Kwame Nkrumah University of Science and Technology

Double Generalization of Integral Transform

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgment had been made in the text.

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Dedication

I dedicate this thesis to my God Almighty; my help in ages past and my hope for years to come. Then to my girls Gyamfuaa, Boadiwaa, Konadu and Aseda

EyiahTweneboa, my parents and many other persons who have been and continue to be a blessing to my life. God Bless you all richly... AMEN

Abstract

In this work, we extended the one dimensional generalization of integral transform to two dimensions. Thus, we introduce double Generalization of integral transform (DGIT), $G_xG_y\{f(x,y)\} = uv\int_0^\infty\int_0^\infty f(ux,vy)e^{-(usx+pvy)}dxdy, \quad \forall(x,y) \in$ $\{0\} \cup R^+$, for solving partial differential equations (PDEs).

In addition, the convolution, linearity, scaling and convergence properties of DGIT are established in this thesis. We then applied the DGIT to solve some PDEs which confirms the solutions of these PDEs obtained by using other integral transforms.

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Chapter 1

Introduction

1.1 Background of Study

Historically, the search for general methods for solving differential equations is believed to have begun when Newton (1964), classified the first order differential equations into three classes:

1. $\frac{dy}{dx} = f(x)$

2. $\frac{dy}{dx} = f(x,y)$

3. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = u$

Newton expressed the right side of each of the equations in powers of the dependent variables and then assumed an infinite series as the solution. He then went ahead to determine the coefficients of the infinite series in Newton (1744). Other mathematicians such as Leibniz, Bernoulli, Euler and Lagrange have contributed immensely in finding ways of solving differential equations. The efforts of these great researchers have given rise to such classical methods for solving differential equations like Cauchy-Euler method, method of undetermined coefficients, variation of parameters, separation of variables and integrating factor method, which unfortunately, are limited in usage these days because to be able to use them, one has to search for an appropriate technique in order to obtain a solution thereby making them tedious and cumbersome to use.

Currently, integral transforms are the main concern of mathematicians and scientist generally. The history of integral transforms is traced back to the celebrated works of Laplace (1820) and Fourier (1822). Laplace, in an attempt to solve a probabilistic problem, introduce the Laplace transform. This transform has effectively been used in finding the solution of linear differential equations and integral equations. On the other hand, Fourier (1822), provided the modern mathematical theory of heat conduction, Fourier series, and Fourier integrals with applications. He later stumbled on the now famous Fourier transform and the inverse Fourier transform when he attempted to extend the Fourier Integral theorem which is defined on an finite interval to an infinite interval.

Even though both the Laplace and the Fourier transforms have been discovered earlier in the nineteenth century, it was Heaviside (1899) who applied them in electrical engineering to solve ordinary differential equations of electrical circuits and systems and later developed it into modern operational calculus

Integral transforms, are therefore, unique mathematical operation methods through which a real or complex-valued function is transformed into another class of function or sequence.

The advantage of integral transforms over other classical methods is that it transforms a difficult mathematical problem to a relatively easy problem, which can easily be solved. In the study of initial-boundary value problem involving differential equations, for instance, the differential operators are replaced by much simpler algebraic operations, which can readily be solved. The solution of the original problem is then obtained in the original variables by the inverse transformation.

1.2 Definitions of some terminologies and theorems

In order to define the integral transform, we give some basic fundamental definitions:

Definition 1. An **Integral Transform** is any transform T of a given function of the following form:

$$(Tf)(u) = \int_{t_1}^{t_2} f(t)K(t,u)dt$$

where $K(t,u)$ is the kernel, u is the transformed variable and t is the independent variable [Zill (2013)].

Definition 2. *(Linear Operator)*

Let X and Y be two real (or complex) linear spaces, then a transformation $T : X \to Y$ is called a *linear operator* if the domain of T, D_T, is a linear subspace of X, and if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

$\forall\, x, y \in D_T$ and all real (or complex) scalars α and β [Hoffman and Kunze (1971)].

Definition 3. *(Linear Integral Operator)*

If $f(t)$ and $g(t)$ are continuous functions on domain Ω, then the integral operator T is said to be linear if

$$T(\alpha f(t) + \beta g(t)) = \alpha Tf(t) + \beta Tg(t) \qquad \forall f(t), g(t) \in \Omega$$

and α, β scalars [Kreyszig (1978)].

Definition 4. *(Convolution Theorem)*

If $f(t)$ and $g(t)$ are piecewise continuous on $[0,\infty)$ and of exponential order, then

$$f * g = \int_0^\infty f(p)g(x-p)dp$$

is the convolution of the functions f and g [see Zill (2013)].

Definition 5. *(Bounded Linear Operator)*

A linear operator $T : \Omega \to Y$ is said to be bounded if there is a positive constant K such that

$$\|T(x)\| \leq K\|x\| \qquad \forall\, x \in \Omega$$

where K is the boundedness constant [Knapp (2005)]

Definition 6. A **Hilbert Space** is a vector space Ω with an inner product $\langle f,g \rangle$ such that the norm defined by

$$\|f\| = \sqrt{\langle f,g \rangle}$$

turns Ω into a complete metric space [Royden and Fitzpatrick (2010)].

Definition 7. *(Convergence of Series)*
Let $S_n = \sum_{k-1}^{n} a_k$ be the *nth* partial sum of the first n terms of a sequence, then its series converges if there exists a number ℓ such that for any arbitrarily $\varepsilon > 0$, there is an integer $\mathbb{N}$ such that $|S_n - \ell| \leq \varepsilon$ for all $n \geq \mathbb{N}$ [Thomson *et al.* (2001)]

Definition 8. *(Euclidean Space)* An euclidean space is a finite-dimensional vector space over the reals $\mathbb{R}$, with an inner product $\langle \cdot,\cdot \rangle$.

Definition 9. *(Linear Space)*
A linear space X over a field F is a set whose elements are called *vectors* and where two operations, *addition* and *scalar multiplication*, are defined:

1. addition, denoted by +, such that to every pair $x,y \in X$ there exists a vector $x + y \in X$ and

$$x + y = y + x, \qquad x + (y + z) = (x + y) + z, \qquad \forall x,y,z \in X$$

2. scalar multiplication of $x \in X$ by elements $k \in F$, denoted by $kx \in X$, and

$$k(ax) = (ka)x, \qquad k(x+y) = kx+ky, \qquad (k+a)x = kx+ax, \qquad x,y \in X, \qquad k,a \in F$$

Definition 10. *(Norm)*

Let X be a vector space over the field R of real numbers (or the field C of complex numbers). Then a *norm* on X is a function that assigns to each vector $x \in X$ a real number x, satisfying the following four conditions:

1. $kxk \geq 0$
2. $kxk = 0$ iff $x = 0$
3. $k\alpha xk = |\alpha|kxk$ for all $x \in X$, $\alpha \in$ R or (C)
4. $kx + yk \leq kxk + kyk$ for all $x,y \in X$

Definition 11. *(Banach Space)*

A Banach space is a normed vector space $(E, k \cdot k\ )$ that is complete, i.e., every Cauchy sequence in E is convergent, where E is equipped with the metric $d(x,y) :=$ $x - y$.

Definition 12. *(Riesz Representation Theorem)*

Let H be a Hilbert space. For any bounded linear functional $f : H \to$ K there is a unique $y \in H$ such that

$$f(x) = hx;yi \quad \text{for all} \quad x \in H$$

Moreover, $kfk_{H^*} = kxk_H$.

Definition 13. *(Fixed Point Theorem)*

An element $x \in X$ is a fixed point of $f : X \, 7\!\to X$ if $f(x) = x$

1.3 Problem Statement

There are many methods used in searching for solutions to differential equations with variable coefficients. The Cauchy-Euler method, for instance, transforms a linear differential equation into an algebraic equation with an appropriate substitution technique. But the problem with some of these classical methods is that, there is no single substitution expression for a single type of differential equation, thereby making the use of these methods in searching for solution to differential equations more tedious and cumbersome.

However, integral transformations, which has been the focus of mathematicians and scientists of late, have successfully been applied to finding solution of many differential equations. These integral transforms convert the partial differential equations with constant coefficient and boundary conditions into algebraic equation which is then converted back by its inverse operator to obtain the desired solution in a suitable functional space.

1.4 Objectives of Study

The general objectives of this thesis is as follows;

1. To extend the Generalization of integral transform to a double generalization of integral transform (DGIT)
2. To obtain the expression for partial derivatives using the double GIT
3. To apply the double GIT for solving partial differential equations.

1.5 Justification

Most physical phenomena in science, engineering and social science undergo change with time. In an attempt to solve these physical problems led to mathematical models involving functions and their derivatives that gave rise to differential equations.

Integral transforms, believed to have been invented by Euler within the context of second-order differential equation (DE) problems [see Deakin (1985)], are useful tools for solving problems involving differential equations, especially when their solutions on the corresponding domains of definition are difficult to deal with using other classical methods of solving differential equations.

For a given differential equation, defined on a domain, the application of a suitable integral transform allows it to be expressed in such a form that its mathematical manipulation is easier than the original one. In this way, if a solution on the transformed domain is found, then an application of the inverse integral transform will give the solution of the original, Klamkin and Newman (1961.).

Integral transforms, therefore, convert a differential equation into an algebraic equation which in turn is easier to solve than using analytic methods.

1.6 Methodology

We introduce the analytic approach of transforming partial differential equations in an algebraic domain. We also capture the derivations of some functions in two variables as well as derivations of partial derivatives in chapter three. The quantitative method (figures) using Matlab will be used to plot the PDEs in three dimensions.

1.7 Organization of Study

The research is organized into five chapters. Chapter one contains the introduction. It also includes statement of the problem, objectives, justification and organization of the study. In chapter two, we review works which are related to double GIT. The third chapter contains the analytic results of the double GIT and its properties. This is followed by chapter four, which features the application of the double GIT to partial differential equations. Finally, in chapter five, we summarize the findings and give recommendations.

Chapter 2

Literature Review

Since its conception about two centuries ago, integral transforms have successfully been used for solving many problems in applied mathematics, mathematical physics, and engineering science. There are different types of integral transforms in literature that are used in solving differential equations. In this chapter we will summarize a few of these integral transforms.

Laplace (as cited in Schiff (1999)), introduced the Laplace transform in an attempt to solve a statistical problem of chance. The Laplace converts differential equations to algebraic equations and then reconvert it back by means of its inverse operator to obtain the desired solution in a suitable functional space. Besides being different and efficient to traditional methods of solving differential equations like variation of parameters, integrating factor, and the method of undetermined coefficients, the Laplace transform method is particularly advantageous for input terms that are piecewise-defined, periodic or impulsive as well as being able to incorporate the boundary conditions from the beginning.

The Laplace transform is obtained by using the kernel $K(t,s) = e^{-st}$. Thus, for a given function $f(t)$ defined for $0 \leq t \leq \infty$, its Laplace transform is defined as

$$F(s) = \mathcal{L}f(t) \doteq \int_0^\infty f(t)e^{-st}dt \quad \text{for } s > 0,$$

where $F(s)$ is a complex-valued function of complex numbers. The Laplace transform maps a real function to a complex one. In this integral transform, the parameter s is assumed to be positive and large enough to ensure the integral converges. The fact is, in more advance applications, s tends to be complex and in such cases the real part of s must be positive and large enough to ensure convergence [see Stroud (2003)].

The Laplace transform of some elementary functions are tabulated below.

$f(t) = L^{-1}\{F(s)\}$	$F(s) = Lf(t)$
	$\frac{a}{s}$ $\quad s > 0$
a	$\frac{1}{s+a}$ $\quad s > -a$
e^{-at}	$\frac{1}{(s+a)^2}$ $\quad s > -a$
te^{-at} t t^2	$\frac{1}{s^2}$ $\quad s > 0$
	2 $s > 0$ den
$\sin at$	$\frac{a}{s^2+a^2}$ $\quad s^2 + a^2 > 0$
$\cos at$	$\frac{s}{s^2+a^2}$ $\quad s^2 + a^2 > 0$

Estrin and Higgins (1951) extended the Laplace transform to double Laplace transform in a real domain. They applied the double Laplace transform to solve problems in electrostatics and heat conduction. They observed that a double Laplace transform of a function *f* of two variables *x,t* is given as

$$\mathcal{L}_t\mathcal{L}_x\{f(x,t)\} = F(p,s) = \int_0^\infty e^{-st}\int_0^\infty e^{-px} f(x,t)\,dx\,dt,$$

where, the improper integral converges and complex numbers.

Coon and Bernstein (1953) observed the properties of the double Laplace transform including conditions for transforming derivatives, integrals and convolution. Dhunde *et al.* (2013) obtained some properties of double Laplace transform.

They worked on the linearity property, change of scale, shifting property, double Laplace transform of partial derivatives and double Laplace transform of integral and a function multiplied or divided by *xt*. Dhunde and Waghmare (2014) presented the convergence, absolute convergence and uniform convergence of double Laplace transform and also used the double Laplace transform to solve Volterra Integro-Partial differential equation

Since its proposition, the double Laplace transform has been applied to many problems in physics, engineering and applied mathematics. Eltayeb and Kilic¸man (2008), applied a double Laplace transform to search for solution of wave equation, heat, and Laplace's equations with convolution terms. They later in Eltayeb and Kili¸cman (2013) applied the double Laplace transform to solve the general linear telegraph and partial integro–differential equations. Elzaki (2012), on the other hand, proposed a new method by combining the double Laplace transform and modified variational iteration method to solve nonlinear convolution partial differential equations.

