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PROJECT SELECTION UNDER MULTI-PERIOD CAPITAL RATIONING

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ALEX KWAKU PEPRAH
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DECLARATION

I, A.K. PEPRAH do hereby declare that the thesis entitled PROJECT SELECTION UNDER MULTI-PERIOD CAPITAL RATIONING-- CASE STUDY: SUNYANI MUNICIPAL ASSEMBLY was done entirely by me and under the supervision of DR. S.K. AMPONSAH and DR. F.T. ODURO.

ALEX KWAKU PEPRAH, 9800(PG9888I06) 05-03-08
(Student's Name & ID) (Signature) (Date)

DR S.K AMPONSAH 05/05/09
(Supervisor's Name) (Signature) (Date)

DR F.T. ODURO 06/05/09
(Supervisor's Name) (Signature) (Date)

PROF BADU 06/05/09
(Dean of IDL) (Signature) (Date)

DR. EDWARD PREMPEH 05/05/09
(Head, Mathematics Dept.) (Signature) (Date)

ABSTRACT

Due to insufficient funds available for investment, Municipal Assemblies are unable to initiate or implement all their viable projects. The dilemma facing the Municipal Assemblies is that, which of the numerous projects the management should invest under the limited investment capital in order to maximize profit.

To solve this project selection problem, data were drawn from sources such as financial statements, monthly and annual reports, daily newspapers and other relevant documents of the Sunyani Municipal Assembly over the period 2001-2006. Financial Ratios, Continuous Probability Analysis, Linear and Integer Programming Models were used to analyze the data. Sensitivity analyses were performed on the project parameters.

The Linear Programming Model designed was suitable for solving large scale project selection under multi-period capital rationing problems, and this produces the optimal solution quantities (i.e., the project to be initiated), the value of the objective function (i.e., the Net Present Value) and the opportunity cost of the binding constraints. On the other hand, the selection of small scale project problems was found to be solved easily by Integer Programming Models.

The Mathematical models such as linear programming and Integer programming were found to be:

- Suitable for selecting potential projects and maximizing the returns from the batch of projects selected by the Sunyani Municipal Assembly,
- Suitable for carrying out sensitivity analysis on the optimal solution to Linear programming problems to see how sensitive the project selection is to the changes in the parameters of the model.

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DEDICATION

Dedicated to my dearest daughter, **Yvonne Peprah**.

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CHAPTER ONE

INTRODUCTION

A recent survey of the capital budgeting practices of large companies by Pike (1995) showed that over seventy-five percent (75%) of companies used payback as an appraisal method, often in conjunction with other techniques. The same survey showed that only seventeen percent (17%) of the companies used Net Present Value (NPV) as their primary evaluation technique in spite of generally acknowledged technical superiority of NPV over payback .This would seem to suggest that much of the academic preoccupation with refining measurement techniques may be misplaced. Nevertheless, investment opportunities are of far greater importance to the companies than the particular appraisal method used, since successful investment appraisal is entirely dependent on the accuracy of the cost and revenue estimates.

The investment appraisal employed in this study is project selection under multi-period capital rationing. Capital rationing is manifested in the situation where the firm or company is unable to initiate all projects, which are apparently profitable because sufficient funds are not available. At a point where investment funds are expected to be limited over several periods of time is called multi-period capital rationing.

The effects of capital rationing may develop for internal purposes. For example, it may be decided that investment should be limited to the amount that can be financed solely from retained earnings or kept within a given capital budget. The external and internal factors, which impose quantitative limits, lead to two opposing view-points developing known as the "hard" and "soft" view of capital rationing.

The "hard" view is that, there is an absolute amount of money a company may borrow or raise externally. The "soft" view on the other hand is that rationing by a quantitative limit such as an arbitrary capital expenditure budget should only be seen as a temporary administrative expedient because such a limit is not determined by the market and such a limit would not be imposed by a profit maximizing company.

Whatever the causes of the limited capital supply available for investment purposes means that, not only must each project cover the cost of the capital but that the project or batch of projects selected must maximize returns from the limited funds available, making some form of ranking necessary.

Ways of achieving this objective include the following rationing possibilities:

- Single period capital rationing with divisible projects.
- Single period capital rationing with indivisible projects.
- Multi-period capital rationing with divisible projects.
- Multi-period capital rationing with indivisible projects.

These entire investment appraisals would be explained in the subsequent chapters.

Background

Missions and Objectives of District and Municipal Assemblies

Within the period of colonial rule and post independence era of Ghana, local government practices have undergone some metamorphosis.

The current structure of Municipal Assemblies (MA) and District Assemblies (DA), for instance, was introduced during the era of the Provisional National Defence Council (PNDC) to make it more responsive and relevant to the modern day demands of democratization, industrialization and development.

Among the main objectives of the MA and DA are as follows:

(i) To promote good governance by way of enhancing grass root participation of democracy.

(ii) To allow the local people a certain degree of local autonomy to identify and solve local problems better. This is because the local people know their own problems.

(iii) To undertake appropriate activities and harnessing resources that would promote well-being of the people by sponsoring economic and developmental projects.

In pursuance of the above stated missions, the Municipal Assemblies perform several functions such as;

(i) Serving as tools to bring government to the people without making the government too distant and alien.

(ii) Making mobilization of human and material resources for development much easier. This is because the policies and project works at the local level are seen by the people as their own creation and hence they become very proud to nurture them to fruition.

(iii) Offering the people who take part in their activities greater opportunities for the acquisition of training, experience and exposure, which prepare them for trade and leadership positions at the higher level of governance. Thus the Municipal Assemblies more or less serve as training and business organizations for the local people.

(iv) Providing social and economic infrastructural facilities to complement the efforts of the central government.

MA Projects and Sources of Revenue

The Sunyani Municipal Assembly (SMA) undertakes variety of economic and developmental projects. Among these projects are; construction of market squares, lorry parks, school buildings, water boreholes, clinics, recreational centre, latrines (KVIP), roads, providing housing facilities, toll booths, sanitation facilities, agricultural projects and electrification. These projects are the main sources of income for the SMA.

Apart from these projects, the SMA like any other MA, derives its sources of income from the fourth, fifth and sixth schedules of the Act 462 of 1993 constitution.

Under schedule four, the SMA has the right to issue vehicle licenses to all vehicles operating within the jurisdiction of the Assembly.

Under the 5th schedule, the SMA derives some of its sources of income from entertainment. Examples of these are concert, musical or theatrical performances, video shows, cinemas, dancing discotheques and others to which admission is obtained by payment of money or reward except where the proceeds are devoted to charity.

The 6th schedule provides the SMA the avenue taxes based on gambling and betting (Act 268, 1965), gambling (Act NRC Decree 174, 1973) as well as Casino Revenue (Casino tax decree NRCD 200, 1973).

Statement of the Problem

It is evident from the above that the cash inflow for projects and running of the functions of the MA comes largely from monies accrued from the projects and common fund. Usually, the limitation of these funds extends over a long period of time. The MA operates under multi-period capital rationing. That is, the MA is unable to initiate all its projects, which are apparently profitable but funds available are insufficient

The vast increasing number of the youth unemployment problem has hit the Sunyani area for some time now. This has compelled the SMA to undertake projects to create employment opportunities for the youth. At the same time, the SMA wishes to invest in the projects that will maximize returns from the batch of projects it would select with regard to capital limitation. The dilemma facing the SMA is about which of the numerous projects they should invest under the limited funds in order to maximize profit and also solve the youth unemployment problem.

The problem statement of this study is to formulate mathematical programming models for solving the MA project selection under the multi-period capital limitation problem.

Purpose of Study

The purpose of this study is:

(a) To formulate Linear programming (LP) and an Integer programming (IP) models for solving MA project selection problems.

(b) To maximize the returns from the batch of the projects selected with regards to the capital limitation.

(c) To carry out sensitivity analysis on the project parameters to see how sensitive the project selection decision is to the data.

Significance of Study

The study is to throw light on the inherent difficulties and problems facing firms, MA and companies in selection of projects with regards to the capital limitation. The models will serve as a tool for solving various problems of multi-period capital rationing in firms. The study can also be a guide for policy and decision makers in the various firms and organizations to maximize profit from the batch of projects they would undertake.

Scope of the Study

As indicated in the background, the MA was introduced during the era of the PNDC, that is, in 1979. However, the study will cover the project works over the period 2001-2006. It is hoped that the models will be of interest to managements and planners of the various MA and firms in Ghana.

Methodology

Data for the study is purely a secondary data. The data shall be obtained from sources such as financial statements, annual reports, monthly reports, daily newspapers and other relevant documents of the SMA.

The data shall critically be examined and classified into tables. The projects with different lives will be compared. The financial ratios and Continuous Probabilistic Analysis (CPA) will be used to analyze the data. The LP and IP models for solving project selection problems from optimization theory perspective will also be presented.

Overview

The remaining work is organized as follows:

Chapter two provides the literature review for the project selection under multi-period capital limitation problem addressed in this thesis.

Chapter three examines the procedure and formulation of linear programming and integer programming models for solving programming problems from optimization theory perspective. In this chapter, the sources, the method of collection of data and comparison of data using financial ratios and Continuous Probabilistic Analysis (CPA) will be given.

The results from chapter three will be used in chapter four to formulate a suitable optimization problem to compute the objective function or decision rule of the programming problems. A MATLAB program for solving the LP and IP problems formulated in chapter three will also be provided. This will be followed by sensitivity analysis of the results and the findings.

Chapter five will be discussions, conclusions and recommendations of the study.

Limitations

There are no doubts that in right circumstances, the LP and the IP can be a useful method of dealing with multi-period capital rationing problems. There are, however, few assumptions and limitations, which are worth mentioning. These include the assumption that;

- All functions which are linear may not be realistic.
- The projects and constraints are all being independent of one another.
- The cash flows, resources and constraints which are known with certainty may also not be realistic.

The researchers are also aware that there are other techniques like payback (PBP) and internal rate of return (IRR) for selecting optimal projects. However, due to the number of serious limitations they present, only NPV will be used in the formulation of the objective function of the models.

CHAPTER TWO

LITERATURE REVIEW

Introduction

This chapter presents the necessary literature review for the project selection under multi-period capital limitation problems addressed in this thesis. It is upon this background information that the programming models will be formulated.

Multi – Period Capital Rationing

The multi-period capital rationing is where investment funds are expected to be limited over several periods. In such circumstances, it becomes difficult to choose the batch of projects, that is, some starting immediately, first or second or others couple of periods which yield the maximum returns and yet which remain within the capital limits.

The problem becomes one of optimizing a factor, (e.g. NPV) where resources are limited and the funds available over the periods are considered. This will be recognized as a situation where linear programming and integer programming could be used. Both IP and LP have been used successfully in solving multi-period capital rationing problems. Specifically, the LP method is usually used to solve divisible projects, that is, where a fractional part of a project is desired to be undertaken. Where the projects are not divisible, the only feasible solution method is IP method (Terry, 1997).

Project Selection

The appropriate choice of investment projects depends primarily on the nature of cash flows generated by the projects, the risk level associated with the cash flows and the budgetary limitations of the corporation over time. For the past four decades, researchers have attempted to present a working investment choice model that considers the various aspects of the budgetary process.

In 1963, the first mathematical programming formulation of the multi-period capital rationing problem was provided (Weingartner, 1963). In his formulation, the net discount cash inflows for projects were maximized while cash outflows and availability of resources were maintained in each period. This formulation withstood many criticisms over the past three decades. The majority of these criticisms were based on three main features of the models. These being the appropriate selection of an objective function, the determination of a suitable discount rate to account for project returns and the inability of the model to deal with uncertain budgetary constraints.

In the seminar work by Weingartner, the author provided a framework using a deterministic linear programming approach. His model uses Net Present Value (NPV) as its objective function. The value associated with the timing of a particular cash flow is adjusted by an appropriate discount rate (KIRA, 2000).

It is evident from the above surveys that the LP model for project evaluation under capital rationing made use of project cash flows, NPV, IRR and other investment evaluation techniques. It is therefore imperative that these methods are examined to see how they could be incorporated into the formulation of LP and IP in the proposed study.

