KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

PREDATOR - PREY MODEL OF SECURITY FORCES VERSUS CRIMINALS IN A CONTEMPORARY GHANAIAN COMMUNITY

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

This work is dedicated to my sweet mother Mabel Awalime, who worked assiduously to make me what I am now.

Abstract

A Predator prey model is developed from the population dynamics of security forces versus criminals in a contemporary community.

Using five major crime data covering the period of 2000 to 2010 from Ghana Statistical Service, two models were developed and a local stability analysis of the model after determination of the equilibrium points was investigated. With some assumptions, the parameters of the model are estimated and the simulation of the model for various scenarios using MATLAB is done.

These simulations give the typical almost sinusoidal trajectories for both the populations of the security forces and criminals. This appears to confirm that the propagation of security forces and criminals follows the predator - prey model. It also shows that the security forces and criminal activities keeps rising and falling with time.

It was observed that, by analysing the same point in sequential phases and finding the time in between them the approximate periods of the phase plane cycle is found to be 12 months.

The results obtained suggest that criminal activities persist if not the introduction of voluntary guards.

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Chapter 1

Introduction

1.1 Background of the Study

The word crime, from the root of latin *cerño* means "I decide, I give judgement". According to Fafa (2010), crime is the act of breaking rule(s) or regulation(s) for which a governing authority (via mechanisms such as legal system) can ultimately prescribe a conviction.

Crime has also been defined in social or non-legal terms. The social definition of crime is that it is behaviour or an activity that offends the social code of a particular community. Mower (1959) has defined it as "an anti-social act".

Karen (1956), has explained it as "an act or a failure to act that is considered to be so detrimental to the well-being of a society, as judged by its prevailing standards, that action against it cannot be entrusted to private initiative or to haphazard methods but must be taken by an organised society in accordance with tested procedures." Sellin (1970) has described crime as "violation of conduct norms of the normative groups". Clinard (1957) has, however, maintained that all deviations from social norms are not crimes. He talks of three types of deviation:

(i) tolerated deviation

(ii) deviation which is mildly disapproved and

(iii) deviation which is strongly disapproved. He perceives the third type of deviation as crime.

Based on the above definitions, crime can simply be defined as a deviant behaviour that violates prescribed norms or values which is frowned upon by society.

A criminal is someone who has committed (or been legally convicted of) a crime.

Criminals are common to all societies. To fight against them, contemporary communities takes different security measures such as, bringing security forces into the community.

Several forms of criminal activities occur in societies which may be described as major or minor crime by virtue of their nature and impact. These are categorised and defined by Levitt (1996) as follows:
Robbery- The taking or attempting to take anything of value from the care, custody or control of a person or persons by force, or threat of force, or violence and or by putting the victim in fear
Burglary- Entering a building unlawfully with intent to commit a felony or to steal valuable property.
Motor Vehicle Theft- The theft or attempted theft of a motor vehicle
Larceny - The unlawful taking of property from possession of another. Examples are thefts of bicycles or auto mobile accessories, shoplifting, pocket - picking, or the stealing of any property or article which is not taken by force and violence or by fraud, attempted larcenies are included. Embezzlement, "con" games, forgery and worthless checks are excluded.
Aggravated Assault- An unlawful attack by one person upon another for the purpose of inflicting severe or aggravated bodily injury. This type of assault usually is accompanied by the use of a weapon or by means likely to produce death or great bodily harm. Simple assaults are excluded Levitt (1996).
Murder- The unlawful killing of one human by another, especially with premeditated malice source.
Forcible Rape- The carnal knowledge of an individual against his or her will.

According to Act 2960, five degrees of offences are recognized in Ghana. Capital offences, for which the maximum penalty is death by hanging, include murder, treason and piracy. First- degree felonies punishable by life imprisonment are limited to manslaughter, rape, and mutiny. Second - degree felonies, punishable by ten years imprisonment include intentional and unlawful harm to persons, perjury and robbery. Misdemeanour's punishable by various terms of imprisonment, include assault, theft, unlawful assembly, official corruption and public nuisances. Increased penalties apply to individuals with a prior criminal record.

All crimes either major or minor have negative effects both on individuals and societies or the nation at large. These negative effects range from destruction and lost of property, lost of innocent lives, fear and panic, security threat, budget constraints and many more.

The security forces including the Police service, the Bureau of National Investigations (BNI) and the Armed Forces. The GPS (Ghana Police Service) under the ministry of interior is responsible for maintaining law and order.

The military continued to participate in law enforcement activities. The Bureau of national investigations handled cases considered critical to the state security and answered directly to the ministry of national security. The police maintained specialized units in Accra for homicide, forensics, domestic violence, trafficking in persons, visa fraud, narcotics, and cybercrimes.

Jane's, Sentinel Country Risk Assessment - Ghana, Security and Foreign Forces, updated 7 December 2011, further observed that the, Police-associated departments of the interior ministry include ... the Criminal Investigations Department (CID); ... Narcotics Control Board (NCB); ... Immigration Service; and Customs and Excise Service, in addition Jane explained that Ghana's Customs and Excise Service operates as part of the Police Service, and that border checkpoints are manned by the Immigration Service (under which there is a Border Patrol Unit) and the Customs and Excise Service. The army also conducts limited border security patrol

Several interventions have been put in place in preventing crime in the country. Some were time-based or discrete whilst others were continuous or permanent interventions. Some specific interventions put in place in Ghana in recent times to combat crime includes the following;

Police partner transport owners in crime combat, Xmas crime combat, Government ordering assemblies to name streets, Ghana and Togo police join forces to combat crime, police launched 'operation calm life', crime education and the establishment of community policing service.

1.2 Statement of the Problem

Crime has become one of the most critical challenges facing Ghana and the world at large. The rate of all criminal activities combined in 1990 and 1996 was 416.32, also, between 1997 and 2000 according to INTERPOL data, the rate of total index offences increased from 416.32 to 461.28, an increase of 10.8 percent.

NTERPOL data for Ghana (2000)

However in recent time, according to Ghana Statistical Service the level of criminal activities in the past 3 years has increased to 75 percent, with the following break down; Home broken and things stolen 56.25 percent; mugged or robbed 41.67 percent; car stolen 50 percent; attacked 50 percent; insult 18.75 percent; subject to physical attack because of skin colour, ethnic, origin, or religion 25 percent; dealing in drugs 37 percent; vandalism and theft 50 percent; violent crimes such as assault and armed robbery 81.25 percent; corruption and bribery 93.75 percent.

Hence there is the need for the security forces to be more proactive in deploying officers into communities to combat the propagation of criminals

The Ghana Police Service is divided into twelve (12) administrative regions. Below the regions, there are: 51 Police Divisions, Commanded by Divisional Commanders, 179 Police Districts Commanded by District Commanders, and 651 Police Stations and posts supervised by Station Officers. The Service has manpower strength of a little over twenty- nine thousand personnel with a male to female ratio of about 4:1 and police civilian ratio of about 1:850

$www.en.wikipedia.org/wiki/Law\_enforcement\_in\_Ghana$.

In an effort to combat the propagation of criminal activities, there is the need for a continuous quantitative monitoring of the criminals by deploying security forces in to contemporary Ghanaian community, and this can be effectively done within the field of mathematical ecology. Hence a predator-prey model of population dynamics of security forces versus criminals could help solve this problem.

1.3 Objectives of the Study

- To study the population or activity level dynamics of security forces versus criminals in a contemporary Ghanaian community using predator prey model.
- To perform a local stability analysis of the model after determination of equilibrium points using phase and trajectory plots.
- To determine the implications of the results in terms of interventions such as the introduction of volunteer reinforcement of the security forces.

1.4 Methodology

The data used for the study is a secondary data collected from Ghana Statistical Service, Accra, from 2000 to 2010. KNUST library, internet and Somanya main library were other sources of vital information to this research.

A predator - prey model of population dynamics of security forces versus criminals was formulated as a system of differential equations and the equilibrium points determined, the stability of the equilibrium points was also determined. Simulation using MATLAB was done. Graphs were plotted to show the trend of the incidence of criminals and security forces activities.

1.5 Justification of the Study

The problem of crime is a national issue. Several programs have been initiated in the country to help reduce crime but based on empirical evidence its prevalence rate is still high. Considering the simultaneous loss of innocent lives, cost in the medication of victims, loss of productive hours, destruction and loss of properties, fear and panic, security threats, budget constraints and many more, it requires renewed commitment from the Government, Non-Governmental Organizations and all in sundry to help fight for complete reduction in crime in the country.

The study seeks to develop a model that will be used to predict the growth of crime and will also help on systematic basis to monitor trends and rates in occurrence and its outcomes.

The study would serve as a guide to security forces and stakeholders in making informed and intelligent policy decisions with regard to the management of crime rates especially in the country. This would spearhead the developmental agenda in terms of peace and total calmness.

The model will be applied to advice the security forces and all those concerned in bringing crime under control, as to which group of the society should be targeted most.

1.6 The Study Area

Ghana is situated in West Africa and its capital city is Accra. It is bordered by Togo to the east, Burkina Faso to the north, Cote d'voire to the west and the Atlantic Ocean to the south. Ghana achieved independence from British rule in 1957. After the 1966 ouster of its independence leader, Kwame Nkrumah, the country was rocked for 15 years by a series of military coups and experienced successive military and civilian governments. The central intelligence, 'Word FactBook' , last updated January 2012, listed the area of the country at 238;533 sq km.

The provisional results of the 2010 population and Housing Census shows that the total population of Ghana is 24,233,431 (11,801,661 males and 12,233,770 females). The males form 48.7 percent of the population and the females constitute 51.7 percent. Additionally, it has growth rate 1.8 percent ; birth rate of 28.0/1000; infant mortality rate of 49.9/1000; life expectancy of 60.5; and density per sq km: 101 (infoplease, 2011).

1.7 Organization of the Thesis

The structure of the thesis is as follows;
Chapter one dealt with the Introduction, which comprises the Background of Study, the Problem Statement, the Objective of the Study, Justification, the study area and the Methodology.
The second chapter has dealt with the review of relevant literature showing the work done previously in the area of crime and its related issues and some differential equation models.
Chapter three dealt with the modelling and data analysis. The concepts of Chapter 2 would be applied in constructing and solving the predator - prey models as well as the model equations in their matrix forms. The equilibrium points were determined and the stability analysis of the points was done.
Chapter four dealt with the empirical results and analysis of model estimation using predator - prey model. The parameters of the differential equations were also determined and simulations of the model equations using MATLAB was done for the phase portrait as well as for the trajectories.
In chapter Five, the final chapter, the conclusion was made and this was followed by recommendations.

Chapter 2

Literature Review

2.1 Introduction

According to Leedy (1989), the purpose of the review of related literature in a study is to discover facts, findings, concerning the area of study and how they can propel the researcher to explore the unknown.

2.2 Crime Overview

Huang (2011), focused on identifying the patterns of the non-gun related crimes in the 21 areas of Los Angeles based on exploratory data analysis, principle component analysis, cluster analysis, as well as Pearson's X2 statistics to discover unusual crimes in areas with the following conclusions: The percentages of total crime in 21 areas are almost the same but the percentages of a specific crime type in 21 areas differ a lot, and each crime type has its own pattern. BTFV is the crime type that happens most. HOM, ARSON and KID were rare. The distributions of 13 crime types' frequency vary according to areas. Most crimes are most frequent in January and less frequent in February, from PCA, part of the variance (up to 50 percent) of frequency of 13 crime types in 60 months can be well explained by 1 or 2 PCs, according to the area.

Fung and Keung (1999) conducted three empirical studies detecting the determinants of crime in England, using time series analyses to look for cointegrating relationships between property crimes and unemployment as well as law enforcement instruments, employing panel data and corresponding techniques to control for area-specific fixed effects as well as the endogeneity of law enforcement variables and allowed crime rate to have spatial spillover effect, in other words, the crime rate in one area is affected by, in addition to its local crime-influential factors, the crime rates and crime-related factors in its neighbouring areas.

Martin and Sherman (1986), for example, conducted an experiment designed to evaluate a repeat offender project (named ROP) carried out by the Metropolitan Police Department of Washington D.C. The objective of ROP was to identify and apprehend active recidivists. To achieve this objective the police involved in the study created a special unit whose specific task was to draw up lists of potential targets and then attempt to gather evidence which would warrant their arrest and prosecution. The experimental design required ROP officers to randomly divide their list of potential targets into two groups, one of which became their focus of interest while the other (control) were designated off-limits to ROP officers but could be investigated, arrested and prosecuted by any other police. Despite some difficulties with the random assignment, the results of the study provided moderately strong evidence that ROP increased the likelihood of arrest of targeted repeat offenders. More importantly, ROP-initiated arrests were shown to be more likely than control group arrests to result in prosecution and conviction as felonies. Furthermore, those convicted were found to be more likely to receive a prison sentence and, if sentenced to prison, were more likely to receive a longer prison term. Against these findings, ROP was found to significantly lower the arrest productivity of officers involved in the project, primarily because police involved in the program generally effected fewer arrests for public order offences. This last result may, of course, have been a positive outcome.