Buschman (1983) also used the double Laplace transform to solve a problem on heat transfer between a plate and a fluid flowing across the plate. Lokenath (2016) dealt with the double Laplace transform and its properties with examples and applications to functional, integral and partial differential equations. He also proved several simple theorems dealing with general properties of the double Laplace transform and discussed the convolution of the double Laplace transform, its properties and the convolution theorem with a proof.
In Dhunde and Waghmare (2017) the double Laplace transform method was used to find the solutions to a wide classes of equations in mathematical physics. With examples, they applied the double Laplace transform to the advection-diffusion equation, the linear dissipative wave equation, the reaction-diffusion equation, the Korteweg-de Vries (KdV) equation, the telegraph equation, the Euler-Bernoulli equation and the Klein-Gordon equation.

The strongest defense for Laplace transform, apart from being able to convert differential equations to algebraic form as stated above, is its ability to directly give the solution of differential equations with given boundary values without necessarily finding the general solution first and then evaluating from it the arbitrary constants.

The Fourier integral transform was introduced in a manuscript and a memoir in Fourier (1807) and Fourier (1811) respectively, deposited in the institute of France and later collected and expanded in a book about the analytic theory of heat in Fourier (1822). In modern notation, Fourier and inverse Fourier transform of a real-valued function over an interval $(-\infty,\infty)$ are expressed as;

$$\mathcal{F}[f(x)] = F_c(u) = \frac{2}{\pi}\int_{-\infty}^{\infty} f(x)\cos(ux)dx$$

$$\mathcal{F}^{-1}[F_c(u)] = f(x) = \int_{-\infty}^{\infty} F_c(u)\cos(ux)du$$

This pair of equations is known today as the cosine fourier transform and the inverse cosine fourier transform, respectively. In a similar way, in his treatise 'The analytical Theory of heat', Fourier derived the sine fourier transform and the inverse sine fourier transform [see Dominguez (2016)] as

$$\mathcal{F}[f(x)] = F_s(u) = \frac{2}{\pi}\int_{-\infty}^{\infty} f(x)\sin(ux)dx$$

$$\mathcal{F}^{-1}[F_s(u)] = f(x) = \int_{-\infty}^{\infty} F_s(u)\sin(ux)du$$

Papoulis (1962) introduced both the Fourier and inverse Fourier transform of a suitable function $f(x)$, defined on the whole real line and is complex-valued as ;

$$\mathcal{F}[f(x)] = F(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixu}dx$$

$$\mathcal{F}^{-1}[F(u)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u)e^{ixu}du$$

Clearly, both the Fourier transform and its inverse are linear integral operators with the forward transform taking an exponential kernel of $K(t,s) = e^{-ixu}$. For any $u \in$ R, integrating $f(x)$ against e^{-ixu} with respect to x produces a *complex valued* function of u, that is, the Fourier transform $F(u)$ is a complex-valued function of $u \in$ R.

In the application of the Fourier transform, especially in applied mathematics and electrical engineering, x represents a space variable in applied mathematics but it is replaced by the time variable t in electrical engineering. Again in applied mathematics, $k = \frac{2\pi}{\lambda}$ is a wavenumber variable where λ is the wavelength but k is replaced by the frequency variable $\omega = 2\pi\nu$ where ν is the frequency in cycles per second in electrical engineering.

The Fourier transform is an ideal transform for analyzing time-varying signals in electrical engineering and seismology because of its ability to map a function or signal of time t to a function of frequency ω. The central philosophy behind Fourier transforms is that almost every imaginable signal can be broken down into a combination of simple waves. A complicated signal can be broken down into simple waves. This break down, and how much of each wave is needed, is the Fourier Transform. Fourier transforms (FT) take a signal and express it in terms of the frequencies of the waves that make up that signal.

Fourier transforms are widely used in many fields of sciences and engineering, including image processing, power distribution system, geoscience, crystallography and quantum mechanics. Most modern technological advances like television, music CDs and DVs, cell phones, movies, computer graphics, and fingerprint analysis and storage, are, in one way or another, founded upon the many ramifications of Fourier theory.

Ezhil (2017), applied the Fourier transform to identify the noise, distortion and interference present in the signal in a power distribution system. Gupta (2013), also discussed how Fourier Transform is used in cell phone networking and in Lenssen and Needell (2014), readers are introduced to a setup to understand how the Discrete Fourier Transform is used to analyze a musical signal for chord structure.

Unlike the Laplace and Fourier transforms that were introduced to solve physical problems, Mellin transform arose in a mathematical context. Riemann (1876) was the first to recognize the Mellin transform in a memoir on prime numbers in which he used it to study the famous Zeta function. Even though Cahen (1894) explicitly formulated the Mellin transform, it is Mellin (1896) and Mellin (1902) who gave its systematic formulation and inverse formula. Using the theory of special functions, Mellin developed applications to the solution of hypergeometric differential equations and to the derivation of asymptotic expansions.

The Mellin transform and its inverse are derived from the complex Fourier transform and its inverse. Kang (1958) introduced the Mellin transform of a function and the inverse Mellin transformed as

$$\mathcal{M}[f(x)] = F(s) = \int_0^\infty f(x)x^{s-1}dx$$

$$\mathcal{M}^{-1}[F(s)] = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}ds$$

where M and M^{-1} are the Mellin transform and the inverse Mellin transform respectively and $f(x)$, a real valued function defined on $[0,\infty)$. The Mellin transform has a polynomial kernel of x^{s-1} with transform variable s being a complex number.

By changing the variable $x = e^{-t}$ shows that the Mellin integral transform is closely related to the Laplace transform and to a large extent the Fourier transform. Despite these connections, there are numerous applications where it is more convenient to directly operate with the Mellin transform instead of the Laplace or Fourier version. For instance, in complex function theory (i.e. asymptotes of Gamma-related functions), in number theory; when working on coefficients of

Dirichlet series, and in the analysis of harmonic sums algorithms, it is more convenient to directly operate in the Mellin transform

Kropivsky and Ben-Naim (1994) extended the Mellin transform in one dimension to two dimensions in a study on fragmentation in two dimensions. The double Mellin transform is defined in Eltayeb and Kilic¸man (2007) as

$$\mathcal{M}_{xy}[f(x,y)\colon p,q] = F(p,q) = \int_0^\infty \int_0^\infty f(x,y)x^{p-1}y^{q-1}dxdy,$$

where, p and q are complex numbers.

Watugala (1993) introduced the Sumudu integral transform for solving differential equations and control engineering problems. Even though it is relatively new, the Sumudu integral transform is one of the powerful integral transforms in literature, especially it's scale and unit-preserving properties makes it a suitable transform for solving problems without resorting to a new frequency domain.

Another very interesting fact about the Sumudu transform is that it is able to maintain the same Taylor coefficient for both the original function and its Sumudu transform with the only exception being the factor (see Zhang (2007)). It has a theoretical duality to the Laplace transform and this was shown in Belgacem *et al.* (2003).

Kili¸cman and Eltayeb (2010) defined the Sumudu transform over the set of the functions

$$A = \{f(t)\colon \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau_j}},\ \text{if } t \in (-1)^j \times (0,\infty]\}$$

by the formula below

$$G(u) = S[f(t);u] =\colon \int_0^\infty f(ut)e^{-t}dt, \quad u \in (\tau_1, \tau_2)$$

The properties of the Sumudu transform were established by Asiru (2001) and subsequently applied to partial differential equations. There have been numerous other applications of the Sumudu transform. In Kadem (2005) for instance, the Sumudu transform was applied to the one-dimensional neutron transport.

Atangana and Kilic¸man (2013) on the other hand applied the Sumudu integral transform to solve nonlinear fractional partial differential equations describing heat-like equation with variable coefficients.

A. Kili¸cman and H. Eltayeb introduced the double Sumudu Transform method for solving linear second order partial differential equations with non-constant coefficient. Jean Tchuenche and Nyimvua (2007) defined double Sumudu transform of a function $f(x,y)$; of two variables $x,y \in R_+$ by

$$F(u,v) = S_2[(x,y);(u,v)] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{x}{u}+\frac{y}{v})} f(x,y)dxdy$$

where S_2 indicates double Sumudu transform and $f(x,y)$ a function which can be expressed as a convergent infinite series. The properties and relationship between the double Sumudu and double Laplace transforms were given in Kili¸cman and Eltayeb (2009). They also used the double Sumudu transform to solve wave equation in one dimension having singularity at initial conditions. Osman and Ali Bashir (2016) solved partial differential equation with variables coefficient by using the double Sumudu transform.

Barnes (2016) introduced both the polynomial integral transform and the double polynomial integral transform to solve differential equations and partial differential equation with little computational effort. Due to its polynomial function kernel the polynomial integral transform ensures rapid convergence of solutions of differential equations The Polynomial Integral Transform of a function $f(x)$, defined for $x \geq 1, x \in [1,\infty)$ is given by the integral equation

$$B(f(x)) = F(s) = \int_1^{\infty} f(\ln x) x^{-s-1} dx$$

The double polynomial integral transform of a function $f(x,t)$ is

$$B_x B_t(f(x,t);(p,s)) = F(p,s) = \int_1^{\infty}\int_1^{\infty} f(\ln x, \ln t) x^{-p-1} t^{-s-1} dx dt$$

The multiplicity relation between the polynomial integral transform and other transform were enumerated in Chaudhary *et al.* (2018). They showed the dual relation between the polynomial integral transform and other famous integral transforms. Specifically, they looked at the duality of the polynomial integral transform and the Natural transform, Sumudu transform, Fourier transform, Laplace transform and the Mellin transform.

Khan and Khan (2008) first introduced the Natural transform, which used to be called N-transform. They applied this integral transform to search for the solution of fluid flow problem and the Maxwell's equations. Later on this transform was applied to many other ordinary differential equations with integer order to find their solutions in Belgacem and Silambarasan (2012).
Baskonus et al (2014) observed the natural integral transform, $f(t)$ over a set of functions

$$A = \{f(t) \colon \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times (0, \infty]\}$$

as:

$$N[f(t)] = R(s,u) = \int_0^{\infty} f(ut) e^{-st} dt \qquad u > 0, s > 0$$

where $N[f(t)]$ is the natural transform of the function $f(t)$ and the variables u and s are the natural transform variables.

The Natural integral transform is derived by taking the product of the Fourier and the Inverse Fourier transforms and then setting $x > 0, f(x) = f(x)H(x)e^{-cx}$ where

$H(x)$ is the Unit step function defined as $H(x) = 1, x \geq 0$ and $0; x < 0$ and finally making the substitution $\frac{s}{u} = c + ik$.

The specialty of Natural transform is that, with a slight change in variables, it can converge to either Laplace transform or Samudu transform.

Kili¸cman and Omran (2017) extended the one dimensional Natural transform to two dimensional Natural transform including some of its properties. They also set a relation between the double Natural transform and double Laplace, double Sumudu transforms in addition to applying the double Natural transform to get the solutions of some general linear telegraphs, wave and partial integrodifferential equations.

Kili¸cman and Omran (2017) defined the double Natural transform of a function *f*(*x*,*y*) as

$$\mathbb{N}_+^2\{f(x,y)\} = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\frac{sx}{u}+\frac{py}{v})} f(x,y)dxdy$$

where $x,y \in \mathrm{R}_+$

The Hankel transform, a self-reciprocal transform, was constructed from the twodimensional Fourier transform with transition to polar coordinates and application of the integral representation of the Bessel functions [See Lokenath and

Dambaru (2015)]. It is self-reciprocal because its inverse transform is just the Hankel transform again.

The Hankel transform of order *v* of a function *f*(*x*) is defined as

$$\mathcal{H}_v[f(x);p] = g(p;v) = \int_0^\infty (px)^{\frac{1}{2}} J_v(px) f(x)dx, \quad p > 0$$

and gave its inverse as

$$f(x) = \int_0^\infty (xp)^{\frac{1}{2}} J_v(xp) g(p)dp, \quad \mathrm{R}(v) > -1$$

(as cited in Mathai *et al.* (2010))

Another transform that is widely used in analyzing discrete signal and discrete linear time-invariant (LTI) system is the Z transform. It came out of a complicated method introduced by Gardner and Barnes in the early 1940s to solve linear, constant-coefficient difference equations by Laplace transform and later simplified and later simplified by W. Hurewicz in 1947 into the Z transform as seen today.

Lokenath and Dambaru (2015) defines the Z transform of a sequence $\{f(n)\}$ denoted by $F(z)$ of a complex variable z as

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

where Z is a linear transformation and can be considered as an operator mapping sequences of scalars into functions of the complex variable $z(= re^{jw})$. Clearly, Z transform is a power series and it exists for only those values of z for which $F(z)$ attains a finite value.

