KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY - KUMASI

THE NATURE OF THE LOGISTIC FUNCTION AS A NONLINEAR DISCRETE DYNAMICAL SYSTEM

By

Patrick Akwasi Anamuah Mensah

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

Patrick Akwasi Anamuah Mensah ....

Signature Date

Student(PG2550214)

Certified by:

Rev.Dr.W.Obeng-Denteh ....

Supervisor Signature Date

Certified by:

Dr. R.K Avuglah ....

Head of Department Signature Date

Dedication

I whole-heartedly dedicate this work to my wonderful sweet wife and children. I also dedicate this work to my one and only sweet mother Mrs. Veronica Bosua and my dear uncle Mr. Albert Kwaku Antobam. Finally, I dedicate to my siblings and family members for their unwavering supports and motivation.

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Abstract

The logistic equation is a model of population growth first introduced by PierreFrancios Verhulst. This model is a continuous form that depends on time and a possible way of restructuring this continuous form of the equation into a discrete equation is known as the logistic map.

The discrete logistic equation as a model is written as: $x_{n+1} = rx_n(1-x_n)$ where n = 0,1,2,3...,x_n is the state at the discrete time n andr is the control parameter which operate within any given range and as a very simple example for nonlinear map in dynamics, it changes in behavior moving from one regime to another regime depends on the adjustment or variation of the control parameter. For some parameter values of r the logistic map display periodic behaviour (period-1 orbits "fixed point", period-2 orbits and period-n orbits), and for others, it displays chaotic behavior. This research seeks to explore and look into the behavior of the equation as r the parameter keeps increasing within a specified interval 1 to 4 inclusive [1, 4]. A geometrical procedure to examine the logistic function behavior through graphical analysis irrespective of the variations of the parameter r gives a pictorial nature of the map.

In the displayed of the logistic map into periodic-orbits points through iterating, it shows the attracting and repelling behavior which depend on the parameter values. Beyond period-2 orbits are various kinds of period doubling that shows the logistic map behavior from the periodic regime into chaotic. Diagrammatically, bifurcation as commonly used in nonlinear dynamics gives a better behavior of the logistic map in dynamics as the control parameter r is varied. So, ideally in this research I seek to determine each periodic solution of range of the parameter r values by performing analysis for the periodic orbits so as to get a very good understanding of the bifurcation that are encountered in the logistic map.

Chaotic regime as the final regime of the logistic map as observed in the working lies on the increase of the parameter r within a certain range or for a particular value. Through higher periodic oscillating unstableness occurs leading to chaos, and it shows that the final behavior of the logistic map is the chaotic regime. In studying the logistic map as a good and a perfect model into chaos, the concepts of iteration and orbits was also studied carefully as a foundation to the build-up of the main work. Work on iterating function and orbits were studied vigorously. Hence concluding that iterations occur when a particular function is evaluated over and over with its outputs starting with an initial inputs, the outputs obtained from each evaluation form the orbits in sequential order.

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Chapter 1

INTRODUCTION

1.1 Introduction

Chapter one is devoted to the introduction of the study. The chapter included the background of the study of dynamics, nonlinear dynamical system, chaos, logistic map, objective, research aim and significance, scope, limitations and the structural of the thesis.

1.1.1 Dynamical System

A system that is made up of an abstract phase space or state space 'where the state phase is simply applying numbers to stand in for the state of the system', thereby describing the actual state at any given time or through it coordinates, and where there is a rule in dynamics that shows the early future of variables in all the state depicting just the current values of the state variable is called a dynamical system. This present values where the first is also called the initial point/value gives a clear indication as to how a state space will behave in the
future.

Mathematically, initial value problem which scientist consider in describing a dynamical system helps in predicting it state of condition is base on time. The consequence is that time cannot be ignored and a state at a particular time erupts to a state or possibly a collection of states at a later time. Thus the states of it condition is been arranged by time, where time is classified as a single entity.

Dynamical system in its generality is made up of the following tools:

1. A "phase space"X whose elements or points represent possible states of the system.

2. And a "Time"which may be discrete or continuous type. It may extend either only into the future which may be termed irreversible or noninvertible Processes or into the past as well as the future and in other words, reversible or invertible processes. The sequence of time moments for a reversible discrete-time process is in a natural correspondence to the set of all integers; irreversibility corresponds to considering only nonnegative integers. Also, for a continuous-time process, time is quite different in the way it is been represented by the set of all real numbers in the reversible case and by the set of non-negative real numbers for the irreversible case (Katok and Hasselblatt, 1995).

Deterministic as in dynamical systems is as the results of every state uniqueness. One may also say that if the consequents of probability distributions are as a result of stochastic or random. Hence deterministic system exhibit randomness (stochastic behaviour) (a very good explanation is when a coin toss has double consequents with equal probability for each initial condition). It is deterministic because of an initial value that gives a definite results of the actual state of the system.[Obeng-Denteh, 2012]

Dynamical system as studies has shown is either discrete or continuous time. Therefore deterministic system as in dynamical system is defined using map because of it discrete time nature that is, $x_1 = f(x_0)$, where the state x1is an outcome from the initial conditionx0 at the preceding time value. So the dynamical system that has discrete-time is seen as what we called the map.

1.1.2 Non-Linear Dynamical System

There are two very important tools that make up dynamical system as stated by many physical scientists:

1. A dynamic or rule,"which is basically about system evolution"

2. An initial state/condition “which begins a system and how it will end in the future or the past ”.

According to Tufillaro, Abbort and Reilly 2013, difference and differential equations are the best rule and good way for explaining and describing natural phenomena. And that differential equations has really contributed in the outcome all important theories of physics in terms of how it is been formulated. Hence V. I. Arnold a mathematicians made a very constructive statement that: in scientific mathematical philosophy differential equations is a basis,“in the scientific world.(Tufillaro, Abbott, Reilly, 2013)

With the help of Newton and Leibniz and the scientific philosophy started with calculus and the present day scientist are also contributing their quota (Tufillaro et al, 2013).

Dynamical system is a system of motion that is model through time as to how changes occur through a reliable rule or state given an initial value. Thereby, using time in which are discrete and also depends on integers, or if continuous then varies over real values. Hence the systems are deterministic in nature, but not random, so previous condition help in determining the future state of the system as functions.[Akin, 2007].

Qualitative study of differential equations has also resulted in dynamical systems theory and nonlinear dynamics, with the reason behind it as a way to understand and predict the movement or in other words motion around us: examples the planets with it orbits, when a string vibrate, when the surface of a pond produces ripples and the lasting pattern of the weather as it evolves are all a result of it implication. Scientific philosophy in it first two hundred years, has really played their part in formulating some of the major rules in the world, and few to mention are Newton, Hamilton, Poincare, Leibniz, Maxwell etc but it results are limited in findings their outcomes.

Nonlinear system as dynamics is so profound in the sense that the system may contain multiple attractors. It also depend on it initial condition or state and it whole new set of condition comes in line with the way the parameters are controlled. (Socolar, 2006)

According to Socolar J. E. S (2006) the strange attractor that comes out of the nonlinear is the state space structure associated with the phenomenon called chaos. Hence chaos do happens when there is an unprecedented pattern where the behavior is abnormal or unpredictable due an increase in the control parameter.

1.1.3 Chaotic

In 1963, at the Massachusetts Institute of Technology a meteorologist named Edwards Lorenz, was attempting to simulate weather patterns on a computer using a model derived from twelve relatively complicated equations. Rather than to repeat the entirety of a computer run from the previous day, he decided to begin the calculation anew using data output from partway through the previous run. However, instead of inputting the data to the six-decimal-place accuracy that was retained by the computer, he utilized the three-decimal-place accuracy that the computer printout provided. Much to his surprise, the results of the run were entirely different from what they had been before. He was seeing sensitivity to initial conditions. Even though the equations were fixed, just a minute difference in the initial states caused a big alteration in the outcome. This is often called the "butterfly effect,"as a butterfly fluttering its wings in Hong Kong might change conditions enough to eventually create a tornado in Texas. This also means, small changes can have dramatic impact. This is why long-term weather prediction is so difficult. In "Deterministic non-periodic flow,"his nowclassic paper about this discovery, Lorenz wrote, 'when one apply an outcome to the atmosphere they

indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly'
[Adams, 2008]. All this was because of the inevitable inaccuracy and incomplete-

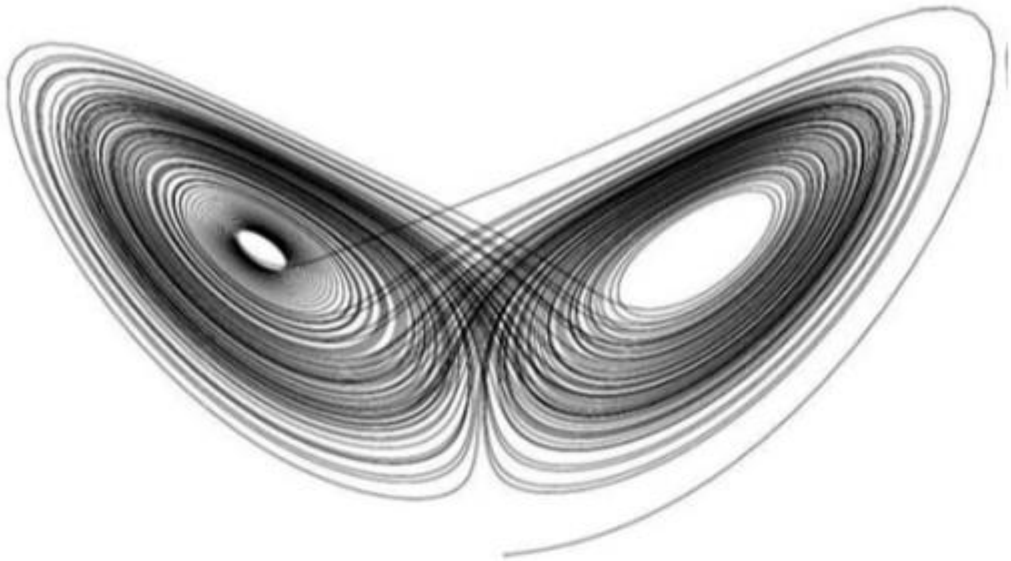


Figure 1.1: The chaotic Lorenz attractor

ness of weather observations; precise very-long-range forecasting would seem to be non-existent. With the aid of a computer, Lorenz was observing a phenomenon that Henri Poincar´*e* had intuitively described sixty years earlier, as a result of his study of the three-body problem. In Poincar´*e* "Science and Method,"easy in 1903, he suggested, If the laws of nature and the conditions of the universe from the onset were known, then it situation could be predicted exactly for the same universe. Scientifically for chaos to occur or happens basing on a small change of an initial state/conditions bring out huge ones in the final phenomena due to the differences.(Adams and Franzosa, 2008)

Between the times of Poincar´*e* and Lorenz, others had also glimpsed this intriguing phenomenon, but it took the introduction of the computer as an experimental tool in mathematics and science for it to be seen and understood broadly enough to be recognized as a robust part of many physical and mathematical systems.

A mathematical and scientific revolution was born in the name of chaos, and over the subsequent decades many scientists and mathematicians worked to identify, describe, and define this phenomenon, its properties, and its consequences. Sensitive dependence on initial condition lead to a system called chaos. Hence Chaos is the name that is widely used for the general structure and behaviors of the result base on this condition. Along with unpredictability of sensitive dependence on initial conditions, it includes an element of regularity. It behaves in an absurd way which is not uniform in nature. Orbits in a chaotic system can appear periodic over a span of time (possibly very long), but eventually they diverge into another realm of behavior, perhaps a different apparent periodicity. The regularity of the Earth orbiting the sun might be an approximate long-term periodicity in an overall chaotic system. (See [Pet] for a further discussion about this possibility)[Adams & Franzosa, 2008].

When a period keeps returning to itself within a time which is time-dependent trajectory then we have a periodic solution of a flow. This in nature is a closed curve and that behave in a balance state but it stableness and unstableness is solely depended on it close orbits as it move nearer or far from the periodic orbits. And as it keeps moving away and toward a particular point a period, a slight increase beyond a required value makes it behave non-periodic.

But a quasi-periodic solution do occur when periodic solutions are been sum up with disproportionate periods. When the ratios of two periods are irrational then it is disproportionate. Therefore in today's mathematical society the need to construct and control periodic and quasi-periodic orbits are very crucial. Therefore physical scientists and so on are now making progress within the usage of this idea. (Tufillaro et al, 2013)

Chaos then is not an equilibrium point, periodic, or quasi-periodic but rather unpredictable in nature but rather an asymptotic motion. In factual sense chaotic regime can see as unbalance behavior of a system that shows irregular patterns in its display. This particular term chaos is very in itself practical but not specific in the sense that, the system if it is graphically displayed, it fills almost the whole space and if it is algebraic the values are not periodic. Moreover, chaotic motion need to be sensitive dependence on initial conditions and it solution must bounded asymptotically: two trajectories that start arbitrarily close toward one another on the chaotic limit set begin to repel so smartly that, they become, for all practical purposes, uncorrelated. In a more precise way, the ability of a chaotic system to exhibits randomly (uncorrelated) behavior is called deterministic.

Note that, stable asymptotic motions is an example that can be use to explain or describe attractors, e.g. stable limit orbits, sinks etc. And the limits sets that unstable (e.g., sources) are examples use in explaining repellers. In describing attracting and repelling the term we normally use is strange attractor and strange repeller that are limit sets that are chaotic. (Tufillaro et al, 2013)

Important Note 1: *Chaos is not the only source of complicated, unpredictable behavior, but the identification of chaos and its associated structures has helped mathematicians and scientists better understand the dynamics of nonlinear systems. And, of course, topological concepts have played an important role along the way. (Adams & Franzosa, 2008)*

1.1.4 The Logistic Map

In the 1844 of November 30, Pierre Francois Verhulst was 40 years when he published in "tome XVIII"with the title "mathematical investigation of the law of population growth", a research he did in contribution to the Memoires de L "Academic"in Belgian nation which was young then. This work he did was the starting point for further research in the field of demography, mathematical biology, and biometry. In his Memories, a model about population growth which is very closed in the living environment with a few resources that maintain its members with a clear cut purpose to predict the demographic evolution was introduced. And to deal with question about the high population size which are sustainable with the scarce resources. One century later chaos theory was the new field in science which is the result that came out of the model he studied as a discrete version. (Verhulst, 1845).

The Verhulst Model which is now called the logistic equation or logistic growth curve $f(x) = rx(1 - x)$ is a population growth model that P.F. Verhulst (1845, 1847) was the first to published and it was a continuous time model, in which a modification of this gave rise to the logistic map which is a discrete recurrence equation. Logistic map was obtained by removing first term from logistic equation. In its more usual form $f(x) = rx(1 - x)$ is how logistic function is stated, where from zero to one are the x values all inclusive i.e [0, 1] and where'r' is a positive real number 'r¿0'. Logistic map applications in pure and applied mathematics in this modern era have been useful.

Mathematicians and Scientists in this era have studied intensively the logistic map as an archetypal one dimensional dynamical system. In the past years the one-dimensional logistic map has been extensively studied. This map describes the typical behavior of many dissipative dynamical systems most especially in chaotic regimes and has application in Physics, biology, electronics, economics and many other different subjects [Chugh & Rani, 2012].

