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A theoretical and numerical analysis of a fractal–fractional two-strain model of meningitis

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ABSTRACT

Meningitis is an inflammation of the membranes that surround and protect the brain and spinal cord. Typically, the enlargement is caused by a bacterial or viral infection of the fluid around the brain and spinal cord. For many years, licensed vaccinations against meningococcal, pneumococcal, and Haemophilus influenzae diseases have been accessible. Vaccines are meant to protect against the most dangerous strains of these germs, which are known as serotypes or serogroups. There have been significant increases in strain coverage and vaccine availability throughout time, but there is no universal vaccine against these illnesses. In this study, we explore the mathematical features of a new six-compartmental fractal–fractional two-strain model of meningitis. With the use of compact functions and $\phi-\psi$-contractions, we establish the existence of solutions. To study the unique solutions, we employ the Banach principle. On the basis of the Hyers-Ulam definition for the fractal–fractional two-strain model of meningitis, stable solutions are examined. From the numerical simulations, we notice that as the fractal–fractional order decreases, the number of infected individuals with strain 1 of meningitis decreases, while the number of infected individuals with strain 2 rises. This means that all serotypes or serogroups need to be controlled effectively for the disease to be closed up.

Introduction

The brain and spinal cord are surrounded by membranes called meninges, which can lead to meningitis if these membranes become inflamed [1]. There are several factors that can contribute to the disease, including bacteria, viruses and even protozoa, but most bacterial agents are common in meningitis in children and adolescents. The disease is more likely to spread in crowded public places where people spend a lot of time together, such as student dormitories, prisons, and military camps [2,3]. Pathogenic microorganisms can spread the disease among people in a community in a very short time. These microorganisms can be classified according to different age groups: Streptococcus pneumonia, Listeria monocytogenes, Neisseria meningitides, Group B streptococcus, and Haemophilias. Bacterial meningitis, if it progresses, can cause damage to the brain and impair learning and even hearing loss. Of course, some of the symptoms that can warn of this disease are sensitivity to natural light, nausea and headache [4]. The disease becomes more dangerous and deadly when the symptoms are not detected immediately and the infection covers the whole brain. In such cases, even with the necessary treatments, the patient may die [5,6].

Because the risk of different diseases is so important, scientists have made mathematical models that can predict how viruses and bacteria diseases spread in a population and how to control them using different optimal control strategies [7–11]. To better understand the subject, we can refer to some of these studies in which researchers use operators and mathematical parameters to design models [12–18]. Recently,

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singular and non-singular fractional operators have played a major role in such an area. The mathematical modeling of anthrax in animals [19], Hepatitis C [20], memristor-based circuit [21–23], dynamics of environmental persistence of infections [24], Langevin equation [25], genetic regulatory networks [26], Mump virus [27], Zika virus [28,29], mosaic disease [30], Computer viruses [31], thermostat control [32–34], pantograph equation [35,36], canine distemper virus [37], Q-fever [38,39], hybrid equation of p-Laplacian operators [40], COVID-19 [41–44], co-dynamics of COVID-19 and diabetes [45], chemical structure of cyclohexane [46,47], Navier systems [48,49], etc. are some instances of the application of mathematical models in simulations of the diseases. In the theoretical study of all these mentioned models, the theoretical results are among the basic parts of the analysis of mathematical models; Because the existence of a solution for a system allows us to study other properties such as stable solutions, equilibrium solutions, numerical solutions and their simulations. Usually, fixed point theory is effective in this field and its role can be observed in boundary and initial value problems. Many researchers have used this technique to prove existence results in [50–58].

In 2017, Abdon Atangana [59] worked on a new form of hybrid operators which is referred as fractal–fractional derivatives, and derived some of their properties numerically and graphically. These hybrid operators deal with some existing notions from fractal calculus and fractional calculus. The fractal–fractional derivatives are the convolution of three types of kernels with fractal derivatives; in other words, these hybrid derivatives have three kinds of kernels of the generalized Mittag–Leffler-law type, exponential-law type and power-law type. The components of the fractal dimension and fractional order play an important role in the simulating of these operators. All differential equations defined by the fractional–fractal derivatives correspond the dimension and order of the given system to a new system of the rational order.

By considering such abilities, we are able to generalize ordinary differential equations to advanced systems having arbitrary values of orders and dimensions. In fact, we can analyze and discuss a wide range of nonlocal boundary value problems in which there are some fractal behaviors. Therefore, a limited number of mathematicians and researchers generalized several phenomena to fractal–fractional models and found interesting and accurate results and simulations in comparison to fractional models. For instances, we mention such generalized fractal–fractional models in some works done by Shah et al. [60], Algehyane et al. [61], Owolabi et al. [62], Gomez-Aguilar et al. [63], Najafi et al. [64], Ali et al. [65], Khan et al. [66], Asamoah [67], Min Li et al. [68] and Addai et al. [69].

The application of mathematical modeling has also been continued by some researchers on dynamics of meningitis. Miller and Shahab [70] emphasized on the immunization in terms of cost constrain for controlling epidemic meningitis in 2005. Broutin et al. [71] applied several mathematical notions to implement wave-length technique to investigate dynamics of meningitis in nine countries of Africa. They announced that international mass collaborations in public health are effective to control the disease. Irving et al. [72] designed a deterministic simple model to analyze the control strategies of meningitis.

Unlike other studies, most models in relation to meningitis have been only designed with respect to one strain. Therefore, we need to investigate new models equipped with multiple strains of meningitis and analyze some relevant qualitative properties.

The main motivation is to present a fractal–fractional multi-strain model of meningitis and discuss it mathematically. Our contribution to this new approach is to design this mathematical structure via the newly-defined hybrid power-law operator.

To follow this study, Section “Preliminaries” is devoted to recalling some definitions. The main structure of the fractal–fractional multi-strain model of meningitis is presented in Section “Description of the meningitis model”. The existence analysis is conducted via the $\phi-\psi$-contractions and also Leray–Schauder theorem in Section “Existence of solutions”. The Banach principle is applied for investigating the unique solutions in Section “Uniqueness result”. Stable solutions are defined and proved in Section “Stability criterion” in the sense of Hyers-Ulam. To simulate the suggested model numerically, we use a new type of two-step Lagrange polynomials in the context of Adams–Bashforth method to derive some algorithms in Section “Simulations”. In Section “Simulations”, by using Matlab, we analyze the behaviors of solutions during a finite time period, and try to investigate the role of some parameters in controlling the disease by varying the values of dimensions and orders. The conclusions are presented in Section “Conclusions”.

Preliminaries

In this section, several required definitions on the hybrid fractional–fractal operators are assembled. Moreover, a special form of contractions is recalled.

Definition 1 (*[59]*)**.** Assume that $\mathcal{M}$ is fractal-differentiable and continuous on (a,b) with fractal dimension ν. Then the fractal–fractional derivative of $\mathcal{M}$ of order ω is given by

$$ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{a,\mathrm{t}}\mathcal{M}(\mathrm{t}) = \frac{1}{\Gamma(n-\omega)}\frac{\mathrm{d}}{\mathrm{d}\mathrm{t}^{\nu}}\int_a^{\mathrm{t}}(\mathrm{t}-\mathrm{m})^{n-\omega-1}\mathcal{M}(\mathrm{m})\,\mathrm{dm}, $$
$$ (n-1<\omega,\nu\le n\in\mathbb{N}), $$

where the kernel is of the power-law type and

$$ \frac{\mathrm{d}\mathcal{M}(\mathrm{m})}{\mathrm{dm}^{\nu}} = \lim_{\mathrm{t}\to\mathrm{m}}\frac{\mathcal{M}(\mathrm{t})-\mathcal{M}(\mathrm{m})}{\mathrm{t}^{\nu}-\mathrm{m}^{\nu}}, $$

is the fractal derivative.

It is notable that by taking $\nu=1$, the fractal–fractional derivative ${}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{a,\mathrm{t}}$ is the same ωth-Riemann–Liouville derivative ${}^{RL}\mathfrak{D}^{\omega}_{a,\mathrm{t}}$.

Definition 2 (*[59]*)**.** Assume that $\mathcal{M}$ is continuous on (a,b). The (ω,ν)-fractal–fractional integral of $\mathcal{M}$ with the power-law type kernel is

$$ {}^{\mathbf{FFP}}\mathfrak{I}^{\omega,\nu}_{a,\mathrm{t}}\mathcal{M}(\mathrm{t}) = \frac{\nu}{\Gamma(\omega)}\int_a^{\mathrm{t}}\mathrm{m}^{\nu-1}(\mathrm{t}-\mathrm{m})^{\omega-1}\mathcal{M}(\mathrm{m})\,\mathrm{dm}. $$

Consider Ψ as a subcategory of increasing mappings $\psi:\mathbb{R}^{\ge 0}\to\mathbb{R}^{\ge 0}$ with $\psi(\mathrm{t})<\mathrm{t}$, $\forall\,\mathrm{t}>0$, and

$$ \sum_{\ell=1}^{\infty}\psi^{\ell}(\mathrm{t})<\infty. $$

Definition 3 (*[73]*)**.** Let $\mathcal{M}:\mathbb{X}\to\mathbb{X}$ and $\phi:\mathbb{X}^2\to\mathbb{R}^{\ge 0}$, where $\mathbb{X}$ is a norm space. Then

(1) if for each $\varkappa_1,\varkappa_2\in\mathbb{X}$,

$$ \phi(\varkappa_1,\varkappa_2)\mathbf{d}(\mathcal{M}\varkappa_1,\mathcal{M}\varkappa_2)\le\psi(d(\varkappa_1,\varkappa_2)), $$

then $\mathcal{M}$ is $\phi-\psi$-contraction.

(2) if $\phi(\varkappa_1,\varkappa_2)\ge 1$ gives $\phi(\mathcal{M}\varkappa_1,\mathcal{M}\varkappa_2)\ge 1$, then $\mathcal{M}$ is ϕ-admissible.

Description of the meningitis model

Baba et al. [74] designed a meningitis model in the context of a six-compartmental system of differential equations. These six compartments are denoted by $\mathcal{S}(\mathrm{t})$, $\mathcal{C}_1(\mathrm{t})$, $\mathcal{C}_2(\mathrm{t})$, $\mathcal{I}_1(\mathrm{t})$, $\mathcal{I}_2(\mathrm{t})$, $\mathcal{R}(\mathrm{t})$. The susceptible individuals at time t is denoted by $\mathcal{S}(\mathrm{t})$, $\mathcal{C}_1(\mathrm{t})$ denotes carrier individuals with strain 1 meningitis at time t, $\mathcal{C}_2(\mathrm{t})$ denotes carrier individuals with strain 2 meningitis at time t, $\mathcal{I}_1(\mathrm{t})$, and $\mathcal{I}_2(\mathrm{t})$ represent the number of infected individuals with stain 1 and strain 2 respectively. The recovered individuals from strain 1 and strain 2 at time t is denoted as $\mathcal{R}(\mathrm{t})$. Because of birth, migration and other factors of growth of population, we consider the recruitment rate as a constant value in the susceptible individuals and there is no double infection. B is the recruitment rate, ζ stands for the loss of immunity, s is the rate of natural death, p_1 and p_2 are the rates of effectiveness contact caused by strain 1 and strain 2, respectively, θ denotes the rate of natural recovery, q_1 and q_2 stand for the rates of progression from $\mathcal{C}_1$ to $\mathcal{I}_1$ and from $\mathcal{C}_2$ to $\mathcal{I}_2$, respectively. The parameter b_1 is the rate of mortality due to disease caused by strain 1 and b_2 is also the same parameter caused by strain 2. Finally, the rate of recovery from disease caused by strains 1 and 2 are given by r_1 and r_2, respectively. In view of the above description the system of equation governing the proposed model in Baba et al. [74] is given as

$$\begin{cases} \dfrac{d\mathcal{S}(t)}{dt} = B + \zeta\mathcal{R}(t) - p_1\mathcal{S}(t)(\mathcal{C}_1(t) + \mathcal{I}_1(t)) - p_2\mathcal{S}(t)(\mathcal{C}_2(t) + \mathcal{I}_2(t)) \\ \qquad -s\mathcal{S}(t), \\ \dfrac{d\mathcal{C}_1(t)}{dt} = p_1\mathcal{S}(t)(\mathcal{C}_1(t) + \mathcal{I}_1(t)) - (q_1 + s + \theta)\mathcal{C}_1(t), \\ \dfrac{d\mathcal{I}_1(t)}{dt} = q_1\mathcal{C}_1(t) - (s + b_1 + r_1)\mathcal{I}_1(t), \\ \dfrac{d\mathcal{C}_2(t)}{dt} = p_2\mathcal{S}(t)(\mathcal{C}_2(t) + \mathcal{I}_2(t)) - (q_2 + s + \theta)\mathcal{C}_2(t), \\ \dfrac{d\mathcal{I}_2(t)}{dt} = q_2\mathcal{C}_2(t) - (s + b_2 + r_2)\mathcal{I}_2(t), \\ \dfrac{d\mathcal{R}(t)}{dt} = r_1\mathcal{I}_1(t) + r_2\mathcal{I}_2(t) + \theta\mathcal{C}_1(t) + \theta\mathcal{C}_2(t) - (\zeta + s)\mathcal{R}(t), \end{cases} \tag{1}$$

The initial conditions are $\mathcal{S}(0) = \mathcal{S}_0 > 0$, $\mathcal{C}_1(0) = \mathcal{C}_{1,0} \geq 0$, $\mathcal{C}_2(0) = \mathcal{C}_{2,0} \geq 0$, $\mathcal{I}_1(0) = \mathcal{I}_{1,0} \geq 0$, $\mathcal{I}_2(0) = \mathcal{I}_{2,0} \geq 0$, and $\mathcal{R}(0) = \mathcal{R}_0 \geq 0$.

