KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

Optimal Spline Based Gas-Lift Allocation Using Lagrange's Multiplier

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M.Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

I dedicate this to my father, Mr. Isaac K. Ntherful and mother, Miss Elizabeth Saim.

Abstract

An accurate prediction of the production rate of fluids from a reservoir to the surface is essential for efficient artificial lift application in an oil field. In an oil production field, a group of wells can be put under gas lift operation. For each well, there will be an optimum point of gas injection and oil production rate where the incremental oil production rate relative to an incremental gas injection rate should be equal for all wells in the field. The optimum point for a given group of wells was found by Jamal (2001) using the application of Lagrange Multipliers. An approach that depends on the functional relationship between oil production and gas injection rates. Two types of functions, quadratic and rational functions were used to fit the gas injection against oil production data. This thesis used spline based functions to fit the gas injection verses oil production data. Gas-in, oil-out data fit with spline based function produced better results than both the rational function and quadratic function with regard to both data fit and resultant total optimum oil rate. The two methods used in this thesis were able to increase the total oil production. However the total optimum oil production rate for data fitting with the spline based function is found to be higher than the total optimum oil production rate for data fitting with rational function. The optimal value of the spline based function was found to be twice that of the rational function.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

This chapter gives a brief introduction on the concept of gas-lift technique and gas allocation into gas injection wells. It also discusses the problem statement and the objective of the study. And finally describes the organization of this thesis.

1.1 Background of the study

Most oil producing wells flow naturally without artificial stimulation when they are first drilled. But as the wells mature the energy level falls and this affects the production rate to fall to a level that is economically no longer profitable. In order to increase production flow rate, artificial processes are applied to either increase the production level or to facilitate the flow of the hydrocarbons. These processes are classified as secondary recovery method or artificial lift. The artificial lift supplements the natural drive effects on pressure maintenance and displacement by employing water injection/water flooding, natural gas injection, rod pumps also called Donkey pumps or Downhole pumps.

Gas lift technique is one of the widely used artificial lift methods in the oil field.

This drive mechanism is identified by the presence of a gas cap over an oil zone in an oil reservoir. The energy to produce the fluid comes from the expansion of the gas in the gas cap and the expansion of the gas liberated from the liquid as the pressure declines.

According to Ayatollahia et al. (2004), gas lift technique involves an injection of a compressed high pressure gas into an injection well in order to lighten the column fluids to allow the reservoir pressure to force the fluid to the surface.

Gas lift can be controlled for a single well to optimize production, and to reduce slugging effects where the gas droplets collect to form large bubbles that can upset production. In a field where a multiple of wells are put under gas lift operation, gas is injected to each of the wells and the total oil production rate from the field would equal to the sum of the individual oil rates. When limited gas is available, the gas is allocated to each well in order to maximize the oil production rate from the field. In this case, the lift gas is optimized over the wells to use available gas in the most efficient way. For each well, there would then be an optimum point of gas injection and oil production rate where the incremental oil production rate relative to an incremental gas injection rate should be equal for all wells in the field. This thesis is an extension on the previous research work by Jamal (2001) and other researchers who applied the Lagrange Multipliers technique to find the optimum point of gas injection seeking to maximize oil production. The research work seeks to improve these previous works by developing a model using the proposed constraint cubic spline algorithm by Kruger (n.d.) for accurate and quality prediction and then applies the Lagrange's multiplier to carry out the optimization for oil production.

1.2 Statement Of Problem

An ideal allocation of limited available gas to a multiple gas injection wells in a petroleum production field has been in a core consideration for years when trying to maximize oil production.

In view of this many optimization techniques have been applied by different researchers. But irrespective of the optimization method, there should be a good approximation method that best estimates a function that relates gas injection to oil production for the formulation of the constrained optimization problem. This is because, the inability to identify an appropriate function have serious repercussion on the optimum point found.

Lagrange is one of the most useful methods for optimizing models of that nature but its efficiency and effectiveness depends on an initial curve fitting method. In literature most researchers have used the rational and the quadratic functions which are global interpolation methods that is construction single equation that fits all the data points. These methods result in smooth curves however they are prone to severe oscillation and overshoot at intermediate points. Although, the methods seem to give good functions for the Lagrange to produce a good optimization result, the error of these methods tend to increase drastically as the order n becomes large. Often, a higher degree polynomial introduces unnecessary oscillations or wiggles. Hence, the polynomial and the rational interpolation in this case have not been accurate.

This work seeks to identify a curve fitting that can give the best fit for the optimization, find a suitable approximation method that will best lead the Lagrange to find the appropriate optimum value.

This research aims at finding an appropriate approximation method that develops a suitable function for the formulation of the constrained optimization problem.

1.3 Research Objectives

The objective of this work is to improve on exiting literature to find an optimum amount of gas to maximize oil production from an oil field on a day-to-day basis. The investigation of developed constrained cubic spline technique will be carried out through application to the Oil-field data which was used by Jamal (2001).

1.3.1 Specific Objectives

The objectives of this study are specifically stated as follows:

- to identify a mathematical model that can best fit real well data.
- to perform the Gas-lift performance curve with the modified cubic spline and compare with an existing one.
- to develop a model from the best fit and carry out an optimization of oil production with the Lagrange's multiplier.

1.4 Outline of the study

This work has been organized into five chapters. Chapter 1 covers the concept of gas lift system and gas allocation, statement of problem on the identification of appropriate approximation function on gas injection and allocation of limited available gas to multiple wells, objectives of the study and the thesis organization. Chapter 2 covers a review of related literature on the study. Chapter 3 deals with the methodology adopted for the study, it comprises the rational function model and cubic spline interpolation for the curve fitting of gas injection and oil output rates of five gas lifted wells and the optimization procedure to find the optimum gas for injection using Lagrange multiplier. Chapter 4 presents data analysis and discussions of the results and Chapter 5 which is the last chapter is devoted for conclusion and recommendations of the study.

Chapter 2

Review of Literature

2.1 Introduction

In this chapter, some basic concepts and notations that relate to the concept of gas lift optimization will be discussed. There will also be a review of some literature in the area of gas lift optimization and rate allocation as well as approximations and interpolations theories. Major theories, arguments, methodologies, approaches and controversies in the existing literature on the subject of this study will be discussed in this chapter.

2.2 Overview of Approximation and Interpolation Theory

Curve fitting is the process of finding a smooth and continuous curve to pass through or closer to a set of data points. According to Won et al. (2005), curve fitting is the process of finding a curve that could best indicate the trend of a given set of data. The curve does not have to go through the data points.

According to Singiresu (2002), curve fitting of a set of data points can be done using two approaches. The first approach called Collocation, that is where the curve is made to pass through every data point. The approach is used either when the data is known to be accurate or the data are known to be generated by evaluating a complicated function at a discrete set of points (Singiresu, 2002). Polynomial, trigonometric or exponential function is used to approximate a set of data point. In some cases, piecewise curve fitting is used where a specified function is made through sub-groups of data points.

The second approach is where the curve is made to represent the general trend of the data. According to Singiresu (2002), this approach is useful when there are more data points than the number of unknown coefficients or when the data appear to have a significant error or noise. As posited by Kruger (n.d.), interpolation is used to estimate the value of a function between known data points without knowing the actual function. According to Henrici. (1982), as cited in Kruger (n.d.), interpolation methods can be divided into two main categories and these are:

Global interpolation. These methods rely on constructing single equation that fits all the data points. This equation is usually a high degree polynomial equation. Although these methods result in smooth curves, they are usually not well suited for engineering applications, as they are prone to severe oscillation and overshoot at intermediate points.

Piecewise interpolation. These methods rely on constructing a polynomial of low degree between each pair of known data points. If a first degree polynomial is used, it is called linear interpolation. For second and third degree polynomials, it is called quadratic and cubic splines respectively. The higher the degree of the spline, the smoother the curve. Splines of degree m, will have continuous derivatives up to degree m-1 at the data points.

Interpolation of a set of data points can be done using polynomial, spline functions or Fourier series. However, polynomial interpolation is commonly used and many numerical methods are based on polynomial approximations.

For a given set of N+1 data points $(x_0, y_0), (x_1, y_1), ..., (x_N, y_N)$, the interest is to find the N^{th} order polynomial function that can match them.

$$P_N = a_0 + a_1 x + a_2 x^2 + ... + a_N x^N \tag{2.1}$$

The coefficients can be obtained by solving a set of algebraic equations.

$$a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_N x_0^N = y_0$$

$$a_0 + a_1 x_1 + a_2 x_1^2 + ... + a_N x_1^N = y_1$$

$$a_0 + a_N x_+ a_2 x_N^2 + ... + a_N x_N^N = y_N$$

But, as the number of data points increases, so as that of the unknown variables and equations. Consequently it may not be so easy to solve.

However, there are a number of alternative forms of expressing an interpolating polynomial beyond the familiar format. Among them are Lagrange, Newton's forward and backward difference, Hermite interpolations.

According to Singiresu (2002), the errors of a single polynomial tend to increase drastically as its order n becomes large. Singiresu (2002) said the higher order polynomial often introduces unnecessary oscillations and wiggles. Because of this polynomial interpolation will not be accurate. Therefore the answer to using

information from more data points and at the same time keeping the function true to the data behaviour is in spline interpolation. The most common spline interpolations used are linear, quadratic, and cubic splines.
To obtain a smoother curve, cubic splines are frequently recommended. They are generally well behaved and continuous up to the second order derivative at the data points. Even though cubic splines are less prone to oscillation or overshooting than global polynomial equations, they do not prevent it. Thus, the use of cubic splines is limited to applications where oscillation and overshoot are acceptable or desirable (Kruger, n.d.).

2.2.1 Linear Spline

According to Singiresu (2002), linear or first order spline represents a straight line between the data points (knots). Let n+1 data points be available as $[x_i, f(x_i)]$, i=0,1,2,....,n. The author explained that considering two neighbouring data points $[x_{i-1}, f(x_{i-1})]$ and $[x_i, f(x_i)]$, the equation of the line joining the two points is defined as $f_i = f(x_{i-1}) + \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}(x - x_{i-1})$; i=1,2,...,n, where the function $f_i(x)$ represents a set of n piecewise linear equations (splines) using the n+1 data points and $\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}$ is the slope between x_{i-1} and x_i.
According to Addendum (n.d.), linear spline interpolation is no different from linear polynomial interpolation. He said linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So this is improved by using quadratic splines.

2.2.2 Quadratic Splines

In quadratic splines, a second order polynomial approximates the data between two consecutive data points or knots. Singiresu (2002) explained the quadratic spline that given $x_i, f(x_i)$, $i = 0, 1, 2, ..., n$ to denote $n+1$ data points, the equation of the quadratic spline between the data points $x_{i-1}, f(x_{i-1})$ and $x_i, f(x_i)$ can be expressed as $f_i(x) = a_i + b_i x + c_i x^2; i = 1, 2, ..., n$ where a_i, b_i and c_i are the unknown coefficients. If there are n intervals, there would be $3n$ coefficients to be evaluated. To find the $3n$ unknowns, $3n$ equations need to be set up and then simultaneously solved. According to Singiresu (2002), these $3n$ equations are found by the following conditions.

1. The function value at the interior knot x_i must be equal to $f(x_i)$ whether it is computed using $f(x_i)$ or $f(x_{i+1})$: That is

$$f_i(x = x_i) = a_i + b_i x_i + c_i x_i^2 = f(x_i); i = 1, 2, ..., n, \tag{2.2}$$

and

$$f_{i+1}(x = x_i) = a_{i+1} + b_{i+1} x_i + c_{i+1} x_i^2 = f(x_i); i = 1, 2, ..., n-1, \tag{2.3}$$

Equations 2.2 and 2.3 give $(2n - 2)$ conditions.

2. The first and last functions, $f_1(x)$ and $f_n(x)$ must pass through the end points x_0 and x_n respectively:

$$f_1(x = x_0) = a_1 + b_1 x_0 + c_1 x_0^2 = f(x_0), \tag{2.4}$$

and

$$f_n(x = x_n) = a_n + b_n x_n + c_n x_n^2 = f(x_n). \tag{2.5}$$

3. The first derivatives or slopes of two quadratic splines are continuous at the interior points. For example, the derivative of equation 2.2 gives the slope as

$$f_i^{'}(x) = b_i + 2x_i c_i; i = 1, 2, ..., n, \tag{2.6}$$

and hence, the continuity of slope leads to

$$f_i^{'}(x = x_i) = f_{i+1}^{'}(x = x_i); \tag{2.7}$$

that is,

$$b_i + 2c_i x_i = b_{i+1} + 2c_{i+1} x_i; i = 1, 2, ..., n - 1. \tag{2.8}$$

Equation 2.8 yields $(n - 1)$ conditions.

4. So far, the total number of equations is $(3n - 1)$ equations. In order to evaluate all $3n$ constants, one more equation is needed. There are several possible conditions that can be used. For instance, the second derivative can be assumed to be zero at the final point (x_n). That is;

$$f_n^{''}(x = x_n) = 2c_n = 0, or c_n = 0$$

.