Levitt (1996) offered evidence that legalized abortion has contributed significantly to recent crime reductions on their study titled The Impact of Legalized Abortion on Crime. Crime began to fall roughly 18 years after abortion legalization. The 5 states that allowed abortion in 1970 experienced declines earlier than the rest of the nation, which legalized in 1973. Roe V. Wade States with high abortion rates in the 1970s and 1980s experienced greater crime reductions in the 1990s. In high abortion states, only arrests of those born after abortion legalization fall relative to low abortion states. Legalized abortion appears to account for as much as 50 percent of the recent drop in crime.

Again, Appiahene (1998) studied and discussed the trends and patterns of robbery, and reactions to it in contemporary Ghana between 1982 and 1993. The study contends that robbery as a crime of opportunity appears to have been prevalent in pre-colonial times as well as during the subsequent period of slavery. Its trends and patterns however, have changed with the introduction of a monetary economy that has resulted in increased opportunities and targets for robbery. The descriptive statistical data derived from official police records concluded that even though the incidence and volume of robbery in Ghana is quantitatively small compared to the rates of other index offenses, and minuscule within the population at large, official reaction to it has been rather swift and merciless. No reason can be assigned to the executions other than deterrence, which raises questions as to its efficacy.

According to Agyemang (2012), in his thesis work Peng, etal used time series ARIMA model to make short-term forecasting of property crime for one city of China. With the given data of property crime for 50 weeks, an ARIMA model is determined and the crime amount of 1 week ahead is predicted. The model fitted and forecast results were compared with the SES and HES. It showed that the ARIMA model had higher fitting and forecasting accuracy than exponential smoothing and therefore would be helpful for the local police stations and municipal governments in decision making and crime suppression.

Also, Groove (1996) made two significant contributions to the advancement of knowledge within crime prevention through a comparison of systematic review and scientific realist evaluation methods for crime prevention. The first of these is to evaluate the success of repeat victimization prevention interventions. Interventions across four crime types are assessed herein, and the context-mechanisms-outcome configurations examined. The second contribution of their thesis assessed two techniques of meta-evaluation: systematic reviews and realist syntheses. Repeat victimization prevention is revealed as an effective way of reducing crime, with a need for further research to apply the principle across further crime types. A requirement is identified for a greater breadth and depth of information to be included in future crime prevention evaluations. The systematic review is shown to be a useful way of assessing the overall effectiveness of the interventions, whilst the realist synthesis fills in the detail of why some interventions work and others fail. It is concluded that both approaches to meta-evaluation have useful contributions to make, and that a third way incorporating the best elements from each method should be developed.

We conceptualize cybercrime as criminal activities or crimes in which computing devices or other forms of ICTs are the target source (Pati, 2003). From the perspective of ICT for development, it is not misplaced to say that cyber crime portends some dangers and have the potential to stall the developmental contributions accruable from a well-harnessed ICT adoption, diffusion and usage in Sub-Saharan Africa. Cyber fraud has a potential to widen the digital divide, crumble the information infrastructure and affect consumer confidence in online transactions (Salifu, 2008; Longe et al., 2009; Oumarou, 2007). Literature is, however, sparse on nation-specific extent of these fraudulent cyber activities as well as nation-specific measures put in place to address them. For instance, Ghana, our country of interest, in this research ranked among the top ten for the source of fraudulent cyber activities in the world with Nigeria ranking 3rd in the 2008 Internet Crime Report (2008). The Ghanaian government has made concerted efforts to create a knowledge-based economy, thereby making Ghana an ICT driven economy. The use of the Internet in Ghana has also seen a significant increase since the liberalization of the telecommunication industry in the 1990s. The country had 43 Internet users per 1,000 people in 2008 as compared to 1 Internet user in 1999 (ITU, 2009). The number of PC ownership doubled to 52 owners per 1,000 people between 1999 and 2005 (ITU, 2007). With these developments also come negative effects and unintended consequences of ICT, particularly, cyber crime.

Mohammed (2000) forecasted residential burglary in Kuala Lumpur. Compared to other crimes in Kuala Lumpur, residential burglary shows high number per year. The econometric and ARIMA model were constructed to develop the forecasting model, it is supported by statistical Software to validate the forecasting model by using 2004 burglary data.

Fernandez (2005), studied crime prevention and the perception of safety in campus design focusing on the outdoor environment on a college campus. The criteria for a safe design was developed from research gathered on crime prevention and the psychological reactions of users to exterior site features as well as crimes reported on the LSU campus were compiled on a crime map and analyzed to determine whether student perceptions of unsafe and safe areas were justified. The results exposed a perceived lack of safety among users in certain areas, the evaluations of both perceived safe and unsafe areas on campus brought about a better understanding of how users see and interact in their surroundings. In order to design or improve an area many factors must be in place to make the area safe for users and deter crime while at the same time being perceived as safe by the users to the site.

Boakye (2012), concluded that the introduction of the Community Policing unit of the Ghana Police Service as a crime combat intervention with regard to the

Eastern region had a significant abrupt impact of reducing the crime cases over the period under study by a monthly average estimate of approximately 16 cases with a long term effect being -16.2339.

According to Donkor and Boateng (2004), after he was elected for his first four-year term in December 2000, Ghanaian President J.A. Kufuor appointed two women to oversee two new ministries created specifically to act on behalf of the country's women and children, The Ministry of Women and Children's Affairs and the Ministry of Education's Girl-Child Education Unit. President Kufuor also established the Women's Endowment Fund to assist women entrepreneurs and affirmed the need for the Women and Juvenile Unit of the country's police service, which was created in 1998 to address an increase in cases of abuse against women and children.

However, both print and electronic media suggest that violence against women is increasing. More than 30 women have been murdered over the last five years by what 19 authorities describe as a serial killer or gang, and no one has been convicted in connection with the slaying. At least seven women were killed in the course of 2002 by their husbands or companions over alleged infidelities. It is on record that the Women and Children's Affairs Minister, Mrs. Gladys Asmah, condemned the killings, describing a dangerous, emerging culture in the country in which men lash out violently against women, not over alleged transgressions, but to control women's sexuality and sexual behaviour. Galvanized by Asmah's remarks, hundreds of women took to the streets in the capital of Accra on April 6, 2003 to protest the killings. According to data gathered by the International Federation of Women Lawyers (FIDA, 2003), a total of 95 percent of the victims of domestic violence are women. These abuses usually go unreported and seldom come before the courts. However, 1998 legislation doubled the mandatory sentence for rape. In late 1998, the police administration established a "women and juvenile unit" to handle cases involving domestic violence, child abuse, and

juvenile offences. The unit works closely with the Department of Social Welfare, FIDA, and the Legal Aid Board.
Tamekloe (1998), the media increasingly report cases of assault and rape. In the late 1998, a series of ”mysterious murders of women began to occur in the Mateheko area of Accra. Three of the 20 murders reportedly involved husbands’ suspicion of their wives’ infidelity. The men subsequently were arrested but not convicted. On several occasions, women of Accra demonstrated in a concerted effort to attract attention to violence against women.

2.3 Predator - Prey Model

Modelling is one of the ways by which formulae are used or deduced to help solve a real life situation, occurrence or a problem.
Model building is not only predictive but can also be descriptive; the important thing, for one to remember is that, the less complicated the system and the more information available, the greater the likelihood of a successful model.
Charkraborty (2006), presented numerical study of biological problem in a predator - prey system. They stated that according to Salvatore (1972) it was not until the Reverend Thomas Robert Malthus in his Essay on the principle of population”, however, that population studies became more quantitative in nature and attempts to express them in mathematical formulas were made. Malthus was an economist and not a mathematician, therefore he did not state the problem in mathematical terms, that is, to reach certain conclusions through mathematical reasoning and symbolism. He stated that, when unchecked, population increases in a geometric ratio whereas resources like food, shelter etc increase in an arithmetic ratio. He was concerned only with the socio-economic implications of the population problem and he did not pursue it mathematically. Nearly half a century later, it was Charles Darwin(1936), and, independently, Alfred R. Wallace(1963), who from these two facts, the geometric ratio of increase and the tendency of the average number of individuals in a given species to remain con-

stant, deduced the struggle for existence. Both Darwin and Wallace recognized that the problem of struggle for existence was quantitative in nature and it needed the methods of mathematics for its development. Darwin also emphasised the difficulties one would face if he were to pursue a mathematical solution of the problem because of the lack of quantitative data, the difficulty of securing it, and the enormous complexities of the problem.

Souvik et al. (2013),model a predator-prey - disease model with immune response in the infected prey. The basic reproduction number of the within - host model is defined and it is found that there are three equilibria: extinction equilibrium, infection-free equilibrium and infection-persistent equilibrium. The stabilities of these equilibria are completely determined by the reproduction number of the within-host model. Furthermore, they define a basic reproduction number of the between host model and two predator invasion numbers: predator invasion number in the absence of disease and predator invasion number in the presence of disease they have predator and infection - free equilibrium, infection-free equilibrium, predator-free equilibrium and a coexistence equilibrium. They determine the local stabilities of these equilibria with conditions on the reproduction and invasion reproduction numbers. Finally, they show that the predator-free equilibrium is globally stable.

Alfred, who independently stated the principle of evolution of the species by natural selection, held similar views concerning the possibility of using mathematics in describing the struggle for existence. He noted that the number of individuals of a given species remains approximately constant and attributed it to the limited supply of food and to the action of the various enemies, but he recognized that the factors present in the struggle are so many and varied that he could not possibly give a quantitative description of the phenomenon. In the nineteenth century problems of population growth began to be viewed in a larger biological context which included the study of variations in the number of individuals of all species. The first authors who attempted to give a quantitative formulation of the theory of natural selection were those, such as Francis Galton and Karl Pearson, who founded the School of biometry and came to be known as the biometricians.

According to Cohen and Murray (1981), "During World War 1, fishing was limited and in the upper Adriatic Sea certain kinds of fish was found to be more abundant than before. In order to understand this phenomenon the Italian mathematician V. Volterra(1926), constructed an analytical model describing a two species predator - prey community. At the same time, the same model, independently of Volterra, was produced by an American ecologist and mathematician Lotka. Several of the two species predator - prey and competition models are now called Lotka - Volterra models in honour of these ecologists who made the first breakthrough in modern mathematical biology.
According to Murray (2002), while testing the Lotka - Volterra model, Gause(1934), used Paramecium caudatum as prey with Didinium nasutum as predator. In his laboratory experiment, the prey was totally consumed by the predator which then starved. He was able to obtain coexistence only by adding a few Paramecium shortly before the predator became extinct. This perpetuated the system for one more cycle until the addition of prey was again necessary. Gause concluded that coexistence in laboratory system can be obtained by the immigration of prey from outside the system. Luckinbill (1973), demonstrated that coexistence can be obtained where prey are afforded a refuge in which some portion of the prey population is exempt from prey predation.

Huffaker (1958), obtained the coexistence of two species of mites, one of which was predatory, by adding a physical complexity that enhances prey dispersal while hindering the predator movement. Hence two important questions arises whether coexistence can be obtained in a physically homogeneous environment, and if so, what are the mechanisms to be followed. The studies of Flanders and others suggested a simple hypothesis behind such coexistence. As the number of predators in a system increases, the prey population decreases and is eventually reduced to a density at which they are too scarce to find. This results in the decline of predator population that allows temporary escape and reproduction of some prey. Thus coexistence may occur in a physically homogeneous environment where prey can be sufficiently scarce that predators are unable to find them all, but still survive. This suggests that coexistence might be obtained in a laboratory environment by reducing the frequency of contact between predator and prey so that, if the prey density is low, prey will be difficult for the predator to find. Hence, without introducing physical complexity, it is possible to obtain the coexistence of both prey and predator for a prolonged time by reducing the frequency of contact between them".

Based on this hypothesis Luckinbill conducted a laboratory experiment with Paramecium aurelia as prey and Didinium nasutum as predator in 6 ml of half - strength cerophyl medium thickened with methyl cellulose. The thickened mixture provided a refuge for the prey by restricting the frequency of contacts between predator and prey. He obtained sustained oscillatory behaviour in populations of Paramecium aurelia and Didinium nasutum. According to Luckinbill, this implied their coexistence even in laboratory conditions for a longer period, which is in qualitative agreement with the predictions of predator - prey models". From his experiment, Luckinbill concluded that in systems with half - strength medium, coexistence was maintained. In systems with full - strength medium, Didinium became extinct and the prey survived. In systems with full - strength medium and additional bacteria as food for prey, the predator captured all the Paramecium and both components became extinct.

According to Aspriha Chakraborty, the logistic Lotka - Volterra predator - prey

equations with diffusion based on Luckinbill's experiment with Didinium nasutum as predator and Paramecium aurelia as prey, have been solved numerically along with a third equation to include prey taxis in the system. The effect of taxis on the dynamics of the population has been examined under three different non - uniform initial conditions and four different response functions of predators. The four response functions are Holling Type 2 Response, Beddington Type Response or Holling Type 3 Response, a response function involving predator interference and a modified sigmoid response function. The operator splitting method and forward difference Euler scheme have been used to solve the differential equations. The stability of the solutions has been established for each model using Routh - Hurwitz conditions, variational matrix. This has been further verified through numerical simulations.
The numerical solutions have been obtained both with and without prey - taxis coefficient. The effect of bifurcation value of prey- taxis coefficient on the numerical solution has been examined. It has been observed that as the value of the taxis coefficient is increased significantly from the bifurcation value chaotic dynamics develops for each model. The introduction of diffusion in predator velocity in the system restores it back to normal periodic behaviour.