Inspired by the mentioned standard model (1), we define another mathematical two-strain model of meningitis via the fractal–fractional hybrid derivatives as

$$\begin{cases} {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{S}(t) = B + \zeta\mathcal{R}(t) - p_1\mathcal{S}(t)(\mathcal{C}_1(t) + \mathcal{I}_1(t)) - p_2\mathcal{S}(t)(\mathcal{C}_2(t) + \mathcal{I}_2(t)) \\ \qquad -s\mathcal{S}(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_1(t) = p_1\mathcal{S}(t)(\mathcal{C}_1(t) + \mathcal{I}_1(t)) - (q_1 + s + \theta)\mathcal{C}_1(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_1(t) = q_1\mathcal{C}_1(t) - (s + b_1 + r_1)\mathcal{I}_1(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_2(t) = p_2\mathcal{S}(t)(\mathcal{C}_2(t) + \mathcal{I}_2(t)) - (q_2 + s + \theta)\mathcal{C}_2(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_2(t) = q_2\mathcal{C}_2(t) - (s + b_2 + r_2)\mathcal{I}_2(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{R}(t) = r_1\mathcal{I}_1(t) + r_2\mathcal{I}_2(t) + \theta\mathcal{C}_1(t) + \theta\mathcal{C}_2(t) - (\zeta + s)\mathcal{R}(t), \end{cases} \tag{2}$$

with initial conditions

$$\begin{aligned} &\mathcal{S}(0) = \mathcal{S}_0 > 0, \quad \mathcal{C}_1(0) = \mathcal{C}_{1,0} \geq 0, \quad \mathcal{C}_2(0) = \mathcal{C}_{2,0} \geq 0, \\ &\mathcal{I}_1(0) = \mathcal{I}_{1,0} \geq 0, \quad \mathcal{I}_2(0) = \mathcal{I}_{2,0} \geq 0, \quad \mathcal{R}(0) = \mathcal{R}_0 \geq 0, \end{aligned}$$

in which ${}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}$ is the fractional–fractal derivative with the fractal dimension $\nu \in (0,1]$ and fractional order $\omega \in (0,1]$. Some necessary assumptions, we impose on the model are that: all the involved parameters in the model (2) are nonnegative, (the non-negativity can be seen in Baba et al. [74]), hence the state functions of the model is defined by

$$\mathcal{N}(t) = \mathcal{S}(t) + \mathcal{C}_1(t) + \mathcal{C}_2(t) + \mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{R}(t),$$

where $\mathcal{N}(t)$ is the total population at the time $t \in \mathbb{J} := [0, T], (T > 0)$.

Existence of solutions

To investigate the qualitative properties, we consider a Banach space as $\mathbb{X} = \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y} \times \mathbb{Y}$, where $\mathbb{Y} = C(\mathbb{J}, \mathbb{R})$ shows the subclass of all continuous functions under the norm

$$\begin{aligned} \|\mathcal{K}\|_{\mathbb{X}} &= \|(\mathcal{S}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{I}_1, \mathcal{I}_2, \mathcal{R})\|_{\mathbb{X}} \\ &= \max\{|\mathcal{S}(t)| + |\mathcal{C}_1(t)| + |\mathcal{C}_2(t)| + |\mathcal{I}_1(t)| + |\mathcal{I}_2(t)| + |\mathcal{R}(t)| : \quad t \in \mathbb{J}\}. \end{aligned}$$

To begin the analysis, at first we define the R.H.S. of the fractal–fractional model (2) via six functions as

$$\begin{cases} \mathcal{F}_1(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)) = B + \zeta\mathcal{R}(t) - p_1\mathcal{S}(t)(\mathcal{C}_1(t) \\ \qquad + \mathcal{I}_1(t)) \\ \qquad - p_2\mathcal{S}(t)(\mathcal{C}_2(t) + \mathcal{I}_2(t)) - s\mathcal{S}(t), \\ \mathcal{F}_2(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)) = p_1\mathcal{S}(t)(\mathcal{C}_1(t) + \mathcal{I}_1(t)) \\ \qquad - (q_1 + s + \theta)\mathcal{C}_1(t), \\ \mathcal{F}_3(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)) = q_1\mathcal{C}_1(t) - (s + b_1 + r_1)\mathcal{I}_1(t), \\ \mathcal{F}_4(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)) = p_2\mathcal{S}(t)(\mathcal{C}_2(t) + \mathcal{I}_2(t)) \\ \qquad - (q_2 + s + \theta)\mathcal{C}_2(t), \\ \mathcal{F}_5(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)) = q_2\mathcal{C}_2(t) - (s + b_2 + r_2)\mathcal{I}_2(t), \\ \mathcal{F}_6(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)) = r_1\mathcal{I}_1(t) + r_2\mathcal{I}_2(t) \\ \qquad + \theta(\mathcal{C}_1(t) + \mathcal{C}_2(t)) \\ \qquad - (\zeta + s)\mathcal{R}(t). \end{cases} \tag{3}$$

Due to the differentiability of integral, the given problem (2) is presented by

$$\begin{cases} {}^{\mathbf{RL}}\mathfrak{D}^{\omega}_{0,t}\mathcal{S}(t) = \nu t^{\nu-1}\mathcal{F}_1(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ {}^{\mathbf{RL}}\mathfrak{D}^{\omega}_{0,t}\mathcal{C}_1(t) = \nu t^{\nu-1}\mathcal{F}_2(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ {}^{\mathbf{RL}}\mathfrak{D}^{\omega}_{0,t}\mathcal{I}_1(t) = \nu t^{\nu-1}\mathcal{F}_3(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ {}^{\mathbf{RL}}\mathfrak{D}^{\omega}_{0,t}\mathcal{C}_2(t) = \nu t^{\nu-1}\mathcal{F}_4(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ {}^{\mathbf{RL}}\mathfrak{D}^{\omega}_{0,t}\mathcal{I}_2(t) = \nu t^{\nu-1}\mathcal{F}_5(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ {}^{\mathbf{RL}}\mathfrak{D}^{\omega}_{0,t}\mathcal{R}(t) = \nu t^{\nu-1}\mathcal{F}_6(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)). \end{cases} \tag{4}$$

By (4), the extended system of initial value problems is reformulated as

$$\begin{cases} {}^{\mathbf{RL}}\mathfrak{D}^{\omega}_{0,t}\mathcal{K}(t) = \nu t^{\nu-1}\mathcal{F}(t, \mathcal{K}(t)), \\ \mathcal{K}(0) = \mathcal{K}_0, \quad \omega, \nu \in (0,1], \end{cases} \tag{5}$$

for $t \in \mathbb{J}$ so that

$$\begin{aligned} \mathcal{K}(t) &= (\mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t))^T, \\ \mathcal{K}_0 &= (\mathcal{S}_0, \mathcal{C}_{1,0}, \mathcal{I}_{1,0}, \mathcal{C}_{2,0}, \mathcal{I}_{2,0}, \mathcal{R}_0)^T, \end{aligned} \tag{6}$$

and

$$\mathcal{F}(t, \mathcal{K}(t)) = \begin{cases} \mathcal{F}_1(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ \mathcal{F}_2(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ \mathcal{F}_3(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ \mathcal{F}_4(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ \mathcal{F}_5(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)), \\ \mathcal{F}_6(t, \mathcal{S}(t), \mathcal{C}_1(t), \mathcal{I}_1(t), \mathcal{C}_2(t), \mathcal{I}_2(t), \mathcal{R}(t)). \end{cases} \tag{7}$$

By operating the hybrid fractional–fractal integral on both sides of (5), we have

$$\mathcal{K}(t) = \mathcal{K}(0) + \frac{\nu}{\Gamma(\omega)}\int_0^t \mathrm{m}^{\nu-1}(t - \mathrm{m})^{\omega-1}\mathcal{F}(\mathrm{m}, \mathcal{K}(\mathrm{m}))\,d\mathrm{m}. \tag{8}$$

In fact, we obtain the following extended form

$$\begin{cases} \mathcal{S}(t) = \mathcal{S}_0 + \dfrac{\nu}{\Gamma(\omega)} \displaystyle\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ \quad \times \mathcal{F}_1\big(m, \mathcal{S}(m), \mathcal{C}_1(m), \mathcal{I}_1(m), \mathcal{C}_2(m), \mathcal{I}_2(m), \mathcal{R}(m)\big)\, dm, \\ \mathcal{C}_1(t) = \mathcal{C}_{1,0} + \dfrac{\nu}{\Gamma(\omega)} \displaystyle\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ \quad \times \mathcal{F}_2\big(m, \mathcal{S}(m), \mathcal{C}_1(m), \mathcal{I}_1(m), \mathcal{C}_2(m), \mathcal{I}_2(m), \mathcal{R}(m)\big)\, dm, \\ \mathcal{I}_1(t) = \mathcal{I}_{1,0} + \dfrac{\nu}{\Gamma(\omega)} \displaystyle\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ \quad \times \mathcal{F}_3\big(m, \mathcal{S}(m), \mathcal{C}_1(m), \mathcal{I}_1(m), \mathcal{C}_2(m), \mathcal{I}_2(m), \mathcal{R}(m)\big)\, dm, \\ \mathcal{C}_2(t) = \mathcal{C}_{2,0} + \dfrac{\nu}{\Gamma(\omega)} \displaystyle\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ \quad \times \mathcal{F}_4\big(m, \mathcal{S}(m), \mathcal{C}_1(m), \mathcal{I}_1(m), \mathcal{C}_2(m), \mathcal{I}_2(m), \mathcal{R}(m)\big)\, dm, \\ \mathcal{I}_2(t) = \mathcal{I}_{2,0} + \dfrac{\nu}{\Gamma(\omega)} \displaystyle\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ \quad \times \mathcal{F}_5\big(m, \mathcal{S}(m), \mathcal{C}_1(m), \mathcal{I}_1(m), \mathcal{C}_2(m), \mathcal{I}_2(m), \mathcal{R}(m)\big)\, dm, \\ \mathcal{R}(t) = \mathcal{R}_0 + \dfrac{\nu}{\Gamma(\omega)} \displaystyle\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ \quad \times \mathcal{F}_6\big(m, \mathcal{S}(m), \mathcal{C}_1(m), \mathcal{I}_1(m), \mathcal{C}_2(m), \mathcal{I}_2(m), \mathcal{R}(m)\big)\, dm. \end{cases} \tag{9}$$

In this position, we transform (2) into a fixed point problem. Define $G : \mathbb{X} \to \mathbb{X}$ by

$$G(\mathcal{K}(t)) = \mathcal{K}(0) + \frac{\nu}{\Gamma(\omega)} \int_0^t m^{\nu-1}(t-m)^{\omega-1}\mathcal{F}\big(m, \mathcal{K}(m)\big)\, dm. \tag{10}$$

In the following, a fixed point theorem for $\phi - \psi$-contractions is given which is useful for our goal.

Theorem 1 *([73]). Let $(\mathbb{X}, \mathbf{d})$ be a metric space with the completeness property, $\psi \in \Psi$, $\phi : \mathbb{X}^2 \to \mathbb{R}$, and $\mathcal{G} : \mathbb{X} \to \mathbb{X}$ an $\phi - \psi$-contraction s.t.*

(1) $\mathcal{G}$ is ϕ-admissible;
(2) $\exists\, \varkappa_0 \in \mathbb{X}$, s.t. $\phi(\varkappa_0, \mathcal{G}\varkappa_0) \geq 1$;
(3) $\forall \{\varkappa_n\} \subseteq \mathbb{X}$ with $\varkappa_n \to \varkappa$ and $\phi(\varkappa_n, \varkappa_{n+1}) \geq 1$, $\forall n \geq 1$, we have $\phi(\varkappa_n, \varkappa) \geq 1$, $\forall n \geq 1$.

Then a fixed point exists for $\mathcal{G}$.

Here, the existence result is established under $\phi - \psi$-contractions.

Theorem 2. *Let $\exists\, \mathfrak{T} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\exists\, \psi \in \Psi$ and $\exists\, \mathcal{F} \in C(\mathbb{J} \times \mathbb{X}, \mathbb{X})$. Also,*

($\mathfrak{P}_1$) *$\forall\, \mathcal{K}_1, \mathcal{K}_2 \in \mathbb{X}$ and $t \in \mathbb{J}$,*

$$|\mathcal{F}(t, \mathcal{K}_1(t)) - \mathcal{F}(t, \mathcal{K}_2(t))| \leq \tilde{\ell}\, \psi(|\mathcal{K}_1(t) - \mathcal{K}_2(t)|),$$

with $\mathfrak{T}\big(\mathcal{K}_1(t), \mathcal{K}_2(t)\big) \geq 0$ *and* $\tilde{\ell} = \dfrac{\Gamma(\nu+\omega)}{\nu T^{\nu+\omega-1}\Gamma(\nu)}$;

($\mathfrak{P}_2$) *$\exists\, \mathcal{K}_0 \in \mathbb{X}$ s.t. $\forall\, t \in \mathbb{J}$,*

$$\mathfrak{T}\big(\mathcal{K}_0(t), G(\mathcal{K}_0(t))\big) \geq 0,$$

and also

$$\mathfrak{T}\big(\mathcal{K}_1(t), \mathcal{K}_2(t)\big) \geq 0 \implies \mathfrak{T}\big(G(\mathcal{K}_1(t)), G(\mathcal{K}_2(t))\big) \geq 0;$$

($\mathfrak{P}_3$) *$\forall \{\mathcal{K}_n\}_{n\geq 1} \subseteq \mathbb{X}$ with $\mathcal{K}_n \to \mathcal{K}$,*

$$\mathfrak{T}\big(\mathcal{K}_n(t), \mathcal{K}_{n+1}(t)\big) \geq 0 \implies \mathfrak{T}\big(\mathcal{K}_n(t), \mathcal{K}(t)\big) \geq 0,$$

for each n and $t \in \mathbb{J}$.

Then, there is a solution for the given fractional–fractal model of meningitis (2).

