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A fractal–fractional order model for exploring the dynamics of Monkeypox disease

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ABSTRACT

This study explores the biological behaviour of the Monkeypox disease using a fractal–fractional operator. We discuss the existence and uniqueness of the solution of the model using the fixed-point concept. We further show that the Monkeypox fractal–fractional model is stable through the Hyers–Ulam and Hyers–Ulam Rassias stability criteria. The epidemiological threshold of the model is obtained. The numerical simulation for the proposed model is obtained using the Newton polynomial. For instance, the disease dies out at lower fractional values. We investigated the effects of some key parameters on the dynamics of the disease. The variation of the parameters shows that quarantine and isolation are effective approaches to managing, controlling, or eradicating the Monkeypox disease.

1. Introduction

Monkeypox is classified to be one of the zoonosis, that is, a disease caused by a virus that spreads between animals and people [1,2]. It is further known to be a double-stranded deoxyribonucleic acid virus in the family of Poxvirridae, specifically belonging to the orthopoxvirus genus [3]. Animals like; diverse species of monkeys, rope squirrels, tree squirrels, Gambian pouched rats, and many others have high potency as carriers of the virus as evidenced in Africa [4]. The disease is also known to spread in the human domain, where a susceptible person becomes exposed or has close contact with the respiratory secretions and skin lesions of an infected person or contaminated objects. An individual who has acquired the virus may have an incubation period of two to three weeks before showing symptoms. The disease's initial stages are classified as the invasion period and later followed by a skin eruption [5–7]. Preventing or reducing the infection rate of the monkeypox disease is very effective, especially with measures like isolation and quarantine. Human monkeypox was first heard of in the Democratic Republic of the Congo from an infant around 1970. During this time, not less than eleven (11) African countries reported the same, such as Cameron, Liberia, Nigeria, Sierra Leone and others [8]. In 2017, Nigeria had a massive outbreak of the monkeypox disease with about 500 suspected cases, of which 200 cases were confirmed with a fatality rate of 3% [9]. Monkeypox has become a worldwide health condition today, as some other countries outside Africa recorded disease cases. For instance, the United state of America reported its first case, about 70 confirmed cases, in 2003, which was linked to an interaction between a human and an infected pet prairie dog [10]. Some other countries like the United Kingdom and Singapore reported confirmed cases in 2018 and 2019, respectively [11]. The United States of America again recorded monkeypox disease cases in July and November 2021. The World health organization has recently reported an ongoing outbreak of the monkeypox disease, first reported in the United Kingdom in May 2022 in a patient who is supposed to have travelled from Nigeria [11]. Mathematical models have been applied in the study of the dynamics of diseases as they have the capabilities to deal with both linear and nonlinear systems, see [12–18]. For instance, the Covid-19 disease was widely studied by the use of mathematical modelling [19–21] and also diseases found in the family of Poxviridae like smallpox [22,23], cowpox [24], and chickenpox [25]. The dynamics of the spread of the monkeypox disease has also been studied through the usage of mathematical modelling [8,26–32]. It has been reported that the integer order models are unable to capture the full complexity of nonlinear systems used to model biological systems [33], and thus, a generalization approach of the integer order model has been introduced and known as fractional order models [34]. Unfortunately, the integer order, which deals with fixed discrete order, fails to capture the memory effect inherently exhibited by these biological systems. Fractional order derivatives have therefore become an insightful method used in epidemiology. This results from its ability to use discrete and continuous value orders in describing the changes in the dynamics of a disease or a biological system. Thus, fractional

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models have an inherent memory capture ability and have become a preferable approach in studying biological systems [33]. Several fractional order approaches have been applied in the study of health diseases [35–49]. The transmission trends of the Monkeypox disease have been widely investigated through mathematical modelling via a non-fractional order. Few papers have studied this disease through a fractional order derivative, which has a robust advantage over the integer models. The non-integer order models can capture the memory effect, thus, describing the biological system using continuous and not only discrete values [49]. For instance, in the work of [50], the Monkeypox disease was studied by means of the Atangana–Baleanu to investigate the driving dynamics of the disease in Ghana. In their work, the fractional order was seen to have a massive influence on the biological trends of Monkeypox disease. Again, in [51], the Monkeypox disease was studied through an integer and a fractional order model in the Caputo–Fabrizio sense. It was observed in their work that the control strategies necessary for the eradication of the disease were explicitly seen at lower fractional orders, which the integer order models failed to capture accurately. Also, [52] studied the dynamics of the Monkeypox disease through the Caputo-fractional order derivatives. It was keenly observed that the memory effect exhibited in the dynamics of the Monkeypox disease has a significant influence on the length of time it may take for a solution trajectory to be in a steady state, see the following works on the impact of fractional derivative on monkeypox disease [53–55]. Recently, [56] defined a novel fractional operator which has the properties of the Mittag-Leffler function and exponential decay referred to as the fractal–fractional order derivative, thus, a non-integer order and a fractal dimension. This robust approach studies the disease dynamics with a non-integer order within a fractal dimension and is best used in studying non-local physical problems that exhibit time fractal behaviours [49,57]. As a result, the combined fractal dimension and fractional order models can accurately describe the physical system compared to the existing fractional order derivatives. So as explained above, the integer order model is a local operator and has major flaws in investigating the dynamics of a disease, whereas the fractional operators also pose some challenges due to the diverse or even inconsistent patterns observed in biological systems. Thus, by comparing the fractional operator model to that of the fractal–fractional operators, the aforementioned models are observed to have a weak memory effect recognition and thus unable to describe completely the exact influence of memory in a disease. The fractal–fractional model, due to its significance over the fractional order derivative has therefore received numerous application in the study of infectious diseases, for instance, see [58–62,62–64]. To list a few, in the work of [65], the fractal–fractional derivative model has been applied to study the dynamics of smoking decisions through the two-step Lagrange polynomial. Also, in [66], the fractal–fractional model was used to investigate the biological trends of the Ebola virus disease and was reported to have a significant influence on the Monkeypox disease population at different fractal–fractional values. Notwithstanding this, very few works have applied this novel fractal–fractional method to understand the transmission trends of the Monkeypox disease. The only paper to apply the fractal–fractional model is in [26], where the fractal–fractional operator modelled the Monkeypox disease. They studied the dynamics of the Monkeypox disease through several scenarios where diverse fractal dimensions and fractional orders were investigated, and their research findings indicated that the fractal–fractional orders play an essential role in determining the dynamics of the disease as all trajectories of the compartments in the population converged uniformly. The novelty of this research is to investigate the dynamics of the Monkeypox disease and how it can be efficiently controlled through the combined fractal and fractional derivatives, the fractal–fractional derivative model, which is known to have a strong memory and repeated effect recognition. Since the transmission dynamics of monkeypox has a long history connection to its spread. Therefore, we extend the work of [67], which investigated the dynamics of the Monkeypox disease, incorporating two control strategies, isolation and quarantine, through a non-fractional order derivative.

We have organized the entire paper into six (6) sections. In Section 2, we give an introduction to investigating the dynamics of Monkeypox disease through a mathematical approach. We also outline some basic and essential definitions of this work. We then presented the non-fractional of the Monkeypox disease in Section 3. In Section 4, we reformulate the integer order equations into fractal–fractional. We further discussed physical properties like the biological threshold and the Monkeypox model's existence, uniqueness and stability. The numerical scheme of the fractal–fractional Monkeypox model via Newton's polynomial is presented in Section 5. Graphical representations of the dynamics of the disease are presented in Section 6 by means of numerical simulations. We finally conclude the work in Section 7.

2. Basic notations

Some essential definitions are stated here for the purpose of the model under study.

Definition 2.1 (*[56,62,68]*). Given $\theta \in C[(\varsigma_1, \varsigma_2), \mathbb{R}]$ and is defined to be fractal derivative on the interval $(\varsigma_1, \varsigma_2)$ of order $0 < \eta_2^* < 1$. We then define θ as a fractal–fractional operator in the Atangana–Baleanu sense with order $0 < \eta_1 < 1$, with the generalized Mittag-Leffler kernel type given as

$$^{FFC}\mathbf{D}^{\zeta_1}_{(0,t)}[\theta(t)] = \frac{\mathcal{G}(\eta_1)}{1-\eta_1}\frac{d}{dt^{\eta_2}} \times \int_0^t \theta(s) B_{\eta_1}\left[-\frac{\eta_1}{1-\eta_1}(t-s)^{\eta_1}\right] ds,$$

where, $\mathcal{G}(\eta_1) = 1 - \eta_1 + \frac{\eta_1}{\Gamma(\eta_1)}$ and also $\frac{d\theta(s)}{ds^{\eta^2}} = \lim_{t \longrightarrow 0} \frac{\theta(t)=\theta(s)}{t^{\eta_2} - s^{\eta_2}}$

Definition 2.2 (*[56,62,68]*). Supposing that $\theta \in C[(\alpha, \beta), \mathbb{R}]$ and also defined to be fractal antiderivative on the interval (α, β) of order $0 < \eta_2^* < 1$. We then define the fractal–fractional function for θ in Atangana–Baleanu sense of order $0 < \eta_1 < 1$, with the generalized Mittag-Leffler kernel type is given as

$$^{FFM}\mathbf{I}^{\eta_1}_{(0,t)}[\theta(t)] = \frac{\eta_1 \eta_2}{\mathcal{G}(\eta_1)\Gamma(\eta_1)} \times \int_0^t S^{\eta_1 - 1}\theta(s)(t-s)^{\eta_1 - 1} ds + \frac{\eta_2(1-\eta_1)t^{\eta_2 - 1}}{\mathcal{G}(\eta_1)}\theta(t),$$

where, $\mathcal{G}(\eta_1) = 1 - \eta_1 + \frac{\eta_1}{\Gamma(\eta_1)}$.

Definition 2.3 (*[69]*). The generalized $\eta_1 th$ Liouville Caputo type derivative $^{GL}\mathbf{D}^{\eta_1,\eta_2}_{b_+}$ is given as :

$$^{GLC}\mathbf{D}^{\eta_1,\eta_2}_{b_+}\mathcal{G}(t) = \frac{\eta_2^{\eta_1 - m + 1}}{\Gamma(m-\eta_1)} \int_b^t \tau^{\eta_2 - 1}[t^{\eta_2} - \tau^{\eta_2}]^{m-\eta_1-1} \left(\tau^{1-\eta_2}\frac{d}{d\tau}\right)^m \mathcal{G}(\tau) d\tau, \quad t > a,$$

with $\eta_2 > 0$ and also $m - 1 < \eta_1 \leq m$.

Definition 2.4 (*[70]*). Given the normed space $\mathcal{X}$, let us define the operator $\mathcal{G} : \mathcal{X} \longrightarrow \mathcal{X}$ and also $\mathcal{X}^8 \in \mathbb{R}^+ \cup \{0\}$. Thus, **(a)** $\forall \alpha_1, \alpha_2 \in \mathcal{X}$, the function $\mathcal{G}$ is a $\phi - \psi$ contraction whenever

$$\phi(\alpha_1, \alpha_2)\mathbf{d}(\mathcal{G}(\alpha_1), \mathcal{G}(\alpha_2)) \leq \psi(\mathbf{d}(\alpha_1, \alpha_2)).$$

(b) $\mathcal{G}$ is a ϕ -admissible if

$$\phi(\alpha_1, \alpha_2) \geq 1 \implies \phi(\mathcal{G}(\alpha_1), \mathcal{G}(\alpha_2)) \geq 1.$$

3. Non-fractional order monkeypox model

This work discusses the monkeypox disease, which is medically described to be phlebotomized condition. In this work, we extend the work of [67], which significantly considers two key control strategies: isolation and quarantine. They, therefore, categorized the Monkeypox disease into seven (7) state variables, that is, Susceptible human (S_w), Exposed human (E_w), Infected human (I_w), Isolated human (J_w), Quarantined human (Q_w), Recovered human (R_w), Susceptible rodents (S_v), and Infected rodents (I_v). This model was studied through an integer derivative order, and we intend to investigate the dynamics of this disease through the novel fractal–fractional operators. In [67], the integer order model was given as;

$$\begin{aligned}
\frac{dS_w}{dt} &= \zeta_h - \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - \sigma_u S_w + \iota\nu Q_w,\\
\frac{dE_w}{dt} &= \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - (\beta + \sigma_u)E_w,\\
\frac{dI_w}{dt} &= \beta(1-\mu)E_w + \iota(1-\nu)Q_w - (\sigma_u + \lambda_h + \kappa)I_w,\\
\frac{dJ_w}{dt} &= \kappa\phi I_w - (\sigma_u + \lambda_h + \pi)J_w,\\
\frac{dQ_w}{dt} &= \beta\mu E_w - (\iota + \sigma_u)Q_w,\\
\frac{dR_w}{dt} &= \kappa(1-\phi)I_w + \pi J_w - \sigma_u R_w,\\
\frac{dS_v}{dt} &= \zeta_r - \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v S_v,\\
\frac{dI_v}{dt} &= \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v I_v,
\end{aligned} \tag{1}$$

with the initial conditions $S_w(0) = S_{u0}, E_w(0) = E_{u0}, I_w(0) = I_{u0}, R_w(0) = R_{u0}, S_v(0) = S_{v0}, I_v(0) = I_{v0}$, where $S_w(0) \geq 0, E_w(0) \geq 0, I_w(0) \geq 0, R_w(0) \geq 0, S_v(0) \geq 0$ and $I_v(0) \geq 0$.

4. Fractal-fractional mathematical model of the monkeypox disease

The integer order Eq. (1) is thus reformulated into a fractal–fractional order system. This is necessary since the fractal–fractional model has a significant and unique nature of systems with non-local memory and hereditary properties, with time fractal behaviours seen in many biological systems. By using the fractal–fractional derivative and integral operators, we reformulate the monkeypox model as such;

$$\begin{aligned}
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} S_w(t) &= \zeta_h - \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - \sigma_u S_w + \iota\nu Q_w,\\
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} E_w(t) &= \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - (\beta + \sigma_u)E_w,\\
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} I_w(t) &= \beta(1-\mu)E_w + \iota(1-\nu)Q_w - (\sigma_u + \lambda_h + \kappa)I_w,\\
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} J_w(t) &= \kappa\phi I_w - (\sigma_u + \lambda_h + \pi)J_w,\\
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} Q_w(t) &= \beta\mu E_w - (\iota + \sigma_u)Q_w,\\
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} R_w(t) &= \kappa(1-\phi)I_w + \pi J_w - \sigma_u R_w,\\
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} S_v(t) &= \zeta_r - \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v S_v,\\
{}^{FFM}\mathcal{D}^{\eta_1,\eta_2}_{0,t} I_v(t) &= \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v I_v,
\end{aligned} \tag{2}$$

where ${}^{\mathcal{FMP}}\mathcal{D}^{\eta_1\eta_2}_{0,t}$ represents the fractal–fractional derivative of fractional order $0 < \eta_1 < 1$ and also with a fractal dimension $0 < \eta_2 < 1$ through the power law type kernel in the Caputo sense. It is essential to note that all the model parameters are nonnegative with the state variables given as $N(t) = S_w(t) + E_w(t) + I_w(t) + J_w(t) + Q_w(t) + R_w(t) + S_v(t) + I_v(t)$ with the time $t \in U := [0, \tau], (\tau > 0)$.