According to Geoffrey,an extensive computer study of the function $f(x)$ and other related maps was done by Paul Stein and Stanislaw Ulam in the early 1950's, but much concerning these maps remain mysterious. (Goodson, 2015)

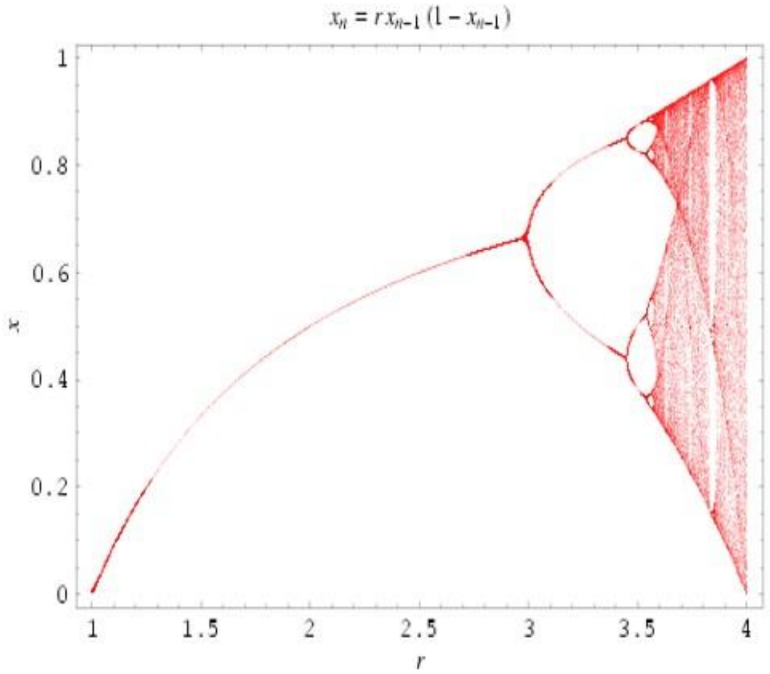


Figure 1.2: The chaotic Lorenz attractor

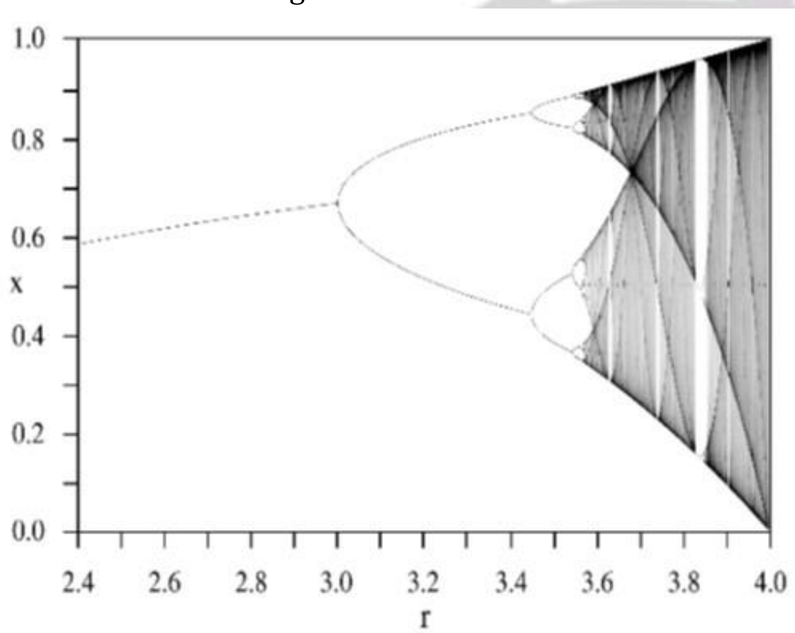


Figure 1.3: The logistic map

Important Note 2: *As the chaos revolution took hold, many mathematicians and scientists began to examine systems for the behaviors seen in the logistic family. The dynamics in this simple model have helped deepen the understanding of phenomena such as turbulence in fluid flow, motions of celestial bodies, outbreaks of diseases,* *and vibrations in machinery, thereby making science and mathematics useful and meaningful (Adams & Franzosa, 2008)*

1.2 Research Aim and Significance

The primary aim is to study the nature/behavior of logistic function on topological dynamics (dynamical systems) and making it within reach to academia and users of dynamical systems described in this research.

1.3 Objective of The Study

The basic objectives of this research intents to;

1. Study the logistic function in dynamical systems through its behavior.
2. Asses /analysis the behavior of the logistic function
3. Study the route of the logistic function into chaos in dynamics.

1.4 Scope

The research will involve a study collection of works in the area under study recently. Research papers and resources written by academicians would be consulted. Well established textbooks and resources in this field would not be left out. The knowledge and use of computer software, MATLAB, and LATEX would be employing in producing this research.

1.5 Limitations

This researcher encounters a number of problems in the process of carrying out the study. A major limitation is that of financial resources. In order to come out with the desired result, appropriate instrument model for the work would be a crucial limitation. Existing works of academicians in the field of study would be the limited scope

1.6 Structure of The Thesis

The thesis has the following chapters:

- Chapter 1: Introduction
- Chapter 2: Literature Review
- Chapter 3: Theoretical Exposition
- Chapter 4: Logistic function
- Chapter 5: The Conclusion
- References

Chapter 2

LITERATURE REVIEW

2.1 Introduction

Under this chapter we consider just a review of some relevant literatures. It also incorporates some summary on dynamical system and chaos.

2.1.1 Dynamical and Chaos

For the past decades, the theory of dynamical systems and chaos has penetrates into many disciplines like philosophy, art, biology, genetics, economy and other engineering fields through periodicity, recurrence, sensitivity, entropy and finally chaos.

A dynamical system is the study of how things change over time that is either into the past or the future state. Examples include the growth of populations, the change in the weather, radioactive decay, mixing of liquids such as the ocean currents, motion of the planets, the interest in a bank account. Some of these dynamical systems are well behaved and predictable, for example, if you know how much money you have in the bank today, it should be possible to calculate how much you will have next month (based on how much you deposit, interest rate etc.)

However, some dynamical systems are inherently unpredictable and uncorrelated and so are called chaotic. An example of this is weather forecasting, which is generally unreliable beyond predicting weather for the next three or four days.

Intuition tells us that chaotic behavior will happen provided we have some degree of randomness in the system. However, chaos can happen even when the dynamical system is deterministic, that is, its future behavior is completely determined by its initial conditions. To quote Edward Lorenz, who was the first to realize that deterministic chaos is present in weather forecasting? In theory, if we could measure exactly the weather at some instant in time at every point in the earth's atmosphere, we could predict how it will behave in the future. But because we can only approximately measure the weather (wind speed and direction, temperature etc.), the future weather is unpredictable. (Goodson, 2015) Chaotic behavior in the three-body problem was discovered by Henry Poincare, which proved that there was a lot of wrong view of about this system. Therefore as stated in Tufillaro et al, 2013 Henry saw that as clever or more intelligent one is it will still be difficult to write down an equations that handle many nonlinear equations and solve. In a (bounded) closed form solution we might expect that any small variation in the initial conditions gives a proportional change in the predicted trajectories. Chaos in nature behave in a way through it differences irrespective of the closeness of two initial state. As clever as Poincare restructured the notion of it solution to differential equation he saw the consequent in it totality and the smartly (Tufillaro et al, 2013)

However, the notion of chaos was used for the first time by Li and Yorke in 1975. Let us recall that a pair of points is called Li-Yorke chaotic if limits superior of distances of their iterations is positive while limits inferior is zero. That means that there are time sections where two orbits are arbitrarily close following time sections where they are distant enough. The systems were originally assumed to be chaotic according to Li and Yorke, 1975 that is the set of those pairs is uncountable. Later on more sophisticated versions of chaos were constructed and compared to discover which one is better. There are still open problems in this area where mathematicians and scientist are developing model to tackle this open problems. How Li and Yorke sense in chaos was, and still is, extensively studied by many authors. (Lampart, 2013) As the father or founder of dynamical system as many people regard Henri Poincar´*e* to be, his published came out with two classical monographs; that is in 1892-1899 'New methods of Celestial Mechanic' and in 1905-1910 Lectures on Celestial Mechanics.(Holmes, P. 1990) According to Tabuada et al, 2007 dynamical system as concept in mathematical science is depends on a constant rule that explains the reliance of a time of a point in space which is geometrical in Mallat, 2009. Examples that really speaks and talk about this concept include the models in mathematics that explains;the movement of a clock pendulum in Holm et al, 2001, Piela et al, 2008 also talked about the water that flows through pipes and then within a lake number of fish each spring in, Feger et al, (2010).

Gottschalk & Hedlund [1955] also defined topological Dynamics says as 'the study of transformation of groups which depends on the same topological properties whose examples appear in classical dynamics. Newtonian mechanics originated the concepts of dynamical system in Sharipov, (2001). The theory of dynamical systems has a branch called topological dynamics which is qualitative in mathematics and it is in only general topology that the asymptotic properties of dynamical system can be learnt. [From Wikipedia,free encyclopedia, 2013] The setting for the study of dynamical systems involves a space, time, and a time evolution (Katok and Hasselblatt, 1995)

Comparably, irrespectively of how huge the domain of topological dynamics is, the physical sciences and other field have found the usefulness of it applications (Hufschmidt et al, 2005).

Links or connectivity of the topological dynamics and dynamical system is on the differential equation as state by King et al, (2010), and difference equation according to Taixiang et al, (2005) or other time scale. Topological

Dynamics is also applicable in biology. Hofbauar and Sigmund, 1988; May, (1973). Dynamical system deals with the repeated applications which start with an initial point also known as the seed to predict changes. The iteration procedure or way has been the preferred way as to integrating the system or solving a particular system (Price et al, 2009). So trajectory or obit is when a particular system that has an initial point, is solvable and it all future points are possible to be determine by collecting those outcomes together (Wells, 2011). In solving a dynamical systems years past was very demanding that is either through high define techniques [Powell, 2007]. Not until the introduction of computing machines [Crouch, 2010].

Yaacov (2008) definition of a dynamical system was basically about compact metric space whose self-map is continuous. He then added to it that iterations of a periodic map is studies through topological dynamics or equivalently the orbits of points of the state space.

Note: transitivity, minimality, recurrence shadowing property, attractors, stability, sensitivity, topological entropy and equi-continuity are all concepts of topological dynamics. (Yaacov, 2008)

The expanding nature of the topological dynamics as a theory has really cover more action of general topological, usually locally compact, countable and discrete, or Polish (admits a complete, separable metric).Therefore to really understand the links between dynamical systems theory and topological dynamics in general is by the idea from general topology, analysis etc (Akins, 2007).

2.2 Logistic Map and Chaos

2.2.1 The Logistic Map

Pierre Verhulst was the first to publish this model which is now called the logistic map. It is quadratic recurrence relation which was used for population growth. This equation was given as;

$$\frac{dx}{dt} = rx(1-x)$$

which is a continuous form and depended on the time which are real < But the discrete form of the Logistic map is given by $x_{n+1} = rx_n(1-x_n)$ with r as positive value known as the control parameter which works within a given intervals and $n = 0,1,2,3,...,x_n$ is the state at the discrete time n. This particular equation can be clear or well understood in this order; example, assuming constant conditions every year, the population (of insects for example) at year n uniquely determines the population at year $n + 1$. Therefore we have a one-dimensional map. Say that there are x_n insects at year n and that every insect lays, on the average, r eggs, each of which hatches at year n+1. At year n+1 there is going to a population yield of $x = rx_n$ when $r > 1$, this yields increases population exponentially. When the population is too large however, the insects will exhaust their food supply as they eat and grow, and not all insects will reach maturity. Hence the average number of eggs laid will become less than r as x_n is increased. A simple assumption is then that the number of eggs lay per insect decreases linearly with the insect population, $r[1 - (x_n/x)]$ where x is the population at which the insects exhaust their food supply. We then have the one-dimensional map $x_{n+1} = rx_n[1-(x_n/x)]$ this then lead us the Logistic map. (Jo Bovy, 2004).

When value $r = 3.5699456$ then the map is always periodic. One will also notice that when the period doubling ends then it getting to chaos. When the control parameter is 3.9, the logistic map moves from periodic hence leading to chaos. Therefore the Logistic map is chaotic when $r = 4$ as a matured value So at $r = 3$ one can notice the occurrence of the first bifurcation and $r = 3.5$ the next one occurs. The value nearby $r = 3.6$ triggering the avalanche process that leads to increased systemic instability, that becoming total at $r = 4$. In this point it establishes chaotic behavior. (Mahalu, 2013).

After, May was able to introduce the iterated logistic map as a simple model for the dynamics of populations (predator-prey). It now stands the best and more efficient example of a nonlinear dynamical system that displays chaos.

The logistic map has provided the basic fundamental insight into aspects of the mathematical theory and the numerical methods relevant to deterministic chaos. In fact, a large proportion of chaotic phenomena in high dimensional models are already observable in the logistic map. (Raghavachari, 1999)

2.2.2 Chaos

Many mathematical literature for the past decades has really discussed the existence of chaotic dynamics with much contributions from Levinson, Poincare, Cartwright and Littlewood, Smale, Birkhoff and Kolmogorov and his students, among others.(Celso, Edwards, James, 1987) Nevertheless, chaos has been recognized recently for it broad influence. Consequently, growth of chaos is expanding massively, and many applications have been made across wider spectrum of scientific areas like; chemistry, ecology, economics, physics, engineering, fluid mechanics, to name a few. Some examples which are specific to chaotic time dependence are; stirred chemical reactor systems, simple model for the yearly variation of insect populations, the determination of limits on the length of reliable weather forecasting, and convection of a fluid heated from below. (Celso et al, 1987)

have been made across wider spectrum of scientific areas like; chemistry, ecology, economics, physics, engineering, fluid mechanics, to name a few. Some examples which are specific to chaotic time dependence are; stirred chemical reactor systems, simple model for the yearly variation of insect populations, the determination of limits on the length of reliable weather forecasting, and convection of a fluid heated from below. (Celso et al, 1987) According to Obeng-denteh,(2012), initial conditions produces widely repelling results for dynamical systems from a small differences which deemed as chaotic (Kellert. S, 1993) accounting for the difficulty in long-term. So chaotic system in dynamical systems is when the system is very highly sensitive to initial states and hence called the butterfly effect. In many field it application is very important for instance, meteorology, economics, physics, engineering and so forth. This abnormal behavior of the chaos occur even the systems are deterministic nature, that is one can determine the future behavior using it initial state/conditions, with no stochastic elements included. (Kellert. S, 1993), the nature of it behaviour is refer as deterministic chaos, or simply chaos and Edward Lorenz summarized the theory. (Danforth, C.M, 2013) [Obeng-denteh, 2012]

Since the dawn time chaos has become an indispensable part of human life that cannot be left unnoticed, most especially human agitations that leads to violence. One of the most famous models with chaotic behavior is the logistic map which is a difference equation. It simple analytic form meant that it has been used in many scientific disciplines such as biology, cryptography, communication, chemical, physics mathematics proper, eco-mathematics and the markets with easy manipulation. One of the mathematical models of chaos was discovered by Mitchell Feigenbaum (1978), he considered ordinary difference equations used for example in biology to describe the development of population in its dependence on time. He discovered that population oscillates in time between stable powers (fixed points), the number of which doubles according to changes in the power of external parameter. The generalization of the Feigenbaum model contains all first-order difference equations $f(x_n) = x_{n+1}$. The condition for the existence of chaos is the single maximum of a function $f(xn)$. Feigenbaum also proved that the transition to chaos is described by two universal constants α and δ, where $\alpha = -2.503...$ and $\delta = 4.669...$ and later named them Feigenbaum numbers. (Rak
& Rak, 2015)

Chapter 3

THEORETICAL EXPOSITION

In this section, we explore some of the useful definitions and examples of topological spaces, topological dynamics, and dynamical systems, chaos etc.