Proof. Consider $\mathcal{K}_1, \mathcal{K}_2$ belonging to $\mathbb{X}$ so that $\mathfrak{T}\big(\mathcal{K}_1(t), \mathcal{K}_2(t)\big) \geq 0$, for each $t \in \mathbb{J}$. Definition of the Beta function and some simple computations yield that

$$\begin{aligned} \Big|G(\mathcal{K}_1(t)) - G(\mathcal{K}_2(t))\Big| &\leq \frac{\nu}{\Gamma(\omega)} \int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ &\quad \times \Big|\mathcal{F}\big(m, \mathcal{K}_1(m)\big) - \mathcal{F}\big(m, \mathcal{K}_2(m)\big)\Big|\, dm \\ &\leq \frac{\nu}{\Gamma(\omega)} \int_0^t m^{\nu-1}(t-m)^{\omega-1}\tilde{\ell}\psi \\ &\quad \times \big(|\mathcal{K}_1(m) - \mathcal{K}_2(m)|\big)\, dm \\ &\leq \frac{\nu\tilde{\ell}T^{\nu+\omega-1}\mathbb{B}(\nu,\omega)}{\Gamma(\omega)}\psi\big(\|\mathcal{K}_1 - \mathcal{K}_2\|_{\mathbb{X}}\big) \\ &= \frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}\tilde{\ell}\psi\big(\|\mathcal{K}_1 - \mathcal{K}_2\|_{\mathbb{X}}\big). \end{aligned}$$

Therefore,

$$\|G(\mathcal{K}_1) - G(\mathcal{K}_2)\|_{\mathbb{X}} \leq \frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}\tilde{\ell}\psi\big(\|\mathcal{K}_1 - \mathcal{K}_2\|_{\mathbb{X}}\big) = \psi\big(\|\mathcal{K}_1 - \mathcal{K}_2\|_{\mathbb{X}}\big).$$

For arbitrary elements $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{X}$, define $\phi : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ by

$$\phi(\mathcal{K}_1, \mathcal{K}_2) = \begin{cases} 1 & \text{if } \mathfrak{T}\big(\mathcal{K}_1(t), \mathcal{K}_2(t)\big) \geq 0, \\ 0 & \text{Otherwise.} \end{cases}$$

Then,

$$\phi(\mathcal{K}_1, \mathcal{K}_2)\mathbf{d}(G(\mathcal{K}_1), G(\mathcal{K}_2)) \leq \psi(\mathbf{d}(\mathcal{K}_1, \mathcal{K}_2)), \qquad \forall\, \mathcal{K}_1, \mathcal{K}_2 \in \mathbb{X}.$$

So, G is an $\phi - \psi$-contraction. In this step, let $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{X}$ with $\phi(\mathcal{K}_1, \mathcal{K}_2) \geq 1$. From the property of ϕ, we find that $\mathfrak{T}\big(\mathcal{K}_1(t), \mathcal{K}_2(t)\big) \geq 0$. Thus ($\mathfrak{P}_2$) implies that $\mathfrak{T}\big(G(\mathcal{K}_1(t)), G(\mathcal{K}_2(t))\big) \geq 0$. Once again, ϕ yields that $\phi(G(\mathcal{K}_1), G(\mathcal{K}_2)) \geq 1$. Therefore G is an ϕ-admissible.

Moreover, ($\mathfrak{P}_2$) confirms that there is some $\mathcal{K}_0 \in \mathbb{X}$. Then, $\mathfrak{T}\big(\mathcal{K}_0(t), G(\mathcal{K}_0(t))\big) \geq 0$, $\forall\, t \in \mathbb{J}$. It is evident that

$$\phi(\mathcal{K}_0, G(\mathcal{K}_0)) \geq 1.$$

Consider $\{\mathcal{K}_n\}_{n\geq 1} \subseteq \mathbb{X}$ s.t. $\mathcal{K}_n \to \mathcal{K}$ and $\forall\, n$, $\phi(\mathcal{K}_n, \mathcal{K}_{n+1}) \geq 1$. Definition of ϕ implies that

$$\mathfrak{T}\big(\mathcal{K}_n(t), \mathcal{K}_{n+1}(t)\big) \geq 0.$$

Thus, ($\mathfrak{P}_3$) gives $\mathfrak{T}\big(\mathcal{K}_n(t), \mathcal{K}(t)\big) \geq 0$. Therefore $\phi(\mathcal{K}_n, \mathcal{K}) \geq 1$, $\forall\, n$. By Theorem 1, we find that $\exists \mathcal{K}^* \in \mathbb{X}$ s.t. $G(\mathcal{K}^*) = \mathcal{K}^*$. Consequently, $\mathcal{K}^* = \big(\mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*\big)^T$ is a solution to the fractal–fractional two-strain model of meningitis (2). ☐

The Leray–Schauder theorem is the next auxiliary theorem for our purpose in relation to the existence result.

Theorem 3 *([75]). Let $\mathbb{X}$ be a Banach space, $\mathcal{E}$ a convex, bounded, and closed set in $\mathbb{X}$ and $\mathbb{G} \subseteq \mathcal{E}$ an open set with $0 \in \mathbb{G}$. If $f : \bar{\mathbb{G}} \to \mathcal{E}$ is continuous and compact, then either:*

(i) $\exists \varkappa^* \in \bar{\mathbb{G}}$ s.t. $f(\varkappa^*) = \varkappa^*$, or
(ii) $\exists\, \varkappa \in \partial\mathbb{G}$ and $\mu \in (0,1)$ s.t. $\varkappa = \mu f(\varkappa)$.

Remark 1. For convenience, we define

$$\Lambda = \mathcal{K}_0, \tag{11}$$

and

$$\Delta = \frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}. \tag{12}$$

Theorem 4. *Assume that $\mathcal{F} \in C(\mathbb{J} \times \mathbb{X}, \mathbb{X})$.*

(C_1): $\exists\, \phi \in L^1(\mathbb{J}, \mathbb{R}^+)$ and $\exists$ an increasing map $B \in C([0,\infty), \mathbb{R}^{>0})$ satisfying, for all $t \in \mathbb{J}$ and $\mathcal{K} \in \mathbb{X}$,

$$|\mathcal{F}(t, \mathcal{K}(t))| \leq \phi(t)B(|\mathcal{K}(t)|);$$

($\mathcal{C}_2$): $\exists\, \gamma > 0$ and

$$\frac{\gamma}{\Lambda + \Delta\phi_0^* B(\gamma)} > 1, \tag{13}$$

with $\phi_0^* = \sup_{\mathfrak{t}\in\mathbb{J}} |\phi((\mathfrak{t}))|$ and Λ, Δ are specified in (11) and (12).

Then a solution exists for the given fractal–fractional two-strain model of meningitis (2) on $\mathbb{J}$.

Proof. Consider $G : \mathbb{X} \to \mathbb{X}$ defined by (10) and

$$\mathcal{N}_\epsilon = \{\mathcal{K} \in \mathbb{X} : \|\mathcal{K}\|_{\mathbb{X}} \le \epsilon\}, \quad \text{for some } \epsilon > 0.$$

From the continuity of $\mathcal{F}$, we yield the continuity of G immediately. Now from ($\mathcal{C}_1$) and for $\mathcal{K} \in \mathcal{N}_\epsilon$, we have

$$\begin{aligned} |G(\mathcal{K}(\mathfrak{t}))| &\le |\mathcal{K}(0)| + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathfrak{t}} \mathfrak{m}^{\nu-1}(\mathfrak{t}-\mathfrak{m})^{\omega-1} |\mathcal{F}(\mathfrak{m}, \mathcal{K}(\mathfrak{m}))| \mathbf{d}\mathfrak{m} \\ &\le \mathcal{K}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathfrak{t}} \mathfrak{m}^{\nu-1}(\mathfrak{t}-\mathfrak{m})^{\omega-1} \phi(\mathfrak{m}) B(|\mathcal{K}(\mathfrak{m})|) \mathbf{d}\mathfrak{m} \\ &\le \mathcal{K}_0 + \frac{\nu T^{\nu+\omega-1}\mathbb{B}(\nu,\omega)}{\Gamma(\omega)} \phi_0^* B(\|\mathcal{K}\|_{\mathbb{X}}) \\ &\le \Lambda + \Delta\phi_0^* B(\epsilon). \end{aligned}$$

Hence

$$\|G\mathcal{K}\| \le \Lambda + \Delta\phi_0^* B(\epsilon) < \infty. \tag{14}$$

The uniform boundedness of G is derived on $\mathbb{X}$. Arbitrarily, choose $\mathfrak{t}, \mathfrak{t}' \in [0, T]$ s.t. $\mathfrak{t} < \mathfrak{t}'$ and $\mathcal{K} \in \mathcal{N}_\epsilon$. By assuming

$$\sup_{(\mathfrak{t},\mathcal{K})\in\mathbb{J}\times\mathcal{N}_\epsilon} |\mathcal{F}(\mathfrak{t}, \mathcal{K}(\mathfrak{t}))| = \mathcal{F}^* < \infty,$$

we have

$$\begin{aligned} |G(\mathcal{K}(\mathfrak{t}')) - G(\mathcal{K}(\mathfrak{t}))| &= \left| \frac{\nu}{\Gamma(\omega)} \int_0^{\mathfrak{t}'} \mathfrak{m}^{\nu-1}(\mathfrak{t}'-\mathfrak{m})^{\omega-1} \mathcal{F}(\mathfrak{m}, \mathcal{K}(\mathfrak{m})) \mathbf{d}\mathfrak{m} \right. \\ &\quad \left. - \frac{\nu}{\Gamma(\omega)} \int_0^{\mathfrak{t}} \mathfrak{m}^{\nu-1}(\mathfrak{t}-\mathfrak{m})^{\omega-1} \mathcal{F}(\mathfrak{m}, \mathcal{F}(\mathfrak{m})) \mathbf{d}\mathfrak{m} \right| \\ &\le \frac{\nu\mathcal{F}^*}{\Gamma(\omega)} \left| \int_0^{\mathfrak{t}'} \mathfrak{m}^{\nu-1}(\mathfrak{t}'-\mathfrak{m})^{\omega-1} \mathbf{d}\mathfrak{m} \right. \\ &\quad \left. - \int_0^{\mathfrak{t}} \mathfrak{m}^{\nu-1}(\mathfrak{t}-\mathfrak{m})^{\omega-1} \mathbf{d}\mathfrak{m} \right| \\ &\le \frac{\nu\mathcal{F}^*\mathbb{B}(\nu,\omega)}{\Gamma(\omega)} [\mathfrak{t}'^{\nu+\omega-1} - \mathfrak{t}^{\nu+\omega-1}] \\ &= \frac{\nu\mathcal{F}^*\Gamma(\nu)}{\Gamma(\nu+\omega)} [(\mathfrak{t}')^{\nu+\omega-1} - \mathfrak{t}^{\nu+\omega-1}], \end{aligned} \tag{15}$$

which is not dependent of $\mathcal{K}$, and as $\mathfrak{t}' \to \mathfrak{t}$, the R. H. S. of above inequality tends to 0. Thus

$$\|G(\mathcal{K}(\mathfrak{t}')) - G(\mathcal{K}(\mathfrak{t}))\|_{\mathbb{X}} \to 0.$$

This proves the equicontinuity of G. Finally, the Arzelá-Ascoli theorem gives the compactness of G on $\mathcal{N}_\epsilon$. The hypotheses of Theorem 3 on the operator G have now been verified. Thus one of the two cases (i) or (ii) will hold. Utilizing ($\mathcal{C}_2$), we construct

$$\mathbb{P} = \{\mathcal{K} \in \mathbb{X} : \|\mathcal{K}\|_{\mathbb{X}} < \gamma\}$$

for some $\gamma > 0$ via

$$\Lambda + \Delta\phi_0^* B(\gamma) < \gamma.$$

Utilizing ($\mathcal{C}1$) and by (14), we estimate

$$\|G\mathcal{K}\|_{\mathbb{X}} \le \Lambda + \Delta\phi_0^* B(\mathcal{K}). \tag{16}$$

Consider the existence of $\mathcal{K} \in \partial\mathbb{P}$ and $\alpha \in (0,1)$ s.t. $\mathcal{K} = \alpha G(\mathcal{K})$. For such α and $\mathcal{K}$, by (16),

$$\gamma = \|\mathcal{K}\|_{\mathbb{X}} = \alpha\|G\mathcal{K}\|_{\mathbb{X}} < \Lambda + \Delta\phi_0^* B(\|\mathcal{K}\|_{\mathbb{X}}) < \Lambda + \Delta\phi_0^* B(\gamma) < \gamma,$$

which is not possible. Therefore, (ii) is invalid and by Theorem 3, G admits a fixed point in $\bar{\mathbb{P}}$. In other words, it is found a solution for the fractal–fractional two-strain model of meningitis (2). $\square$

Uniqueness result

Now, we are ready to discuss on the unique solutions of the model.