4.1. Positivity and boundedness of the monkeypox model

We discuss and thus explicitly establish that the monkeypox model is positive and bounded.

Theorem 4.1. *Given the initial conditions* $\{S_w(0), E_w(0), I_w(0), J_w(0), Q_w(0), R_w(0), S_v(0), I_v(0)\} \subset \mathcal{J}$ *and further supposing that there exist the solutions* $\{S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v\}$, *then it suffices that all of the solutions are positive for all* $t \geq 0$.

Proof. In this section, we will prove the theorem by following the approach as used in [49,71–73]. Let us state some essential and basic properties that establish the positivity of the monkeypox model. Considering the norm

$$\|\Phi\|_{\infty} = \sup_{t\in D_{\Phi}} |\Phi(t)|, \tag{3}$$

where D_{Φ} represent the domain of Φ. We start by considering first the susceptible human class, $S_w(t)$, $\forall t \geq 0$, this yields; ☐

$$\begin{aligned}
{}^{\mathcal{FFM}}D_{0,t}^{\eta_1\eta_2} S_w(t) &=\geq \zeta_h - \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - \sigma_u S_w + \imath\nu Q_w,\\
&\geq \zeta_h - (\varrho_1 I_v + \varrho_2 I_w) S_w - \sigma_u S_w + \imath\nu Q_w,\\
&\geq -(\sigma_u + (\varrho_1 + \varrho_2)|\mathcal{X}|) S_w,\\
&\geq -(\sigma_u + (\varrho_1 + \varrho_2) \sup_{t\in\mathcal{D}_\lambda} |\mathcal{X}|) S_w,\\
&\geq -(\sigma_u + (\varrho_1 + \varrho_2)\|\mathcal{X}\|_{\infty}) S_w.
\end{aligned} \tag{4}$$

We therefore have;

$$S_w(t) \geq S_w(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\eta_2}\upsilon(\sigma_u + (\varrho_1+\varrho_2)\|\mathcal{J}\|_{\infty})t^{\upsilon}}{AB(\upsilon) - (1-\upsilon)(\sigma_u + (\varrho_1+\varrho_2)\|\mathcal{J}\|_{\infty})}\right], \tag{5}$$

where τ represents the time component. Therefore the susceptible human state variable $S_w(t)$ of the monkeypox model is positive for all $t \geq 0$.

Again, for the Exposed human class, $E_w(t)$, we have:

$$\begin{aligned}
{}^{\mathcal{FFP}}D_{0,t}^{\eta_1\eta_2} E_w(t) &= \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - (\beta + \sigma_u) E_w,\\
&\geq -(\beta + \sigma_u) E_w,\\
E_w(t) &\geq E_w(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\eta_2}\upsilon(\beta+\sigma_u)t^{\upsilon}}{AB(\upsilon) - (1-\upsilon)(\beta+\sigma_u)}\right].
\end{aligned} \tag{6}$$

It is therefore obvious to assert that the Exposed human compartment, $E_w(t)$ is also positive for all $t \geq 0$. We have, in similar manner, proved the positivity of the other state variables and stated them below;

$$\begin{aligned}
I_w(t) &\geq I_w(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\eta_2}\upsilon(\sigma_u+\lambda_h+\kappa)t^{\upsilon}}{AB(\upsilon)-(1-\upsilon)(\sigma_u+\lambda_h+\kappa)}\right],\\
J_w(t) &\geq J_w(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\zeta_2}\upsilon(\sigma_u+\lambda_h+\pi)t^{\upsilon}}{AB(\upsilon)-(1-\upsilon)(\sigma_u+\lambda_h+\pi)}\right],\\
Q_w(t) &\geq Q_w(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\zeta_2}\upsilon(\imath+\sigma_u)t^{\upsilon}}{AB(\upsilon)-(1-\upsilon)(\imath+\sigma_u)}\right],\\
R_w(t) &\geq R_w(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\zeta_2}\upsilon(\sigma_u)t^{\upsilon}}{AB(\upsilon)-(1-\upsilon)(\sigma_u)}\right],\\
S_v(t) &\geq S_v(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\zeta_2}\upsilon(\sigma_v+\varrho_3\|\mathcal{X}\|_{\infty})t^{\upsilon}}{AB(\upsilon)-(1-\upsilon)(\sigma_v+\varrho_3\|\mathcal{X}\|_{\infty})}\right],\\
I_v(t) &\geq I_v(0)\mathcal{Q}_{\upsilon}\left[-\frac{\tau^{1-\zeta_2}\upsilon(\sigma_v+\varrho_3\|\mathcal{X}\|_{\infty})t^{\upsilon}}{AB(\upsilon)-(1-\upsilon)(\sigma_v+\varrho_3\|\mathcal{X}\|_{\infty})}\right].
\end{aligned} \tag{7}$$

Therefore, we have established the positivity of the Monkeypox model $\forall t \geq 0$.

Theorem 4.2. *Given the solution of the Monkeypox model (1) $S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*$, with positive initial conditions. Then all solutions of the model are bounded.*

Proof. We have established explicitly that the solutions to the Monkeypox model are all positive for every $t \geq 0$; by following the approach in [26], we prove the boundedness of the model. Let $\mathcal{J} = S_w + E_w + I_w + J_w + Q_w + R_w$, this results in

$$^{FFM}\mathcal{D}_{0,t}^{\eta_1}\mathcal{J}_h(t) = \zeta_h - \sigma_u J_w - \imath\upsilon Q_w + \delta Q_w - \delta\upsilon Q_w - \lambda_h I_w - \imath Q_w,$$

which implies that

$$^{FFM}\mathcal{D}_{0,t}^{\eta_1} J(t) \leq \zeta_h - \sigma_u\mathcal{J}.$$

Thus, we have

$$\Psi_w = \{S_w, E_w, I_w, J_w, Q_w, R_w \in \mathbb{R}_+^6 | \mathcal{J} \leq \frac{\zeta_h}{\sigma_u}\}.$$

Also for the rodent population, we have $\mathcal{K} = S_v + I_v$, and this results in;

$$^{FFM}\mathcal{D}_{0,t}^{\eta_1}\mathcal{K}(t) \leq \zeta_r - \sigma_v\mathcal{K},$$

which leads to

$$\Psi_v = \{S_v, I_v \in \mathbb{R}_+^2 | \mathcal{K} \leq \frac{\zeta_r}{\sigma_v}\}.$$

Therefore we say that all solutions of the model are positively invariant with the given initial conditions in the domain Ψ for every $t \geq 0$. ☐

Theorem 4.3. *Now we say that the Monkeypox model is thus a dynamical system on the compact set;*

$$\mathcal{S} = \{S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t) \in \mathbb{R}^6_+\},$$

and

$$\mathcal{S} = \{S_v(t), I_v(t) \in \mathbb{R}^2_+\}.$$

4.2. Existence criteria of the monkeypox model

In this section, we discuss the existence of the monkeypox disease model. This is then established through the fixed point theory on the fractal–fractional system (3). Considering the Banach space $\mathbb{H} = \mathbb{T}^8$ with $\mathbb{T} = C(\mathbb{U}, \mathbb{R})$ and also $\|\mathcal{M}\|_{\mathbb{H}} = max\{|S_w(t)| + |E_w(t)| + |I_w(t)| + |J_w(t)| + |Q_w(t)| + |R_w(t)| + |S_v(t)| + |I_v(t)|\}$ such that $t \in \mathbb{H}$. With the assumptions above, and also considering our prior knowledge of the differentiability of integrals, we can reformulate the fractal fractional system (2) as

$$\begin{aligned}
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} S_w(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_1(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} E_w(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_2(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} I_w(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_3(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} J_w(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_4(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} Q_w(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_5(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} R_w(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_6(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} S_v(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_7(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}^{\eta_1}_{0,t} Ir(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}_8(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),
\end{aligned} \tag{8}$$

where we define;

$$\begin{aligned}
&\mathcal{V}_1(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \zeta_h - \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - \sigma_u S_w + \iota\nu Q_w,\\
&\mathcal{V}_2(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - (\beta + \sigma_u) E_w,\\
&\mathcal{V}_3(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \beta(1-\mu)E_w + \delta(1-\nu)Q_w - (\sigma_u + \lambda_h + \kappa) I_w,\\
&\mathcal{V}_4(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \kappa\phi I_w - (\sigma_u + \lambda_h + \pi) J_w,\\
&\mathcal{V}_5(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \beta\mu E_w - (\iota + \sigma_u) Q_w,\\
&\mathcal{V}_6(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \kappa(1-\phi) I_w + \pi J_w - \sigma_u R_w,\\
&\mathcal{V}_7(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \zeta_r - \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v S_v,\\
&\mathcal{V}_8(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v I_v.
\end{aligned} \tag{9}$$

With the above system, the fractal–fractional system (8) can be represented with the initial value problem

$$\begin{cases} {}^{RL}\mathcal{D}^{\eta_1}_{0,t} \mathcal{O}(t) &= \eta_2 t^{\eta_2 - 1} \mathcal{V}(t, \mathcal{O}(t)), \quad \eta_1, \eta_2 \in (0, 1],\\ \mathcal{O}(0) &= \mathcal{O}_0, \end{cases} \tag{10}$$

where we have $\mathcal{O}(t) = (S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t))^T$ with

$$\mathcal{O}_0 = (S_{u0}, E_{u0}, I_{u0}, J_{u0}, Q_{u0}, R_{u0}, S_{v0}, I_{v0})^T$$

and also

$$\mathcal{V}(t, \mathcal{O}(t)) = \begin{cases}
\mathcal{V}_1(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
\mathcal{V}_2(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
\mathcal{V}_3(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
\mathcal{V}_4(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
\mathcal{V}_5(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
\mathcal{V}_6(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
\mathcal{V}_7(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
\mathcal{V}_8(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)).
\end{cases} \tag{11}$$

By applying the definition of the fractal–fractional integral on the compact form of Eq. (10) yields;

$$\mathcal{O}(t) = \mathcal{O}(0) + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2 - 1} (t - \varphi)^{\eta_1 - 1} \mathcal{V}(\varphi, \mathcal{O}(\varphi)) d\varphi. \tag{12}$$

from Eq. (12), the fractal–fractional monkeypox model could be reformulated as

$$
\begin{aligned}
S_w(t) &= S_{u0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi,\\
E_w(t) &= E_{u0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi,\\
I_w(t) &= I_{u0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi,\\
J_w(t) &= J_{u0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi,\\
Q_w(t) &= Q_{u0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi,\\
R_w(t) &= R_{u0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi,\\
S_v(t) &= S_{v0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi,\\
I_v(t) &= I_{v0} + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))d\varphi.
\end{aligned}
\tag{13}
$$

We can then reformulate the fractal–fractional system (3) as a fixed point problem. We then define $\mathbb{N} = \mathbb{H} \longrightarrow \mathbb{H}$ by

$$
\mathcal{V}[\mathcal{O}(t)] = \mathcal{O}(0) + \frac{\xi}{\Gamma(\eta_1)} \int_0^t \eta_2\varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}(\varphi, \mathcal{O}(\varphi))d\varphi. \tag{14}
$$

Let us recall the fixed point theorem and use it to establish the existence of our model.

Theorem 4.4 ([70]). *Let us suppose* $(\mathbb{H}, d)$ *be a complete metric space,* $\tau : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}, \rho \in \pi$ *and* $\mathcal{V} : \mathbb{H} \times \mathbb{H}$ *be a* $\tau - \rho$ *contractive map. We further assume that*
P_1*:* $\mathcal{V}$ *is* $\tau-$ *admissible on* $\mathbb{H}$*;*
P_2 *:* $\forall h_0 \in \mathcal{H},\ \tau(h_0, \mathcal{V})h_0 \geq 1$*;*
P_3*: for any sequence* $\{h_n\} \in \mathbb{H}$ *with* $h_n \longrightarrow h$ *and* $\tau(h_n, h_{n+1}) \geq 1, \forall n \geq 1$ *we then have* $\tau(h_n, h) \geq 1, \forall n \geq 1$*. Then there exists an* $h^* \in \mathbb{H}$ *such that* $\mathcal{V}h^* = h^*$*.*

To prove this, let us state some special operators.