Projects and Cash Flows

Every decision the company makes is a capital budgeting decision whenever it changes the company's cash flows and considers launching a new project. This involves a phase where the new product is advertised and distributed. Hence the company will have cash outflows for paying advertising agencies, distributors, transportation services etc. Then, for the period of time the company may have cash inflows from the sale of the products in the future. Thus, two types of cash flows are identified; cash inflow and cash outflow from the project. Cash flow items include:

Cash inflows

- The project revenues
- Government grants
- Resale or scrap value of assets
- Tax receipts
- Any other cash inflows caused by accepting the project.

Cash outflows

- Initial investment in acquiring the assets
- Project costs, labour, materials etc.
- Tax payments
- Any other cash outflow caused by accepting a project.

The difficulty with making these decisions are that, typically many cash flows are affected and they usually extend over a long period of time. Investment appraisal criteria could be employed in analyzing capital budgeting decisions by aggregating the multiple of the cash flows into one number. Thus, all cash flows have to be included in the analysis whenever they are affected by the decision (Smith & Whaley, 2004).

Determination of Cash Flows in the NPV analysis

The incremental net cash flow of an investment proposal is defined as the difference between the company's cash flows if the investment project is undertaken and the company's cash flows if the investment project is not undertaken. That is, the net cash flow generated by certain assets is given by the equation:

Net Cash Flow = Cash Inflow – Cash Outflow

= Revenue – Expenses – Capital Expenditure – Taxes ... . (1)

The income tax paid is determined by:

Taxes = t (Revenue – Expenses – Depreciation) .... (2)

Where t is the corporate tax rate. The depreciation is not a cash expenses and it only affects cash flows through its effect on taxes. Substituting equation (1) into equation (2) yields an expression for the company's cash flow:

Cash Flow = (1-t) (Revenue – Expenses) + t (Depreciation) – Capital expenditure.... (3)

The term t (Depreciation) is sometimes known as the depreciation tax shield.

Investment Appraisal Techniques

The investment evaluation techniques under consideration in this study include:

- Net Present Value (NPV)
- Discount Rate (DR)
- Payback Period (PBP)
- Internal Rate of Return (IRR)

alternatives, the one yielding the higher NPV should be selected. The following sums up the NPVs in various situations.

If	It means	Then
NPV > 0	the investment would add value to the firm	the project may be accepted
NPV < 0	the investment would subtract value from the firm	the project should be rejected
NPV = 0	the investment would neither gain nor lose value for the firm	It should be indifferent in the decision whether to accept or reject the project. This project adds no monetary value. Decision should be based on other criteria, e.g. strategic positioning or other factors not explicitly included in the calculation.

Source: Baker, 2007

However, NPV = 0 does not mean that a project is only expected to break, even in the sense of undiscounted profit or loss (earnings). It will show net total positive cash flow and earnings over its life.

In sum, it is optimal to make a decision that generates positive NPV of their incremental cash values. If there are more than two alternatives, it is optimal to choose the alternative that generates the highest NPV.

Illustration

X Corporation must decide whether to introduce a new product line. The new product will have startup costs, operational costs, and incoming cash flows over six years. This project will have an immediate (t=0) cash outflow of GH¢100,000 (which might include machinery, and employee training costs). Other cash outflows for years 1-6 are expected to be GH¢5,000 per year. Cash inflows are expected to be GH¢30,000 per year for years 1-6. All cash flows are after-tax, and there are no cash flows expected after year 6. The required rate of return is 10%. The present value (PV) can be calculated for each year:

T=0 -GH¢100,000 / 1.10^0 = -GH¢100,000 PV.

T=1 (GH¢30,000 - GH¢5,000) / 1.10^1 = GH¢ 22,727 PV.

T=2 (GH¢30,000 - GH¢5,000) / 1.10^2 = GH¢20,661 PV.

T=3 (GH¢30,000 - GH¢5,000) / 1.10^3 = GH¢18,783 PV.

T=4 (GH¢30,000 - GH¢5,000) / 1.10^4 = GH¢17,075 PV.

T=5 (GH¢30,000 - GH¢5,000) / 1.10^5 = GH¢15,523 PV.

T=6 (GH¢30,000 - GH¢5,000) / 1.10^6 = GH¢14,112 PV.

The sum of all these present values is the net present value, which equals GH¢8,881. Since the NPV is greater than zero, the corporation should invest in the project.

Discount Rate

The rate used to discount future cash flows to their present values is a key input of this process. Most firms have a well defined policy regarding their capital structure, so the weighted average cost of capital (after tax) is used with all projects. Some people believe that it is appropriate to use higher discount rates to adjust for risk for riskier projects. Another method is to use a variable discount rate with higher rates applied to cash flows occurring further along the time span, (reflecting the yield curve premium for long-term debt).

Another approach for the discount rate is to decide the rate, which the capital needed for the project could return if invested in an alternative venture. If, for example, the capital required for Project A can earn five percent elsewhere, use this discount rate in the NPV calculation to allow a direct comparison to be made between Project A and the alternative. Obviously, NPV value obtained using variable discount rates with the years of the investment duration better reflects the real situation than that calculated from a constant discount rate for the entire investment duration. (Baker, 2007).

For some professional investors, their investment funds are committed to target a specified rate of return. In such cases, that rate of return should be selected as the discount rate for the NPV calculation. In this way, a direct comparison can be made between the profitability of the project and the desired rate of return.

To some extent, the selection of the discount rate is dependent on the use to which it will be put. If the intent is simply to determine whether a project will add value to the company, using the firm's weighted average cost of capital may be appropriate. If trying to decide between alternative investments in order to maximize the value of the firm, the corporate investment rate would probably be a better choice. Using variable rates over time or discounting "guaranteed" cash flows which is different from "at risk" cash flows may be a superior methodology, but it is seldom used in practice. Using the discount rate to adjust for risk is often difficult to do in practice (especially internationally), and is really difficult to do well. An alternative to using discount factor to adjust for risk is to explicitly correct the cash flows for the risk elements and then discount at the firm's rate (Philip.1997).

Payback Period

Numerous surveys have shown that payback is a popular technique for appraising projects either on its own or in conjunction with other methods. Payback can be defined as the period, usually expressed in years, which it takes for a project's net cash inflows to recoup the original investment. The usual decision rule is to accept the project with the shortest payback period. The payback has several advantages and disadvantages. Among these are:

Advantages

i. Uses project cash flows rather than accounting profits and hence is more objectively based.

ii. Favours quick return projects which may produce faster growth for company and hence liquidity

Disadvantages

i. Payback does not measure overall project worth because it does not consider cash flows after payback period.

ii. It provides only a crude measure of timing of project cash flows

In spite of any theoretical disadvantages payback is undoubtedly the most popular appraisal criterion in practice (Pike, 1995).

Internal Rate Return (IRR)

The IRR of a project is the rate which equates the NPV of the project's cash flows to zero; or equivalently the rate of return which equates the PV of inflows to the PV of the outflows.

Internal rate of return rule

IRR is return that equates initial investment with PV of cash flows

$$O = -C_o + \sum_{t=1}^{T} C_t \left[\frac{1}{(1 + IRR)^t} \right],$$

where;

t - the time of the cash flow,

0- zero,

C_t - the net cash flow (the amount of cash) at time t, and

C_0 - the capital outlay at the beginning of the investment time ($t = 0$).

The decision rules include:

Accept projects with $IRR>r$

Reject projects with $IRR<r$

The problems with IRR are:

a. Ignores Value Creation (Scale).
b. Assumes cash flows being reinvested at IRR.
c. Multiple IRRs if later cash flows are negative.

Profitability Index (PI)

Another investment appraisal technique, the PI, is used when the companies or firms have only a limited supply of capital with which to invest is positive NPV projects. This type of problem is referred to as capital rationing. Given that the objective is to maximize shareholder wealth, the objective in the capital rationing problem is to identify that subset of projects that collectively have the height aggregate NPV. To assist in that evaluation, this method requires that each project's PI is computed using:

Profitability Index (PI) $= \frac{NPV}{I}$ where I = Initial investment.

The project's PI is then ranked from highest to lowest and then selected from the top of the list until the capital budget is exhausted. The idea behind the PI method is that it provides the subset of projects that can maximize the aggregate NPV.

In general the PI is of limited usefulness and the use of NPV is considered safer.

Project Valuation

Several authors employed the single period capital asset pricing model (CAPM) (Lintner and Sharpe, 1965) to value long-lived capital assets. The study by Bogue and Rol, (1960) was an early attempt in this direction. They demonstrated that the single period CAPM has limitations in valuing long-term capital projects. Their approach was later explored and extended by Brennah, (1998) which used the continuous time version of the CAPM and were able to derive the project's value assuming simple expectation formulation of the future cash flows.

In the study reported by Harrell and Baker (2005) they investigated the valuation of a capital investment project stipulating that a time-varying normal stochastic process with unknown means generates the project's cash flows. Investors are assumed to be Bayesian decision makers under uncertainty in the sense that they combine their prior information with evidence from the observed project's cash flows to sequentially about unknown mean cash flows. It is assumed that the CAPM of Lintner and Sharpe, (1965) is valid now, and in the future periods, as it provides the basis of valuation framework in the capital budgeting problem. In this context they derived the valuation formulae and betas of capital investment projects under different scenarios regarding the behaviour of the projects cash flows. Also, they investigated the inter-temporary behaviour of the project's betas for various parameter values, and examined the validity of the traditional textbook valuation formulae. The application of the conventional NPV technique in capital budgeting under uncertainty is criticized.

The stochastic capital rationing (SCR) model developed by KIRA, (2000) does not directly consider the issue of uncertain project cash flows in its analysis. Rather, they developed a procedure for the capital budgeting problem wherein both uncertainty in budgetary constraints and returns can be addressed simultaneously. This is realized by utilizing the SCR model and by considering varying standard deviations of project returns in generating the optimal composition of projects. Many authors and researchers consider the possibility of information upgrading in project valuation, but do not present a specific model delineating how learning can be formally embedded into the multi-period capital rationing problems. This study is a step in bridging this gap.

The purpose of this study is to develop or formulate LP and IP models for solving multi-period capital rationing problems. Specifically, the LP model will be designated to solve multi-period capital rationing (MCR) with divisible project problems while IP will be used to solve MCR with indivisible project problems. The models seek to produce optimal solution quantities (i.e. the projects to be initiated), the value of the objective function (i.e. the total NPV) and the shadow costs (i.e. opportunity costs of the binding constraints).

CHAPTER THREE

REVIEW OF BASIC METHODS

Introduction

This chapter examines the sources and method of collection of data. It is also proposed to provide the statistical analysis, basic methods and formulation of IP and LP models for solving MA project selection under multi- period capital rationing problems.

Sources and Data Collection

A capital budgeting is the process of considering alternative capital projects and selecting those alternatives that provide the most profitable return on available funds, within the framework of the company's goals and objectives. A capital project is any available alternative, to purchase, build, lease or renovate buildings, equipment, or other long range major items of property or projects. The alternative selected usually involves large sums of money and results in a large increase on fixed assets for several years. Once a company builds a plant or undertakes some other capital expenditure, the company becomes less flexible regarding future plans (Hermanson and Maher, 1992).

Based on this fact, most of the necessary information about cash inflows and cash outflows of the SMA small scale and large scale projects were extracted from the SMA financial statements, monthly and annual reports for the period 2001-2006. The annual net cash flow which is the difference between the cash inflows and cash outflows during each period for the under listed projects were then estimated and recorded (Tables 1)

The concepts of capital budgeting can be applied to not-for-profit organizations, such as universities, school districts, cities and not-for-profit hospitals. Since these organizations are not subject to as many taxes as profit-making organizations, the cash flows related to taxes are usually zero or near zero (Maher and Hermanson 1992, p.125). Due to this, the tax factor in the estimation of the annual net cash flows for the SMA projects was ignored. (Equation 3, Chapter 2).