The Z transform is used in many areas of applied mathematics as signal processing, control theory, economics and other fields. The infinite Z-transform technique, for instance, is used to derive the solutions of boundary-value problems characterized by linear difference equations such as discrete electrostatic field problems and ladder type networks. It is also used to analyze digital filters, simulate continuous systems and find frequency response.

Hilbert (1912), introduced the Hilbert transform in his famous paper on integral equations but it is Hardy (1924) and Titchmarsh (1925) who simultaneously developed the properties of the Hilbert transform. The Hilbert transform is used in signal processing, fluid mechanics and in aerodynamics. This transform is able to extend real functions into analytic functions, an advantage it has over other signal processing transforms.

The Hilbert transform, denoted by $\hat{f}_H(x)$, for a function $f(t)$ defined on the real line $-\infty < t < \infty$ is given as

$$\mathbf{H}\{f(t)\} = \hat{f}_H(x) = \frac{1}{\pi} = \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt$$

where x is real and the integral is treated like Cauchy principal value [see Saff and Snider (1976)]. and its inverse transform is given by

$$f(t) = H^{-1}\left\{\hat{f}(x)\right\} = -H\left\{\hat{f}(x)\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{f}_H(x)dx}{x-t}$$

The Hilbert transform of $f(t)$ is the convolution of $f(t)$ with the signal $1/\pi t$. There is a relation between a real function $f(t)$ and its Hilbert transform $\hat{f}(t)$ in such a way that they together create a strong analytic signal which can be written with an amplitude and a phase, where the derivative of the phase can be identified as the instantaneous frequency.

Titchmarsh (1967), gives a more classical treatment of the Hilbert transform with Kober (1943) giving further properties and its application

The Legendre transform, named after Adrien-Marie Legendre, uses a Legendre polynomials $P_n(x)$ as kernel of the transform. It is used to provide the connection between the Lagrangian L($\dot{q}$) and the Hamiltonian H(p) in classical mechanics. Churchill (1954) observed that, the *Legendre transform* of a function $f(x)$ defined in $-1 < x < 1$ by the

$$\mathcal{J}_n\{f(x\}) = \tilde{f}(n) = \int_{-1}^{1} P_n(x) f(x) dx$$

provided the integral exists and where $P_n(x)$ is the *Legendre polynomial* of degree $n(\geq 0)$. Also, the inverse *Legendre polynomial* is given by

$$f(x) = \mathcal{J}^{-1}\left\{\tilde{f}(n)\right\} = \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)\tilde{f}(n)P_n(x)$$

Stieltjes transform is another transform that arises in many problems, especially in moment problems, in applied mathematics, mathematical physics, and engineering science . This transform was introduced by Stieltjes (1894) in a study on continued fractions.

The Stieltjes transform of a function f(t) is defined by using the Laplace transform of F(s)= L f(t) with respect to s. That is, taking the Laplace transform of F(s), clearly give

$$\begin{aligned}\mathcal{L}\{F(s)\} = F(z) &= \int_0^\infty e^{-sz} F(s)ds \\ &= \int_0^\infty e^{-sz}ds \int_0^\infty e^{-st} f(t)dt\end{aligned}$$

Now, interchanging the order of integration and evaluation the inner integral gives

$$F(z) = \int_0^\infty \frac{f(t)}{t+z}dt$$

Therefore, Stieltjes transform of a locally integrable function $f(t)$ is defined as

$$\mathcal{T} = F(z) = \int_0^\infty \frac{f(t)}{t+z}dt$$

where z is a complex variable in the cut plane $|argz| < \pi$ [see Widder (1971)].
Lokenath and Dambaru (2015) gives some basic properties of the Stieltjes transform, its inversion theorem and some applications. Erd´elyi *et al.* (1954) gives the generalized Stieltjes transform and some basic properties of the Stieltjes transform.

Another transform that is very effective in solving heat conduction problems in a semi-infinite medium with a variable thermal conductivity in the presence of a heat source within a medium is the Laguerre transform. The Laguerre transform uses a Laguerre polynomial as its kernel.

Debnath (1960) defines the *Laguerre transform* of a function $f(x)$ defined in

$0 \le x < \infty$ as

$$L\{f(x)\} = \bar{f}_\alpha(n) = \int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)f(x)dx$$

and the inverse Laguerre transform as

$$f(x) = L^{-1}\{\bar{f}_\alpha(n)\} = \sum_{n=0}^{\infty}(\delta_n)^{-1}\bar{f}_\alpha(n)L_n^\alpha(x)$$

where $L_n^\alpha(x)$ is the Laguerre polynomial of degree $n(\geq 0)$ and of order $\alpha(> -1)$. When $\alpha = 0$ the Laguerre transform and its inverse are respectively given by the pair of equations below

$$L\{f(x)\} = \bar{f}_0(n) = \int_0^\infty e^{-x}L_n(x)f(x)dx$$

$$f(x) = L^{-1}\{f_0\bar{(n)}\} = \sum_{n=0}^{\infty}\bar{f}_0(n)L_n(x)$$

where $L_n(x)$ is the Laguerre polynomial of degree n and order 0 [see McCully (1960)].

Debnath (1964) introduced the Hermite transform with the Hermite polynomial $H_n(x)$ as its kernel and also proved some of its basic operational properties. He defined the Hermite transform of a function $F(x)$ over the interval $-\infty < x < \infty$ and its inverse by the integrals

$$H\{F(x)\} = f_H(n) = \int_{-\infty}^{\infty} e^{-x^2}H_n(x)F(x)dx$$

and

$$H^{-1}\{f_H(n)\} = F(x) = \sum_{n=0}^{\infty}(\delta_n)^{-1}f_H(n)H_n(x)$$

respectively, where $\delta_n = \sqrt{\pi}n!2^n$. Dimovski and Kalla (1988) gives the extension of the convolution theorem of Hermite transform for odd numbers, first proved by Debnath (1968) to cover both odd and even functions.

The Jacobi transform was introduced by Debnath (1963) with applications to physical problems described by differential equations, including the problem of heat conduction in a finite domain with variable thermal conductivity. It uses a Jacobi Polynomial as its kernel. The Jacobi transform of a function $F(x)$ defined in $-1 < x < 1$ is given by the integral

$$J\{F(x)\} = f^{(\alpha,\beta)}(n) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) F(x)dx \quad \alpha > -1, \beta > -1,$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of order n. The Jacobi transform is a generalization of the Legendre and Gegenbauer transforms. In Debnath (1967), the author applied the Jacobi transform to search for the solution of partial differential equations

Rognlie (1969) introduced the generalized integral transforms by using the method of generalizing an analytic function of a single complex variable to an analogous function of several complex variables, introduced by Carlson (1969). He created a generalized transform of

$$\int_E \int_\lambda \psi(x)\phi(\sum_{i=1}^{k} u_i y_i, x)P(b,u)dxdu'$$

where λ is a path of integration in the complex plane C^1, φ is the kernel of the transform and the y_i's and x may be real or complex. He used this method to generalize the Fourier, Laplace and Stieltjes transforms to functions of several variables by replacing the kernel of the transform by the generalized kernel φ. For instance, he gives the generalized Fourier transform of a real variable and a complex variable as

$$\begin{aligned}\overline{H}(b,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \int_E exp\left(-it\sum_{i=1}^{k} u_i x_i\right) P(b,u)du'dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)S(b,-ixt)dt, \quad Re(b) > 0\end{aligned}$$

and

$$\overline{H}(b,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \int_E exp\left(-it\sum_{i=1}^{k} u_i s_i\right) P(b,u)du'dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)S(b,-ist)dt, \quad Re(b) > 0$$

respectively. Rognlie (1969) also gives the operational Properties of the generalized transforms, relation of the generalized transforms to other transforms and applications of the generalized Laplace transform and the generalized Stieltjes transform.

Barnes *et al.* (2018) introduced the generalization of integral transform (GIT) of the function $f(t)$ for solving both differential and interodifferential equations.
This transform generalizes integral transforms with exponential kernel.
The generalization of integral transform of a function $f(t)$ is

$$G\{f(t)\} = G(s) = u\int_0^{\infty} f(ut)e^{-ust}dt, \quad \forall t \geq 0$$

Below is a table that shows the generalization of integral transform of some functions.

$f(t) = G^{-1}\{F(s)\}$	$F(s) = Gf(t)$
a $e_{at}\, t_n$ $sinat$ $cosat$ $sinhat$ $coshat$	$\frac{a}{s}$ $\frac{u}{us-a}$ $\frac{n!}{u^{n-1}s^{n+1}}$ $\frac{au}{(us)^2+a^2}$ $\frac{u^2s}{(us)^2+a^2}$ $\frac{au}{(us)^2-a^2}$ $\frac{u^2s}{(us)^2-a^2}$

Barnes *et al.* (2019) have extended the GIT to fractional GIT, for solving fractional differential equations. They established properties associated with fractional GIT and also showed that the solution of fractional differential equations given by this method is unique.

The fractional GIT of a function $f(t)$ is given by

$$G\{D^{\alpha}f(t)\} = u\int_{0}^{\infty} D^{\alpha}e^{-ust}f(ut)dt$$

$$= u^{(\alpha+1)}s^{\alpha}G\{f(t)\} - \sum_{k=1}^{n} u^{(\alpha+2-k)}s^{(\alpha-k)}f^{(k-1)}(0)$$

However, one of the interesting feature of the GIT is its incapability for solving the differential equations in two dimensions. Thus, both the GIT and its fractional form cannot be used to ascertain the solution of partial differential equations (PDEs). In most practices in engineering and applied sciences, the problems are usually formulated in two or more dimensions which depict the actual mechanisms on the ground. For example, the flow of oil through the rock bearing oil is in three dimensions but not in one dimension. In this case, one have to consider the flow of oil in $x-$direction, $y-$direction, as well as $z-$direction. The formulation of flow of oil in one dimension is not only unrealistic, but also it does not provide the full underlying mechanisms in the understanding of the flow of oil through the bedrocks. Also, the problem of flow of traffic or the creation of traffic jams on roads is usually multi-dimensional. Since one have to consider the intensity of the traffic jam by vehicles with respect to various directions in which these cars are transversing.

Chapter 3

The Double Generalization of Integral Transform

The Generalization of Integral Transform, proposed by Barnes *et al.* (2018) solves ordinary differential and integro differential equations by converting these equations into the *us* domain and then reconverting the result by its inverse operator.

However, the generalization of integral transform cannot solve partial differential and integro differential equations. It only solves differential equations in one dimension. In most practices, problems involving differential equations comes in two dimensions. It is of this reason that we extend the generalization of integral transform to double generalization of integral transform, for solving partial differential equations (PDEs). In addition, some of the properties of the double Generalization of Integral Transform will be provided in this chapter. Moreover, we will look at the convergence of the double GIT and its derivatives.

To begin with, we state the following definitions of integral transforms that will enable us to achieve our results.

Definition 14 (The Double Polynomial Integral Transform)**.** Let $f(x)$ be a function defined for $x \geq 1$. Then the integral

$$B_x B_t(f(x,t);(p,s)) = F(p,s) = \int_0^\infty \int_0^\infty f(\ln x, \ln t) x^{-p-1} t^{-s-1} dx dt \qquad (3.1)$$

is the double polynomial integral transform of $f(x,t)$ for $x,t \in (1,\infty]$ provided the integral converges, [see, Barnes (2016)].

Definition 15 (The Double Sumudu Integral Transform)**.** If $f(w,y)$ is a function which can be expressed as a convergent infinite series, then the double Sumudu transform of a function $f(w,y);w,y \in R_+$ is

$$F(s,p) = S_2[(w,y);(s,p)] = \int_0^\infty \int_0^\infty e^{-(sw+py)} f(sw,py)dwdy \tag{3.2}$$

where S_2 indicates the double Sumudu transform, [see Jean Tchuenche and Nyimvua (2007)].

3.1 The Derivation of Double Generalization of Integral Transform

In this section, we provide the double generalization of integral transform in *Theorem 1* below.