Theorem 4.5. *Let us suppose that there exists* $\Xi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$*, and there also exist* $\mathcal{V} \in C(U \times \mathbb{H}, \mathbb{H})$*. There also exists an operator* $\phi \in \Phi$ *such that*
G_1 : *for every* $T_1, T_2 \in \mathbb{H}$ *and* $t \in U$*, we say that;* $|\mathcal{V}(t, T_1(t)) - \mathcal{V}(t, T_2(t))| \leq \psi\pi(|T_1(t) - T_2(t)|)$*, with* $\Xi(T_1(t), T_2(t)) \geq 0$*, where* $\psi = \frac{\Gamma(\eta_1+\eta_2)}{\eta^2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}$*.*
G_2 : *there exist* $T_0 \in \mathbb{H}$ *so that* $\forall t \in U$*, we have* $(T_0(t), \mathcal{V}(T_0(t))) \geq 0$*, and also* $\Xi(T_1(t), T_1(t)) \geq 0$ *yields* $\Xi\mathcal{V}[(T_1(t)), \mathcal{V}(T_2(t))] \geq 0, \forall T_1, T_2 \in \mathbb{H}$ *and* $t \in U$*,*
G_3 : *for all* $\{T_n\}_{n\geq 1} \subseteq \mathbb{H}$ *with* $T_n \longrightarrow T$ *and* $[T_n(t), T_{n+1}(n)] \geq 0, \quad \forall n \in \mathbb{N}, \forall t \in U$*, we have* $\Xi(T_n(t), T(t))$*. If this holds, we then say that a solution exists for the fractal–fractional system (5) and hence a solution is found for the fractal–fractional system (3).*

Proof. We apply the beta function's definition by first taking some two arbitrary constant values of the operator $\mathbb{H}$ as T_1 and T_2 by the property $\Xi(T_1(t), T_2(2))$, for every $t \in U$. Now by the Beta function, we have;

$$
\begin{aligned}
|\mathcal{V}(T_1(t)) - \mathcal{V}(T_2(t))| &\leq \frac{\xi}{\Gamma(\eta_1)} \int_0^t \eta_2\varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}|\mathcal{V}(\varphi, \mathcal{O}_1(\varphi)) - \mathcal{V}(\varphi, \mathcal{O}_2(\varphi))|d\varphi,\\
&\leq \frac{\xi}{\Gamma(\eta_1)} \int_0^t \eta_2\varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\psi\phi(|T_1(\varphi) - T_2(\varphi)|)d\varphi,\\
&\leq \frac{\eta_2\mu^* M^{\eta_1+\eta_2-1}\mathbb{B}(\eta_1, \eta_2)}{\Gamma(\eta_1)}\phi(\|T_1 - T_2\|_{\mathbb{H}}),\\
&= \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1)}\phi(\|T_1 - T_2\|_{\mathbb{H}}).
\end{aligned}
\tag{15}
$$

This will therefore lead to

$$
\begin{aligned}
|\mathcal{V}(T_1(t)) - \mathcal{V}(T_2(t))| &\leq \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1)}\phi(\|T_1 - T_2\|_{\mathbb{H}}),\\
&= \phi(\|T_1 - T_2\|_{\mathbb{H}}).
\end{aligned}
\tag{16}
$$

This yields the nonnegative map $\tau : \mathbb{H} \times \mathbb{H} \longrightarrow [0, \infty)$ which is best defined by

$$
\tau(T_1, T_2) = \begin{cases} 0, & \Xi(T_1(t), T_2(t)) < 0,\\ 1, & \Xi(T_1(t), T_2(t)) \geq 0, \end{cases} \tag{17}
$$

for every given element $T_1, T_2 \in \mathbb{H}$. We thus derive

$$
\tau(T_1, T_2)d(\mathcal{V}(T_1), \mathcal{V}(T_2)) \leq \phi(d(T_1, T_2)),
$$

for every $T_1, T_2 \in \mathbb{H}$. This therefore suffices $\mathcal{V}$ to be a $\tau - \phi$ contraction. To further show that $\mathcal{V}$ is also $\tau-$ admissible, we let $T_1, T_2 \in \mathbb{H}$ with the condition that $\tau(T_1(t), T_2(t)) \geq 1$. Now the definition of τ results in $\Xi(T_1(t), T_2(t)) \geq 0$. Also,

$$
\Xi[\mathcal{V}(T_1(t)), \mathcal{V}(T_2(t))] \geq 0,
$$

is satisfied. It is again proven that

$$\tau(\mathcal{V}(T_1(t)), \mathcal{V}(T_2(t))) \geq 1,$$

by the definition of τ. This therefore proofs that $\mathcal{V}$ is a $\tau-$admissible.

The existence of $T_0 \in \mathbb{H}$ is assured by condition G_2 which obviously establish that $\tau(T_0, \mathcal{V}(T_0)) \geq 1$. Hence the conditions P_1 and P_2 are satisfied. We further establish condition P_3. Let us suppose $\{T_n\}_{n\geq 1} \subseteq \mathbb{H}$ be any sequence with the limit $T_n \longrightarrow T$ and $\tau(T_n, T_{n+1}) \geq 1$, $\forall n$. This establishes the nonnegative map τ given by

$$\Xi(T_1(t), T_{n+1}(t)) \geq 0.$$

Therefore condition G_3 suffices that

$$\Xi(T_n(t), T(t)) \geq 0.$$

This also establishes condition P_3 of theorem (3.4). To conclude, theorem (3.4) ensures the existence of some fixed point for $\mathcal{V}$ as $T^* \in \mathbb{H}$. This, therefore, suffices that $T^* = (S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, S_v^*, I_v^*)$ is a solution to the fractal–fractional monkeypox model, and this ends the argument. ☐

Theorem 4.6 (*Leray–Schauder*).[74] *Let us assume that $\mathbb{H}$ is a Banach space and also Π be a bounded convex and closed set in $\mathbb{H}$, Θ an open set in Π such that $0 \in \Theta$. Furthermore, the operator $\mathcal{Q} : \Theta \longrightarrow \Pi$, then either;*

$N1$ *there exist a fixed point $y \in \Theta$ such that $\mathcal{Q}(y) = y$, or*
$N2$ *there exist $f \in \partial\Theta$ and β lies in the domain $(0, 1)$ so that $y = \beta\mathcal{Q}(y)$.*

Theorem 4.7. *Let there be $\mathcal{V} \in C(U \times \mathbb{H}, \mathbb{H})$, then;*

$D1$ *then there exist both $\varrho \in J^1(U, [0, \infty))$ and a monotonic increasing function $\Upsilon \in C([0, \infty), (0, \infty))$ such that $\forall t \in U$ and $T \in \mathbb{H}$, we have*

$$|C(t, T(t))| \leq \varrho(t)C(|T(t)|);$$

$D2$ *there exist $\lambda > 0$ such that*

$$\lambda > T_0 + \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2 + 1)}{\Gamma(\eta_1 + \eta_2)}\varrho_0^* C(\lambda), \tag{18}$$

given that

$$\varrho_0^* = \sup_{t\in U} |\varrho(t)|,$$

a solution will therefore exist for the fractal–fractional Monkeypox model (2).

Theorem 4.8. *Here, we consider the map $\mathcal{V}$ as defined in Eq. (9) and the closed ball*

$$\mathcal{N}_J = \{T \in \mathbb{H} : \|T \leq J\|\}. \tag{19}$$

The continuity of $\mathcal{V}$ suffices that of the map $\mathcal{N}_J$, now from $D1$, we have;

$$\begin{aligned} |\mathcal{Q}(T(t))| &\leq |T(0)|\frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}|\mathcal{V}(\varphi, T(\varphi)) - \mathcal{V}(\varphi, \mathcal{O}_2(\varphi))|d\varphi, \\ &\leq T_0 + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \eta_2\varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\varrho(\varphi)C(|T(\varphi)|)d\varphi, \\ &\leq T_0 + \frac{\eta_2 M^{\eta_1+\eta_2-1}\mathbb{B}(\eta_1, \eta_2)}{\Gamma(\eta_1)}\varrho_0^* C(\|T\|_{\mathbb{H}}), \\ &= T_0 + \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1 + \eta_2)}\varrho_0^* C(\mu), \end{aligned} \tag{20}$$

for $T \in \mathbb{N}_J$ which yields

$$|\mathcal{V}(T(t))| \leq T_0 + \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1 + \eta_2)}\varrho_0^* C(\mu) < \infty. \tag{21}$$

Therefore $\mathcal{V}$ is observed to be uniformly bounded on $\mathbb{H}$. Thus, sufficing the equicontinuity of $\mathcal{V}$. We arbitrarily take $t, t_* \in [0, T]$ where $t < t_*$ and $T \in \mathcal{N}_J$.

Supposing further that

$$\sup_{(t,T)\in U\times\mathbb{N}_J} |\mathcal{V}(t, T(t))| = C^* < \infty,$$

and this yields;

$$\begin{aligned} |\mathcal{Q}(T(t_*)) - \mathcal{Q}(T(t))| &\leq \frac{\eta_2}{\Gamma(\eta_1)}\int_0^{t_*} \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}|\mathcal{V}(\varphi, T(\varphi))|d\varphi - \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\psi\phi(|T_1(\varphi) - T_2(\varphi)|)d\varphi, \\ &\leq \frac{\eta_2\mathcal{V}^*}{\Gamma(\eta_1)}|\int_0^{t_*} \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1} - \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}, \\ &\leq \frac{\eta_2\mathcal{V}^*\mathbb{B}(\eta_1, \eta_2)}{\Gamma(\eta_1)}|t_*^{\eta_2+\eta_1-1} - t^{\eta_2+\eta_1-1}|, \\ &= \frac{\eta_2\mathcal{V}^*\Gamma(\eta_2)}{\Gamma(\eta_2 + \eta_1)}|t_*^{\eta_2+\eta_1-1} - t^{\eta_2+\eta_1-1}|, \end{aligned} \tag{22}$$

where we observe that the right-hand side of Eq. (22) is independent of T and also continuous when $t_* \longrightarrow t$, thus,

$$\|\mathcal{Q}(T(t_*)) - \mathcal{Q}(T(t))\|_{\mathbb{H}} \longrightarrow 0,$$

and this establishes the equicontinuity of the map $\mathcal{V}$ as $t_* \longrightarrow t$. Therefore, $\mathcal{Q}$ is said to be compact on $\mathcal{N}_J$ by the Arzela–Ascoli theorem. We, therefore, assert that theorem (3.4) is validated on $\mathcal{Q}$ since we have one of the consequences. Now, from D2, we have $\lambda > 0$ such that

$$\lambda > T_0 + \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2+1)}{\Gamma(\eta_1+\eta_2)}\varrho_0^* C(\lambda), \tag{23}$$

By considering

$$\Theta := \{T \in \mathbb{H} : \|T\|_{\mathbb{H}} < \lambda\},$$

we assume further that $T \in \partial\Theta$ and $0 < \beta < 1$ exist subject to the condition $T = \beta\mathcal{Q}(T)$, we then posit that

$$\begin{aligned}
\lambda &= \|T\|_{\mathbb{H}} = \beta\|\mathcal{Q}T\|_{\mathbb{H}},\\
&\leq \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2+1)}{\Gamma(\eta_1+\eta_2)}\varrho_0^* C(\|T\|_{\mathbb{H}}),\\
&< \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2+1)}{\Gamma(\eta_1+\eta_2)}\varrho_0^* C(\lambda),\\
&< \lambda.
\end{aligned} \tag{24}$$

Thus, case 2 cannot be validated since the expression above cannot occur. This implies that the operator $\mathcal{Q}$ has a fixed point in Θ. This thus suffices that the fractal–fractional monkeypox model has a solution, and this completes the proof.

4.3. Uniqueness results

By the use of the Lipschitz condition under the function $\mathcal{V}_i$ where $i = 1, 2, \dots, 8$, we establish the uniqueness of the solution of the fractal–fractional monkeypox model understudy.

Lemma 4.1. *Let us define the functions $S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v, S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^* \in \mathbb{X} := (U, \mathbb{R})$ and further assume that $(Z1)$ for some $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8 > 0$ there exists $\|S_w\| \leq \zeta_1, \|E_w\| \leq \zeta_2, \|I_w\| \leq \zeta_3, \|J_w\| \leq \zeta_4, \|Q_w\| \leq \zeta_5, \|R_w\| \leq \zeta_6, \|S_v\| \leq \zeta_7$ and $\|I_v\| \leq \zeta_8 > 0$.*

Then the operators $\mathcal{V}_1, \mathcal{V}_2.\mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_8$ defined in theorem (3.4) meets the Lipschitz condition with respect to the components when $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8 < 1$ and we have;

$$\begin{aligned}
\gamma_1 &= \varrho_1\zeta_8 + \varrho_2\zeta_3 - \sigma_u\\
\gamma_2 &= \beta + \sigma_u,\\
\gamma_3 &= \sigma_u + \lambda_h + \kappa,\\
\gamma_4 &= \sigma_u + \lambda_h + \pi,\\
\gamma_5 &= \imath + \sigma_u,\\
\gamma_6 &= \sigma_u,\\
\gamma_7 &= \varrho_3\zeta_7 - \sigma_v,\\
\gamma_8 &= \varrho_3\zeta_7 - \sigma_v.
\end{aligned} \tag{25}$$

Proof. Considering the first operator $\mathcal{V}_1$, we say that for every $S_w, S_w^* \in \mathbb{X} : C(U, \mathbb{R})$we have the results;

$$\begin{aligned}
&\|\mathcal{V}_1(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_1(t, S_w^*(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t))\|,\\
&= \|\left(\zeta_h - \frac{\varrho_1 I_v(t) + \varrho_2 I_w(t)}{N_u}S_w(t) - \sigma_u S_w(t) + \imath\nu Q_w(t)\right) - \left(\zeta_h - \frac{\varrho_1 I_v(t) + \varrho_2 I_w(t)}{N_u}S_w^*(t) - \sigma_u S_w^*(t) + \imath\nu Q_w(t)\right)\|,\\
&\leq [\varrho_1\|I_v(t)\| + \varrho_2\|I_w(t)\| - \sigma_u]\|S_w(t) - S_w^*(t)\|,\\
&\leq [\varrho_1\zeta_8 + \varrho_2\zeta_3 - \sigma_u]\|S_w(t) - S_w^*(t)\|,\\
&= \gamma_1\|S_w(t) - S_w^*(t)\|.
\end{aligned} \tag{26}$$

It is thus observed that the kernel $\mathcal{V}_1$ satisfy the Lipschitz condition in relation to S_w with the constant $\gamma_1 < 1$. We then consider also the next operator $\mathcal{V}_2$, for every $E_w, E_w^* \in \mathbb{X} : C(U, \mathbb{R})$ yields;

$$\begin{aligned}
&\|\mathcal{V}_2(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_2(t, S_w(t), E_w^*(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t))\|,\\
&= \|\left(\frac{\varrho_1 I_v + \varrho_2 I_w(t)}{N_u}S_w(t) - (\beta + \sigma_u)E_w(t)\right) - \left(\frac{\varrho_1 I_v(t) + \varrho_2 I_w(t)}{N_u}S_w(t) - (\beta + \sigma_u)E_w^*(t)\right)\|,\\
&\leq [\beta + \sigma_u]\|E_w(t) - E_w^*(t)\|,\\
&= \gamma_2\|E_w(t) - E_w^*(t)\|.
\end{aligned} \tag{27}$$

So the second kernel $\mathcal{V}_2$ also satisfy the Lipschitz condition in relation with E_w with the constant $\gamma_2 < 1$. Moreso, we also consider the third operator $\mathcal{V}_3$, for every $I_w, I_w^* \in \mathbb{X} : C(U, \mathbb{R})$ yields;

$$\begin{aligned}
&\|\mathcal{V}_3(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_3(t, S_w(t), E_w(t), I_w^*(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t))\|,\\
&= \|\left(\beta(1-\mu)E_w(t) + \delta(1-\nu)Q_w(t) - (\sigma_u + \lambda_h + \kappa)I_w(t)\right) - \left(\beta(1-\mu)E_w(t) + \delta(1-\nu)Q_w(t) - (\sigma_u + \lambda_h + \kappa)I_w^*(t)\right)\|,\\
&\leq [\sigma_u + \lambda_h + \kappa]\|I_w(t) - I_w^*(t)\|,\\
&= \gamma_3\|I_w(t) - I_w^*(t)\|.
\end{aligned} \tag{28}$$