Siekmann (2008), studied and discussed that ecosystems are characterised by a wealth of highly complex interdependencies. Mathematical models have already contributed considerably to gaining a better understanding of prey-predator interactions and epidemic spread; however often, such models also lead to new problems: A classical example from prey-predator dynamics is the paradox of enrichment, a very recent branch of research is the extension of epidemiological models by a pathogen population: Whereas infection transmission is usually modelled by direct contacts of infected and susceptibles, in the new approach the infection is spread by a pathogen population that moves freely in the environment the replication process in the host is integrated. In fact, the results lead to much more general conclusions about competition in ecosystems.

Azugah and Oduro (2012), apply the predator - prey model to come out with a model for the propagation of HIV, taking into consideration the population of newly infected males and newly infected females at a given time. This particular model is conjectured because it has been observed that there is an almost sinusoidal rising and falling of the time series trajectories of newly infected male and female cases of a data collected in Ghana. With some few assumptions made the model is formulated and the analysis shows that it conforms to the predator - prey model. Even though parameters in respect of newly infected males and females were not directly available for the simulation, with some assumptions, the parameters of the model are estimated and the simulation of the model for various scenarios using MATLAB is done. These simulations give the typical almost sinusoidal trajectories for both the populations of the newly infected males and newly infected females. This appears to confirm that the propagation of HIV follows the predator - prey model. It also shows that the rate of infections of HIV keeps rising and falling with time. The curves also show that more females are infected at any given time than males.

Again Azugah and Oduro (2012), in their research, stated that Aggarwala (2001) came out with two models on the spread of HIV. In first part, he discussed a ratio dependent predator - prey model and applied it to the spread of HIV/AIDS in a society. For this model, he divided the population into two classes; the HIV positive individuals and the HIV negative individuals. The model was then applied to the data available for the Canadian society obtained from Health Canada and statistics Canada, it was predicted that the number of HIV positive as individuals would go up for the next five years. The results were compared with actual numbers and the comparison was satisfactory. In the second part, a considerably more detailed density dependent model for the propagation of HIV/AIDS was developed. This model divides the society into three classes; HIV negative individuals, HIV positive individuals who have not developed AIDS disease and those who have developed AIDS. This model was also applied to the data available from Health Canada. It was also established that the model was physically relevant by showing that in the model the number of both HIV positive and HIV negative people stay positive and finite for all $t > 0$

2.3.1 Predator - Prey Model: Lotka - Volterra Systems

When species interact the population dynamics of each species is affected. In general there is a whole web of interacting species, sometimes called a trophic web, which makes for structurally complex communities. According to Murray there are three main types of interaction.

(i) If the growth rate of one population is decreased and the other increased the populations are in a predator - prey situation.

(ii) If the growth rate of each population is decreased then it is competition.

(iii) If each population's growth rate is enhanced then it is called mutualism or symbiosis.

Volterra (1926) first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. If N(t) is the prey population and P(t) that of the predator at time t then Volterra's model is

$$\frac{dN}{dt} = aN - bNP$$
$$\frac{dP}{dt} = cPN - dP$$

where a, b, c and d are positive constants. The assumptions in the model are:

(i) The prey in the absence of any predation grows unboundedly in a Malthusian way; this is the aN term in.

(ii) The effect of the predation is to reduce the preys per capita growth rate by a term proportional to the prey and predator populations; this is the bNP term.

(iii) In the absence of any prey for sustenance the predators death rate results in exponential decay, that is, the dP term in.

(iv) The preys contribution to the predators growth rate is cNP; that is, it is proportional to the available prey as well as to the size of the predator population. The NP terms can be thought of as representing the conversion of energy from one source to another.

2.4 Concluding Remarks

Predator - Prey model has been the basis of some interesting ecological studies. In particular, Volterra and D'Anconna successfully used the model to explain the increase in the population of sharks in the Mediterranean during world war 1 when the fishing "prey" species decreased.

Based on the above literature reviewed by previous authors on criminal activities. It will be concluded that predator - prey model is a powerful tool in evaluating the propagation of criminals and the security forces in any contemporary community

Chapter 3

MODEL FORMULATION

3.1 Introduction

In this chapter, certain key concepts and methods of analyzing differential equation that are central to the project have been discussed and the model developed to predict the propagation of crime in contemporary Ghanaian community. The adequacy of the model will be judged by empirically fitting it to the data collected from Criminal Investigation Department (CID) Headquarters Accra.

3.2 Definitions

3.2.1 Differential Equation

Differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

The order of a differential equation is the largest derivative present in the differential equation. In the differential equations listed below (1) is a first order differential equation, (2) and (3) are second order differential equations, (4) is a fourth order differential equation. (Dawkins, 2007).

$$m\frac{dv}{dt} = F(t, v) \tag{1}$$

$$sin(y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2e^{-5y} \tag{2}$$

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{3}$$

$$y^{(4)} + 10y'' - 4y' + 2y = cos(t) \tag{4}$$

3.2.2 Ordinary and Partial Differential Equations

A differential equation is called an ordinary differential equation, abbreviated by ode, if it has ordinary derivatives in it. Likewise, a differential equation is called a partial differential equation, abbreviated by pde, if it has differential derivatives in it. In the differential equations above (1), (2) and (4) are ode's and (3) is a pde. (Dawkins, 2007).

3.2.3 Steady state

A steady state or critical point of a system of differential equations is the set of point for $\dot{x} = 0$ (Morris et al, 1974).

3.2.4 Stability of equilibrium point

Suppose that the system of differential equations $\dot{x} = u(x, y)$ has an equilibrium points at $x = x_e, y = y_e$. The equilibrium point is said to be stable when all points in the neighbourhood of the equilibrium point remain in the neighbourhood of the equilibrium point as time increases (see Figure 3.1) (Leah, 2005).

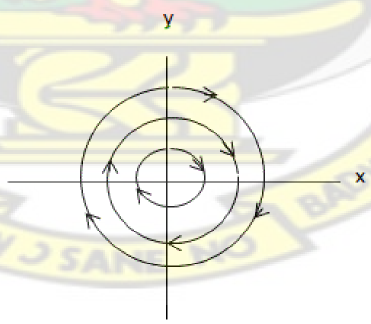

Figure 3.1: Stable centre

3.2.5 Unstability of equilibrium point

Suppose that the system of differential equations $\dot{x} = u(x, y)$ has an equilibrium points at $x = x_e, y = y_e$. The equilibrium point is said to be unstable when all points in the neighbourhood of the equilibrium point moves away from the neighbourhood of the equilibrium point as time increases (see Figure 3.2) (Leah, 2005).

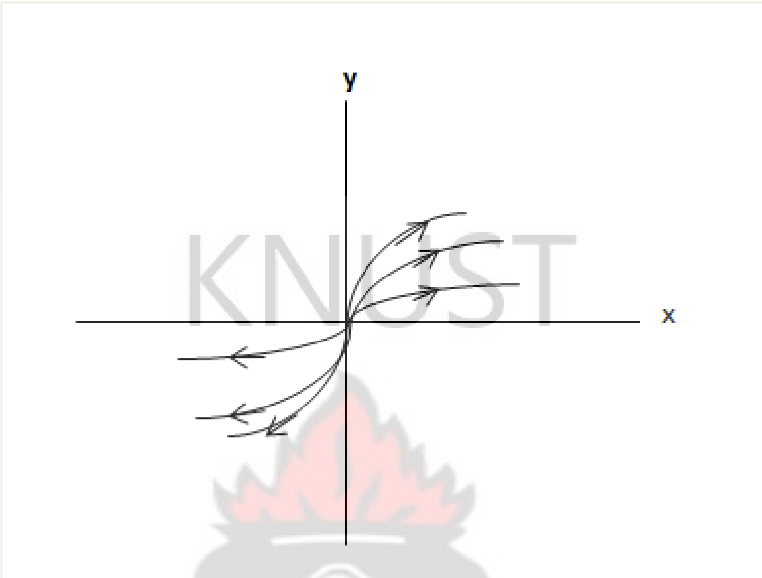

Figure 3.2: Unstable

3.2.6 Asymptotically stable

If and only if all eigenvalues of a trivial solution have negative real parts then the solution is said to be asymptotically stable. (Morris et al, 1974).

3.2.7 Linearisation and stability

Stability properties depend on the behaviour of the system near the equilibrium point, hence in conducting analysis of stability, it is convenient to replace the full non-linear description by a simpler description that approximates the system near the equilibrium point, often a linear approximation is enough to clarify the stability properties, this idea of checking stability by examinations of a linearised version of the system is referred to as Liapunov's First method. This is usually the first step in the analysis of any equilibrium point. The linearization of the non linear system is based on linearization of the non linear function F in its description. An nth-order system is defined by n function, each of which depends on the n variables. In this case each function is approximated by the relations. (Waltman, 1991).

$f_1(\bar{x}_1+\bar{y}_1, \bar{x}_2+\bar{y}_2, ..., \bar{x}_n+\bar{y}_n) = f_1(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n)+\partial/\partial x_1 f_1(\bar{x}_1, \bar{x}_2, ...\bar{x}_n)y_1+ \partial/\partial x_2 f_2(\bar{x}_1, \bar{x}_2, ...\bar{x}_n)y_2 + ... + \partial/\partial x_1 f_i(\bar{x}_n, \bar{x}_n, ...\bar{x}_n)y_n + ...$ The linear approximation for the vectors f(x) is made up of the n separate approximations for each component function. The complete result is expressed compactly in a vector notation as $f(\bar{x} + y) = F(\bar{x}) + F(y)$

In this expression F is $n \times n$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & ....\frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & ....\frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & ....\frac{\partial f_1}{\partial x_1} \end{pmatrix}$$

This is called the Jacobian matrix To determine the stability properties of a linear system, we determine the location of the eigenvalues of the system matrix and the stability properties of the linear version of a non linear system.

The importance of this technique is that except for the boundary situation, the eigenvalues of the linearised system completely exposes the stability properties of an equilibrium point of a system. This is because, for small deviations from the equilibrium point, the performance of the equilibrium is approximately governed by the linear terms. These terms dominate and it also determine the stability provided the linear terms do not vanished.(Waltman, 1991)

3.2.8 The Phase Plane

Consider the linear system with constant coefficients

$x^1 = ax + by$

$y^1 = cx + dy$..(1)

They can be solved explicitly by linear system; this can be regarded as the first approximation of the nonlinear system

$x^1 = f(x, y)$

$y^1 = g(x, y)$...(2)

Where $f(x, y)$ and $g(x, y)$ satisfy $f(0, 0) = g(0, 0)$ =0 and have continuous partial derivatives, which at the origin are labelled as

$\frac{\partial f(0,0)}{\partial x} = a, \frac{\partial f(0,0)}{\partial y} = b, \frac{\partial g(0,0)}{\partial x} = c, \frac{\partial g(0,0)}{\partial y} = d,$

It can be observed that exact knowledge of the behaviour of solution (1) can often give qualitative knowledge of the behaviour of solutions of (2) near the origin. To avoid complications, we assume that $ad - bc \neq 0$ (that is the jacobian of the RHS of (2) is not zero). The assumption that f and g have continuous derivatives implies that if a set of initial conditions $x(t_0) = \alpha$ and $y(t_0) = \beta$ is added to the system (2) then the existence of a unique solution is guaranteed. (Morris et al. 1974)

Theorem

We let $f(x, y)$ and $g(x, y)$ be continuously differentiable. Then there is a solution of initial value problem

$x^1 = f(x, y)$

$y^1 = g(x, y)$

where $x(t_0) = \alpha$ and $y(t_0) = \beta$, valid on the interval $I = (t_0 - y, t_0 + y)$ if this is denoted by $x(t, \alpha, \beta)$, $y(t, \alpha, \beta)$ are continuous function for α and β. The solution above is defined for all $t \in R$.

Points along the solution of (2) can be viewed as a triple in $R^3(x(t), y(t), t)$, a path traced out in three dimensions consisting of a time coordinate (x,y). The absence of the independent variable t in the RHS of (2) makes another interpretation useful. Solutions may be regarded in the plane as a parametric curve given by x(t),y(t) with t as the parameter. This curve is simply the projection of the triple $(t, x(t), y(t))$ in three dimensional space onto the plane of the space variables. The curve x(t), y(t) is called a trajectory on an orbit and the plane is called the PHASE PLANE

We shall explore the bases of the highly geometric approach with a view toward the appreciation that will follow later on. To see how the phase plane is a useful concept, let us note first an elementary property of the solution (2), (Morris et al., 1974).