2.2.3 Cubic Splines

According to Won et al. (2005) the piecewise-quadratic curve is not smooth enough to please the eyes, since the second-order derivatives of quadratic polynomials for adjacent subintervals can not be made to conform with each other. However, cubic splines have proven to be a good compromise between accuracy and complexity. Consequently, the concept of cubic splines is developed as followed. A cubic spline through a set of data points is a curve obtained by joining each point to the next with a cubic polynomial, where adjoining cubic spline must a have matching first and second derivative at their common point. The equation of the cubic spline in the i^th interval, $[x_{i-1}, x_i]$ is defined as

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3; i = 1, 2, ..., n. \tag{2.9}$$

where the $4n$ coefficients a_i, b_i, c_i and d_i for $i = 1, 2, .., n$.

According to Hoffman (2001) these $4n$ coefficients a_i, b_i, c_i and d_i can be evaluated using the following conditions:

1. The function values, $f(x_i) = f \ (i = 2, 3.....n)$, must be the same in the two splines on either side of xi at all of the $n-1$ interior points. This constraint yields $2(n-1)$ conditions.

2. The slope or first order derivative of the two splines on either side of point x_i; must be equal at all of the $n-1$ interior points.

   $$f_i^{'}(x_i) = f_{i+1}^{'}(x_i)$$

   . This constraint yields $(n-1)$ conditions.

3. The second derivative of the two splines on either side of point x_i must be equal at all of the n - 1 interior points. This constraint yields $(n-1)$ conditions.

4. The first and last spline must pass through the first (i.e., x_1) and last (i.e.,

X_{n+1}) points. That is, $f_1(X_1) = f_1$ and $f_n(x_{n+l}) = f_{n+1}$. This yields 2 conditions.

5. Two more equations are needed to obtain the polynomial uniquely. There are various types of conditions that can be prescribed to obtain the two equations;

    - We shall consider the case of a natural spline, in which we set the two conditions as $f^{''}(x_0) = 0$, $f^{''}(x_n) = 0$.

    the curvature [i.e. the second derivative] must be specified at the first x_1 and last x_{n+l} points. That is, $f_l^{''}(x_1) = f_1^{''}$ and $f^{''}(x_{n+1}) = f_{n-l}^{''}$. This constraint yields 2 conditions.

2.3 Injection

There are two main groups of wells; these are injection and production wells. The former is drilled to inject gas or water into the reservoir to boost the flow of hydrocarbons unto the surface whiles the latter is for production of oil and gas. Injection is done to maintain overall hydrostatic reservoir pressure and force the fluid toward the production wells.

2.4 Artificial Lift

According to Havard (2006), production well can be free flowing or lifted. A well that has enough down hole pressure to reach a suitable well-head production pressure and maintain an acceptable well-flow is known as a free flowing oil well. The one whose formation pressure is too low is termed as lifted or artificial lift. The artificial lift is put into two groups which are pumps and gas lift.

Pumps

The pumps comprises those that use pumps and this is:

Rod Pumps

Sucker Rod Pumps are commonly used in land based operations. These are also called Donkey pumps or Beam pumps. A motor drives a reciprocating beam, connected to a polished rod passing into the tubing via a stuffing box. The sucker rod continues down to the oil level and is connected to a plunger with a valve (Havard, 2006).

2.4.1 Types of Gas Lift Operation

A well can be placed on continuous or intermittent lift. In a continuous flow gas lift, the flowing bottom hole pressure remains constant for a particular set of conditions. A continuous injection of volume of high pressure gas into a rising stream of well fluids in such a way that useful work is done in lifting the well fluids. Continuous lift is suitable for wells with high productivity index and reasonably high bottom hole pressure relative to the well depth. It is a very flexible form of artificial lift.

According to Ibrahim (2007), Tools (n.d.) wrote that continuous gas flow is a very flexible form of artificial lift and can be used to produce liquid rates in excess of 75000 barrels per day in larger tubing or casing flow application down to 50 barrels per day or less in smaller tubing sizes.

Figure 2.1 shows a typical continuous flow gas lift installation.

Figure 2.1: Continuous Gas lift Flow.

In intermittent gas lift operation, the flowing bottom hole pressure differs with a particular period of a production cycle. In this method, gas is injected to the well in periodic intervals coinciding with fluid fill-in rate from the producing formation into the well bore. Intermittent lift is designed to produce the oil at the actual rate that the fluid enters the well bore from the reservoir. The system enables the fluids to build up in the production tubing at the bottom of the well. High pressure injection gas is quickly injected into the production tubing underneath the fluids which has built up and this rapidly propels it to the surface. Then the gas injection ceases until the build up of a new fluid and the cycle continues. Figure 2.2 shows the principles of the intermittent gas lift cycle. Lift type is used on wells with relatively low fluid volumes.

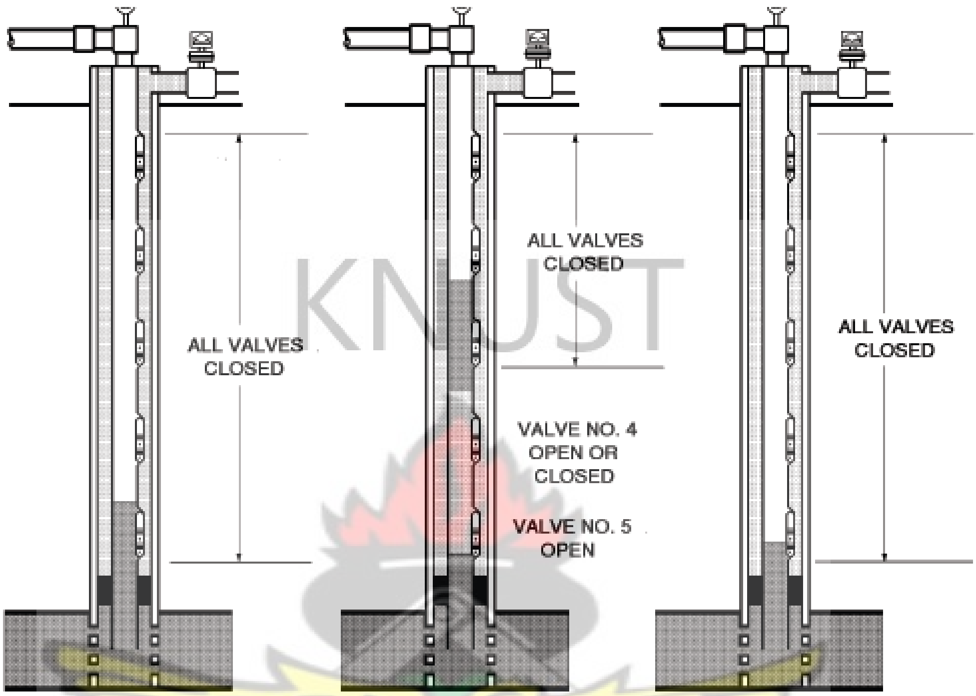

Figure 2.2: Intermittent Flow Gas Lift Installation.

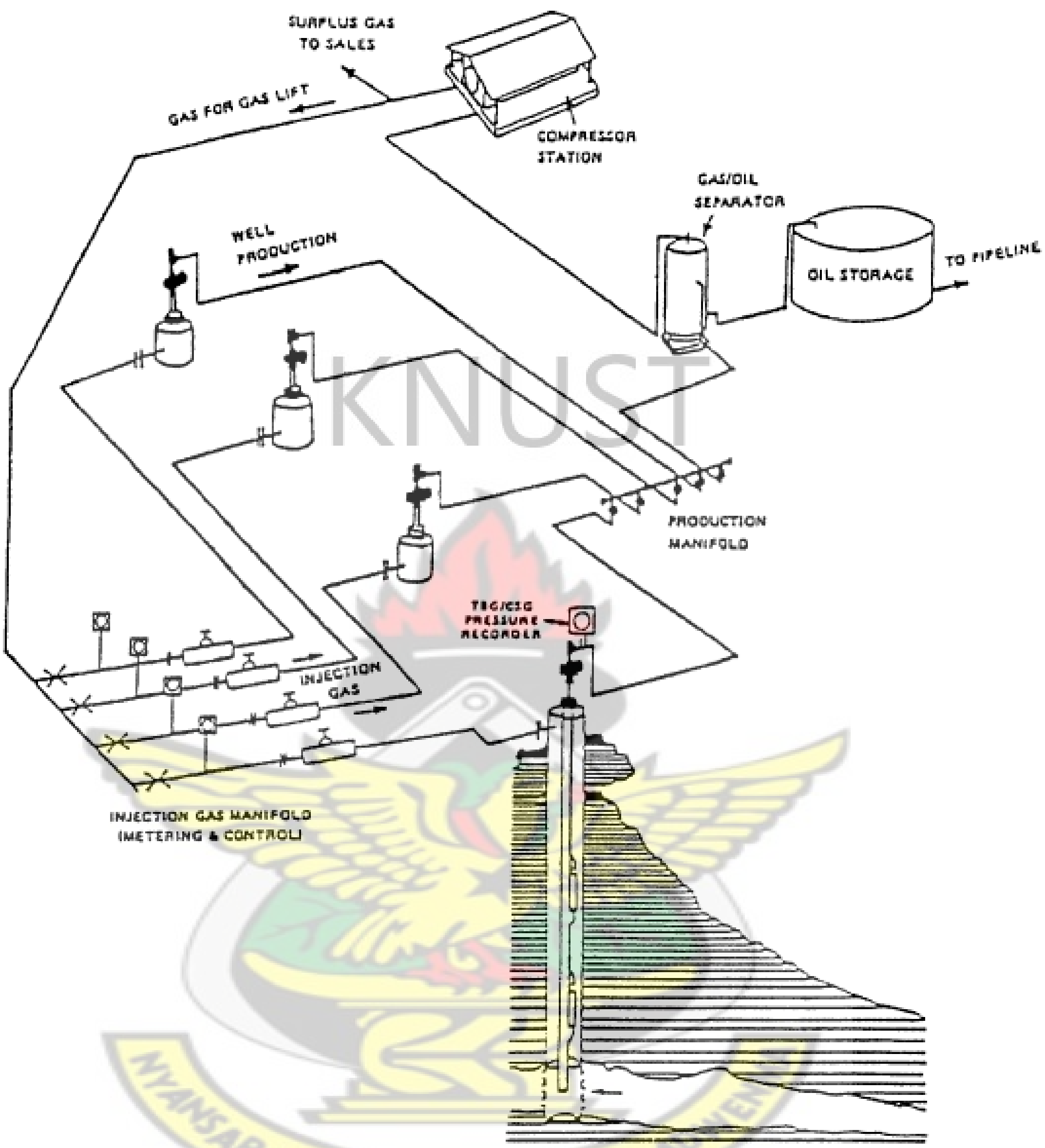

Figure 2.3: A Typical Gas Lift System.

Drive mechanisms

According to Dake (2002), the basic types of drive mechanisms are:

2.4.2 Water drive

A majority of petroleum reservoirs are surrounded by aquifers on the portion of their boundaries. The driving force for this recovery mechanism is the movement of water into the pore spaces to replace the oil that has been produced. This create a vast quantity of water which provides a great store of energy and that aids the production of oil and gas. This mechanism is classified into two depending on the geometry of the reservoir namely bottom water drive and edge water drive. The mechanism is illustrated as 'Water Drive' in the figure 2.4

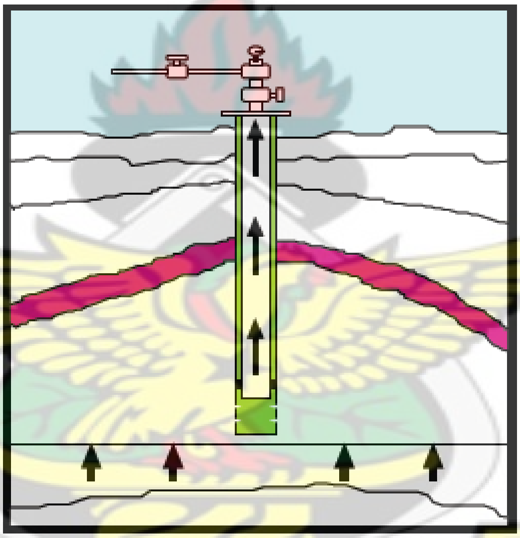

Figure 2.4: A Typical Water Drive.

Schlumberger (2000) in work said that the water drive is one of the most efficient of the primary drive mechanisms, capable of yielding up to 50% of the original oil in place. This kind of mechanism is often supplemented by the injection of high pressure treated salt water into the reservoir to maintain the pressure and 'sweep' the oil toward the well bore.

2.4.3 Solution gas drive

This drive mechanism is characterized with the presence of gas dissolved in oil. Here the bubble point of gas expands as the gas escapes from the oil. This expels the oil from porous media driving it through the reservoir toward the well and assists in lifting it to the surface. Figure 2.5 illustrates a solution gas drive.

