x}}.$$

The particular solution is obtained as,

$$y_p(x) = \int_0^x G(x,\tau) r(\tau) d\tau = \int_0^x \left[\frac{e^{-3\tau} \cdot e^{2x} - e^{-3x} \cdot e^{2\tau}}{5e^{-3x}} \right] 0 d\tau$$

Since $r(\tau) = 0$, it implies that

$$y_p(x) = 0.$$

Putting in the initial conditions

$$y_h(0) = k_1 e^{-3(0)} + k_2 e^{2(0)}$$

Yielded

$$k_1 + k_2 = 4 \cdots\cdots\cdots (4.8)$$

$$y_h' = -3k_1 e^{-3(0)} + 2k_2 e^{2(0)}$$

$$-3k_1 + 2k_2 = -6 \cdots\cdots\cdots (4.9)$$

Solving (4.8) and (4.9) simultaneously we obtain

$$k_1 = \frac{14}{5} \quad k_2 = \frac{6}{5}$$

Hence, the general solution

$$y(x) = \frac{14}{5} e^{-3x} + \frac{6}{5} e^{2x}$$

Example 5 (Kanwal, 1971)

Solve the the differential eqaution of the form $y'' + 6y' + 13y = 0$ with initial conditions $y(0) = 4 \quad y'(0) = -6$

Assume

$$y = e^{\lambda x}, \quad y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}.$$

Substituting y, y', y'' into the differential equation, we obtain the auxiliary equation

$$\lambda^2 + 16\lambda + 13 = 0.$$

For solutions to the auxiliary equation, we obtain the roots of the auxiliary equation as

$$\lambda_1 = -3 + 2i \quad \lambda_2 = -3 - 2i$$

Since the auxiliary equation has complex roots then

$$y_1 = e^{-3x}cos2x \quad y_2 = e^{-3x}sin2x$$

Hence, the corresponding solution to the homogeneous differential equation is of the form

$$y_h(x) = Ae^{-3x}cos2x + Be^{-3x}sin2x$$

For the particular solution we need to derive the Wronskian and the Green's functions. The Wronskian

$$\omega = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = \begin{bmatrix} e^{-3x}cos2x & e^{-3x}sin2x \\ 3e^{-3x}cos2x - 2e^{-3x}sin2x & 2e^{3x}cos2x - 3e^{-3x}sin2x \end{bmatrix}$$

$$\omega = 2e^{-6x}cos^2 2x - 6e^{-6x}sin2xcos2x + 2e^{-6x}sin^2 2x$$

Now the Green's function

$$G(x,\tau) = \frac{e^{-3\tau}cos2\tau e^{-3x}sin2x - e^{-3x}cos2xe^{-3\tau}sin\tau}{2e^{-6x}cos^2 2x - 6e^{-6x}sin2xcos2x + 2e^{-6x}sin^2 2x}$$

For the particular solution

$$y_p = \int_0^x \left[\frac{e^{-3\tau}cos2\tau e^{-3x}sin2x - e^{-3x}cos2xe^{-3\tau}sin\tau}{2e^{-6x}cos^2 2x - 6e^{-6x}sin2xcos2x + 2e^{-6x}sin^2 2x}\right] d\tau(0)$$

Since $r(\tau) = 0$, it implies that the particular solution

$$y_p = 0$$

Putting in the initial condition we obtain

$$y(0) = e^{-3(0)}[Acos(2(0)) + Bsin(2(0))]$$

Then

$$A = 4$$

For

$$y_h'(x) = -3e^{-3x}[Acos2x + Bsin2x] + e^{-3x}[-2Asin2x + 2Bcos2x]$$

$$y_h'(0) = -3e^{-3(0)}[Acos2(0) + Bsin2(0)] + e^{-3x}[-2Asin2(0) + 2Bcos2(0)]$$

Since

$$y'(0) = -6$$

We obtain

$$-3A + 2B = -6$$

. Putting $A = 4$ into the equation

$$-6 = -3(4) + 2B \quad , \quad B = 3$$

Hence the general solution

$$y = e^{-3x}[4cos2x + 3sin^2 x]$$

Properties of Greens function

In chapter three Green's theorem properties was presented .In this segment how the knowledge of the properties of the Green's functions allows one to quickly construct Green's functions is shown.