The SMA projects were classified into small scale and large scale projects. The discount factors were also estimated at cost of capital of ten percent (10%) for each cash inflow for each project and the corresponding NPV at ten percent (10%) (Table1). The distribution of cash outlay for SMA small scale projects are shown in Table 2. Table 2 also shows the capital requirements for each project, available capital at each period and the corresponding capital returns.

Table 1.

Annual Net Cash Flow for SMA Large Scale Projects for 2001-2006 in GH¢ × 10^2

Year	2001	2002	2003	2004	2005	2006	NPV at	P.I
Project/ Period	0	1	2	3	4	5	10%	
Market Circle	100	-100	-200	-400	600	500	264	0.82
School Building	400	500	-1000	1200	-1400	1200	719	0.80
Health Post	250	200	-360	500	-400	0	237	0.92
Electrification	30	50	60	60	150	-90	217	0.33

Agriculture	10	20	-10	0	30	50	72	0.72
Discount Factors	1.00	0.909	0.826	0.751	0.683	0.621		
Capital Limitation Q_j	550	500						

Source: SMA

Table 2.

Distribution of Capital Requirement for SMA Small Scale Projects for 2001-2006 in GH¢ $\times 10^2$

Year	2001	2002	2003	2004	2005	2006	Capital
Project/ Period	0	1	2	3	4	5	Returns
KVIP (X_1)	50	30	20	20	60	40	50
Refuse Facility (X_2)	10	80	20	20	30	60	30
Water Borehole (X_3)	15	15	30	40	60	0	50
Taxi Circle (X_4)	10	40	10	10	0	10	10
Toll Booth (X_5)	10	0	10	20	50	10	20
Available Capital R_j	80	145	90	100	165	80	

Source: SMA

Statistical Analysis

The Continuous Probabilistic Analysis (CPA) was used for the analysis of the cash flows and NPV of the various projects. This shows the variability of the project outcomes, which results from the variability of the individual project cash flows. This enabled the researcher to make probability assessment of the likelihood of the various project cash flows and variability (risk) of using the project's NPV in the proposed models.

The most useful measure for statistical purposes is the standard deviation. Initially, the mean (NPV) is computed, followed by the dispersion and variance of the period's cash flows. The projects' standard deviation is then obtained by combining the discounted standard deviations of the individual cash flows, using what is known as the statistical sum. Having calculated the means and standard deviations of the various projects, the relative variability of the distribution of the project cash flows were then computed, using the formula;

$$\text{Coefficient of Variation} = \frac{\sigma i}{\chi} \text{ x } 100\%, \text{ where } \chi \text{ is the mean of the cash flows.}$$

The results were then compared and recorded (Table 3 & 4).

Table 3

Distribution of Large Scale Projects Cash flows (2001-2006)

Project	Mean ($\bar{x}$)	Standard deviation (σi)	Coefficient of variation/%
Market Circle	266.67	205.48	77
School Building	250.00	373.05	39
Health Post	285.00	160.59	56
Electrification	73.33	38.59	53
Agriculture	20.00	16.32	82

Table 4

Distribution of Small Scale Projects Cash flows (2001-2006)

Project	Mean (x)	Standard deviation (σi)	Coefficient of variation/%
KVIP (X_1)	36.67	14.91	41
Refuse Facility (X_2)	36.67	24.94	68
Water Borehole (X_3)	26.67	19.51	73
Taxi Circle (X_4)	13.33	12.47	93
Toll Booth (X_5)	16.67	15.99	96

The results in tables 3 and 4 will enable the researcher to make probability statements about the projects outcomes which reflect the variability's expected in each period's cash flows and the distribution of the competing projects to be compared favourably.

Linear Programming (LP)

A Linear Programming (LP) is one of the most widely used optimization techniques and perhaps the most effective. The term Linear Programming was coined by George Dantzig in 1947 to refer to problems in which both the objective function and constraints are linear (Dantzig, 1998, Martin, 1999).

A Linear Programming is the problem of optimizing linear objective in the decision variables $X_1, X_2, \ldots\ldots\ldots X_n$ subject to linear equality or inequality constraints on the X's.

Standard Form of Linear Programming

In the standard form, Linear Programming is expressed as;

Maximize $F = \sum_{j=1}^{n} C_j X_j$ ………………………………………………(1)

Subject to $\sum_{j=1}^{n} a(i,j) X_j = b_i$, $i = 1, 2, \ldots, m$ ………………………(2)

$l_j \le X_j \le u_j$, $j = 1, 2 \ldots, n$, ………………………(3)

where C_j are the objective function coefficients, $a(i, j)$ and b_i are parameters in the m linear inequality constraints and l_j and u_j are lower and upper bounds with $l_j \le u_j$. Both l_j and u_j may be positive or negative.

A linear programming can be expressed more conveniently using matrices;

Minimize $F = C^1 X$ ……………………………(4)

Subject to: $\mathbf{A}X = b$ ……………………………(5)

$l \le X \le u$ ……………………………(6)

A is $m \times n$ matrix whose (i, j) element is the constraint coefficient $a(i, j)$ and C, b, l, u are vectors whose complements are C_j, b_j, u_j respectively. If any of the equations (1- 5) were redundant, that is, linear combinations of the others, they could be detected without changing any solutions of the system. If there is no solution or if there is one solution for equation (5), there can be no optimization. Thus the case of greatest interest is where the system of equation (5) has more than unknown equations and has at least two and potentially and infinite number of solutions. This occurs if and only if

$n > m$, and

Rank $(\mathbf{A}) = m$

These conditions are assumed to be true in what follows. The problem of linear programming is to first detect whether solutions exist, and if so, to find one yielding the minimum F.

Basic Definitions

1. A feasible solution to LP problem is a vector $X = (X_1, X_2, \ldots\ldots X_n)$ that satisfies the equation **AX** $= b$ and the bounds $l \le X \le u$.
2. A linear programming (LP) is feasible if there exists a feasible solution otherwise it is said to be infeasible.
3. A basic feasible solution is a basic solution in which variables satisfy their bounds $l_j \le X_j \le u_j$.
4. A non degenerate basic feasible solution in which all basic variables X_j are strictly between their bounds, that is, $l_j \le X_j \le u_j$.
5. An optimal solution $X = (X_1, X_2, \ldots\ldots X_n)$ is a feasible solution subject to $C^T X$, **AX** $= b$ and $X \le 0$. (Lasdon and Powell, 1998).

Equivalent Forms of LP

A linear programming can take on several forms. It might be maximizing instead of minimizing. The LP can have a combination of equality and inequality constraints. Some variables may be restricted to be non-positive instead of non-negative, or be unrestricted in sign. Two forms are said to be equivalent if they have the same set of optimal solutions or are either infeasible or unbounded.

1. A maximization problem can be expressed as a minimization problem;

   I.e. maximize $C^T X \equiv$ minimize $-C^T X$

2. An equality can be represented as a pair of inequalities;

$$a_i^T x = b_i \equiv \begin{cases} a_i^T x \le b_i \\ a_i^T x \ge b_i \end{cases} \qquad \text{OR}$$

$$\begin{cases} a_i{}^T x \geq b_i \\ -a_i{}^T x \leq -b_i{}^T \end{cases}$$

3. By adding a slack variable, an inequality can be represented as a combination of equality and non-negativity constraints. For example,

$$\sum_{j=1}^{n} a(i,\ j)x_j \leq b_j$$

Then a slack variable is defined as $s_i \geq o$ such that

$$\sum_{j=i}^{n} a(i,j)x_i + s_i = b_i\ ,$$

and the inequality becomes equality.

Similarly, if the inequality is $\sum_{j=1}^{n} a(i,j)_j j \geq b_i$, it is written as;

$$\sum_{j=1}^{n} a(i,j)X_j - s_j = b_i$$

4. Non-positive constraint can be expressed as non-negative constraints. To express X_j, replace X_j everywhere with $-y_j$ and impose the $y_j \geq 0$.

5. X may be unrestricted in sign. In such a case, X_j is replaced everywhere by $\overset{+}{X}_j - \overset{-}{X}_j$, adding the constraints $\overset{+}{X}_k$, $\overset{-}{X}_j \geq 0$.

In general, an inequality can be represented using a combination of equality and non-negative constraints, and vice versa. Using these rules;

Minimize $\{C^T X$, st $\mathbf{A}X = b\}$ can be transformed into Minimize $\{\mathbf{A}\overset{+}{X} - \mathbf{A}\overset{-}{X} - s = b$, $\overset{+}{X}$, $\overset{-}{X}$, $s \geq 0\}$. The former LP is said to be in canonical form, the latter in standard form.

Duality

For any given Linear programming problem called Primal, there is an associated Linear Programming called Dual problem. Duality is an important concept in Linear Programming problem since it is algorithmic and allows a proof of optimality.

Rules for Taking Dual Problems

If PRIMAL problem (P) is minimal problem, then the DUAL (D) problem is maximum problem and vice versa.

In general, using the rules of transforming a Linear Programming in Standard Canonical form, the Dual (D) of Primal (P) is;

Minimize $Z = C^T X$

Subject to $\mathbf{A} X \le b,$

$X_j \ge 0,$

where $X = (X_1, X_2, ..., X_n)^T$ is any *n*-vector,

$C = (C_1, C_2, ..., C_n)^T$ is any *n*-vector,

$\mathbf{A} = a(i, j)$ is $m \times n$ matrix and

$b = (b_1, b_2, ..., b_n)^T$ is any m-vector is DUAL (D).

Maximize $W = b^T y$

Subject to $\mathbf{A}^{\mathrm{T}} \; y \ge C, y \ge 0.$

The variables in the P are called Primal variables and the variables in the Dual problem are called Dual variables.

The general rules for converting primal problem of any form into dual problem can be summarized as shown in table 3.1:

Table 3.1

PRIMAL PROBLEM *Maximization*	DUAL PROBLEM *Minimization*
• Coefficients of objective function	• Right hand sides of constraints
• Coefficients of i^{th} constraints	• Coefficients of i^{th} variable
• i^{th} constraint is an inequality of the form ≤	• i^{th} variable satisfies ≥ 0
• i^{th} constraint is an equality (=)	• i^{th} variable is unrestricted
• i^{th} variable is unrestricted	• i^{th} constraint is an equality (=)
• i^{th} variable satisfies ≥ 0	• i^{th} constraint is an inequality of the type ≥
• Number of variables	• Number of constraints
• Number of constraints	• Number of variables
• If inequality of type ≥ occurs in maximization problem convert to type ≤ by multiplying through by -1	• If inequality of type ≤ occurs in minimization problem convert to type ≥ by multiplying through by -1

Solution Techniques for Linear Programming

There are several approaches for solving the linear programming problems. Among these techniques are:

(i) Graphical Approach

(ii) Simplex Algorithms

(iii) LINDO Software

(iv) QSB Package

(v) MATLAB

(vi) YE's Interior Point Algorithms

(vii) Microsoft Excel 2003

The most convenient and effective technique in use now is the Simplex Algorithm. Essentially, the Simplex Algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged), the objective function as it does so, until it reaches the vertex corresponding to the optimal linear programming solution.

The Simplex Algorithm for solving LP's was developed by Dantzig, (1940). A number of different versions of the algorithms have now been developed. One of these later versions, called the Reversed Simplex Algorithms (Appendix B) forms the basis of the most modern computer packages for solving LP's. (Arsham, 1999).

Sensitivity Analysis

Sensitivity analysis (or post-optimal analysis) allows the researcher to observe the effect of changes in the parameters of linear programming problems on the optimal solution. Given the LP package, it is easier to change the data to see how the solution changes (if at all) as certain key data items change.

As a by-product of using the Simplex Algorithm, researchers automatically get sensitivity information;

(a) for the variables, the Reduced Cost (also known as Opportunity Cost) column which give each variable an estimate of how the objective function will change if that variable is made non-zero. This is called "Reduced Cost" for the variables. The

Reduced Cost can also be interpreted as the amount by which the objective function coefficient for a variable needs to change before that variable becomes non-zero.