So the third kernel $\mathcal{V}_3$ also satisfy the Lipschitz condition in relation with E_w with the constant $\gamma_3 < 1$. Considering the further the next kernel $\mathcal{V}_4$, we say that for every $J_w, J_w^* \in \mathbb{X} : C(U, \mathbb{R})$we have the results;

$$\begin{aligned}
&\|\mathcal{V}_4(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_4(t, S_w(t), E_w(t), I_w(t), J_w^*(t), Q_w(t), R_w(t), S_v(t), I_v(t))\|,\\
&= \| \left(\kappa\phi I_w - (\sigma_u + \lambda_h + \pi)J_w\right) - \left(\kappa\phi I_w - (\sigma_u + \lambda_h + \pi)J_w\right) \|,\\
&\leq [\sigma_u + \lambda_h + \pi]\|J_w(t) - J_w^*(t)\|,\\
&= \gamma_4\|J_w(t) - J_w^*(t)\|.
\end{aligned} \tag{29}$$

It is thus observed that the kernel $\mathcal{V}_4$ satisfy the Lipschitz condition in relation to J_w with the constant $\gamma_4 < 1$. We also derived the rest of the state variables in the fractal–fractional monkeypox model as below;

$$\begin{aligned}
&\|\mathcal{V}_5(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_5(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w^*(t), R_w(t), S_v(t), I_v(t))\|,\\
&= \| \left(\beta\mu E_w(t) - (\iota + \sigma_u)Q_w(t)\right) - \left(\beta\mu E_w(t) - (\iota + \sigma_u)Q_w^*(t)\right) \|,\\
&\leq [\iota + \sigma_u]\|Q_w(t) - Q_w^*(t)\|,\\
&= \gamma_5\|Q_w(t) - Q_w^*(t)\|.
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\|\mathcal{V}_6(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_6(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w^*(t), R_w(t), S_v(t), I_v(t))\|,\\
&= \| \left(\kappa(1-\phi)I_w(t) + \pi J_w(t) - \sigma_u R_w(t)\right) - \left(\kappa(1-\phi)I_w(t) + \pi J_w(t) - \sigma_u R_w^*(t)\right) \|,\\
&\leq [\sigma_u]\|R_w(t) - R_w^*(t)\|,\\
&= \gamma_6\|R_w(t) - R_w^*(t)\|.
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\|\mathcal{V}_7(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_7(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v^*(t), I_v(t))\|,\\
&= \| \left(\zeta_r - \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v S_v\right) - \left(\zeta_r - \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v S_v\right) \|,\\
&\leq [\varrho_3\|I_v(t)\| - \sigma_v]\|S_v - S_v^*\|,\\
&\leq [\varrho_3\zeta_7 - \sigma_v]\|S_v(t) - S_v^*(t)\|,\\
&= \gamma_7\|S_v(t) - S_v^*(t)\|.
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\|\mathcal{V}_8(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) - \mathcal{V}_8(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v^*(t))\|,\\
&= \| \left(\frac{\varrho_3 S_v I_v}{N_v} - \sigma_v I_v\right) - \left(\frac{\varrho_3 S_v I_v}{N_v} - \sigma_v I_v\right) \|,\\
&\leq [\varrho_3\|S_v(t)\| - \sigma_v]\|I_v - I_v^*\|,\\
&\leq [\varrho_3\zeta_7 - \sigma_v]\|I_v(t) - I_v^*(t)\|,\\
&= \gamma_8\|I_v(t) - I_v^*(t)\|.
\end{aligned} \tag{33}$$

The results derived therefore suffices that all the functions $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_8$ satisfy the Lipschitz condition with respect to the related constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8 > 0$ respectively. □

Theorem 4.9. *Given that condition $Z1$ from lemma 1 is valid, we state that the fractal–fractional monkeypox model yields a linearly independent solution if*

$$\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_i < 1, \qquad i \in \{1, 2, \ldots, 8\}. \tag{34}$$

Proof. Let us provide indirect proof to establish the uniqueness of the solution to the fractal–fractional monkeypox model. We, therefore, use a proof-by-contradiction approach. In this manner, let us assume that our theorem's conclusion is invalid. That is, there exists at least one more solution of the fractal–fractional monkeypox model. Supposing that $S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t)$ is also a solution to the fractal–fractional monkeypox model and having the initial conditions $S_{u0}, E_{u0}, I_{u0}, J_{u0}, Q_{u0}, R_{u0}, S_{v0}, I_{v0}$ provided that by Eq. (13) we have

$$\begin{aligned}
S_w^*(t) &= S_{u0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_1(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
E_w^*(t) &= E_{u0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_2(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
I_w^*(t) &= I_{u0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_3(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
J_w^*(t) &= J_{u0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_4(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
Q_w^*(t) &= Q_{u0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_5(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
R_w^*(t) &= R_{u0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_6(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
S_v^*(t) &= S_{v0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_7(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
I_v^*(t) &= I_{v0} + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_8(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi.
\end{aligned} \tag{35}$$

We, therefore, do the following estimations.

$$\begin{aligned}
|S_w(t) - S_w^*(t)| &\leq \frac{\eta_2}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1} |\mathcal{V}_1[S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi)], \\
&- \mathcal{V}_1[S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi)]| d\varphi, \\
&\leq \frac{\eta_2}{\Gamma(\eta_1)} \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1} \|S_w - S_w^*\| d\varphi, \\
&\leq \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)} \gamma_1 \|S_w - S_w^*\|,
\end{aligned} \tag{36}$$

therefore,

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|S_w - S_w^*\|, \leq 0. \tag{37}$$

It therefore follows from Eq. (37) that the inequality derived is valid if $\|S_w - S_w^*\| = 0$ which implies that $S_w = S_w^*$. Similar observations are made in the other state variables and are given below;

$$|E_w(t) - E_w^*(t)| \leq \left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|E_w - E_w^*\|,$$

this results in

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|E_w - E_w^*\|, \leq 0.$$

Also, the inequality obtained is valid if $\|E_w - E_w^*\| = 0$ which implies that $E_w = E_w^*$. Again,

$$|I_w(t) - I_w^*(t)| \leq \left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|I_w - I_w^*\|,$$

we then obtain;

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|I_w - I_w^*\|, \leq 0.$$

Also, the inequality obtained is valid if $\|I_w - I_w^*\| = 0$ which implies that $I_w = I_w^*$. We also have;

$$|J_w(t) - J_w^*(t)| \leq \left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|J_w - J_w^*\|,$$

this results in

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|J_w - J_w^*\|, \leq 0.$$

We again observe that the inequality obtained is valid if $\|J_w - J_w^*\| = 0$ which implies that $J_w = J_w^*$. We also have;

$$|Q_w(t) - Q_w^*(t)| \leq \left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|Q_w - Q_w^*\|,$$

This leads to the following;

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|Q_w - Q_w^*\|, \leq 0.$$

We again observe that the inequality obtained is valid if $\|Q_w - Q_w^*\| = 0$ which implies that $I_w = I_w^*$. We also have;

$$|R_w(t) - R_w^*(t)| \leq \left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|R_w - R_w^*\|,$$

this yields;

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|R_w - R_w^*\|, \leq 0.$$

We again observe that the inequality obtained is valid if $\|R_w - R_w^*\| = 0$ which implies that $R_w = R_w^*$. Again,

$$|S_v(t) - S_v^*(t)| \leq \left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|S_v - S_v^*\|,$$

this results in the following;

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|S_v - S_v^*\|, \leq 0.$$

We again observe that the inequality obtained is valid if $\|S_v - S_v^*\| = 0$ which implies that $S_v = S_v^*$. Finally, we have;

$$|I_v(t) - I_v^*(t)| \leq \left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|I_v - I_v^*\|,$$

this results in the following;

$$\left[1 - \frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\right] \|I_v - I_v^*\|, \leq 0.$$

We again observe that the inequality obtained is valid if $\|I_v - I_v^*\| = 0$ which implies that $I_v = I_v^*$. The results obtained therefore suggest a contradiction since we have instead proven that.

$$(S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = (S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t)).$$

We have therefore shown that the fractal–fractional Monkeypox model yields only one solution, that is, a unique solution and the proof is completed. □

4.4. Stability analysis of the fractal-fractional monkeypox model

An essential aspect of modelling a biological system is to study how stable the solution to the model will be. Solutions to such systems are not always expressed as exact but could be approximate. How sure will we be that the solution obtained to our biological system is stable, that is, a small perturbation in the system will lead to a small change in the solution? In this section, we analyse the fractal–fractional monkeypox model's stability through the popular Ulam-Hyers stability criterion and Hyers–Ulam–Rassias stability.

Definition 4.1. The fractal–fractional monkeypox model is said to be Hyers–Ulam stable if there exists $0 < \mathcal{L}_{\mathcal{V}_i} \in \mathbb{R}$ where $i = 1, 2, 3, \ldots, 8 : \forall \mu_i > 0$ and also for all $(S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t)) \in \mathbb{H}$ satisfying.

$$\begin{aligned}
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_2(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_3(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_4(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_5(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_6(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_7(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) - \mathcal{V}_8(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1,
\end{aligned} \tag{38}$$

then there exist $(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v) \in \mathbb{H}$ that satisfies the fractal–fractional monkeypox model (2) such that

$$\begin{aligned}
|S_w^*(t) - S_w(t)| &\leq \mathcal{L}_{\mathcal{V}_1}\mu_1,\\
|E_w^*(t) - E_w(t)| &\leq \mathcal{L}_{\mathcal{V}_2}\mu_2,\\
|I_w(t) - I_w^*(t)| &\leq \mathcal{L}_{\mathcal{V}_3}\mu_3,\\
|J_w^*(t) - J_w(t)| &\leq \mathcal{L}_{\mathcal{V}_4}\mu_4,\\
|Q_w^*(t) - Q_w(t)| &\leq \mathcal{L}_{\mathcal{V}_5}\mu_5,\\
|R_w^*(t) - R_w(t)| &\leq \mathcal{L}_{\mathcal{V}_6}\mu_6,\\
|S_v^*(t) - S_v(t)| &\leq \mathcal{L}_{\mathcal{V}_7}\mu_7,\\
|I_v^*(t) - S_v(t)| &\leq \mathcal{L}_{\mathcal{V}_8}\mu_8.
\end{aligned}$$

Definition 4.2. The given fractal–fractional monkeypox model (2) is said to be Hyers–Ulam–Rassias stable if there exists $\mathcal{L}_{\mathcal{V}_i} \in C(\mathbb{R}^+, \mathbb{R}^+)$ where $i = 1, 2, 3, \ldots, 8$ with $\mathcal{L}_{\mathcal{V}_i}(0)$ such that for every $\mathcal{L}_i$ being positive and also for every $(S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*) \in \mathbb{H}$ that satisfies (38), there exist $(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v) \in \mathbb{H}$ as a solution of the fractal–fractional monkeypox model (3) such that;

$$\begin{aligned}
|S_w^*(t) - S_w(t)| &\leq \mathcal{L}_{\mathcal{V}_1}(\mu_1),\\
|E_w^*(t) - E_w(t)| &\leq \mathcal{L}_{\mathcal{V}_2}(\mu_2),\\
|I_w(t) - I_w^*(t)| &\leq \mathcal{L}_{\mathcal{V}_3}(\mu_3),\\
|J_w^*(t) - J_w(t)| &\leq \mathcal{L}_{\mathcal{V}_4}(\mu_4),\\
|Q_w^*(t) - Q_w(t)| &\leq \mathcal{L}_{\mathcal{V}_5}(\mu_5),\\
|R_w^*(t) - R_w(t)| &\leq \mathcal{L}_{\mathcal{V}_6}(\mu_6),\\
|S_v^*(t) - S_v(t)| &\leq \mathcal{L}_{\mathcal{V}_7}(\mu_7),\\
|I_v^*(t) - S_v(t)| &\leq \mathcal{L}_{\mathcal{V}_8}(\mu_8).
\end{aligned} \tag{39}$$

It is essential to note that definition (3.2) is obtained from definition (3.1).

Remark 4.1. Notice that $(S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*) \in \mathbb{H}$ is a solution for (3.1) if and only if

$$m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8 \in C([0, T], \mathbb{R})$$

(depending on $(S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*)$, respectively) so that $\forall t \in U$.

a. $|m_i(t)| < \mu_i$ where $i = 1, 2, 3, \ldots, 8$,

b. We have

$$\begin{aligned}
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_2(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_3(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_4(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_5(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_6(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_7(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) = \mathcal{V}_8(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t).
\end{aligned}\tag{40}$$

Definition 4.3. We again state that the given fractal–fractional monkeypox model (2) is said to be Hyers–Ulam–Rassias stable with respect to Ψ_i where $i = 1, 2, 3, \ldots, 8$ if there is an $\mathcal{L}_{(\mathcal{V}_i,\Psi_i)}$ such that for every positive μ_i and $(S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*) \in \mathbb{H}$ that satisfies.

$$\begin{aligned}
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1\Psi_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_2(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_2\Psi_2,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_3(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_3\Psi_3\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_4(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_4\Psi_4,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_5(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_5\Psi_5,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_6(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_6\Psi_6,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_7(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_7\Psi_7,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_8(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_8\Psi_8,
\end{aligned}\tag{41}$$

Then there exist $(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v) \in \mathbb{H}$ which is a solution to the fractal–fractional monkeypox model (2) with

$$\begin{aligned}
|S_w^*(t) - S_w(t)| &\leq \mu_1\mathcal{L}_{(\mathcal{V}_1,\Psi_1)}\Psi_1, \quad \forall t \in U,\\
|E_w^*(t) - E_w(t)| &\leq \mu_2\mathcal{L}_{(\mathcal{V}_2,\Psi_2)}\Psi_2, \quad \forall t \in U,\\
|I_w(t) - I_w^*(t)| &\leq \mu_3\mathcal{L}_{(\mathcal{V}_3,\Psi_3)}\Psi_3, \quad \forall t \in U,\\
|J_w^*(t) - J_w(t)| &\leq \mu_4\mathcal{L}_{(\mathcal{V}_4,\Psi_4)}\Psi_4, \quad \forall t \in U,\\
|Q_w^*(t) - Q_w(t)| &\leq \mu_5\mathcal{L}_{(\mathcal{V}_5,\Psi_5)}\Psi_5, \quad \forall t \in U,\\
|R_w^*(t) - R_w(t)| &\leq \mu_6\mathcal{L}_{(\mathcal{V}_6,\Psi_6)}\Psi_6, \quad \forall t \in U,\\
|S_v^*(t) - S_v(t)| &\leq \mu_7\mathcal{L}_{(\mathcal{V}_7,\Psi_7)}\Psi_7, \quad \forall t \in U,\\
|I_v^*(t) - S_v(t)| &\leq \mu_8\mathcal{L}_{(\mathcal{V}_8,\Psi_8)}\Psi_8, \quad \forall t \in U.
\end{aligned}\tag{42}$$