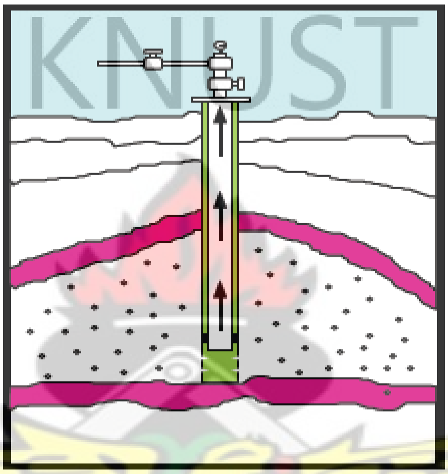

Figure 2.5: A Typical Solution Gas Drive.

According to Schlumberger (2000), it is generally considered as the least effective type of drive, yielding only 15% to 25% of the oil originally contained in the reservoir.

2.4.4 Compaction drive

This drive mechanism might occur during depletion when rock grains are subjected to stress beyond elasticity limit. It leads to a re-compaction of partially deformed or even destroyed rock grains that might result in gradual or abrupt reduction of the reservoir pore volume, posited by Stanghelle (2009).

2.4.5 Gas-cap drive

This is identified by the formation of gas over the oil layer in an oil reservoir. Since gas lighter than oil, it rises to the top of the reservoir and forms a gas cap over the oil.

When the gas in the gas-cap expands, the energy to produce the fluids also increases. The expansion of the gas liberated from the liquid as the pressure declines drives the oil toward the wellbore.

High gas compressibility and the extended gas cap size ensure a long lasting and efficient field performance. Up to 35% of the original oil in place can be recovered under a gas-cap drive as stated in Stanghelle (2009). The effectiveness of the recovery also depends on the on the geometry of the reservoir. Extremely thin oil may result in very small recovery factor. Very steep dip angle allows a good drainage of oil at the bottom of the structure. About 60% oil can be recovered. A typical gas cap drive is shown in the figure 2.6.

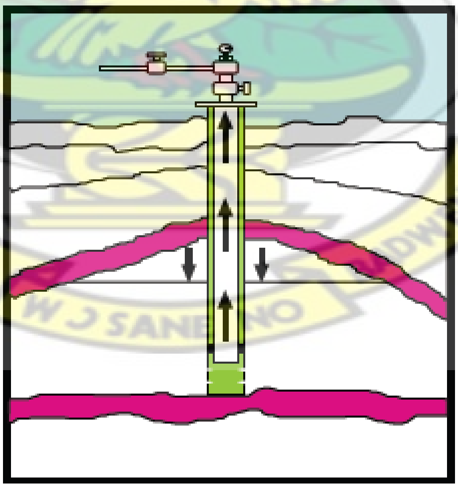

Figure 2.6: A Typical Gas Cap Drive.

The reservoir geometry is an important factor on the efficiency of the gas cap drive. The efficiency of the drive also depends on the relative size of the gas-cap compared to the volume of the oil, the viscosity and the vertical permeability.

2.4.6 History of Gas Lift

The first use of gas lift was to remove water from mines in Chemnitz, Hungary in the mid 18*th* century by (Shaw, 1939). But according to Cloud (1937), gas lift was first used in the oil industry in 1864 for wells in Pennsylvania. This was called a 'well blower' which was a system that consisted of an air-filled pipe connected to the tubing that blew compressed air into the bottom of the well to reduce oil density and increase well production rates. In Texas around 1900 gas lift with air was first used in large-scale oilfield applications, and in 1920 natural gas replaced air as the lifting gas of choice because it had a lower risk of explosion.

At first, gas was injected essentially uncontrolled into the bottom of the well and gas lift application was limited to shallow wells because of low injection pressures attainable (Gabor, 2005). In the mid 1930 the invention of a spring-operated differential gas lift valve and the development of a stepwise unloading process consisting of multiple well injection points allowed gas lift to be used for wells of even greater depths. The valve opened if there was enough pressure difference between casing and tubing, and allowed a more controlled gas injection. These valves were fixed in place on the tubing. Some other valves were developed that could be mechanically opened from the surface but all these had reliability problems and if they failed the entire tubing and valve had to be replaced presented by Brown (1980).

According to King (1944), the first pressure-operated gas lift valve which uses a pressurized bellows instead of a mechanical spring to control gas injection was presented in 1944. Later there was an invention of wire line retrievable valves which allows the valve to be replaced in the event of malfunction without replacing the entire tubing.

2.4.7 Gas Lift Optimization, Rate Allocation and Well selection

Many researchers have conducted various studies on gas lift optimization system. A description of a non-linear optimization problem with constraints associated with the optimal distribution of the lift gas was given by Sharma et al. (2012). A non-linear objective function was developed using a simple dynamic model of the oil field where the decision variables represent the lift gas flow rate set points of each oil well of the field. The lift gas optimization problem was solved using the 'fmincon' solver found in MATLAB. As an alternative and for verification, hill climbing method was used in solving the optimization problem. Using these two methods, it was shown that after the optimization, the total oil production is increased by about 4%. For multiple oil wells sharing lift gas from a common source, a cascade control strategy along with a non-linear steady state optimizer behaves as a self-optimizing control structure when the total supply of lift gas is assumed to be the only input disturbance present in the process. Simulation results showed that repeated optimization performed after the first time optimization under the presence of the input disturbance has no effect in the total oil production.

Hatton & Potter (2011) used SAS/OR optimization techniques to provide quick results using a scalable (from 1 to n well) solution. Background on artificial injection to orient the reader, theory on the mathematical formulation of the optimization, and the SAS code with results was described.

A study of the implementation of improving traditional method by using portfolio selection theory that generates several combination of gas lift injection rate was done by Piyawat (2010). Piyawat (2010) introduced a new approach for well selection, portfolio theory method, and reviewed what the portfolio theory method is, how it can be applied and how the oil field operation in Thailand can gain more value through the use of portfolio theory method.

Schlumberger-Doll Research engaged in a study to streamlining the complexity and speed of the optimization process. The approach focused on reducing the number of mathematical steps, and new optimization algorithms (Schlumberger, n.d.)

Lo and Holden applied linear programming to find which wells is to be opened, partially opened or closed. They assumed that each well could produce any oil rate between zero and the maximal oil rate, and that the water cut and gas-oil ratio were the same for all rates (i.e. no gas or water coning)as indicated in Jamal (2001). The method used is able to handle multiple constraints on oil, water, liquid, and gas production for groups of wells (or all) Bieker et al. (2006). Allah (2007) discussed the use of a Multi-phase Flow Meter to optimize gas lift field operations. This in particular compares analysis methods individual well's performance using multi flow meter versus Standard Nodal Analysis. It also tackles GUPCO's field experience of gas lift offshore operations. Sugiarso (1995) conducted a study on gas lift optimization for limited gas supply using gas lift optimization allocation and nodal analysis software. Accurate measurement of field and well test data and close monitoring of the gas lift system resulted in a total oil gain of 1250 barrels per day.

Use of programmable logic controllers was reported by Knott (1991) in offshore oil field for controlling production from gas lifted wells, with well data being transmitted through radio link between the controllers and the on-board control room. An optimum production flow rate was maintained for each well by controlling the operation of the well-head choke valves and valves on the lift gas lines Knott (1991). Stewart et al. (1989) decreased orifice sizes of the gas lift valves and redesigned the gas lift headers to remove the problems of slugging and hydrate formation. The compressor suction pressure had to be lowered due to increasing water cuts. Computer-based distributed control systems were used for determining optimum gas injection rate for each well. Back-pressure on gas lift wells was minimized by replacing flow line chokes with elbow spool pieces effecting an increase of 3000 barrels per day of oil production.

Well evaluation software was used by Stinson (1988) for analysing the design of a gas lift system. Effects of tubing size, gas injection pressure, separator pressure and gas injection volume on production rates and compressor horsepower requirements for gas lift wells were studied using this software. Laing (1986) described production optimization from gas lift wells by conducting special training for operators, analysing flowing pressure and temperature surveys, replacing defective gas lift valves, measuring correctly injection gas flow rate to each well, twinning surface flow line and improving gas lift design techniques. Injection points were bracketed for future or uncertain conditions. Leonard (1984) described a closed-loop gas lift system where software was used to generate well performance curves for optimization calculations. The top valves were changed to spring-loaded valves in order to avoid the effects of excessive temperature differentials caused by the permafrost during unloading.

A strategy for gas lift allocation and production well selection based on the incremental gas oil ratio and formation gas oil ratio concepts was implemented by Stoisits et al. (1994) in order to increase the oil production from a compression capacity limited field. The total oil production from the field increased by approximately 10,000 barrels per day as a result of the optimization strategy. Coltharp

& Khokhar (1984) described a computer based control system to monitor and control the operation of the gas lifted wells. The control system maximized the oil production by automatically controlling the gas injection rates within specified limits and also by maintaining constant supply of available gas to high producing wells following the shut down of any compressor.

An economic approach to oil production and gas allocation in continuous gas lift was discussed by Clegg (1982) The optimal gas injection rate, defined as the rate resulting in the highest present value profit after tax, would be less than the rates achieving maximum oil production and maximum current operating cash income. The study indicated that the maximum oil flow is usually not the best economic point of operation. Everitt (1994) showed that gas-lift optimization efforts in a large mature field could reduce the gas lift requirements by 50%. Everitt (1994) significantly reduced the compression costs and markedly improved the field's financial performance. The lift gas requirements were minimized by elimination of heading problems by installing a smaller orifice at the point of injection. Adjunta & Majek (1994) reported uses of high-performance personal computer and intelligent remote terminal unit for optimizing the remote control of gas injection and surveillance of offshore gas lifted wells. The steady-state production curve of a well was generated with the use of a solar-powered well-head monitor. A computer program fitted this curve with a third degree polynomial to determine the optimal gas injection rate which was used as the set point in the control system network. The flow/no-flow condition of the well was determined by installing a nozzle on the well discharge side.

Systems analysis techniques were applied by Amondin & Jackson (1996) for optimizing gas lift allocation in a group of gas lifted wells. An optimization software was used to determine the optimal gas allocation rates from the analysis of well performance curves modelled with a polynomial function. The optimization algorithm also handled the choke settings in surrounding naturally flowing wells. **?** applied non-linear optimization algorithms to a field model that was composed of a reservoir model, a well model with gas lift, a choke model and a separator model. The combination of the production parameters such as tubing diameter, separator pressure, gas injection depth and volume of gas injection was looked for optimizing the net present value of the model. The genetic algorithm optimization techniques were found to be both stable and efficient to address these sorts of optimization problems. Programmable logic controller was used by Lematayer & Miret (1991) to increase the gas-lift efficiency with an increase in oil production and a decrease in gas injection. Tubing head temperature was used as the production rate indicator, which was maximized by adjusting the gas injection rate.

Application of a genetic algorithm was described by Mantecon (1983) to the problem of assigning optimum gas injection rates to a group of wells given an available total gas supply for the field. The productivity of the field increased by approximately 20% with respect to the approach based on individual well optimization. Mantecon (1983) presented a multi-stage optimization program giving emphasis on individual well optimization, improving gas lift design technique, converting to other artificial lift methods, improving system diagnostics, field personnel training and enhancing engineering communication. Significant improvements were found in individual well profitability, optimizing injection gas usage, well monitoring and data acquisition. Cooksey & Pool (1995) discussed the application of automated control system in optimizing continuous flow gas lift operations. The controllers could be used with centralized master station direction or as stand-alone products. Local controllers when they were not continuously dependent on the central master station performed better in optimizing the operations of gas lifted wells.

An artificial lift allocation program, based on a personal computer was used by A. Woodyard (1989) to improve the productivity of gas lifted wells. The wells were identified for shut-in or reallocation based on their injection gas-oil ratios. The program provided comparison between a correlation-based well gradient and the latest survey of pressure at depth. Changes in wellhead pressures during reallocation of injected gas were not considered by the program. Kanu (1992) proposed gas lift production optimization with data gathering, systems analysis, gas allocation, gas lift valve placement and evaluation and implementation. Well performance was reviewed by plotting tubing and casing pressures before implementing allocation calculations. The author also recommended that one team should be responsible for both analysis and implementation of gas lift optimization. Walsh (1994) emphasized gas lift valve quality, a workshop for setting and testing valves, well modelling and optimization, consistent design methodology and training of operations personnel for successful gas lift production optimization. Methods for accurately measuring the gas injection rates and guidelines for well re-testing were also presented. El-M. & Price (1995) developed a gas lift allocation model simulating the combined performance of the reservoir, production wells, flow lines and gas lift system. A multi phase fluid flow simulator was used to generate a system performance curve for each well taking into account any changes in reservoir pressure, well productivity index, water cut and gas lift entry points. Bottom hole pressure surveys and pipeline profiles were checked for making appropriate choices of multi phase flow correlations. The performance curves were fitted with a polynomial function and optimal allocation of available gas to each well was determined considering equal gradients in tubing or the slope of gas-in, oil-out curves of all producing wells.