Theorem 1. *Let $f(x,y)$ be a function of two variables* $(w,y) \in \{0\} \cup R^+$ *then the double generalization of integral transform is*

$$G_xG_y\{f(x,y)\} = F(s,p) = uv\int_0^\infty \int_0^\infty f(ux,vy)e^{-(usx+pvy)}dxdy \tag{3.3}$$

where G_xG_y indicates the double generalization of integral transform and u and v the transformed variables. The double GIT exists provided the double integral in equation (3.3) *exists.*

Proof: Comparing the kernels in equations (3.1) and (3.2), we obtain

$$x^{-(s+1)}t^{-(p+1)} = e^{-(sw+py)}$$

$$x^{-(s+1)}t^{-(p+1)} = e^{-sw} \cdot e^{-py} \tag{3.4}$$

Comparing the transformed variable *s* on both sides of equation (3.4), we obtain

$$x^{-(s+1)} = e^{-sw}$$

$$\ln x^{-(s+1)} = \ln e^{-sw}$$

$$-(s+1)\ln x = -sw$$

$$\ln x = \frac{sw}{(s+1)}$$

$$x = e^{\frac{sw}{(s+1)}} \qquad (3.5)$$

Also, we can see from equation (3.4) that

$$t^{-(p+1)} = e^{-py}$$

$$\ln t^{-(p+1)} = \ln e^{-py}$$

$$-(p+1)\ln t = -py$$

$$\Rightarrow \qquad t = e^{\frac{py}{(p+1)}} \qquad (3.6)$$

Also, we can see from equations (3.1) and (3.2) that

$$dxdt = dwdy \cdot \frac{\partial(x,t)}{\partial(w,y)}$$

But

$$\frac{\partial(x,t)}{\partial(w,y)} = \frac{s}{(s+1)}e^{\frac{sw}{(s+1)}} \cdot \frac{p}{(p+1)}e^{\frac{py}{(p+1)}}$$

Therefore

$$dxdt = \frac{s}{(s+1)}e^{\frac{sw}{(s+1)}} \cdot \frac{p}{(p+1)}e^{\frac{py}{(p+1)}} \cdot dwdy \qquad (3.7)$$

Substituting equations (3.5) - (3.7) into equation (3.1) yields

$$G_wG_y\{f(w,y)\} = \int_0^\infty \int_0^\infty f\left(\frac{sw}{s+1}, \frac{py}{p+1}\right) \cdot e^{-sw} \cdot e^{-py} \cdot \frac{s}{(s+1)}e^{\frac{sw}{(s+1)}} \cdot \frac{p}{(p+1)}e^{\frac{py}{(p+1)}} \cdot dwdy$$

$$G_wG_y\{f(w,y)\} = \frac{s}{(s+1)} \cdot \frac{p}{(p+1)} \int_0^\infty \int_0^\infty f\left(\frac{sw}{s+1}, \frac{py}{p+1}\right) \cdot e^{-sw} \cdot e^{-py} \cdot e^{\frac{sw}{(s+1)}} \cdot e^{\frac{py}{(p+1)}} \cdot dwdy$$

$$G_wG_y\{f(w,y)\} = \frac{s}{(s+1)} \cdot \frac{p}{(p+1)} \int_0^\infty \int_0^\infty f\left(\frac{sw}{s+1}, \frac{py}{p+1}\right) \cdot e^{-\frac{s^2w}{s+1}} \cdot e^{-\frac{p^2y}{p+1}} \cdot dwdy$$

Setting $u = \frac{s}{s+1}$ and $v = \frac{p}{p+1}$, we obtain

$$G_wG_y\{f(w,y)\} = uv \int_0^\infty \int_0^\infty f(uw,vy) \cdot e^{-suw} \cdot e^{-pvy} \cdot dwdy$$

$$G_xG_y\{f(x,y)\} = F(s,p) = uv \int_0^\infty \int_0^\infty f(ux,vy) \cdot e^{-(sux+pvy)} \cdot dxdy \qquad (3.8)$$

3.2 Convergence of the Double Generalization of Integral Transform

To show the convergence of the double generalization of integral transform for $(x,y) \in \{0\} \cup R^+$, we state the lemmas which are relevant in proving the existence of double GIT.

Theorem 2. *Let $\phi(x,y)$ be a function of two variables continuous in the positive quadrant of the wy-plane. If the integral*

$$uv \int_0^\infty \int_0^\infty \varphi(ux,vy) \cdot e^{-sux} \cdot e^{-pvy} \cdot dxdy \qquad (3.9)$$

converges at $p = p_0$, $s = s_0$ then the integral (3.9) converges for $p > p_0$, $s > s_0$ Proof

Before we prove Theorem 1, we state the following lemmas:

Lemma 1. *If the integral*

$$u \int_0^\infty e^{-sux} \varphi(x,y)dx \qquad (3.10)$$

converges at $s = s_0$, then the integral (3.10) converges for $s > s_0$ Proof :

Setting

$$\alpha(x,y) = \int_0^y e^{-s_0 ut}\varphi(x,t)dt, \qquad 0 < x < \infty \tag{3.11}$$

Clearly $\alpha(x,0) = 0$ and $\lim_{y\to\infty} \alpha(x,y)$ exists because integral $\int_0^\infty e^{-sux}\varphi(x,y)dx$

converges at $s = s_0$.

From the first fundamental theorem of calculus

$$\alpha_y(x,y) = e^{-usx_0}\phi(x,y)$$

Choose ε_1 and R_1 so that $0 < \varepsilon_1 < R_1$.

$$\int_{\varepsilon_1}^{R_1} e^{-sux}\varphi(x,y)dx = \int_{\varepsilon_1}^{R_1} e^{-sux}e^{-usx_0}\alpha_y(x,y)dx$$

$$\int_{\varepsilon_1}^{R_1} e^{-sux}\varphi(x,y)dx = \int_{\varepsilon_1}^{R_1} e^{-(s-s_0)ux}\alpha_y(x,y)dx.$$

Using the integration by parts, we have

$$\int_{\varepsilon_1}^{R_1} e^{-usx}\varphi(x,y)dx = [e^{-(s-s_0)ux}\alpha(x,y)]_{\varepsilon_1}^{R_1} - \int_{\varepsilon_1}^{R_1} e^{-(s-s_0)ux}[-(s-s_0)]\alpha(x,y)dx$$

$$\begin{aligned}\int_{\varepsilon_1}^{R_1} e^{-usx}\phi(x,y)dx &= e^{-(s-s_0)uR_1}\alpha(R_1,y) - e^{-(s-s_0)u\varepsilon_1}\alpha(\varepsilon_1,y)\\ &\quad +(s-s_0)\int_{\varepsilon_1}^{R_1} e^{-(s-s_0)ux}\alpha(x,y)dx.\end{aligned}$$

Now let $\varepsilon_1 \to 0$ and $R_1 \to \infty$. If $s > s_0$, then

$$\int_{\varepsilon_1}^{R_1} e^{-usx}\varphi(x,y)dx = (s-s_0)\int_0^\infty e^{-(s-s_0)ux}\alpha(x,y)dx \tag{3.12}$$

The theorem is proved if the integral on the right converges. By using the Limit test for convergence, we have

$$\lim_{x\to\infty} x^2 e^{-(s-s_0)ux}\alpha(x,y)dx = \lim_{x\to\infty} \frac{x^2}{e^{-(s-s_0)ux}}[\lim_{x\to\infty} \alpha(x,y)]$$

$$= 0 \times [\lim_{x\to\infty} \alpha(x,y)] = 0 =< +\infty$$

Therefore, the integral on the right of (3.12) converges for $s > s_0$. Hence the integral $u\int_0^\infty e^{-usx}\varphi(x,y)dx$ converges for $s > s_0$.

Lemma 2. *If (a) integral*

$$h(y,s) = \int_0^\infty e^{-usx}\varphi(x,y)dw \tag{3.13}$$

converges fo $s \geq s_0$ and (b) integral

$$\int_0^\infty e^{-pvy}h(y,s)dy \tag{3.14}$$

converges at $p = p_0$, then the integral (3.14) converges for $p > p_0$.

Proof: Let

$$\beta(y,s) = \int_0^y e^{-vp_0x}h(x,s)dx, \qquad 0 < y < \infty \tag{3.15}$$

Therefore $\beta(0,s) = 0$ and $\lim_{y\to 0} \beta(y,s)$ exists because integral $\int_0^\infty e^{-pvy}h(y,s)dy$ converges at $p = p_0$.

It can be seen from (3.14) that $\beta_y(y,s) = e^{-p_0vy}h(y,s)$. We choose ε_2 and R_2 so that $0 < \varepsilon_2 < R_2$.

$$\int_{\varepsilon_2}^{R_2} e^{-pvy}h(y,s)dy = \int_{\varepsilon_2}^{R_2} e^{-pvy}e^{-p_0vy}\beta_y(y,s)dy$$

$$= \int_{\varepsilon_2}^{R_2} e^{-(p-p_0)vy}\beta_y(y,s)dy$$

$$= [e^{-(p-p_0)vy}\beta(y,s)]_{\varepsilon_2}^{R_2} - \int_{\varepsilon_2}^{R_2} e^{-(p-p_0)vy}[-(p-p_0)]\beta(y,s)dy$$

$$\int_{\varepsilon_2}^{R_2} e^{-pvy}h(y,s)dy = e^{-(p-p_0)vR_2}\beta(R_2,s) - e^{-(p-p_0)v\varepsilon_2}\beta(\varepsilon_2,s)$$
$$+(p-p_0)\int_{\varepsilon_2}^{R_2} e^{-(p-p_0)vy}\beta_y(y,s)dy$$

Now let $\varepsilon_2 \to 0$ and $R_2 \to \infty$. If $p > p_0$, then

$$\int_{\varepsilon_2}^{R_2} e^{-pvy}h(y,s)dy = (p-p_0)\int_{\varepsilon_2}^{R_2} e^{-(p-p_0)vy}\beta(y,s)dy \quad \text{for } p > p_0. \tag{3.16}$$

The theorem is proved if the integral on the right converges. By using the limit test for convergence, we have

$$\lim_{y\to\infty} y^2 e^{-(p-p_0)vy}\beta_y(y,s) = \lim_{y\to\infty}\frac{x^2}{e^{-(p-p_0)vy}}[\lim_{y\to\infty}\beta_y(y,s)]$$
$$= 0 \times [\lim_{y\to\infty}\beta_y(y,s)] = 0 =< +\infty$$

Therefore, integral on the right of (3.16) converges for $p > p_0$. Hence the integral $\int_0^\infty e^{-pvy}h(y,s)dy$ converges for $p > p_0$

It follows from *Theorem 1* that:

$$uv\int_0^\infty\int_0^\infty \varphi(x,y)\cdot e^{-usx-pvy}dxdy = v\int_0^\infty e^{-pvy}\left\{u\int_0^\infty e^{-usx}\varphi(x,y)dx\right\}dy$$

$$\therefore uv\int_0^\infty\int_0^\infty \varphi(x,y)\cdot e^{-usx-pvy}dxdy = v\int_0^\infty e^{-pvy}h(y,s)dy \tag{3.17}$$

where $h(y,s) = u\int_0^\infty e^{-usx}\varphi(x,y)dx$.

By Lemma 3.3.2, integral

$$u\int_0^\infty e^{-usx}\varphi(x,y)dw$$

converges for $s > s_0$.

We can also see from Lemma 3.3.3 that, the integral

$$v\int_0^\infty e^{-pvy}h(y,s)dy$$

converges for $p > p_0$.

Therefore, the integral in RHS of (3.17) converges for $p > p_0, s > s_0$.
Hence the integral

$$uv \int_0^\infty \int_0^\infty \varphi(x,y) e^{-usx-pvy} dxdy$$

converges for $p > p_0$, $s > s_0$.

3.3 Existence of a Double Generalization of Integral Transform

In this subsection, we show that a double generalization of integral transform exists for $w,y \in [0,\infty)\times[0,\infty)$. Notwithstanding, we show that the double GIT exists for all $w,y \in [0,\infty) \times [0,\infty)$

Theorem 3. *Let $f(x,y)$ be a continuous function in every finite intervals $(0,X)$ and $(0,Y)$ and of exponential order $(e^{aux+bvy})$, then the double generalization of integral transform of $f(x,y)$ exists for all s and p provided $Re\ s > a$ and $Re\ p > b$*

Proof: Since the function $f(x,y)$ is of exponential order, then any scalars $a,b \in \mathbb{R}; a > 0, b > 0, x,y \in [0,\infty) \times [0,\infty)$, there exists a positive constant K such that for all $x > X$ and $y > Y$

$$|f(x,y)| \leq Ke^{aux+bvy}$$

We observe that

$$f(x,y) = O(e^{aux+bvy}) \quad \text{as} \quad x \to \infty, y \to \infty$$

Thus $\lim_{x\to\infty,y\to\infty} e^{-\alpha ux-\beta vy} = K \lim_{x\to\infty,y\to\infty} e^{-(\alpha-a)ux} e^{-(\beta-b)vy} = 0 \quad \alpha > a, \beta > b$

We can see that

$$|F(s,p)| = \left| uv \int_0^\infty \int_0^\infty f(ux,vy) \cdot e^{-(usx+pvy)} \cdot dxdy \right| \leq$$

$$K \int_0^\infty e^{(s-a)ux} dx \int_0^\infty e^{(p-b)vy} dy$$

$\therefore$ $$|F(s,p)| = \frac{K}{uv(s-a)(p-b)}$$ for *Res > a,Rep > b* (3.18)

It follows from (3.18) that:

$$\left\| \frac{1}{uv(s-a)(p-b)} \right\| \to 0,$$

$\forall$ *Re s > a*, *Re p > b*

$\Rightarrow$ $F(s,p) = 0$

Hence, $f(x,y)$ exists for all s and p

3.4 Convolution Theorem of the double generalization of integral transform

In this section, we show that the double GIT operator convolve with two functions $f(x,y)$ and $g(x,y)$. Thus, the finding is summarized in Theorem 4.