Definition 4.4. Our fractal–fractional Monkeypox model (2) is Hyers–Ulam–Rassias stable with respect to the operators Ψ_i if and only there is a positive $\mathcal{L}_{(\mathcal{V}_i,\Psi_i)} \in \mathbb{R}$ such that for every $(S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*) \in \mathbb{H}$ that satisfies

$$\begin{aligned}
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_1,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_2(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_2,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_3(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_3,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_4(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_4,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_5(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_5,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_6(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_6,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_7(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_7,\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2}S_w^*(t) - \mathcal{V}_8(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \Psi_8,
\end{aligned}\tag{43}$$

then there exist $(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v) \in \mathbb{H}$ which is a solution to the fractal–fractional monkeypox model (2) with

$$\begin{aligned}
|S_w^*(t) - S_w(t)| &\leq \mathcal{L}_{(\mathcal{V}_1,\Psi_1)}\Psi_1,\\
|E_w^*(t) - E_w(t)| &\leq \mathcal{L}_{(\mathcal{V}_2,\Psi_2)}\Psi_2,\\
|I_w(t) - I_w^*(t)| &\leq \mathcal{L}_{(\mathcal{V}_3,\Psi_3)}\Psi_3,\\
|J_w^*(t) - J_w(t)| &\leq \mathcal{L}_{(\mathcal{V}_4,\Psi_4)}\Psi_4,\\
|Q_w^*(t) - Q_w(t)| &\leq \mathcal{L}_{(\mathcal{V}_5,\Psi_5)}\Psi_5,\\
|R_w^*(t) - R_w(t)| &\leq \mathcal{L}_{(\mathcal{V}_6,\Psi_6)}\Psi_6,\\
|S_v^*(t) - S_v(t)| &\leq \mathcal{L}_{(\mathcal{V}_7,\Psi_7)}\Psi_7,\\
|I_v^*(t) - S_v(t)| &\leq \mathcal{L}_{(\mathcal{V}_8,\Psi_8)}\Psi_8.
\end{aligned}\tag{44}$$

It is again important to note that definition (3.4) is obtained from definition (3.3), then we say that definition (3.3) established the Ulam-Hyers stability criterion of solutions.

Remark 4.2. It is further observed that $(S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*) \in \mathbb{H}$ is a solution for (3.2) if and only if $m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8 \in C([0,T], \mathbb{R})$ (depending on $(S_w^*, E_w^*, I_w^*, J_w^*, Q_w^*, R_w^*, S_v^*, I_v^*)$, respectively) such that $\forall t \in U$.
(a) $|m_i(t)| < \mu_i \Psi_i(\varphi)$ where $i = 1, 2, 3, \dots, 8$,
(b) We have

$$
\begin{aligned}
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_2(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_3(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_4(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_5(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_6(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_7(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),\\
&|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w^*(t) = \mathcal{V}_8(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t).
\end{aligned}
\tag{45}
$$

Given the definitions and remarks above, we discuss establishing the Ulam-Hyers of our fractal–fractional Monkeypox model.

Theorem 4.10. *Given that the assumption* $(Z1)$ *in lemma (3.1) is satisfied, then the given fractal–fractional Monkeypox model on* $U := [0,T]$ *and also is generalized Ulam-Hyers stable such that*

$$
\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_1 < 1, \qquad i = 1, 2, 3, \dots, 8,
$$

Where γ_i *are already defined in lemma (3.1).*

Proof. Let μ_1 be positive and $S_w^* \in \mathbb{X}$ be arbitrary such that

$$
|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1} S_w^*(t) - \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1.
$$

It is then observed that from Remarks (6.3), we can find the operator $m_1(t)$ that satisfies.

$$
|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1} S_w^*(t) = \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),
$$

where $|m_1(t)| \le \mu_1$. IT then follows that;

$$
\begin{aligned}
S_w^*(t) = S_{w0} &+ \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_1(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
&+ \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1} m_1(\varphi)d\varphi.
\end{aligned}
$$

Now by the uniqueness theorem, let us assume $S_w \in \mathbb{X}$ be the only solution of our model understudy, that is, the fractal–fractional monkeypox model. Then we define $S_w(t)$ by;

$$
S_w^*(t) = S_{w0} + \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_1(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,
$$

Which can further be given as

$$
\begin{aligned}
|S_w^*(t) - S_w(t)| \le{}& \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}|m_1(\varphi)|d\varphi + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1},\\
&\times |\mathcal{V}_1(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi)),\\
&- \mathcal{V}_1(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))|d\varphi,\\
&\le \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\mu_1 + \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_1\|S_w^* - S_w\|.
\end{aligned}
$$

This leads to

$$
\|S_w^* - S_w\| \le \frac{\frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\mu_1}{\frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_1}.
$$

If we then let $\mathcal{L}_{\mathcal{V}_i} = \frac{\frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\mu_1}{\frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_1}$, then we say that $\|S_w^* - S_w\| \leq \mathcal{L}_{\mathcal{V}_1}\mu_1$, similarly we have the following results for the other state variables as;

$$
\begin{aligned}
\|E_w^* - E_w\| &\leq \mathcal{L}_{\mathcal{V}_2}\mu_2,\\
\|I_w^* - I_w\| &\leq \mathcal{L}_{\mathcal{V}_3}\mu_3,\\
\|J_w^* - J_w\| &\leq \mathcal{L}_{\mathcal{V}_4}\mu_4,\\
\|Q_w^* - Q_w\| &\leq \mathcal{L}_{\mathcal{V}_5}\mu_5,\\
\|R_w^* - R_w\| &\leq \mathcal{L}_{\mathcal{V}_6}\mu_6,\\
\|S_v^* - S_v\| &\leq \mathcal{L}_{\mathcal{V}_7}\mu_7,\\
\|I_v^* - I_v\| &\leq \mathcal{L}_{\mathcal{V}_8}\mu_8,
\end{aligned}
\tag{46}
$$

where

$$
\frac{\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\mu_i}{\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_i}, \quad i = 2, 3, \dots, 8.
$$

This, therefore, suffices that the Ulam-Hyers stability criterion of the fractal–fractional monkeypox model is well fulfilled. By further assuming that

$$
\mathcal{L}_{\mathcal{V}_i} = \frac{\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\mu_i}{\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_i}, \quad i = 1, 2, \dots, 8.
$$

Where $\mathcal{L}_{\mathcal{V}_i}(0) = 0$, then the generalized Hyers–Ulam stability criterion of the fractal–fractional monkey model is fully established. In so doing, the Ulam–Hyers–Rassias stability of our fractal–fractional monkeypox model is studied this way. □

Theorem 4.11. *Let us suppose that the condition* $(Z1)$ *is true and* $(Z2)$*, there exist some increasing functions* $\Psi_i \in C([0,T], \mathbb{R}^+)$ *where* $i = 1, 2, 3, \dots, 8$ *and there is* $\chi_{\Psi_i} > 0$ *provided that*

$$
|{}^{FFP}\mathcal{J}_{0,t}^{\eta_1}\Phi^*(t) < \chi_{\Psi_i}(t), \forall t \in U,\ i = 1, 2, 3, \dots, 8.
$$

Then our fractal–fractional monkeypox model is said to be Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stable.

Proof. Now, for all μ_1 be positive and $S_w^* \in \mathbb{X}$ that satisfies

$$
|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1}S_w^*(t) - \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| < \mu_1\Psi_1(t).
$$

It is further observed that from Remarks (6.3), we can find the operator $m_1(t)$ that satisfies

$$
|{}^{FFP}\mathcal{D}_{0,t}^{\eta_1}S_w^*(t) = \mathcal{V}_1(t, S_w^*(t), E_w^*(t), I_w^*(t), J_w^*(t), Q_w^*(t), R_w^*(t), S_v^*(t), I_v^*(t))| + m_1(t),
$$

where $|m_1(t)| \leq \mu_1\Psi_1(t)$. It then follows that;

$$
\begin{aligned}
S_w^*(t) = S_{u0} &+ \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_1(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,\\
&+ \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}m_1(\varphi)d\varphi.
\end{aligned}
$$

It is observed that by the uniqueness theorem, we can assume that $S_w \in \mathbb{X}$ be the only solution of our model understudy, that is, the fractal–fractional monkeypox model. Then we define $S_w(t)$ by;

$$
S_w^*(t) = S_{u0} + \int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\mathcal{V}_1(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi))d\varphi,
$$

which can further be obtained as

$$
\begin{aligned}
|S_w^*(t) - S_w(t)| \leq{} & \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}|m_1(\varphi)|d\varphi + \frac{\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1},\\
& \times |\mathcal{V}_1(\varphi, S_w^*(\varphi), E_w^*(\varphi), I_w^*(\varphi), J_w^*(\varphi), Q_w^*(\varphi), R_w^*(\varphi), S_v^*(\varphi), I_v^*(\varphi)),\\
& - \mathcal{V}_1(\varphi, S_w(\varphi), E_w(\varphi), I_w(\varphi), J_w(\varphi), Q_w(\varphi), R_w(\varphi), S_v(\varphi), I_v(\varphi))|d\varphi,\\
& \leq \frac{\mu_1\eta_2}{\Gamma(\eta_1)}\int_0^t \varphi^{\eta_2-1}(t-\varphi)^{\eta_1-1}\Psi_1(\varphi)d\varphi + \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_1\|S_w^* - S_w\|,\\
& \leq \mu_1\chi_{\Psi_1}\Psi_1(t) + \frac{\eta_2 M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_1\|S_w^* - S_w\|.
\end{aligned}
$$

We therefore have;

$$
\|S_w^* - S_w\| \leq \frac{\mu_1\chi_{\Psi_1}\Psi_1(t)}{1 - \frac{\eta_2 M^{\eta_2+\eta_1-1}\Gamma(\eta_2)}{\Gamma(\eta_2+\eta_1)}}.
$$

Table 1
Model parameters Values.

Parameters	Description	Value	Source
ζ_h	Rate of Recruitment into the Susceptible Human Compartment	64850	[51]
ζ_r	Rate of Recruitment into the Susceptible Rodents Compartment	0.2	[51]
σ_u	Natural death Rate of Human	0.000303	[51]
σ_v	Natural death Rate of Rodents	0.00200	[51,75]
ϱ_1	Rodents to Humans Contact Rate	0.052466	[51]
ϱ_2	Humans to Humans Contact Rate	0.022325	[51]
ϱ_3	Rodents to Rodents Contact Rate	0.012458	[51]
ι	Rate of Movement from Quarantine	0.003286	[51]
υ	Proportion of Persons without infection after Quarantine	$0 < \varrho_3 < 1$	Assumed
ϕ	Proportion of Infected Persons in Isolation	$0 < \upsilon < 1$	Assumed
λ_h	monkeypox related death rate in humans	0.003286	[51]
κ	Recovery Rate from Infectious Compartment	0.0088366	[51]
μ	Proportion of exposed persons in Quarantine	$0 < \mu < 1$	Assumed
π	Recovery rate of Persons that have received Treatment during Isolation	0.036246	[51]
β	Rate of Movement from Exposed Compartment	0.016744	[51]

If we then let $\mathcal{L}_{(\mathcal{V}_i,\Psi_i)} \leq \frac{\mu_1 \chi_{\Psi_1} \Psi_1(t)}{1-\frac{\eta_2 M^{\eta_2+\eta_1-1}\Gamma(\eta_2)}{\Gamma(\eta_2+\eta_1)}}$, then we say that $\|S_w^* - S_w\| \leq \mathcal{L}_{(\mathcal{V}_1,\Psi_1)}\mu_1\Psi_1$, similarly we have the following results for the other state variables as;

$$
\begin{aligned}
\|E_w^* - E_w\| &\leq \mathcal{L}_{(\mathcal{V}_2,\Psi_2)}\mu_2\Psi_2,\\
\|I_w^* - I_w\| &\leq \mathcal{L}_{(\mathcal{V}_3,\Psi_3)}\mu_3\Psi_3,\\
\|J_w^* - J_w\| &\mathcal{L}_{(\mathcal{V}_4,\Psi_4)}\mu_4\Psi_4,\\
\|Q_w^* - Q_w\| &\leq \mathcal{L}_{(\mathcal{V}_5,\Psi_5)}\mu_5\Psi_5,\\
\|R_w^* - R_w\| &\leq \mathcal{L}_{(\mathcal{V}_6,\Psi_6)}\mu_6\Psi_6,\\
\|S_v^* - S_v\| &\leq \mathcal{L}_{(\mathcal{V}_7,\Psi_7)}\mu_7\Psi_7,\\
\|I_v^* - I_v\| &\leq \mathcal{L}_{(\mathcal{V}_8,\Psi_8)}\mu_8\Psi_8,
\end{aligned}
\tag{47}
$$

where

$$
\frac{\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\mu_i}{\frac{M^{\eta_1+\eta_2-1}\Gamma(\eta_2)}{\Gamma(\eta_1+\eta_2)}\gamma_i}, \quad i = 2, 3, \ldots, 8.
$$

In conclusion, this suffices that the fractal–fractional monkeypox model is a Hyers–Ulam–Rassias stable. Giving further that μ_i where $i = 1, 2, 3, \ldots, 8$, we say that the fractal–fractional monkeypox model is a generalized Hyers–Ulam–Rassias stable. This completes the stability analysis. □

5. Numerical scheme

This section concentrates on deriving an approximate solution to the fractal–fractional monkeypox model. This goal describes a numerical scheme through Newton's polynomial approach; see [76,77] for more details. The numerical scheme applies new differential and integral operators, substituting the classical differential operator with the Mittag-Leffler kernel.