3.1 Useful Definitions and Theorems from General Topology and Dynamical system

3.1.1 Continuos function

In the topological arena, a continuous function is often called a map. Basically, two ideas which are not the same comes into mind that is; the calculus definition and the topology definition. In view of the area of interest, we will focus our attention on the definition as in topology.

A map f of a topological space (X,τ) into $s(Y,S)$ is continuous if and only if the inverse of each open set is open. Our first formal exposure to continuity usually comes in a calculus or analysis course, focusing on functions mapping the real line $<$ to $<$ Typically, it is defined as follows:

Calculus Definition 3.1.1 (Adams and Franzosa, 2008)

A function $f : <^n \to <^n$ is continuous in the sense that if for every > 0 then, there exist $\delta > 0$ such that, when $|x - x_0| < \delta$ and then $|f(x) - f(x_0)| < \epsilon$This is what we called the “,δ”definition of continuity or continuous function. Hence there is a general definition of continuity for functions that map one topological space to another. This topological definition of continuity is very simple to state and, it is equivalent to the ,δ definition for functions that map $< \to <$ **Topology Definition 3.1.2 (Adams,2008)**

Given two topological spaces X and Y , then a function $f : X \to Y$ is continuous if $f^{-1}(V)$ is open in X for every open set V in Y . This is call the open set definition of continuity.

EXAMPLE 3.6

Let X and Y be topological spaces.

(i) *The identity function id : $X \to X$*

(ii) *Pick $y_0 \in Y$, and consider and the constant function, $C : X \to Y$ defined by $C(x) = y_0$ for every $x \in X$. We show that C is continuous. Suppose that V is open in Y ; then $C^{-1}(V) = X$ if $y_0 \in V$, and $(V) = 0$, if $y_0 \notin V$. In either case $C^{-1}(V)$ is open in X and therefore C is continuous*

Theorem 3.5: *(Adams &Franzosa, 2008). Any function $f : X \to Y$ is continuous in the open set definition of continuity if and only for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$.*

Proof 3.4 *suppose the open set definition holds for functionsf: $X \to Y$ Then given $x \in X$ and an open set $U \subset Y$ containing $f(x)$ and then let $V = f^{-1}(U)$. It then implies that $x \in V$ and that V is open in X since f is continuous by the open set definition. Clearly $f(V) \subset U$, as required.*

Definition 3.1.3: *A function f: $X \to Y$ is continuous if and only if the preimage of any open set in Y is open inX. For closed sets to open sets, you can use the following equivalent definition:*

Definition 3.1.4: *A function $f : X \to Y$ is continuous if and only if the preimage of any closed set in Y is closed in X.*

Important Note 3:

The composite of 2 continuous functions is continuous.

Not all continuous function necessarily map open sets to open sets or close sets to close sets. A typical example, is the function $f : < \to <$ *,define by* $f(x) = x^2$ *is continuous, but the image of the open set* $(-1,1)$ *is* $[0,1)$, *which is not open.*

3.2 Topological Dynamics

3.2.1 Topological Dynamics and Orbits (Kolyada, 2013)

By a (topological) dynamics system we mean a pair (X,T) consisting of a compact metric space (X,d) and a continuous map $T : X \to X$. If X is a singleton, then we say that (X,T) is trivial. If $K \subset X$ is a non-empty closed subset satisfying $T(K) \subset K$, then we say that (K,T) is a subsystem of (X,T) and (X,T) is minimal if it has no proper subsystem. The (positive) orbit of x under T is the set; $Orb(X,T) = Tnx : nZ$ clearly $(Orb(X,T),T)$ is the closure and is the subsystem of (X,T) and (X,T) is minimal if the closure of $Orb(X,T) = X$, for every $x \in X$ then a point is: $x \in X$

1. Minimal, if x belongs to some minimal subsystem of (X,T)
2. Recurrent, if lim inf $n \to d(Tnx,x) = 0$
3. Transitive, if the closure of $Orb(X,T) = X$

Definition 3.2.2 A dynamical system (X,T) *is usually called minimal if the closure of* $xT = x$ *for all* $x \in X$*. i.e. if all point is a transitive point. "When a system with its entire Orbit is dense"*

Definition 3.2.3 A point $x \in X$ *is termed as transitive if the closure of* $xT = x$*. The dynamical system* (X,T) *is said to be point transitive if there exist a transitive point x in X.*

Important Note 4: when the Orbit is dense or the system is with a dense Orbit then it is transitive. For the purpose of this work the Orbit will always be assumed to be dense.

Definition 3.2.4

If(X, T) and (Y,T) be a dynamical system and f:X → Y be a continuous mapping. Then fis homeomorphism if

a. *f is continuous and*

b. *f(xt) = f(x)t for all xT = x ∈ X and t ∈ T*

3.3 One-Dimensional Map

3.3.1 Discrete-time dynamical systems in one dimension

Let f : R → R be real valued function, called a map. Taking x0 as the initial point, f maps to the sequence $x0, f(x0), f^2(x0), ... fn(x0), ...$, where $f^n(x_0)$ stands for the nth iteration or composition of f with itself. The sequence $x0, f(x0), f^2(x0), ... fn(x0), ...$ is called the orbit of x_0 under f.

3.3.2 Fixed Point: (Tufillaro, Tyler and Jeemiah, 2013)

A simple, linear map f : R → R of the Real line R to itself is given by $f(x) = mx$. This linear map can have stretching, but folding. The period one point of a map (points that map to themselves after one iteration) are also called fixed point(Tufillaro et al, 2013).

Important Note 5:

1. *The origin m is an attracting fixed point or sink, Ifm* < 1.

2. *If m > 1, then the origin is fixed point but a repelling fixed point or source. Since it move away from it. If m = 1, then all initial condition lead immediately to a period one orbit defined by y = x.*

Important Note 6 neutral stability is when all the periodic cycles/orbits are in equilibrium or lie on a line. An orbit of a map is periodic if it repeats itself after a finite number of iteration. By a careful look, a point on a period two orbit has the property that; $f^2(x_0) = x_0$, and a period three point is true for $f^3(x_0) = x_0$ that is, it repeats itself after three iterations.

Important Note 7 : in general a period n point repeats itself after n iteration and is a solution to the equation, $f^n(x_0) = x_0$ i.e. a period n point is a fixed point of the nth composite function of f.

Definition 3.3: (Tufillaro et al, 2013) If f : R → R if $f(x_0) = x_0$, then the point x0 is fixed point forf. Also if fn(x0) = x_0 then the point x_0 is a periodic point of period n forf but $f^i(x0)$ 6= x_0 for 0 ≤ i ≤ n. The point x_0 is eventually periodic if $f^m(x_0) = f^{m+n}(x0)$ but x_0 is not itself periodic.

3.3.3 Periodic Point

From the definition 3.3 of a period n point,$f^n(x_0) = x_0$ Givenf be a map on R, x_0 is call a periodic point of n if $f^n(x_0) = x_0$ and n is the smallest such positive integer. The orbit with initial point x_0 i.e. $x_0, f(x_0), f^2(x_0), ..., f^{n-1}(x_0)$ is called a periodic orbit of periodn. Basically x_0 is called a period-n point and its orbit a period-n orbit.

Theorem: (Ya Yan Lu)

Let fbe a smooth map, p1, p2... pk be a period-k orbit off.

- *If $|f^0(pk)...f^0(p2),f^0(p1)| < 1$, the period−k orbit is a periodic sink.*
- *If $|f^0(pk)...f^0(p2),f^0(p1)| > 1$, the period-k orbit is a periodic source.*

Definition 3.4.0

i. *periodic point p is attracting if* $|(f^n)^0(p)| < 1$

ii. *periodic point p is repelling if* $|(f^n)^0(p)| > 1$
iii *point p is neutral if* $|(f^n)^0(p)| = 1$

Important Note 8: the prime denote differentiation with respect to p

3.3.4 The Mean Value Theorem

Given F as a differentiable function on $a \leq x < b$. Then there exist c between a and b for which the following equation is true: $F'(c) = \frac{F(b) - F(a)}{(b - a)}$ The

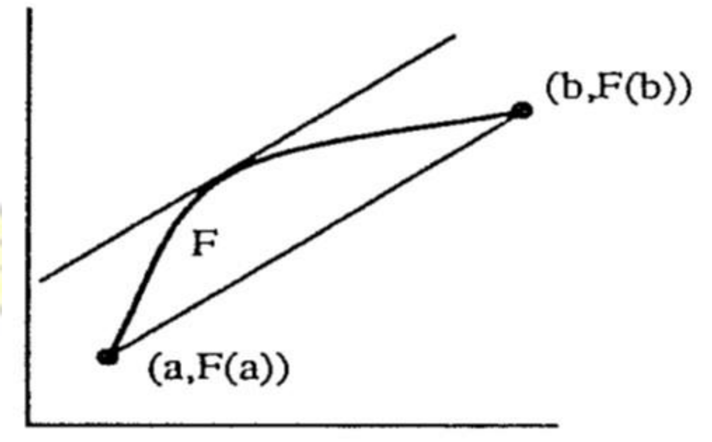


Figure 3.1: the mean value theorem (adapted from Robert Devaney, 2003)

content of the mean value theorem is best exhibited geometrically. The quantity,

$M = \frac{F(b) - F(a)}{(b - a)}$ is basically the gradient of the straight line connecting two points $(a,F(a))$ and $(b,F(b))$ on the graph ofF. This theorem simply means that, provided Fis differentiable on $a \leq x < b$,c is a point which lies withina andb at which the slope of the tangent line, $F^0(c)$, is exactly equal to M.

Important Note 9: In the course of this work the mean value theorem will be used in showing effect of the fixed point in the main work.

3.3.5 Sensitive Dependence On Initial Conditions

In a one-dimensional map, we can begin with x_0 and x'_0 such that x'_0 is very close to x_0, then by considering the orbits $\{x_0, f(x_0), f^2(x'_0)...\}, \{x_0, f(x_0), f^2(x'_0)...\}$. Sensitive dependence on initial conditions occurs when two orbits are not close to each other and when this occur we the deterministic. Let $f : \mathrm{R} \to \mathrm{R}$, if $\delta > 0$ is a constant such that $\delta > 0$, then there is x satisfying $|x - x_0| < \delta$ and an integer n, such that $|f^n(x) - f^n(x_0)| \geq \epsilon$. Where the point x_0 is called sensitive point,

hence the point x0 has sensitive dependence on the initial condition x.

Bifurcation Diagram: (Neville, 2014). By focusing on long term behavior of the control parameter of the logistic map diagram below, we can see where the logistic map is periodic and where it is chaotic. In general, when there is a bifurcation there is qualitative change in the behavior for a long term as control parameter varies for a particular map, we say that the system underwent a bifurcation that is by splitting. In fact, the bifurcation diagram captures the orbit of points after the system has stabilized. This is the asymptotic behavior represented base on the control parameter. The point where the splitting does

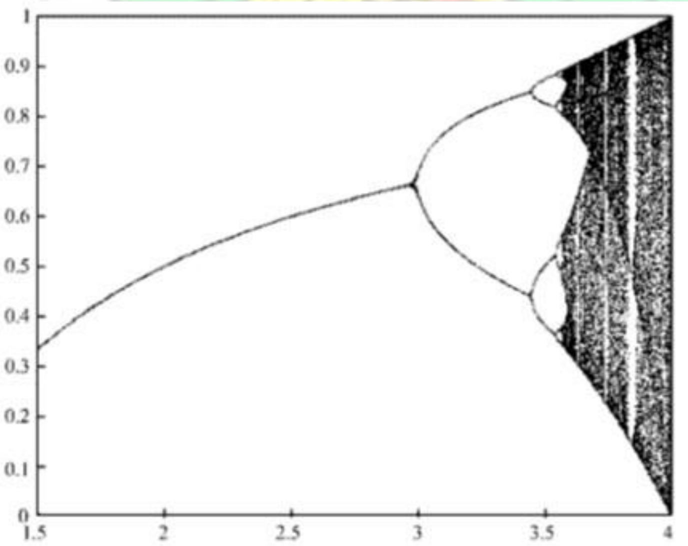


Figure 3.2: logistic map (bifurcation diagram)

occur gives the bifurcation value and is neutral at that point. The bifurcation diagram enables the visualization of the true nature of the function.

3.4 Chaos

Definition 3.6.0: (Adams &Franzosa, 2008)

A function $f : Y \to Y$ ischaotic orchaos if

1. *The set of periodic points off is dense in Y,*
2. *For every U, V open in Y, there exists $y \in U$ and n belong to positive integers such that$f^n(y) \in V$.*

For one, there is regular periodic behavior which distributed densely throughout the domain. And this happens irrespective of a point we use from the domain, arbitrarily there are periodic points which are close by.

The second condition is basically about transitivity in topology i.e. (topological transitivity). And this explains that for every pair of regions in the domain is put together by the system. Also given any open sets which are pair, at least one point is in the first set and by iteration is mapped into the next set/ second set.

In a paper presented by Berg, he based his facts on the logistic map, $f(x) = \alpha x - \alpha x^2$, then by iterating the function to its third composition, $f^3(x) = x$. So in his attempt for looking for values of afor which there will be a diagonal at 45 for the third iteration graph of $f(x)$. He saw these Two points to obvious; the fixed points 0 and $\frac{(a-1)}{a}$. He also saw that it was prime period one, so they also have period three (he then proposed this question, after one iteration what then is fixed will also remain fixed after three as well?). But at $a = 3.828...$

Another period three orbit exists, and it was, remarkably enough, to be stable (Berg, N.D).

Note, in the left picture the little box is what is been enlarged in the right picture

(Adapted from Daan van den Berg, N.D)

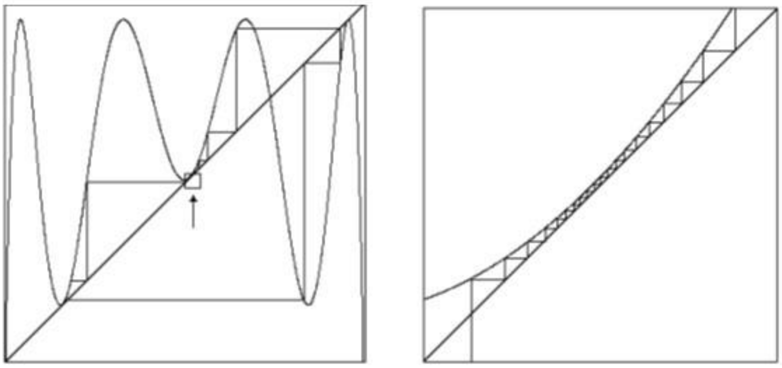


Figure 3.3: : Daan van den Berg Graphical display of the 3rd-iteration of the logistic map at (a=3.828)

As the chaos revolution took hold, many mathematicians and scientists began to examine systems for the behaviors seen in the logistic family. The dynamics in this simple model have helped deepen the understanding of phenomena such as turbulence in fluid flow, motions of celestial bodies, outbreaks of diseases, and vibrations in machinery as already stated in the introduction.