(b) for each constraint column headed shadow price indicates by how much the objective function will change if the right hand side of the corresponding constraint is changed. This is known as the "Marginal Value" or "Dual Value" for the constraint.

LP Formulation and Applications

The conditions for a Mathematical model to be a Linear Programming;

(a) all variables must be continuous (i.e. can take fractional values)

(b) a single objective (minimum or maximum) must be indicated

(c) the objective and constraints are linear (i.e. any term is either a constant or constant multiples of an unknown).

The LP are important in everyday life because, many practical problems can be formulated as LP and also there is an algorithm (called the Simplex Algorithm) which enables the researcher to solve LP numerically relatively easy (Padberg & Hoffman, 1995).

LP Model for Project Selection under Multi-Capital Rationing Problem

Based on the above information about LP, a formal formulation of LP model for solving MA Project Selection problems is presented below.

Conditions

- The capital available for investment is denoted by Q_j.
- The problem facing the MA is that, which project or portions of a project it should initiate with Q_j.

Steps.

i. The project NPV's, are determined using

$$\beta_j(NPV) = \sum_{t=1}^{n}\left[\frac{C_t}{(1+r)^t}\right],$$

where t = 0, 1,

$j = 1, 2\ldots 5$ and C is the cash flows.

Assuming the results, NPV of the projects to be; Market Circle (A) = β_1, School Building (B) = β_2, Health Post (C) = β_3, Electrification (D) = β_4 and Agriculture (E) = β_5.

ii. Formulating the problem as LP means defining the objective function, decision variables and constraints. The objective function of MA is to maximize NPV. That is;

Maximize $Z = \beta_1 X_A + \beta_2 X_B + \beta_3 X_C + \beta_4 X_D + \beta_5 X_E$,

where the decision variables (X_j);

X_A is proportion of project A to be initiated (j=1),

X_B is proportion of project B to be initiated (j=2),

X_C is proportion of project C to be initiated (j=3),

X_D is proportion of project D to be initiated (j=4),

X_E is proportion of project E to be initiated (j=5).

iii. Subject to the constraints in the MA problem are the budgetary limitations in periods 0 and 1 (table 1). Taking the periods separately, gives;

Capital at time (t) = 0,

$$a_{(1,1)}X_A + a_{(1,2)}X_B + a_{(1,3)}X_C + a_{(1,4)}X_D + a_{(1,5)}X_E \leq Q_1$$

Capital at time (t) = 1,

$$a_{(2,1)}X_A + a_{(2,2)}X_B + a_{(2,3)}X_C + a_{(2,4)}X_D + a_{(2,5)}X_E \leq Q_2.$$

To ensure that a project is not accepted more than once or **negative** projects are accepted, the constraints regarding the proportions of the projects are specified as shown below;

$$X_{A,}X_{B,}X_{C,}X_{D,}X_{E} \leq 1$$
$$X_{A,}X_{B,}X_{C,}X_{D,}\ X_{E} \geq 0.$$

where $a_{(i,j))}$ = cash flow for each period for each project.

The whole formulation in a compact form is thus;

Maximize Z = $\beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5$

Subject to

$$\begin{array}{lr}
a_{(1,1)}X_1 + a_{(1,,2)}X_2 + a_{(1,3)}X_3 + a_{(1,4)}X_4 + a_{(1,5)}X_5 & \leq Q_1 \\
a_{(2,1)}X_1 + a_{(2,2)}X_2 + a_{(2,3)}X_3 + a_{(2,4)}X_4 + a_{(2,5)}X_5 & \leq Q_2 \\
X_1 & \leq 1 \\
X_2 & \leq 1 \\
X_3 & \leq 1 \\
X_4 & \leq 1 \\
X_5 & \leq 1 \\
X_1 & \geq 0 \\
X_2 & \geq 0 \\
X_3 & \geq 0 \\
X_4 & \geq 0 \\
X_5 & \geq 0
\end{array}$$

A more general form is;

$$\text{Maximize } Z = \sum_{j=1}^{N} \beta_j X_j \qquad \text{(objective function)}$$

$$\text{Subject to } \sum_{j=1}^{N} a_{(i,j)} X_j \le Q_i \qquad \text{(constraints)}$$

$$i = 1, 2.... m$$

$$0 \le X_j \le 1$$

$$j = 1, 2... N,$$

where $\{a_{(i,j)}, Q_j \, and \, \beta_j\}$ are given and N is the number of projects to be invested.

Assumptions

The assumptions made in formulating the LP model include:

(i) All the projects and constraints are independent on one another.

(ii) Equal investment opportunities are assumed for the projects for each period.

(iii) The cash flows, resources and constraints are known with certainty.

Integer Linear Programming (IP)

In many of the LP problems, certain variables should have been regarded as Integer values. But for the sake of convenience, the variables are allowed to take fractional values reasoning that the variables are likely to be so large that any fractional part could be neglected. Whilst this is acceptable in some situations, in many cases, a numeric solution must be found such that the variables take integer values. The problems in which this is the case are called Integer Programs (IP) and the subject of solving such programs is called Integer Programming. Thus Integer Programming is the subset of Linear Programming in which all the variables are required to be non- negative integers.

General Forms of Integer Programming (IP)

1. An IP in which all the variables are required to be integers is called Pure IP problem,

i.e. maximize $Z = \sum_{j=1}^{n} a_j X_j$ (objective function)

Subject to $\sum_{j=1}^{n} a_{(i,j)} X_j \leq b_j$ (constraints)

$$X_j \leq 0, X_j \text{ Integer}$$

$$j = 1, 2 \ldots n$$

2. An IP in which only some of the variables are required to be integers is called Mixed Integer Programming,

i.e. Maximize $Z = \sum Q_j X_j$

Subject to $\sum_{j=1}^{n} a_{(i,j)} X_j \leq P_j$

$$\sum_{j=1}^{n} X_j + \sum y_j \geq W$$

$$X_j \geq 0, \quad y = 0, \text{ or } 1$$

$$j = 1, 2 \ldots.. n$$

3. An IP problem in which all the variables must be equal to 0 or 1 is called a Zero-one Integer Programming,

i.e. Maximize $Z = \sum Q_j X_j$

Subject to $\sum_{j=1}^{n} d_j X_j$ less than D_j

$$X_j = 0 \text{ or } 1$$

4. The LP obtained by omitting all integers or 0-1 constraints as variables is called the LP Relaxation of IP.

For example, the LP relaxation of (1) is

Maximize $Z = 3X_1 + 2X_2$

Subject to $X_1 + X_2 \le 6$

$X_1, X_2 \ge 0$

And the LP relaxation of (2) is

Maximize $Z = X_1 + X_2$

Subject to $X_1 + 2X_2 \le 2$

$2X_1 + X_2 \le 1$

$X_1, X_2 \ge 0$

Any IP may be viewed as LP relaxation plus some additional constraints, the constraints that state which variables must be integers or be 0 or 1. Hence the LP relaxation is less constrained or more relaxed version of the LP. This means that the feasible region for an IP must be contained in the feasible region for the corresponding LP relaxation. LP relaxation for any IP is a maximize problem. This implies that;

Optimal value for LP relaxation ≥ Optimal Z value for LP.

This result plays a key role in the discussion of the solution of IP (Whaley and Smith, 2002).

Solution Techniques for IP

There is general purpose (independent of LP being solved) and computationally effective (able to solve large LP) algorithms (Simplex or interior point) for solving LP problems. However, there is no similar general purpose and computationally effective algorithm exists for solving IP problems. This means that IP problems are harder to solve than LP problems. There are, at least three different approaches for solving IP problems, although they are frequently combined into "hybrid" solution procedures in computational practice. They are;

- Enumerative techniques
- Relaxation and decomposition techniques
- Cutting planes approaches based on polyhedral combinatory.

The most effective and simplest approach to solving a pure IP problem is to numerate all finitely many possibilities. However, due to the "combinatorial explosion" resulting from the parameter "size" only the smallest instances could be solved by such an approach. Sometimes one can implicitly eliminate many possibilities by domination or feasibility arguments. Besides straight-forward or implicit enumeration, the most commonly used enumerative approach is called BRANCH and BOUND, where the "branching" refers to the enumeration part of the solution technique and "bounding" refers to the fathoming of possible solutions by comparison to a known upper or lower bound on the solution value (Land and Doig, 1960).

To obtain an upper bound on the problem (e.g. in maximization problem), the problem is relaxed in a way which makes the solution to the relaxed problem, relatively easy to solve. All commercial branch-and-bound codes relax the problem by dropping the integrality conditions and solve the resultant continuous LP problem over the set P. If the solution to the relaxed linear programming problem satisfies the integrality restrictions, the solution obtained is optimal. If the LP is infeasible, then so is the integer program. Otherwise, at least one of the integer variables is fractional in the LP solution. One chooses one or such fractional variables and "branches" to create two or more sub problems which exclude the prior solution but do not eliminate any feasible integer solutions. These new problems constitute "nodes" on a branching tree, and an LP problem is solved for each node created. Nodes can be fathomed if the solution to the sub problem is infeasible, satisfies all of the integrality restrictions, or has an objective function value worse than a known integer solution (Powell and Brennah, 1998).

Illustration

Consider the LP problem, (parameters measured in $m);

$$\text{Maximize } Z = 0.2X_1 + 0.3X_2 + 0.5X_3 + 0.1X_4$$

$$\text{Subject to} \qquad 0.5X_1 + 1.0X_2 + 0.1.5X_3 + 0.1X_4 \leq 3.1$$

$$0.3X_1 + 0.8X_2 + 1.5X_3 + 0.4X_4 \leq 2.5$$

$$0.2X_1 + 0.2X_2 + 0.3X_3 + 0.1X_4 \leq 0.4$$

$$X_j = 0 \text{ or } 1, \quad j = 1, 2, 3, 4.$$

What makes this problem difficult is the fact that the variables are restricted to integers (zero or one). If the variables are allowed to be fractional (takes all values between zero and one for example) then LP would be obtained which can easily be solved.

To solve this LP relaxation of the problem, the $x_j = 0$ or 1, j =1, 2, 3, 4 is replaced by $0 \leq x_j \leq 1$, j =1, 2, 3, 4. Then using MATLAB package gives the solution $x_2 = 0.5$, $x_3 = 1$, $x_1 = x_4 = 0$ of value 0.65 (i.e. the objective function value of the optimal linear programming solution is 0.65).

Now, the optimal integer solution is ≤ 0.65, i.e. this value of 0.65 is an upper bound on the optimal integer solution. This is because when the integrality constraint is relaxed, the solution value ends up with at least that of the optimal integer solution (and may be better).

Consider this LP relaxation solution. The variable x_2 which is fractional needs to be an integer. To remove this troublesome fractional value, two new problems can be generated:

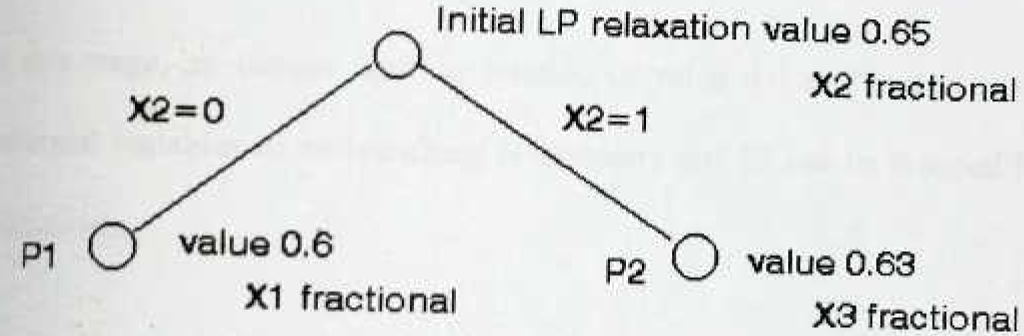
- original LP relaxation plus $x_2 = 0$
- original LP relaxation plus $x_2 = 1$

Then the optimal integer solution to the original problem is contained in one of these two new problems. This process of taking a fractional variable (a variable which takes a fractional value in the LP relaxation) and explicitly constraining it to each of its integer values is known as branching. It can be represented diagrammatically as below (in a tree diagram, which is how the name tree search arises).