Lemma

According to Morris et al. (1974), if $(\omega_1(t), \omega_2(t))$ is a solution of (2), then $\rho_1(t-\eta)$, $\rho_2(t-\eta)$ belongs to any real number η

Proof

We define $\psi_1(t) = \psi(t-\eta)$ and $\psi_2(t) = \psi(t-\eta)$, then

$\theta_1(t)$=$\omega_1(t-\eta) = f(\omega_1(t-\eta)), \omega_2(t-\eta) = f(\theta_1(t), \theta_2(t))$

$\theta_2(t) = \omega_2(t-\eta) = g(\omega_1(t-\eta)), \omega_2(t-\eta) = g(\theta_1(t), \theta_2(t))$, and therefore $\omega_1(t)$, $\omega_2(t)$ solves (2) where $\theta_1(t), \theta_2(t) \in$ R and $\psi_1(t-\eta)$ and $\psi_2(t-\eta)$, t $\in$ R describe the same of points in the plane and hence the same trajectory. Now, if the solution is viewed as points in R^3 as $(t, x(t), y(t))$ representing time and two spaces coordinates, there is a unique solution through each points.

If we project these solutions onto phase plane by using only $x(t), y(t)$ as coordinates, we might not have a tangle of result. The fact that this is not the case when t does not appear explicitly in f and g is stated in the theorem below. (William and Richard, 1992)

Theorem

let $f(x, y)$ and $g(x, y)$ be continuously differentiable. Then there is a solution of initial value problem. (Morris et al, 1974).

$x^1 = f(x, y)$

$y^1 = g(x, y)$

Proof

Assuming to the contrary, there are two different trajectories $(\omega_1(t), \omega_2(t)$, passing through (x_0, y_0) that is $\omega(t_0) = x_0 = \omega_1(t_1), \omega_2(t_0)y_0 = \omega_2(t_1)$, where necessarily $t_0 \neq t_1$ [By the uniqueness of the solutions of initial value theorem] by the above lemma, the functions $x_1(t) = \omega_1(t - t_1 + t_0)$ and $x_2(t) = \omega_2(t - t_0 + t_0)$ form a solution of (1) yet $x_1(t_1) = \omega_1(t_0) = x_0 = \omega(t_1)$ and $x_2(t_1) = \omega_2(t_0) = y_0 = \omega(t_1)$ for all t. Hence $\omega_1(t), \omega_2(t)$ are the same trajectories (uses different parameterization).

On the other hand, consider that if $f(x_0, y_0) \neq 0$ in (2), then the initial value problem $\frac{d_y}{d_x} = \frac{g(x,y)}{f(x,y)}$..(3)

$y(x_0) = y_0$ has a unique solution since $\frac{d_y}{d_x} = \frac{y^1(t)}{x^1(t)} = \frac{g(x(t),y(t))}{f(x(t),y(t))}$. We now use the plane technique to analyze the system in (1). This system is in the form $x^1 = AX$ that makes computation of the eigenvalues and eigenvectors and the conversion to polar coordinates easy. The analysis of this simple system provides guidelines as to what sorts of behaviour are possible in the following cases

3.2.9 Example 1(Matrices with real distinct eigenvalues of the same sign)

Let the eigenvalues of A be real distinct and of the same sign: take as a representative

$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ the system in (1) is then $x^1 = \lambda x$ and $y^1 = \mu y$. This can be solve to obtain $x(t) = x_0 e^{\lambda t}$ and $y(t) = y_0 e^{\mu t}$, lets note the following,

a) If λ and μ are negative $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$ since the coverage is monotone, the origin is an asymptotically stable critical point.

b) If λ and μ are positive then $\lim_{t\to\infty} x(t) = \pm\infty$ and $\lim_{t\to\infty} y(t) = \pm\infty$ since the limiting behaviour is the same no matter how close, the initial conditions are to the origin, this is sufficient to show that the origin is unstable for a non-linear system, we are interested only in the behaviour near the critical point and such detail global behaviour will not generally be known, so the following idea is useful. The instability of the origin follows from the fact that the trajectory tends to the origin as time runs backward.(Leah, 2005).

3.2.10 Example II (Matrices with Real eigenvalues of opposite signs)

Assuming without loss of generality that $\lambda < 0 < \mu$ then the solution are $x(t) = x_0e^{\lambda t}, y(t) = y_0e^{\mu t}$ and $r(t) = (x_0e^{2\lambda t} + y_0e^{2\mu t})^{\frac{1}{2}}$
μ greater if $y_0 \neq 0$ then r(t) satisfies $\lim_{t\to 0} r(t) = \infty$ again if$y_0 = 0, \theta(t) = tan^{-1}(\frac{y_0e\mu t}{x_0e^{\lambda t}})$ satisfies $(\lim_{t\to\infty} \theta(t) = \pm\frac{\pi}{2})$ if $y_0 = 0$, then $\lim_{t\to\infty} r(t) = 0$ and the trajectory approaches the origin with $\theta(t) = 0$ for all t, in this case the origin is said to be a SADDLE POINT. (Leah, 2005).

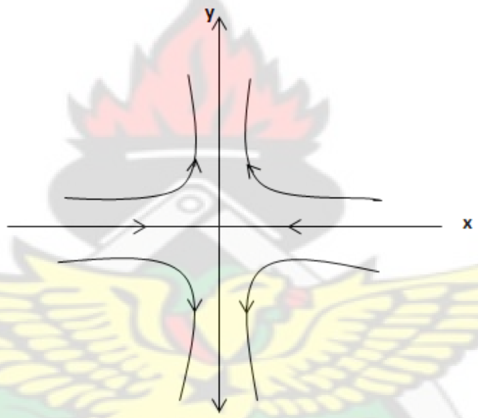

Figure 3.3: Saddle

3.2.11 Example III (Matrices with complex conjugate eigenvalues of non zero real parts)

Taking the matrix A $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\alpha\beta \neq 0$ so that the eigenvalues are $\lambda = \alpha \pm \beta i$ where without loss of generality, we take β the system is given by
$x^1 = \alpha x + \beta y$
$y^1 = \beta x + \alpha y$ polar coordinates are especially useful. The transformation to polar coordinates functions yield $r^1 = \alpha r$, $\theta^1 = -\beta$ this system may be solve to obtain $r(t) = r_0e^{\alpha t}$,$\theta(t) = \theta_0 - \beta t$ as $t \to \infty, \theta(t) \to \infty$ so that the solution wind around

the origin arbitrary many times. The polar radius tends to zero as $t \to \infty$, α is negative and in this case the critical point is asymptotically stable, the polar radius tends monotonically to $+\infty$ as $t \to \infty$ and to zero as $t \to -\infty$ if α is positive. Hence , n in this case is the critical point and is unstable.

The shape of the curve can be easily obtained in polar coordinate $\frac{\partial r}{\partial \theta} = \frac{r^1}{\theta^1} = \frac{\alpha r}{-\beta}$ it follows that $\log(\frac{r}{r_0}) = (\frac{\alpha}{-\beta})(\theta - \theta_0)$ this curve is a logarithmic spiral so that trajectory in the phase plane is logarithmic spiral. This type of critical point is called a spiral or focus point. (Leah, 2005).

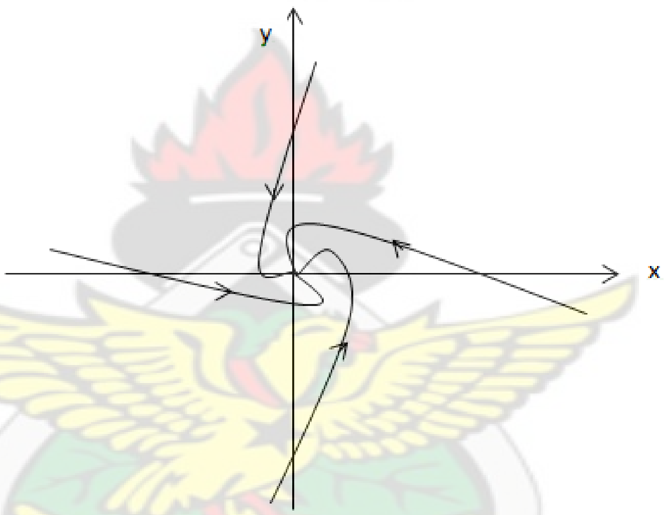

Figure 3.4: Stable spiral sink

3.2.12 Example IV (Purely imaginary eigenvalues)

If the eigenvalues are purely imaginary. This is the same as the previous case except that $\alpha = 0$

The corresponding representative of the class is

A $\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$ The equation for the polar functions are as follow $r_1 = 0$, $\theta_1 = -\beta$ and may be solved to obtain $r = r_0$, $\theta = -\beta t + \theta_0$ the trajectories are circles of radius r_0 about the critical point. This type of critical point is called a centre.

Since the trajectories in the phase plane are closed curves, the corresponding solutions are periodic since the trajectories circles that begin near the origin remain there, the centre is stable, but not asymptotically stable (Leah, 2005).

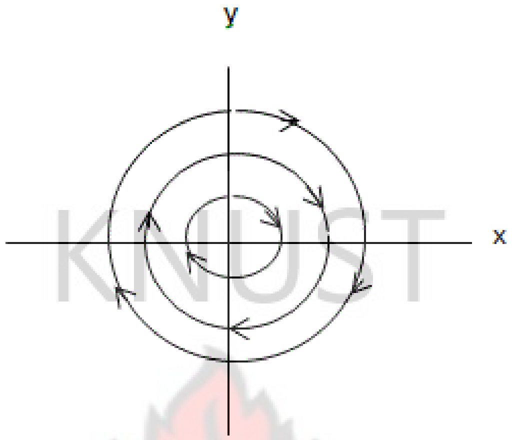

Figure 3.5: Stable centre

3.2.13 Example V (matrices with repeated eigenvalues)

If the eigenvalues are equal, then there are two representative elements(depending on whether there are one or two linearly independent eigenvectors corresponding to the repeated eigenvalues). We have seen that when the matrix of coefficients has two real identical positive eigenvalues and two linearly independent eigenvectors, all the paths are straight lines radiating away from the origin, as in figure 3.6. The equilibrium point is called unstable star source. But if the two identical eigenvalues are negative (but there are still two independent eigenvectors), the arrows on the paths in figure 3.6 are reversed and is called a Stable star sink

First lets consider $A \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ the system becomes $x^1 = \lambda x$ and $y^1 = \lambda y$ here we referred to the equation as uncoupled, hence a solution is given by $x(t) = x_0 e^{\lambda t}$, $y(t) = y_0 e^{\lambda t}$ thus $r(t) = (x^2(t) + y^2(t))^{\frac{1}{2}}$ and $\lambda = 0$ asymptotic stability and if $\lambda > 0, \lim_{t \to -\infty} r(t) = +\infty$ the polar angle is $\theta(t) = tan^{-1}(y_0 e^{\lambda t}/x_0 e^{\lambda t}) =$

$tan^{-1}(\frac{y_0}{x_0}) = \theta_0$ and the direction is constant, that is, the trajectories are half way approaching or leaving the origin.

The solutions are depicted for $\lambda < 0$. (Leah, 2005).

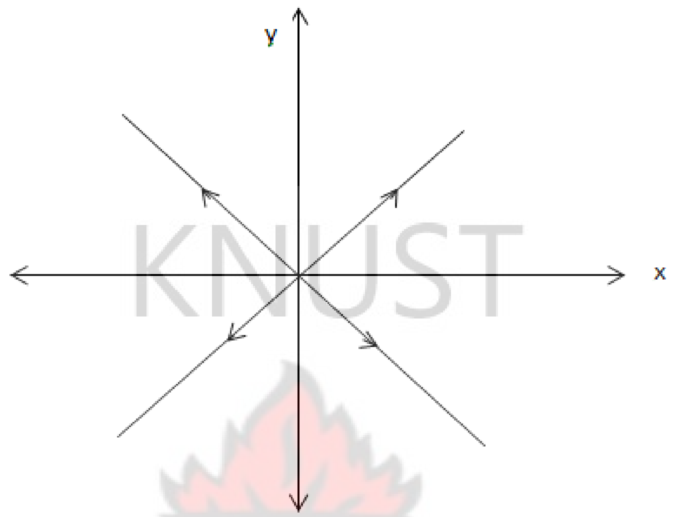

Figure 3.6: Unstable star source

3.2.14 Liapunov function

Earlier stability and the various types of stability including asymptotic stability of an equilibrium x of a dynamic system $x^1 = f(x)$..(4)

F:W $\rightarrow R^n$ is a map on an open set W $\subset R^n$. If $\bar{x}$ is a sink, stability can be detected by examining the eigenvalues of the part $Df(x)$, other than finding all solutions to (1) which may be difficult if not impossible.

The Russian Mathematician A.M Liapunov, in his 1892 doctoral thesis, found a very useful criterion for stability. It is a generalization that for a sink there is a norm on R^n such that $|x(t) - \bar{x}|$ decreases for solutions $x(t)$ near x. He showed that certain other functions could be used instead of the norm to guarantee stability

Let $V : U \rightarrow R$ be a differentiable function defined by a neighbourhood $U \subset W$ of $\bar{x}$ we denote $V : U \rightarrow R$ the function defined by $V(x) = DV(x)(f(x))$. Here the right hand side is simply the operator DV(x) applied to the vector f(x), then if $\phi_t(x)$ is the solution to (1) passing through x when $t = 0, V(x) = \frac{d}{dt}, V(\phi_t(x))$ by the chain rule consequently, if V(x) is negative, then V decreases along the solution of (1) through x. We can state the Liaponuv's theorem. Let $x \in W$ be an equilibrium for

$V : U \to R$ be continuous function defined on a neighbourhood $U \subset W$ of x, differentiable on u - x, such that,

(a) $V(x) = 0$ and $V(x) > 0$ if $\bar{x} \neq x$

(b) $V(x) < 0$ in $u - \bar{x}$, (all trajectories flow "downhill" toward $\bar{x}$) then $\bar{x}$ is globally asymptotically stable, for all initial conditions $x(t) \to \bar{x}$ as $t \to \infty$.