Lemma 1. *Let $\mathcal{S}, \mathcal{C}_1, \mathcal{I}_1, \mathcal{C}_2, \mathcal{I}_2, \mathcal{R}, \mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^* \in \mathbb{Y} = C(\mathbb{J}, \mathbb{R})$ and (H1): $\|\mathcal{S}\| \le \lambda_1$, $\|\mathcal{C}_1\| \le \lambda_2$, $\|\mathcal{I}_1\| \le \lambda_3$, $\|\mathcal{C}_2\| \le \lambda_4$, $\|\mathcal{I}_2\| \le \lambda_5$, $\|\mathcal{R}\| \le \lambda_6$ for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 > 0$, where the norms are sup-norm with respect to $\mathfrak{t}$.*

In this case, $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6$ given in (3) are Lipschitz w. r. t. the corresponding components whenever $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6 > 0$, where

$$\begin{aligned} &\kappa_1 = p_1(\lambda_2 + \lambda_3) + p_2(\lambda_4 + \lambda_5) + s, \quad \kappa_2 = p_1\lambda_1 + (q_1 + s + \theta), \\ &\kappa_3 = s + b_1 + r_1, \\ &\kappa_4 = p_2\lambda_1 + (q_2 + s + \theta), \quad \kappa_5 = s + b_2 + r_2, \quad \kappa_6 = \zeta + s. \end{aligned} \tag{17}$$

Proof. Starting from the function $\mathcal{F}_1$, for each $\mathcal{S}, \mathcal{S}^* \in \mathbb{Y}$, we estimate

$$\begin{aligned} &\| \mathcal{F}_1 \big(\mathfrak{t}, \mathcal{S}(\mathfrak{t}), \mathcal{C}_1(\mathfrak{t}), \mathcal{I}_1(\mathfrak{t}), \mathcal{C}_2(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \\ &\quad - \mathcal{F}_1 \big(\mathfrak{t}, \mathcal{S}^*(\mathfrak{t}), \mathcal{C}_1(\mathfrak{t}), \mathcal{I}_1(\mathfrak{t}), \mathcal{C}_2(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \| \\ &\quad = \| (B + \zeta\mathcal{R}(\mathfrak{t}) - p_1\mathcal{S}(\mathfrak{t})(\mathcal{C}_1(\mathfrak{t}) + \mathcal{I}_1(\mathfrak{t})) - p_2\mathcal{S}(\mathfrak{t})(\mathcal{C}_2(\mathfrak{t}) \\ &\quad + \mathcal{I}_2(\mathfrak{t})) - s\mathcal{S}(\mathfrak{t})) \\ &\quad - (B + \zeta\mathcal{R}(\mathfrak{t}) - p_1\mathcal{S}^*(\mathfrak{t})(\mathcal{C}_1(\mathfrak{t}) + \mathcal{I}_1(\mathfrak{t})) - p_2\mathcal{S}^*(\mathfrak{t})(\mathcal{C}_2(\mathfrak{t}) + \mathcal{I}_2(\mathfrak{t})) \\ &\quad - s\mathcal{S}^*(\mathfrak{t})) \| \\ &\quad \le \| -p_1(\mathcal{C}_1(\mathfrak{t}) + \mathcal{I}_1(\mathfrak{t}))(\mathcal{S} - \mathcal{S}^*) - p_2(\mathcal{C}_2(\mathfrak{t}) + \mathcal{I}_2(\mathfrak{t}))(\mathcal{S} - \mathcal{S}^*) \\ &\quad - s(\mathcal{S} - \mathcal{S}^*) \| \\ &\quad \le (p_1(\|\mathcal{C}_1(\mathfrak{t})\| + \|\mathcal{I}_1(\mathfrak{t})\|) + p_2(\|\mathcal{C}_2(\mathfrak{t})\| + \|\mathcal{I}_2(\mathfrak{t})\|) + s)\|\mathcal{S} - \mathcal{S}^*\| \\ &\quad \le (p_1(\lambda_2 + \lambda_3) + p_2(\lambda_4 + \lambda_5) + s)\|\mathcal{S} - \mathcal{S}^*\| \\ &\quad \le \kappa_1\|\mathcal{S} - \mathcal{S}^*\|. \end{aligned} \tag{18}$$

This means that $\mathcal{F}_1$ is Lipschitz w. r. t. $\mathcal{S}$ with constant $\kappa_1 > 0$. Regarding the function $\mathcal{F}_2$, for each $\mathcal{C}_1, \mathcal{C}_1^* \in \mathbb{Y} := C(\mathbb{J}, \mathbb{R})$, we have

$$\begin{aligned} &\| \mathcal{F}_2 \big(\mathfrak{t}, \mathcal{S}(\mathfrak{t}), \mathcal{C}_1(\mathfrak{t}), \mathcal{I}_1(\mathfrak{t}), \mathcal{C}_2(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \\ &\quad - \mathcal{F}_2 \big(\mathfrak{t}, \mathcal{S}(\mathfrak{t}), \mathcal{C}_1^*(\mathfrak{t}), \mathcal{I}_1(\mathfrak{t}), \mathcal{C}_2(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \| \\ &\quad = \| (p_1\mathcal{S}(\mathfrak{t})(\mathcal{C}_1(\mathfrak{t}) + \mathcal{I}_1(\mathfrak{t})) - (q_1 + s + \theta)\mathcal{C}_1(\mathfrak{t})) \\ &\quad - (p_1\mathcal{S}(\mathfrak{t})(\mathcal{C}_1^*(\mathfrak{t}) + \mathcal{I}_1(\mathfrak{t})) - (q_1 + s + \theta)\mathcal{C}_1^*(\mathfrak{t})) \| \\ &\quad \le \|p_1\mathcal{S}(\mathfrak{t})(\mathcal{C}_1(\mathfrak{t}) - \mathcal{C}_1^*(\mathfrak{t})) - (q_1 + s + \theta)(\mathcal{C}_1(\mathfrak{t}) - \mathcal{C}_1^*(\mathfrak{t}))\| \\ &\quad \le (p_1\|\mathcal{S}(\mathfrak{t})\| + (q_1 + s + \theta))\|\mathcal{C}_1 - \mathcal{C}_1^*\| \\ &\quad \le (p_1\lambda_1 + (q_1 + s + \theta))\|\mathcal{C}_1 - \mathcal{C}_1^*\| \\ &\quad \le \kappa_2\|\mathcal{C}_1 - \mathcal{C}_1^*\|. \end{aligned}$$

This leads that $\mathcal{F}_2$ is Lipschitz w. r. t. $\mathcal{C}_1$ with constant $\kappa_2 > 0$. Now $\forall\, \mathcal{I}_1, \mathcal{I}_1^* \in \mathbb{Y}$, we write

$$\begin{aligned} &\| \mathcal{F}_3 \big(\mathfrak{t}, \mathcal{S}(\mathfrak{t}), \mathcal{C}_1(\mathfrak{t}), \mathcal{I}_1(\mathfrak{t}), \mathcal{C}_2(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \\ &\quad - \mathcal{F}_3 \big(\mathfrak{t}, \mathcal{S}(\mathfrak{t}), \mathcal{C}_1(\mathfrak{t}), \mathcal{I}_1^*(\mathfrak{t}), \mathcal{C}_2(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \| \\ &\quad = \|(q_1\mathcal{C}_1(\mathfrak{t}) - (s + b_1 + r_1)\mathcal{I}_1(\mathfrak{t})) - (q_1\mathcal{C}_1(\mathfrak{t}) - (s + b_1 + r_1)\mathcal{I}_1^*(\mathfrak{t}))\| \\ &\quad \le (s + b_1 + r_1)\|\mathcal{I}_1 - \mathcal{I}_1^*\| \\ &\quad = \kappa_3\|\mathcal{I}_1 - \mathcal{I}_1^*\|. \end{aligned}$$

Thus $\mathcal{F}_3$ is Lipschitz w. r. t. $\mathcal{I}_1$ with constant $\kappa_3 > 0$. Now for two arbitrary members $\mathcal{C}_2, \mathcal{C}_2^* \in \mathbb{Y}$, we estimate

$$\begin{aligned} &\| \mathcal{F}_4 \big(\mathfrak{t}, \mathcal{S}(\mathfrak{t}), \mathcal{C}_1(\mathfrak{t}), \mathcal{I}_1(\mathfrak{t}), \mathcal{C}_2(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \\ &- \mathcal{F}_4 \big(\mathfrak{t}, \mathcal{S}(\mathfrak{t}), \mathcal{C}_1(\mathfrak{t}), \mathcal{I}_1(\mathfrak{t}), \mathcal{C}_2^*(\mathfrak{t}), \mathcal{I}_2(\mathfrak{t}), \mathcal{R}(\mathfrak{t}) \big) \| \\ &= \| (p_2\mathcal{S}(\mathfrak{t})(\mathcal{C}_2(\mathfrak{t}) + \mathcal{I}_2(\mathfrak{t})) - (q_2 + s + \theta)\mathcal{C}_2(\mathfrak{t})) - (p_2\mathcal{S}(\mathfrak{t})(\mathcal{C}_2^*(\mathfrak{t}) + \mathcal{I}_2(\mathfrak{t})) \\ &- (q_2 + s + \theta)\mathcal{C}_2^*(\mathfrak{t})) \| \\ &\le \|p_2\mathcal{S}(\mathfrak{t})(\mathcal{C}_2(\mathfrak{t}) - \mathcal{C}_2^*(\mathfrak{t})) + (q_2 + s + \theta)(\mathcal{C}_2(\mathfrak{t}) - \mathcal{C}_2^*(\mathfrak{t}))\| \\ &\le (p_2\|\mathcal{S}(\mathfrak{t})\| + (q_2 + s + \theta))\|\mathcal{C}_2 - \mathcal{C}_2^*\| \end{aligned}$$

$$
\begin{aligned}
&\le (p_2\lambda_1 + (q_2 + s + \theta))\|\mathcal{C}_2 - \mathcal{C}_2^*\| \\
&\le \kappa_4\|\mathcal{C}_2 - \mathcal{C}_2^*\|.
\end{aligned}
$$

Thus $\mathcal{F}_4$ is Lipschitz w. r. t. $\mathcal{F}$ with $\kappa_4 > 0$. For every $\mathcal{I}_2, \mathcal{I}_2^* \in \mathbb{Y}$, we may write

$$
\begin{aligned}
\| &\mathcal{F}_5\big(\mathrm{t}, \mathcal{S}(\mathrm{t}), \mathcal{C}_1(\mathrm{t}), \mathcal{I}_1(\mathrm{t}), \mathcal{C}_2(\mathrm{t}), \mathcal{I}_2(\mathrm{t}), \mathcal{R}(\mathrm{t})\big) \\
&\quad - \mathcal{F}_5\big(\mathrm{t}, \mathcal{S}(\mathrm{t}), \mathcal{C}_1(\mathrm{t}), \mathcal{I}_1(\mathrm{t}), \mathcal{C}_2(\mathrm{t}), \mathcal{I}_2^*(\mathrm{t}), \mathcal{R}(\mathrm{t})\big) \| \\
&\quad = \|(q_2\mathcal{C}_2(\mathrm{t}) - (s + b_2 + r_2)\mathcal{I}_2(\mathrm{t})) - (q_2\mathcal{C}_2(\mathrm{t}) - (s + b_2 + r_2)\mathcal{I}_2^*(\mathrm{t}))\| \\
&\quad \le (s + b_2 + r_2)\|\mathcal{I}_2 - \mathcal{I}_2^*\| \\
&\quad = \kappa_5\|\mathcal{I}_2 - \mathcal{I}_2^*\|.
\end{aligned}
$$

This shows that $\mathcal{F}_5$ is Lipschitz w. r. t. $\mathcal{F}$ with constant $\kappa_5 > 0$. Now for each $\mathcal{R}, \mathcal{R}^* \in \mathbb{Y}$, we get

$$
\begin{aligned}
\| &\mathcal{F}_6\big(\mathrm{t}, \mathcal{S}(\mathrm{t}), \mathcal{C}_1(\mathrm{t}), \mathcal{I}_1(\mathrm{t}), \mathcal{C}_2(\mathrm{t}), \mathcal{I}_2(\mathrm{t}), \mathcal{R}(\mathrm{t})\big) \\
&\quad - \mathcal{F}_6\big(\mathrm{t}, \mathcal{S}(\mathrm{t}), \mathcal{C}_1(\mathrm{t}), \mathcal{I}_1(\mathrm{t}), \mathcal{C}_2(\mathrm{t}), \mathcal{I}_2(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big) \| \\
&\quad = \| (r_1\mathcal{I}_1(\mathrm{t}) + r_2\mathcal{I}_2(\mathrm{t}) + \theta\mathcal{C}_1(\mathrm{t}) + \theta\mathcal{C}_2(\mathrm{t}) - (\zeta + s)\mathcal{R}(\mathrm{t})) \\
&\quad - (r_1\mathcal{I}_1(\mathrm{t}) + r_2\mathcal{I}_2(\mathrm{t}) + \theta\mathcal{C}_1(\mathrm{t}) + \theta\mathcal{C}_2(\mathrm{t}) - (\zeta + s)\mathcal{R}^*(\mathrm{t})) \| \\
&\quad \le \| - (\zeta + s)(\mathcal{R}(\mathrm{t}) - \mathcal{R}^*(\mathrm{t}))\| \\
&\quad \le (\zeta + s)\|\mathcal{R} - \mathcal{R}^*\| \\
&\quad = \kappa_6\|\mathcal{R} - \mathcal{R}^*\|.
\end{aligned}
$$

Therefore $\mathcal{F}_6$ is Lipschitz w. r. t. $\mathcal{R}$ with $\kappa_6 > 0$. From the above, we conclude that $\mathcal{F}_i, i = 1, 2, 3, 4, 5, 6$, are Lipschitz w. r. t. the corresponding component with constants $\kappa_i, i = 1, 2, 3, 4, 5, 6$, respectively. ☐

Theorem 5. *Consider (H1). Then the given fractal–fractional two-strain model of meningitis* (2) *admits a unique solution if*

$$\Delta\kappa_i < 1, \quad i \in \{1, 2, 3, 4, 5, 6\},$$

where Δ is introduced in (12).