Chaos is not the only source of complicated, unpredictable behavior, but the identification of chaos and its associated structures has helped mathematicians and scientists better understand the dynamics of nonlinear systems. And, of course, topological concepts have played an important role along the way.

Chapter 4

THE MAIN WORK

This chapter is made of some useful introductions and definitions in logistic function, iteration, orbits and the main work.

4.1 The Logistic Function

In 1976, May defined the logistic map or function as a polynomial mapping with second degree (equivalently, recurrence relation) which is a simple non-linear dynamical equations serving as an example of chaos. The function became famous by Pierre Francois Verhulst (May, 1976) in a seminal 1976 paper which he presented, logistic equation part as a discrete- time demographic model. Mathematically this is given as,

$$T(x) = kx(1 - x)$$

Where x takes the numbers from zero and one inclusive, $x = [0,1]$, and k is the value of interest which is a parameter $k > 0$.

However, the logistic map as a model solely depends on initial condition and parameter values has some problem in the sense that it at time lead to negative population size. But in the older Ricker model this problem cannot be seen, which also exhibits chaotic dynamics. The quadratic difference equation explaining the logistic map can be regarded as operation which folding and stretching on the range (0,1).

The concepts or study of the theory chaos is now considered to be easy through the simplicity of the logistic map.

Chaos can be describe roughly as a chaotic system that shows a high sensitivity to the initial condition. Sensitivity to initial conditions commonly be seen as just representing the map as repeated folding and stretching of the space on which it is defined.

This function finds applications in a number of fields including biology, artificial neural networks, ecology, economics, chemistry, statistics, demography, and biomathematics which are of great important in the sciences and the social sciences.

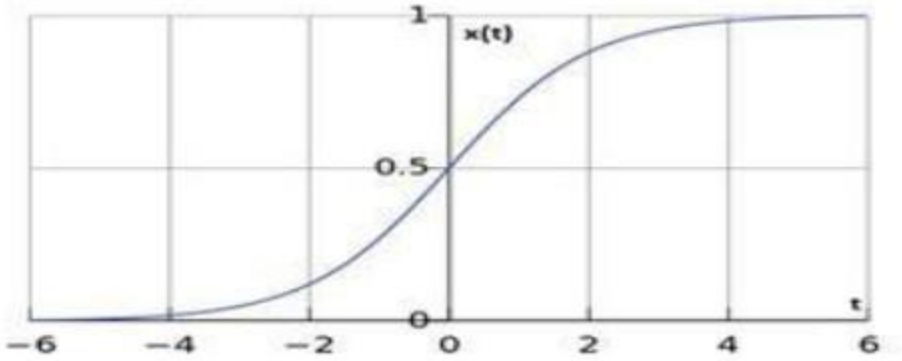


Figure 4.1: Logistic function (Adapted from George Mahalu, 2013)

Important Note 10: The logistic map provides a rich example to explore periodic regions, complex chaotic behavior and self-similarity. For the interest of this work, where the logistic map changes in behavior due to the parameter introduced into the map and allowed to vary continuously so that changes in the dynamics of the logistic function can be noticed.

4.2 One-Dimensional Map

4.2.1 Iterating Function

There are many kinds of problems in science and mathematics that involve iteration. Iteration means to repeat a process over and over. In dynamics, the process that is repeated is the application of a function. To iterate a function means to evaluate the function over and over using the results obtained from the previous as the input for the next application. This is the same process as typing a number into a scientific calculator, the repeatedly striking one of the function keys such as “sin”or “cos”. Mathematically, this is the process of repeatedly composing the function with itself. (Devaney, 2003)

Definition 4.10: (Adams and Franzosa, 2008) Let R be a topological space and f : R → R be a function mappingRtoR. For every positive integernand a ∈ R then it composition is define as;

$$f(a) = fof...f(a)$$

Important Note 11: the composition of n copies of the functionf, the idea/notion is that we start with x, then applying f to x, then applyf to $f(x)$, and continue this iterative process until we obtain $f^n(x)$

For instance, for a function f, $f^2(x)$ is the second iterate off, namely $f(f(x))$,$f^3(x)$ the third $f(f(f(x)))$,in general, $f_n(x)$ is the n-fold composition of f with itself.

Important Note 12: The dynamical system defined by$f : X \to X$ is the family of function fn for n belongs to positive integers with each $f^n(x)$ maps from X to itself.

Example 4.10

Let $f : Y \to Y$ be a mapping define by $f(y) = y^2 + 1$ Then, by iterating;

$$f^2(y) = (y^2 + 1)^2 + 1$$

$$f^3(y) = ((y^2 + 1)^2 + 1)^2 + 1$$

Similarly, if $f(x) = \sqrt{x}$

Then, by iterating

$$f^2(x) = \sqrt{\sqrt{x}}$$

$$f^3(x) = \sqrt{\sqrt{\sqrt{x}}}$$

Important Note 13: one need to realize that $f^n(x)$ does not mean raise $f(x)$ to the nth power but n^{th} iterate.

Example 4.11: Let $f : \mathrm{R} \rightarrow \mathrm{R}$ be a function given by $f(x) = \frac{x}{3}$

Therefore, by the dynamical system definition f is the family of function given by;

$$f(x) = \frac{x}{3} ... f^2(x) = \frac{x}{3^2} ... f^3(x) = \frac{x}{3^3} ... f^n(x) = \frac{x}{3^n}$$

. *Important Note 14: In iterating a function of a dynamical system, we describe it, $f(x)$ by it new state by unit after unit of time.*

Important Note 15 By considering the following function and its phase diagram below;

i. If $f : \mathrm{R} \rightarrow \mathrm{R}$, defined by $f(x) = -2x$

By solving, $f(x) = -2x$, $f^n(x) = (-2)^n x$ to it third iteration,

$n = 1,$ $\quad f(x) = -2x$

$n = 2,$ $\quad f^2(x) = 4x$

$n = 3,$ $\quad f^3(x) = -8x$

ii. $g : \mathrm{R} \rightarrow \mathrm{R}$, defined by $g(x) = \frac{1}{4}x$

By iterating the function; $(x) = \frac{1}{4}x$, $g^n(x) = (\frac{1}{4})^n$ to third iteration, that is

$$n = 1 \qquad g(x) = \frac{1}{4}x$$

$$n = 2 \qquad g^2(x) = \frac{1}{16}x$$

$$n = n \qquad g(x) = \frac{1}{4}^n x$$

iii. $h : \mathrm{R} \rightarrow \mathrm{R}$, defined by $h(x) = -x$

By iterating, $h(x) = -x$, $h^n(x) = (-1)^n x$

Taking $n = 1,2,n$

$n = 1$ $h(x) = -x$

$n = 2$ $h^2(x) = x$

$n = 3$ $h^n(x) = (-1)^n(x)$

iv. $k : R \rightarrow R$, defined by $k(x) = 0$

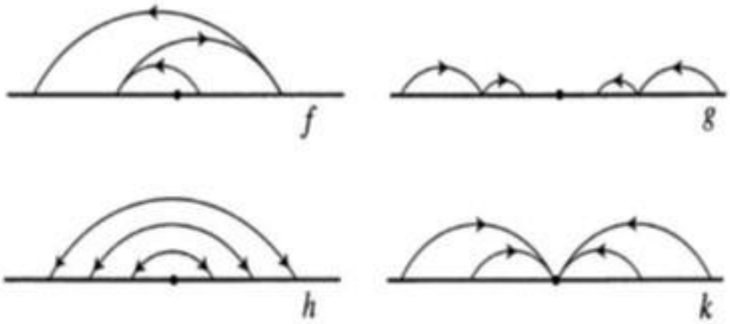


Figure 4.2: The phase diagram for f,g,h,k

Important Note 16: now for the function $f^n(x) = (-2)^n x$ *, the iterations keeps moving away from zero (0) and bounces in between positive and negative values. In the function* $g^n(x) = (\frac{1}{4})^n x$ *the iteration of g on a non-zero value results in values that approaches a zero (0) limit. But the function* $h^n(x) = (-1)^n x$ *shows a dynamical picture which is different, clearly the iteration of h keeps on oscillating between positive and negative constant values, that is −x and x.*

4.2.2 Orbits

Definition 4.11: (Adams &Fransoza, 2008) Given $a_0 \in R$*, we define the Orbit of a0 under f to be the points of sequence,*

$$x_0 = f(a_0), x_1 = f(a_0), x_2 = f^2(a_0) ... a_n = f^n(a_{n-1})$$

$$O(a_0) = a_0, f(a_0), f^2(a_1) ...$$

Where $f2(a_0) = f(f(a_0)), f3(a_0) = f(f2(a_0))$, and continuing indefinitely, so that $fn(x) = f \circ f \circ f \circ \ldots f(a)$; (n-times composition). Important Note 17: the point a_0 is called the seed of the orbits.

Definition 4.12: (Robert Devaney, 2003)

Let $f: M \to M$, and assume $m \in M$, the orbit of m under f is the sequence $\{m, f(m), f2(m) \ldots f^n(m) \ldots\}$ and it denoted by $O(m)$.

Example 1.12: some first few points on the orbit of x given $f(x) = \sqrt{x}$ are ;

$$X_0 = 6561$$

$$X_1 = f(x) = \sqrt{6561} = 81$$

$$X_2 = f^2(x_1) = \sqrt{81} = 9$$

$$X_3 = f^3(x_2) = \sqrt{9} = 3$$

$$X_4 = f^4(x_3) = \sqrt{3} = 1.73$$

$$6561, 81, 9, 3, \ldots f^n(x_{n-1})$$

Important Note 18: Clearly, as iteration occurs based on evaluating a particular function over with its outputs, the outputs obtained from each evaluation form what we call the Orbit in sequential order.

Important Note 19: many are the types of Orbits in dynamical systems

(a) Fixed point

The point a_0 is a fixed point if it satisfies the condition; $f(a_0) = a_0$

That is; $f(f(a_0)) = f^2(a_0) = f(f(f(a_0))) = f^3(a_0) = f(a_0) = a_0$

Hence, $f^n(a_0) = a_0$

Important Note 20: a fixed point x_0 *gives a constant output after going through all the iterations. Its orbit gives a constant sequence,*$x_0,x_0,x_0,x_0,\ldots$ *which never moves. On the Real line,* $0,-1,1$ *are fixed point for* $f(x) = x^3$ *and for* $f(x) = x^2$, 0 *and* 1 *are fixed point.*

(b) **Periodic Orbit or Cycle**

If x_0 is an initial point then it is a periodic point when; $f^n(x_0) = x_0$ is true for some $n > 0$

Important Note 21 n is called the prime of the orbit. If x_0 is periodic with prime n, then the orbit of x_0 is just a repeating sequence of number; x_0, $f(x_0)\ldots fn - 1(x_0)$ If x_0 is the first value or initial point for period-n point for a function $f : X \to X$, then the points $x_0, f(x_0)\ldots f^{n-1}(x_0)$ are all period-n points and are all distinct.

Example 1.13:

let $f : M \to M$ be a mapping where $f(x) = x^2 - 1$. Let x = 0 $X_0 = f(0) = (0)^2 - 1 = -1$ $X_1 = f(-1) = (-1)^2 - 1 = 0$ $X_2 = f(0) = (0)^2 - 1 = -1$

$X_3 = f(-1) = (-1)^2 - 1 = 0$ Hence, the orbit of zero (0) is, $0,-1,0,-1,0,-1,\ldots$

Therefore, by periodic we say that 0 and -1 form a 2-cycle. **Important Note 22:** a period-1 point is also known as a fixed point which is within the set of all periodic points of a function.

4.3 The Logistic Map

Logistic map is defined as $x_{(n+1)} = rx_n(1 - x_n)$, where $n = 0,1,2,3,\ldots$n but for purpose of this work we will let $x_{(n+1)} = f(x_n)$ then, $f(x_n) = rx_n(1 - x_n)$, with domain $[0,1]$. i.e. $x_n \in [0,1]$. We call this the logistic growth function. We think of the parameter r as a growth rate (control parameter), and we distinguish different functions with different growth rates by writing them as f_r rather thanf. That is $f_r(x_n) = rx_n(1 - x_n)$ So by setting the parameter $r = 1$, the logistic equation becomes $f(x_n) = x_n(1 - x_n) = x_n - x_n^2$

4.3.1 Characteristics

There are some characteristics of the logistic function that we can explore for the purposes of this work. They are the graphical nature of the map, periodic points (period-n points), the bifurcation graph (cobweb) of the logistic function/map and finally the chaotic nature. Therefore to meet the interest of the work concerning the logistic map changes in behavior? The parameters introduced into the map are allowed to vary continuously so that the changes that bring about the characteristics in the dynamics of the logistic function can be seen/ noticed.

4.4 The Graphical Analysis

Under this section we examine how the geometrical procedure within the logistic function is. It allows one to study, analyze and determine the behavior of orbit of the function/map in many cases. For the purpose of this research we will Take the growth rates1 $\leq r \leq 4$ such that f maps [0,1] back [0,1] as our interest. Basically, the family of logistic function corresponds to the parameter value $r > 1$.
Graphical display of equation 4.2

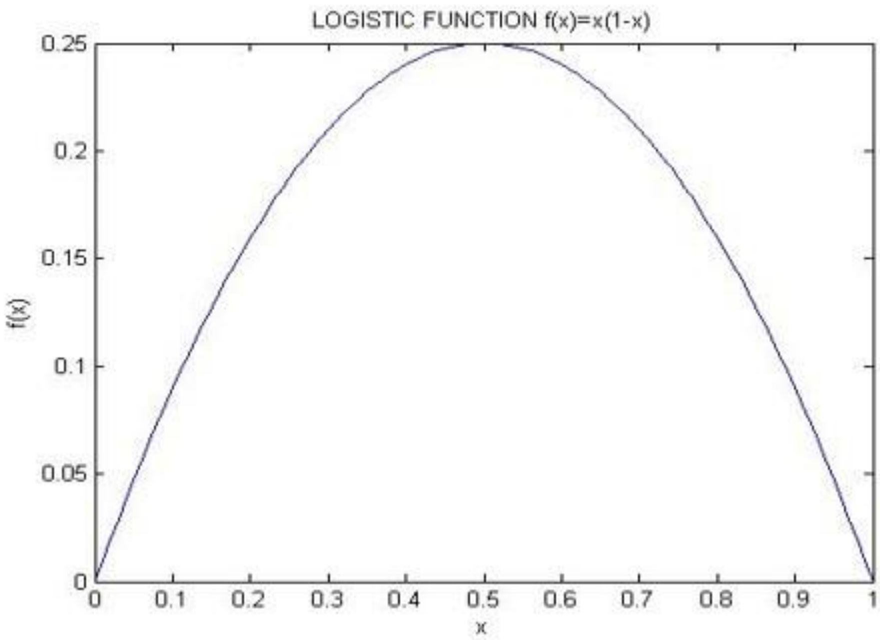


Figure 4.3: Graph of $f(x) = x_n - x_n^2$

The logistic function (equation) $f(x_n) = x_n(1 - x_n)$, is made up of two common features which are

1. The root of x in the function $f(x_n) = rx - rx^2$ If $f(x_n) = 0$,

   then $rx - rx^2 = 0$ $rx(1 - x) = 0$

   Implies, $rx = 0$ and $1 - x = 0$

   Then, $x = 0$ and $x = 1$

2. The maxima and minima of x depend on the parameter r, and to obtain this we work by, $f(x_n) = rx - rx^2$ $df = r - 2rx$ $df = 0$ implies that, $r - 2rx = o$ $r(1 - 2x) = 0$ $r = 0$

   $1 - 2x = 0$

   $$x = \frac{1}{2}$$

As stated above in feature (2), to determine that $x = \frac{1}{2}$ is the maxima, the determination of the hollow/concave nature is of important. Therefore, the second derivate or the rate of change of the function is used and the final conclusion is based on the sign. Thus, $d^2 f(\frac{1}{2}) = -2r$

Important Note 23: since it is open down at $x = \frac{1}{2}$*, the point* $(\frac{1}{2}, \frac{r}{4})$ *is the maximum point for the map, and also the sign being negative. Hence from the two main features of the logistic function it is clear that it passes through x at0 and1 and maximum at* $x = \frac{1}{2}$ *since it is concave down as shown in figure 4.4. df and* $d^2 f$ *are the first and second derivatives of the logistic function*

4.4.1 The solutions of the logistic function

A well detailed graphical representation of the logistic function will pave way for us to study it by showing the Orbit of a given point x when we impose a diagonal (dash) line y = x on the logistic function. Graphical representation of $f(x_n) = rx_n - rx^2_n$ where $r = 3.9$ and $y = x$ Clearly, the diagonal (dash) line and the logistic curve intersect at a point xas shown in the graphical display (figure 4.4) below and to find this point we equate the two functions and solve it algebraically. Thus, $y = f(x_n)$ implies

$x = rx - rx^2$ $rx^2 - rx + x = 0$

$x(rx - r + 1) = 0$, $x = 0$ and

$rx - r - 1 = 0$

Hence, $x = \frac{r-1}{r}$ or $x = 0$ are the two main solutions of the logistic function f(x).