The intuition is that all trajectories move monotonically down the graph of V(x) toward $\bar{x}$

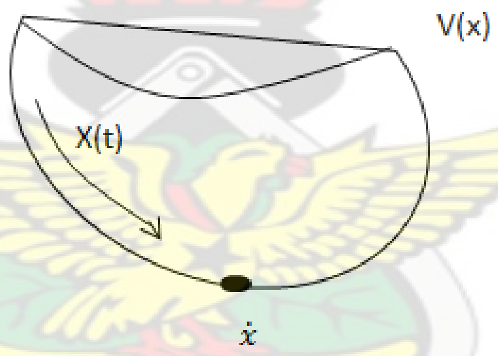

Figure 3.7:

Furthermore , if

(c) $V(x) < 0$ in $u - x$, then x is asymptotically stable.

A function v satisfying (a) and (b) is called Liapunov function for x, if (c) also holds, we called v a strict Liapunov function. The only equilibrium is the origin $x = y = 0$. (Waltman, 1995).

3.2.15 Malthusian Model

According to the Malthusian model (Unlimited population growth) an elementary model of population growth is based on the assumption that; the rate of growth of the population is proportional to the size of the population. This implies that the rate of change of a population depend only on the size of the population and nothing else.
Exponential functions come into play in situations in which the rate at which some quantity grows or decays (i.e. increases or decreases over time) is proportional to the quantity present. The quantities evolving from the assumption are as follows; $\frac{dP}{dt} = kP(1 - \frac{P}{K})$.
Where P is Population at time t, k is a constant of proportionality and K is the carrying capacity.

3.3 Predator - Prey model of Security forces versus Criminals

3.3.1 Model Assumptions

The following assumptions are made.

- If no security forces are present in the community, criminals operate in the community at a rate proportional to their population.
- The number of encounters between the security forces and criminals is proportional to the product of security forces and criminals
- In the absence of criminals in the community, the security forces population will decline at a rate proportional to itself.
- An increase in the security forces in a criminal prone community will reduce

the activities of the criminals.

- As the security forces grows in number, the interaction rate between the security forces and criminals also increases.

Table 3.1: Parameters and their definitions

Parameter	Parameter definition
μ	The growth rate coefficient of criminals
β	the rate at which the security forces population declines
α	the constant of proportionality that measures the effect on the criminals population
γ	The growth rate coefficient of the security forces.

3.4 Model A

3.4.1 Model formulation

We let

$H(t)$ be the number of criminals at time t

$P(t)$ be the number of security forces at time t

N be the population at time t

Then;

$$H(t) + P(t) = N \tag{3.1}$$

equation (3.1) represent the security forces and criminals cases at time t

Linear term

In the absence of security forces, criminals tend to increase exponentially to their current population

$$\frac{dH}{dt} = \mu H,\ \mu > 0, \text{ when P} = 0 \tag{3.2}$$

In the absence of criminals the security forces population tends to decrease exponentially

$\frac{dP}{dt} = -\beta P$, $\beta > 0$, when H $= 0$ (3.3)

Non - linear term

Are obtained through the mass action law, that is, as the security forces grows in number the interaction rate between the security forces and criminals also increases.

$\frac{dP}{dt} = \gamma HP$ where $\gamma > 0$ (3.4)

Increase in the security forces in a criminal prone community will reduce the activity of the criminals.

$\frac{dH}{dt} = -\alpha HP$ (3.5)

Combining the linear and non-linear parts of the model, we have

$$\frac{dH}{dt} = \mu H - \alpha HP$$
$$\frac{dP}{dt} = -\beta P + \gamma HP \quad (3.6)$$

From our equation (3.1) we have,

$$H(t) + P(t) = N$$
$$P(t) = N - H$$
$$H(t) = N - P$$

substituting P(t) and H(t) into equation (3.6)

$$\frac{d(N-P)}{dt} = \mu(N-P) - \alpha(N-P)(N-H)$$
$$\frac{d(N-H)}{dt} = -\beta(N-H) + \gamma(N-P)(N-H) \quad (3.7)$$

The constants μ, α, β and γ are all positive where μ and γ are the growth rate constants and α and β are measures of effect of their interactions.

3.4.2 Analysis of the model

Taking the first linear term equation (3.2)

$\frac{dH}{dt} = \mu H$

Equation (3.2) models the criminals that are not affected by the security forces, hence growing exponentially according to a rule of the form

$H(t) = Ce^{\mu t}$

Where C is a positive constant representing the criminals population when $t = 0$, $H = C$

For the second linear term, equation (3.3)

$\frac{dP}{dt} = -\beta P$

Equation (3.3) represents a declining population of security forces in the community exponentially due to the absence of criminals given by

$P(t) = De^{-\beta t}$

where D is a positive constant representing initial security forces population

However for interaction populations where encounters are unavoidable we assume that the number of encounters between security forces and criminals is proportional both to the population of P security forces and the population of H criminals in the community.

The growth rate of criminals decreases by a factor proportional to the number of encounters between security forces and the criminals that is by a factor HP

We revise our first model to include this extra term

$$\frac{dH}{dt} = \mu H - \alpha HP \tag{3.8}$$

for some positive constant α

From (a) we let $\alpha = \frac{\mu}{K}$, where $K > 0$, we now have

$$\frac{dH}{dt} = \dot{H} = \mu H(1 - \frac{P}{K}) \tag{3.9}$$

which is a non-linear term.

Similarly for a population of security forces in a criminal free environment we have from equation (3.3)

$\frac{dP}{dt} = -\beta P$

where β is a positive constant. This represents exponential decay. However if there is a population H criminals for the security forces to overpower, we expect the growth rate $\frac{dP}{dt}$ of the security forces to increase and a simple assumption is that the growth rate $\frac{dP}{dt}$ of security forces increases by a factor proportional to the number of encounters between the criminals and the security forces. Our revised model for the security forces is given by

$$\frac{dP}{dt} = \dot{P} = -\beta P + \gamma HP \qquad (3.10)$$

for some positive constant γ.

Again it is convenient to let $\gamma = \frac{\beta}{Q}$, where $Q > 0$, thus ,

$$\frac{dP}{dt} = \dot{P} = -\beta P(1 - \frac{H}{Q}) \qquad (3.11)$$

which is also a non - linear term

We now have our two equations as

$$\frac{dH}{dt} = \dot{H} = \mu H(1 - \frac{P}{K}) \qquad (3.9)$$

$$\frac{dP}{dt} = \dot{P} = -\beta P(1 - \frac{H}{Q}) \qquad (3.10)$$

Which is in the form of Lotka - Volterra equations where $P \geq 0$ and $H \geq 0$

Rearranging equation (3.9), we have $\frac{\dot{H}}{H} = \mu - \frac{\mu}{K}P$.

The graph of the proportionate growth rate $\frac{\dot{H}}{H}$ of criminals as a function of the population P of security forces is given by

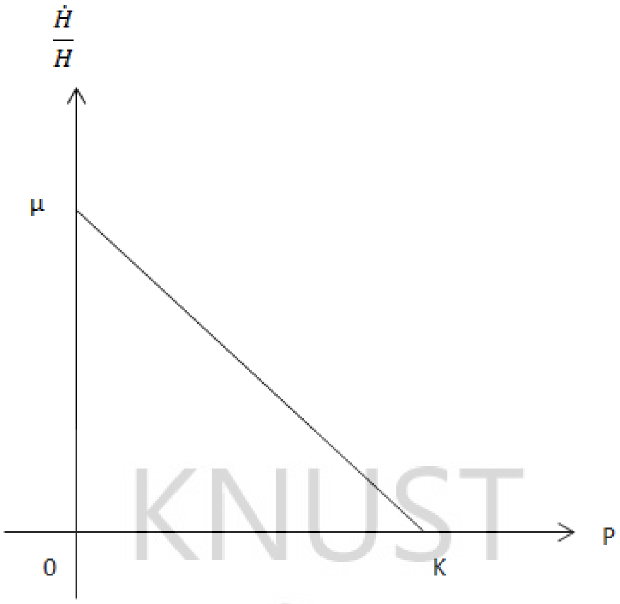

Figure 3.8: Proportionate growth rate of criminals as a function of the population of security forces

The proportionate growth rate $\frac{\dot{H}}{H}$ of the criminals decreases as the population P of security forces increases, becomes zero when $P = K$. The population H of criminals will increase if the population P of the security forces is less than K but will decline if $P > K$

Similarly, from equation (3.11) we rearrange to have

$\frac{\dot{P}}{P} = -\beta + \frac{\beta H}{Q}$

and the graph $\frac{\dot{P}}{P}$ of the security forces as a function of the population H of criminals is given by figure 3.8.

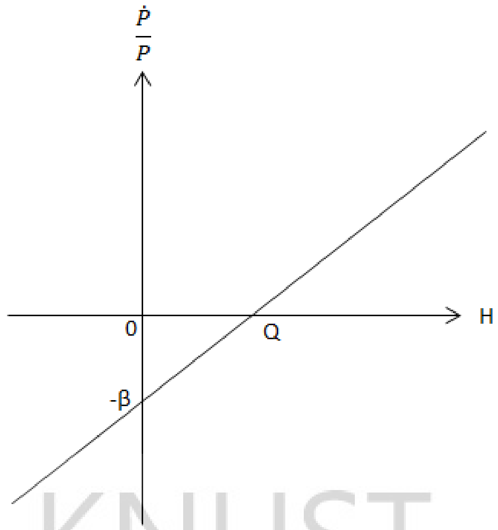

Figure 3.9: Proportionate growth rate of the security forces as a function of the population of criminals

The proportionate growth rate $\frac{\dot{P}}{P}$ of security forces increases linearly as the population H of criminal's increases. The security forces population will decrease if the population of the criminals is less than Q, but will increase if H is greater than Q

3.4.3 The equilibrium points

To find the equilibrium points of the system of differential equation $\dot{x} = u(x, y)$ for some vector field u we solve $u(x, y) = 0$, from equation (3.9) and (3.11) we have,

$\frac{dH}{dt} = 0$ and $\frac{dP}{dt} = 0$,

that is

$$\mu H(1 - \frac{P}{K}) = 0 \tag{3.12}$$

$$-\beta P(1 - \frac{H}{Q}) = 0 \tag{3.13}$$

From equation (3.12), we deduce that either $H = 0$ or $P = K$, if $H = 0$, equation (3.13) reduces to

$-\beta P = 0$

$P = 0$

Hence (0,0) is an equilibrium point

If $P = K$, equation (3.13) becomes $-\beta K(1 - \frac{H}{Q}) = 0$, so $H = Q$

Hence (Q, K) is the other equilibrium point

3.4.4 Stability of equilibrium point

Using the Jacobian matrix

$$J(x,y) = \begin{pmatrix} \frac{\partial u(x,y)}{\partial x} & \frac{\partial u(x,y)}{\partial y} \\ \frac{\partial v(x,y)}{\partial x} & \frac{\partial v(x,y)}{\partial y} \end{pmatrix}$$

from equation (4) and (5), we have

$$J(H,P) = \begin{pmatrix} \mu(1 - \frac{P}{K}) & -\frac{\mu H}{K} \\ \frac{\beta P}{Q} & -\beta(1 - \frac{H}{Q}) \end{pmatrix}$$

At the critical point (0,0)

$$J(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & -\beta \end{pmatrix}$$

$$det|J - \lambda I| = \begin{pmatrix} \mu - \lambda & 0 \\ 0 & -\beta - \lambda \end{pmatrix}$$

$\lambda_1 = \mu$ and $\lambda_2 = -\beta$

Hence the origin is a saddle point and therefore unstable, that is if a perturbation results in a catastrophe change with the population of security forces or criminals collapsing to zero or increasing without limit, we say that the equilibrium point is unstable.

For the second critical point (Q, K)

$$J(Q,K) = \begin{pmatrix} 0 & \dfrac{-\mu Q}{K} \\ \frac{\beta K}{Q} & 0 \end{pmatrix}$$

$$det|J - \lambda I| = \begin{pmatrix} 0-\lambda & \frac{-\mu Q}{K} \\ \frac{\beta K}{Q} & 0-\lambda \end{pmatrix}$$

$\lambda^2 + \mu\beta = 0, \Rightarrow \lambda = \pm i\sqrt{\mu\beta}$

From the analysis of this we arrive at a number of results.

The criminals (prey) depends on parameter associated with the security forces (predators) $P = K$. A similar result holds for steady state levels of security forces (predators) $H = Q$, it is the particular coupling of the variables that leads to this effect. To paraphrase, the presence of security forces (predator) $P \neq 0$, means that the available criminals (prey) has to just suffice to make growth rate due to predation, $\frac{\beta H}{Q}$ equal security forces (predator) rate β for a steady predator population to persists.

similarly, when criminals (prey) are present ($H \neq 0$) security forces (predators) can only keep them under control when prey growth rate μ and predation rate $\frac{\mu}{K}$ are equal.