Theorem 4. *Let f(x,y) and g(x,y) be double generalization of integral transformable. Then double generalization of integral transform of the double convolution of the functions f(x,y) and g(x,y),*

$$(f * g) = \int_0^x \int_0^y f(\zeta,\eta) g(x-\zeta, y-\eta) d\zeta d\eta$$

is given by

$$G_x G_y \left[(f * g)(u,v) \right] = F(s,p) \frac{1}{uv} G(s,p) \qquad (3.19)$$

Proof: By using the definition of the double generalization of integral transform and double convolution, we have

$$G_{xy}\left[(f*g)(x,y);u,v\right] = uv\int_0^\infty\int_0^\infty e^{-usx-pvy}(f**)(x,y)dxdy$$

Hence

$$G_xG_y\left[(f*g)(x,y)\right] = uv\int_0^\infty\int_0^\infty e^{-usx-pvy}\left(\int_0^x\int_0^y f(\zeta,\eta)g(x-\zeta,y-\eta)d\zeta d\eta\right)dxdy$$

let $\alpha = x - \zeta$ and $\beta = y - \eta$ and using the valid extension of upper bound of integrals to $x \to \infty$ and $y \to \infty$, it yields

$$\begin{aligned} G_xG_y\left[(f*g)\right] &= uv\int_0^\infty\int_0^\infty e^{-su(\alpha+\zeta)-pv(\beta+\eta)} \times \\ &\quad \left(\int_{-\zeta}^\infty\int_{-\eta}^\infty f(x-\alpha,y-\beta)g(\alpha,\beta)(-d\zeta)(-d\eta)\right)d\alpha d\beta \\ &= uv\left[\int_0^\infty\int_0^\infty e^{-\zeta su-\eta pv}f(x-\alpha,y-\beta)d\zeta d\eta\right] \times \\ &\quad \left[\int_{-\zeta}^\infty\int_{-\eta}^\infty e^{-\alpha su-\beta pv}g(\alpha,\beta)d\alpha d\beta\right] \end{aligned}$$

where both $f(x,y)$ and $g(x,y)$ have zero value for $x < 0$ and $y < 0$, it then follows with respect to the lower limits of integration that

$$\begin{aligned} G_xG_y\left[(f*g)\right] &= uv\left[\int_0^\infty\int_0^\infty e^{-\zeta su-\eta pv}f(x-\alpha,y-\beta)d\zeta d\eta\right] \times \\ &\quad \left[\int_0^\infty\int_0^\infty e^{-\alpha su-\beta pv}g(\alpha,\beta)d\alpha d\beta\right] \\ &= \left[uv\int_0^\infty\int_0^\infty e^{-\zeta su-\eta pv}f(\zeta,\eta)d\zeta d\eta\right]\left[\int_0^\infty\int_0^\infty e^{-\alpha su-\beta pv}g(\alpha,\beta)d\alpha d\beta\right] \\ &= F(s,p)\frac{1}{uv}G(s,p) \end{aligned}$$

$$\Rightarrow \qquad G_xG_y\left[(f*g)\right] = F(s,p)\frac{1}{uv}G(s,p)$$

3.5 Properties of the Double Generalization of Integral Transform

Next, we present some properties of a double generalization of integral Transform.

Theorem 5. *(Linearity Property)*

The double generalization of integral transform is a linear operator.

Proof: Suppose $f(x,y)$ and $h(x,y)$ are double generalization of integral transformable functions and α_1 and α_2 are real constants, then

$$
\begin{aligned}
G_xG_y\left[\alpha_1 f(x,y) + \alpha_2 h(x,y)\right] = & \; uv\int_0^\infty\int_0^\infty \left[(\alpha_1 f(x,y) + \alpha_2 h(x,y))e^{-usx-pvy}\right] dxdy \\
&= uv\int_0^\infty\int_0^\infty \alpha_1 f(x,y)e^{-usx-pvy}dxdy \\
&+ uv\int_0^\infty\int_0^\infty \alpha_2 h(x,y))e^{-usx-pvy}dxdy \\
&= \alpha_1 uv\int_0^\infty\int_0^\infty f(x,y)e^{-usx-pvy}dxdy \\
&+ \alpha_2 uv\int_0^\infty\int_0^\infty h(x,y))e^{-usx-pvy}dxdy
\end{aligned}
$$

$$\therefore \qquad G_xG_y\left[\alpha_1 f(x,y) + \alpha_2 h(x,y)\right] = \alpha_1 G_{xy}(f(x,y)) + \alpha_2 G_{xy}(h(x,y))$$

as required.

Theorem 6. *(Inverse Operator)*

The Inverse double generalized integral transform is also a linear operator.

Proof: Taking the inverse integral transform of both sides of equation (3.20), we have

$$
\begin{aligned}
\alpha_1 f(x,y) + \alpha_2 h(x,y) &= G_{xy}^{-1}(\alpha_1 f(x,y) + \alpha_2 h(x,y)) \\
&= G_{xy}^{-1}(\alpha_1 f(x,y)) + G_{xy}^{-1}(\alpha_2 h(x,y)) = G_{xy}^{-1} \\
&(\alpha_1 F(u,v) + \alpha_2 H(u,v))
\end{aligned}
$$

where $G_{xy}f(x,y) = F(u,v)$ and $G_{xy}h(x,y) = H(u,v)$, respectively.

Theorem 7. *(Heaviside Unit Step Function)*

If the double generalized integral transform of a function $f(x,y)$ exists, then

$$G_{xy}\{f(x-\eta, y-\theta)H(x-\eta, y-\theta)\} = e^{-su\eta-pv\theta}G_{xy}f(x-\eta, y-\theta) \qquad (3.20)$$

where $H(x,y)$ is a Heaviside unit step function defined by $H(x-\eta, y-\theta) = 1$, when $x > \eta$ and $y > \theta$; and $H(x-\eta, y-\theta) = 0$, when $x < \eta$ and $y < \theta$.

Proof: By definition, we have

$$\{f(x-\eta, y-\theta)H(x-\eta, y-\theta)\} = uv\int_0^\infty\int_0^\infty e^{-usx-pvy}f(x-\eta, y-\theta) \times H(x-\eta, y-\theta)dxdy$$

for $x > \eta$ and $y > \theta$, we have

$$G_{xy}\{f(x-\eta, y-\theta)H(x-\eta, y-\theta)\} = uv\int_0^\eta\int_0^\theta \left[e^{-usx-pvy}f(x-\eta, y-\theta)\right]dxdy$$

Putting $\alpha = x-\eta$ and $\beta = y-\theta$, gives

$$\begin{aligned} G_{xy}f(\alpha,\beta) &= uve^{-su\eta-pv\theta}\int_0^\infty\int_0^\infty \left[e^{-su\alpha-pv\beta}f(\alpha,\beta)\right]d\alpha d\beta \\ &= e^{-su\eta-pv\theta}G_{xy}f(\alpha,\beta) \\ &= e^{-su\eta-pv\theta}G_{xy}f(x-\eta, y-\theta) \end{aligned}$$

Theorem 8. *(Periodic Function)*

If the double generalized integral transform of the function $f(x,y)$ exists where $f(x,y)$ is a periodic function of periods a and b, that is $f(x+a, y+b) = f(x,y)$, $\forall x,y$ then

$$G_{xy}f(x,y) = uv[1-e^{-(asu+bpv)}]^{-1}\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy \tag{3.21}$$

Proof: By definition,

$$G_{xy}\{f(x,y)\} = uv\int_0^\infty\int_0^\infty f(x,y)e^{-(usx+pvy)}dxdy$$

$$= uv\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy + uv\int_a^\infty\int_b^\infty f(x,y)e^{-(usx+pvy)}dxdy$$

By setting $x = \alpha + a$ and $y = \beta + b$ in the second double integral, we have

$$G_yG_x\{f(x,y)\} = uv\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy$$

$$+\,uve^{-(asu+bpv)}\int_0^\infty\int_0^\infty f(\alpha+a,\beta+b)e^{-(\alpha su+\beta pv)}d\alpha d\beta$$

$$= uv\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy$$

$$+\,uve^{-(asu+bpv)}\int_0^\infty\int_0^\infty f(\alpha,\beta)e^{-(\alpha su+\beta pv)}d\alpha d\beta$$

$$= uv\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy + e^{-(asu+bpv)}G_{xy}\{f(x,y)\}$$

This implies that

$$G_{xy}\{f(x,y)\} - e^{-(asu+bpv)}G_{xy}\{f(x,y)\} = uv\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy$$

$$G_{wy}\{f(x,y)\}(1-e^{-(asu+bpv)}) = uv\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy$$

$$G_{xy}\{f(x,y)\} = uv[1-e^{-(asu+bpv)}]^{-1}\int_0^a\int_0^b f(x,y)e^{-(usx+pvy)}dxdy$$

This proves the theorem of double generalized integral transform of a periodic function

Theorem 9. *(First Shifting Property)*

If $G_yG_x\{f(x,y)\} = F(s,p)$ *then* $G_yG_x\left\{e^{aux+bvy}f(x,y)\right\} = F(s-a,p-b)$ *where a and b are constants.*

Proof: From equation (3.3), we can see that:

$$\begin{aligned}
G_yG_x\left\{e^{aux+bvy}f(x,y)\right\} &= uv\int_0^\infty\int_0^\infty e^{aux+bvy}f(x,y)e^{-(usx+pvy)}dxdy \\
&= uv\int_0^\infty e^{-(pvy-bvy)}\int_0^\infty f(w,y)e^{-(usx-aux)}dxdy \\
&= uv\int_0^\infty e^{-(p-b)vy}\int_0^\infty f(x,y)e^{-(s-a)ux}dxdy
\end{aligned}$$

0 0 $F(s-a,p-b)$ **Theorem 10.** *(Change of Scale Property)*

If $G_yG_x\{f(x,y)\} = F(s,p)$ then $G_yG_x\{f(ax,by)\} = \frac{1}{(au)(bv)}F\left(\frac{s}{au},\frac{p}{bv}\right)$ *where a and b are constants.*

Proof: From equation (3.3)

$$G_yG_x\{f(ax,by)\} = uv\int_0^\infty\int_0^\infty f(aux,bvy)e^{-(usx+pvy)}dxdy \tag{3.22}$$

We put $r = auw$ and $t = bvy$ in equation (3.23) where r and t takes the limit from 0 to ∞. Hence, we get

$$\begin{aligned} G_yG_x\{f(ax,by)\} &= uv\int_0^\infty\int_0^\infty f(r,t)e^{-(\frac{r}{au}su+\frac{t}{bv}pv)}\frac{dr}{au}\frac{dt}{bv} \\ &= \frac{uv}{(au)(bv)}\int_0^\infty e^{-\frac{pvt}{bv}}\int_0^\infty e^{-\frac{urs}{au}}f(r,t)drdt \\ &= \frac{1}{(au)(bv)}F\left(\frac{s}{au},\frac{p}{bv}\right) \end{aligned}$$

Now, we apply the double Generalization of Integral transform on some special functions as follows:

1. Let $f(x,y) = 1$ for $x > 0$ and $y > 0$, then
$$G_{xy}\{f(x,y)\} = uv\int_0^\infty\int_0^\infty e^{-(usx+vpy)}dxdy = \frac{1}{sp}$$

2. Let $f(x,y) = e^{ax+by}$. where a, b are constants, then
$$G_{xy}\{f(x,y)\} = uv\int_0^\infty\int_0^\infty e^{-(usx+vpy)}e^{ax+py}dxdy = \left(\frac{u}{us-a}\right)\left(\frac{v}{vp-b}\right)$$

3. Let $f(x,y) = e^{i(ax+by)}$. where a, b are constants, then
$$\begin{aligned} G_{xy}\left\{e^{i(ax+by)}\right\} = \left(\frac{u}{us-ia}\right)\left(\frac{v}{vp-ib}\right) &= \frac{(us+ia)(vp+ib)}{(u^2s^2+a^2)(v^2p^2+b^2)} \\ &= \frac{uv(uvsp-ab)+iuv(avp+bus)}{(u^2s^2+a^2)(v^2p^2+b^2)} \end{aligned}$$

   consequently,

$$G_{xy}\{cos(ax+by)\} = \frac{uv(uvsp-ab)}{(u^2s^2+a^2)(v^2p^2+b^2)}$$

$$G_{xy}\left\{sin(ax+by)\right\} = \frac{uv(avp+bus)}{(u^2s^2+a^2)(v^2p^2+b^2)}$$

4. Let $f(x,y) = cosh(ax + by)$,where a, b are any constants, then

$$\begin{aligned} G_{xy}\left\{cosh(ax+by)\right\} &= \frac{1}{2}\left[G_{xy}\left\{e^{ax+by}\right\} + G_{xy}\left\{e^{-(ax+by)}\right\}\right] \\ &= \frac{1}{2}\left[\frac{uv}{(us-a)(vp-b)} + \frac{uv}{(us+a)(vp+b)}\right] \end{aligned}$$

Similarly we can obtain

$$G_{xy}\left\{sinh(ax+by)\right\} = \frac{1}{2}\left[\frac{uv}{(us-a)(vp-b)} - \frac{uv}{(us+a)(vp+b)}\right]$$