Important Note 24: In Figure 4.12 the points of intersection of the diagonal line y = x with logistic map f(x) produce the two main solutions. And this give us the clue that the logistic model has $x = \frac{r-1}{r}$ *and* $x = 0$ *as the two fixed point, which also serve as solution.*

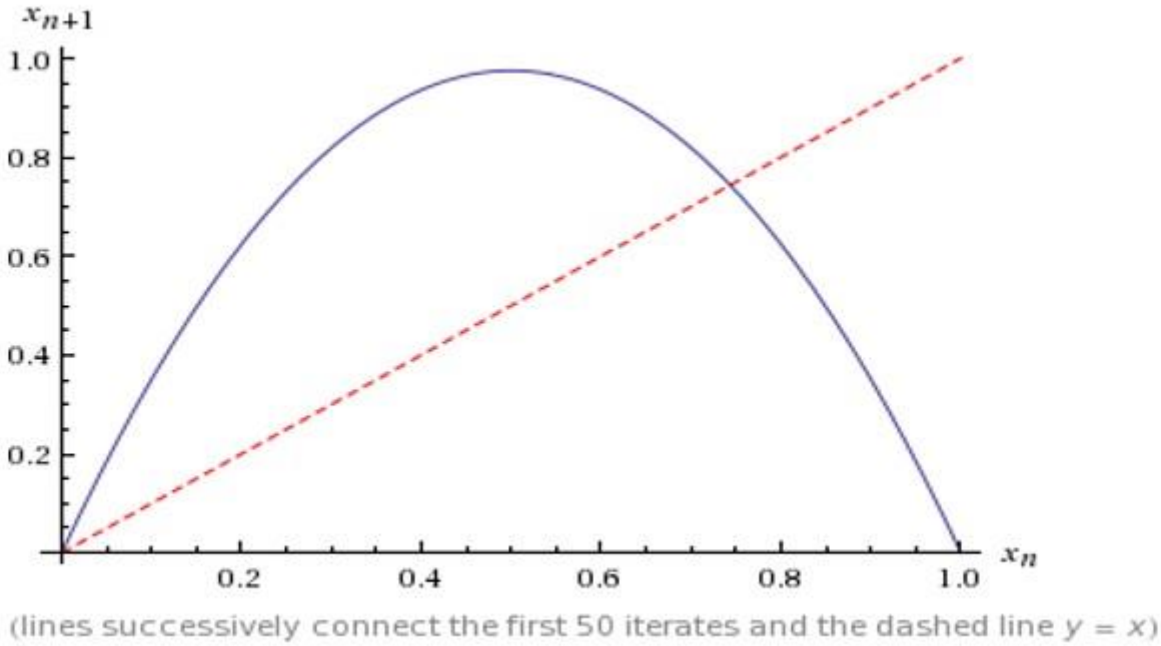


Figure 4.4: Graphical display of imposing a diagonal line on the graph of $f(x_n) = 3.9x_n - 3.9x_n^2$

4.4.2 Graphical Display of the Iteration of the Logistic Map

Example 4.14: Considering a function $f(x_n) = 2x_n(1 - x_n)$ *we can iterate and represent it graphically and analyze it behavior. Taking* x_0 *as the seed and starting on x-axis, we a vertical line can be drawn to* $f(x_n)$*. A horizontal line is drawn to meet the diagonal (dash) line* $y = x$ *for the next iteration and then vertically to meet* x_0*. So iterating the function,* $f(x_n) = 2x_n(1-x_n)$ *algebraically, with the seed* $x_0 = 0.1$*. At* $x_0 = 0.1$ $f(x_0) = 2(0.1)[1 - 0.1] = 0.18$ $f(x_1) = 0.30$ $f(x_2) = 0.42$ $f(x_3) = 0.50$

Iterating $f(x_n)$ with WolframAlpha Computational knowledge engine with graphical display

n	1	2	3	4	5
x_0	0.10000	0.18000	0.29250	0.41611	0.48593

Table 4.1: iteration of $f(x_n) = 2x_n(1 - x_n)$ with $x_0 = 0.10000$

Then the graphical display of this iterating function will be;

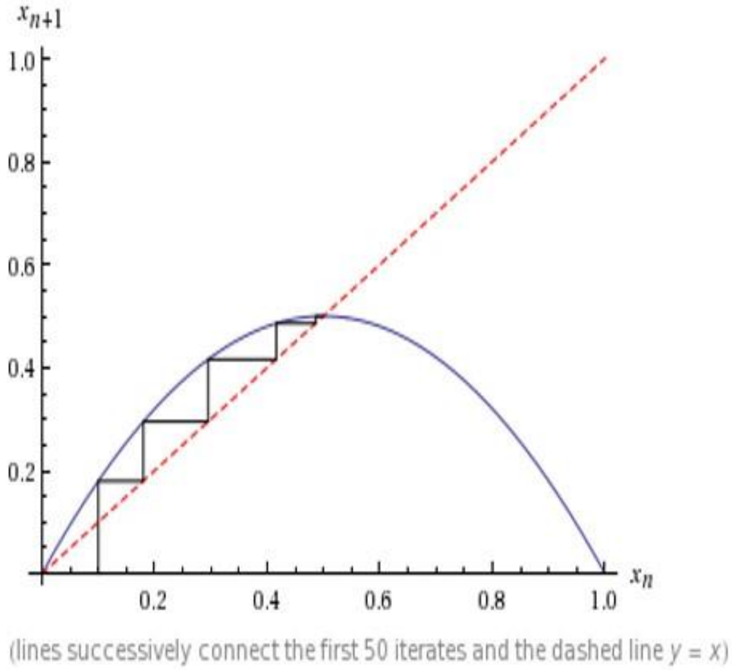


Figure 4.5: The graphical representation of the iteration of $f(x_n) = 2x_n(1 - x_n), x_0 = 0.1$

4.4.3 Determination of the fixed point of the logistic map

Since, we have already know the intersection points of the logistic map and the diagonal (dash) line y= x. We can then assume that the point of contact of the diagonal (dash)f(x)=y = xand $f(x_n) = x_n(1 - x_n)$, are the fixed points for this map $f(x_n)$. We can then evaluate the logistic map at the points of intersections $x_0 = 0$ and$x_0 = \frac{r-1}{r}$ By evaluating the function at $x_0 = 0$, $f(0) = r(0)(1 - 0)$ $f(0) = 0$

Similarly, evaluating using intersection$x_0 = \frac{r-1}{r}$

$$\begin{aligned}
f\left(\frac{r-1}{r}\right) &= r\left(\frac{r-1}{r}\right)\left(1-\frac{r-1}{r}\right)\\
&= (r-1)\left(1-\frac{r-1}{r}\right)\\
&= (r-1)\left(\frac{r-r+1}{r}\right)\\
&= (r-1)\frac{1}{r}\\
&= \frac{r-1}{r}
\end{aligned}$$

Clearly, from the above deductions both points of intersection $x_0 = 0$ and $x_0 = \frac{r-1}{r}$ are fixed points of the logistic function and serving as solutions.

Example 4.15: By considering the control parameter r = 2.7, and the fixed point $x_0 = (2.7 - 1)/2.7 = 17/27$ *on the function* $f(x_n) = 2.7x_n(1 - x_n)$ *it gives back the same point after several iterating, therefore serving as the fixed point for the function.*

Illustration 4.10: Thus if the fixed point $x_0 = 17/27$, then the fixed point of the function will be, $f\left(\frac{17}{27}\right) = 2.7\left(\frac{17}{27}\right)\left[1 - \frac{17}{27}\right] = \frac{17}{27}$.

Also by considering the graphical representation of orbits of the fixed point $x_0 = \frac{17}{27}$. The vertical black line in the figure 4.15 below is directed from the point of intersection to the x-axis x_0 of the logistic function $f(x_n) = 2.7x_n(1 - x_n)$.

n	0	1	2	3	4
0.62963	0.62963	0.62963	0.62963	0.62963	

Table 4.2: iteration of $f(x_n) = 2.7x_n(1 - x_n)$ with $x_0 = 0.62963$

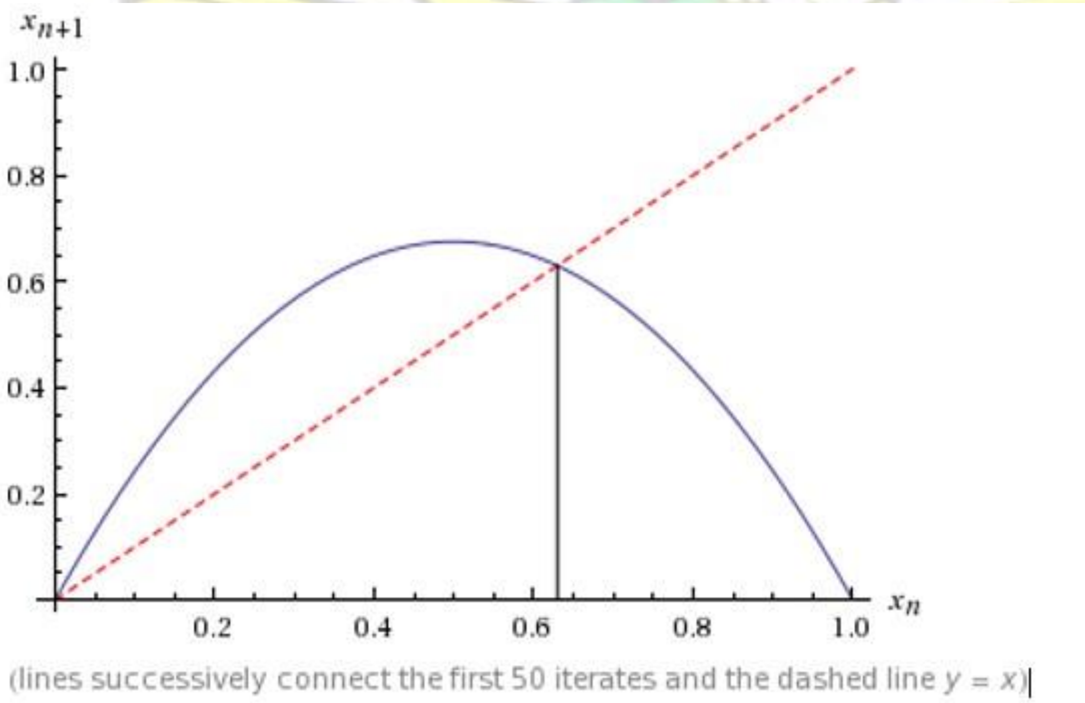


Figure 4.6: The graph of fixed point $x_0 = \frac{17}{27}$

Important Note 25: mathematically fixed points as in orbit are grouped within attracting, repelling or neutral.

4.4.4 Attracting fixed points and repelling fixed points of logistic map

Definition 4.13

1. *If* $|f^0(p)| < 1$*, then pis known as attracting fixed point(stable)*
2. *If* $|f^0(p)| > 1$*, then pointp is called a repelling fixed point (unstable).*
3. *If* $|f^0(p)| = 1$*, then fixed point pis called neutral or indifferent.*

Naturally,pas a fixed point is either converging to itself or diverging from itself as one increases iterations without bound where the point are closer Then, by investigating into the convergence and the divergence of case [1] and [2] of definition 4.13 using the mean value theorem.

Proof of (1): Clearly from the definition, since $|f^0(p)| < 1$, then a number $M > 0$ such that, $|f^0(p)| < M < 1$, then taking $\delta > 0$ so that $|f^0(p)| < M$ provided $b \in I = |p - \delta, +\delta|$

Let b be a point in I, then by the Mean Value Theorem $|f(b)-f(p)|/|b-p| < M$, then $|f(b) - f(p)| < M|b - p|$ $\frac{|f(b)-f(p)|}{|b-p|} < M, then |f(b) - f(p)| < M|b - p|$

Now, p as a fixed point, it implies that, $|f(b) - f(p)| < M|b - p|$

This simply means that, from $f(b)$ to pthe distance is smaller as compare to the distance from b to p, although $M|b - p|$ is small since $0 < M < 1$

Now, let us consider n¿ 0, then $|f^n(b)-p| < M^n|b-p|$ Now $M^n \to 0$ as $n \to \infty$, then $f^n(b) \to p$ as $n \to$ Then, from the inequality $|f^n(b) - p| < M^n|b - p|$ the points (nearby orbits) converge to the fixed point p **Proof of (2)**: similarly, if M ¿ 0, provided $I = |p - \delta, p + \delta| \in b, |f^0(p)| > M > 1$. By the Mean Value Theorem, let b be any point in I. Since $|f^0(p)| > 1$ then, $\frac{|f(b)-f(p)|}{|b-p|} > M$ implying that $|f(b) - f(p)| > M|b - p|$ Because the fixed point is p, it implies that, then $|f(b) - f(p)| > M|b - p|$ for $M > 1$ shows $f(b)$ is far from p than b.

For $n > 1$, will hold for or $|f^2(b) - f(p)| > M^2|b - p|$ meaning that the point f2 (b) and p are further away than that of $f(b)$ and p by deduction. If $n > 1$ is true then, $n > 0$ holds, hence $|f^n(b) - f(p)| > M^n|b - p|$ is proved. Therefore the points diverge from the fixed point p So it is clear that as n decreases we get **attracting fixed point** since fn(b) moves very close to p and as n increases **repelling fixed point** is obtain since $f^n(b)$ go further away from p

Example 4.16: Now we determine if the fixed point of the logistic function are attracting and repelling fixed point base on the definition above. We take the logistic function, $f(x_n) = rx_n(1 - x_n)$ Then, we take the derivative and solve the absolute.