The solution to these two new LP relaxations problems are given below:

- P1 - original LP relaxation plus $x_2 = 0$, solution $x_1 = 0.5$, $x_3 = 1$, $x_2 = x_4 = 0$ of value 0.6
- P2 - original LP relaxation plus $x_2 = 1$, solution $x_2 = 1$, $x_3 = 0.67$, $x_1 = x_4 = 0$ of value 0.63

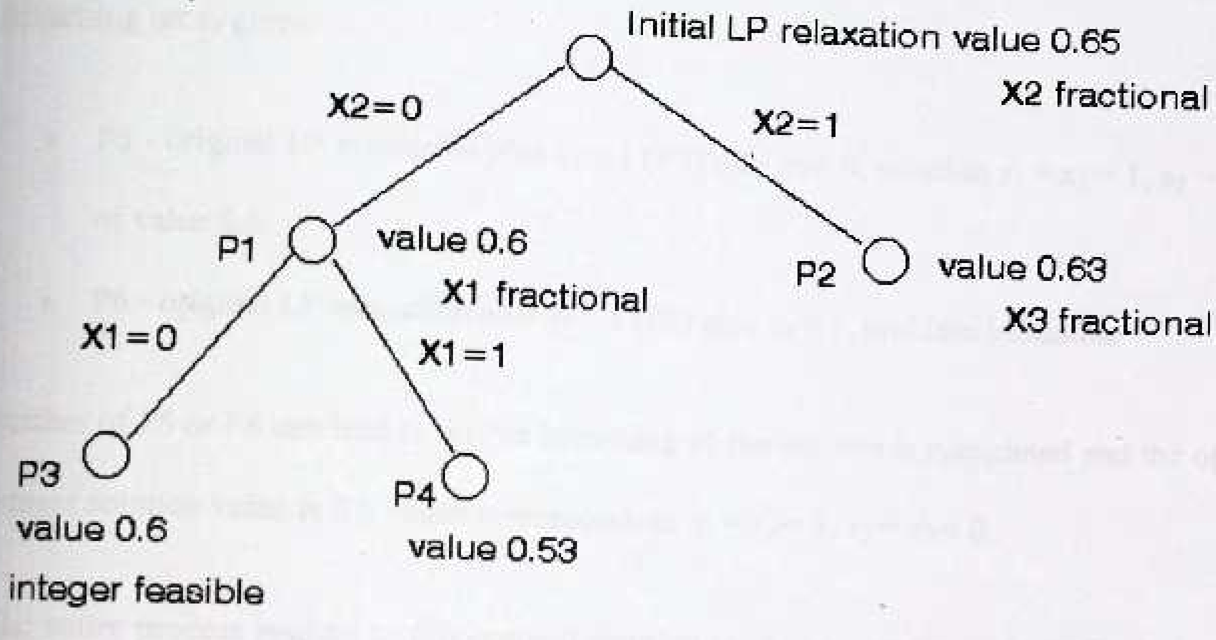
This can be represented diagrammatically as below.



To find the optimal integer solution, the process is repeated, i.e. choosing one of these two problems, choosing one fractional variable and generating two new problems to be solved.

Choosing problem P1 means branch on x_1 to get a list of LP relaxations as:

- P3 - original LP relaxation plus $x_2 = 0$ (P1) plus $x_1 = 0$, solution $x_3 = x_4 = 1$, $x_1 = x_2 = 0$ of value 0.6
- P4 - original LP relaxation plus $x_2 = 0$ (P1) plus $x_1 = 1$, solution $x_1 = 1$, $x_3 = 0.67$, $x_2 = x_4 = 0$ of value 0.53
- P2 - original LP relaxation plus $x_2 = 1$, solution $x_2 = 1$, $x_3 = 0.67$, $x_1 = x_4 = 0$ of value 0.63. This can again be represented diagrammatically as below.



At this stage, an integer feasible solution of value 0.6 at P3 is identified. There are no fractional variables so no branching is necessary and P3 can be dropped from the list of LP relaxations.

Hence, the new information about the optimal (best) integer solution is that, it lies between 0.6 and 0.65 (inclusive).

Considering P4, it has value 0.53 and has a fractional variable (x_3). However if the branching were to be on x_3, any objective function solution values that would be obtained after branching can never be better (higher) than 0.53. As already the integer feasible solution value is 0.6, P4 can be dropped from the list of LP relaxations since branching from it could never find an improved feasible solution. This is known as **bounding** - using a known feasible solution to identify that some relaxations are not of any interest and can be discarded.

Hence it is just left with:

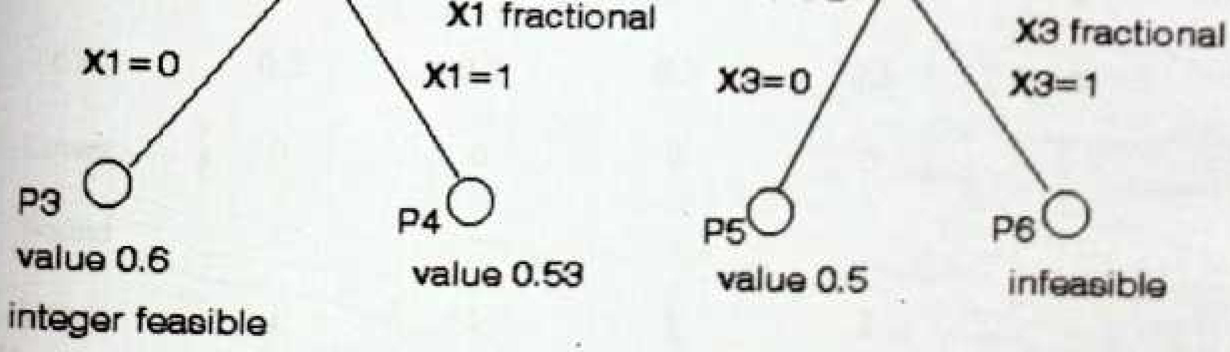
- P2 - original LP relaxation plus $x_2 = 1$, solution $x_2 = 1$, $x_3 = 0.67$, $x_1 = x_4 = 0$ of value 0.63

Branching on x_3 gives;

- P5 - original LP relaxation plus $x_2 = 1$ (P2) plus $x_3 = 0$, solution $x_1 = x_2 = 1$, $x_3 = x_4 = 0$ of value 0.5
- P6 - original LP relaxation plus $x_2 = 1$ (P2) plus $x_3 = 1$, problem infeasible

Neither of P5 or P6 can lead to further branching so the process is completed and the optimal integer solution value is 0.6 which corresponds to $x_3 = x_4 = 1$, $x_1 = x_2 = 0$.

The entire process leading to this optimal solution (and to prove that it is optimal) is shown graphically below.

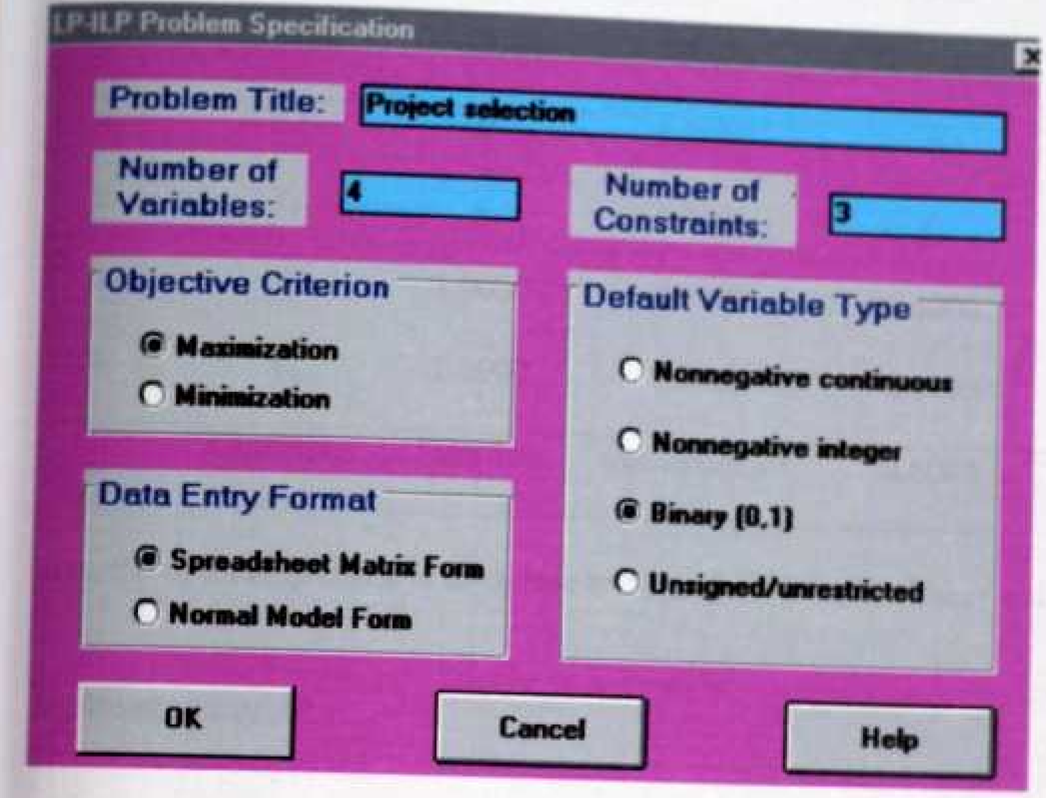


Thus, the optimal integer solution for this problem is $0.6m.

Alternative Solution to the Problem using QSB Software

The branch and bound method is the method used by the package.

The package input for the problem presented above is given below.



LP-ILP Problem Specification

Problem Title: Project selection

Number of Variables: 4

Number of Constraints: 3

Objective Criterion

- ● Maximization
- ○ Minimization

Default Variable Type

- ○ Nonnegative continuous
- ○ Nonnegative integer
- ● Binary (0,1)
- ○ Unsigned/unrestricted

Data Entry Format

- ● Spreadsheet Matrix Form
- ○ Normal Model Form

Variable	X_1	X_2	X_3	X_4	Direction	R. H. S
Maximize	0.2	0.3	0.5	0.1		
C_1	0.5	1.0	1.5	0.1	≤	3.1
C_2	0.3	0.8	1.5	0.4	≤	2.5
C_3	0.2	0.2	0.3	0.1	≤	0.4
Lower Bound	0	0	0	0		
Upper	1	1	1	1		

Bound				
Variable Type	Binary	Binary	Binary	Binary

The solution is shown below:

Date	Decision Variable	Solution Value	Unit Cost or Profit	Total Contribution	Reduced Cost	Basis Status
1	X_1	0	0.2000	0	0.2000	At bound
2	X_2	0	0.3000	0	0.3000	At bound
3	X_3	1.0000	0.5000	0.5000	0	basic
4	X_4	1.0000	0.1000	0.1000	0	basic
	Objective	Function	(maximum) =	0.6000		

Here, the optimal decision is to choose to do projects 3 and 4 (Baker and Harrell, 2007).

Applications of IP

The major application areas which IP can be applied are;

(a) manpower scheduling problems concerned with security personnel.

(b) church location problems

(c) capital budgeting problems

(d) traveling salesperson problems

(e) the cutting stock problems

(f) vehicle routing problems

IP Model for Project Selection under Multiperiod capital Rationing Problem

Based on the background information about IP, a formal formulation of an IP model for solving MA project selection problems is presented below.

Conditions

i. The problem facing MA is that, it cannot invest in all N-projects suitable for investment which run n-years.

ii. The project characteristics shows that $\sum d_{(i,j)}$ is greater than R_j where $d_{(I,j)}$ is the least capital requirement for j projects and R_j is the capital for investment.

The decision problem is that, which projects MA should select in order to maximize the total returns. To formulate IP for the MA problem, the following steps were used.