Proof. The outcome of theorem is assumed to be invalid. That is, another solution exists for the fractal–fractional two-strain model of meningitis (2). Let $(\mathcal{S}^*(\mathrm{t}), \mathcal{C}_1^*(\mathrm{t}), \mathcal{I}_1^*(\mathrm{t}), \mathcal{C}_2^*(\mathrm{t}), \mathcal{I}_2^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))$ be other solution furnished with the initial conditions $(\mathcal{S}_0, \mathcal{C}_{1,0}, \mathcal{I}_{1,0}, \mathcal{C}_{2,0}, \mathcal{I}_{2,0}, \mathcal{R}_0)$ s.t. by (10), we obtain

$$
\begin{aligned}
\mathcal{S}^*(\mathrm{t}) &= \mathcal{S}_0 + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1} \\
&\quad \times \mathcal{F}_1(\mathfrak{m}, \mathcal{S}^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_1^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_2^*(\mathfrak{m}), \mathcal{R}^*(\mathfrak{m}))\mathbf{d}\mathfrak{m}, \\
\mathcal{C}_1^*(\mathrm{t}) &= \mathcal{C}_{1,0} + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1} \\
&\quad \times \mathcal{F}_2(\mathfrak{m}, \mathcal{S}^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_1^*(\mathfrak{m}), \mathcal{C}_2^*(\mathfrak{m}), \mathcal{I}_2^*(\mathfrak{m}), \mathcal{R}^*(\mathfrak{m}))\mathbf{d}\mathfrak{m}, \\
\mathcal{I}_1^*(\mathrm{t}) &= \mathcal{I}_{1,0} + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1} \\
&\quad \times \mathcal{F}_3(\mathfrak{m}, \mathcal{S}^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_1^*(\mathfrak{m}), \mathcal{C}_2^*(\mathfrak{m}), \mathcal{I}_2^*(\mathfrak{m}), \mathcal{R}^*(\mathfrak{m}))\mathbf{d}\mathfrak{m}, \\
\mathcal{C}_2^*(\mathrm{t}) &= \mathcal{C}_{2,0} + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1} \\
&\quad \times \mathcal{F}_4(\mathfrak{m}, \mathcal{S}^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_1^*(\mathfrak{m}), \mathcal{C}_2^*(\mathfrak{m}), \mathcal{I}_2^*(\mathfrak{m}), \mathcal{R}^*(\mathfrak{m}))\mathbf{d}\mathfrak{m}, \\
\mathcal{I}_2^*(\mathrm{t}) &= \mathcal{I}_{2,0} + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1} \\
&\quad \times \mathcal{F}_5(\mathfrak{m}, \mathcal{S}^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_1^*(\mathfrak{m}), \mathcal{C}_2^*(\mathfrak{m}), \mathcal{I}_2^*(\mathfrak{m}), \mathcal{R}^*(\mathfrak{m}))\mathbf{d}\mathfrak{m},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}^*(\mathrm{t}) &= \mathcal{R}_0 + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1} \\
&\quad \times \mathcal{F}_6(\mathfrak{m}, \mathcal{S}^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_1^*(\mathfrak{m}), \mathcal{C}_2^*(\mathfrak{m}), \mathcal{I}_2^*(\mathfrak{m}), \mathcal{R}^*(\mathfrak{m}))\mathbf{d}\mathfrak{m}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|\mathcal{S}(\mathrm{t}) - \mathcal{S}^*(\mathrm{t})| \le \frac{\nu}{\Gamma(\omega)}&\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1} \\
&\times |\mathcal{F}_1(\mathfrak{m}, \mathcal{S}(\mathfrak{m}), \mathcal{C}_1(\mathfrak{m}), \mathcal{I}_1(\mathfrak{m}), \mathcal{C}_2(\mathfrak{m}), \mathcal{I}_2(\mathfrak{m}), \mathcal{R}(\mathfrak{m})) \\
&- \mathcal{F}_1(\mathfrak{m}, \mathcal{S}^*(\mathfrak{m}), \mathcal{C}_1^*(\mathfrak{m}), \mathcal{I}_1^*(\mathfrak{m}), \mathcal{C}_2^*(\mathfrak{m}), \mathcal{I}_2^*(\mathfrak{m}), \mathcal{R}^*(\mathfrak{m}))|\mathbf{d}\mathfrak{m} \\
&\le \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathfrak{m}^{\nu-1}(\mathrm{t} - \mathfrak{m})^{\omega-1}\kappa_1\|\mathcal{S} - \mathcal{S}^*\|\mathbf{d}\mathfrak{m} \\
&\le \Delta\kappa_1\|\mathcal{S} - \mathcal{S}^*\|.
\end{aligned}
$$

Thus

$$[1 - \Delta\kappa_1]\|\mathcal{S} - \mathcal{S}^*\| \le 0.$$

This is true whenever $\|\mathcal{S} - \mathcal{S}^*\| = 0$, and so $\mathcal{S} = \mathcal{S}^*$. Again, from

$$\|\mathcal{C}_1 - \mathcal{C}_1^*\| \le [1 - \Delta\kappa_2]\|\mathcal{C}_1 - \mathcal{C}_1^*\|,$$

we obtain

$$[1 - \Delta\kappa_2]\|\mathcal{C}_1 - \mathcal{C}_1^*\| \le 0.$$

This gives $\|\mathcal{C}_1 - \mathcal{C}_1^*\| = 0$ and so $\mathcal{C}_1 = \mathcal{C}_1^*$. Moreover, we have

$$\|\mathcal{I}_1 - \mathcal{I}_1^*\| \le [1 - \Delta\kappa_3]\|\mathcal{I}_1 - \mathcal{I}_1^*\|.$$

This yields that

$$[1 - \Delta\kappa_3]\|\mathcal{I}_1 - \mathcal{I}_1^*\| \le 0.$$

This implies that $\|\mathcal{I}_1 - \mathcal{I}_1^*\| = 0$ and thus $\mathcal{I}_1 = \mathcal{I}_1^*$. Similarly, from

$$\|\mathcal{C}_2 - \mathcal{C}_2^*\| \le [1 - \Delta\kappa_4]\|\mathcal{C}_2 - \mathcal{C}_2^*\|,$$

we get

$$[1 - \Delta\kappa_4]\|\mathcal{C}_2 - \mathcal{C}_2^*\| \le 0.$$

This is true if $\|\mathcal{C}_2 - \mathcal{C}_2^*\| = 0$. Thus $\mathcal{C}_2 = \mathcal{C}_2^*$. Similarly,

$$\|\mathcal{I}_2 - \mathcal{I}_2^*\| \le [1 - \Delta\kappa_5]\|\mathcal{I}_2 - \mathcal{I}_2^*\|,$$

gives

$$[1 - \Delta\kappa_5]\|\mathcal{I}_2 - \mathcal{I}_2^*\| \le 0.$$

This implies that $\|\mathcal{I}_2 - \mathcal{I}_2^*\| = 0$ and hence $\mathcal{I}_2 = \mathcal{I}_2^*$. Finally, from

$$\|\mathcal{R} - \mathcal{R}^*\| \le [1 - \Delta\kappa_6]\|\mathcal{R} - \mathcal{R}^*\|,$$

we get

$$[1 - \Delta\kappa_6]\|\mathcal{R} - \mathcal{R}^*\| \le 0.$$

This implies that $\|\mathcal{R} - \mathcal{R}^*\| = 0$ and so $\mathcal{R} = \mathcal{R}^*$. Therefore, we obtain

$$(\mathcal{S}(\mathrm{t}), \mathcal{C}_1(\mathrm{t}), \mathcal{I}_1(\mathrm{t}), \mathcal{C}_2(\mathrm{t}), \mathcal{I}_2(\mathrm{t}), \mathcal{R}(\mathrm{t})) = (\mathcal{S}^*(\mathrm{t}), \mathcal{C}_1^*(\mathrm{t}), \mathcal{I}_1^*(\mathrm{t}), \mathcal{C}_2^*(\mathrm{t}), \mathcal{I}_2^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})).$$

From these results, we find that the fractal–fractional two-strain model of meningitis (2) admits a solution uniquely. ☐

Stability criterion

In terms of fractional calculus, stability plays an important role. Ulam–Hyers and Ulam–Hyers–Rassias type stability are one of the most attractive of the various types of stability. The Ulam–Hyers stability was first introduced by [76] and afterwards generalized by Rassias in his published paper [77]. This form of stability is important in a range of natural phenomena where searching for an exact or accurate solution is difficult. In modeling techniques, where it might be difficult to obtain accurate solutions, Ulam–Hyers stability is used to obtain an approximation of the solution in order to govern the proposed model's dynamics in a functional manner. In this section, we will examine the stable solutions in the sense of Ulam–Hyers and Ulam–Hyers–Rassias for the fractal–fractional two-strain model of meningitis (eqmodel2).

Definition 4. The fractal–fractional two-strain model of meningitis (2) is Ulam–Hyers stable if $\exists\ 0 < M_{\mathcal{F}_i} \in \mathbb{R}, i = 1,2,3,4,5,6$ s.t. for all $\epsilon_i > 0$ and for all $(\mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*) \in \mathbb{X}$ fulfilling

$$\begin{cases} |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{S}^*(t) - \mathcal{F}_1(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_1, \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_1^*(t) - \mathcal{F}_2(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_2. \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_1^*(t) - \mathcal{F}_3(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_3, \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_2^*(t) - \mathcal{F}_4(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_4, \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_2^*(t) - \mathcal{F}_5(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_5, \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{R}^*(t) - \mathcal{F}_6(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_6, \end{cases} \tag{19}$$

there exists $(\mathcal{S}, \mathcal{C}_1, \mathcal{I}_1, \mathcal{C}_2, \mathcal{I}_2, \mathcal{R}) \in \mathbb{X}$ satisfying the fractal–fractional two-strain model of meningitis (2) with

$$\begin{cases} |\mathcal{S}^*(t) - \mathcal{S}(t)| \le M_{\mathcal{F}_1}\epsilon_1, \\ |\mathcal{C}_1^*(t) - \mathcal{C}_1(t)| \le M_{\mathcal{F}_2}\epsilon_2, \\ |\mathcal{I}_1^*(t) - \mathcal{I}_1(t)| \le M_{\mathcal{F}_3}\epsilon_3, \\ |\mathcal{C}_2^*(t) - \mathcal{C}_2(t)| \le M_{\mathcal{F}_4}\epsilon_4, \\ |\mathcal{I}_2^*(t) - \mathcal{I}_2(t)| \le M_{\mathcal{F}_5}\epsilon_5, \\ |\mathcal{R}^*(t) - \mathcal{R}(t)| \le M_{\mathcal{F}_6}\epsilon_6. \end{cases} \tag{20}$$

Remark 2. $(\mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*) \in \mathbb{X}$ is a solution of (19) iff $\exists\, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6 \in C([0,T], \mathbb{R})$ (depending upon $\mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*$, respectively) s.t. $\forall\, t \in \mathbb{J}$, (i). $|\eta_i(t)| < \epsilon_i$, $(i = 1,2,3,4,5,6)$, and

$$(ii).\begin{cases} {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{S}^*(t) & = \mathcal{F}_1(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_1(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_1^*(t) & = \mathcal{F}_2(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_2(t). \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_1^*(t) & = \mathcal{F}_3(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_3(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_2^*(t) & = \mathcal{F}_4(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_4(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_2^*(t) & = \mathcal{F}_5(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_5(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{R}^*(t) & = \mathcal{F}_6(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_6(t). \end{cases} \tag{21}$$

Definition 5. The fractal–fractional two-strain model of meningitis (2) is Ulam–Hyers–Rassias stable w. r. t. functions Ψ_i, $i = 1,2,3,4,5,6$ whenever $\exists\ 0 < M_{\mathcal{F}_i,\Psi_i} \in \mathbb{R}, i = 1,2,3,4,5,6$ s.t. for all $\epsilon_i > 0$ and for all $(\mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*) \in \mathbb{X}$ satisfying

$$\begin{cases} |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{S}^*(t) - \mathcal{F}_1(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_1\Psi_1(t), \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_1^*(t) - \mathcal{F}_2(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_2\Psi_2(t). \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_1^*(t) - \mathcal{F}_3(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_3\Psi_3(t), \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_2^*(t) - \mathcal{F}_4(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_4\Psi_4(t), \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_2^*(t) - \mathcal{F}_5(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_5\Psi_5(t), \\ |{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{R}^*(t) - \mathcal{F}_6(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t))| & < \epsilon_6\Psi_6(t), \end{cases} \tag{22}$$

$\exists\, (\mathcal{S}, \mathcal{C}_1, \mathcal{I}_1, \mathcal{C}_2, \mathcal{I}_2, \mathcal{R}) \in \mathbb{X}$ satisfying the fractal–fractional two-strain model of meningitis (2) with

$$\begin{cases} |\mathcal{S}^*(t) - \mathcal{S}(t)| & \le M_{\mathcal{F}_1,\Psi_1}\epsilon_1\Psi_1(t), \\ |\mathcal{C}_1^*(t) - \mathcal{C}_1(t)| & \le M_{\mathcal{F}_2,\Psi_2}\epsilon_2\Psi_2(t), \\ |\mathcal{I}_1^*(t) - \mathcal{I}_1(t)| & \le M_{\mathcal{F}_3,\Psi_3}\epsilon_3\Psi_3(t), \\ |\mathcal{C}_2^*(t) - \mathcal{C}_2(t)| & \le M_{\mathcal{F}_4,\Psi_4}\epsilon_4\Psi_4(t), \\ |\mathcal{I}_2^*(t) - \mathcal{I}_2(t)| & \le M_{\mathcal{F}_5,\Psi_5}\epsilon_5\Psi_5(t), \\ |\mathcal{R}^*(t) - \mathcal{R}(t)| & \le M_{\mathcal{F}_6,\Psi_6}\epsilon_6\Psi_6(t). \end{cases} \tag{23}$$

Remark 3. $(\mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*) \in \mathbb{X}$ is a solution of (22) iff $\exists\, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6 \in C([0,T], \mathbb{R})$ (depending upon $\mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*$, respectively) s.t. for all $t \in \mathbb{J}$, (i). $|\eta_i(t)| < \Psi_i(t)\epsilon_i$, $(i = 1,2,3,4,5,6)$, and

$$(ii).\begin{cases} {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{S}^*(t) & = \mathcal{F}_1(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_1(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_1^*(t) & = \mathcal{F}_2(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_2(t). \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_1^*(t) & = \mathcal{F}_3(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_3(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{C}_2^*(t) & = \mathcal{F}_4(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_4(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{I}_2^*(t) & = \mathcal{F}_5(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_5(t), \\ {}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{R}^*(t) & = \mathcal{F}_6(\mathcal{S}^*(t), \mathcal{C}_1^*(t), \mathcal{I}_1^*(t), \mathcal{C}_2^*(t), \mathcal{I}_2^*(t), \mathcal{R}^*(t)) + \eta_6(t). \end{cases} \tag{24}$$

Theorem 6. *The fractal–fractional two-strain model of meningitis (2) is Ulam–Hyers stable on $\mathbb{J} := [0,T]$ s.t.*

$$\Delta\kappa_i < 1, \qquad i \in \{1,2,3,4,5,6\},$$

where κ_i and Δ are given by (12) and (17), respectively, if assumption (H1) is valid.