At the point $x_0 = 0, f^0(x) = r - 2rx$,

1. *Attracting ,*

$$|f^0(0)| < 1$$

$$|r - 2r(0)| < 1$$

$$-1 < r - 2r(0) < 1 \; -1 <$$

$$r < 1$$

2. *Repelling,*

$$|f^0(0)| > 1$$

$$|r - 2r(0)| > 1, then \; r - 2r(0) > 1$$

$$or \; r - 2r(0) < 1 \; r > 1 or r < 1$$

Clearly, for proof 1, $r \in [0,1]$ is inside the domain of $r \in [1,4]$, hence $x_0 = 0$ is attracting but repelling for proof 2 since $r \in (1,4]$

At the fixed point $x_0 = \frac{r-1}{r}, f'(x) = r - 2rx$

1. Attracting, $\left| f\left(\frac{r-1}{r}\right) \right| < 1$

   $|-r+2| < 1$

   $-1 < -r + 2 < 1$

   $-3 < -r < -1$

   $1 < r < 3$

   Clearly $x_0 = \dfrac{r-1}{r}$ is asymptotically stable for $r \in [1,3]$ that is attracting fixed point as in Figure 4.7 below.

   Example: Algebraic and graphical illustration of the attracting fixed point of the logistic function, when r = 2.3 < 3 then the fixed point will be 0.57. *Algebraic; taking x = 0.10 as an initial point and the control parameter r =* 2.3 < 3. *Then for* $f(x_n) = 2.3x_n - 2.3x_n^2$ *at* $x_0 = 0.10, x_1 = f(x_0) =$ 0.207, $x_2 = f^2(x_1) = 0.378$, $x_3 = f^3(x_2) = 0.541$, $x_4 = f^4(x_3) = 0.571$, $x_5 = f^5(x_4) = 0.563$, $x_6 = f^6(x_5) = 0.570$, $x_7 = f^7(x_6) = 0.570$

   It is obvious that using $x_0 = 0.10$ as initial point and $r = 2.3 < 3$, the algebraic iterations shows that the outcomes get closer to the fixed point 0.57.

n	0	1	2	3	4
x_n	0.10000	0.20700	0.3775	0,54051	0.57123

Table 4.3: Iteration for $f(x_n) = 2.3x_n - 2.3x^2_n$ at $x_0 = 0.10000$

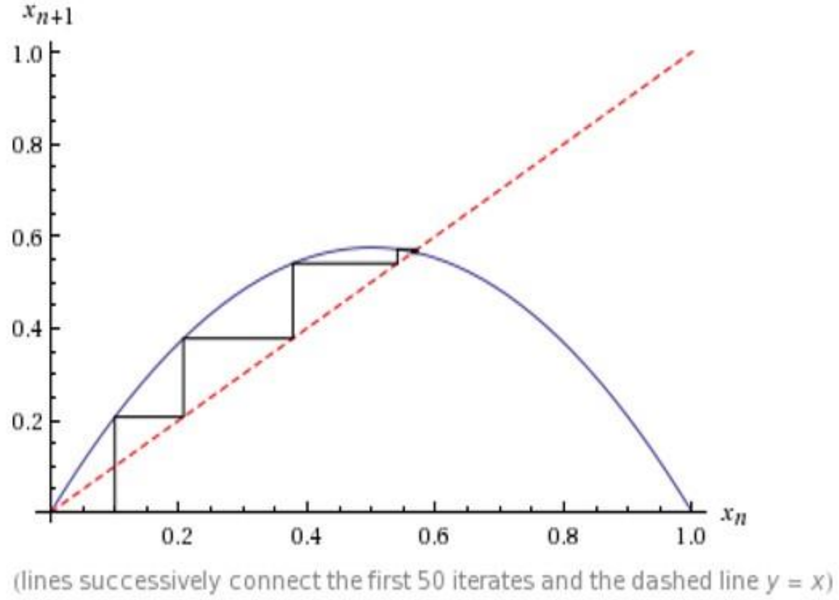


Figure 4.7: The graph of fixed point $x_0 = \frac{17}{27}$

Remarks 1:

From the graph the point of intersection as display in Figure 4.17 is 0.57 as indicated with a black line. By iterating the function using $x_0 = 0.1$ *as an initial point for* $f(x_n)$*, upon continues iteration it move gradually and converges to 0.57 as indicated by the black line, hence attracting when r lie within 0 and 3.*

2 Repelling, $\left|f'\left(\frac{r-1}{r}\right|\right) > 1$ $|-r+2| >$ 1, then $-r+2 >$ *or* $-r+2 < -1$

Therefore, $r < 1$ or $r > 3$ Hence, $x_0 = \frac{r-1}{r}$ is a repelling fixed point when $r \in (0,1) U (3,4)$ as shown in Figure 4.8 below.

Example:

Illustration of the repelling fixed point through algebraic and graphical representations when $r = 3.5 > 3$ *with a fixed point* $x_0 = 0.71$

Algebraically, taking $x = 0.10$ *as an initial point and the control parameter* $r = 3.5 > 3$.

Then for $f(x_n) = 3.5x - 3.5x^2$ *at* $x_0 = 0.10 x_1 = f(x_0) = 0.315, x^2 = f_2(x_1) = 0.755$, $x_3 = f_3(x_2) = 0.647, x_4 = f_4(x_3) = 0.799, x_5 = f_5(x_4) = 0.561, x_6 = f_6(x_5) = 0.862, x_7 = f_7(x_6) = 0.417$

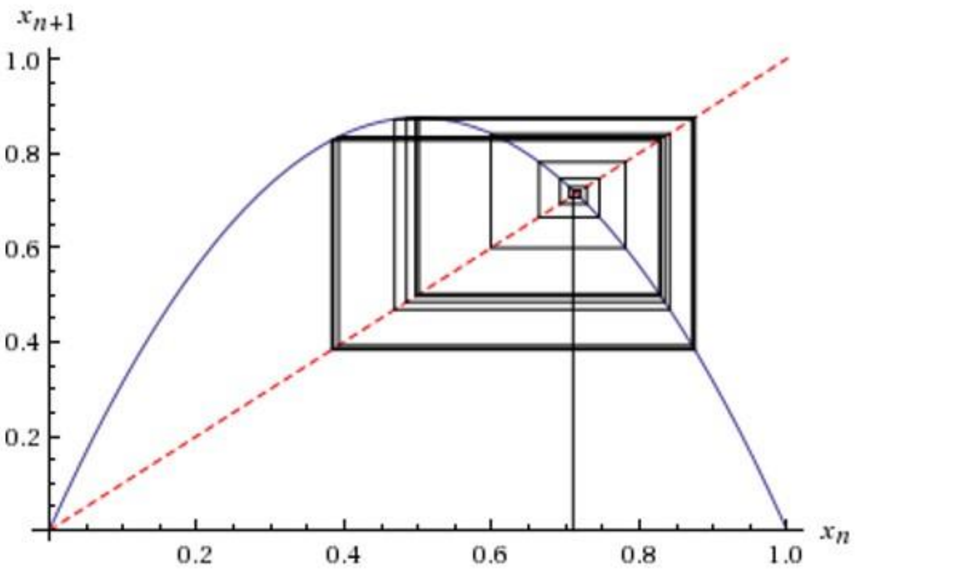
It can be noticed that frequently iterating the function, the outcomes keep moving away from the fixed point $x_0 = 0.71$.

n	0	1	2	3	4
x_n	0.71000	0.72065	0.70460	0.72849	0.69227

Table 4.4: the iteration of $f(x_n) = 3.5x - 3.5x^2$ at $x_0 = 0.71000$

Graphical representation of repelling state of the logistic function and the diagonal

line $y = x$ when $r = 3.5 > 3$ with a fixed point $x_0 = 0.71$



(lines successively connect the first 50 iterates and the dashed line $y = x$)

Figure 4.8: The graph of fixed point $x_0 = \frac{17}{27}$

Remarks 2: Clearly, the point of intersection of the graph in Figure 4.8 is 0.71*. Taking* $x_0 = 0.71$ *as an initial point, frequently iterating the function it keeps moving away from 0.71 and then diverges as shown clearly with the black line.*

Hence repelling when r is greater than 3 or less than 1.

4.4.5 The periodic orbits (period-2) and the bifurcation diagram of the logistic map

From the logistic function $f(x_n) = rx_n - rx_n^2$, r as a parameter of interest can lie within 0 and 3 i.e. 0 < *r* >< 3. If we iterate the function base on this interval it gives birth to an Orbit that alternate between two values (twice the period). The second iterate of the logistic function with the help of the fixed point gives the period-2 Orbits.

Example:

Considering the function or map $f(x_n) = 3.2x_n - 3.2x_n^2$ *for* $x_n \in (0,1)$*, let* $x_0 = 0.5$ *By iteration of the function f(x) the following sequence was obtain At*

$x_0 = 0.5, x_1 = f(x_0) = 0.80, x_2 = f^2(x_1) = 0.51, x_3 = f^3(x_2) = 0.80, x_4 = f^4(x_3) = 0.51, x_5 = f$

n	0	1	2	3	4
x_n	0.50000	0.80000	0.51200	0.79954	0.51288

Table 4.5: iteration of $f(x_n) = 3.2x_n - 3.2x_n^2$ with $x_0 = 0.50000$

Clearly, the iteration of the function $f(x_n) = 3.2x_n - 3.2x_n^2$ is a repeat of numbers that alternate between two values. Thus Orb = 0.51, 0.80 as the Orbits for the function $f(x_n) = 3.2x_n - 3.2x_n^2$ with $x_0 = 0.5$ as the initial point. This point x_0 is a period-2 points for the map.

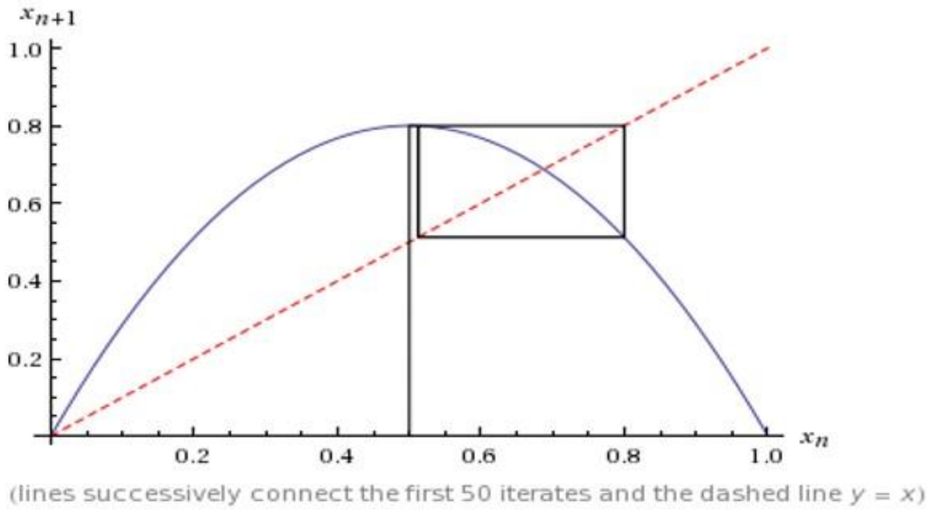


Figure 4.9: graphical display of $f(x_n) = 3.2x_n - 3.2x^2_n$

Definition 4.14

1. *periodic point p is attracting/converging if* $|(f^n)^0(p)| < 1$
2. *The periodic point p is repelling if* $|(f^n)^0(p)| > 1$
3. *The point p is neutral if* $|(f^n)^0(p)| = 1$

   Important Note 26: the prime denote differentiation with respect to p

Important Note 27: It is obvious that by the definition of the periodic n point as the iteration of the fixed point p in n time, thus $f^n(p) = p$ *for instance,* $f(f(p)) = p$*, then the conditions for a fixed point p to be attracting fixed point also hold for attracting periodic point p for period n point. So it is also true for the repelling periodic point p if a fixed point p is a repelling fixed point p.*

Now as it was clearly worked out for the fixed point of the logistic function for the period-1as $x = 0$ and $x = \frac{r-1}{r}$ algebraically, we can also do same for the second period. We then find fixed points for the logistic function for the period-2 point. We can then go ahead by finding the fixed points of the logistic functionf2(x) from

$f(x_n) = rx_n - rx^2_n$ for the period-2 point using the second iteration f2(x) of the logistic function. By evaluating $f^2(x) = f(f(x))$,

$$f^2(x) = r(rx(1-x))[1-(rx(1-x))]$$
$$= r^2x(1-x)[1-rx+rx^2]$$
$$= (r^2x - r^2x^2)[1-rx+rx^2]$$
$$= r^2x[1-x-rx+2rx^2-rx^3] \quad (4.1)$$

But for period-2 point,

$$f2(x) = x \quad (4.1)$$

Then by equating 4.1 and 4.2

$$x = r^2x[1-x-rx+2rx^2-rx^3]$$

$$0 = r^2x[1-x-rx+2rx^2-rx^3] - x$$
$$0 = x(r^2)[1-x-rx+2rx^2-rx^3]-1)$$
$$0 = -x\left(x-1+\frac{1}{r}\right)(r^2x^2-(r^2+r)x+r+1)$$

This implies $0 = -x$, $0 = x - 1 + \frac{1}{r}$ and $0 = r^2x^2 - (r^2 + r)x + r + 1$

Therefore $x = 0$, $x = \frac{r-1}{r}$ and $x = \frac{\pm\sqrt{r^2+2r-3+r+1}}{2r}$ are the solutions or the fixed points for period-2

But our interest is that $r > 0$ for x takes real. So by putting $r^2 - 2r - 3 \geq 0$

implies that , $(r-3)(r+1) \geq 0 \quad \Rightarrow \quad r \geq 3$ Or $r \leq -1$

Hence $r \geq 3$ will be our interest for this work at this section since our interest was that $r > 0$ for x to be real. At exactly $r = 3$ and beyond, the behavior of the logistic map begins to change and it is as the results of the increasing nature of the control parameter r, and this bring about bifurcation (splitting), bifurcation is simply when a system begins to change in behavior as a result of the control parameter been varied for a longer time. Now by carefully looking at the bifurcation diagram below the first bifurcation starts at exactly $r = 3$ and that is a period-2 periodic points.