$$
\begin{aligned}
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_w(t) &= \zeta_h - \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - \sigma_u S_w + \iota v Q_w,\\
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} E_w(t) &= \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - (\beta + \sigma_u) E_w,\\
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} I_w(t) &= \beta(1-\mu)E_w + \delta(1-\nu)Q_w - (\sigma_u + \lambda_h + \kappa)I_w,\\
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} J_w(t) &= \kappa\phi I_w - (\sigma_u + \lambda_h + \pi)J_w,\\
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} Q_w(t) &= \beta\mu E_w - (\iota + \sigma_u)Q_w,\\
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} R_w(t) &= \kappa(1-\phi)I_w + \pi J_w - \sigma_u R_w,\\
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} S_v(t) &= \zeta_r - \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v S_v,\\
{}^{FFM}\mathcal{D}_{0,t}^{\eta_1,\eta_2} I_v(t) &= \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v I_v.
\end{aligned}
\tag{48}
$$

We then reformulate the model in the form

$$
\begin{aligned}
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} S_w(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_1(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} E_w(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_2(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),
\end{aligned}
$$

$$
\begin{aligned}
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} I_w(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_3(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} J_w(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_4(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} Q_w(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_5(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} R_w(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_6(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} S_v(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_7(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),\\
{}^{RL}\mathcal{D}_{0,t}^{\eta_1} Ir(t) &= \eta_2 t^{\eta_2-1}\mathcal{V}_8(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)),
\end{aligned}
\tag{49}
$$

noting that;

$$
\begin{aligned}
&\mathcal{V}_1(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \zeta_h - \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - \sigma_u S_w + \iota\nu Q_w,\\
&\mathcal{V}_2(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \frac{\varrho_1 I_v + \varrho_2 I_w}{N_u} S_w - (\beta + \sigma_u) E_w,\\
&\mathcal{V}_3(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \beta(1-\mu)E_w + \delta(1-\nu)Q_w - (\sigma_u + \lambda_h + \kappa) I_w,\\
&\mathcal{V}_4(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \kappa\phi I_w - (\sigma_u + \lambda_h + \pi) J_w,\\
&\mathcal{V}_5(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \beta\mu E_w - (\iota + \sigma_u) Q_w,\\
&\mathcal{V}_6(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \kappa(1-\phi) I_w + \pi J_w - \sigma_u R_w,\\
&\mathcal{V}_7(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \zeta_r - \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v S_v,\\
&\mathcal{V}_8(t, S_w(t), E_w(t), I_w(t), J_w(t), Q_w(t), R_w(t), S_v(t), I_v(t)) = \frac{\varrho_3 S_v I_v}{N_v} - \sigma_v I_v.
\end{aligned}
\tag{50}
$$

The above system is reformulated in a fractal–fractional integral form with a Mittag-Leffler kernel; this leads to;

$$
\begin{aligned}
S_w(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} S_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} S_{h1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau,\\
E_w(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} E_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} E_{h1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau,\\
I_w(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} I_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} I_{h1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau,\\
J_w(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} J_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} J_{h1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau,\\
Q_w(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} Q_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} Q_{h1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau,\\
R_w(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} R_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} Q_{h1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau,\\
S_v(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} S_{r1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} S_{r1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau,\\
I_v(t_n+1) &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} I_{r1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} I_{r1}(S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v)\tau^{1-\eta_2}(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned}
\tag{51}
$$

Let us recall that Newton's polynomial is given as;

$$
\begin{aligned}
\mathcal{V}(t, S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v) \approxeq\ & \mathcal{V}(t_{n-2}, S_{h(n-2)}, E_{h(n-2)}, I_{h(n-2)}, J_{h(n-2)}, Q_{h(n-2)}, R_{h(n-2)}, S_{r(n-2)}, I_{r(n-2)}),\\
&+ \frac{1}{\nabla t}[\mathcal{V}(t_{n-1}, S_{h(n-1)}, E_{h(n-1)}, I_{h(n-1)}, J_{h(n-1)}, Q_{h(n-1)}, R_{h(n-1)}, S_{r(n-1)}, I_{r(n-1)}),\\
&- \mathcal{V}(t_{n-2}, S_{h(n-2)}, E_{h(n-2)}, I_{h(n-2)}, J_{h(n-2)}, Q_{h(n-2)}, R_{h(n-2)}, S_{r(n-2)}, I_{r(n-2)})](\tau - t_{n-2})
\end{aligned}
$$

$$\begin{aligned}
&+\ \frac{1}{2\nabla t^2}\mathcal{V}[(t, S_w, E_w, I_w, J_w, Q_w, R_w, S_v, I_v),\\
&-\ 2\mathcal{V}(t_{n-1}, S_{h(n-1)}, E_{h(n-1)}, I_{h(n-1)}, J_{h(n-1)}, Q_{h(n-1)}, R_{h(n-1)}, S_{r(n-1)}, I_{r(n-1)}),\\
&-\ \mathcal{V}(t_{n-2}, S_{h(n-2)}, E_{h(n-2)}, I_{h(n-2)}, J_{h(n-2)}, Q_{h(n-2)}, R_{h(n-2)}, S_{r(n-2)}, I_{r(n-2)})],\\
&\times\ (\tau - t_{n-2})(\tau - t_{n-1}).
\end{aligned}\tag{52}$$

we then substitute Eq. (52) into Eq. (51), this yields;

$$\begin{aligned}
S_w^{n+1} = {} & \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} S_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
& + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} S_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})\tau^{1\eta_2}(t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2},\\
& \times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t}[t_{j-1}^{1-\eta_1} S_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& - t_{j-1}^{1-\eta_1} S_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
& \times \sum_{j=2}^{n} \frac{1}{2\nabla t^2}[t_j^{1-\eta_2} S_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} S_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& + t_{j-1}^{1-\eta_2} S_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned}\tag{53}$$

$$\begin{aligned}
E_w^{n+1} = {} & \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} E_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
& + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} E_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})\tau^{1\eta_2}(t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2},\\
& \times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t}[t_{j-1}^{1-\eta_1} E_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& - t_{j-1}^{1-\eta_1} S_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
& \times \sum_{j=2}^{n} \frac{1}{2\nabla t^2}[t_j^{1-\eta_2} E_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} E_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& + t_{j-1}^{1-\eta_2} S_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned}\tag{54}$$

$$\begin{aligned}
I_w^{n+1} = {} & \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} I_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
& + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} I_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})\tau^{1\eta_2}(t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2},\\
& \times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t}[t_{j-1}^{1-\eta_1} I_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& - t_{j-1}^{1-\eta_1} I_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
& \times \sum_{j=2}^{n} \frac{1}{2\nabla t^2} \times [t_j^{1-\eta_2} I_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} S_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& + t_{j-1}^{1-\eta_2} I_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned}\tag{55}$$

$$\begin{aligned}
J_w^{n+1} = {} & \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} J_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
& + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} J_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})\tau^{1\eta_2}(t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2},\\
& \times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t}[t_{j-1}^{1-\eta_1} J_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& - t_{j-1}^{1-\eta_1} J_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
& \times \sum_{j=2}^{n} \frac{1}{2\nabla t^2}[t_j^{1-\eta_2} J_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} S_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& + t_{j-1}^{1-\eta_2} J_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned}\tag{56}$$

$$\begin{aligned}
Q_w^{n+1} &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} Q_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} Q_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2}) \tau^{1\eta_2} (t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2}\\
&\times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t} [t_{j-1}^{1-\eta_1} Q_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&- t_{j-1}^{1-\eta_1} Q_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
&\times \sum_{j=2}^{n} \frac{1}{2\nabla t^2} [t_j^{1-\eta_2} Q_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} Q_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&+ t_{j-1}^{1-\eta_2} Q_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned} \tag{57}$$

$$\begin{aligned}
R_w^{n+1} &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} R_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} R_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2}) \tau^{1\eta_2} (t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2},\\
&\times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t} [t_{j-1}^{1-\eta_1} R_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&- t_{j-1}^{1-\eta_1} R_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
&\times \sum_{j=2}^{n} \frac{1}{2\nabla t^2} [t_j^{1-\eta_2} R_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} R_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&+ t_{j-1}^{1-\eta_2} R_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned} \tag{58}$$

$$\begin{aligned}
S_v^{n+1} &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} S_{r1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} S_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2}) \tau^{1\eta_2} (t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2},\\
&\times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t} [t_{j-1}^{1-\eta_1} S_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&- t_{j-1}^{1-\eta_1} S_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
&\times \sum_{j=2}^{n} \frac{1}{2\nabla t^2} [t_j^{1-\eta_2} S_{r1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} S_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&+ t_{j-1}^{1-\eta_2} S_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned} \tag{59}$$

$$\begin{aligned}
I_v^{n+1} &= \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} I_{r1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} I_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2}) \tau^{1\eta_2} (t_{n+1}-\tau)^{\eta_1-1} t_{j-2}^{1-\eta_2},\\
&\times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)} \sum_{j=2}^{n} \frac{1}{\nabla t} [t_{j-1}^{1-\eta_1} I_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&- t_{j-1}^{1-\eta_1} I_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau + \frac{\eta_1}{AB(\eta_1)\Gamma(\eta_1)},\\
&\times \sum_{j=2}^{n} \frac{1}{2\nabla t^2} [t_j^{1-\eta_2} I_{r1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) + 2t_{j-1}^{1-\eta_2} I_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&+ t_{j-1}^{1-\eta_2} I_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})] \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1}-\tau)^{\eta_1-1} d\tau.
\end{aligned} \tag{60}$$

The integrals in the above equations are then evaluated as;

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\eta_1-1} d\tau &= \frac{(\nabla t)^{\eta_1}}{\eta_1} [(n-j+)^{\eta_1} - (n-j)^{\eta_1}],\\
\int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1}-\tau)^{\eta_1-1} d\tau &= \frac{(\nabla t)^{\eta_1+1}}{\eta_1(\eta_1+1)} [(n-j+1)^{\eta_1}(n-j+3+2\eta_1) - (n-j)^{\eta_1}(n-j+3+3\eta_1)],
\end{aligned}$$

$$\begin{aligned}\int_{t_j}^{t_{j+1}}(\tau-t_{j-2})(\tau-t_{j-1})(t_{n+1}-\tau)^{\eta_1-1}d\tau=\frac{(\nabla t)^{\eta_1+2}}{\eta_1(\eta_1+1)(\eta_1+2)}&[(n-j+1)^{\eta_1}(2(n-v)^2+(3\eta_1+10)(n-j)+2\eta_1^2+9\eta_1+12),\\ &-(n-j)^{\eta_1}(2(n-j)^2+(5\eta_1+10)(n-v)+6\eta_1^2+18\eta_1+12)].\end{aligned}\tag{61}$$

We then substitute Eq. (61) into Eqs. (53)–(60) and that yields;

$$\begin{aligned}S_w^{n+1}=&\frac{1-\eta_1}{AB(\eta_2)}t_n^{1-\eta_2}S_{h1}(t_n,S_w(t_n),E_w(t_n),I_w(t_n),J_w(t_n),Q_w(t_n),R_w(t_n),S_v(t_n),I_v(t_n)),\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)}\sum_{j=2}^{n}t_{j-2}^{1-\eta_2}S_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})[(n-j+1)^{\eta_1}-(n-j)^{\eta_1}],\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)}\sum_{j=2}^{n}[t_{j-1}^{1-\eta_2}S_{h1}(t_{j-1},S_w^{j-1},E_w^{j-1},I_w^{j-1},J_w^{j-1},Q_w^{j-1},R_w^{j-1},S_v^{j-1},I_v^{j-1}),\\ &-t_{j-2}^{1-\eta_2}S_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})][(n-j+1)^{\eta_1},\\ &\times(n-j+3+2\eta_1)-(n-j)^{\eta_1}(n-j+3+3\eta_1)]+\frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)}\sum_{j=2}^{n}[t_j^{1-\eta_2}S_{h1}(t_j,S_w^j,E_w^j,I_w^j,J_w^j,Q_w^j,R_w^j,S_v^j,I_v^j),\\ &-2t_{j-1}^{1-\eta_2}S_{h1}(t_{j-1},S_w^{j-1},E_w^{j-1},I_w^{j-1},J_w^{j-1},Q_w^{j-1},R_w^{j-1},S_v^{j-1},I_v^{j-1})+t_{j-2}^{1-\eta_2},\\ &\times S_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})],\\ &\times[(n-v+1)^{\eta_1}\{2(n-j)^2+(3\eta_1+10)(n-j)+2\eta_1^2+9\eta_1+12\}-(n-j)^{\eta_1}\{2(n-j)^2,\\ &+(5\eta_1+10)(n-j)+6\eta_1^2+18\eta_1+12\}].\end{aligned}\tag{62}$$

$$\begin{aligned}E_w^{n+1}=&\frac{1-\eta_1}{AB(\eta_2)}t_n^{1-\eta_2}E_{h1}(t_n,S_w(t_n),E_w(t_n),I_w(t_n),J_w(t_n),Q_w(t_n),R_w(t_n),S_v(t_n),I_v(t_n)),\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)}\sum_{j=2}^{n}t_{j-2}^{1-\eta_2}E_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})[(n-j+1)^{\eta_1}-(n-j)^{\eta_1}],\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)}\sum_{j=2}^{n}[t_{j-1}^{1-\eta_2}E_{h1}(t_{j-1},S_w^{j-1},E_w^{j-1},I_w^{j-1},J_w^{j-1},Q_w^{j-1},R_w^{j-1},S_v^{j-1},I_v^{j-1}),\\ &-t_{j-2}^{1-\eta_2}E_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})][(n-j+1)^{\eta_1}(n-j+3+2\eta_1)-(n-j)^{\eta_1},\\ &\times(n-j+3+3\eta_1)]+\frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)}\sum_{j=2}^{n}[t_j^{1-\eta_2}E_{h1}(t_j,S_w^j,E_w^j,I_w^j,J_w^j,Q_w^j,R_w^j,S_v^j,I_v^j)-2t_{j-1}^{1-\eta_2},\\ &\times E_{h1}(t_{j-1},S_w^{j-1},E_w^{j-1},I_w^{j-1},J_w^{j-1},Q_w^{j-1},R_w^{j-1},S_v^{j-1},I_v^{j-1})+t_{j-2}^{1-\eta_2},\\ &\times E_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})],\\ &\times[(n-v+1)^{\eta_1}\{2(n-j)^2+(3\eta_1+10)(n-j)+2\eta_1^2+9\eta_1+12\}-(n-j)^{\eta_1}\{2(n-j)^2,\\ &+(5\eta_1+10)(n-j)+6\eta_1^2+18\eta_1+12\}].\end{aligned}\tag{63}$$