Proof. Let $\epsilon_1 > 0$ and $\mathcal{S}^* \in \mathbb{Y}$ s.t.

$$\left|{}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{S}^*(t) - \mathcal{F}_1\left(t, \mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*\right)\right| < \epsilon_1.$$

Then, in view of Remark 2, $\exists\, \eta_1(t)$ so that

$${}^{\mathbf{FFP}}\mathfrak{D}^{\omega,\nu}_{0,t}\mathcal{S}^*(t) = \mathcal{F}_1\left(t, \mathcal{S}^*, \mathcal{C}_1^*, \mathcal{I}_1^*, \mathcal{C}_2^*, \mathcal{I}_2^*, \mathcal{R}^*\right) + \eta_1(t),$$

and $|\eta_1(t)| \le \epsilon_1$. Hence

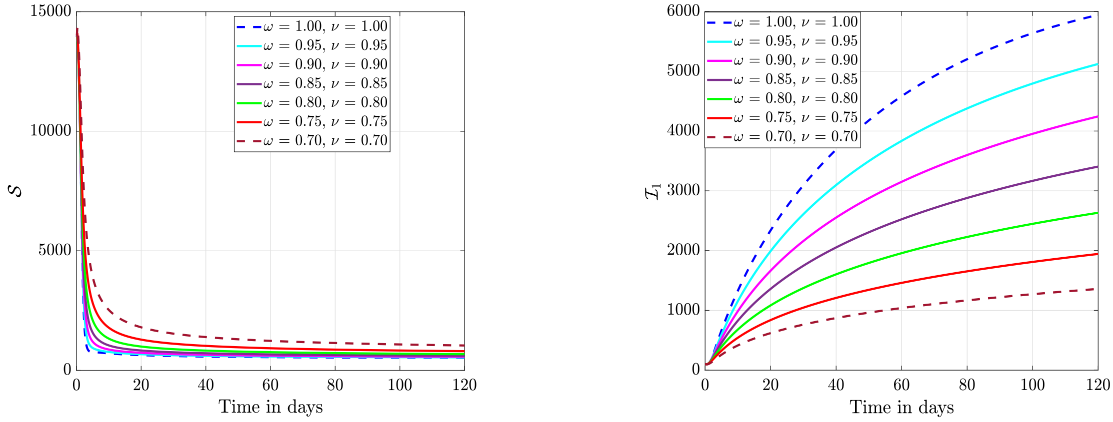
$$\begin{aligned} \mathcal{S}^*(t) = & \mathcal{S}_0 + \frac{\nu}{\Gamma(\omega)}\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ & \times \mathcal{F}_1\left(m, \mathcal{S}^*(m), \mathcal{C}_1^*(m), \mathcal{I}_1^*(m), \mathcal{C}_2^*(m), \mathcal{I}_2^*(m), \mathcal{R}^*(m)\right)\mathbf{d}m \\ & + \frac{\nu}{\Gamma(\omega)}\int_0^t m^{\nu-1}(t-m)^{\omega-1}\eta_1(m)\mathbf{d}m. \end{aligned}$$

From Theorem 5, we consider $\mathcal{S} \in \mathbb{Y}$ as the unique solution of the fractal–fractional two-strain model of meningitis. Then $\mathcal{S}(t)$ is formulated by

$$\begin{aligned} \mathcal{S}(t) = & \mathcal{S}_0 + \frac{\nu}{\Gamma(\omega)}\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ & \times \mathcal{F}_1\left(m, \mathcal{S}^*(m), \mathcal{C}_1^*(m), \mathcal{I}_1^*(m), \mathcal{C}_2^*(m), \mathcal{I}_2^*(m), \mathcal{R}^*(m)\right)\mathbf{d}m. \end{aligned}$$

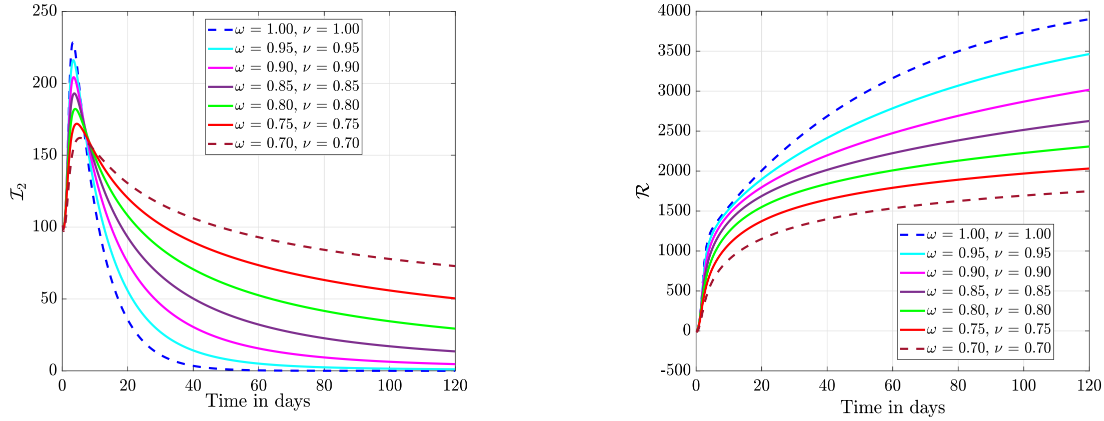
Then

$$\begin{aligned} |\mathcal{S}^*(t) - \mathcal{S}(t)| \le & \frac{\nu}{\Gamma(\omega)}\int_0^t m^{\nu-1}(t-m)^{\omega-1}|\eta_1(m)|\mathbf{d}m \\ & + \frac{\nu}{\Gamma(\omega)}\int_0^t m^{\nu-1}(t-m)^{\omega-1} \\ & \times \left|\mathcal{F}_1\left(m, \mathcal{S}^*(m), \mathcal{C}_1^*(m), \mathcal{I}_1^*(m), \mathcal{C}_2^*(m), \mathcal{I}_2^*(m), \mathcal{R}^*(m)\right)\right. \end{aligned}$$



(a) Susceptible humans $\mathcal{S}(\mathrm{t})$

(b) Infected humans $\mathcal{I}_1(\mathrm{t})$ with strain 1



(c) Infected humans $\mathcal{I}_2(\mathrm{t})$ with strain 2

(d) Recovered humans $\mathcal{R}(\mathrm{t})$

Fig. 1. A fractal and fractional trajectories with different orders of $\omega = \nu$.

$$- \mathcal{F}_1\Big(\mathrm{m}, \mathcal{S}(\mathrm{m}), \mathcal{C}_1(\mathrm{m}), \mathcal{I}_1(\mathrm{m}), \mathcal{C}_2(\mathrm{m}), \mathcal{I}_2(\mathrm{m}), \mathcal{R}(\mathrm{m})\Big)\Big|\mathbf{d}\mathrm{m}$$

$$\le \Delta\epsilon_1 + \Delta\kappa_1\|\mathcal{S}^* - \mathcal{S}\|.$$

Therefore

$$\|\mathcal{S}^* - \mathcal{S}\| \le \frac{\Delta\epsilon_1}{1 - \Delta\kappa_1}.$$

If $M_{\mathcal{F}_1} = \frac{\Delta}{1-\Delta\kappa_1}$, then $\|\mathcal{S}^* - \mathcal{S}\| \le M_{\mathcal{F}_1}\epsilon_1$. Similarly,

$$\|\mathcal{C}_1^* - \mathcal{C}_1\| \le M_{\mathcal{F}_2}\epsilon_2, \quad \|\mathcal{I}_1^* - \mathcal{I}_1\| \le M_{\mathcal{F}_3}\epsilon_3, \quad \|\mathcal{C}_2^* - \mathcal{C}_2\| \le M_{\mathcal{F}_4}\epsilon_4,$$

$$\|\mathcal{I}_2^* - \mathcal{I}_2\| \le M_{\mathcal{F}_5}\epsilon_5, \quad \|\mathcal{R}^* - \mathcal{R}\| \le M_{\mathcal{F}_6}\epsilon_6,$$

where

$$M_{\mathcal{F}_i} = \frac{\Delta}{1 - \Delta\kappa_i}, \qquad (i \in \{2,3,4,5,6\}).$$

Thus, the Ulam–Hyers stability of the fractal–fractional two-strain model of meningitis (2) is fulfilled. ☐

Theorem 7. *Let* (H'): $\exists$ *increasing mappings* $\Psi_i \in C([0,T], \mathbb{R}^+), (i \in \{1,2,3,4,5,6\})$ *and* $\exists\ \Lambda_{\Psi_i} > 0$ *s.t.* $\forall \mathrm{t} \in \mathbb{J}$,

$$^{\mathbf{FFP}}\mathfrak{J}^{\omega,\nu}_{0,\mathrm{t}}\Psi_i(\mathrm{t}) < \Lambda_{\Psi_i}\Psi_i(\mathrm{t}), (i \in \{1,2,3,4,5,6\}). \tag{25}$$

Then the fractal–fractional two-strain model of meningitis (2) *is Ulam–Hyers–Rassias stable whenever* $(H1)$ *is satisfied.*

Proof. For every $\epsilon_1 > 0$ and $\forall \mathcal{S}^* \in \mathbb{Y}$ fulfilling

$$\left|^{\mathbf{FFP}}\mathcal{D}^{\omega,\nu}_{0,\mathrm{t}}\mathcal{S}^*(\mathrm{t}) - \mathcal{F}_1(\mathrm{t}, \mathcal{S}^*(\mathrm{t}), \mathcal{C}_1^*(\mathrm{t}), \mathcal{I}_1^*(\mathrm{t}), \mathcal{C}_2^*(\mathrm{t}), \mathcal{I}_2(\mathrm{t}), \mathcal{R}(\mathrm{t}))\right| < \epsilon_1\Psi_1(\mathrm{t}),$$

$\exists\ \eta_1(\mathrm{t})$ s.t.

$$^{\mathbf{FFP}}\mathcal{D}^{\omega,\nu}_{0,\mathrm{t}}\mathcal{S}^*(\mathrm{t}) = \mathcal{F}_1(\mathrm{t}, \mathcal{S}^*(\mathrm{t}), \mathcal{C}_1^*(\mathrm{t}), \mathcal{I}_1^*(\mathrm{t}), \mathcal{C}_2^*(\mathrm{t}), \mathcal{I}_2(\mathrm{t}), \mathcal{R}(\mathrm{t})) + \eta_1(\mathrm{t}),$$

and $|\eta_1(\mathrm{t})| \le \epsilon_1\Psi_1(\mathrm{t})$. Hence

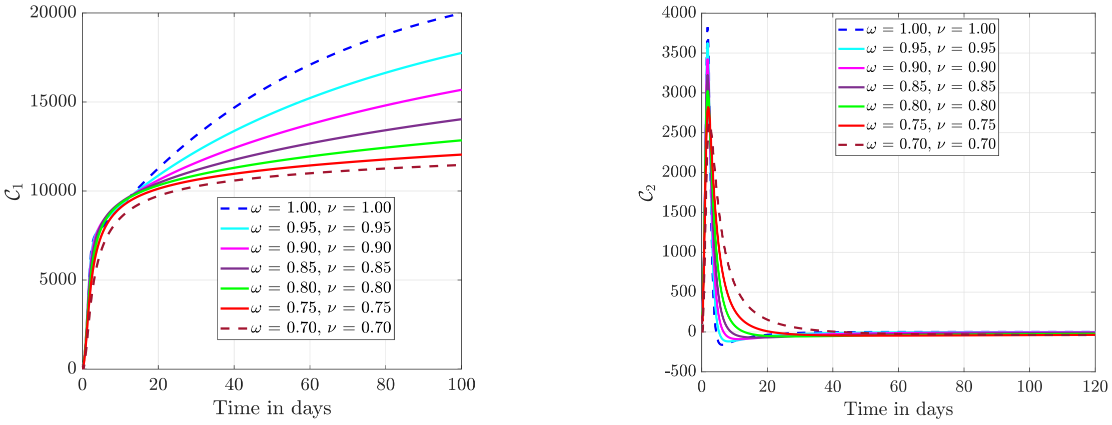
$$\begin{aligned}\mathcal{S}^*(\mathrm{t}) =& \mathcal{S}_0 + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathrm{m}^{\nu-1}(\mathrm{t}-\mathrm{m})^{\omega-1} \\ &\times \mathcal{F}_1(\mathrm{m}, \mathcal{S}^*(\mathrm{m}), \mathcal{C}_1^*(\mathrm{m}), \mathcal{I}_1^*(\mathrm{m}), \mathcal{C}_2^*(\mathrm{m}), \mathcal{I}_2(\mathrm{m}), \mathcal{R}(\mathrm{m}))\mathbf{d}\mathrm{m} \\ &+ \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathrm{m}^{\nu-1}(\mathrm{t}-\mathrm{m})^{\omega-1}\eta_1(\mathrm{m})\mathbf{d}\mathrm{m}.\end{aligned}$$

From Theorem 5, we consider $\mathcal{S} \in \mathbb{Y}$ as a unique solution for the fractal–fractional two-strain model of meningitis (2). Hence $\mathcal{S}(\mathrm{t})$ is

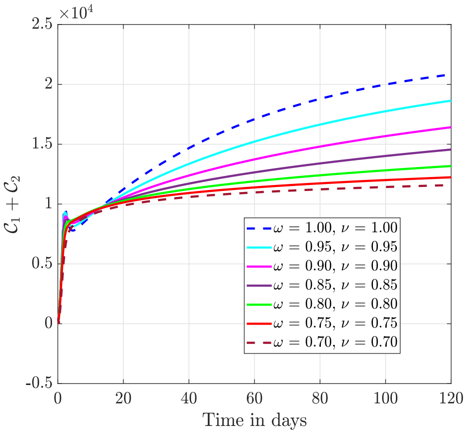
$$\begin{aligned}\mathcal{S}(\mathrm{t}) = \mathcal{S}_0 + \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathrm{m}^{\nu-1}(\mathrm{t}-\mathrm{m})^{\omega-1} \\ \times \mathcal{F}_1(\mathrm{m}, \mathcal{S}(\mathrm{m}), \mathcal{C}_1(\mathrm{m}), \mathcal{I}_1(\mathrm{m}), \mathcal{C}_2(\mathrm{m}), \mathcal{I}_2(\mathrm{m}), \mathcal{R}(\mathrm{m}))\mathrm{d}\mathrm{m}\end{aligned}$$