The second result we have is that the steady state (Q, K) is neutrally stable (a centre), the eigenvalues are pure imaginary and the steady state is not spiral point the off - diagonal terms $\frac{-\mu Q}{K}$ and $\frac{-\mu Q}{Q}$ are of opposite sign and the diagonals term evaluated are zero. Stability analysis predicts oscillations about the steady state. The factor $\sqrt{\mu\beta}$ governs the frequency of these oscillation.

3.4.5 Phase path to the system

To find the phase path to the system, we consider the critical points $(0,0)$ and (Q,K)

The critical point at (0,0) is a saddle point and the Figure below shows a typical profile of solutions that are in the first quadrant and near (0,0)

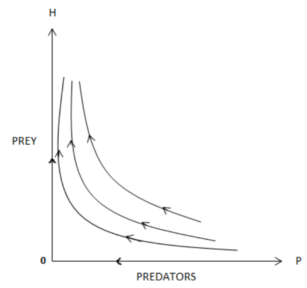

Figure 3.10:

Because the matrix of the second critical point has pure imaginary eigenvalues $\lambda = \pm i\sqrt{\mu\beta}$, the critical point (Q, K), is a centre. We investigate the possibility using the phase - plane method.

From equation (3.9) and equation (3.10)

$$\dot{H} = H(\mu - \alpha P)$$

$$\dot{P} = P(-\beta + \gamma H)$$

$$\frac{dH}{dP} = \frac{H(\mu-\alpha P)}{P(-\beta+\gamma H)}$$

$$\int(\frac{(-\beta+\gamma H)}{H})dH = \int(\frac{(\mu-\alpha P)}{P})dP$$

$$-\beta lnH + \gamma H = \mu lnP - \alpha P + c$$

$$(P^{\mu}e^{-\alpha P})(H^{\beta}e^{-\gamma H}) = c_0$$

$$F(P) = P^{\mu}e^{-\alpha P} \text{ and}$$
$$G(H) = H^{\beta}e^{-\gamma H}$$

A typical graphs of the nonnegative functions $F(P) = P^{\mu}e^{-\alpha P}$ and $G(H) = H^{\beta}e^{-\gamma H}$ are shown in the figures below

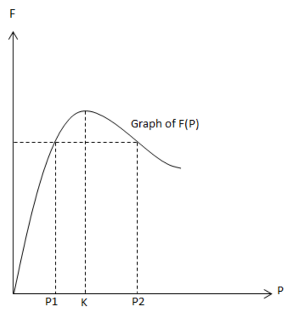

Figure 3.11:

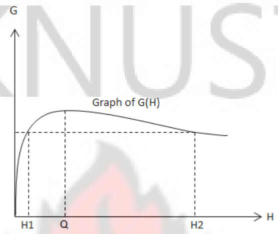

Figure 3.12:

$F(P)$ has an absolute maximum at $P = K$ whereas $G(H)$ has an absolute maximum at $H = Q$.

We note also that with the exception of 0 and the absolute maximum, F and G each take on all values in their range precisely twice.

These graphs were used to established the following properties in the figure below.

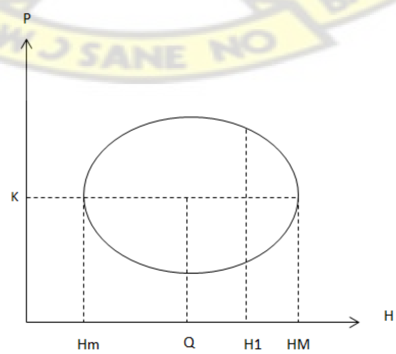

Figure 3.13: A periodic graph of the security forces and the criminals

we deduce that

- if $P = K$, the equation $F(P)G(H) = C_0$, has exactly two solution H_m and H_M that satisfy $H_m < Q < H_M$
- $H_m < H_1 < H_M$ and $H = H_1$, then $F(P)G(H) = C_0$, has two solutions $P_1 < K < P_2$

We can define the trajectories,and the constants can be changed and varied for different trajectories. A sample plot of the phase portrait of the Model representing several different trajectories is shown below

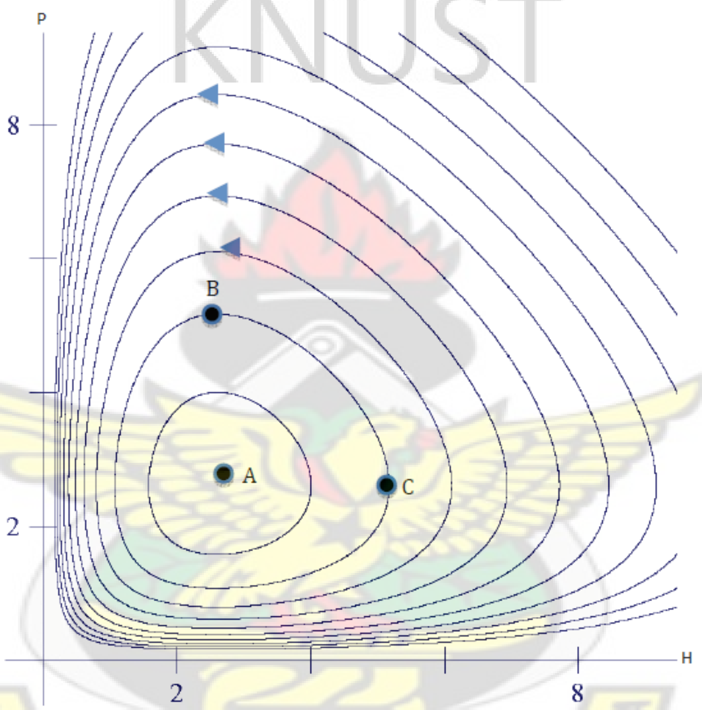

Figure 3.14: Sample Lotka-Volterra Phase Portrait

Two points B and C were included in order to discuss phase portrait function. The purpose of this phase portrait is to show the cyclic fluctuations of the security forces and criminals with respect to each other without showing the change in time. Let C=(5,3) and let B=(3,5) (not perfectly to scale as shown)

These two points are on the same trajectory of the system, and with the direction of the trajectory, point C advances to point B as time goes on. One must understand that each point does not just represent arbitrary numbers, but each point represents the state of security forces and criminals population.

Another extremely important plot stemming from the model is the predator-prey cycle chart, representing periodic activity in the population fluctuation as shown in figure below.

It is seen that as time progresses in years, security forces population and criminals population clearly fluctuate at cyclic interval. Notice that as the criminal population peaks, security forces population begins to rise rapidly, yet as the security forces population rises, the criminals population falls rapidly

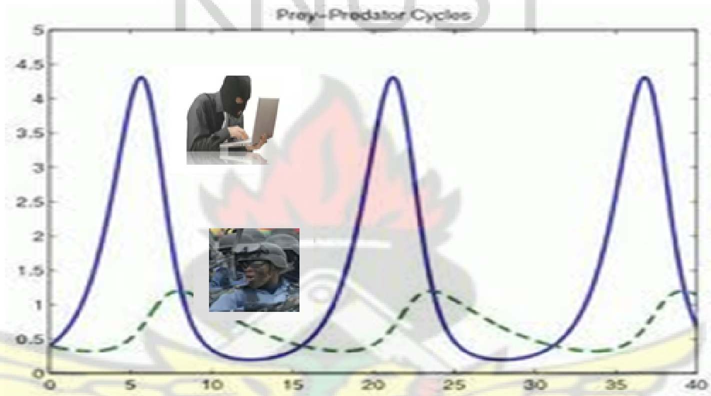

Figure 3.15:

3.5 Model B

3.5.1 Introduction of Volunteer Guards

The model that we consider here is a slight modification of model A, where we introduce two more parameters (i) The recruitment rate of volunteer guards and (ii) The rate at which the volunteer guards come into contact with the criminals.

Table 3.2: Parameters and their definitions

Parameter	Parameter definition
μ	The growth rate coefficient of criminals
β	The rate at which the security forces population declines
α	The constant of proportionality that measures the effect on the criminals population
γ	The growth rate coefficient of the security forces
τ	The rate at which the criminal population come into contact with the volunteer guards
u_1	The rate at which the volunteer guards are recruited

$$\frac{dH}{dt} = \mu H - \alpha HP - \tau u_1 H \tag{3.14}$$

$$\frac{dP}{dt} = -\beta P + \gamma HP + \tau u_1 H \tag{3.15}$$

3.6 Analysis of Model B

3.6.1 The equilibrium points

We equate equation (3.14) and (3.15) to zero to have;

$\frac{dH}{dt} = 0$ and $\frac{dP}{dt} = 0$

$$\frac{dH}{dt} = \mu H - \alpha HP - \tau u_1 H = 0 \tag{3.16}$$

$$\frac{dP}{dt} = -\beta P + \gamma HP + \tau u_1 H = 0 \tag{3.17}$$

From equation (3.16) we let $H = 0$ and substitute the value into equation (3.17), to have $P = 0$, Hence (0,0) is an equilibrium point

Again from equation (3.16) we make P the subject to have;

$P = \frac{\mu - \tau u_1}{\alpha}$,

we substitute P into equation (3.17), thus

$-\beta(\frac{\mu-\tau u_1}{\alpha}) + \gamma(\frac{\mu H-\tau u_1 H}{\alpha}) + \tau u_1 H = 0$

$\frac{-\beta\mu+\beta\tau u_1}{\alpha} + \frac{\gamma H\mu-\gamma H\tau u_1}{\alpha} + \tau u_1 H = 0$

$\frac{-\beta\mu+\beta\tau u_1+\gamma H\mu-\gamma H\tau u_1+\tau u_1 H\alpha}{\alpha} = 0$

$-\beta\mu + \beta\tau u_1 + \gamma H\mu - \gamma H\tau u_1 + \tau u_1 H\alpha = 0$

$\gamma H\mu - \gamma H\tau u_1 + \tau u_1 H\alpha = \beta\mu - \beta\tau u_1$

Hence $H = \frac{\beta\mu-\beta\tau u_1}{\gamma\mu-\gamma\tau u_1+\tau u_1\alpha}$

the second equilibrium point is given by $(\frac{\mu-\tau u_1}{\alpha}, \frac{\beta\mu-\beta\tau u_1}{\gamma\mu-\gamma\tau u_1+\tau u_1\alpha})$

The equilibrium points are;

$(0,0)$ and $(\frac{\mu-\tau u_1}{\alpha}, \frac{\beta\mu-\beta\tau u_1}{\gamma\mu-\gamma\tau u_1+\tau u_1\alpha})$

3.6.2 Stability of equilibrium points

$\frac{dH}{dt} = \mu H - \alpha HP - \tau u_1 H..........f_1$

$\frac{dP}{dt} = -\beta P + \gamma HP + \tau u_1 H.........f_2$

$$J(H,P) = \begin{pmatrix} \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial P} \end{pmatrix}$$

$$J(H,P) = \begin{pmatrix} \mu - \alpha P - \tau u_1 & -\alpha H \\ \gamma P + \tau u_1 & -\beta + \gamma H \end{pmatrix}$$

at the critical point (0,0)

$$J(0,0) = \begin{pmatrix} \mu - \tau u_1 & 0 \\ \tau u_1 & -\beta \end{pmatrix}$$

$$det|J - \lambda I| = \begin{pmatrix} (\mu - \tau u_1) - \lambda & 0 \\ \tau u_1 & -\beta - \lambda \end{pmatrix}$$

$$((\mu - \tau u_1) - \lambda)(-\beta - \lambda) = 0$$

$$\lambda_1 = \mu - \tau u_1 \text{ and } \lambda_2 = -\beta$$

For the second equilibrium point

$$(\frac{\mu H - \tau u_1 H}{\alpha H}, \frac{\beta\mu - \beta\tau u_1}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha})$$

$$J(\frac{\mu H - \tau u_1 H}{\alpha H}, \frac{\beta\mu - \beta\tau u_1}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha}) = \begin{pmatrix} \mu - \alpha P - \tau u_1 & -\alpha H \\ \gamma P + \tau u_1 & -\beta + \gamma H \end{pmatrix}$$

$$J(\frac{\mu H - \tau u_1 H}{\alpha H}, \frac{\beta\mu - \beta\tau u_1}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha}) = \begin{pmatrix} \mu - \alpha(\frac{\beta\mu - \beta\tau u_1}{\alpha\mu - \gamma\tau u_1 + \tau u_1 \alpha}) - \tau u_1 & -\alpha(\frac{\mu H - \tau H u_1}{\alpha H}) \\ \gamma(\frac{\beta\mu - \beta\tau u_1}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha}) + \tau u_1 & -\beta + \gamma(\frac{\mu H - \tau H u_1}{\alpha H}) \end{pmatrix}$$

$$J(\frac{\mu H - \tau u_1 H}{\alpha H}, \frac{\beta\mu - \beta\tau u_1}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha}) = \begin{pmatrix} \frac{\mu(\gamma\mu - \gamma\tau u_1 + \alpha) - (\alpha\beta\mu + \alpha\beta\tau u_1) - \tau u_1(\gamma\mu - \gamma\tau u_1 \alpha)}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha} & \frac{-\alpha\mu H + \tau\alpha H u_1}{\alpha H} \\ \frac{\gamma\beta\mu - \gamma\beta\tau u_1 + \tau u_1 + \tau u_1 \gamma\mu - \tau^2 u_1^2 \gamma + \tau^2 u_1^2 \alpha}{\gamma\mu - \gamma\tau_1 + \tau u_1 \alpha} & \frac{-\beta\alpha H + \gamma\mu H - \gamma\tau H u_1}{\alpha H} \end{pmatrix}$$

$$J(\frac{\mu H - \tau u_1 H}{\alpha H}, \frac{\beta\mu - \beta\tau u_1}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$det|J - \lambda I| = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - \lambda(a + d) + (ad - bc) = 0$$

$$\text{Trace} = a + d$$

$$\text{Det} = ad - bc$$

$$\text{Trace} = (\frac{\mu(\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha) - (\alpha\beta\mu + \alpha\beta u_1) - \tau u_1(\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha)}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha}) + (\frac{-\beta\alpha + \gamma\mu - \gamma\tau u_1}{\alpha})$$

$$\det = (\frac{\mu(\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha) - (\alpha\beta\mu + \alpha\beta\tau u_1) - \tau u_1(\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha)}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha})(\frac{\gamma\mu - \beta\alpha - \gamma\tau u_1}{\alpha}) - (\frac{\tau\alpha u_1 - \alpha\mu}{\alpha})(\frac{\gamma\beta\mu - \gamma\beta\tau u_1 + \tau u_1 \gamma\mu - \tau^2 \mu_1^2 \gamma + \tau^2 u_1^2 \alpha}{\gamma\mu - \gamma\tau u_1 + \tau u_1 \alpha})$$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