The importance of performance analysis of gas lift valve was demonstrated by Laing (1986) in maximizing production from gas lifted wells. When used in conjunction with downhole pressure and temperature surveys, the performance analysis of gas lift valve helped identify leaking valves, correct mandrel spacing, predict the consequences of increasing surface injection pressure and develop flexible design procedures for optimizing production. Blann & Williams (n.d.) defined the most economical gas injection pressure for a gas lift installation as one, which resulted in the lowest compression horsepower per barrel of fluid lifted. The other advantages associated with it were higher production rates, lower injection gas volumes, smaller sizes of gas distribution lines and less downhole equipment. Major factors such as bubble point pressure and solution gas oil ratio of the produced fluid, well productivity, water-cut, wellhead back pressure, injected gas properties, well design facilities and type of gas lift equipment were considered to determine the most economical gas injection pressure. Redden et al. (1974) calculated optimum distribution of available lift gas to a group of gas lifted wells based on each well's contribution to the profit of the system. The optimum gas injection rate was the rate at which the expense for an added increment of gas injection was equal to the increment of revenue returned. In case of total system gas requirements exceeding total gas available, the gas injection rate was reduced according to a priority ranking of the wells in such a manner that minimized the loss of revenue.

Systems analysis techniques were applied by Brown et al. (1982) in order to optimize production rate and gas consumption of a continuous flow gas lift system. Effects of well capability, tubing size, flow line size, separator and gas injection pressures were analyzed for optimizing production from a group of wells placed on continuous gas lift. Increase in flow line size was recommended over lowering of separator pressure to obtain higher production rate. Also pressure loss in the flow line was considered in determining optimum gas injection rate. Kanu (1992) showed that the application of systems analysis techniques to gas lift design improved production from gas lifted wells by at least 50%. The analysis led to changes in casing and wellhead pressures, gas oil ratios and gas lift valve settings. With the high cost of gas compression, the improvement in gas-lift well performance brought about by system analysis also marked an automatic improvement in economics. Simmons (1972) defined gas lift optimization as the process of determining gas injection rates for a group of oil wells that would result in the highest present value operating cash income over the life of the wells. Accurate measurements of optimum gas injection rates were very much dependent on the use of reliable multiphase flow correlations and the analysis of the effects of various parameters such as tubing size, separator pressure or gas lift system pressure. Computer programs were used for both unlimited and limited gas supply. Errors in well and cost data, inaccuracies in rate measurements and multiphase flow correlations, well heading and mechanical problems all made it extremely difficult to achieve the profits predicted by optimization programs.

Two methods were presented by Kleyweg & Dalziel (1983) to find optimal distribution of the available lift-gas for a group of wells on a platform. In one method they applied a linear programming technique to the polynomials representing the well performance curves. They also used a step-by-step method, where the well performance curves were scanned to find the curve with the maximum slope after all the wells had been kicked off with enough gas. The gas lift rate to the corresponding well was then increased by one step and so on until all the available gas was distributed. Total oil rate for both methods were almost the same although gas allocations to individual wells varied. They suggested that step-by-step method was more appropriate to be incorporated in an automatic gas lift optimization system with computer control of gas lift chokes. A simulation and optimization method based on a mathematical technique called sequential linear programming was developed by Handley-S. et al. (2000) to determine the optimal lift-gas allocation to networks of gas-lifted wells. The lift-gas in this case is the associated gas produced from a group of wells in an oil field. It was shown that apart from its use as lift-gas, the associated gas can also be used optimally for other purposes such as for compression, for re-injection into reservoirs and for sale.

Gas lift optimization is crucial to sustain production as oil fields mature. Several gas lift optimization tools such as nodal analysis, gas lift optimisation allocation model, gas lift surveillance databases and gas lift monitoring system were used by Chia & Hussain (1999) in gas lift optimization efforts like production system pressure reduction, well mix optimisation, gas lift training and four-point tests. The challenges faced by them during the gas lift implementation and optimization were dual completion gas lift operation, retrieval of tight dummy valves, emulsions and sand production. A pressure-balance based multi phase flow network solving technique, coupled with sequential quadratic programming was used by Dutta-R. & Kattapuram (1997) to solve network based gas-lift allocation optimization problems. The primary constraint in gas-lift networks was the availability of injection gas. The other constraints included compressor operation limits, production ceiling contracts, water handling facilities and allowable operating pressures. The overall field production improved reasonably while considerable savings could be made in terms of time and analysis by solving the field wide allocation problem simultaneously. Buitrago et al. (1996) used a global optimization technique for determining the optimum gas injection rate for a group of wells in order to maximize the total oil production rate for a given total amount of gas without restriction in the well response and the number of wells in the system.

Optimization of a continuous flow gas-lift system was carried out by Zheng-G. & Ramstad (1988) by maximizing the daily cash income from the productions of the gas lifted wells subject to various system constraints such as limited total liquid production rate, limited total gas production rate, limited individual well liquid production rates and limited lift-gas supply. Sensitivity analysis of the optimum results with respect to changes in system parameters were also presented in the work. Lo (n.d.) also considered multiple production constraints such as field rates of gas, water and liquid to find optimal allocation of continuous lift gas to wells for maximizing the total oil rate from the field. The optimum marginal gas oil ratio, at which a well had to operate depended on its water oil ratio and was different for each well. Edwards et al. (1990) established a gas-lift optimization and production allocation model for manifolded sub sea wells. A multi-phase simulator was used to generate well performance curves, which were installed in the data base of the model. Optimal gas allocation from limited gas to each well was found using the model by maximizing total oil rate from the field. The analysis was subject to the constrained optimum that the gradients of gas-in, oil-out curves of all producing wells were equal.

A personal computer based gas lift workstation was presented by A. H. Woodyard (1988) in order to perform gradient computations, historical survey matches, find the proper injection depth for current well conditions and determine the corresponding optimum injection rate for each well. The workstation contained an artificial lift allocation program based on a database and analytical programs. Using the program response curves were generated for each well between their gas injection and oil output rates and the optimal gas allocation to each well was found according to the principal of equal slope. Kanu (1992) presented the formulation of an economic slope based on the concept that the profit from incremental recovery of oil should be equal to the cost of additional gas injected to effect that production. This economic slope was used to allocate a total amount of gas at the optimal economic point for a group of wells.

Chapter 3

Methodology

Theoretical background and review of related literature in the second chapter provides more understanding of this chapter. This thesis provides new methodology to optimize rate allocation in gas-lift system and helps related organizations to make decision about selecting the best performance group of gas lift well and amount of gas to be injected in each gas injection well to ensure maximum oil production. The methodology is based on Rational function with Least squares and Cubic Spline interpolations for curve fitting and Lagrange's Multiplier for the optimization. This chapter emphasizes the use of LaGrange's Multipliers to find equal point for gas allocation considering limited gas available. In addition, the effects of the type of function used to fit the curve are demonstrated. In order to test for the effectiveness of the method, the value of the square of the correlation coefficient, R^2 is found to indicate the goodness of the data fit. The following algorithm outlines the steps involve in the calculation.

1. Tabulation of Gas-In Oil-Out Data for each well.

2. Curve fit well data with trial functions using least Square Method of fitting and the cubic spline method.

3. Calculation of LaGrange Multiplier and Optimum Gas injection rates

4. Calculation of Optimum Oil production rates

   The weighted average squared correlation coefficient is calculated to find the contributions made by types of interpolation used to fit the curve.

3.1 Curve fitting of Data

The data, gas injection (α) and oil production (σ) from each individual well are fit by two approximation methods: (1) Least Square Rational Function and (2) Cubic Spline interpolations:

3.1.1 Curve Fitting of Data using Rational Function Method

The gas injection and oil production rates are fit by using Rational function for each well. The oil production rate (σ) measured in STB/D is considered as dependent variable where the gas injection rate (α) measured in MSCF/D is the independent variable. Coefficients of the function a, b and c are determined by the least square method. The rational function is defined as

$$\sigma = \frac{a + c\alpha}{1 + b\alpha}. \tag{3.1}$$

With the least square coefficients (a, b, and c) and the oil output rates for the function for the fit are calculated for each well. The oil production rates from the data calculated from the rational function fits are plotted against the gas injection rates.

Using the gas injection and oil out put in well one the least square derivation for rational function is presented as below: Since the rational function is non-linear, an iterative procedure is used to determine the coefficients of the function.

This is illustrated below:

Step one: The function to be minimized to determine the least square coefficients, is defined as follows:

$$\begin{aligned} SMIN &= \sum_{i=1}^{5}\left(\sigma_i - \frac{(a+c\alpha_i)}{(1+b\alpha_i)}\right)^2 \\ &= \sum_{i=1}^{5}\frac{(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)^2}{(1+b\alpha_i)^2} \end{aligned} \tag{3.2}$$

Step two: The squared denominator at the right hand side of Equation 3.2 is initially set to unity. So the Equation 3.2 reduces to:

$$SMIN = \sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)^2 \tag{3.3}$$

The necessary conditions for minimizing SMIN are obtained from Equation 3.3 as follows:

$$\frac{\partial SMIN}{\partial a} = 2\sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)(-1) = 0 \tag{3.4}$$

$$\frac{\partial SMIN}{\partial b} = 2\sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)(\alpha_1\sigma_1) = 0 \tag{3.5}$$

and

$$\frac{\partial SMIN}{\partial c} = 2\sum_{i=1}^{5}(\sigma_i + b\alpha_i\sigma_i - a - c\alpha_i)(-\alpha_1) = 0 \tag{3.6}$$

Rearrangement of these conditions leads to the following equations:

$$5a - b\sum_{i=1}^{5}\alpha_i\sigma_i + c\sum_{i=1}^{5}\alpha_i = \sum_{i=1}^{5}\sigma_i \tag{3.7}$$

$$a\sum_{i=1}^{5}\alpha_1 - b\sum_{i=1}^{5}\alpha_i^2\sigma_i + c\sum_{i=1}^{5}\alpha_i^2 = \sum_{i=1}^{5}\alpha_i\sigma_i \tag{3.8}$$

and

$$a\sum_{i=1}^{5}\alpha_1\sigma_i - b\sum_{i=1}^{5}\alpha_i^2\sigma_i^2 + c\sum_{i=1}^{5}\alpha_i^2\sigma_i = \sum_{i=1}^{5}\alpha_i\sigma_i^2 \tag{3.9}$$

The above three equations are solved simultaneously to obtain the values of the least square coefficients as follows:

$$a = \frac{D_a}{D} \tag{3.10}$$

$$b = \frac{D_b}{D} \tag{3.11}$$

and

$$c = \frac{D_c}{D} \tag{3.12}$$

where the determinants (D) are defined as follows:

$$D_a = \begin{vmatrix} \sum_{i=1}^{5}\sigma_i & -\sum_{i=1}^{5}\alpha_i\sigma_i & \sum_{i=1}^{5}\alpha_i \\ \sum_{i=1}^{5}\alpha_i\sigma_i & -\sum_{i=1}^{5}\alpha_i^2\sigma_i & \sum_{i=1}^{5}\alpha_i^2 \\ \sum_{i=1}^{5}\alpha_i\sigma_i^2 & -\sum_{i=1}^{5}\alpha_i^2\sigma_i^2 & \sum_{i=1}^{5}\alpha_i^2\sigma_i \end{vmatrix}$$

$$D_b = \begin{vmatrix} 5 & -\sum_{i=1}^{5}\sigma_i & \sum_{i=1}^{5}\alpha_i \\ \sum_{i=1}^{5}\alpha_i & -\sum_{i=1}^{5}\alpha_i\sigma_i & \sum_{i=1}^{5}\alpha_i^2 \\ \sum_{i=1}^{5}\alpha_i\sigma_i & -\sum_{i=1}^{5}\alpha_i\sigma_i^2 & \sum_{i=1}^{5}\alpha_i^2\sigma_i \end{vmatrix}$$

and

$$D_c = \begin{vmatrix} 5 & -\sum_{i=1}^{5}\alpha_i\sigma_i & \sum_{i=1}^{5}\sigma_i \\ \sum_{i=1}^{5}\alpha_i & -\sum_{i=1}^{5}\alpha_i^2\sigma_i & \sum_{i=1}^{5}\alpha_i\sigma_i \\ \sum_{i=1}^{5}\alpha_i\sigma_i & -\sum_{i=1}^{5}\alpha_i^2\sigma_i^2 & \sum_{i=1}^{5}\alpha_i\sigma_i^2 \end{vmatrix}$$

Calculating the summation terms from the data in well one that is in Table (A.1), the result is presented in the table below

Table 3.1: Summation terms for well one

$\sum_{i=1}^{5} \alpha_i$	2526.75
$\sum_{i=1}^{5} \alpha_i^2$	2365778.063
$\sum_{i=1}^{5} \alpha_i \sigma_i^2$	2602341880
$\sum_{i=1}^{5} \alpha_i^2 \sigma_i^2$	3.08322E4-12
$\sum_{i=1}^{5} \alpha_i^2 \sigma_i$	722696955.2
$\sum_{i=1}^{5} \alpha_i \sigma_i$	740627.25
$\sum_{i=1}^{5} \sigma_i$	1122

After a number of iterations the least square coefficients a,b and c finally obtained as follows: a = 66.37952624 b = 0.006947628 and c= 2.34116541. In the nutshell the least square coefficients in the five wells by the rational function fit are tabulated as below

Table 3.2: Rational function fit between gas injection and oil output rates in all wells