3.6 The Double Generalization of Integral Transform of Derivatives

In this subsection, we present the Double Generalization of Integral Transform of derivatives of the function $f(x,y)$ with respect to x,y . The double generalization of integral transform is defined as

$$G_{xy}\{f(x,y)\} = F(s,p) = uv\int_0^\infty\int_0^\infty f(x,y)e^{-(usx+pvy)}dxdy$$

and the double generalized integral transform of second partial derivative with respect to w is of the form

$$\begin{aligned} G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial x^2}\right] &= uv\int_0^\infty\int_0^\infty e^{-(usx+pvy)}\frac{\partial^2 f(x,y)}{\partial x^2}dxdy \\ &= v\int_0^\infty e^{-pvy}\left(u\int_0^\infty e^{-usx}\frac{\partial^2 f(x,y)}{\partial x^2}dx\right)dy \end{aligned}$$

By using integration by parts to compute the integral inside the brackets, we have

$$u\int_0^\infty e^{-usx}\frac{\partial^2 f(x,y)}{\partial x^2}dx = (us)^2F(u,v) - (us)^2f(x,0) - u\frac{\partial^2 f(0,y)}{\partial x} \qquad (3.23)$$

By taking the generalization of integral transform with respect to y for equation 3.23, we get double generalized integral transform of the form

$$G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial x^2}\right] = (us)^2F(x,y) - u^2sf(0,y) - u\frac{\partial F(0,y)}{\partial x} \tag{3.24}$$

Similarly, the double generalization of integral transform of $\frac{\partial^2 f(x,y)}{\partial y^2}$ is given by

$$\begin{aligned} G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial y^2}\right] &= uv\int_0^\infty \int_0^\infty e^{-(usx+pvy)}\frac{\partial^2 f(x,y)}{\partial x^2}dxdy \\ &= u\int_0^\infty e^{-usx}\left(v\int_0^\infty e^{-pvy}\frac{\partial^2 f(x,y)}{\partial y^2}dy\right)dx \end{aligned}$$

Again by using integration by parts to compute the integral inside the brackets, we have

$$v\int_0^\infty e^{-pvy}\frac{\partial^2 f(x,y)}{\partial y^2}dy = (vp)^2F(u,v) - (vp)^2f(0,y) - u\frac{\partial^2 f(x,0)}{\partial y} \tag{3.25}$$

Finally, by taking the generalization of integral transform with respect to *x* for equation 3.23, we get double generalization of integral transform of the form

$$G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial y^2}\right] = (vp)^2F(x,y) - v^2pf(x,0) - v\frac{\partial F(x,0)}{\partial y} \tag{3.26}$$

Also the double generalization of integral transform of second partial derivative with respect to *x* and *y* is of the form

$$G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial x\partial y}\right] = G_{xy}\left\{f_{xy}(x,y)\right\}$$

By definition

$$
\begin{aligned}
G_{xy}\left\{f_{xy}(x,y)\right\} &= uv\int_0^\infty\int_0^\infty e^{-(usx+pvy)} f_{xy}(x,y)dxdy \\
&= uv\int_{y=0}^\infty e^{-pvy}\left\{\int_{x=0}^\infty e^{-usx} f_{xy}(x,y)dx\right\}dy \\
&= uv\int_{y=0}^\infty e^{-pvy}\left\{f_y(x,y)e^{-usx}\ |_{x=0}^\infty + (us)\int_{x=0}^\infty e^{-usx} f_y(x,y)dx\right\}dy \\
&= uv\int_{y=0}^\infty e^{-pvy}\left\{-f_y(0,y) + (us)\int_{x=0}^\infty e^{-usx} f_y(x,y)dx\right\}dy \\
&= -uv\left\{f(0,y)e^{-pvy}\ |_{y=0}^\infty + vp\int_{y=0}^\infty f(0,y)e^{-pvy}dy\right\} \\
&\quad + \left\{(us)(uv)\int_{x=0}^\infty e^{-usx}\int_{y=0}^\infty e^{-pvy} f_y(x,y)dx\right\}dy \\
&= -uv\left\{-f(0,0) + vp\int_{y=0}^\infty f(0,y)e^{-pvy}dy\right\} \\
&+ \left\{(us)(uv)\int_{x=0}^\infty e^{-usx}\left[f(x,y)e^{-pvy}\ |_{y=0}^\infty + \int_{y=0}^\infty e^{-pvy} f(x,y)dy\right]\right\}dx
\end{aligned}
$$

which gives

$$
\begin{aligned}
G_{xy}\left\{f_{xy}(x,y)\right\} = (uv)f(0,0) + (vp)(uv)\int_{y=0}^\infty f(0,y)e^{-pvy}dy+ \\
(us)(uv)\int_{x=0}^\infty e^{-usx} f(x,0) + (us)(vp)(uv) \\
\int_0^\infty\int_0^\infty e^{-(usx+pvy)} f(x,y)dydx
\end{aligned}
$$

$$
\therefore \quad G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial x\partial y}\right] = (uv)f(0,0) + (vp)F(0,p) + (us)F(s,0) + (us)(vp)F(s,p)
$$

Hence the double generalization of integral transform of derivative yields the equations below;

$$
G_{xy}\left[\frac{\partial f(x,y)}{\partial x}\right] = usF(s,p) - uF(0,p) \tag{3.27}
$$

$$
G_{xy}\left[\frac{\partial f(x,y)}{\partial y}\right] = vpF(s,p) - vF(s,0) \tag{3.28}
$$

$$
G_{xy}\left[\frac{\partial f(x,y)}{\partial x\partial y}\right] = (us)(vp)F(s,p) - usF(s,0) - vpF(0,p) + uvf(0,0) \tag{3.29}
$$

$$
G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial x^2}\right] = (us)^2F(s,p) - u^2sf(0,p) - u\frac{\partial F(0,p)}{\partial x} \tag{3.30}
$$

$$
G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial y^2}\right] = (vp)^2F(s,p) - v^2pF(s,0) - v\frac{\partial F(s,0)}{\partial y} \tag{3.31}
$$

3.7 Double Generalization of integral transform of double integral

We now find the double GIT of a double integral.

Theorem 11. *If* $G_{xy}\{f(x,y)\} = F(s,p)$ *and*

$$g(x,y) = \int_0^x \int_0^y f(r,t)dtdr \tag{3.32}$$

then

$$G_x G_y \left\{ \int_0^x \int_0^y f(r,t)dtdr \right\} = \frac{F(s,p)}{(us)(vp)} \tag{3.33}$$

Proof: We denote $h(x,y) = \int_0^y f(x,t)dt$

By the fundamental theorem of calculus

$$h_y(x,y) = f(x,y) \tag{3.34}$$

and

$$h(x,0) = 0 \tag{3.35}$$

Taking the double GIT of equation (3.34), we get

$$G_x G_y\{h_y(x,y)\} = G_x G_y\{f(x,y)\}$$

$$uv \int_0^\infty \int_0^\infty h_y(x,y) e^{-(usx+vpy)} dxdy = uv \int_0^\infty \int_0^\infty f(x,y) e^{-(usx+vpy)} dxdy$$

$$uv \int_{x=0}^\infty e^{-usx} \left\{ \int_{y=0}^\infty h_y(x,y) e^{-vpy} dy \right\} dx = F(s,p)$$

$$uv \int_{x=0}^\infty e^{-usx} \left\{ (-vp) h(x,y) e^{-vpy} \mid_{y=0}^\infty + (vp) \int_{y=0}^\infty h(x,y) e^{-vpy} dy \right\} dx = F(s,p)$$

$$(uv)(vp) \int_{x=0}^\infty e^{-usx} h(x,0) dx + (vp)(uv) \int_{x=0}^\infty \int_{y=0}^\infty h(x,y) e^{-(usx+vpy)} dydx = F(s,p)$$

$$\therefore \quad (vp)H(s,0) + (vp)H(s,p) = F(s,p)$$

Taking the single GIT of equation (3.35) gives $H(s,0) = 0$

$$\Rightarrow \qquad H(s,p) = \frac{F(s,p)}{vp} \tag{3.36}$$

From equation (3.32)

$$g(x,y) = \int_0^x h(r,y)dr$$

Again, by fundamental theorem of calculus

$$g_x(x,y) = h(x,y) \tag{3.37}$$

and

$$g(0,y) = 0 \tag{3.38}$$

Taking double GIT of equation (3.37), we get

$$G_xG_y\{g_x(x,y)\} = G_xG_y\{h(x,y)\}$$

$$uv\int_0^\infty\int_0^\infty g_x(x,y)e^{-(usx+vpy)}dxdy = uv\int_0^\infty\int_0^\infty h(x,y)e^{-(usx+vpy)}dxdy$$

$$uv\int_{x=0}^\infty e^{-usx}\left\{\int_{y=0}^\infty g_x(x,y)e^{-vpy}dy\right\}dx = H(s,p)$$

$$uv\int_{y=0}^\infty e^{-vpy}\left\{(-us)g(x,y)e^{-usx}\,|_{x=0}^\infty + (us)\int_{x=0}^\infty g(x,y)e^{-usx}dx\right\}dy = H(s,p)$$

$$(uv)(us)\int_{x=0}^\infty e^{-vpy}g(0,y)dy + (us)(uv)\int_{x=0}^\infty\int_{y=0}^\infty g(x,y)e^{-(usx+vpy)}dydx = H(s,p)$$

$$\therefore \quad (us)G(0,p) + (us)G(s,p) = H(s,p)$$

Taking the single GIT of equation (3.38) gives $G(0,p) = 0$

$$\Rightarrow \qquad (us)G(s,p) = H(s,p)$$

From equation (3.

36) $\Rightarrow$

$$(us)G(s,p) = \frac{F(s,p)}{VP}$$

$$\therefore \quad G(s,p) = \frac{F(s,p)}{(us)(vp)}$$

$$G_x G_y \left\{ \int_0^x \int_0^y f(r,t)dtdr \right\} = \frac{F(s,p)}{(us)(vp)}$$

Chapter 4

Illustration of the Double Generalization of Integral Transform

In this section, we apply the double generalization of integral transform (DGIT) method to partial differential equations.

Example 1: We obtain the solution of the following partial differential equation,

$$f_x = 2f_y + f, \qquad f(w,0) = e^{-3x} \quad \text{for} \quad x > 0, y > 0.$$

Taking the double generalization of integral transform (DGIT) of both sides gives

$$G_{xy}[f_x] = G_{xy}[2f_y + f]$$

$$\Rightarrow \qquad usF(s,p) - uF(0,p) = 2vpF(s,p) - 2vF(s,0) + F(s,p)$$

$$F(s,p) = \frac{u}{[us - 2vp - 1]}F(0,p) - \frac{2v}{[us - 2vp - 1]}F(s,0) \tag{4.1}$$

Putting the initial condition $f(x,0) = e^{-3x}$, $\Rightarrow F(s,0) = \frac{u}{us+3}$ into equation (4.1) gives

$$F(s,p) = \frac{u}{[us - 2vp - 1]}F(0,p) - \frac{v}{vp+2}\left(\frac{u}{us - 2vp - 1} - \frac{u}{us+3}\right) \tag{4.2}$$

Taking the inverse of the double generalization of integral transform of equation (4.2) gives

$$G_{xy}{}^{-1}\{F(s,p)\} = G_{xy}{}^{-1}\left\{\frac{u}{[us - 2vp - 1]}F(0,p) - \frac{v}{vp+2}\left(\frac{u}{us - 2vp - 1} - \frac{u}{us+3}\right)\right\}$$

$$\therefore \quad f(x,y) = e^{-3x}e^{-2y}$$

is the solution of the PDE in the Example

Example 2: $f_{yy} = kf_{xx}$, $f(x,0) = \sin\pi x$, $f(0,y) = 0$, $f(1,y) = 0$, $0 < x < 1$, $y > 0$

Taking the double GIT of both sides, we have

$$G_{xy}[f_y] = kG_{xy}[f_{xx}]$$

$$\Rightarrow \qquad vpF(s,p) - vF(s,0) = k[(us)^2F(s,p) - u^2sF(0,p) - uF_x(0,p]$$