We consider the following sufficient and necessary conditions;

- $\lambda_1 < \lambda_2 < 0$: Stable improper node
- $\lambda_1 = \lambda_2 < 0$: Stable node
- $\lambda_1 < 0 < \lambda_2$: Unstable saddle point
- $\lambda_1 = \lambda_2 > 0$: Unstable node
- $\lambda_1 > \lambda_2 > 0$: Unstable improper node
- $\lambda_1, \lambda_2 = p \pm iq, p < 0$: Spiral sink
- $\lambda_1, \lambda_2 = p \pm iq, p > 0$: Spiral source
- $= \pm iq$: center

If $\lambda_1 < 0$ and $\lambda_2 < 0$. then for $\lambda_1 < 0$ it is essential that $T < 0$,
this will always make $\lambda_2 < 0$.
However, it is also necessary that $|T| > \sqrt{T^2 - 4D}$. Otherwise λ_1 would be positive, since the radical would dominate over T. Squaring both sides and rewriting, we see that $T^2 > T^2 - 4D$, or $D > 0$.

Chapter 4

Analysis

4.1 Introduction

This chapter deals with the analysis of the data collected from Ghana Statistical Service on major criminal activities from 2000 to 2010. The determination of parameters, the phase portrait, the identification of types of trajectories, simulation using Matlab is also done to see how the model works practically.

4.2 Display of Data

For the purpose of the flow of the analysis, the five major categories of crime in Ghana from 2000 to 2010 is displayed in Appendix.

4.3 Parameter Determination

The crime data was entered into Microsoft Excel and the solver application was used in order to discover the best possible fit parameters to the model. The security forces parameters were taken based on similar analysis used in other research.

Table 4.1: Parameters and Their Values

Parameter	Parameter definition	Values
μ	The Growth rate coefficient of criminals	0.4
β	The rate at which the security forces population declines	0.9
α	The constant that measures the effect on the criminals	0.001
γ	The growth rate coefficient of the security forces	0.001
τ	The rate of contact between criminals and volunteer guards	1×10^{-4}
u_1	Recruitment rate of volunteer guards	0.03
H_0	Initial number of criminals	600
P_0	Initial number of security forces	400
N_0	Initial number of total population	1000

Source: Lucas (2011) and Ghana Statistical Service Report 2010

4.4 Numerical Results

4.4.1 Model A

$$\frac{dH}{dt} = \mu H(1 - \frac{P}{K}) \tag{4.1}$$

$$\frac{dP}{dt} = -\beta P(1 - \frac{H}{Q}) \tag{4.2}$$

From the parameter values given in Table 4.1, we estimate that the eigenvalues for the first equilibrium point is given by,

$\lambda_1 = 0.4$ and $\lambda_2 = -0.9$. From this it is clear that the origin is a saddle and therefore unstable.

Again from the second equilibrium point we have the eigenvalues

$\lambda_{1,2} = \pm i\sqrt{0.36}$

$\lambda_{1,2} = \pm 6i$, hence the steady state is purely a centre and therefore stable.

4.4.2 Model B

From equation (3.16) and (3.17),

$$\frac{dH}{dt} = \mu H - \alpha HP - \tau u_1 H \tag{4.3}$$

$$\frac{dP}{dt} = -\beta P + \gamma HP + \tau u_1 H \tag{4.4}$$

The stability analysis of the second model.

The stability analysis of the first equilibrium point. We found out that

$$((\mu - \tau u_1) - \lambda)(-\beta - \lambda) = 0$$

$$\lambda_1 = \mu - \tau u_1 \text{ and } \lambda_2 = -\beta$$

$\lambda_1 = 0.4 - (1 \times 10^{-4})(0.03)$

$\lambda_1 = 0.4, \qquad \lambda_2 = -0.9$

The equilibrium point is a saddle point, therefore unstable

The stability analysis of the second equilibrium point. We find the Trace and the determinant to have;

Trace $= (\frac{\mu(\gamma\mu - \gamma\tau u_1 + \tau u_1\alpha) - (\alpha\beta\mu + \alpha\beta u_1) - \tau u_1(\gamma\mu - \gamma\tau u_1 + \tau u_1\alpha)}{\gamma\mu - \gamma\tau u_1 + \tau u_1\alpha}) + (\frac{-\beta\alpha + \gamma\mu - \gamma\tau u_1}{\alpha})$

Trace $= -3.5 \times 10^{-8}$

det $= (\frac{\mu(\gamma\mu - \gamma\tau u_1 + \tau u_1\alpha) - (\alpha\beta\mu + \alpha\beta\tau u_1) - \tau u_1(\gamma\mu - \gamma\tau u_1 + \tau u_1\alpha)}{\gamma\mu - \gamma\tau u_1 + \tau u_1\alpha})(\frac{\gamma\mu - \beta\alpha - \gamma\tau u_1}{\alpha}) - (\frac{\tau\alpha u_1 - \alpha\mu}{\alpha})(\frac{\gamma\beta\mu - \gamma\beta\tau u_1 + \tau u_1\gamma\mu - \tau^2\mu_1^2\gamma + \tau^2 u_1^2\alpha}{\gamma\mu - \gamma\tau u_1 + \tau u_1\alpha})$

det$= 3.0 \times 10^{-7}$

$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

$\lambda_{1,2} = \frac{(-3.5\times10^{-7}) \pm \sqrt{(-3.5\times10^{-7})^2 - 4(3.0\times10^{-7})}}{2}$

$\lambda_{1,2} = \frac{-3.5\times10^{-7} \pm (1.01\times10^{-3})i}{2}$

Since $T < 0$ and $D > 0$ we have a spiral sink, hence asymptotically stable

4.5 Simulation

Model A

We find the numerical solution of the general model

$$\frac{dH}{dt} = \mu H(1 - \frac{P}{K}) \tag{4.1}$$

$$\frac{dP}{dt} = -\beta P(1 - \frac{H}{Q}) \tag{4.2}$$

Model B

$$\frac{dH}{dt} = \mu H - \alpha HP - \tau u_1 H \tag{4.3}$$

$$\frac{dP}{dt} = -\beta P + \gamma HP + \tau u_1 H \tag{4.4}$$

Taking equation (4.1) and equation (4.3)

Considering the initial value problem $H(t) = f(t, H(t)), H(0) = H_0, t = t_0 =$

$0, t > 0$.

The general formula for the Euler's method of the above initial value problem is

$t_{i+1} = t_i + h$

$H(t_{i+1}) = H(t_i) + hf(t_i, H(t_i)) \qquad$ for $i = 0, 1, 2, ..., z - 1$.

We let $[0, a]$, be the interval over which we want to find the solution of the problem.

We generate a set of points $(t_i, H(t_i))$ for our approximation. For convenience we subdivide the interval $[0, a]$ in to z subintervals $[t_i, t_{i+1}]$ of equal width, $h = \frac{a}{z}$, by using the nodes $t_i = ih$, for $i = 0, 1, 2, ..., z$.

suppose that $H(t)$ is continuous on $[0, a]$ and by using the generalize Taylor's formula to expand

$\frac{dH}{dt} = f(t, H(t))$, where

$f(t, H(t)) = \mu H(1 - \frac{P}{K})$ and

$f(t, H(t)) = \mu H - \alpha HP - \tau u_1 H$

$H(0) = H_0 = 600, P(0) = P_0 = 400, N_0 = 1000, t = t_0 = 0, t > 0$

$H = H_0 + \delta t f(t_0, H(t_0))$

let $\delta t = h$

$H(t_1) = H(t_0) + h^1 f^1(t_0, H(t_0)) + \frac{h^2}{2!} f^{11}(t_0, H(t_0))...$

If the step size h is chosen small enough, then we neglect the higher order and have

$H(t_1) = H(t_o) + h^1 f(t_0, H(t_0))$

The process is repeated and generates a sequence of points that approximates the solution $H(t)$ then the general formula for the Euler method for the model of criminals is given by

$t_{i+1} = t_i + h$

$H(t_{i+1}) = H(t_i) + h^1 f(t_i, H(t_i))$

but $f(t_i, H(t_i)) = \mu H(t_i)(1 - \frac{P(t_i)}{K})$

and

$f(t_i, H(t_i)) = \mu H(t_i) - \alpha H(t_i)P(t_i) - \tau u_1 H(t_i)$

substituting we have

$$H(t_{i+1}) = H(t_i) + h^1[\mu H(t_i)(1 - \frac{P(t_i)}{K})] \quad (4.5)$$

$$H(t_{i+1}) = H(t_i) + h^1[\mu H(t_i) - \alpha H(t_i)P(t_i) - \tau u_1 H(t_i)] \quad (4.6)$$

From equation (4.2) and (4.4)

$\frac{dP}{dt} = f(t, P(t))$, where

$f(t, P(t)) = -\beta P(1 - \frac{P}{Q})$ and

$f(t, P(t)) = -\beta P + \gamma HP + \tau u_1 H$

$H(0) = H_0 = 600, P(0) = P_0 = 400, N_0 = 1000, t = t_0 = 0, t > 0$

$P = P_0 + \delta t f(t_0, Pt_0)$, let $\delta t = h$

We let $[0, a]$ be the interval over which we want to find the solution of the problem, we generate a set points $t_i, P(t_i)$ for our approximation.

For convenience we subdivide the interval $[0, a]$ into z subintervals $[t_i, t_{i+1}]$ of equal width $h = \frac{a}{z}$, by using the nodes $t_i = ih$, for each value t there is a value a so that

$P(t_1) = P(t_0) + h^1 f^1(t_0, P(t_0)) + \frac{h^2}{2!} f^{11}(t_0, H(t_0))$...

Now if the step size h is chosen enough then we may neglect the higher order term to get

$P(t_1) = P(t_0) + h^1 f^1(t_0, P(t_o))$

The process is repeated and generates a sequence of points that approximate the solution $P(t)$. Then the general formula for Euler's method for security forces is given by

$t_{i+1} = t_{i+h}$

$P(t_{i+1}) = P(t_i) + h^1 f(t_i, P(t_i))$ but

$f(t_i, P(t_i)) = -\beta P(t_i)(1 - \frac{H(t_i)}{Q})$ and $f(t_i, P(t_i)) = -\beta P(t_i) + \gamma H(t_i)P(t_i) + \tau u_1 H(t_i)$