Parameter	Well one	Well two	Well three	Well four	Well five
a	66.3795262	13.70669684	18.73088482	41.46434637	78.8900435
b	0.006947628	0.035507572	0.033123421	0.020440884	0.009741759
c	2.34116541	6.535830025	5.61731063	5.183795835	3.596152748

The sum of the squares of the errors between the data and the function is calculated for wells as follows:

$$S = \sum_{i=1}^{5}(\sigma_i - \frac{a + c\alpha_i}{1 + b\alpha_i})^2$$

The sum of the squares of the errors between the data and the mean oil output rate is calculated for the wells is also defined as:

$$S_B = \sum_{i=1}^{5}(\sigma_i - \frac{\sum_{i=1}^{5}\sigma_i}{5})^2$$

Hence, the square of the correlation coefficient for the wells is calculated as:

$$R^2 = 1 - \frac{S}{S_B} \tag{3.13}$$

The result is tabulated in Table 4.6 The weighted average value of R^2 is calculated in order to find the effects of the rational function data fit on total output from the five wells, this is defined as:

$$\overline{R^2} = \frac{\sum_{i=1}^{5} \sigma_i R_i^2}{\sum_{i=1}^{5} \sigma_i} \tag{3.14}$$

where $\overline{R^2}$ is the weighted average value of the square of correlation coefficient, the numerator is the sum of the products of optimum oil output rate and square of correlation coefficient of each well and the denominator is the total optimum rate of oil output. The product term in the numerator takes account of the contributions made by each well to the total production. The results are shown in Table 4.7

3.1.2 Cubic Spline Interpolation

The cubic spline is considered as an alternative to the least square rational function for fitting well data. According to Kruger (n.d.), the cubic spline interpolation is a useful technique to interpolate between known data points where a third degree polynomial is constructed between each point. Due to its stable and smooth characteristics, it fits the data very well and represents true well behaviour. So the cubic spline is an appropriate choice to model the well gas injection and oil output data. In his definition of cubic spline Kruger (n.d.) posited that having a collection of known data points (α_0, σ_0), (α_1, σ_1), ... (α_{i-1}, σ_{i-1}), (α_i, σ_i), (α_{i+1}, σ_{i+1}), ... (α_n, σ_n), to interpolate between these data points using cubic splines, a third degree polynomial is constructed between each point. The equation to the left of point (α_i, σ_i) is indicated as f_i with a σ value of $f_i(\alpha_i)$ at point α_i. Similarly, the equation to the right of point (α_i, σ_i) is indicated as f_{i+1} with a σ value of $f_{i+1}(\alpha_i)$ at point α_i. Kruger (n.d.)further said that even though cubic splines are well behaved for many applications, it sometimes does not prevent overshoot at intermediate points, hence his Kruger (n.d.)'s proposed Constrained Cubic Splines is applied in this work. The principle behind the proposed constrained cubic spline is to prevent overshooting by sacrificing smoothness. This is achieved by eliminating the requirement for equal second order derivatives at every point and replacing it with specified first order derivatives. Thus, Kruger (n.d.) proposed Constrained Cubic Spline is as follows:

$$f'(\alpha_i) = \frac{2}{\frac{\alpha_{i+1}-\alpha_i}{\sigma_{i+1}-\sigma_i} + \frac{\alpha_i-\alpha_{i-1}}{\sigma_i-\sigma_{i-1}}}$$

=0 if slope changes sign at point. The slope at the end points are defined as:

$$f_1'(\alpha_0) = \frac{3(\sigma_1 - \sigma_0)}{2(\alpha_1 - \alpha_0)} - \frac{f'(\alpha_1)}{2}$$

$$f_n'(\alpha_n) = \frac{3(\sigma_n - \sigma_{n-1})}{2(\alpha_n - \alpha_{n-1})} - \frac{f'(\alpha_{n-1})}{2}$$

Because the slope at each point is known, each spline function,can be calculated based on the two adjacent points on each side. This is summarized in the equations below.

$$f_i''(\alpha_{i-1}) = -\frac{2[f_i'(\alpha_i) + 2f_i'(\alpha_{i-1})]}{(\alpha_i - \alpha_{i-1})} + \frac{6(\sigma_i - \sigma_{i-1})}{(\alpha_i - \alpha_{i-1})^2}$$

$$f_i''(\alpha_{i-1}) = \frac{2[2f_i'(\alpha_i) + f_i'(\alpha_{i-1})]}{(\alpha_i - \alpha_{i-1})} - \frac{6(\sigma_i - \sigma_{i-1})}{(\alpha_i - \alpha_{i-1})^2}$$

$$d_i = \frac{(f_i''(\alpha_i) - f_i''(\alpha_{i-1})}{6(\alpha_i - \alpha_{i-1})}$$

$$c_i = \frac{\alpha_i(f_i''(\alpha_{i-1}) - \alpha_{i-1} f_i''(\alpha_i)}{2(\alpha_i - \alpha_{i-1})}$$

$$b_i = \frac{(\sigma_i - \sigma_{i-1}) - c_i(\alpha_i^2 - \alpha_{i-1}^2) - d_i(\alpha_i^3 - \alpha_{i-1}^3)}{(\alpha_i - \alpha_{i-1})}$$

$$a_i = \sigma_{i-1} - b_i\alpha_{i-1} - c_i\alpha_{i-1}^2 - d_i\alpha_{i-1}^3$$

Hence, a third degree polynomial constructed between each point is generally defined as

$$f_i(\alpha) = a_i + b_i\alpha + c_i\alpha^2 + d_i\alpha^3 \tag{3.15}$$

Where the actual parameters (a_i, b_i, c_i and d_i) for each of the cubic spline equations are found directly without solving a system of equations and this permits analytical integration of the data. The implementation of C.J.C. Kruger's proposed Constrained Cubic Spline for the actual parameters a_i, b_i, c_i and d_i for each of the cubic spline equations for the various wells is presented in the table below

Table 3.3: Cubic Spline fit between gas injection and oil output rates in all wells

Parameter	Well one	Well two	Well three	Well four	Well five
a_1	$\frac{14547}{226}$	$\frac{6685}{297}$	$\frac{1428}{59}$	$\frac{8185}{178}$	$\frac{17877}{223}$
b_1	$\frac{1241}{825}$	$\frac{1795}{1441}$	$\frac{429}{266}$	$\frac{1406}{839}$	$\frac{2309}{1259}$
c_1	$\frac{40}{261093}$	$\frac{2}{20721}$	$\frac{12}{43067}$	$\frac{19}{105418}$	$\frac{25}{180208}$
d_1	$\frac{-11}{376953}$	$\frac{-1}{62163}$	$\frac{-5}{94209}$	$\frac{-11}{329570}$	$\frac{-25}{540624}$
a_2	$\frac{10683}{76}$	$\frac{21287}{163}$	$\frac{16207}{151}$	$\frac{11832}{79}$	$\frac{14403}{95}$
b_2	$\frac{522}{911}$	$\frac{402}{1627}$	$\frac{767}{2325}$	$\frac{385}{894}$	$\frac{578}{781}$
c_2	$\frac{-331}{353507}$	$\frac{-15}{29108}$	$\frac{-23}{34732}$	$\frac{-31}{44937}$	$\frac{-27}{46529}$
d_2	$\frac{1}{1648448}$	$\frac{1}{2502508}$	$\frac{1}{2805837}$	$\frac{1}{3570510}$	$\frac{-1}{1259637}$
a_3	$\frac{44729}{258}$	$\frac{18647}{121}$	$\frac{3483}{25}$	$\frac{20247}{100}$	$\frac{21688}{87}$
b_3	$\frac{787}{2313}$	$\frac{307}{4500}$	$\frac{121}{1462}$	$\frac{194}{1985}$	$\frac{338}{1585}$
c_3	$\frac{-38}{97145}$	$\frac{-5}{80068}$	$\frac{-15}{151546}$	$\frac{-13}{1663579}$	$\frac{-17}{113908}$
d_3	$\frac{1}{5375471}$	$\frac{1}{55683357}$	$\frac{1}{24322566}$	$\frac{1}{48530397}$	$\frac{1}{43380272}$
a_4	$\frac{32753}{153}$	$\frac{10009}{61}$	$\frac{23623}{156}$	$\frac{20161}{92}$	$\frac{23483}{80}$
b_4	$\frac{1271}{9175}$	$\frac{162}{4877}$	$\frac{20}{677}$	$\frac{141}{3002}$	$\frac{436}{4847}$
c_4	$\frac{-17}{248896}$	$\frac{-3}{117358}$	$\frac{-2}{84247}$	$\frac{-9}{299188}$	$\frac{-26}{460565}$
d_4	$\frac{1}{56660441}$	$\frac{1}{129093750}$	$\frac{1}{132688988}$	$\frac{1}{134634607}$	$\frac{1}{73336120}$

Using the coefficients in Table 3.3 obtained from the C.J.C Kruger's proposed Constrained Cubic Spline interpolation, oil output rates for the fit are calculated for each well from Equation (3.14). The data points and the oil output rates calculated from the cubic spline functions are tabulated in Tables 4.1 through 4.5

3.1.3 Lagrange Multiplier with Rational Function

To find the optimum point of gas injection rate and oil output rate for each well after fitting the gas in and oil out data, the Lagrange optimization is carried out. The optimum oil rate in each well is expressed as a rational function of its optimum gas injection rate as follows

$$\sigma_i = \frac{a_i + c_i\alpha_i}{1 + b_i\alpha_i} \tag{3.16}$$

where i = 1,2,3,4,5 is the i^{th} well, σ_i are the optimum oil output rates and α_i are the optimum gas injection rate for wells 1,2,3,4 and 5. Whiles the respective least square coefficients a_i, b_i and c_i are also determined. The optimization is subject to a linear equality constraint regarding the availability of limited gas for injection. The total amount of gas available is N measured in MSCF/D. Hence the following defines the constraint equation:

$$\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n = N. \tag{3.17}$$

where n is the number of wells. And a constraint function defined as

$$\emptyset = \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n - N = 0 \tag{3.18}$$

The LaGrange Multiplier relating the partial derivative of the oil rate in each well to the partial derivative of the constraint function is as follows:

$$\frac{\partial\sigma_i}{\partial\alpha_i} = \lambda\frac{\partial\emptyset}{\partial\alpha_i} \tag{3.19}$$

, where i represents the well number and (λ) is the LaGrange Multiplier.

Carrying out the partial derivatives for each well in equation 3.19 yields the following algebraic set of non-linear equations:

$$\lambda b_i^2\alpha_i^2 + 2\lambda b_i\alpha_i + \lambda - c_i + a_ib_i = 0 \tag{3.20}$$

where α_i is the expression of gas injection rates of the i^{th}, well. This set of non-linear equation is solved using Broyden's method of solving algebraic set of equations. The method is able to find better optimal point.

The total optimum gas injection rate is

$$\sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n \tag{3.21}$$

which is equal to the total available gas for injection.

Using the least square coefficients and the optimum gas injection rate for the wells the optimum oil output rate of the wells are calculated from the following equation

$$\sigma_i = \frac{a_i + c_i\alpha_i}{1 + b_i\alpha_i}. \tag{3.22}$$

Hence the total optimum oil output rates of the wells is calculated from

$$\sum_{i=1}^{n} \sigma_i = \sigma_1 + \sigma_2 + \sigma_3 + ... + \sigma_n \tag{3.23}$$

Thus the optimum gas injection and oil production rates of all wells for rational function data fit and the value of the LaGrange Multiplier are tabulated in Table 4.13. As well as the total optimum oil output rate.