$$\begin{aligned}I_w^{n+1}=&\frac{1-\eta_1}{AB(\eta_2)}t_n^{1-\eta_2}I_{h1}(t_n,S_w(t_n),E_w(t_n),I_w(t_n),J_w(t_n),Q_w(t_n),R_w(t_n),S_v(t_n),I_v(t_n)),\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)}\sum_{j=2}^{n}t_{j-2}^{1-\eta_2}I_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})[(n-j+1)^{\eta_1}-(n-j)^{\eta_1}],\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)}\sum_{j=2}^{n}[t_{j-1}^{1-\eta_2}I_{h1}(t_{j-1},S_w^{j-1},E_w^{j-1},I_w^{j-1},J_w^{j-1},Q_w^{j-1},R_w^{j-1},S_v^{j-1},I_v^{j-1}),\\ &-t_{j-2}^{1-\eta_2}I_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})][(n-j+1)^{\eta_1},\\ &\times(n-j+3+2\eta_1)-(n-j)^{\eta_1}(n-j+3+3\eta_1)],\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)}\sum_{j=2}^{n}[t_j^{1-\eta_2}I_{h1}(t_j,S_w^j,E_w^j,I_w^j,J_w^j,Q_w^j,R_w^j,S_v^j,I_v^j)-2t_{j-1}^{1-\eta_2},\\ &\times I_{h1}(t_{j-1},S_w^{j-1},E_w^{j-1},I_w^{j-1},J_w^{j-1},Q_w^{j-1},R_w^{j-1},S_v^{j-1},I_v^{j-1})+t_{j-2}^{1-\eta_2},\\ &\times I_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})],\\ &\times[(n-v+1)^{\eta_1}\{2(n-j)^2+(3\eta_1+10)(n-j)+2\eta_1^2+9\eta_1+12\}-(n-j)^{\eta_1}\{2(n-j)^2,\\ &+(5\eta_1+10)(n-j)+6\eta_1^2+18\eta_1+12\}].\end{aligned}\tag{64}$$

$$\begin{aligned}J_w^{n+1}=&\frac{1-\eta_1}{AB(\eta_2)}t_n^{1-\eta_2}J_{h1}(t_n,S_w(t_n),E_w(t_n),I_w(t_n),J_w(t_n),Q_w(t_n),R_w(t_n),S_v(t_n),I_v(t_n)),\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)}\sum_{j=2}^{n}t_{j-2}^{1-\eta_2}J_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})[(n-j+1)^{\eta_1}-(n-j)^{\eta_1}],\\ &+\frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)}\sum_{j=2}^{n}[t_{j-1}^{1-\eta_2}J_{h1}(t_{j-1},S_w^{j-1},E_w^{j-1},I_w^{j-1},J_w^{j-1},Q_w^{j-1},R_w^{j-1},S_v^{j-1},I_v^{j-1}),\\ &-t_{j-2}^{1-\eta_2}J_{h1}(t_{j-2},S_w^{j-2},E_w^{j-2},I_w^{j-2},J_w^{j-2},Q_w^{j-2},R_w^{j-2},S_v^{j-2},I_v^{j-2})][(n-j+1)^{\eta_1},\end{aligned}\tag{65}$$

$$
\begin{aligned}
&\times (n-j+3+2\eta_1) - (n-j)^{\eta_1}(n-j+3+3\eta_1)],\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)} \sum_{j=2}^{n} [t_j^{1-\eta_2} J_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) - 2t_{j-1}^{1-\eta_2},\\
&\times J_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}) + t_{j-2}^{1-\eta_2},\\
&\times J_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})],\\
&\times [(n-v+1)^{\eta_1}\{2(n-j)^2 + (3\eta_1+10)(n-j) + 2\eta_1^2 + 9\eta_1 + 12\} - (n-j)^{\eta_1}\{2(n-j)^2,\\
&+ (5\eta_1+10)(n-j) + 6\eta_1^2 + 18\eta_1 + 12\}].
\end{aligned}
$$

$$
\begin{aligned}
Q_w^{n+1} = {}& \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} Q_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)} \sum_{j=2}^{n} t_{j-2}^{1-\eta_2} Q_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})[(n-j+1)^{\eta_1} - (n-j)^{\eta_1}],\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)} \sum_{j=2}^{n} [t_{j-1}^{1-\eta_2} Q_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&- t_{j-2}^{1-\eta_2} Q_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})][(n-j+1)^{\eta_1}\\
&\times (n-j+3+2\eta_1) - (n-j)^{\eta_1}(n-j+3+3\eta_1)],\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)} \sum_{j=2}^{n} [t_j^{1-\eta_2} Q_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) - 2t_{j-1}^{1-\eta_2},\\
&\times Q_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}) + t_{j-2}^{1-\eta_2},\\
&\times Q_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})],\\
&\times [(n-v+1)^{\eta_1}\{2(n-j)^2 + (3\eta_1+10)(n-j) + 2\eta_1^2 + 9\eta_1 + 12\} - (n-j)^{\eta_1}\{2(n-j)^2,\\
&+ (5\eta_1+10)(n-j) + 6\eta_1^2 + 18\eta_1 + 12\}].
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
R_w^{n+1} = {}& \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} R_{h1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)} \sum_{j=2}^{n} t_{j-2}^{1-\eta_2} R_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})[(n-j+1)^{\eta_1} - (n-j)^{\eta_1}],\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)} \sum_{j=2}^{n} [t_{j-1}^{1-\eta_2} R_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&- t_{j-2}^{1-\eta_2} R_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})][(n-j+1)^{\eta_1},\\
&\times (n-j+3+2\eta_1) - (n-j)^{\eta_1}(n-j+3+3\eta_1)],\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)} \sum_{j=2}^{n} [t_j^{1-\eta_2} R_{h1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) - 2t_{j-1}^{1-\eta_2},\\
&\times R_{h1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}) + t_{j-2}^{1-\eta_2},\\
&\times R_{h1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})],\\
&\times [(n-v+1)^{\eta_1}\{2(n-j)^2 + (3\eta_1+10)(n-j) + 2\eta_1^2 + 9\eta_1 + 12\} - (n-j)^{\eta_1}\{2(n-j)^2,\\
&+ (5\eta_1+10)(n-j) + 6\eta_1^2 + 18\eta_1 + 12\}].
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
S_v^{n+1} = {}& \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} S_{r1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)} \sum_{j=2}^{n} t_{j-2}^{1-\eta_2} S_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})[(n-j+1)^{\eta_1} - (n-j)^{\eta_1}],\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)} \sum_{j=2}^{n} [t_{j-1}^{1-\eta_2} S_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
&- t_{j-2}^{1-\eta_2} S_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})][(n-j+1)^{\eta_1},\\
&\times (n-j+3+2\eta_1) - (n-j)^{\eta_1}(n-j+3+3\eta_1)],\\
&+ \frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)} \sum_{j=2}^{n} [t_j^{1-\eta_2} S_{r1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) - 2t_{j-1}^{1-\eta_2},\\
&\times S_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}) + t_{j-2}^{1-\eta_2},\\
&\times S_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})],\\
&\times [(n-v+1)^{\eta_1}\{2(n-j)^2 + (3\eta_1+10)(n-j) + 2\eta_1^2 + 9\eta_1 + 12\} - (n-j)^{\eta_1}\{2(n-j)^2,\\
&+ (5\eta_1+10)(n-j) + 6\eta_1^2 + 18\eta_1 + 12\}].
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
I_v^{n+1} = {} & \frac{1-\eta_1}{AB(\eta_2)} t_n^{1-\eta_2} I_{r1}(t_n, S_w(t_n), E_w(t_n), I_w(t_n), J_w(t_n), Q_w(t_n), R_w(t_n), S_v(t_n), I_v(t_n)),\\
& + \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+1)} \sum_{j=2}^{n} t_{j-2}^{1-\eta_2} I_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})[(n-j+1)^{\eta_1} - (n-j)^{\eta_1}],\\
& + \frac{\eta_1(\nabla t)^{\eta_1}}{AB(\eta_1)\Gamma(\eta_1+2)} \sum_{j=2}^{n} [t_{j-1}^{1-\eta_2} I_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}),\\
& - t_{j-2}^{1-\eta_2} I_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})][(n-j+1)^{\eta_1},\\
& \times (n-j+3+2\eta_1) - (n-j)^{\eta_1}(n-j+3+3\eta_1)],\\
& + \frac{\eta_1(\nabla t)^{\eta_1}}{2AB(\eta_1)\Gamma(\eta_1+3)} \sum_{j=2}^{n} [t_j^{1-\eta_2} I_{r1}(t_j, S_w^j, E_w^j, I_w^j, J_w^j, Q_w^j, R_w^j, S_v^j, I_v^j) - 2t_{j-1}^{1-\eta_2},\\
& \times I_{r1}(t_{j-1}, S_w^{j-1}, E_w^{j-1}, I_w^{j-1}, J_w^{j-1}, Q_w^{j-1}, R_w^{j-1}, S_v^{j-1}, I_v^{j-1}) + t_{j-2}^{1-\eta_2},\\
& \times I_{r1}(t_{j-2}, S_w^{j-2}, E_w^{j-2}, I_w^{j-2}, J_w^{j-2}, Q_w^{j-2}, R_w^{j-2}, S_v^{j-2}, I_v^{j-2})],\\
& \times [(n-v+1)^{\eta_1}\{2(n-j)^2 + (3\eta_1+10)(n-j) + 2\eta_1^2 + 9\eta_1 + 12\} - (n-j)^{\eta_1}\{2(n-j)^2,\\
& + (5\eta_1+10)(n-j) + 6\eta_1^2 + 18\eta_1 + 12\}].
\end{aligned}
\tag{69}
$$

6. Numerical simulations

In this section, we present the numerical solutions to the fractal–fractional Monkeypox model by means of the numerical scheme discussed above within the fractal dimension $0 < \eta_2 \leq 1$ and a fractional order $0 < \eta_1 \leq 1$. To explicitly exemplify the results in this work, the following initial conditions were used, that is, $S_w(0) = 214026402$, $E_w(0) = 250$, $I_w(0) = 500$, $J_w(0) = 100$, $Q_w(0) = 100$, $R_w(0) = 10$, $S_v(0) = 50000$, $I_v(0) = 3000$ with the parameter values given in Table 1. We thus investigated the dynamics of the Monkeypox disease by choosing diverse fractional and fractal values to see the impact of the fractal–fractional derivative model on the disease. In addition, the work seeks to identify key parameters that significantly influence the dynamics of the Monkeypox disease. We, therefore, conducted a graphical illustration by perturbing some parameter values to see their impact on some compartments and generally on the disease, which will be useful in determining some control strategies for the disease.

In Figs. 1 and 2, we present a graphical representation of the monkeypox fractal–fractional model where we chose several fractal and fractional values but $\eta_1 = \eta_2$. We observe in Fig. Fig. 1(a) - Fig. 1(d) that all trajectories converge towards different limit points and are in a steady state. It is seen that many people become susceptible to the disease at the least fractal–fractional value $\eta_1 = \eta_2 = 0.80$, whereas less number of infections are observed at that same value. We make similar observations in Fig. 2a–Fig. 2(d) as all trajectories are steady. The different values of fractal dimension and fractional orders show the lasting effect of the fractional orders on the dynamics of the disease.

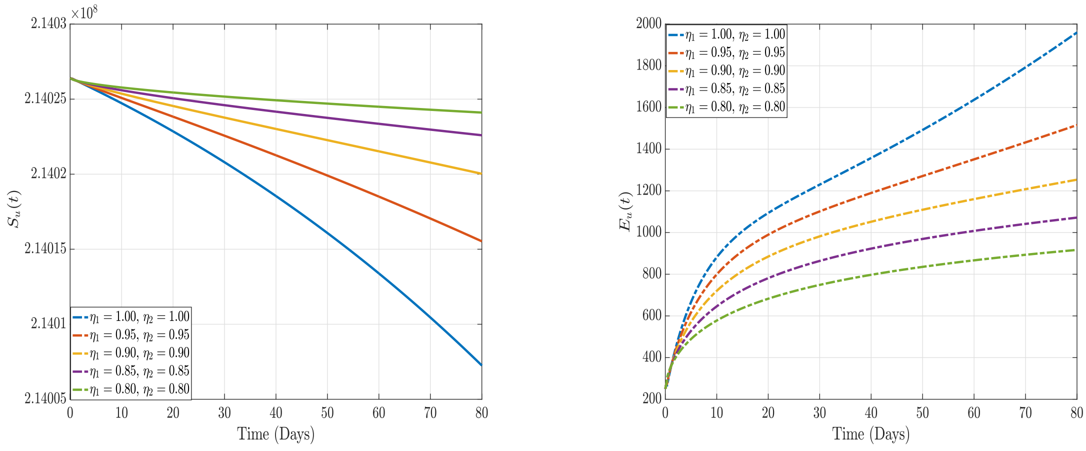
In Figs. 3 and 4, we study the dynamics of the disease at different fractional order values with the same fractal dimension $\eta_1 = 1$. Interestingly, we observe in Fig. 3a that, as we decrease the fractional order from $\eta_1 = 1$ to $eta_1 = 0.95$, susceptibility to the disease declines massively but suddenly starts to rise again as we continue to decrees the fractional order value from $\eta_1 = 0.95$ to $eta_1 = 0.90$ and followed suit with the other fractional values. This observation is seen in the graphs of the other figures, as in Fig. 3b–Fig. 3d, we see the least fractal order value $\eta_1 = 0.80$ trajectory being closer to the integer order than the others. Fig. 4a–Fig. 4d have similar observations as each trajectory has a different limit point in a steady state.

In Fig. 5, We consider different parameter values of the rate of human-to-human contact, σ_2, at $\eta_1 = \eta_2 = 0.8$ to realize its influence on the dynamics of the disease, especially with the Exposed human, Infected human, Isolated human and Quarantined human compartments. It was observed in Fig. 5a–Fig. 5d that, as the rate of human-to-human contact is increased towards unity, many individuals in the ecosystem become exposed to the monkeypox disease, and also, the number of infections appreciates. Consequently, more individuals are isolated due to infections and quarantined as they become exposed to the disease. This indicates that σ_2 significantly affects the dynamics of the disease.

Moreso, in Fig. 6, we again chose values in an increasing order to investigate the effect of the parameter mu, thus, the fraction of quarantined exposed individuals. It is observed in Figs. 6a and 6b that as we increase the parameter value to $\mu = 3$, fewer people become exposed, and similarly, the number of infections reduces. This indicates the significant effect of μ on the disease dynamics.