We get

$$\begin{aligned}\left|\mathcal{S}^*(\mathrm{t}) - \mathcal{S}(\mathrm{t})\right| \le& \frac{\nu}{\Gamma(\omega)}\int_0^{s} \mathrm{m}^{\nu-1}(\mathrm{t}-\mathrm{m})^{\omega-1}|h_1(\mathrm{m})|\mathrm{d}\mathrm{m} \\ &+ \frac{\nu}{\Gamma(\omega)}\int_0^{\mathrm{t}} \mathrm{m}^{\nu-1}(\mathrm{t}-\mathrm{m})^{\omega-1} \\ &\times \Big|\mathcal{F}_1(\mathrm{m}, \mathcal{S}^*(\mathrm{m}), \mathcal{C}_1^*(\mathrm{m}), \mathcal{I}_1^*(\mathrm{m}), \mathcal{C}_2^*(\mathrm{m}), \mathcal{I}_2(\mathrm{m}), \mathcal{R}(\mathrm{m}))\end{aligned}$$



(a) Carrier humans $\mathcal{C}_1(\mathrm{t})$ with strain 1   (b) Carrier humans $\mathcal{C}_2(\mathrm{t})$ with strain 2



(c) Carrier humans $(\mathcal{C}_1(\mathrm{t})+\mathcal{C}_1(\mathrm{t}))$ with strain 1 and strain 2

Fig. 2. A fractal and fractional trajectories with different orders $\omega = \nu$.

$$
\begin{aligned}
&- \mathcal{F}_1(\mathrm{m}, \mathcal{S}(\mathrm{m}), \mathcal{C}_1(\mathrm{m}), \mathcal{I}_1(\mathrm{m}), \mathcal{C}_2(\mathrm{m}), \mathcal{I}_2(\mathrm{m}), \mathcal{R}(\mathrm{m}))\mathrm{dm}\Big| \\
&\le \frac{\varepsilon_1 \nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{m}^{\nu-1}(\mathrm{t}-\mathrm{m})^{\omega-1}\Psi_1(\mathrm{m})\mathrm{dm} + \Delta\kappa_1 \|\mathcal{S}^* - \mathcal{S}\| \\
&\le \varepsilon_1 \Lambda_{\Psi_1}\Psi_1(\mathrm{t}) + \Delta\kappa_1\|\mathcal{S}^* - \mathcal{S}\|.
\end{aligned}
$$

Therefore

$$\|\mathcal{S}^* - \mathcal{S}\| \le \frac{\epsilon_1 \Lambda_{\Psi_1}\Psi_1(\mathrm{t})}{1-\Delta\kappa_1}.$$

If

$$M_{\mathcal{F}_1,\Psi_1} = \frac{\Lambda_{\Psi_1}}{1-\Delta\kappa_1},$$

then $\|\mathcal{S}^* - \mathcal{S}\| \le \epsilon_1 M_{\mathcal{F}_1,\Psi_1}\Psi_1(\mathrm{t})$. In the similar manner, we have

$$
\begin{aligned}
&\|\mathcal{C}_1^* - \mathcal{C}_1\| \le \epsilon_2 M_{\mathcal{F}_2,\Psi_2}\Psi_2(\mathrm{t}), \quad \|\mathcal{I}_1^* - \mathcal{I}_1\| \le \epsilon_3 M_{\mathcal{F}_3,\Psi_3}\Psi_3(\mathrm{t}), \\
&\|\mathcal{C}_2^* - \mathcal{C}_2\| \le \epsilon_4 M_{\mathcal{F}_4,\Psi_4}\Psi_4(\mathrm{t}), \\
&\|\mathcal{I}_2^* - \mathcal{I}_2\| \le \epsilon_5 M_{\mathcal{F}_5,\Psi_5}\Psi_5(\mathrm{t}), \\
&\|\mathcal{R}^* - \mathcal{R}\| \le \epsilon_6 M_{\mathcal{F}_6,\Psi_6}\Psi_6(\mathrm{t}),
\end{aligned}
$$

where

$$M_{\mathcal{F}_i,\Psi_i} = \frac{\Lambda_{\Psi_i}}{1-\Delta\kappa_i}, (i \in \{2,3,4,5,6\}).$$

Hence the fractal–fractional two-strain model of meningitis (2) is Ulam–Hyers–Rassias stable. □

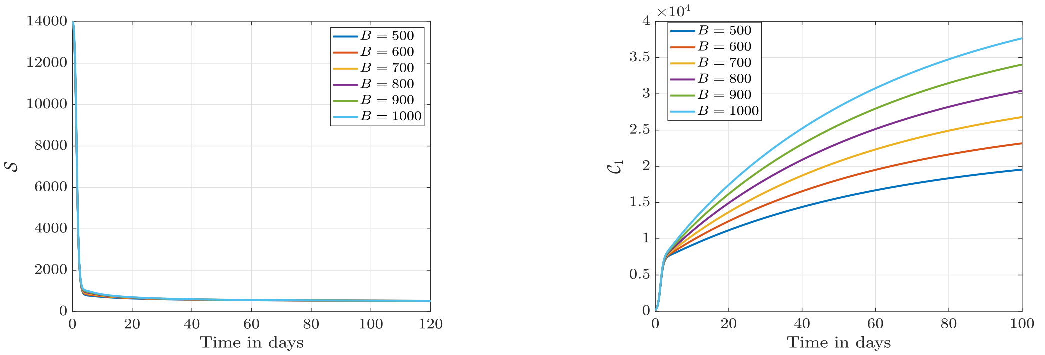
Simulations

In this part of the paper, we shall see the convergence and stability of solutions in a time period by simulating them based the Lagrangian polynomials (Predictor–corrector) [78,79]. To start the process. The following values for the parameters are assumed $B = 500$, $\zeta = 0.5$, $p_1 = 0.000174$, $p_2 = 0.000174$, $s = \frac{1}{65\times 365}$, $q_1 = 0.0263$, $q_2 = 0.0263$, $b_1 = 0.0747$, $b_2 = 0.0747$, $\theta = 0.0896$, $r_1 = 0.0153$, $r_2 = 0.0153$. All parameter values are assumed to be in days. Further, the initial values for state functions are

$$\mathcal{S}(0) = 14\,000, \quad \mathcal{C}_1(0) = 200, \quad \mathcal{I}_1(0) = 100, \quad \mathcal{C}_2(0) = 200, \quad \mathcal{I}_2(0) = 100, \quad \mathcal{R}(0) = 0.$$

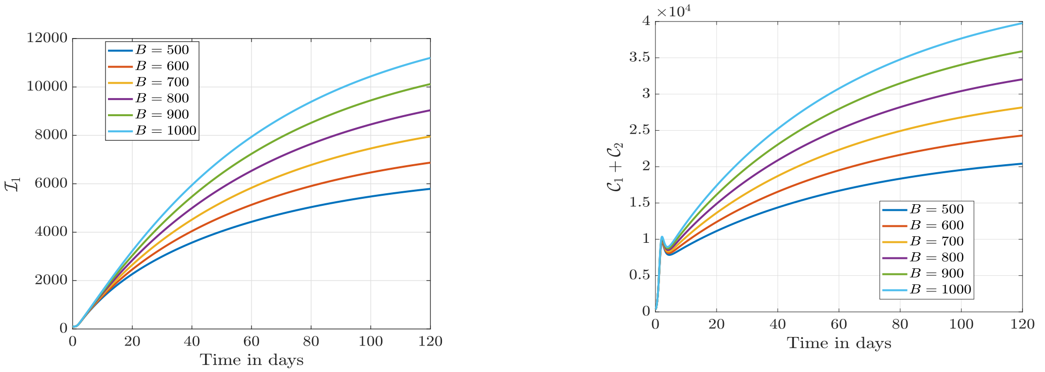
In our simulations for six state functions $\mathcal{S}, \mathcal{C}_1, \mathcal{I}_1, \mathcal{C}_2, \mathcal{I}_2, \mathcal{R}$, the values $\omega = \nu = 0.95, 0.90, 0.85, 0.80, 0.75, 0.70$ are considered for the fractal dimensions and fractional orders.

Fig. 1 shows the fractal–fractional dynamics of the two-stain model of meningitis. In Fig. 1(a), it can be seen that as the fractal–fractional order reduces, the number of susceptible individuals gradually increases. Figs. 1(b) and 1(c) show that as the fractal–fractional order reduces, the number of infected humans with strain 1 of meningitis



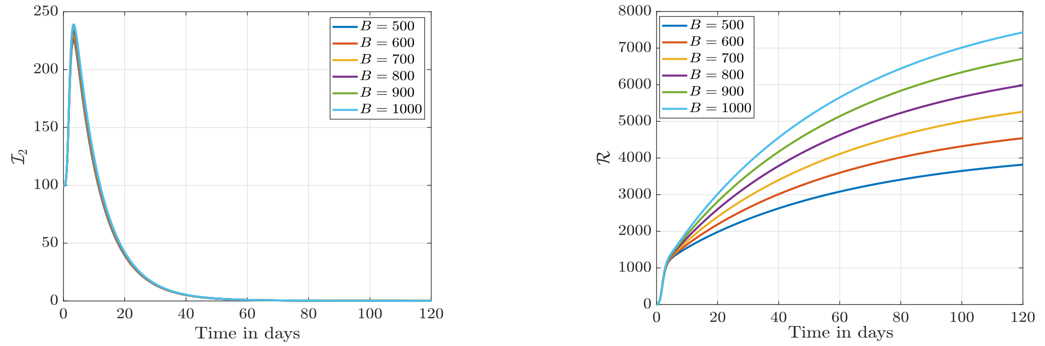
(a) Susceptible humans $\mathcal{S}(\mathrm{t})$

(b) Carrier humans $\mathcal{C}_1(\mathrm{t})$ with strain 1



(c) Infected humans $\mathcal{I}_1(\mathrm{t})$ with strain 1

(d) Carrier humans $(\mathcal{C}_1(\mathrm{t}) + \mathcal{C}_1(\mathrm{t}))$ with strain 1 and strain 2



(e) Infected humans $\mathcal{I}_2(\mathrm{t})$ with strain 2

(f) Recovered humans $\mathcal{R}(\mathrm{t})$

Fig. 3. The effect of varying recruitment rate on susceptible, carriers, infected and recovered humans when the fractal–fractional orders is 0.99.

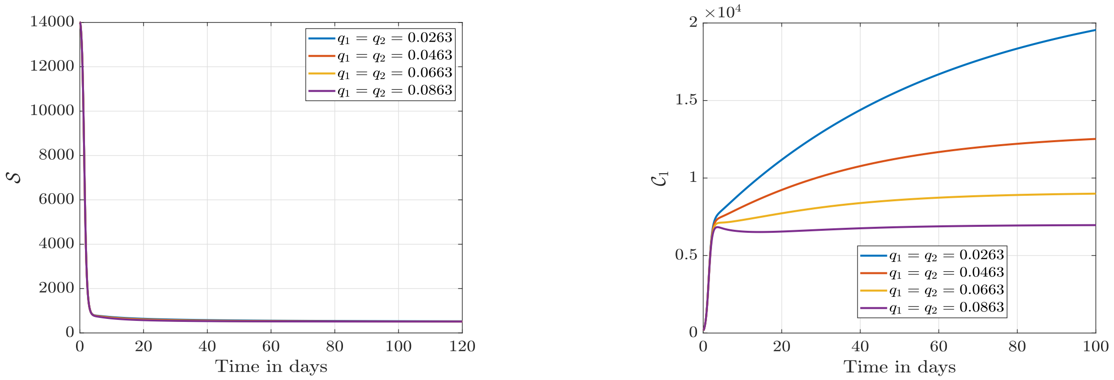
reduces while that of infected humans with strain 2 increases. Fig. 1(d) shows the dynamics of the recovered individuals. It can be seen that the decline in the fractal–fractional orders reduces the number of recovered individuals.

In Fig. 2, we show the dynamics of the carrier compartments. Fig. 2(a) shows that fractal–fractional trajectories have close relationships for 19 days, then the trajectories decrease as the fractal–fractional order decreases. Fig. 2(b) shows that the fractal–fractional order $\omega = \nu = 0.95, 0.90, 0.85, 0.80, 0.75, 0.70$ does not capture the dynamics of carrier humans with strain 2 as that of carrier humans with strain 1. This may be as a result that carrier individuals of particular strains of meningitis cannot be easily detected. Hence, Fig. 2(c) shows the dynamics of the combined dynamics of both strains. It is noticed that the trajectories take the same path for 20 days and then reduce as the fractal–fractional orders reduce.

Fig. 3 shows the effect of perturbing the recruitment rate on the fractal–fractional two-strain model. It is noticed that an increase in the recruitment rate increases the number of carriers, recovered and individuals infected with strain 1, but has no effect or has minimal effect on the number of susceptible humans and individuals infected with strain 2.

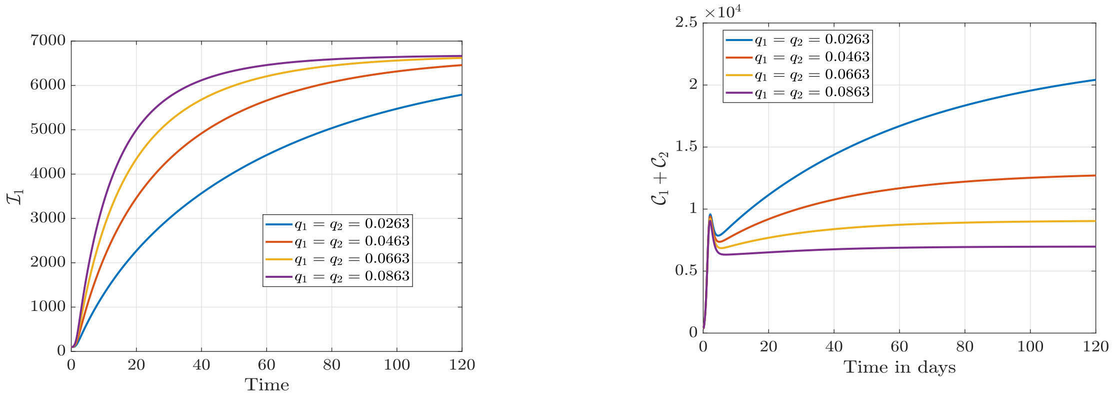
Fig. 4 depicts the fractal–fractional dynamics as the rate of progression from $\mathcal{C}_i$ to $\mathcal{I}_i, i = 1, 2$ is varied. Figs. 4(b), 4(d), and 4(f) show that increasing the progression rate reduces the number of carriers and recovered individuals, whereas Figs. 4(c) and 4(e) show that increasing the progression rate increases the number of individuals infected with strain 1 rather than strain 2. Fig. 4(a) also shows that increasing the progression rate from $\mathcal{C}_i$ to $\mathcal{I}_i, i = 1, 2$ has little effect on the number of susceptible individuals, as expected.