3.1.4 Lagrange Multiplier Cubic Spline Function

After fitting the gas injection and oil output rates of the wells with Cubic Spline functions, the optimum rates of gas injection and oil output in each of the wells are determined using the LaGrange Multiplier method. The optimum oil rate in each well is expressed as a cubic spline function of its optimum gas injection rate as follows:

Cubic Spline Functions for Well one

$$\sigma_1 = a_1 + b_1\alpha_1 + c_1\alpha_1^2 + d_1\alpha_1^3 \tag{3.24}$$

$$\sigma_1 = a_2 + b_2\alpha_1 + c_2\alpha_1^2 + d_2\alpha_1^3 \tag{3.25}$$

$$\sigma_1 = a_3 + b_3\alpha_1 + c_3\alpha_1^2 + d_3\alpha_1^3 \tag{3.26}$$

$$\sigma_1 = a_4 + b_4\alpha_1 + c_4\alpha_1^2 + d_4\alpha_1^3 \tag{3.27}$$

Cubic Spline Functions for Well two

$$\sigma_2 = a_1 + b_1\alpha_1 + c_1\alpha_2^2 + d_1\alpha_2^3 \tag{3.28}$$

$$\sigma_2 = a_2 + b_2\alpha_2 + c_2\alpha_2^2 + d_2\alpha_2^3 \tag{3.29}$$

$$\sigma_2 = a_3 + b_3\alpha_2 + c_3\alpha_2^2 + d_3\alpha_2^3 \tag{3.30}$$

$$\sigma_2 = a_3 + b_3\alpha_2 + c_3\alpha_2^2 + d_3\alpha_2^3 \tag{3.31}$$

$$\sigma_2 = a_4 + b_4\alpha_2 + c_4\alpha_2^2 + d_4\alpha_2^3 \tag{3.32}$$

Cubic Spline Functions for Well three

$$\sigma_3 = a_1 + b_1\alpha_3 + c_1\alpha_3^2 + d_1\alpha_3^3 \tag{3.33}$$

$$\sigma_3 = a_2 + b_2\alpha_3 + c_2\alpha_3^2 + d_2\alpha_3^3 \tag{3.34}$$

$$\sigma_3 = a_3 + b_3\alpha_3 + c_3\alpha_3^2 + d_3\alpha_3^3 \tag{3.35}$$

$$\sigma_3 = a_4 + b_4\alpha_3 + c_4\alpha_3^2 + d_4\alpha_3^3 \tag{3.36}$$

Cubic Spline Functions for Well four

$$\sigma_4 = a_1 + b_1\alpha_4 + c_1\alpha_4^2 + d_1\alpha_4^3 \tag{3.37}$$

$$\sigma_4 = a_2 + b_2\alpha_4 + c_2\alpha_4^2 + d_2\alpha_4^3 \tag{3.38}$$

$$\sigma_4 = a_3 + b_3\alpha_4 + c_3\alpha_4^2 + d_3\alpha_4^3 \tag{3.39}$$

$$\sigma_4 = a_4 + b_4\alpha_4 + c_4\alpha_4^2 + d_4\alpha_4^3 \tag{3.40}$$

Cubic Spline Functions for Well five

$$\sigma_5 = a_1 + b_1\alpha_5 + c_1\alpha_5^2 + d_1\alpha_5^3 \tag{3.41}$$

$$\sigma_5 = a_2 + b_2\alpha_5 + c_2\alpha_5^2 + d_2\alpha_5^3 \tag{3.42}$$

$$\sigma_5 = a_3 + b_3\alpha_5 + c_3\alpha_5^2 + d_3\alpha_5^3 \tag{3.43}$$

$$\sigma_5 = a_4 + b_4\alpha_5 + c_4\alpha_5^2 + d_4\alpha_5^3 \tag{3.44}$$

$$\sigma_5 = a_5 + b_5\alpha_5 + c_5\alpha_5^2 + d_5\alpha_5^3 \tag{3.45}$$

where $\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ and\, \sigma_5$ are the optimum oil output rates and $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ and\, \alpha_5$ are the optimum gas injection rates for wells one, two, three, four and five respectively. The respective coefficients (the actual parameters) a_1 through a_4, b_1 through b_4, c_1 though c_4 and d_1 through d_4 for each of the cubic spline equations for each well was determined by cubic spline interpolation method from the well data. Since each well contains five (5) data points, there was four (4) segments that is four piecewise cubic polynomials for each well, hence there was (4×4) sixteen (16) coefficients to be determined by the cubic spline method for each of the wells. and the were simplified to the following algebraic set of nonlinear equations.

$$\sigma_1 = \sum_{j=1}^{4} a_j + \sum_{j=1}^{4} b_j\alpha_1 + \sum_{j=1}^{4} c_j\alpha_1^2 + \sum_{j=1}^{4} d_j\alpha_1^3 \tag{3.46}$$

$$\sigma_2 = \sum_{j=1}^{4} a_j + \sum_{j=1}^{4} b_j\alpha_2 + \sum_{j=1}^{4} c_j\alpha_2^2 + \sum_{j=1}^{4} d_j\alpha_2^3 \tag{3.47}$$

$$\sigma_3 = \sum_{j=1}^{4} a_j + \sum_{j=1}^{4} b_j\alpha_3 + \sum_{j=1}^{4} c_j\alpha_3^2 + \sum_{j=1}^{4} d_j\alpha_3^3 \quad (3.48)$$

$$\sigma_4 = \sum_{j=1}^{4} a_j + \sum_{j=1}^{4} b_j\alpha_4 + \sum_{j=1}^{4} c_j\alpha_4^2 + \sum_{j=1}^{4} d_j\alpha_4^3 \quad (3.49)$$

$$\sigma_5 = \sum_{j=1}^{4} a_j + \sum_{j=1}^{4} b_j\alpha_5 + \sum_{j=1}^{4} c_j\alpha_5^2 + \sum_{j=1}^{4} d_j\alpha_5^3 \quad (3.50)$$

The analysis is subject to a constraint regarding the availability of limited gas for injection. The total amount of gas available for injection is N MSCF/D. So the constraint equation is defined as follows:

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = N \quad (3.51)$$

and a constraint function is defined as

$$\emptyset = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - N = 0 \quad (3.52)$$

The LaGrange Multiplier relating the partial derivative of the oil rate in each well to the partial derivative of the constraint function are defined as follow:

$$\frac{\partial\sigma_1}{\partial\alpha_1} = \lambda\frac{\partial\emptyset}{\partial\alpha_1}$$

$$\frac{\partial\sigma_2}{\partial\alpha_2} = \lambda\frac{\partial\emptyset}{\partial\alpha_2}$$

$$\frac{\partial\sigma_3}{\partial\alpha_3} = \lambda\frac{\partial\emptyset}{\partial\alpha_3}$$

$$\frac{\partial\sigma_4}{\partial\alpha_4} = \lambda\frac{\partial\emptyset}{\partial\alpha_4}$$

and

$$\frac{\partial \sigma_5}{\partial \alpha_5} = \lambda \frac{\partial \emptyset}{\partial \alpha_5}$$

where λ is the Lagrange Multiplier. After carrying out the partial derivatives for the wells in the equations above, the following non-linear equations are yielded:

$$3\sum_{j=1}^{4} d_j\alpha_1^2 + 2\sum_{j=1}^{4} c_j\alpha_1 + \sum_{j=1}^{4} b_j - \lambda = 0 \quad (3.53)$$

$$3\sum_{j=1}^{4} d_j\alpha_2^2 + 2\sum_{j=1}^{4} c_j\alpha_2 + \sum_{j=1}^{4} b_j - \lambda = 0 \quad (3.54)$$

$$3\sum_{j=1}^{4} d_j\alpha_3^2 + 2\sum_{j=1}^{4} c_j\alpha_3 + \sum_{j=1}^{4} b_j - \lambda = 0 \quad (3.55)$$

$$3\sum_{j=1}^{4} d_j\alpha_4^2 + 2\sum_{j=1}^{4} c_j\alpha_4 + \sum_{j=1}^{4} b_j - \lambda = 0 \quad (3.56)$$

$$3\sum_{j=1}^{4} d_j\alpha_5^2 + 2\sum_{j=1}^{4} c_j\alpha_5 + \sum_{j=1}^{4} b_j - \lambda = 0 \quad (3.57)$$

After the substitution of the of the cubic spline coefficients stated in table 3.3, an algebraic set of n nonlinear equations in 5 unknowns are formed which can be written in the general form as

$$f_i(\alpha_1, \alpha_2, ..., \alpha_5) = 0 \quad (3.58)$$

for i = 1,2,...5. As usual, subscripts denote vector components. To calculate for the Lagrange's multiplier and the optimum gas injection rates of the wells, the Broyden's iterative method for solving algebraic set of nonlinear equations. The optimum rates of gas injection and oil output are presented in Table 3.14.

Chapter 4

Analysis and Results

4.1 Introduction

Wells are grouped into production and injection wells. The production wells are for production of oil and gas while injection wells are drilled to inject gas or water into the reservouir. The purpose of injection is to maintain overall and hydrostatic reservoir pressure and force the oil toward the production wells.

The efficient allocation of the injected gas into a group of continuous flow gas lifted wells immensely enhances production of oil and gas. Obtaining the optimum gas injection rate is important because excessive gas injection rate reduces oil production rate and increases operation cost. To obtain the optimum gas injection and oil production rate, all wells had been modelled properly.

This chapter presents the analysis of the study, the experimental findings and results in optimization process have been discussed.

4.2 Analysis and Results

For a gas lifted oil field where multiple oil wells share the lift gas supplied by the common source, proper allocation and distribution of the available gas is an important issue for maximizing the total oil produced from the oil production wells. For the field with n number of gas lifted oil wells, the objective is to distribute the available gas ensuring optimal production of oil. The amount of lift gas available is assumed to be limited. Thus, optimization for the oil field for this case is the task of finding out the optimal set points of the five wells. This paper describes a non-linear optimization problem with constraints associated with the optimal distribution of the limited lift gas. A non-linear objective function is developed using Lagrange model of the oil field where the decision variables represent the lift gas flow rate set points of each oil well of the field. The lift gas optimization problem is solved using the LaGrange multiplier.

The production system with gas lift is subject to one or more constraints. Availability of injection gas is a very important factor. For limited supply of injection gas it is very important to allocate gas to the wells according to their productivity so that maximum benefit can be obtained from the available gas. In case of gas shortage the wells with low productivity may even be required to be shut down so that gas supply to higher productive wells can remain uninterrupted. Apart from limited gas supply there may be other constraints such as water cut, formation and producing gas oil ratios etc.

4.2.1 Results of Curve Fitting

In order to find the optimum gas allocation it is very important to obtain performance curve, which is the plot of well gas injection rate versus the liquid production rate for a given surface gas injection pressure and shows the producing systems response to continuous flow gas lifting. For field operations, determining these response curves from well tests is much preferred. Proper choice of multi-phase flow pressure drop programs and fluid property correlations is required to obtain accurate well performance curve. Also systems analysis should be carried out to analyze various components such as well capability, tubing size, flow-line size and separator pressure etc. to accurately predict well performance.

In this work, well data are fit with rational function by least square method and cubic spline interpolation method. The data points and the oil output rates calculated from the rational functions are tabulated in 4.1 through 4.5. The oil production rates from data and calculated oil output rate from the rational function fits and cubic spline interpolation model are plotted against gas injection rates in figures 4.1 through 4.5.

As depicted by the calculated values in tables 4.1 through 4.5 and the performance curves in the figures 4.1 through 4.5, the cubic spline interpolation model is the best alternative to the rational function for fitting the well data since they are flexible to handle and do represent the true well behaviour and fits the data very well this is shown in Tables 4.1 through 4.5.

Table 4.1: Oil output rates from data and the two interpolation methods for well one.

α, Gas Injection rate (MSCF/D)	σ, Oil output rate (STB/D)	σ, rational (STB/D)	σ, cubic spline (STB/D)
1.75	67	69.62998011	67
115	195	186.5579756	195
385	257	263.3390997	257
735	286	292.660964	286
1290	317	309.8119404	317

Table 4.2: Oil output rates from data and the two interpolation methods for well two.

α, Gas Injection rate (MSCF/D)	σ, Oil output rate (STB/D)	σ, rational (STB/D)	σ, cubic spline (STB/D)
2	25	25.00278081	25
155	158	157.8739039	158
380	172	172.3137542	172
650	177	176.993773	177
1100	180	179.8157803	180

Table 4.3: Oil output rates from data and the two interpolation methods for well three.

α, Gas Injection rate (MSCF/D)	σ, Oil output rate (STB/D)	σ, rational (STB/D)	σ, cubic spline (STB/D)
1.75	27	26.99631064	27
95	133	133.2076266	133
290	156	155.3633103	156
575	162	162.0617496	162
1050	165	165.3710036	165

Table 4.4: Oil output rates from data and the two interpolation methods for well four.

α, Gas Injection rate (MSCF/D)	σ, Oil output rate (STB/D)	σ, rational (STB/D)	σ, cubic spline (STB/D)
1.8	49	48.99256641	49
20	192	192.1627265	192
350	228	227.5843113	228
780	241	241.0795404	241
1350	246	246.1808397	246

Table 4.5: Oil output rates from data and the two interpolation methods for well five.

α, Gas Injection rate (MSCF/D)	σ, Oil output rate (STB/D)	σ, rational (STB/D)	σ, cubic spline (STB/D)
1	82	81.69038817	82
95	216	218.4013186	216
320	303	298.6520641	303
750	335	334.2039381	335
1380	346	337.775674	346

Due to the goodness of fit from the cubic spline model it is insignificant to calculate the weighted average squared correlation coefficient, R^2. However, in order to see the goodness of data fit of the rational function the value of R^2, the square of the correlation coefficient is calculated and the resulted is shown in table 4.6. The weighted average value of R^2 is also calculated from equation 3.14.

Table 4.6: Square of the correlation coefficient for Rational function.

Parameter	Well one	Well two	Well three	Well four	Well five
S	214.4054118	0.148325274	0.589952874	0.238361978	34.71676169
S_B	39071.2	17513.2	13533.2	27058.8	48425.2
R^2	0.99451244	0.999991531	0.999956407	0.999991191	0.999283085

Table 4.7: Effects of data fit on calculated total oil output.