Step 1: Decision Variables

To define decision variables,

$$\text{let } X_j = \begin{cases} 1, & \textit{if MA invest in project } j \\ 0, & \textit{if MA does not invest in project } j \end{cases}$$

$j = 1, 2, \ldots N$

That is, the X_j are integer variables which must take one of the two possible values (zero or one). This represents Binary decision (e.g. do project = 1 or not to do project = 0)

Step 2: Constraints

To define the constraints,

let $d_{(i,j)}$ = capital requirements for j projects,

R_j = available capital for j projects for each year.

Then the constraints relating to availability of capital funds each year are;

$$\sum_{j=1}^{N} d_{(i,j)} X_j \leq R_j$$

$$i = 1, \quad 2, \ldots, m$$

$$X_j = 0 \text{ OR } 1$$

$$j = 1, \quad 2, \ldots, N$$

Step 3: Objective Function

Let the total profit be $\sum P_j X_j$

Then the total Return is maximized as;

Maximize $Z = \sum_{j=1}^{N} P_j X_j$

This gives a complete IP model for MA problem as;

` Maximize $Z = \sum_{j=1}^{N} P_j X_j$ (objective function)

Subject to $\sum_{j=1}^{N} d_{(i,j)} X_j \le R_j$ (constraints)

$$i = 1, \quad 2, \ldots, m$$

$X_j = 0 \text{ OR } 1$ (non-negative constraints) $j = 1, \quad 2, \ldots, N$

A more compact form is;

Maximize $Z = P_1 X_1 + P_2 X_2 + \ldots\ldots + P_N X_N$

Subject to

$$d_{(1,1)} X_1 + d_{(1,2)} X_2 + \ldots\ldots + d_{(1,N)} X_N \le R_1$$

$$d_{(2,1)} X_1 + d_{(2,2)} X_2 + \ldots\ldots + d_{(2,N)} X_N \le R_2$$

- -

- -

- -

$$d_{(m,1)}X_1 + d_{(m,2)}X_2 + \dots\dots + d_{(m,N)}X_N \le R_N$$

$$X_j = 0 \text{ or } 1$$

$$j = 1, \quad 2, \dots\dots, N$$

This completes the IP model.

Comments

- In writing down the complete IP, the inclusion of the information $x_j = 0$ or 1 (j = 1,...,N) serves as a reminder that the variables are integers
- The zero-one nature of the decision variable means that always a single term or item will be captured, i.e. a project is either accepted or rejected.
- The objective and constraints are linear (i.e. any term in the constraints/objective is either a constant or a constant multiplied by an unknown).Here, only a linear integer programming (IP with a linear objective and linear constraints) is under consideration. Non-linear integer programmings are, however, outside the scope of this thesis.

Extensions to the LP and IP Models

The extensions to the models include;

- projects of different lengths
- projects with different start/end dates
- adding capital inflows from completed projects
- projects with staged returns
- carrying unused capital forward from year to year
- mutually exclusive projects (can have one or the other but not both)
- projects with a time window for the start time.

For the amendment of LP and IP models to deal with these extensions, see Appendix A.

CHAPTER FOUR

ANALYSIS AND INTERPRETATION OF RESULTS

Introduction

In this chapter, the secondary data extracted from the published financial statements, monthly and yearly reports of SMA from 2001-2006 and other supporting documentary evidence obtained from the SMA department, will be analyzed with the help of the LP and IP models formulated in the third chapter. The data will be used to formulate suitable LP and IP objective function and decision variables that will assist the SMA to select viable projects within its capital limitation.

Implementation of LP Model

Applying LP model;

Maximize $Z = \beta_j X_j$

Subject to $\sum_{j=1}^{n} a_{(i,j)} X_j \le b_j$

$$i = 1, \quad 2, \ldots, m$$

$$0 \le X_j \le 1,$$

$$j = 1, \quad 2, \ldots, n,$$

to the SMA capital rationing data (Table 1), the Assembly's project selection problem can be formulated as LP as shown below:

Maximize $Z = 264X_1 + 719X_2 + 237X_3 + 217X_4 + 72X_5$

Subject to $100X_1 + 400X_2 + 250X_3 + 30X_4 + 10X_5 \leq 550$

$$100X_1 + 500X_2 + 200X_3 + 50X_4 + 20X_5 \leq 500$$

$$0 \leq X_j \leq 1, \; j = 1, \quad 2,\ldots,5$$

In this LP problem the assumption is that, fractional variable values will be acceptable. Since the amount required for each period is reasonably large, this will not cause too many problems.

Solution to the LP, using MATLAB Package

There are more than two variables in the LP problem, hence it is considered as large scale LP problem that can be solved easily by software package like MATLAB. The LP module in the MATLAB package is called "Linprog".

To solve an LP problem,

$$\min_x f^T X \text{ such that}$$

$$\mathbf{A} X \leq b$$

$$\mathbf{A}_{eq} X = b_{eq}$$

$$lb \leq X \leq ub$$

where *f, x, b, beq, lb,* and *ub* are vectors and A and Aeq are matrices, using linprog.

the Syntax is;

x = linprog *(f, A, b, Aeq, beq)*

x = linprog *(f, A, b, Aeq, beq, lb, ub)*

x = linprog *(f, A, b, Aeq, beq, lb, ub, x0)*

x = linprog *(f, A, b, Aeq, beq, lb, ub, x0,* options

[X fval] = linprog (...)

[X, fval, exitflag] = linprog (...)

[X, fval, exitflag, output] = linprog (...)

[X, fval, exitflag, output, lambda] = linprog (...)

Description

Linprog solves linear programming problems.

x = linprog *(f, A, b)* solves min $f(x)$ such that $Ax \leq b.$

x = linprog *(f, A, b, Aeq, beq)* solves the problem above while additionally satisfying the equality constraints Aeq.x = *beq*. Set *A= []* and *b= []* if no inequalities exist.

x = linprog *(f, A, b, Aeq, beq, lb, ub)* defines a set of lower and upper bounds on the design variables, x, so that the solution is always in the range $lb \leq x \leq ub$. Set *Aeq= []* and *beq= []* if no equalities exist.

x = linprog *(f, A, b, Aeq, beq, lb, ub, x0)* sets the starting point to *x0*. This option is only available with the medium-scale algorithm (the LargeScale option is set to 'off' using optimset). The default large-scale algorithm and the simplex algorithm ignore any starting point.

x = linprog *(f, A, b, Aeq, beq, lb, ub, x0,* options) minimizes with the optimization options specified in the structure options. Use optimset to set these options.

[X fval] = linprog (...) returns the value of the objective function fun at the solution *x: fval* = *f'(x).*

[X, lambda, exitflag] = linprog (...) returns a value exitflag that describes the exit condition.

[X, lambda, exitflag, output] = linprog (...) returns a structure output that contains information about the optimization.

[X, fval, exitflag, output, lambda] = linprog (...) returns a structure lambda whose fields contain the Lagrange multipliers at the solution x.

Input the parameters of the LP into the "linprog" as

$$f = [-264, -719, -237, -217, -72]$$

$$A = \begin{bmatrix} 100 & 400 & 250 & 30 & 10 \\ 100 & 500 & 200 & 50 & 20 \end{bmatrix}$$

$b = [550{,}500]$

$lb = zeros(5{,}1)$

$ub = ones(5{,}1)$

$b_{eq} = []$

$A_{eq} = []$

[*X*, *fval*, exitflag, output, lambda] = linprog *(f, A, b, Aeq, beq, lb, ub, [], [])*;

Enter

X, *fval*, lambda.ineqlin, lambda.lower

Enter

The solution to the LP problem is shown below;

Decision Variables	Solution Variables	Unit Cost or Profit	Total Contribution	Shadow Price	Reduction Cost
Market (X_1)	1.00	264.00	264.00	0	0
School (X_2)	0.66	719.00	474.54	1.438	0
Health (X_3)	0	237.00	0	0	50.60
Electrification (X_4)	1.00	217.00	217.00	0	0
Agriculture (X_5)	1.00	72.00	72.00	0	0

Optimal solution (Max. objective function) = *GH* ₵ 1027.5×10^2

Interpretation of Solution

The solution indicates the construction of a market place $(X_1 = 1)$, providing Agricultural inputs $(X_3 = 1)$, and electrification projects $(X_4 = 1)$ can be done by the SMA within its capital limitation. At the same time, 0.66 portions of a unit classroom block (X_2) can be initiated while $(X_3 = 0)$ indicates that the Assembly cannot put up health posts within the time frame with the limited capital. This investment plan uses all the funds available in year zero and year one.

The shadow prices indicate that the amount by which the NPV of the optimal plan (i.e. $GH₵\ 1027.5 \times 10^2$) could be increased if the budgetary constraints increases.

For every $GH₵\ 1 \times 10^2$ relaxation of the constraint in period two, $GH₵\ 1.438 \times 10^2$ extra NPV could be obtained. The shadow prices also indicate that extra funds in period zero are not required. That is, the marginal value or dual value is not needed in the first constraint.

The Reduced Cost of 50.6 in X_3 indicates the amount by which the objective coefficient for a variable X_3 needs to change before becoming non-zero is $GH₵\ 50.6 \times 10^2$.

The slack or surplus column gives, for a particular constraint, the difference between the Left Hand side of the constraint when evaluated at the LP optimal (i.e. evaluated at X_j) and the Right Hand side of the constraint. However, in the LP solution, all the constraints are tight or binding (i.e. have zero surplus or slack). None of the constraints is loose (i.e. have non-zero surplus). All these facts may give SMA management some guidance in their considerations of the various alternative sources of capital.

Sensitivity Analysis

As explained earlier on, sensitivity analysis or post optimal analysis permits the MA to observe the effect of changes in the parameters of the LP problem on the optimal solution. At this point, we shall study the impact of changing:

(a) the objective function coefficient (cost coefficient of our LP for multi capital rationing problem)

(b) the Right Hand Side (RHS) coefficient of a constraint of our LP model.

The MATLAB LP package provides the sensitive information (the Reduced Cost) and shadow prices. Hence, the data items concerns are varied and the LP resolved to see how the solution changes as certain parameters change.

Changing the RHS Coefficient of Constraints

The shadow price of the j^{th} constraint gives the amount by which the optimal Z-value is increased in the maximize problem if the RHS coefficient of constraints is increased by one. Hence, the b_2 is increased by one (i.e. 500 +1=501) and the LP resolved to get;

Decision Variables	Solution Variables	Unit Cost or Profit	Total Contribution	Shadow Price	Reduction Cost
Market (X_1)	1.00	264.00	264.00	0	0
School (X_2)	0.66	719.00	474.54	1.438	0
Health (X_3)	0	237.00	0	0	50.60
Electrification (X_4)	1.00	217.00	217.00	0	0
Agriculture (X_5)	1.00	72.00	72.00	0	0

Optimal Solution (maximum objective function) = GH¢ 1029

Interpretation of Solution

At the optimal solution, a unit change in the constraint X_3 does not affect the solution values and reduced cost or opportunity cost.

Also, a unit increased in the capital funds for the constraint (Health Post (X_3)) has created a marginal value of 1.5 in the optimal objective function value (NPV) for a project investment. That is, the optimal objective function value increased from 1027.6 to 1029, with a difference of 1.5 which is approximately equal to the shadow price of 1.438 for the constraint X_3.

From this observation, it can be deduced that in the maximization problem, if RHS of the j^{th} constraint is increased by an amount Δb_j, then assuming the current optimal solution, the new optimal Z-value can be found from;

New optimal Z-value = Old optimal Z-value + Δb_j (constraint j shadow prices)

This analysis is very useful in planning because it enables the management to identify the most sensitive parameters or elements in the projects. Once the elements of the projects are identified, further analysis and study can take place on these elements trying to establish the likelihood of the variability and range of values that might be expected to make a more reasoned decision whether or not to proceed with the project.

Changing the Coefficient of the Objective Function

The Reduced Cost, 50.6 in X_3 row shows the amount by which the objective function coefficient for the variable X_3 should change to make it non-zero. Hence the coefficient of X_3 in the objective function is altered by +50.6 and the LP problem resolved.