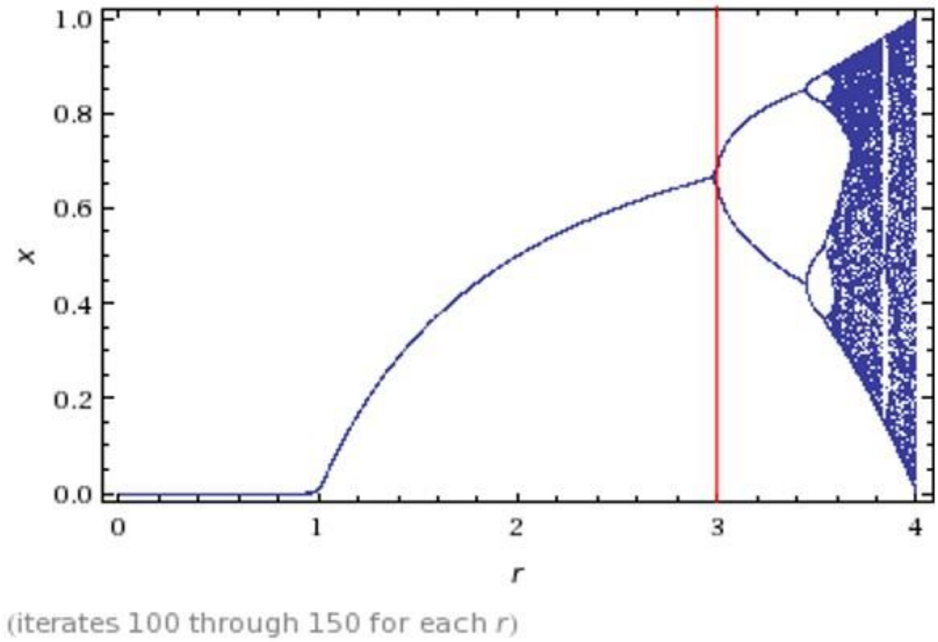


Figure 4.10: Bifurcation diagram of r and $f(x)$

It can be seen that, from the diagram; the logistic function $f(x)$ approaches 0 when r lies between 0 and 1. That is mathematically, $f(x) \to 0$ when $0 \leq r \leq 1$ and converges/attracting to a single point scaling from 0 to approximately 0.625.

This is to show that when we change the parameter, there is a change in behavior in the logistic map.

We can also notice in the figure 4.10 below, that when $r > 3$, there is a bifurcation that is a split of the function. Bifurcation at this point represents the number of periods an initial conditions has when r is an unknown value/number.

Important Note 28: one can notice that the moment period-4 cycles/orbits occur there is another bifurcation. This calls for the need to investigate $f^4(x) = 0$ *and the process at each bifurcation as r gets higher/bigger. And purposely because of this* *bifurcation nature of the logistic function as r keeps increasing, Mitchell Feigenbaum* (1978) *worked on this process and then arrived with the following table. What he discovered was*

$$\lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.6692016\ldots$$

which is now called the Feigenbaum Constant.

n	Bifurcation 2^n - cycle	r_n	$r_{n-1} - r_{n-2}$	$\frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}}$
1	1	3	-	-
2	4	3.449490	-	-
3	8	3.544090	0.44949	4.7515
4	16	3.564407	0.09460	4.6562
5	32	3.568759	0.020317	4.6684
6	64	3.56989	0.004352	4.6692
7	128	3.56993	0.001131	4.6694

Table 4.6: Bifurcation of the logistic function as r keeps increasing

It can be seen that the distance between successive bifurcations shrinks by a constant factor. Feigenbaum Constant/ Number is now used to forecast the next needed control parameterrat exactly the point the bifurcations is happening or occurring.

Important Note 29: the sequence $\{r_n\}$ *is an infinite series which also called a period doubling cascade: this when the control parameter of given system is been adjusted further and further produce sequence of doublings and further doublings of the repeating period, where* 2^n*-cycle exist for every positive integer n. From Feigenbaum computations the location of* r_n *numerically appear closer and closer together through successive period doubling bifurcation.*

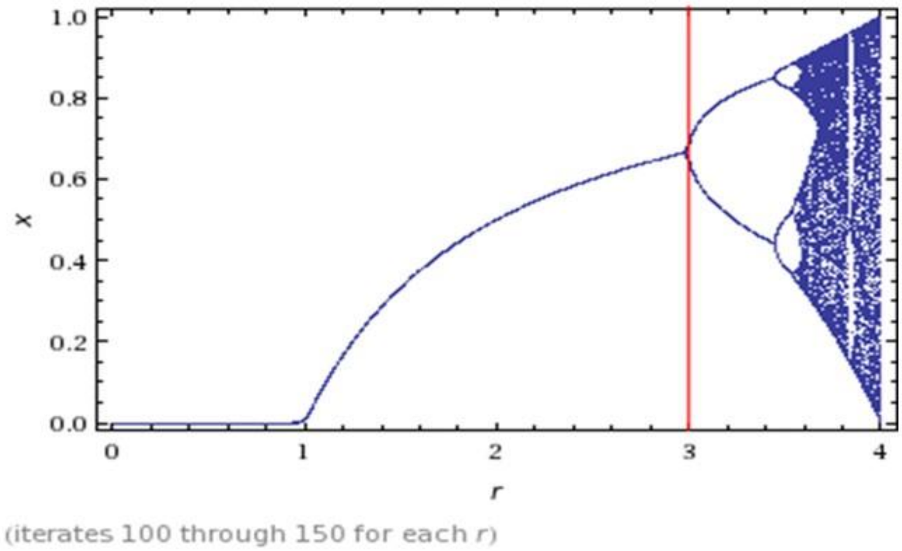


Figure 4.11: First bifurcation at 3

It can be deduce from the table and the bifurcation diagrams above that periodic orbits with period-2 points of $f(x)$ occur when $3 < r < 3.44$ and that of period-4 points happens at $3.44 < r < 3.54$. But period-3 points is seen right in a strip of small open space that occur just before $r = 4$. It can also be noticed that the map/function turns to be unstable when period-2 occurs as a results of bifurcation two stable orbits is attain,which corresponds to the two stable

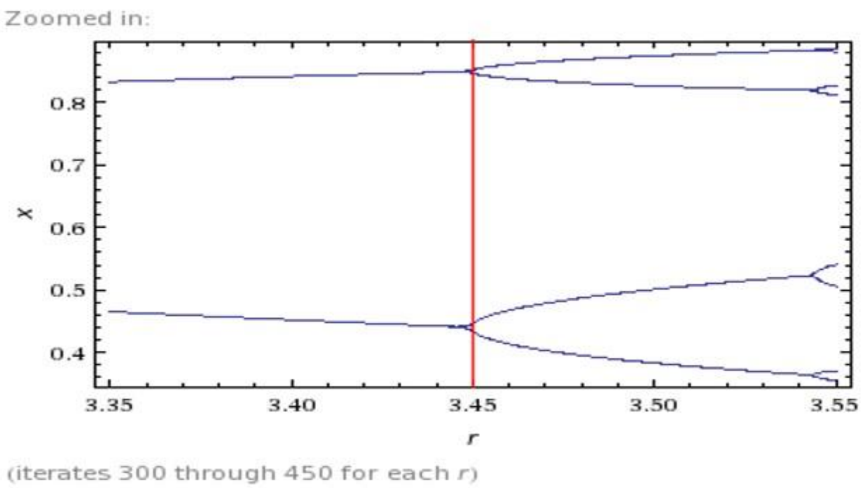


Figure 4.12: Second bifurcation at 3.45

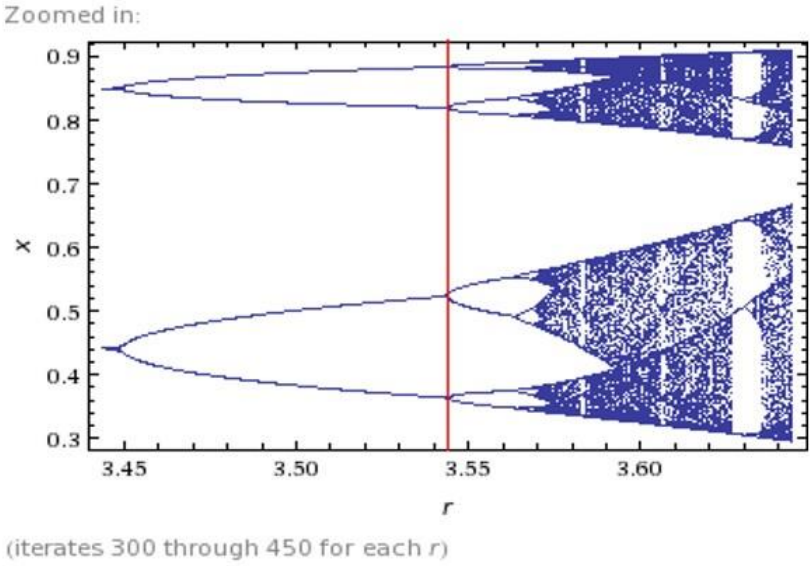


Figure 4.13: Third bifurcation diagram at 3.544

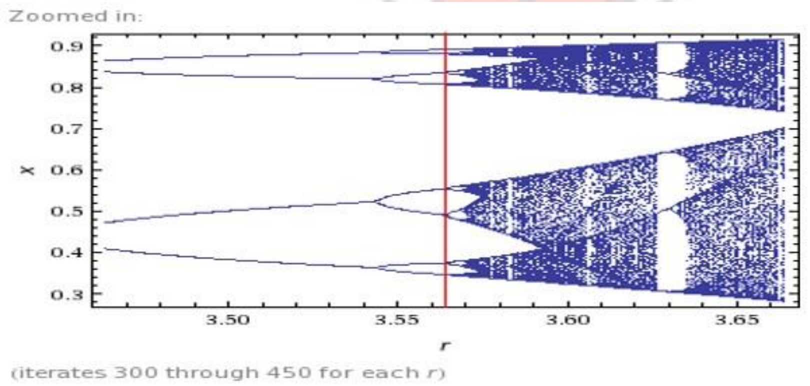


Figure 4.14: Fourth bifurcation diagram at 3.56 period-1 in the second iteration of $f(x)$. When r is slightly higher than 3.54 the function alternate between 8,16,32,64 as shown in Table 1 and figure 4.23 below. Also the lengths $r_{n-1} - r_{n-2}$ of the control parameter distances/gabs producing the same values of alternation reduce speedily. The ratio $\frac{r_{n-1}-r_{n-2}}{rn-rn-1}$ between the lengths $r_{n-1} - r_{n-2}$ of two successive bifurcation distances get closer to the value 4.6692016. And when $r = 4$, chaotic behavior of the map occurs.

4.5 Chaotic

4.5.1 The chaotic nature of the logistic map

The last nature/characteristic of this functionf $(x_n) = rx(1 - x)$, is the chaotic regime. To arrive at this chaotic regime, it has been shown in the various bifurcation diagrams and the Feigenbaum computations that the map moves faster or closer as r is been increase.

In our journey to the chaotic regime by logistic map, it will interest us to consider Alexander Sharkovsky and Li and Yorke theorems about the existence of period-3.

It is very clear that for period-3 points there are some indications of small open space which break beyond a certain point hence periodic leading to chaos. Alexander Sharkovsky constructed a series of numbers in his journey to chaos.

Theorem: for arbitrary continuous function $f : I \to I$ existence cycles of order m, involve the existence cycles of order n in the following order: $3 >> 5 >> 7 >> \ldots > 2\cdot 3 >> 2\cdot 5 >> 2\cdot 7 \ldots >> 2^n\cdot 3 \ldots >> 2^n\cdot 7 \ldots >> 2^n >> 8 >> 4 >> 2 >> 1$'Where '$>$ 'means 'implies 'm is to the left if 1 is read as $m > 1$.

Important Note 30:: the above ordering means that in logistic map/function, if period 2^n cycles are present then it also implies all periodic orbits of period 2i for i < n are also present.

That is, a period-7 orbit existence means that all periodic orbits exist except a period-5 and a period-3 orbit. Whereas period three orbit existences implies the existence of all possible periods (in one-dimensional map).

Theorem 5.10: (Sharkovsky, 1965).

Consider continuous map $x \to f(x)$ of the interval U into itself and assume that it has a k-periodic point. Then f has m-periodic points for all m such that k > m. In particular, if f has a 3-periodic point, it has orbits of any period

Important Note 31: Sharkovsky's theorem was rediscovered in a famous paper by Li and Yorke, in which the term "chaos"was used for the first time to indicate that the behavior of the orbits is very irregular. If periodic orbits of all possible periods exist it clearly gives an indication of a complicated behavior; however more was shown in the paper:

Theorem 5.11: (Li and Yorke, 1975).

Consider continuous map $x \to f(x)$ of the interval U into itself and assume that it has a 3-periodic point.

For the purpose of our work we will redefine both theorems for easy use.

Theorem 5.12 :(Sharkovsky, 1965).

If the function $f : \mathrm{R} \to \mathrm{R}$ is continuous. If fhas a periodic point with period-m and m > 1, then f also has a periodic point with period-1. Theorem 5.13: (Li and Yorke, 1975). Let the function $f : \mathrm{R} \to \mathrm{R}$ be continuous. If there exists period-3 point of the periodic orbit inf, then all periods in the periodic exist in f.

It is evidently clear that, period-3 points in logistic map will make our work very easy hence the need to be investigated if it exists. In figure 4.15 it shows that there is a fairly open gap exists shortly after $r = 3.8$ and by enlarging this bifurcation diagram period-3 points are seen in this gap. Note: it turns to indicate that a

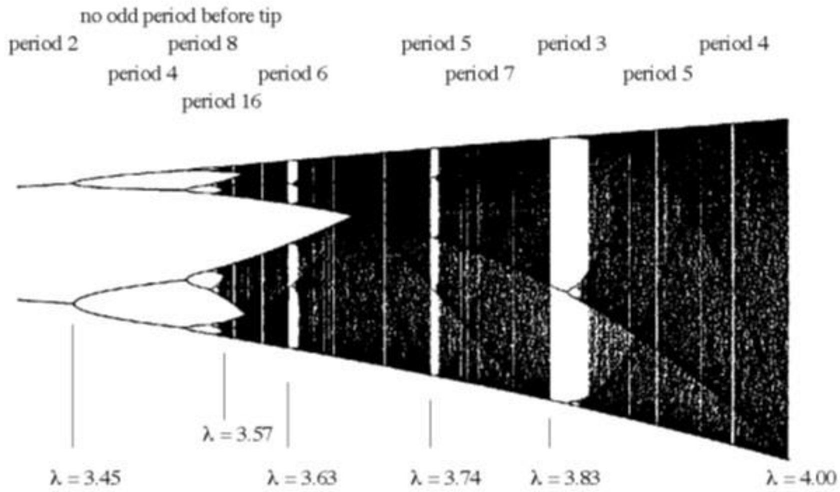


Figure 4.15: Bifurcation diagram for the quadratic/logisticmap(adapted from Tufillaro et al, 2013)

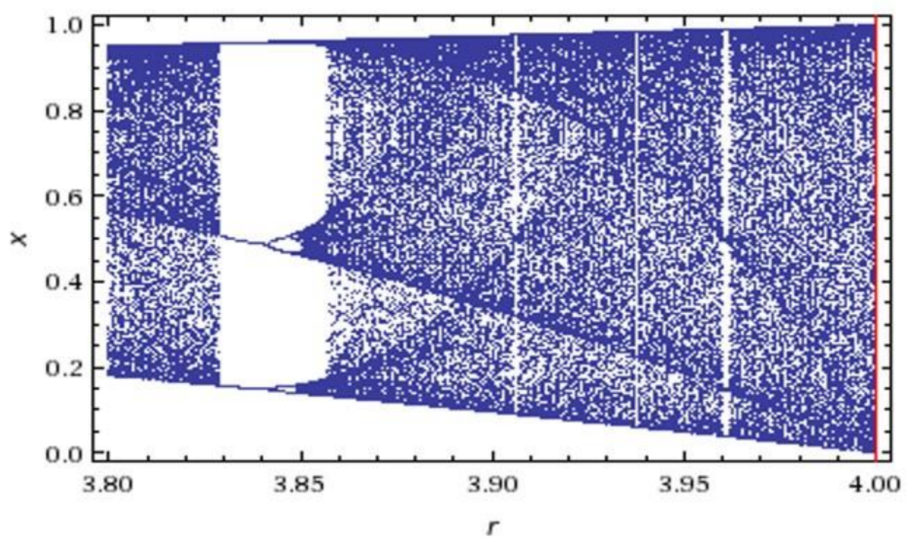


Figure 4.16: Bifurcation diagram for the quadratic/logisticmap(adapted from Tufillaro et al, 2013)

period-3 point exists ifr is lies approximately between 3.83 and 3.84 as shown in the figure 4.23 and figure 4.16 above. It will be much better and easier if we use algebraic approach. Then, by considering the iterations of the logistic function $f(x_n) = rx_n(1 - x_n)$ where $r = 3.83$, implies $f(x_n) = 3.83x_n(1 - x_n)$ Let $x_0 = 0.5$, then;

$x_1 = f(x_0) = 0.9575$,$x_2 = f^2(x_1) = 0.1559$,$x_3 = f^3(x_2) = 0.5039$,$x_4 = f^4(x_3) = 0.9574$ $x_5 = f^5(x_4) = 0.1561$,$x_6 = f^6(x_5) = 0.5044$,$x_7 = f^7(x_6) = 0.9574$,$x_8 = f^8(x_7) = 0.1561$

$$x_9 = f^9(x_8) = 0.5046$$

Now after successive iteration of the function $f(x_n) = 3.83x_n(1 - x_n)$, the sequence we are obtaining are a repeat of numbers that alternate between three values as shown above, thus 0.96, 0.16, 0.50.