Function	$\overline{R^2}$, Average R^2	Total oil output (STB/D)
Rational	0.998479948	1039.802114

Figures 4.1 through 4.5 show a the comparison of results from the two approximation methods (The Rational function and the Cubic Spline). Though it can be seen that the results in some of the figures are virtually identical there is a significant difference in the results shown in tables 4.1 through 4.5. Besides, some of the figures show a noticeable differences between the two methods. In figure

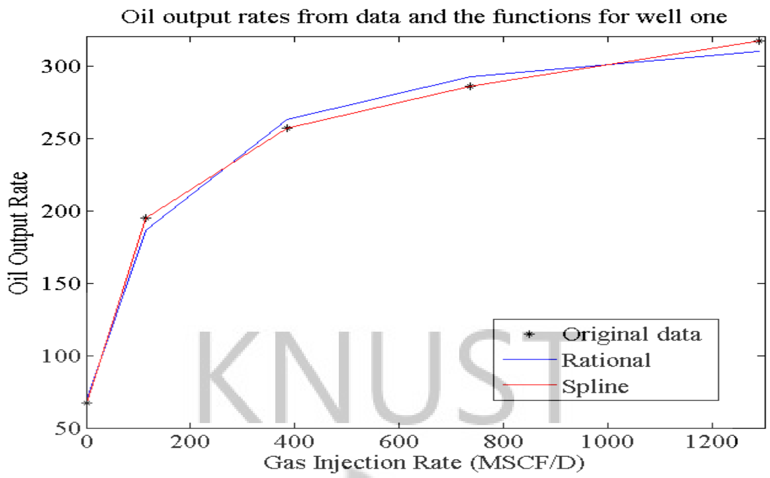

Figure 4.1: Plot of Oil Output versus Gas Injection for Well One.

4.1, the Rational function undergoes an abrupt change from the second point to the last point of the interpolation. This introduces oscillation.

Alternatively, the cubic spline shows a much superior and more acceptable fit as the curve passes through all the data points.

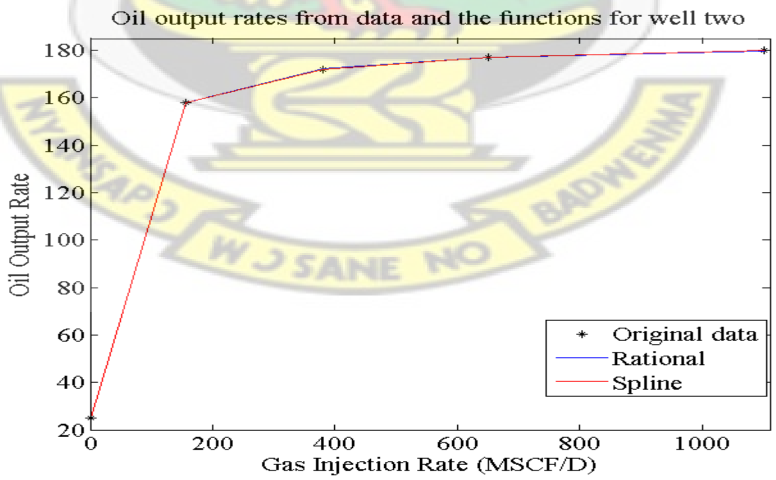

Figure 4.2: Plot of Oil Output versus Gas Injection for Well Two.

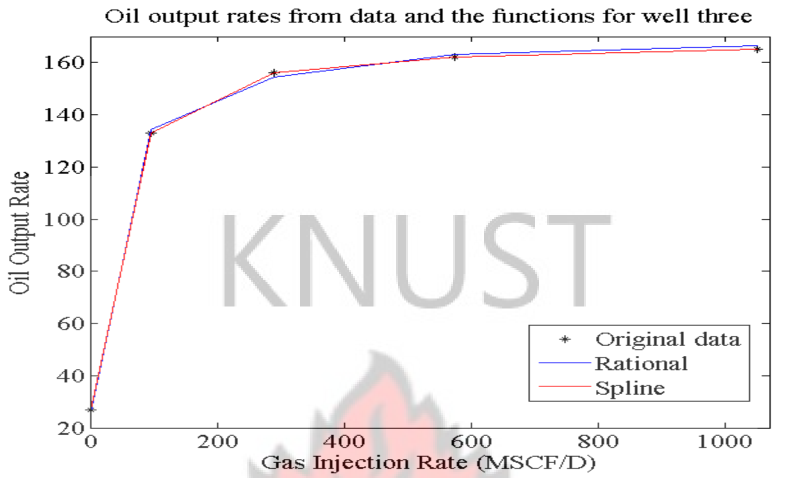

Figure 4.3: Plot of Oil Output versus Gas Injection for Well Three.

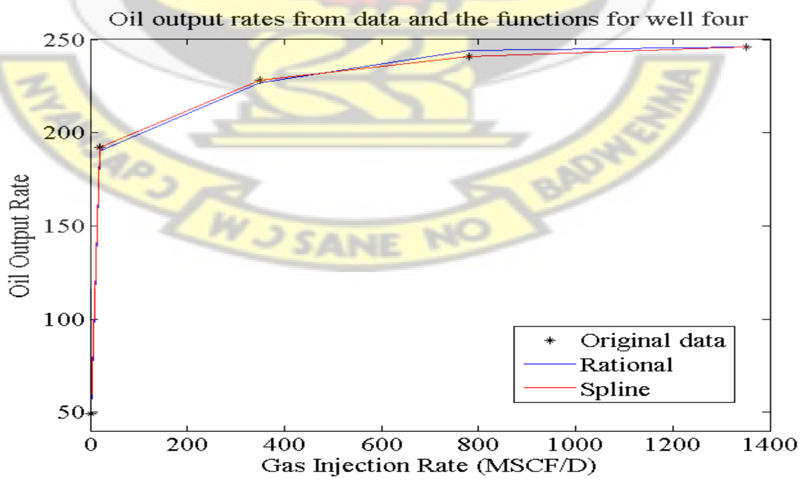

Figure 4.4: Plot of Oil Output versus Gas Injection for Well Four.

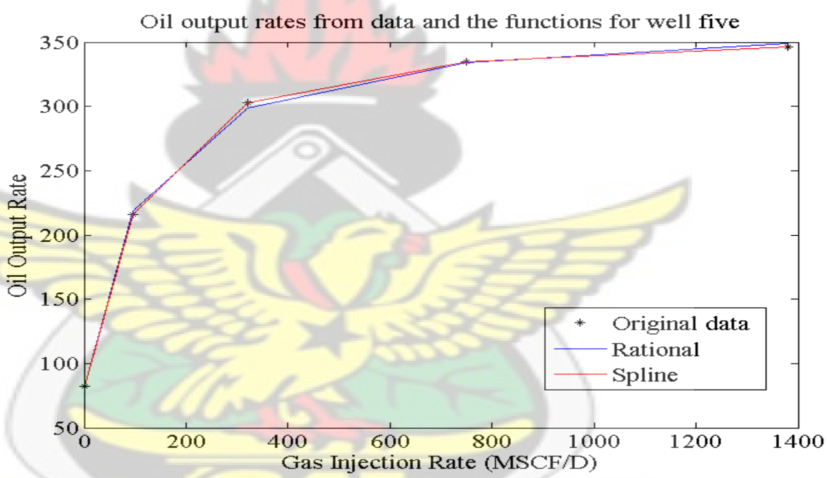

Figure 4.5: Plot of Oil Output versus Gas Injection for Well Five.

In order to check the validity of the mathematical model, the continuous gas lift well in tables A.1 through table A.5 were compared with the results obtained by the interpolation methods in tables 4.1 through 4.5 and the figures 4.1 through 4.5. As shown in these tables and figures, it is noticed that though the rational function does not give wrong predictions of oil output, that of the cubic spline is far better, it match sa the well data. Hence, the productions will take place according to the cubic spline function since it represents real data of the wells.

4.2.2 Results of the Optimization

The production optimization of continuous flow gas lifted wells is carried out using the LaGrange Multiplier method subject to limited gas supply. Well data are generated by using a well performance software program as stated by Jamal (2001) and fit with cubic spline and rational functions by the least square method. The optimum operating conditions are based on the concept that for each well incremental oil production due to an unit increase in gas injection should be equal. So the optimum operating points are nothing but equal slope points.

The optimum gas injection and oil production rates of all wells for rational function data fit and the value of the LaGrange Multiplier are tabulated in Table 4.8.

Table 4.8: Optimum gas injection and oil output rates for rational function data fit.

Lambda λ	0.202070231			
Well	Optimum gas injection rates (MSCF/D)		Optimum oil output rates (STB/D)	
One	α_1	295.091	σ_1	248.2594091
Two	α_2	125.9270512	σ_2	152.9316134
Three	α_3	119.9383631	σ_3	139.2507759
Four	α_4	177.7020761	σ_4	207.8055049
Five	α_5	281.34129	σ_5	291.5548112
	Total	1000	Total	1039.802114

The optimum gas injection and oil production rates of all wells for cubic spline function data fit and the value of the LaGrange Multiplier are tabulated in Table 4.9

Table 4.9: Optimum gas injection and oil output rates for cubic spline function data fit.

Lambda λ	-5.6569			
Well	Optimum gas injection rates (MSCF/D)		Optimum oil output rates (STB/D)	
One	α_1	296.3784	σ_1	502.1615
Two	α_2	382.2095	σ_2	132.1152
Three	α_3	-192.4427	σ_3	386.3273
Four	α_4	272.3827	σ_4	516.4365
Five	α_5	241.4726	σ_5	560.4919
	Total	1000	Total	2097.5324

4.2.3 Lost Production

Curve fitting of well data with the cubic spline function gives accurate predictions since it gives the same results as the well data. The other model, the rational function though does not give poor fit, its results have some variations with the real well data. So productions do not represent optimum operating conditions for the rational model. This creates some differences between the cubic spline optimum oil outputs and the actual oil outputs at the rational function optimum gas injection rates. These differences which are defined as lost productions, are presented in Table 4.10

Table 4.10: Lost productions due to wrong predictions.

Well	Rational optimum gas rates (MSCF/D)	Actual outputs (STB/D)	Cubic Spline optimum oil rates (STB/D)	Lost production (STB/D)
One	295.091	248.2594091	502.1615	253.9021
Two	125.9270512	152.9316134	132.1152	-20.8164
Three	119.9383631	139.2507759	386.3273	247.0765
Four	177.7020761	207.8055049	516.4365	308.6310
Five	281.34129	291.5548112	560.4919	268.9371
Total	1000	1039.802114	2097.5324	1057.7303

Chapter 5

Conclusions and Recommendations

For optimal distribution of the available lift gas among a group of five oil wells on continuous gas lift in order to maximize the total oil production, two different non-linear optimization problems with a linear constraint were formulated using cubic spline function and rational function for least Squares models. The optimization problems were then solved using LaGrange multiplier method subject to the constraint of limited gas supply.

Well data were fitted with cubic spline interpolation method and rational function for least squares method and optimum gas injection and oil output rates were determined for both functions. The two methods used by the researcher were able to increase the total oil production. Total optimum oil production rate for data fitting with cubic spline function is found to be 1057.7303 higher than the total optimum oil production rate for data fitting with rational function. The optimal value of the spline based function was found to be twice that of the rational function.

By visual inspection, examining the figures in chapter 4, the overall performance of the cubic spline seems most proper. As close as all of the curves do follow the data quite well. The rational function is less predictable although the rational function seems to have a good fit to some of the production data. Hence, from visual inspection the cubic spline function had the best match.

Though the rational function gave results closer to real data of the well, these productions do not represent optimum operating conditions for the rational model. Rather the productions will take place according to the cubic spline since it represents real data of the wells. So there were differences between the cubic spline optimum oil outputs and the actual oil outputs at the rational optimum gas injection rates. These differences are shown in Table 4.10.

The cubic spline algorithm was the most suitable for adaptation and is to have good potential and has therefore proven to be a fast algorithm suitable for the purpose of the thesis.

Data or curve fitting plays a major role in production optimization of continuous flow gas lift wells. Some basic powerful advantages for using the proposed constrained cubic spline are: It is a relatively smooth curve; It never overshoots intermediate values; Interpolated values can be calculated directly without solving a system of equations. But problem with spline functions is that the function is forced to fit the data exactly. In general, it can be assumed that the data include error this can still fit the this data with error.

B-Splines which also belong to the same family as the cubic Splines and have the same smoothness conditions as the are therefore recommended for further studies. As they are not forced through the data points exactly which simply have to come close to the data point.

This study considers only one equality constraint regarding the availability of limited lift gas. For future studies it is suggested that more constraints are added to the optimization by taking into consideration changes in choke position, tubing size.

References

Addendum. (n.d.). *Interpolation polynomial.* (Spline Method of Interpolation)

Adjunta, J., & Majek, A. (1994). Wellhead monitors automate lake maricaibo gas lift. *Oil & Gas J.*, 64-67.

Allah, A. (2007). Field experience to optimize gas lift well operations. *Academic EOG Newspaper*, 12-13.

Amondin, M. C., & Jackson, M. A. (1996). A systems approach to production management: Beryl field case study. *European Production Operations Conference,*, 16-17.

Ayatollahia, S., Mostafa, N., & Moshfeghiana, M. (2004). Intermittent gas lift in aghajari oil field, a mathematical study. *Journal of Petroleum Science and Engineering*, *42*, 245 - 255.

Bieker, H. P., Slupphaug, O., & Johansen, T. A. (2006). Real time production optimization of offshore oil and gas production systems: A technology survey. *Society of Petroleum Engineers*.