Example 3

Derive the Greens function using the its stated properties for the problem

$$y'' + x^2 y = r(t), \quad 0 < x < 1.$$

$$y(0) = 0 = y(1).$$

1. Greens functions satisfies the homogeneous form of the second order linear differential Equations. Example ,Finding solutions to the homogeneous equation.

   $$y_h(x) = Asinxt + Bcosxt,$$

   for $x \neq t$ then,

   $$G(x, \tau) = A(\tau)sinxt + B(\tau)cosxt.$$

2. Boundary conditions.

   First we have

   $$G(0, \tau) = B(\tau)cosxt = 0$$

   For

   $$0 \leq t \leq \tau,$$

   so

   $$G(x, \tau) = A(\tau)sinxt$$

   Secondly we have $G(1, \tau)$ then $G(1, \tau) = 0$ for $\tau \leq t \leq 1$ So

   $$G(1, \tau) = A(\tau)sinx + B(\tau)cosx = 0$$

dividing through by $\tau cosx$

$$\frac{A(\tau)sinx}{\tau cosx}+\frac{B(\tau)cosx}{\tau cosx}=\frac{0}{\tau cosx}$$

$$A(\tau)tanx+B=0$$

$$B=-A(\tau)tanx$$

This gives

$$G(x,\tau)=A(\tau)sin(xt)-A(\tau)tanxcos(xt)$$

we can simplify this by factoring $A(\tau)$ and expressing the remaining terms over a common denominator

$$G(x,\tau)=A(\tau)[sin(xt)-A(\tau)tanxcos(xt)$$

$$\frac{A(\tau)}{cosx}[sinxtcosx-sinxcosxt]$$

$$-\frac{A(\tau)}{cosx}sinx(1-t)$$

Let

$$-\frac{A(\tau)}{cosx}=C$$

the results is as

$$G(x,\tau)=C(\tau)sinx(1-t) \quad , \quad r\leq x\leq 1$$

since the co-efficient is arbitrary at this point It was realized that $y_2(x)=sinx(1-t)$ part of the linearly independent solution of the homogeneous part of the differential equation.

3. Symmetrical or The reciprocal property: imposing that $G(x,\tau)=G(\tau,x)$

as a result we have that

$$G(x,\tau) = \begin{cases} A(\tau)sinxt \quad , \quad 0 \leq t \leq \tau \\ C(\tau)sinx(1-t) \quad , \quad \tau \leq t \leq 1 \end{cases}$$

Making it symmetric by picking the forms of $A(\tau)$ and $C(\tau)$. Choosing

$$A(\tau) = Ksinx(1-\tau) \quad and \quad C(\tau) = Ksinx\tau$$

Then

$$G(x,\tau) = \begin{cases} Ksinx(1-\tau)sinxt \quad , \quad 0 \leq t \leq \tau \\ Ksinx(1-t)sinx\tau \quad , \quad \tau \leq t \leq 1 \end{cases}$$

4. The Green's function $G(x,\tau)$ is continous at $x = \tau$. It has been realized that continuity have already been established by benefit of the symmetrical property imposed on it during the last step.

5. The Greens functions satisfies the jump discontinuity property:
The value of K needs to be determined. this can be achieved by taking the jump discontinuity of the derivatives.

$$\frac{\delta}{\delta x}G(x,\tau)$$

So inserting the Green's functions we obtain

$$\frac{\delta G(\tau^{+},\tau)}{\delta x} - \frac{\delta G(\tau^{-},\tau)}{\delta x} = \frac{1}{p(\tau)}$$

From Example

$$p(x) = 1$$

$$= \frac{\delta}{\delta x}[Ksinx(1-t)sinx\tau]_{x=\tau} - \frac{\delta y}{\delta x}[Ksinx(1-t)sinxt]_{x=\tau}$$

$$-kxcosx(1-\tau)sinx\tau - kxsinx(1-t)cosx\tau$$

$$= -kxsinx(\tau + 1 - \tau)$$

$$= -kxsinx$$

Therefore

$$k = \frac{-1}{xsinx}$$

Finally the derived Green's functions is written as

$$G(x,\tau) = \begin{cases} \dfrac{-sin(1-\tau)}{xsinx} & 0 \le t \le \tau \\ \dfrac{-sin(1-\tau)}{xsinx} & \tau \le t \le 1 \end{cases}$$

CHAPTER 5

Summary and Recommendation

5.1 Summary

In this thesis a number of initial value problems have been solved in an integral form, which has a function as a kernel of the integral.This function is called the Green's functions.The Green's functions technique was adopted and used in finding solutions to questions of a linear differential equations of the second order,since the approach provided an accurate results as compared to some approximation methods that are used to solve differential equations.A side providing accurate results,it is sometimes cumbersome and complex when the Green's function is being derived. Green's theorem approach contributed in achieving the objectives of the research.The integral concept was also used to depict how the initial value problem can be solved in a well defined Green's function for the initial value problems that were solved. In the research it was realized that a solution is assured or has been found once the Green's function of the initial value problem of the linear differential equation is known or can be determined .Also it was identified that for one to derive the Green's function a definition, the meaning of the function will be needed and the knowledge of its properties will be important in your determination of the Green's functions.