This clearly, shows that there is the existence of period-3 in logistic function. Hence period-3 point exists implying that all other periods also exist. This affirms the theorems of (Sharkovsky, 1965) and (Li and Yorke, 1975).

Note:

- Since we are now convinced that period 3 also exist in the map as shown in the various figures and the above algebraic iterations. Hence period-3 point exists and if period-3 exits then periodic doubling leading to chaos.
- So the route to chaos can be seen through the existence of period-3, the doubling nature of the periodic orbits and all this routes relies on the strength of the valuer.

One natural question is how large does r have to be in order for chaos to be possible?

From the manipulations of the logistic map,$f(x_n)$is clearly a periodic points of period-2 point at intervals $3 < r < 3.448$ for the map, and Feigenbaum number shows the regular rate of the period doubles. By the approximation of rwhich $3.83 < r < 3.84$, $f(x_n)$ is a periodic points of period-3 whereperiod doubling starts when $r > 3.84$, hence chaotic at $r = 4$? This is as the results of $f(x_n)$ having an infinite period for the logistic map.

Now, by considering a concept that leads to chaos called sensitive dependence on initial conditions and setting $r = 4$. The logistic map becomes;

$$f(x_n) = 4x_n(1 - x_n)$$

Definition 5.10: Let $f : R \to R$, if $\delta > 0$ is a constant such that > 0, then there is x satisfying $|x - x_0| < \delta$ and an integer n, such that $|f^n(x) - f^n(x_0)| \geq \epsilon$.

Where the point x_0 is called sensitive point, andx is the initial condition. **Illustration:** Taking logistic function $f(x_n) = 4x_n(1 - x_n)$ and setting $x_0 = 0.3333$ as the approximation of $\frac{1}{3}$.

Then, the iteration of logistic function with initial value $\frac{1}{3}$. and its approximation 0.3333

When $x_0 = 0.3333$

$x_1 = f(x_0) = 0.8888, x_2 = f^2(x_1) = 0.3952, x_3 = f^3(x_2) = 0.9561, x_4 = f^4(x_3) = 0.1680, x_5 = f^5(x_4) = 0.5591, x_6 = f^6(x_5) = 0.9860, x_7 = f^7(x_6) = 0.0552$

When $\frac{1}{3}$

$x1 = f(x) = 8/9, x2 = f^2(x_1) = 32/81, x3 = f^3(x_2) = 6272/6561, x4 = f^4(x_3) = 0.1684 x_5 = f^5(x_4) = 0.5602, x6 = f^6(x_5) = 0.9855, x_7 = f^7(x_6) = 0.0572$

Note: increase in iterations increases the distance between each successive number. For the chaotic regime we base our argument on the definition and the above
iterations.

Then by setting $\delta = 0.000333$. We choose $\epsilon = 0.0001$, it can be seen that at f^4 the difference is 0.0004 which is more than . And for x and x_0 to get closer let $x_0 =$

0.3333, then 0.0000333 as the difference between x and x_0 due to the iterations and for δ = 0.000333 and our fixed $\epsilon = 0.0001$. Clearly, $|x - x_0| < \delta$ implying that $|0.0000333| < 0.000333$ and at f^7, the resulting difference between the values is 0.0020 which also exceed our fixed $\epsilon = 0.0001$.

Thus if $|f^{(7)}(x) - f^{(7)}(x_{(0)})| \geq \epsilon$ implying $|0.0020| \geq 0.0001$

Therefore since this hold for sensitive condition the map is chaotic at $r = 4$

Finally we can accept the fact that period-3 lead to chaos since it exits by the algebraic analysis and also through the zooming of the bifurcation diagram of figure 4.17 when r lies between 3.83 and 3.84 which are less than r equal to 4. And beyond this period-3 subsequent period occurs called the period doubling cascade into chaos. Also at r=4 the function is sensitive to initial condition therefore showing chaotic behavior.

4.5.2 Graphical iteration of the logistic maps $f(x)$, when $3.8 \leq r \leq 4$ into a chaotic orbit

The graphical nature of the logistic map $f(x_n)$ when $r = 3.8$ into chaotic regime

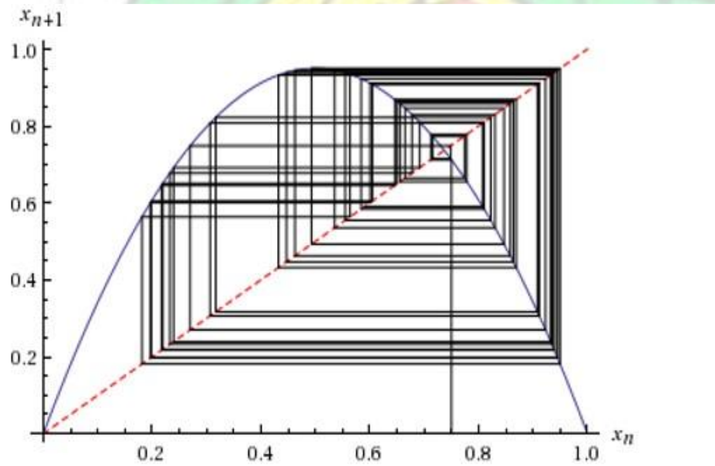


Figure 4.17: $f(x_n) = 3.8x_n(1 - x_n)$

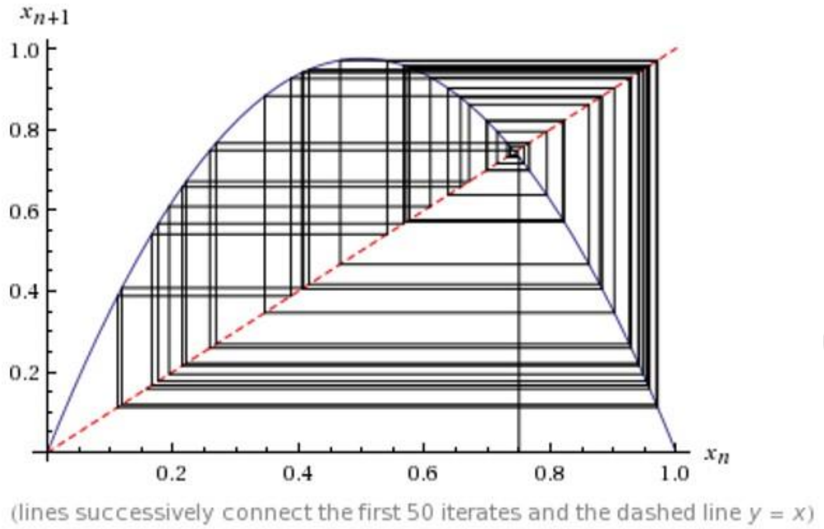
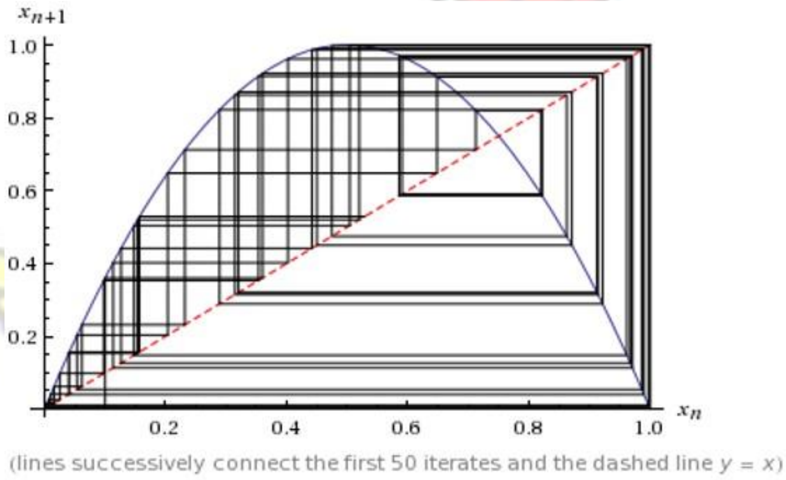


Figure 4.18: $f(x_n) = 3.9x_n(1 - x_n)$



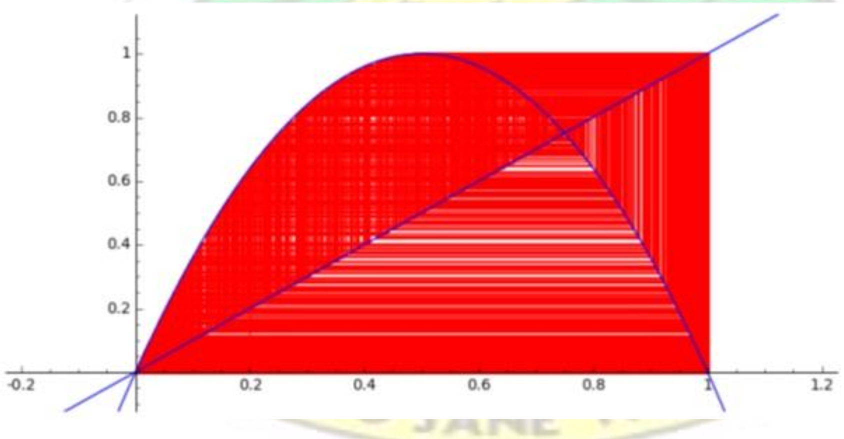


Figure 4.19: $f(x_n) = 4x_n(1 - x_n)$

Remark 3: Clearly the visual display of the above graphs of the logistic map shows how it is leading to a chaotic orbit on a successive number of iterations seen in Figure 4.25, Figure 4.18 and Figure 4.5.2 for the logistic function $f(x_n)$ *at* $r = 3.8$, $r =$

3.9 *and* $r = 4$ *with* $x_0 = 0.75$ *as the initial condition. In the graphical analysis of the logistic map, chaos keeps on occurring in way of filling almost the whole image upon a maximum number of iteration beyond 50 as displayed in figure 4.5.2. Hence it clearly shown in the figures above that; chaotic regime has no correlations, unpredictable and non-periodic. Therefore at* $3.8 \leq r \leq 4$ *the logistic function is chaos, most especially at* $r = 4$

Chapter 5

CONCLUSION

5.1 Conclusion

This work was mainly focused on the logistic map also known as the quadratic function as a non-linear dynamics by analyzing it to see it behavior and it route to chaos. A brief introduction of the logistic function was given for the bases of the main work. This then provided an opportunity for us to know that orbits (fixed points and periodic points) and complex chaotic behavior can be obtained from this map. The logistic map changes in behavior due to the control parameter introduce into the map and when allowed to vary continuously it dynamism begins to show.

It is then obvious that the concepts of chaos can be studied through logistic function/map. This then, gives a clear indication and true picture to study chaos through the logistic map as a perfect model. In studying the logistic map as a good and a perfect model into chaos, the concepts of iteration and orbits was also studied carefully as a foundation to the build-up of the main work. Work on iterating function and orbits from Roberts Devaney (2003) and Adams& Franzosa (2008) was studied vigorously. Hence concluding that iterations occurs when a particular function is evaluated over and over with its outputs starting with an initial inputs, the outputs obtained from each evaluation form the orbits in sequential order.

In the analysis of the logistic function/map as a model, it was shown that the behaviour of the logistic function which moves from periodic into chaos was based on the variation of the control parameter r. clearly the behavior of the logistic function into a period-1 points also known as the fixed points was as a results of the control parameter $r < 3$ having to two main solutions $x = 0$ and $x = \frac{r-1}{r}$ and will only results into periodic point when $r = 3$ or beyond. It was found that the period-1 point (fixed points) of the logistic function was attracting and repelling, that is converging and diverging when r lies within 0 and 3 and $r < 1$ and $r > 3$ respectively. On the issue of period-2 points of the logistic map, it was shown that the bifurcation diagram (figure 4.12) gives a better and transparent solution than that of the algebraic iteration of the function.

In the various bifurcation diagrams and the Feigenbaum computations, we found out that the route to chaos moves faster or closer when the control parameter r keeps on increasing. This makes the logistic function/map exhibits various types and kinds of periodicity or periodic behaviour at a higher points/orbits. It was shown that for period-3 points, there are some open spaces which break beyond a certain points and since it exits then periodic leading to chaos.

On the issue of the chaotic regime as an outcome of the logistic function, when the control parameter $r = 4$ in the function for an initial input, the chaotic region was sensitive to initial condition. Hence chaotic regime is deterministic, sensitive to initial conditions and non-periodic per the outcome of the research.

Graphically the function moves faster to chaos as shown in Figure 4.17: $f(x_n) = 3.8x_n(1 - x_n)$, Figure 4.18: $f(x_n) = 3.9x(1 - x)$ and Figure 4.5.2: $f(x_n) = 4x_n(1 - x_n)$. When the parameter r=4 when we keep on iterating beyond 50. Clear chaos fills almost the whole image.

It can also be concluded that, from the research when the parameter keeps increasing the nature or behavior of the logistic function changes, making it unstable. It then moves from the fixed points/ orbits to the period-2 periodic points which also gives as another solutions $x = 0,\ x = \frac{r-1}{r}$ and $x = \frac{\pm\sqrt{r^2+2r-3+r+1}}{2r}$. But an increase in *r* makes the solutions unstable and higher periodic oscillating occur. When the cycles keep on becoming unstable, period doubling gives way to a different regime hence chaos then occurs.

Finally, an increase in r makes the solutions unstable and higher periodic oscillating occur. When the cycles keep on becoming unstable, period doubling gives way to a different regime hence chaos then occurs. So the route to chaos can be seen through the existence of period-3, the doubling nature of the periodic orbits and sensitive to initial condition. These routes relies on the strength of the control parameter *r*

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