Putting the conditions $f(0,y) = 0 \quad \Rightarrow F(0,p) = 0$ and $f(x,0) = \sin\pi x$ $\Rightarrow F(s,0) = \frac{\pi u}{(us)^2-\pi^2}$ gives

$$vpF(s,p) - \frac{\pi uv}{(us)^2 - \pi^2} = k[(us)^2F(s,p) - u^2sF(0,p) - uF_x(0,p)]$$

$$F(s,p) = \frac{ku}{k(us)^2 - vp}F_x(0,p) - \frac{\pi uv}{[(us)^2 - \pi^2][k(us)^2 - vp]}$$

$$\begin{aligned}
F(s,p) &= \frac{u}{[(us)^2 - \frac{vp}{k}]}F_x(0,p) - \frac{\pi uv}{k[(us)^2 - \pi^2][(us)^2 - \frac{vp}{k}]} \\
&= \frac{u}{[(us)^2 - \frac{vp}{k}]}F_x(0,p) - \frac{\pi v}{k[\frac{vp}{k} + \pi^2]}\left\{\frac{u}{[(us)^2 - \frac{vp}{k}]} - \frac{u}{[(us)^2 + \pi^2]}\right\} \\
&= \frac{u}{[(us)^2 - \frac{vp}{k}]}F_x(0,p) - \frac{\pi uv}{[vp + k\pi^2][(us)^2 - \frac{vp}{k}]} + \frac{\pi uv}{[vp + k\pi^2][(us)^2 + \pi^2]} \\
&= \frac{u}{[(us)^2 - \frac{vp}{k}]}\left(F_x(0,p) - \frac{\pi v}{[vp + k\pi^2]}\right) + \frac{\pi uv}{[vp + k\pi^2][(us)^2 + \pi^2]} \\
&= \frac{u}{\left(us - \sqrt{\frac{vp}{k}}\right)\left(us + \sqrt{\frac{vp}{k}}\right)}\left(F_x(0,p) - \frac{\pi v}{[vp + k\pi^2]}\right) + \frac{\pi uv}{[vp + k\pi^2][(us)^2 + \pi^2]}
\end{aligned}$$

$$\begin{aligned}
F(s,p) = &\left\{\frac{u}{(us - \sqrt{\frac{vp}{k}})} - \frac{u}{(us + \sqrt{\frac{vp}{k}})}\right\}\frac{k}{2vp}\left(F_x(0,p) - \frac{\pi v}{vp + k\pi^2}\right) \\
&+ \frac{\pi uv}{[vp + k\pi^2][(us)^2 + \pi^2]}
\end{aligned} \tag{4.3}$$

Taking the inverse of the double generalization of integral transform of both sides of equation 4.3, we have

$$\begin{aligned}
G_{xy}^{-1}\{F(s,p)\} = G_{xy}^{-1}&\left\{\frac{u}{(us - \sqrt{\frac{vp}{k}})} - \frac{u}{(us + \sqrt{\frac{vp}{k}})}\right\}\frac{k}{2vp}\left(F_x(0,p) - \frac{\pi v}{vp + k\pi^2}\right) \\
&+ \frac{\pi uv}{[vp + k\pi^2][(us)^2 + \pi^2]}
\end{aligned}$$

Putting the boundary condition $f(1,y) = 0 \Rightarrow F(1,p) = 0$ and taking limit as $x \to 1$ gives

$$f(x,y) = \sin\pi x e^{-k\pi y}$$

Example 3: Consider the following wave equation $\frac{\partial^2 w}{\partial y^2} = c^2\frac{\partial^2 w}{\partial x^2}$, where $c^2 = \frac{T}{\rho}$, for positive x and y with the following boundary conditions

$$w(0,y) = f(y) = \begin{cases} siny & \text{if } 0 \leq y \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$, $\lim_{x\to\infty} w(x,y) = 0$, $(y \geq 0)$ and the initial

conditions $w(x,0) = \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0.$

Applying the double GIT on the wave equation gives

$$(vp)^2 W(s,p) - v^2 p W(s,0) - v\frac{\partial W(0,p)}{\partial y} = c^2\left\{(us)^2 W(s,p) - u^2 s W(0,p) - u\frac{\partial W(0,p)}{\partial x}\right\}$$

Substituting the boundary condition $w(x,0) = 0$ which $\Rightarrow W(s,0) = 0$ and $\left.\frac{\partial w}{\partial y}\right|_{y=0} = 0$ which $\Rightarrow W_y(s,0) = 0$ into the above equation we obtain

$$(vp)^2 W(s,p) = c^2\left\{(us)^2 W(s,p) - u^2 s W(0,p) - uW_x(0,p)\right\}$$

$$\therefore \quad W(s,p) = \frac{c^2u^2 s}{c^2(us)^2 - (vp)^2} W(0,p) + \frac{c^2 u}{c^2(us)^2 - (vp)^2} W_x(0,p) \qquad (4.4)$$

Now, $W(0,p) = G_y[w(0,y)] = G_y[f(y)] = F(y)$ and

$$\begin{aligned} W_x(0,p) &= \lim_{s\to 0} W(s,p) \\ &= \lim_{s\to 0} G_{xy}\{w_x(x,y)\} \\ &= \lim_{s\to 0}\int_0^\infty e^{-vsy}\left\{\int_0^\infty e^{-usx} w_x(x,y)dx\right\}dy \\ &= \int_0^\infty e^{-vsy}\left\{\int_0^\infty e^{-usx} w_x(x,y)dx\right\}dy \end{aligned}$$

Thus

$$\begin{aligned} W_x(0,p) &= \int_0^\infty e^{-vsy}\left\{\lim_{x\to\infty}(x,y) - w(0,y)\right\}dy \\ &= \int_0^\infty e^{-vsy} w(0,y)dy \\ &= -F(p) \end{aligned}$$

We can see from equation (4.4) that:

$$
\begin{aligned}
W(s,p) &= \frac{c^2u^2s}{c^2(us)^2-(vp)^2}F(p) - \frac{c^2u}{c^2(us)^2-(vp)^2}F(p)\\
&= \left\{\frac{c^2u^2s}{c^2(us)^2-(vp)^2} - \frac{c^2u}{c^2(us)^2-(vp)^2}\right\}F(p)\\
&= \left\{\frac{1}{2}\left[\frac{u}{us-\frac{vp}{c}} + \frac{u}{us+\frac{vp}{c}}\right] - \frac{c}{2vp}\left[\frac{u}{us-\frac{vp}{c}} - \frac{u}{us+\frac{vp}{c}}\right]\right\}
\end{aligned}
$$

Applying G_x^{-1} of both sides of the above equation, we obtain

$$
\begin{aligned}
W(x,p) &= \left\{\frac{1}{2}\left[e^{\frac{vp}{c}x} + e^{-\frac{vp}{c}x}\right] - \frac{c}{2vp}\left[e^{\frac{vp}{c}x} + e^{-\frac{vp}{c}x}\right]\right\}F(p)\\
&= \left\{\frac{1}{2}e^{\frac{vp}{c}x} - \frac{c}{2vp}e^{\frac{vp}{c}x} + \frac{1}{2}e^{-\frac{vp}{c}x} + \frac{c}{2vp}e^{-\frac{vp}{c}x}\right\}F(p)
\end{aligned}
$$

$$
W(x,p) = \left\{A(p)e^{\frac{vp}{c}x} + B(p)e^{-\frac{vp}{c}x}\right\}F(p) \tag{4.5}
$$

where $A(p) = \frac{1}{2}\left(1-\frac{c}{vp}\right)$ and $B(p) = \frac{1}{2}\left(1+\frac{c}{vp}\right)$

Since $\lim_{x\to\infty} w(x,y) = 0$ when $y \geq 0$ then

$$
\lim_{x\to\infty} W(x,p) = \lim_{x\to\infty}\int_0^\infty e^{-vpy}w(x,y)dy = \int_0^\infty e^{-vpy}\lim_{x\to\infty} w(x,y)dy = 0
$$

This implies that $A(p) = 0$ in (4.5) because $c > 0$, so that for every fixed positive vp the function $e^{\frac{vp}{c}x}$ increases as x increases.

$$
\Rightarrow \qquad W(x,p) = B(p)e^{-\frac{vp}{c}x}F(p)
$$

But if $A(p) = 0$ $\Rightarrow c = vp$ and $B(p) = \frac{1}{2}\left(1+\frac{c}{vp}\right) = 1$

Therefore from equation (4.5)

$$
W(x,p) = e^{\frac{-vpx}{c}}F(p) \tag{4.6}
$$

Taking G_y^{-1}, of both sides of (4.6) from the second shift theorem, we get

$$
w(x,y) = f\left(y-\frac{x}{c}\right)g\left(y-\frac{x}{c}\right)
$$

i.e $w(x,y) = sin\left(y-\frac{x}{c}\right)$ if $\frac{x}{c} < y < \frac{x}{c} + 2\pi$ or $cy > x > (y-2\pi)c$ and zero otherwise

Example 4: Suppose we are given the non-homogeneous telegraph equation that is given as

$$w_{xx}(x,y) - w_{yy}(x,y) - w_y(x,y) - w(x,y) = -2e^{x+y} \tag{4.7}$$

with boundary conditions

$$w(0,y) = e^y, \qquad w_x(0,y) = e^y \tag{4.8}$$

and initial conditions

$$w(x,0) = e^x, \qquad w_y(x,0) = e^x \tag{4.9}$$

By applying the double generalization of integral transform on equation (4.7) and single generalization of integral transform on equations (4.8) and (4.9), we obtain

$$\begin{aligned} W[(s,p);(u,v)] &= \frac{-uv + u^3vs^2 - uv^3p^2 - uv^2p}{(us-1)(vp-1)[(us)^2 - (vp)^2 - vp - 1]} \\ &= \frac{uv}{(us-1)(vp-1)} \end{aligned} \tag{4.10}$$

Taking double inverse of the generalization of integral transform of equation (4.10), gives

$$w(x,y) = e^{x+y}$$

Example 5: Considering the Volterra Integro Partial Differential Equation below,

$$\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} = -1 + e^x + e^y + e^{x+y} + \int_0^w \int_0^y f(r,t)dtdr \tag{4.11}$$

Subject to the initial conditions:

$$f(x,0) = e^x \quad \text{and} \quad f(0,y) = e^y \tag{4.12}$$

Applying double GIT to equation (4.11) we get

$$usF(s,p) - uF(0,p) + vpF(s,p) - vF(s,0) = -\frac{1}{sp} + \frac{u}{p(us-1)} + \frac{v}{s(us-1)} + \frac{uv}{(us-1)(vp-1)} + \frac{F(s,p)}{(us)(vp)} \tag{4.13}$$

and single GIT of equation (4.12) we get

$$F(s,0) = \frac{u}{(us-1)} \quad \text{and} \quad F(0,p) = \frac{v}{(vp-1)} \tag{4.14}$$

Substituting (4.14) into (4.13) we have

$$usF(s,p) - \frac{uv}{(vp-1)} + vpF(s,p) - \frac{uv}{(us-1)} = -\frac{1}{sp} + \frac{u}{p(us-1)} + \frac{v}{s(us-1)} + \frac{uv}{(us-1)(vp-1)} + \frac{F(s,p)}{(us)(vp)}$$

$$F(s,p)\left[us + vp - \frac{1}{(us)(vp)}\right] = -\frac{1}{sp} + \frac{u}{p(us-1)} + \frac{v}{s(us-1)} + \frac{uv}{(vp-1)} + \frac{uv}{(us-1)(vp-1)} + \frac{uv}{(us-1)}$$

$$= \frac{(uv)(us)(sp) + (uv)(vp)(sp) - 1}{sp(us-1)(vp-1)}$$

$$\therefore \quad F(s,p) = \frac{uv}{(us-1)(vp-1)}$$

(4.15) By taking the double inverse GIT of equation (4.15), we obtain the solution of (4.11) as follows

$$f(x,y) = e^{x+y}$$

Example 6: We consider the wave equation below

$$f_{xx}(x,y) - f_{yy}(x,y) = 3(e^{2x+y} - e^{x+2y}), \qquad x,y \in R_+ \tag{4.16}$$

with boundary conditions

$$f(0,y) = e^y + e^{2y}, \quad f_x(0,y) = 2e^y + e^{2y}, \tag{4.17}$$

and initial conditions

$$f(x,0) = e^{2x} + e^x, \quad f_y(x,0) = e^{2x} + 2e^x, \tag{4.18}$$

Taking double Generalization of Integral transform of equation (4.16) and single Generalization of Integral transform of conditions (4.17) and (4.18), we get

$$\begin{aligned}\frac{3uv}{(us-2)(vp-1)} + \frac{3uv}{(us-1)(vp-2)} &= u^2v^2F(s,p) - \frac{u^2vs}{vp-1} - \frac{u^2vs}{vp-2} - \frac{2uv}{vp-1} \\ &- \frac{uv}{vp-2} - v^2p^2F(s,p) + \frac{uv^2p}{us-2} + \frac{uv^2p}{us-1} \\ &+ \frac{uv}{us-2} + \frac{2uv}{us-1}\end{aligned}$$

$$\Rightarrow \quad F(s,p) = \frac{uv}{(us-2)(vp-1)} + \frac{uv}{(us-1)(vp-2)} \tag{4.19}$$

Taking double inverse Generalization of Integral transform of equation (4.19), we get the following solution

$$f(x,y) = e^{2x+y} + e^{x+2y}$$

Chapter 5

Conclusion Summary and Recommendations

5.1 Conclusion

We have introduced the double generalization of integral transform (DGIT) of a function $f(w,y)$

$$G_xG_y\{f(x,y)\} = uv\int_0^\infty\int_0^\infty f(ux,vy)e^{-(usx+pvy)}dxdy, \quad \forall(x,y) \in \{0\} \cup \mathbb{R}^+$$

We have also established some of the properties of the double generalization of integral transform such as linearity, convolution, step function and double generalization of integral transform for derivatives.

The illustration of the double generalization of integral transform to some partial differential equations confirms the solutions obtained using other integral transforms.