Substituting we have

$$P(t_{i+1}) = P(t_i) + h^1[-\beta P(t_i)(1 - \frac{H(t_i)}{Q})] \quad (4.7)$$

$$P(t_{i+1}) = P(t_i) + h^1[-\beta P(t_i) + \gamma H(t_i)P(t_i) + \tau u_1 H(t_i)] \quad (4.8)$$

for $i = 0, 1, ..., z - 1$

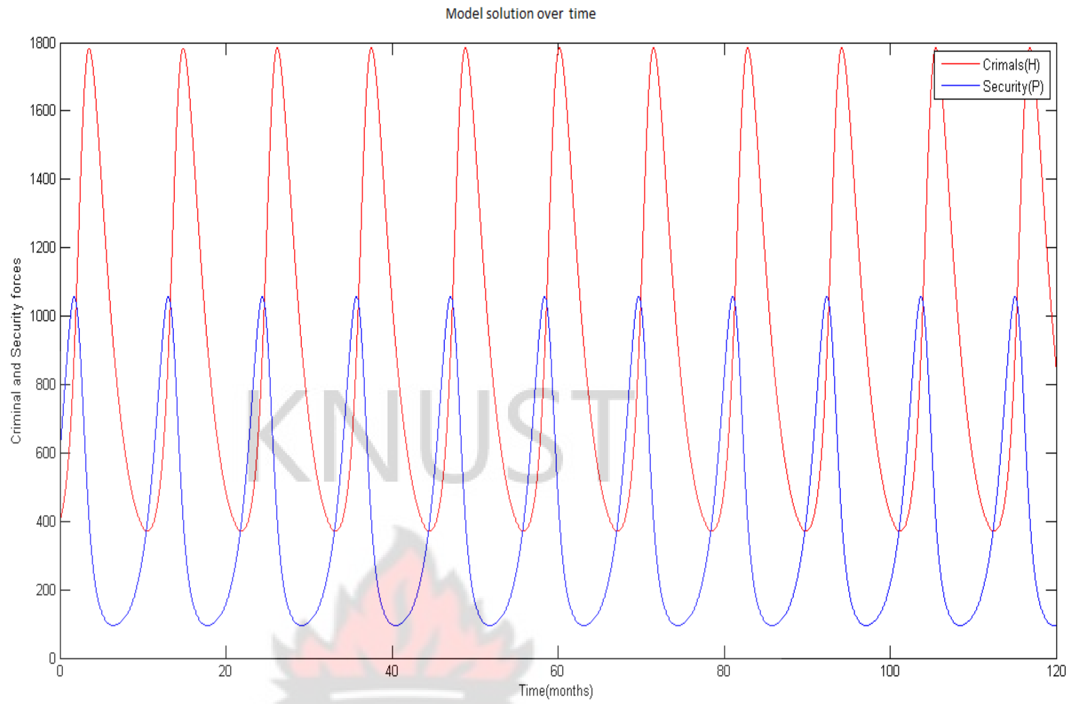

Figure 4.1: TRAJECTORIES 0F MODEL A

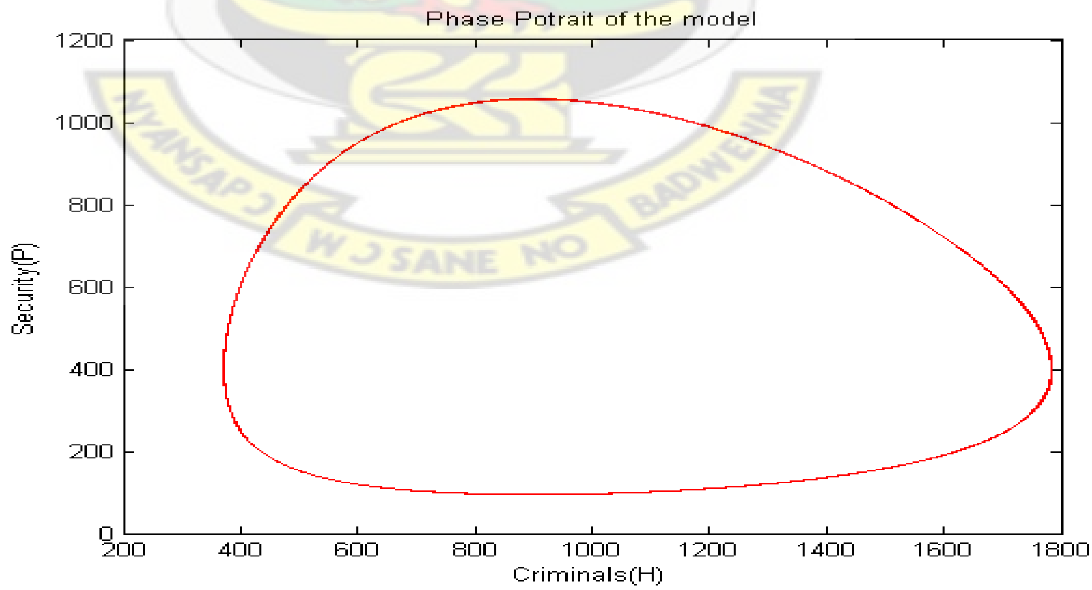

Figure 4.2: PHASE PORTRAIT OF MODEL A

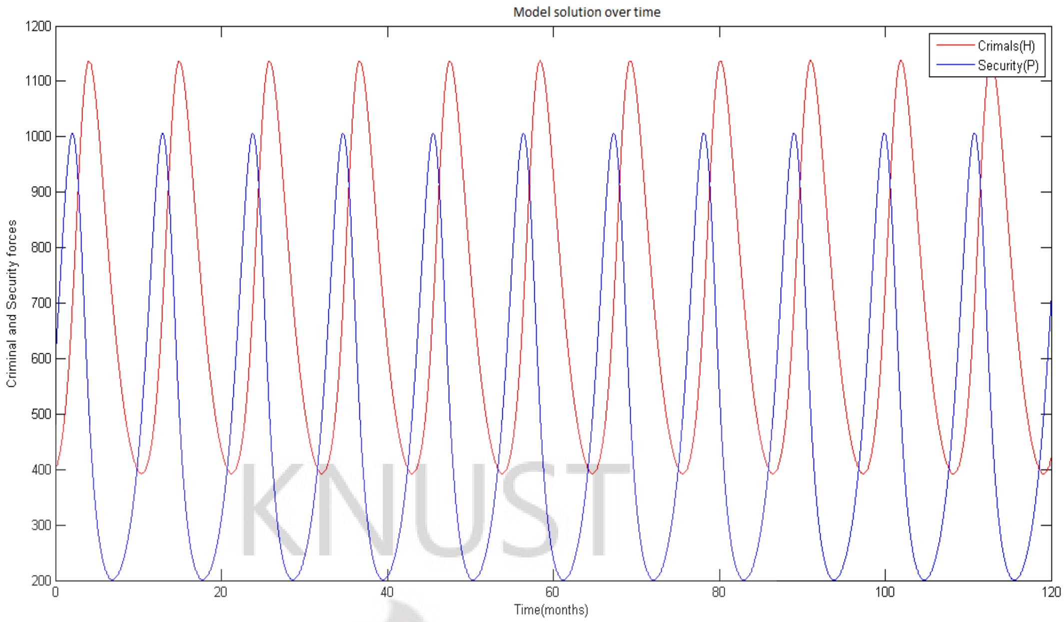

Figure 4.3: TRAJECTORIES 0F MODEL B

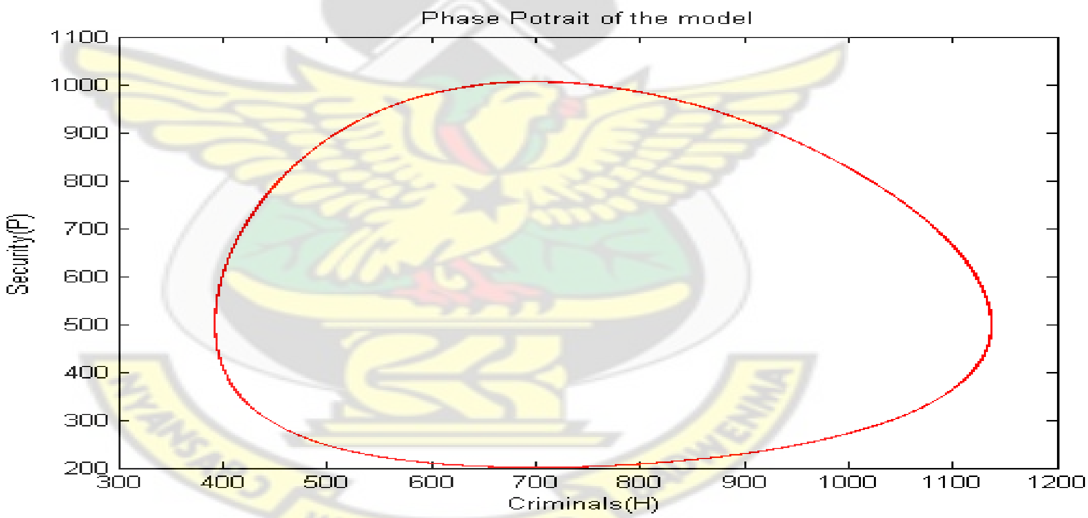

Figure 4.4: PHASE PORTRAIT OF MODEL B

Chapter 5

Conclusion

Based on the objectives, as well as the literature reviewed it is desirable to draw the following conclusions.

The analysis shows that the two models conforms to the predator - prey model.

Secondly the predator - prey cycle chart depicts population (in thousands versus time in years) of both criminals and security forces. This is quite useful in order to visualize the population fluctuation of criminals and security forces with respect to time. The average time of the periodic oscillation can be determined graphically in this way and general population variation characteristics can be determined. Notice that by analysing the same point in sequential phases and finding the time in between them the periodic oscillation for the criminals and security forces is 12 months.

The numerical results confirmed that the introduction of volunteer guards helps the security forces in apprehending criminals at a faster rate.

Fourthly , the simulation shows the successive curve for the change in the population of criminals and security forces with time, it also shows that the curve for criminals is higher than that of the security forces. The curve also shows that although new cases of criminals keeps on rising and falling, it never gets to zero (that is total eradication of criminals).

5.1 Recommendation

It is worthy to explore further the application of the predator - prey model of security forces versus criminals in a contemporary community by considering global stability of the equilibrium point. This will give a more comprehensive idea on security forces and criminals activities.

Other numerical methods such as Runge - Kutta can also be used to approximate the solution of the model.

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Appendix A

Criminal offences, 2000 – 2010

Criminal Offence	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Violent crime	125,811	127,894	130,809	130,448	131,083	119,614	130,411	142,675	140,128	135,501	137,621
Property crime	75,933	76,933	74,900	73,128	72,634	69,007	73,677	82,516	79,236	78,623	80,171
Drug possession	532	674	594	559	506	485	576	708	667	651	712
Other	18,789	20,672	22,112	22,533	20,136	18,230	23,609	29,513	25,123	24,613	25,679
Total	221,065	226,173	228,415	226,668	224,359	207,336	228,273	255,412	245,154	239,388	243642

Figure 5.1: Criminal Data From 2000 to 2010

Appendix B

MATLAB CODE FOR MODEL A

function $euler - sys()$ % clear; clc; clear global

% Definition of parameters in the model

global $\mu\beta KQ$

$\mu = 0.4$

$\beta = 0.9$

$K = 400$

$Q = 900$

$h = 0.0001;$

$t_final = 120;$

% Intial Solution made up $sol = [400; 600];$

$m = length(sol);$

% The various time steps

$t = 0 : h : t_final;$

$n = length(t);$

% Sloution set $set_of_sol = zeros(m, n);$

$set_of_sol(:, 1) = sol;$

%The euler's method $fori = 2 : n$

%Evaluate the function f(t,x)

$fun_out = fun_eval(sol);$

$sol = sol - h * fun_out;$

%Assigning solution to solution set $set_of_sol(:, i) = sol(:);$

end

%Plot of solution sets

figure(1)

$plot(t, set_of_sol(1, :),' r', t, set_of_sol(2, :),' b')$

$ylabel('CriminalandSecurityforces')$

$xlabel('Time(months)')$

$title('SolutionDynamics')$

$legend('Crimals(H)',' Security(P)')$

figure(2)

$plot(set_of_sol(1,:), set_of_sol(2,:),' r')$

$ylabel('Security(P)')$

$xlabel('Criminals(H)')$

$title('PhasePotraitofthemodel')$

end $functionfun_out = fun_eval(sol)$ %Definition of global variables used as parameters in the model

global mu beta K Q

%Model equations as vector used to evaluate

%the functional values at each discrete point.

$fun_out = [mu * sol(1) * (1 - sol(2)/K); -beta * sol(2) * (1 - sol(1)/Q)];$

end

****

MATLAB CODE FOR MODEL B

function $euler - sys()$ % clear; clc; clear global % Definition of parameters in the model

global $\mu\beta\tau u_1\gamma\alpha$

$\mu = 0.4$

$\beta = 0.9$

$\tau = 0.0001$

$u_1 = 0.03$

$\alpha = 0.001$ $\gamma = 0.001$

$h = 0.0001;$

$t_final = 120;$

% Intial Solution made up

$sol = [400; 600];$

$m = length(sol);$

% The various time steps

$t = 0 : h : t_final;$

$n = length(t);$

% Sloution set

$set_of_sol = zeros(m, n);$

$set_of_sol(:, 1) = sol;$

%The euler's method

$fori = 2 : n$

%Evaluate the function f(t,x)

$fun_out = fun_eval(sol);$

$sol = sol - h * fun_out;$

%Assigning solution to solution set

$set_of_sol(:, i) = sol(:);$

end %Plot of solution sets

figure(1)

$plot(t, set_of_sol(1, :),' r', t, set_of_sol(2, :),' b')$

$ylabel('CriminalandSecurityforces')$

$xlabel('Time(months)')$

$title('SolutionDynamics')$ $legend('Crimals(H)',' Security(P)')$

figure(2)

$plot(set_of_sol(1, :), set_of_sol(2, :),' r')$

$ylabel('Security(P)')$

$xlabel('Criminals(H)')$

$title('PhasePotraitofthemodel')$

end $functionfun_out = fun_eval(sol)$ %Definition of global variables used as parameters in the model

global $\mu\beta\tau u_1\gamma\alpha$

%Model equations as vector used to evaluate %the functional values at each discrete point. $fun_out = [mu*sol(1)-alpha*sol(1)*sol(2)-tau*u*sol(1); -beta*sol(2)-gamma*sol(1)*sol(2)+tau*u*sol(1)]$;

end

****