5.2 Recommendation

The researcher recommends that further studies should be done on Green's functions approach to solving partial differential equations and Green's functions approach to differential equations of Higher orders other than two

References

Abushammala, M., 2014. iterative methods for numerical solutions of boundary value problems. united Arab Emirates ,sharjah: s.n.

Adam, A., 2017. Green's functions solutions of non-homogenous Regular sturm liouville problem. j.appl comptatu, pp. so-60.

Alberto cabada, L. C., 2018. Green's functions for even order boundary value problems. Reearch gate, pp. 1- 55.

Alberto Cabana, J. A. C. and Somoza, L. L., 2015. Greens Function and spectral theory for the Hills Equation. Galicia: s.n.

Ameling, H. W., 1983. Numerical Greens function solutions. Wright -Petterson Air force base ,Ohio: Air force institute of technology.

Al-Refai, A. W., 2017. Discrete Greens fuctions. Nicosia : s.n.

Ayedemir, K. O., 2013. Greens function Method for self - adjoint Realization Boundary -value problems with interior singularities. tokat: s.n.

Bo Lu, Y. T., 2015. Study on Green's function on. Nagoya: Nagoya university.

Brown.R.C, 1974. Genaralized Greens functions and generalized inverses for linear differential systems with stieltyles boundary conditions. j differential equations, pp. 16:335-341.

Chris Ding, H. w. & Huang, H., 2007. Image Anotation using Multi-label correlated Greens Function. Arlington: University of Texas.

Deng, X., 1996. Description Of Heterogeneous using pressure and a tracer data. standford university: Xianfa Deng.

Folkow, P. D. & Krusten, G. O. p., 1996. Time domain Green Functions for the homogeneous Timosheno beam, lund University: s.n.

Guerra, D., 2013. Greens Function Diakoptics to Photonic Crystal Modelling. Eindhoven: s.n.

Harwood,R. G. A., 2014. functions, Numerical Evaluation of Acoustic Greens. Manchester: university of Manchester.

Hein, R. O., 2010. Greens Functioen and integral equation for laplace and Helmholtz operators in impedance half spac. s.l.:s.n.

Herron, I. H., 2013. Solving singular boundary value problems for ordinary differential equations. Carrib J.math, pp. 1-30.

James, D. L., 2001. Multiresolution Green's function Method for interactive simulation of large-scale Elastostatic objects and other physical systems in equilibrium, columbia: university of British.

Kanwal, P. R., 1971. linear integral Equations. New York: Academic press.

Kazuhisa Tomita, M. T., 1963. Greens Function Theory of Magnetic Relation. Progree of theoretical physics , p. 29.

Lindell, I. V. , F., 2001. polynomial operators and greens functions. ghent: s.n.

Marco Frasca, A. Z., 2018. A General Representation for Greens function of second oder nonlinear Differential equations. Roma : s.n.

Mariana M. Odashina, B. G. P. & vernek, E., 2016. Pedagogical introduction to equilibrum Greens function Condensed-matter .. brasil: s.n.

Markus Rosenkranz, G. R., 2008. Solving and factoring boundary problems for linear ordinary differential equations. journal symbolic computation, pp. 515-544.

Mehdi Delkhosh, M. D. J., 2012. Greens functions and its Applications. Journal of Basic and applied scientific Research, pp. 1-20.

Mushkhelish, I. N., 1946. Singular Integral Equation. p.noordhoff: s.n.

Ortiz, J. p., 2004. Boundary integral Equations for viscous flows non-newtonian and solid incluions. wisconsin madiso: s.n.

Rahman, M., 2007. integral Equation and their Application. southampton: wit press.

Rizzo, J. F., 1997. bopundary ellements strategies and discretized Green's functions Applications in composite materials and wave mechanics. iowa: s.n.

Rosen, R. c., 2009. Numerical Greens Function modeling of Dimensional Quantum Transport. wolfson college: university of cambridge.

santos, M. N. .. B., 2010. Geens Function Method and the first-order linear differential equation. j math chem, pp. 1-10.

Sheremet, V., 2009. new integral representation in the dynamic uncoupled thermoelasticity. journal thermal stresses.

Shivani Dubey, A. k. & Mishra, M. M., 2012. Greens Function For certain Domains in Heisen Bery Group H. india: University of Delhi.

Steffem Borm, S. C., 2015. Approximation of integral operators by green's quadrature and nested cross approximation. s.l.:s.n.

Teeravara, P., 1977. introduction Greens Function. pp. 1-10.

Tugce Akkaya, w. T. & Horseen, V., 2016. on constructing a greens function for a semi-infinte beam with boundary damping. s.l.:s.n.

Victor, s., 2014. Recent Integral Representations for Thermoelastic Green's Functions of Their Exact Analytical Expressions. journal of thermal streesses, pp. 561-584.