Fig. 5 depicts the fractal–fractional dynamics when the recovery rate is varied. Figs. 5(b), 5(d), and 5(f) show that increasing the recovery



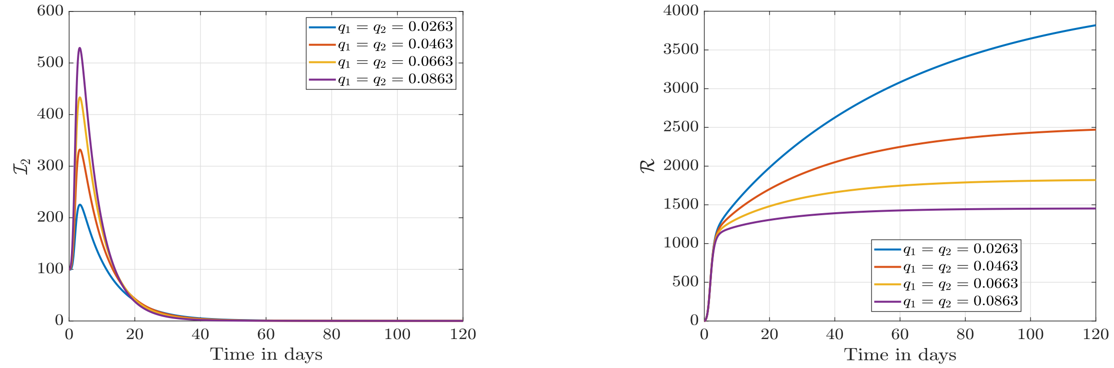
(a) Susceptible humans $\mathcal{S}(\mathfrak{t})$

(b) Carrier humans $\mathcal{C}_1(\mathfrak{t})$ with strain 1



(c) Infected humans $\mathcal{I}_1(\mathfrak{t})$ with strain 1

(d) Carrier humans $(\mathcal{C}_1(\mathfrak{t}) + \mathcal{C}_1(\mathfrak{t}))$ with strain 1 and strain 2



(e) Infected humans $\mathcal{I}_2(\mathfrak{t})$ with strain 2

(f) Recovered humans $\mathcal{R}(\mathfrak{t})$

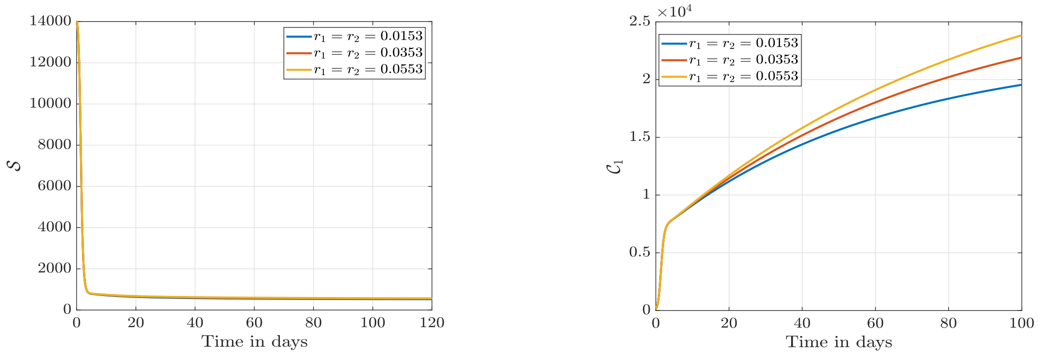
Fig. 4. The effect of varying progression rate from C_1 to I_1, C_2 to I_2 on susceptible, carriers, infected and recovered humans when the fractal–fractional orders is 0.99.

rate increases the number of carries and recovered individuals, whereas in Figs. 5(c) and 5(e), it decreases the number of individuals infected with strain 2 compared to strain 1. It is also noticed in Fig. 5(a) that an increase in the recovery rate has minimal effect on the number of susceptible individuals.

Fig. 6 depicts the fractal–fractional dynamics when the effective contact rate is changed. It is noticed in Figs. 6(b) and 6(c) that an increase in the contact rate increases the number of carries and infected individuals with strain 1, while in Figs. 5(a) and 5(e) it reduces the number of susceptible humans and individuals infected with strain 2. It is also noticed in Fig. 5(f) that the total number of recovered individuals is greater when the contact rate is lower at the end of the simulation time.

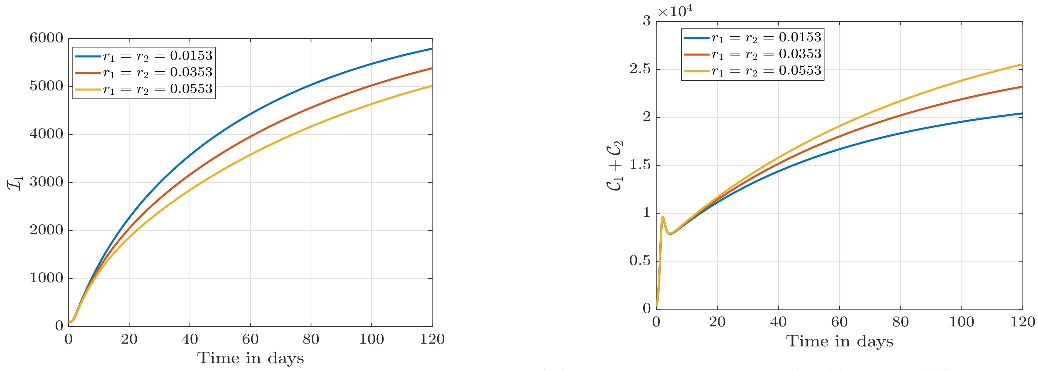
Conclusions

In this paper, we proposed the fractal–fractional dynamics of two strains of meningitis model. The existence of solutions was investigated using the Banach principle, which shows the subclass of all continuous functions under a given norm. It was further shown that the fractal–fractional model of meningitis has a unique solution using the Lipschitz property. We proceeded to the stability criterion in the context of the Ulam–Hyers and Ulam–Hyers–Rassias and their generalized versions for solutions of the given fractal–fractional model. After theoretical analysis of the unique and stable solutions, the numerical scheme for the fractal–fractional two-strain model of meningitis disease was given in an approximate manner using the Lagrangian polynomials



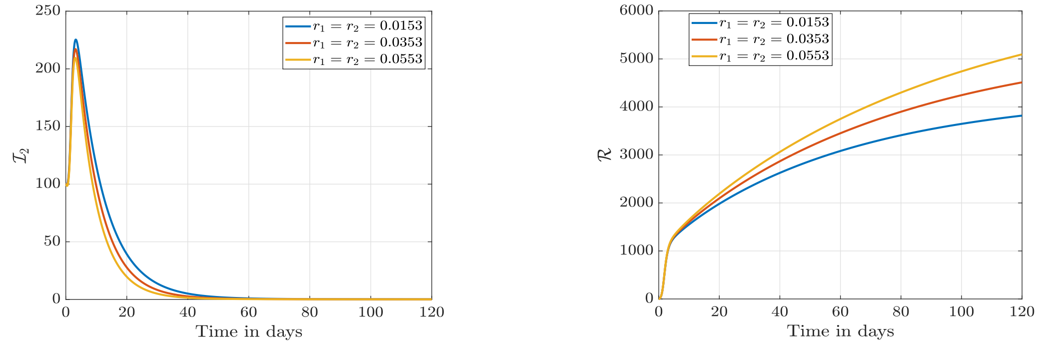
(a) Susceptible humans $\mathcal{S}(\mathrm{t})$

(b) Carrier humans $\mathcal{C}_1(\mathrm{t})$ with strain 1



(c) Infected humans $\mathcal{I}_1(\mathrm{t})$ with strain 1

(d) Carrier humans $(\mathcal{C}_1(\mathrm{t}) + \mathcal{C}_1(\mathrm{t}))$ with strain 1 and strain 2



(e) Infected humans $\mathcal{I}_2(\mathrm{t})$ with strain 2

(f) Recovered humans $\mathcal{I}_2(\mathrm{t})$ with strain 2

Fig. 5. The effect of varying recovery rate on susceptible, carriers, infected and recovered humans when the fractal–fractional orders is 0.99.

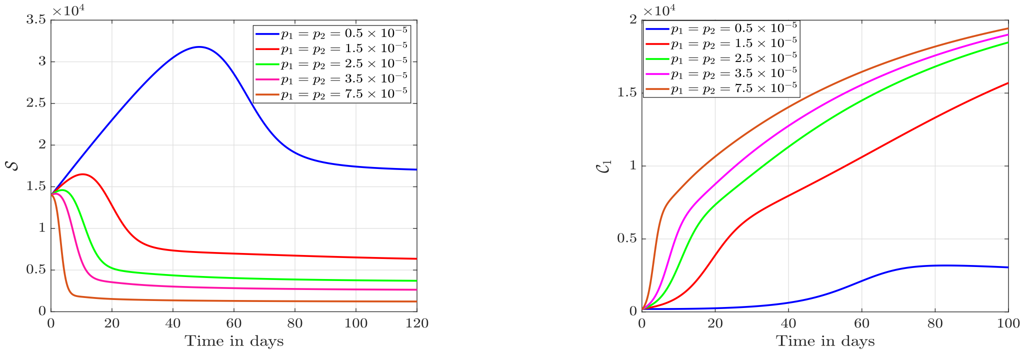
based on the Adams–Bashforth technique. The simulations for six state functions $\mathcal{S}, \mathcal{C}_1, \mathcal{I}_1, \mathcal{C}_2, \mathcal{I}_2, \mathcal{R}$ with fractal–fractional values $\omega = \nu = 0.95, 0.90, 0.85, 0.80, 0.75, 0.70$ are considered using the given numerical scheme. We noticed that the number of susceptible individuals gradually increases as the fractal–fractional values reduce. In some instances, it shows that as the fractal–fractional order reduces, the number of infected humans with strain 1 of meningitis reduces while that of infected humans with strain 2 increases. We again noticed that the dynamics of the recovered individuals decline as the fractal–fractional orders reduce. Finally, we looked at the sensitiveness of some parameters when the fractal–fractional values are kept at 0.99. From that, we noticed a crossover behavior in the carrier compartment, the infectious compartment of strain 2, and the recovered compartment when the effective contact rate is varied. In future work, researchers can compare the given model to real data since this is main limitation in the current work. Other researchers can consider different recovered classes for strain 1 and strain 2 meningitis dynamics with the Mittag-Leffler operator.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

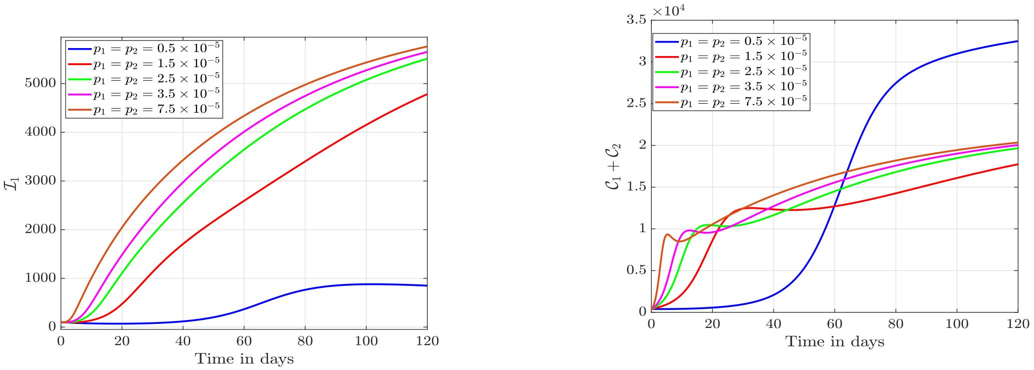
Data availability

Data will be made available on request.



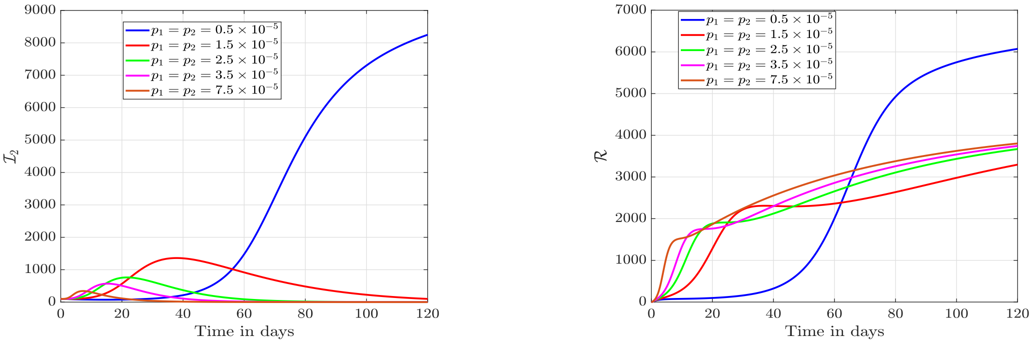
(a) Susceptible humans $\mathcal{S}(t)$

(b) Carrier humans $\mathcal{C}_1(t)$ with strain 1



(c) Infected humans $\mathcal{I}_1(t)$ with strain 1

(d) Carrier humans $(\mathcal{C}_1(t) + \mathcal{C}_1(t))$ with strain 1 and strain 2



(e) Infected humans $\mathcal{I}_2(t)$ with strain 2

(f) Recovered humans $\mathcal{R}(t)$

Fig. 6. The effect of varying the effective contact rate when the fractal–fractional orders is 0.99.

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