Solving by the MATLAB software gives:

Decision Variables	**Solution Variables**	**Unit Cost or Profit**	**Total Contribution**	**Shadow Price**	**Reduction Cost**
Market (X_1)	1.00000	264.00	264.00	0	0
School (X_2)	0.38042	719.00	273.52	1.438	0
Health (X_3)	0.69895	287.60	201.20	0	0
Electrification (X_4)	1.00000	217.00	217.00	0	0
Agriculture (X_5)	1.00000	72.00	72.00	0	0

Optimal solution (maximum objective function) =GH¢ 1027.5×10^2.

Explanation

Addition of the reduced cost of the 50.6 on the row of variable (X_2) to its corresponding coefficient in the objective function effects no changes in the shadow prices with solution values for variables X_1, X_4, X_5 and the optimal objective function. However, there were sharp variations in some the optimal solution values. The coefficient of variable (X_2) decreased from 0.662 to 0.38042 while variable X_3 increased from 0 to 0.69895. Thus, increasing the NPV per unit on variable X_3 impacts a sharp change on the optimal solution. Given the sensitivity analysis of one or more of the key factors of projects like this, the management's task is to decide whether the project is worthwhile.

Implementation of IP Model

IP works reasonably well where there is a hierarchy of decisions to be made. For instance, building a new factory enables various consequential activities to take place. Although the solution depends on the values of all the decision variables, setting the values of the most important ones restricts the values of the decision variables representing the consequential activities. In such a case, the IP code will usually be worked out for itself which are the most important decisions and determine those first or it can be assisted by specifying the hierarchy of decisions explicitly.

Based on these IP principles, the IP model for solving multiperiod capital rationing problems;

$$\text{Maximize } Z = \sum_{j=1}^{N} P_j X_j$$

$$\text{Subject to} \quad \sum_{j=1}^{N} d_{(i,j)} X_j \le R_j \,,$$

$$X_j = 0 or 1 \qquad j = 1, \; 2, \ldots, N$$

is applied to the SMA small scale project selection problem data (table 2) as follows:

Purpose

The SMA project manager wishes to select from N-potential projects for investments so that by the end of n-years, the project selected will maximize returns from these batch of projects with regards to his capital limitation. This problem can be formulated, (using the data in table 2) as zero-one IP problem (do a project = 1, do not project = 0).

Putting the data into the IP model gives:

$$\text{Maximize } Z = 50X_1 + 30X_2 + 50X_3 + 10X_4 + 20X_5$$

$$\text{Subject to } 50X_1 + 10X_2 + 15X_3 + 10X_4 + 10X_5 \le 80$$

$$30X_1 + 80X_2 + 15X_3 + 40X_4 + 0X_5 \le 145$$

$$20X_1 + 20X_2 + 30X_3 + 10X_4 + 10X_5 \le 90$$

$$20X_1+20X_2+40X_3+10X_4+20X_5 \le 100$$

$$60X_1+30X_2+60X_3+0X_4+50X_5 \le 165$$

$$40X_1+60X_2+0X_3+10X_4+10X_5 \le 80$$

$$X_j = 0 \text{ or } 1,\ j=1,\ \ 2,\dots,5$$

where KVIP = X_1, Refuse = X_2, Water = X_3, Taxi = X_4 and Toll Booth = X_5

This is a Binary decision problem and can be solved easily by MATLAB software as shown below:

Solution to IP, using MATLAB

The IP module in the MATLAB package is called "bintprog". To solve an IP problem of the form;

$\min\limits_{X} f^T X$ such that

$AX \le b$

$A_{eq} = b_{eq}$

All variables are integer;

where *f*, *b*, and *beq* are vectors, *A* and *Aeq* are matrices, and the solution *x* is required to be a binary integer vector -- that is, its entries can only take on the values 0 or 1. Using "bintprog", the syntax is

x = bintprog *(f)*

x = bintprog *(f, A, b)*

x = bintprog *(f, A, b, Aeq, beq)*

x = bintprog *(f, A, b, Aeq, beq, x0)*

x = bintprog (f, *A*, *b*, *Aeq*, *beq*, *x0*, options)

[X, fval, exitflag] = bintprog (...)

[X, fval, exitflag, output] = bintprog (...)

Description

x = bintprog (f) solves the binary integer programming problem

$\min_{x} imize\ fX$

x = bintprog *(f, A, b)* solves the binary integer programming problem

$\min_{x} imize\ fX$ such that $AX \le b$

x = bintprog *(f, A, b, Aeq, beq)* solves the preceding problem with the additional equality constraint. $A_{eq} = b_{eq}$

x = bintprog *(f, A, b, Aeq, beq, x0)* sets the starting point for the algorithm to *x0*. If *x0* is not in the feasible region, bintprog uses the default initial point.

x = bintprog *(f, A, b, Aeq, Beq, x0*, options) minimizes with the default optimization options replaced by values in the structure options, which you can create using the function optimset.

[x, fval] = bintprog (...) returns fval, the value of the objective function at *x*.

[X, fval, exitflag] = bintprog (...) returns exitflag that describes the exit condition of bintprog.

[X, fval, exitflag, output] = bintprog (...) returns a structure output that contains information about the optimization.

Input the parameters of IP into the "bintprog" as: $f = [-50,-30,-50,-10,-20]$

$$A = \begin{bmatrix} 50 & 10 & 15 & 10 & 10 \\ 30 & 80 & 15 & 40 & 0 \\ 20 & 20 & 30 & 10 & 10 \\ 20 & 20 & 40 & 10 & 20 \\ 60 & 30 & 60 & 0 & 50 \\ 40 & 60 & 0 & 10 & 10 \end{bmatrix}$$

$$b = [80,145,90,100,165,80]$$

[X, fval, exitflag, output] = bintprog *(f, A, b)*

The solution to the IP problem (output from bintprog) is shown below;

Decision Variable	Solution Value	Unit Cost or Profit	Total Contribution	Reduced Cost
KVIP (X_1)	1	50	50	0
Refuse Facility (X_2)	0	30	0	30
Water Borehole (X_3)	1	50	50	0
Taxi Circle (X_4)	1	10	10	0
Toll Booth (X_5)	0	20	0	20

Optimal objective function value = *GH¢110 X* 10^2.

Interpretation of Solution

The optimal decision is to choose to do projects; KVIP, Water Borehole and Taxi Circle whilst the SMA cannot provide Refuse Facilities and Toll Booths within its capital limitation for the next five years unless the capital investment is reviewed.

The optimal decision achieves maximum returns of *GH¢110 X* 10^2 instead of *GH¢160 x* 10^2. That is, the SMA has been able to recover about sixty-nine percent (69%) of the targeted returns from the batch of projects selected. It is evident that the model has assisted the project manager to select a large number of the viable projects that can maximize profit. This is better than relying on an ad-hoc judgmental approach to the project investment.

The model can be used for sensitivity analysis, for instance, to examine how sensitive the project selection decision is to changes in the parameters of the model.

Findings

(i) The LP and IP models revealed the following facts about project investment parameters. The models;

- Enabled the SMA to maximize profit rather than depending on an ad-hoc judgmental approach to project investment.
- Enabled the SMA to deal with a larger number of viable investment opportunities.
- Assisted the SMA project manager to see that a unit change in the capital funds constraint x_j could create marginal value in the optimal objective function (NPV).
- A change in the coefficient of the objective function by the reduced cost could cause a sharp variation in some of the coefficient of constraints x_j. A unit increase in the NPV would impact some changes in the optimal solution.

(ii) The study has shown that in the project selection problems where proportions or fractional parts of a projects and a whole project(s) are desired to be initiated within a period, the best decision tool is the LP model. Whenever only a whole project is desired to be invested, the IP model is the suitable decision making tool.

(iii) It has also been found that the following computer software packages:

(a) MATLAB

(b) QSB package

(c) Microsoft Excel 2003,

could be used to analyze and solve the problem at stake.

CHAPTER FIVE

SUMMARRY, CONCLUSIONS AND RECOMMENDATIONS

Introduction

The results from the previous chapters are summarized in this chapter. Conclusions, discussions and recommendations are made about the findings.

Summary

The primary objective in this thesis was to design linear programming and integer programming models for solving Sunyani Municipal Assembly's projects selection under multi-period capital rationing problems.

A second objective was to maximize the return from the batch of the projects selected with regard to the capital limitation, and also use the models to carry out sensitivity analysis on the project parameters in order to assist project managements decide effectively which projects are worth undertaking.

Data used for this thesis is purely secondary data extracted from financial statements, annual reports, monthly reports, daily newspapers and other relevant documents from the Assembly. The financial ratios such as NPV, Profitability Index, and also LP and IP models were used in the analysis of the data. The discount factors at cost of capital at ten percent (10%) for each cash flow for each project, NPV's at percent (10%) and relative profitability index (PI) were computed and the results tabulated (Table 1).

The LP model is designed to solve SMA large scale project selection problems and this produces the optimal solution quantities (i.e. the projects to be initiated), the value of the objective function (i.e. the total NPV) and opportunity costs of the binding constraints. On the other hand, selection of the small scale project problems is solved effectively by IP model or quantitative modelling techniques.

Factors which come into play in choosing which IP solution method is appropriate are:

i. The size of the IP (variables and constraints)
ii. Time available to build the model (formulation plus solution algorithm)
iii. Time available for computer solution once the model has been built
iv. Experience

The solutions to both LP and IP models can be given by software such as:

i. MATLAB
ii. QSB Package
iii. Microsoft Excel 2003

The model can be used to carry out sensitivity analysis, for instance, to examine how sensitive the project selection decision is to change in the parameters of the model. This invariably helps the project management to decide effectively on the projects which are worth undertaking.

Discussions and Conclusions

As explained earlier on, the idea behind the profitability index (PI) is that the PI provides the subset of projects that maximizes the aggregate NPV. However, this is not always the case. The projects were ranked with the most highly PI first i.e. Health Post ninety two percent (92%) viable, followed by Market Circle eighty two percent (82%), School Building eighty percent (80%) and Agriculture seventy seven percent (77%). This provided the highest aggregate NPV value of GH¢1392. This value is much higher than the LP model optimal objective function of GH¢1027.5. The total capital available for investment is GH¢1050. The PI thus leads to wrong conclusion, a decision that could have reduced the SMA wealth. The larger coefficient of variation of Agriculture eighty two percent (82%) and Market Circle seventy seven percent (77%) means that the probability of selecting these projects are more certain than others.

The LP model assisted the project manager to accept the hundred percent (100%) construction of market place, hundred percent (100%) agriculture inputs, hundred percent (100%) electrification projects and sixty six percent (66%) construction of classroom blocks but failed to accept the construction of health posts at the total cost of GH¢ 1027.5x 10^2 against the investment capital of GH¢1050 x 10^2 (Table 1).

After post-analysis of the projects' parameters using the LP model, the trend changed. The SMA tend to accept hundred percent (100%) construction of market place, thirty eight percent (38%) construction of classroom block, 69.9% of construction of health post, hundred percent (100%) of both electrification and agriculture inputs at the cost of GH¢ 1027.5 x 10^2. This yielded an acceptable optimal decision.

Much of the information obtainable as by-product of the solution of the LP model can be useful to management in estimating the effect of changes without going to the expense of resolving the LP. Similarly, using the IP model, the optimal decision for maximizing the returns is to do the projects: KVIP, Water Boreholes and Taxi Circle at the cost of GH¢ 110 x 10^2.

One helpful observation is that, the mathematical models support project management in the following ways. The models will enable the project mangers in:

(i) Problem identification, that is, diagnosis of the problem from its symptoms if not obvious, delineation of the subproblem to studies, establishment of objectives, limitation and requirements.

(ii) Formulation of project selection problem as LP or IP.

(iii) Model validation. This involves running the algorithm for the model on the computer in order to ensure that the input data is free from errors.

(iv) Solution of the model, that is, standard computer packages or especially developed algorithms can be used to solve the model. The solutions are many under varying assumptions to establish sensitivity.