Blann, J., & Williams, J. (n.d.). Determining the most profitable gas injection pressure for a gas lift installation. In *Annual technical conference and exhibition, san francisco.*

Brown, K. (1980). The technology of artificial lift method. *The Petroleum Publishing Company, V2A.*

Brown, K., Mach, J., & Proano, E. (1982). Application of systems analysis techniques in optimizing gas lift installations. *Energy Resources Tech*, 157-161.

Buitrago, S., Rodriguez, E., & Espin, D. (1996). Global optimization techniques in gas allocation for continuous flow gas lift systems. In *Gas technology conference, calgary.*

Chia, Y., & Hussain, S. (1999). Gas lift optimization efforts and challenges systems. In *Asia pacific improved oil recovery conference, kuala lumpur.*

Clegg, J. (1982). Discussion of economic approach to oil production and gas allocation in continuous gas lift. *JPT*, 301-302.

Cloud, W. F. (1937). *Petroleum production* (N. O. K, Ed.). University of Oklahoma Press.

Coltharp, E., & Khokhar, M. (1984). Dubai gas lift automation. *Annual Technical Conference and Exhibition*, 16-19.

Cooksey, A., & Pool, M. (1995). Production automation system for gas lift wells. *Production Operations Symposium*, 2-4.

Dake, L. (2002). *Fundamentals of reservoir engineering.* Elsvevier Science B.V.

Dutta-R., K., & Kattapuram, J. (1997, June). A new approach to gas-lift allocation optimisation. In *Spe western regional meeting, long beach.*

Edwards, R., Marshall, D., & Wade, K. (1990). A gas-lift optimization and allocation model for manifolded subsea well. In *European petroleum conference, the hague.*

El-M., Y., & Price, A. (1995). Development of a network and gas lift allocation model for production optimization in the ras budran field. In *Middle east oil show, bahrain.*

Everitt, T. (1994). Gas-lift optimization in a large, mature gom field. *Annual Technical Conference and Exhibition*, 25-28.

Gabor, T. (2005). *Gas lift manual.* Pennwel Corporation.

Handley-S., S., Mckie, C., & Quintero, N. (2000). New mathematical techniques for the optimisation of oil & gas production systems. In *European petroleum conference, paris,.*

Hatton, N., & Potter, K. (2011). Optimization of gas-injected oil wells. *Science Applications International Corporation (SAIC)*, 1-4.

Havard, D. (2006). *An introduction to oil and gas production.* ABB.

Henrici., P. (1982). *Essential of numerical analysis, new york.* John Wiley & Sons.

Hoffman, J. D. (2001). *Numerical methods for engineers and scientists.* Marcel Dekker, Inc.

Ibrahim, A. T. M. (2007). *Optimization of gas lift system in varg field.* Unpublished master's thesis, University of Stavanger.

Jamal, A. S. (2001). *Production optimization via lagrange multipliers.* Unpublished master's thesis, Texas Tech University.

Kanu, E. (1992). Systems analysis, quality data ensure gas-lift optimization. *Pet. Eng. Intl.*, 72-76.

King, W. R. (1944). *Time and volume control for gas intermitters.* In Us Patent Number 2339487.

Kleyweg, T. W., D., & Dalziel, S. (1983). Gaslift optimisation - claymore field. In *Offshore europe conference, aberdeen.*

Knott, T. (1991). Grondin's gas lift switches to remote control. *Offshore Engineer*, 57-58.

Kruger, C. J. C. (n.d.). *Constrained cubic spline interpolation for chemical engineering applications.*

Laing, C. M. (1986). Gas lift design and production optimisation offshore trinidad. *Annual Technical Conference and Exhibition*, 3-6.

Lematayer, P., & Miret, P. (1991). Tool of the 90's to optimize gas-lift efficiency in the conelle field. In *Offshore europe conference.*

Leonard, J. (1984). World's largest closed-loop gas lift system will operate at north slope's prudhoe bay. *Oil & Gas J.*, 71-74.

Lo, K. (n.d.). Optimum lift-gas allocation under multiple production constraints.

Mantecon, J. (1983). Gas-lift optimization on barrow island, western austarlia. In *Asia pacific oil & gas conference.*

Piyawat, T. (2010). *Improved gas-lift optimization by using portfolio theory.* Unpublished master's thesis, Asian Institute of Technology, School of Engineering and Technology, Thailand.

Redden, J., Sherman, T., & Blann, J. (1974). Optimizing gas-lift systems. In *Annual fall meeting, houston.*

Schlumberger. (n.d.). *Avocet gas lift optimization module.* sisinfo@slb.com.

Schlumberger. (2000). *Gas lift design and technology.* Chevron Main Pass 313 Optimization Project.

Sharma, S., Kjetil, F., & Bjrn, G. (2012). Optimization of lift gas allocation in a gas lifted oil field as non-linear optimization problem. *Modeling, Identification and Control*, *33*, 13-25.

Shaw, S. F. (1939). *Gas lift principles and practices.* Houston, TX: Gulf Publishing Co.

Simmons, W. (1972). Optimizing continuous flow gas lift well. *Pet. Eng. Intl.*.

Singiresu, S. R. (2002). *Appied numerical methods for engineers and scientists* (M. J. Horton, Ed.). Tom Robbins Prentice Hall, Inc. Upper Saddle River, New Jersey 07458.

Stanghelle, K. U. (2009). *Evaluation of artificial lift methods on the gyda field.* Unpublished master's thesis, University of Stavanger.

Stewart, D. J., Goodacre, J., & Cruickshank, C. (1989). Bass strait: Production decline mitigation. *SPE Asia-Pacific Conference*, 13-15.

Stinson, R. (1988). The use of systems analysis in the design of a gas-lift system. *Intemational Meeting on Petroleum Engineering*, 1-4.

Stoisits, R. F., Scherer, P. W., & Schmidt, S. E. (1994). Gas optimization at the kupamk river field. *Annual Technical Conference and Exhibition*, *25-28*.

Sugiarso, G. (1995). Novel approach to optimizing gas lift in the jene field. *lATMI Intemational Symposium on Production Optimization*, 24-26.

Tools, B. O. (n.d.). *Gas lift system.* http://www.bakerhughes.com/BOT/completions/gaslift/index.htm.

Walsh, P. (1994). Gaslift operation in bass strait. In *Asia pacific oil & gas conference, melboume.*

Won, Y., Wenyu, C., Tae-Sang, C., & Morris, J. (2005). *Applied numerical methods using matlab.* John Wiley & Sons, Inc.

Woodyard, A. (1989). Gas-lift workstation improves productivity. *Oil & Gas J.*, 76-79.

Woodyard, A. H. (1988). Gas lift allocation from workstation database. In *Petroleum computer conference, san antonio,.*

Zheng-G., H. H., X., & Ramstad, H. (1988). Economical distribution of gas in a continuous gas-lift system subject to variable cost and system constrain. In *Petroleum society annual technical meeting of the petroleum society of cim, calgary.*

Appendix A

Tables

Table A.1: Gas injection and Oil output rate in well one

α, Gas injection rate (MSCF/D)	σ, Oil output rate (STB/D)
1.75	67
115	195
385	257
735	286
1290	317

Table A.2: Gas injection and Oil output rate in well two

α, Gas injection rate (MSCF/D)	σ, Oil output rate (STB/D)
2	25
155	158
380	172
650	177
1100	180

Table A.3: Gas injection and Oil output rate in well three

α, Gas injection rate (MSCF/D)	σ, Oil output rate (STB/D)
1.75	27
95	133
290	156
575	162
1050	165

Table A.4: Gas injection and Oil output rate in well four

α, Gas injection rate (MSCF/D)	σ, Oil output rate (STB/D)
1.8	49
120	192
350	228
780	241
1350	246

Table A.5: Gas injection and Oil output rate in well five

α, Gas injection rate (MSCF/D)	σ, Oil output rate (STB/D)
1	82
95	216
320	303
750	335
1380	346

Appendix B

Constrained Cubic spline Interpolation

```
function solutionSet = gSpline11_02_2013()
clc, clear
solutionSet = [];
%dataSet = [1.75 67; 115 195; 385 257; 735 286; 1290 317];
%dataSet = [2 25; 155 158; 380 172; 650 177; 1100 180];
%dataSet = [1.75 27; 95 133; 290 156; 575 162; 1050 165];
%dataSet = [1.8 49; 120 192; 350 228; 780 241; 1350 246];
dataSet = [1 82; 95 216; 320 303; 750 335; 1380 346];
 \alpha = dataSet(:,1);
\sigma = dataSet(:,2);
n = length( \alpha);

%% section for the first derivative
for i = 2:n-1
```

```
ffd(i) = 2/(( \alpha(i+1) - \alpha(i))/(\sigma(i+1)- \sigma(i))
 +( \alpha(i) - \alpha(i-1))/(\sigma(i) - \sigma(i-1)));
end
ffd(1) = 3*(\sigma(2) - \sigma(1))/(2*( \alpha(2) - \alpha(1)))
 - ffd(2)/2;
ffd(n) = 3*(\sigma(n) - \sigma(n-1))/(2*( \alpha(n) -
\alpha(n-1))) - ffd(n-1)/2;

for i = 2:n % counter for the number of segment
    %% section for the second derivative
fsd_back = -2*(ffd(i) + 2*ffd(i-1))/(\alpha(i)-\alpha(i-1))
 +6*(\sigma(i) - \sigma(i-1))/( \alpha(i)-\alpha(i-1))^2;
fsd_current = 2*(2*ffd(i) + ffd(i-1))/( \alpha(i)-\alpha(i-1))
 -6*(\sigma(i) - \sigma(i-1))/( \alpha(i) - \alpha(i-1))^2;
d(i-1)=(fsd_current - fsd_back)/(6*( \alpha(i)-\alpha(i-1)));
c(i-1) = ( \alpha(i)*fsd_back - \alpha(i-1)*fsd_current)/
(2*( \alpha(i) -\alpha(i-1)));
b(i-1) = ((\sigma(i) - \sigma(i-1)) - c(i-1)*( \alpha(i)^2 -
\alpha(i-1)^2) - d(i-1)*( \alpha(i)^3 - \alpha(i-1)^3))/
( \alpha(i) - \alpha(i-1));
a(i-1) = \sigma(i-1) - b(i-1)* \alpha(i-1) - c(i-1)*
\alpha(i-1)^2 - d(i-1)* \alpha(i-1)^3;
end
format rat
solutionSet = [[1:n-1]' a' b' c' d'];
fprintf('\n %f %f %f %f %f \n',solutionSet');
```

Appendix C

Broyden's Method


```
function [ \alpha ,f \alpha , \alpha \alpha] =
 newtons2(f , \alpha0 ,TolX ,Ma \alphaIter , varargin )
%newtons.m to solve a set of nonlinear eqs
% f1( \alpha)=0, f2( \alpha)=0,..
%input: f = 1^st-order vector ftn equivalent to
%a set of equations
%  \alpha0 = the initial guess of the solution
% TolX = the upper limit of |\alpha(k)-\alpha(k - 1)|
% Ma \alphaIter = the ma \alphaimum # of iteration
%output:  \alpha = the point which the
%algorithm has reached
% f \alpha = f( \alpha(last))
%  \alpha \alpha = the history of  \alpha
h = 1e-4; TolFun = eps; EPS = 1e-6;
f \alpha = feval(f, \alpha0 ,varargin{:});
Nf = length(f \alpha); N \alpha = length( \alpha0);
```

```
if Nf ~= N \alpha, error('Incompatible dimensions
 of f and  \alpha0!'); end
if nargin < 4, Ma \alphaIter = 100; end
if nargin < 3, TolX = EPS; end
 \alpha \alpha(1,:) =  \alpha0(:).';
%Initialize the solution as the initial row vector
%f \alpha0 = norm(f \alpha); %(1)
for k = 1: Ma \alphaIter
d \alpha = -jacob(f, \alpha \alpha(k,:),
h,varargin{:})\f \alpha(:);%-[dfd \alpha]  -1*f \alpha
%for l = 1: 3 %damping to avoid divergence %(2)
%d \alpha = d \alpha/2; %(3)
\alpha \alpha(k + 1,:)=\alpha \alpha(k,:)+d\alpha.';
f \alpha = feval(f, \alpha \alpha(k + 1,:),
\varargin{:}); f \alphan = norm(f \alpha);
% if f \alphan < f \alpha0, break; end %(4)
%end %(5)
if
f \alphan < TolFun || norm(d \alpha) < TolX, break;
 end
%f \alpha0 = f \alphan; %(6)
end
 \alpha =  \alpha \alpha(k + 1,:);
if k == Ma \alphaIter, fprintf('The best in
%d iterations\n',Ma \alphaIter),
end
```

```
function g = jacob(f, \alpha,h,varargin)
%Jacobian of f( \alpha)
if nargin < 3, h = 1e-4; end
h2 = 2*h; N = length( \alpha); \alpha =
\alpha(:).'; I = eye(N);
for n = 1:N
g(:,n) = (feval(f, \alpha + I(n,:)*h,varargin{:}) ...
-feval(f, \alpha - I(n,:)*h,varargin{:}))'/h2;
end
```