We have also shown that, the following equations are the expression for partial derivatives using the double GIT;

$$G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial x^2}\right] = (us)^2F(x,y) - u^2sf(0,y) - u\frac{\partial F(0,y)}{\partial x}$$

$$G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial y^2}\right] = (vp)^2F(x,y) - v^2pf(x,0) - v\frac{\partial F(w,0)}{\partial y}$$

$$G_{xy}\left[\frac{\partial^2 f(x,y)}{\partial x\partial y}\right] = (uv)f(0,0) + (vp)F(0,p) + (us)F(s,0) + (us)(vp)F(s,p)$$

$$G_{xy}\left[\frac{\partial f(x,y)}{\partial x}\right] = usF(s,p) - uF(0,p)$$

$$G_{xy}\left[\frac{\partial f(x,y)}{\partial y}\right] = vpF(s,p) - vF(s,0)$$

5.2 Recommendations

Researchers and students can extend the double generalization of integral transform to fractional double generalization of integral transform.

Researchers and students can also extend the double generalization of integral transform (DGIT) to $p - DGIT$ and $q - DGIT$

REFERENCES

Asiru, M. A. (2001). Sumudu transform and solution of integral equations of convolution type, *International Journal of Mathematics, Ed. Sci. Tech.* **32**: 906–910.

Atangana, A. and Kilic¸man, A. (2013). The use of sumudu transform for solving certain nonlinear fractional heat-like equations, *Abstract and Applied Analysis* **2013**: 12 pages.

Barnes, B. (2016). Polynomial integral transform for solving differential equations, *European Journal of Pure and Applied Mathematics* **9**(6): 140–151.

Barnes, B., Adjei, I. A., Sebil, C. and F., D. N. (2019). Fractional generalized integral transform, *Journal of Mathematics and Statistics Research* **1**(1): 1–4.

Barnes, B., Sebil, C. and Quaye, A. (2018). A generalization of integral transform, *European Journal of Pure and Applied Mathematics* **11**(4): 1130–1142.

Belgacem, F. B. M., Karaballi, A. A. and Kalla, S. L. (2003). Analytical investigations of the sumudu transform and applications to integral production equations, *Mathematical Problems in Engineering* (3): 103–118.

Belgacem, F. B. M. and Silambarasan, R. (2012). Theory of natural transform, *Mathematics in Engineering, Science and Aerospace journal* **3**: 99–124.

Buschman, R. G. (1983). Heat transfer between a fluid and a plate: multidimensional laplace transformation methods, *Interat. J. Math. and Math. Sci.* **6**(3): 589–596.

Cahen, E. (1894). Sur la fonction $\zeta(s)$ de riemann et sur des functions analogues, *Ann de l'Ec. Norm* **11**: 75–164.

Carlson, B. C. (1969). A connection between elementary functions and higher transcendental functions., *SIAM Journal on Applied Mathematic* **17**.

Chaudhary, P., Chanchal, P., Khandelwal, Y. and Singh, H. (2018). Duality of ”some famous integral transforms” from the polynomial integral transform, *International Journal of Mathematics Trends and Technology* **55**(5): 345–348.

Churchill, R. V. (1954). The operational calculus of legendre transforms, *Jour. Math. and Physics* **33**: 165–178.

Coon, G. A. and Bernstein, D. L. (1953). Some properties of the double laplace transformation, *American Mathematical Society* **74**: 135 – 176.

Deakin, M. A. B. (1985). Euler’s invention of the integral transforms, *Archive for History of Exact Sciences* **33**: 307–319.

Debnath, L. (1960). On laguerre transform, *Bull. Calcutta Math. Soc.* **55**: 69 – 77.

Debnath, L. (1963). On jacobi transformations, *Bull. Calcutta Math. Soc.* **55**: 113–120.

Debnath, L. (1964). On hermite transforms, *Mathematicki Vesnik* **1**: 285–292.

Debnath, L. (1967). Solution of partial differential equations by the jacobi transform, *Bull. Calcutta Math. Soc.* **159**: 155–158.

Debnath, L. (1968). Some operational properties of the hermite transform, *Math. Vesnik* **5**(20): 29–36.

Dhunde, R. R., Bhondge, N. M. and Dhongle, P. R. (2013). Some remarks on the properties of double laplace transforms, *International Journal of Applied Physics and Mathematics* **3**(4): 293–295.

Dhunde, R. R. and Waghmare, G. L. (2014). On some convergence theorems of double laplace transform, *Journal of Informatics and Mathematical Sciences* **6**(1): 45–54.

Dhunde, R. R. and Waghmare, G. L. (2017). Double laplace transform method in mathematical physics, *International Journal of Theoretical and Mathematical Physics* **7**(1): 14–20.

Dimovski, I. H. and Kalla, S. L. (1988). Convolution for hermite transforms, *Math. Japonica* **33**: 345–351.

Dominguez, A. (2016). Highlights in the history of the fourier transform, *IEEE Pulse Retrospectroscope* **Jan/Feb**: 53–61.

Eltayeb, H. and Kili¸cman, A. (2007). A note on mellin transform and partial differential equations, *International Journal of Pure and Applied Mathematics* **34**(4): 457–467.

Eltayeb, H. and Kili¸cman, A. (2008). A note on solutions of wave, laplace's and heat equations with convolution terms by using a double laplace transform., *Applied Mathematics Letter* **21**: 1324–1329.

Eltayeb, H. and Kili¸cman, A. (2013). A note on double laplace transform and telegraphic equations, *Abstract and Applied Analysis* **2013**.

Elzaki, T. M. (2012). Double laplace variational iteration method for solution of nonlinear convolution partial differential equations, *Archives Des Sciences* **65**(12): 588–593.

Erd´elyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. (1954). *Tables of Integral Transforms*, Vol. 1 and 2, McGraw-Hill, New York.

Estrin, A. and Higgins, T. J. (1951). The solution of boundary value problems by multiple laplace transformation,, *Journal of the Franklin Institute* **252**(2): 153–167.

Ezhil, S. S. (2017). Real time application of fourier transforms, *Indonesian Journal of Electrical Engineering and Computer Science* **8**(2): 574–577.

Fourier, J. (1822). *La Th´eorie Analytique de la Chaleur,*, Dover Publications, 1955.

Fourier, J. B. J. (1807). *Th´eorie de la propagation de la chaleur dans les solides*, Institute of France.

Fourier, J. B. J. (1811). *Th´eorie du mouvement de la chaleur dans les corps solides*, l'Acad´emie royale des sciences de l'Institute de France.

Gupta, A. (2013). Fourier transform and its application in cell phones, *International Journal of Scientific and Research Publications* **3**: 2 pp.

Hardy, G. (1924). On hilbert transforms, *Messenger of Math.* **54**: 20–27, 81–88.

Heaviside, O. (1899). *Electromagnetic Theory*, Dover Publications, New York.

Hilbert, D. (1912). *Grundzu¨ge einer allgemeinen th´eorie der linearen Integralgeichungen*.

Hoffman, K. and Kunze, R. (1971). *Linear Algebra*, Prentice-Hall, Inc. Englewood Cliffs, New Jersey.

Jean Tchuenche, M. and Nyimvua, S. M. (2007). An application of the double sumudu transform, *Applied Mathematical Sciences* **1**: 31.

Kadem, A. (2005). Solving the one-dimensional neutron transport equation using chebyshev polynomials and the sumudu transform, *Annals of University of Oradea, Fascicle of Mathematics* **12**: 153–171.

Kang, F. (1958). Generalized mellin transforms 1, *Sc. Sinica* **7**: 582–605.

Khan, Z. H. and Khan, W. A. (2008). N-transform properties and applications, *NUST Journal of Engineering Sciences* **1**: 127–133.

Kili¸cman, A. and Eltayeb, H. (2009). An application of double laplace transform and double sumudu transform, *Lobachevskii Journal of Mathematics* **30**(3): 214–223.

Kili¸cman, A. and Eltayeb, H. (2010). On a new integral transform and differential equations, *Mathematical Problems in Engineering* **2010**: 13 pages.

Kili¸cman, A. and Omran, M. (2017). On double natural transform and its applications, *Journal of Nonlinear Sciences and Applications* **10**: 1744–1754.

Klamkin, M. S. and Newman, D. J. (1961.). The philosophy and applications of transform theory, *SIAM Rev* **3**: 10–36.

Knapp, A. W. (2005). *Basic real analysis, with an appendix "elementary complex analysis"*, Birkhauser, Boston.

Kober, H. (1943). A note on hilbert transforms,, *Journal of the London Mathematics Society* **18**: 66–71.

Kreyszig, E. (1978). *Introductory functional analysis with applications*, John Wiley & Sons. Inc., New York.

Kropivsky, P. L. and Ben-Naim, E. (1994). Scaling and multiscaling in models of fragmentation, *American Physical Society* **50**(5).

Laplace, P. S. (1820). *Th´eorie Analytique des Probabiliti´es*, Vol. Vol. I, Part 2, Lerch, Paris.

Lenssen, N. and Needell, D. (2014). An introduction to fourier analysis with applications to music, *Journal of Humanistic Mathematics* **4**: 72–91.

Lokenath, D. (2016). The double laplace transforms and their properties with applications to functional, integral and partial differential equations, *International Journal of Applied and Computational Mathematics* **2**: 223 – 241.

Lokenath, D. and Dambaru, B. (2015). *Integral transforms and their applications*, third edn, CRC Press.

Mathai, A. M., Saxena, R. K. and Haubold, H. J. (2010). *The H-function,;*

Theory and Applications, Springer Science+Business Media.

McCully, J. (1960). The laguerre transform, *SIAM Review* **2**: 185 – 191.

Mellin, H. (1896). Uber die fundamentale wichtgkeit des satzes von cauchy fu¨r¨ die theorien der gamma-und der hypergeometrischen funk-tionen,, *Acta Soc. Fennicae* **21**: 1–115.

Mellin, H. (1902). Uber den zusammenhang zwischen den linearen differential-¨ und differezengleichugen, *Acta Math.* **25**: 139–164.

Newton, I. (1744). *Opuscula mathematica, philosophica et philologica*, Vol. 3 vols, Lausannae and Genevae.

Newton, I. (1964). *The Mathematical Works*, Vol. 2 Vols, Johnson Reprint Corp.

Osman, M. and Ali Bashir, M. (2016). Solution of partial differential equations with variables coefficients using double sumudu transform, *International Journal of Scientific and Research Publications* **6**(6): 37–46.

Papoulis, A. (1962). *The Fourier Integral and Its Applications*, McGraw-Hill, New York.

Riemann, B. (1876). Uber die anzahl der primzahlen unter eine gegebe-nen gr¨osse,¨ *Gesammelte Math. Werke* pp. 136–144.

Rognlie, D. M. (1969). *Generalized integral transform*, phdthesis.
URL: *https://lib.dr.iastate.edu/rtd/3598*

Royden, H. L. and Fitzpatrick, P. M. (2010). *Real analysis*, Pearson Education, Inc, New Jersey.

Saff, E. B. and Snider, A. D. (1976). *Complex analysis for mathematics, science and engineering*, Prentice-Hall, Inc. New York.

Schiff, J. L. (1999). *The Laplace Transform: Theory and Applications*, Springer.

Stieltjes, T. S. (1894). Recherches sur les fractions continues, *Annales de la Facult´e des Sciences de Toulouse* **18**: 1–123.

Stroud, K. A. (2003). *Advance Engineering Mathematics*, 4th edn, Palgrave Macmillan, New York.

Thomson, B. S., Bruckner, J. B. and Bruckner, A. M. (2001). *Elementary real analysis*, 2 edn, Pearson Education Inc., New Jersey.

Titchmarsh, E. (1967). *Introduction to the Theory of Fourier Integrals*, Oxford University Press, Glassgow.

Titchmarsh, E. C. (1925). Conjugate trigonometrical integrals, *Proc. London Math. Soc.* **24**: 109–130.

Watugala, G. K. (1993). Sumudu transform: a new integral transform to solve differential equations and control engineering problems, *International J. Math.* **Sci. Tech. 24**: 35–43.

Widder, D. V. (1971). *An Introduction to Transform Theory*, Academic Press, Boston.

Zhang, J. (2007). A sumudu based algorithm for solving differential equations, *J. Comput. Sci. J. Moldova* **15**: 303–313.

Zill, D. G. (2013). *A First Course in Differential Equations with Modeling Applications*, 9th edn, Brooks/Cole, Boston.

Appendix A

Summary of Double GIT for some functions

$f(x,y)$	$F(s,p) = G_{xy}f(x,y)$

a	$\frac{a}{sp}$
e^{ax+by}	$\frac{uv}{(us-a)(vp-b)}$
$sin(ax+by)$	$\frac{uv(avp+bus)}{(u^2s^2+a^2)(v^2p^2+b^2)}$
$cos(ax+by)$	$\frac{uv(uvsp-ab)}{(u^2s^2+a^2)(v^2p^2+b^2)}$
$sinh(ax+by)$	$\frac{1}{2}\left[\frac{uv}{(us-a)(vp-b)} - \frac{uv}{(us-a)(vp-b)}\right]$
$cosh(ax+by)$	$\frac{1}{2}\left[\frac{uv}{(us-a)(vp-b)} + \frac{uv}{(us-a)(vp-b)}\right]$