(v) Implementation, that is, the implementation of the results of the study or algorithm for solving the models serves as an operational tool.

The advantages of using a software package to solve LP and IP models, rather than a judgmental approach to project selection problem are:

(i) Actually to maximize profit, rather than believing that the judgmental solutions maximize profit. This may end up having bad judgment.

(ii) Making the project selection decision one that can be solved in a routine operational manner on a computer, rather than having to exercise judgment each and every time solution to a problem is desired.

(iii) Those that can be appropriately formulated as LP are almost always better solved by computers than by people.

(iv) Carry out sensitivity analysis very easily using a computer.

Recommendations

To encourage the use of LP and IP models to solve project selection problems:

- The decision rule where capital rationing exists is to maximize the return from the project(s) selected rather than simply accepting or rejecting decisions of the projects in isolation
- Studies have shown that in the project selection problems where proportions or fractional parts of a projects and a whole project(s) are desired to be initiated within a period, the best decision tool is the LP model. Whenever only a whole project is desired to be invested, the IP model or quantitative modelling techniques is the suitable decision making tool.
- Due to the large amount of data involved and the complexity of the mathematical techniques involved, project managers are highly implored to use computer software packages such as Matlab, QSB and Microsoft excel to solve capital rationing problems, particularly for risk evaluation (NPV) and sensitivity analysis.
- The models can be modified to deal with the following extensions:
    i. projects of different lengths
    ii. projects with different start/end dates
    iii. adding capital inflows from completed projects
    iv. projects with staged returns
    v. carrying unused capital forward from year to year
    vi. mutually exclusive projects (can have one or the other but not both)
    vii. projects with a time window for the start time.

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APPENDIX A

EXTENSIONS TO THE MATHEMATICAL MODELS OF PROJECT SELECTION

Consider the IP model for MA project selection problems:

Maximize $Z = \sum_{j=1}^{N} P_j X_j$ (objective function)

Subject to $\sum_{j=1}^{N} d_{(i,j)} X_j \le R_j \qquad i = 1, \ 2, \ldots, m$

$X_j = 0 \ \text{ OR } \ 1$ (non-negative constraints)

$j = 1, \ 2, \ldots, N$

This basic model can be modified to deal with extensions or projects with the following characteristics:

Projects of different lengths

Projects with different lengths are easily dealt with, just set their capital requirement in any year in which the project does not exist (i.e. has not started or has already ended) to zero. For example if project X_1 only runs for 2 years (instead of 3 years) and hence finishes in year 2 then the capital requirement constraint for year 3 becomes (let $j = 1,2,3,4$):

$$0X_1 + d_{(3,2)} X_2 + d_{(3,3)} X_3 + d_{(3,4)} X_4 \le R_3$$

Projects with different start/end dates

Projects with different start/end dates are dealt with in a similar manner as projects of different lengths, just set their capital requirements in any year in which they do not exist (i.e. they have not started or have already ended) to zero. For example if project X_1 starts and ends in year 2; project X_2 starts in year 2 and project X_4 ends in year 2 then the capital requirement constraints become:

$$0X_1 + 0X_2 + d_{(1,3)}X_3 + d_{(1,4)}X_4 \le R_1 \text{ (Year 1)}$$

$$d_{(2,1)}X_1 + d_{(2,2)}X_2 + d_{(2,3)}X_3 + d_{(2,4)}X_4 \le R_2 \quad \text{(Year 2)}$$

$$0X_1 + d_{(3,2)}X_2 + d_{(3,3)}X_3 + 0X_4 \le R_3 \quad \text{(Year 3)}$$

Adding capital inflows from completed projects

If project X_1 finishes in year 2, and all of the return from project X_1 is available as capital in year 3 then this can be formulated by changing the capital requirement constraint for year 3 to:

$$0X_1 + d_{(3,2)}X_2 + d_{(3,3)}X_3 + d_{(3,4)}X_4 \le R_3 + d_{(3,2)}X_1$$

It can be observed that, the $0X_1$ above as project X_1 finishes in year 2 and hence have no capital requirement in year 3.

A question arises here in that the return from a project is put into the capital for future years then should the counting the return comes from the project in the objective function as well? One way to address this for project X_1 here is to say that the return is split into two - one part y_1 (say) that is counted as return taken in the objective and one part y_2 (say) that is taken as return used for future capital in year 3. Then amending the formulation gives

$$y_1 + y_2 = d_{(3,1)} X_1$$

a balancing equality equation to correctly account for the return (since the chose may be not to do project X_1)

$$0X_1 + d_{(3,2)}X_2 + d_{(3,3)}X_3 + d_{(3,4)}X_4 \leq R_3 + y_2$$

to account for the capital added in year 3

Maximize $Z = y_1 + P_2X_2 + P_3X_3 + P_4X_4$

to account for the return declared as profit.

It is observed that, y_1 and y_2 (both ≥ 0) are continuous (fractional) variables so adding them in this way yields a mixed-integer program (MIP).

Solving this MIP numerically, would yield optimal split between the return from project X_1 as return in the objective (y_1) and reinvesting it as available capital in year 3 (y_2).

Projects with staged returns

In this extension, a project gives a return at various stages over its lifetime, and this return can (perhaps) be used as capital to fund ongoing (or new) projects. To illustrate how this can be formulated consider project X_1 which gives a total return of P_1. Suppose now that this project gives a return of q (such that $q + S = P_1$) at the end of year 2, and the remaining return of S in year 3. Suppose further that all of this "early" return can be used as available capital in year 3. Then the capital requirement constraint in year 3 becomes:

$$d_{(3,1)}X_1 + d_{(3,2)}X_2 + d_{(3,3)}X_3 + d_{(3,4)}X_4 \leq R_3 + q$$

Carrying unused capital forward from year to year

In the example as currently formulated there is capital available in each year (R_1 in year 1, R_2 in year 2 and R_3 in year 3). In any particular year all of this capital may not be consumed by the projects that will be chose to do. Suppose that it is allowed to carry forward (from year to year) $r\%$ of any capital that is unused. To formulate this, introduce linear (fractional) variables C_1 and C_2 (≥ 0) with C_1 being the unused capital in year 1 and C_2 being the unused capital in year 2. Then the constraints of the problem become:

$$d_{(1,1)}X_1 + d_{(1,2)}X_2 + d_{(1,3)}X_3 + d_{(1,4)}X_4 + C_1 = R_1 \text{ (Year 1)}$$

$$d_{(2,1)}X_1+d_{(2,2)}X_2+d_{(2,3)}X_3+d_{(2,4)}X_4+ C_2 = R_2 + 0.01C_1r \text{ (Year 2)}$$

$$d_{(3,1)}X_1 + d_{(3,2)}X_2 + d_{(3,3)}X_3 + d_{(3,4)}X_4 = R_3 + 0.01C_2r \text{ (Year 3)}$$

Note that, years 1 and 2 have been employed in the equality relationship:

Capital used + unused capital = capital available

The introduction of additional variables (C_1 and C_2) makes the task of formulating the problem easier.

Mutually exclusive projects (can have one or the other but not both)

Suppose that projects X_3 and X_4 are mutually exclusive, i.e. the MA can choose to do one, or other, of these projects but not both. Then this can be formulated by adding to the problem the constraint:

$X_3 + X_4 \leq 1$

This allows the MA to do neither of the projects $(X_3 = X_4 = 0)$. If the managements wish to insist that they should do exactly one of these projects then

$$X_3 + X_4 = 1$$

Projects with a time window for the start time

Suppose that project X_1 can start either in year 1, when it has the characteristics given above, or in year 2, when it has a different characteristic - still the same return of P_1 but a capital requirement of k_1 in year 2 and k_2 in year 3, where k_1, k_2 are different capital requirements. Then this can be formulated by introducing a new zero-one variable y with $y = 1$ representing choosing to do project X_1 starting in year 2, $y = 0$ representing not choosing to do project X_1 starting in year 2. Then the capital requirement constraints for years 2 and 3 become:

$$d_{(2,1)}X_1 + d_{(2,2)}X_2 + d_{(2,3)}X_3 + d_{(2,4)}X_4 + k_1 y \leq R_2 \text{ (Year 2)}$$

$$d_{(3,1)}X_1 + d_{(3,2)}X_2 + d_{(3,3)}X_3 + d_{(3,4)}X_4 + k_2 y \leq R_3 \text{ (Year 3)}$$

The MA also needs to add a constraint to prevent project X_1 being started at more than one start time (i.e. project X_1 starting in year 1 and project X_1 starting in year 2 are mutually exclusive)

$$X_1 + y \leq 1$$

and the objective becomes

$$\text{Maximize } Z = P_1X_1 + P_2X_2 + P_3X_3 + P_4X_4 + P_1y.$$

APPENDIX B

THE REVISED SIMPLEX ALGORITHM

Consider the LP,

Maximize $\sum_{j=1}^{N} C_j X_j$

Subject to $\sum_{j=1}^{N} a_{(i,j)} X_j \le b_j$, $i = 1, 2\ldots m$

$$X_j \ge 0 \,, j = 1, 2 \ldots N$$

To show how to create a tableau for any set of basic variables, *BV*, we first describe the following notation (assumed the LP has *m* constraints).

BV = any of basic variables (the first element of *BV* is the basic variable in the first constraint, the second variable *BV* is the basic variable in the second constraint, and so on. Thus BV_j is the basic variable for constraint *j* in the desired tableau).

b = right hand side vector of the original tableau's constraints.

a_j = column for X_j in the constraints of the original problem.

$B = m \times m$ matrix where the *jth* constraint is the column for BV_j in the original constraints.

C_j = coefficient of X_j in the objective function.

C_{BV} = $1 \times m$ row vector whose *jth* element is the objective function coefficient for *BVj*

u_i = $m \times 1$ row vector with *ith* element 1 and all other elements equal to zero.

Now it can be deduced that:

$B^{-1}a_i$ = column for X_j in *BV* tableau. .... (1)

$C_{BV}\,B^{-1}a_i - C_j$ = coefficient of X_j in row zero. .... (2)

$B^{-1}b$ = right hand side of constraints in *BV* tableau. .... (3)

$C_{BV}\,B^{-1}u_i$ = coefficient of slack variable S_j in *BV* in row zero .... (4)

$C_{BV}\,B^{-1}(-u_i)$ = coefficient of excess variable e_i in *BV* row. .... (5)

$M + C_{BV}\,B^{-1}u_i$ = coefficient of artificial variable a_i in *BV* in row zero (in maximum problem). .... (6)

$C_{BV}\,B^{-1}\,b$ = right hand side of *BV* row. .... (7)

If the *BV*, B^{-1} and the original tableau are known, formulae (1) – (7) will enable the reader to compute any part of the simplex tableau for any set of basic variables *BV*. This means that if a computer is programmed to perform the simplex algorithm, all the computer needs to store on any pivot is the current set of basic variables, B^{-1}, and the initial tableau. Then (1) – (7) can be used to generate any portion of the simplex tableau. This idea is the basis of the Revised Simplex Algorithm.

APPENDIX C

SUMMARY OF IMPORTANT FORMULAS

Payback Period (PBP)

This is the cost of investment divided by the cash flow period.

i.e. $PBP = \frac{cost\ of\ investment}{cash\ flow\ period}$ .... (1)

Net Present Value (NPV)

The Net Present Value (including the time of money) of initial and future flows is given by the equation

$NPV = \sum_{t=1}^{N}\left(\frac{C_t}{(1+r_p)^t}\right)$ .... (2)

Internal Rate of Return

This is the interest or discount rate of which the future net cash flows equals the initial cash outlay.

$O = \sum_{t=1}^{N}\left[\frac{C_t}{(1+IRR)^t}\right] - C_o$ .... (3)

Profitability Index (P.I)

This is the NPV per unit initial investment.

i.e. $\mathrm{Pr}\,ofitability\ Index = \frac{NPV}{Initial\ \ Investment}$ .... (4)

Net Cash Flow = Cash Inflow – Cash Outflow ....(5)