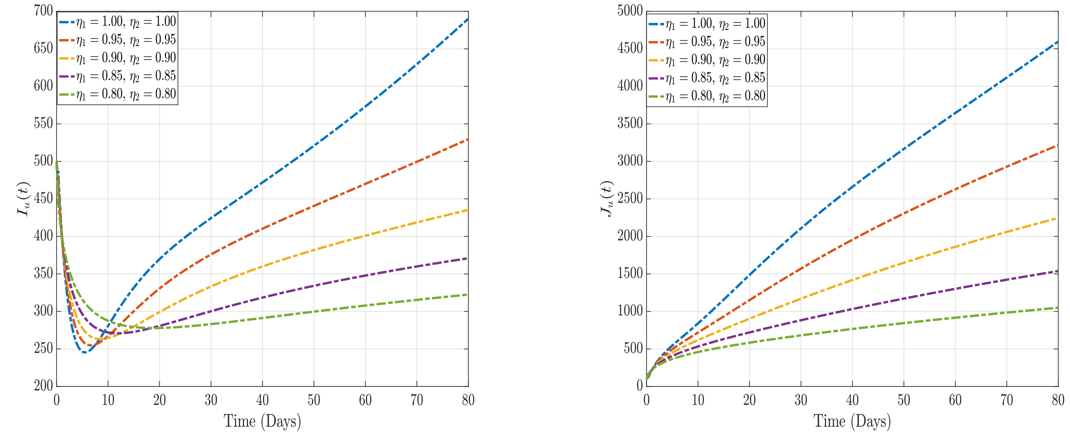
In Fig. 7, we showed the graphical representation of the parameters ζ and κ, that is, the fraction of individuals isolated and the rate of movement from the infected class, respectively. It is again observed that these parameters significantly influence the disease dynamics; as their values increase, fewer individuals in the ecosystem become exposed and infected, as seen in Figs. 7a and 7b.

Finally, in Fig. 8, we further consider how significant human-to-rodent and human-to-human contact rates are in the ecosystem. With no doubt, Figs. 8a–8d shows that, as we perturb the values of these parameters in increasing order, many individuals become exposed and infected, leading to a rise in the isolation and quarantine compartments. This exemplifies that these parameters significantly influence the steady-state nature of the disease.



(a) Fractal-fractional dynamics of the Susceptible Human Compartment $S_w(t)$ at different fractional order η_1 and fractal η_2 values

(b) Fractal-fractional dynamics of the Exposed Human Compartment $E_w(t)$ at different fractional order η_1 and fractal η_2 values



(c) Fractal-fractional dynamics of the Infected Human Compartment $I_w(t)$ at different fractional order η_1 and fractal η_2 values

(d) Fractal-fractional dynamics of the Isolated Human Compartment $J_w(t)$ at different fractional order η_1 and fractal η_2 values

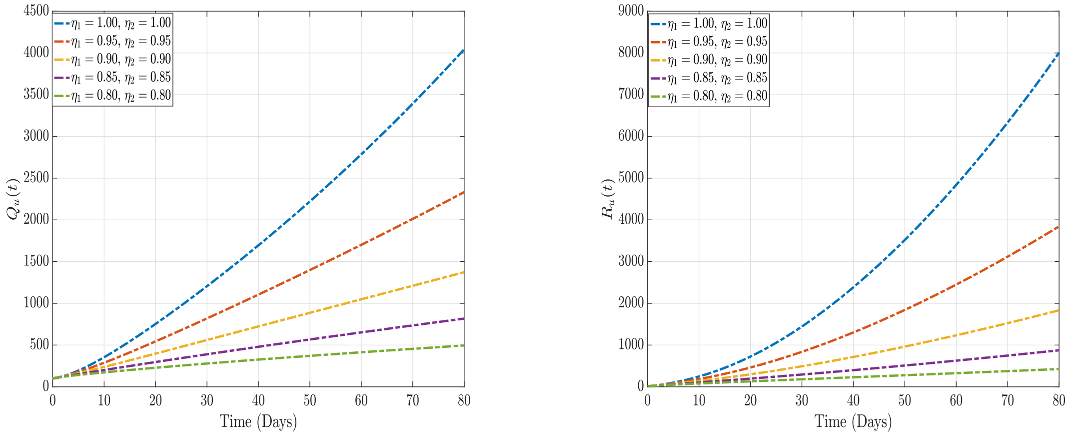
Fig. 1. Fractal–fractional dynamics of the Susceptible Human ($S_w(t)$), the Exposed Human $E_w(t)$, the Infected Human ($I_w(t)$), and the Isolated Human ($J_w(t)$), at different fractal–fractional values.

7. Conclusion

This research work concentrated on the transmission dynamics of monkeypox disease by means of the novel fractal–fractional. In this work, we discussed some physical axioms of the fractal–fractional monkeypox model involving the existence property and uniqueness by means of fixed point theory. Also, the model was shown to be Hyers–Ulam and Hyers–Ulam–Rassias stable. The epidemiological threshold of the model was also established. We further presented an extensive numerical scheme for the fractal–fractional monkeypox model by using the novel Newton's polynomial, and in addition, we showed that the fractal dimension and fractional order have a significant effect on the trajectories of the model. It was observed that many individuals become infected with the monkeypox disease as the fractional order and fractal dimension get very close to unity. In other words, fewer infections of the disease are recorded at lower fractal–fractional values, indicating we will have a Monkey-free state at lower fractal–fractional values. The studies further presented a graphical representation of the influence of some key parameters on the dynamics of the disease; for instance, we realized that the monkeypox disease persists as many individuals become infected when the rate at which human-to-human contact increases. This indicates that individuals exposed to the Monkeypox disease need to be quickly quarantined, and also, those infected should be isolated to control and manage or even eradicate the spread of the disease. Policymakers are therefore encouraged to enhance quarantine and isolation techniques as we seek to eradicate monkeypox. Researchers are invited to conduct an optimal control analysis on the monkeypox disease and the spatial dynamics of the disease using partial differential equations.

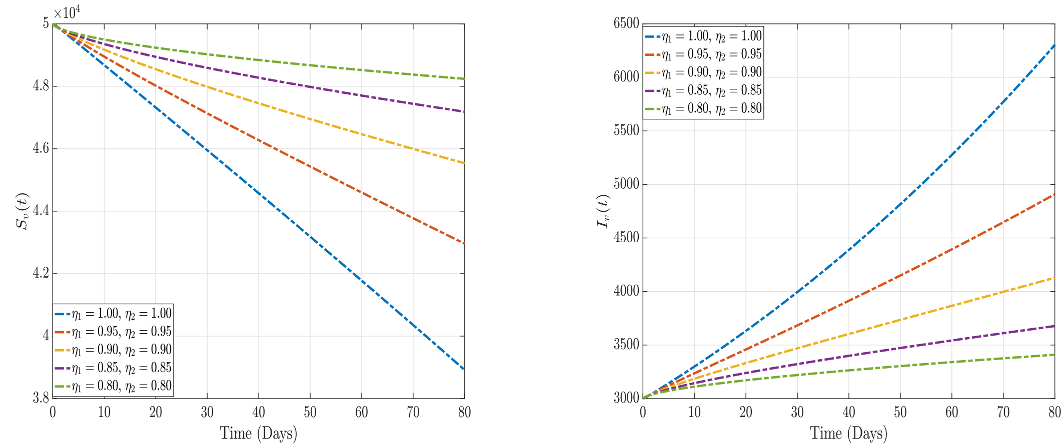
Research funding

None declared.



(a) Fractal-fractional dynamics of the Quarantined Human Compartment $Q_w(t)$ at different fractional order η_1 and fractal η_2 values

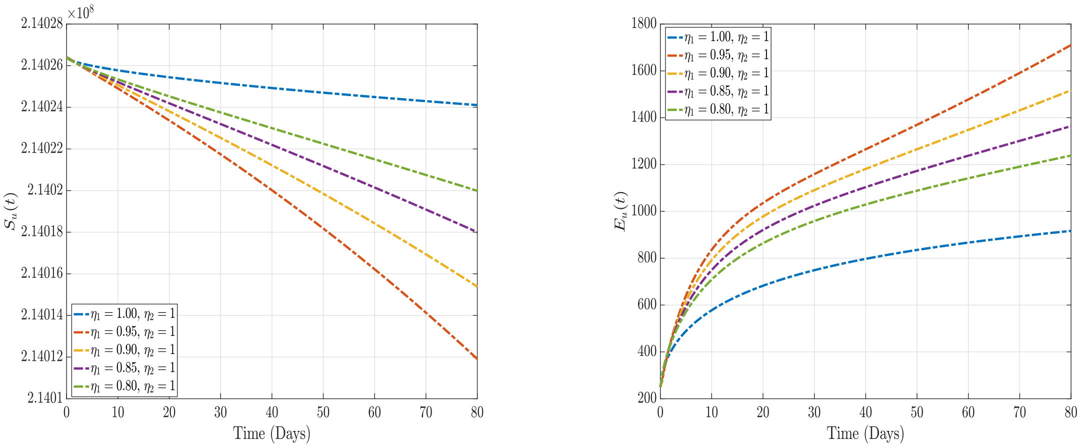
(b) Fractal-fractional dynamics of the Recovered Human Compartment $R_w(t)$ at different fractional order η_1 and fractal η_2 values



(c) Fractal-fractional dynamics of the Susceptible Rodents Compartment $S_v(t)$ at different fractional order η_1 and fractal η_2 values

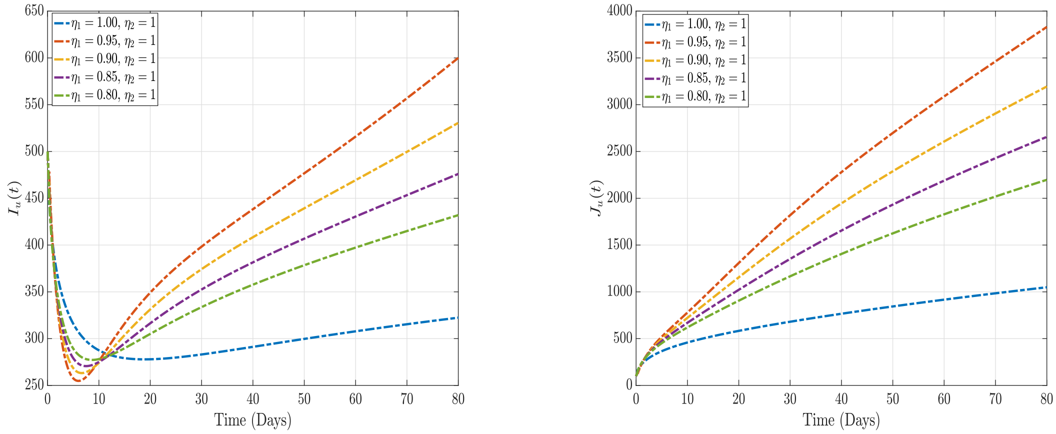
(d) Fractal-fractional dynamics of the Infected Rodents Compartment $I_v(t)$ at different fractional order η_1 and fractal η_2 values

Fig. 2. Fractal–fractional dynamics of the Quarantined Human ($Q_w(t)$), the Recovered Human $R_w(t)$, the Susceptible Rodents ($S_v(t)$), and the Infected Rodents ($I_v(t)$), at different fractal–fractional values.



(a) Fractal-fractional dynamics of the Susceptible Human Compartment $S_w(t)$ at different fractional orders η_1 and a constant fractal $\eta_2 = 1$ value

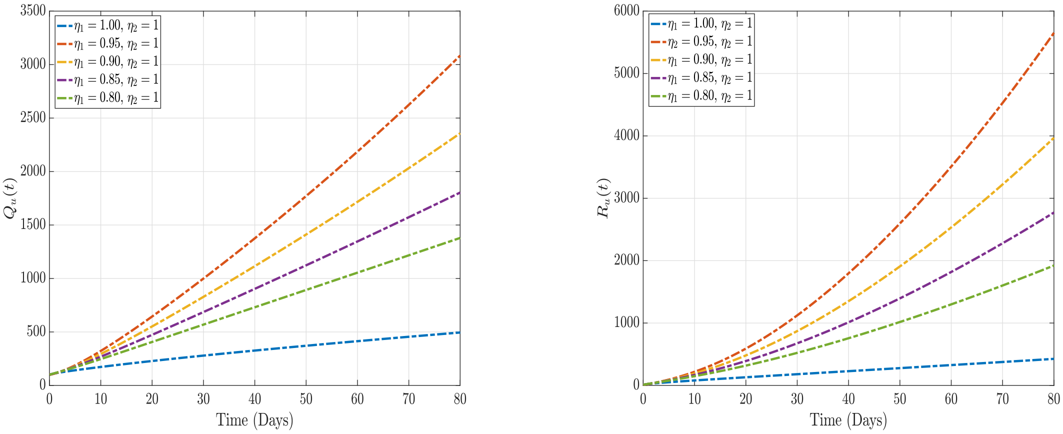
(b) Fractal-fractional dynamics of the Exposed Human Compartment $E_w(t)$ at different fractional orders η_1 and a constant fractal $\eta_2 = 1$ value



(c) Fractal-fractional dynamics of the Infected Human Compartment $I_w(t)$ at different fractional orders η_1 and a constant fractal $\eta_2 = 1$ value

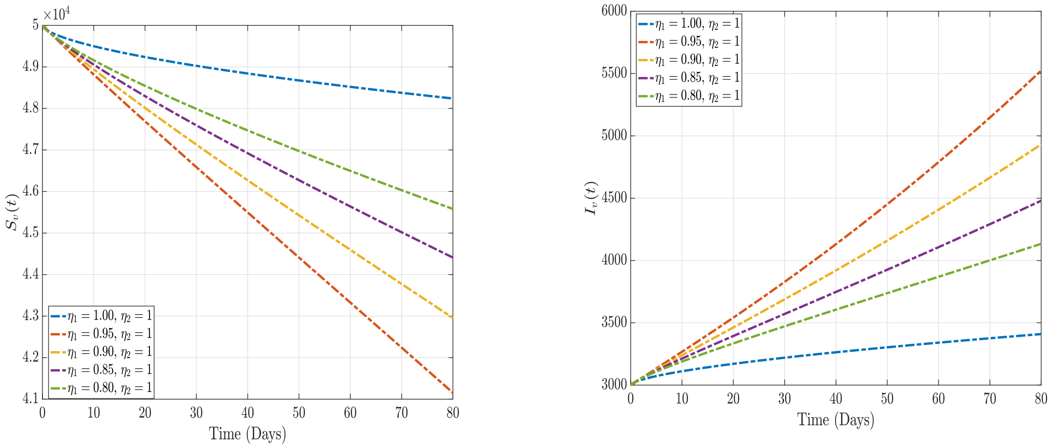
(d) Fractal-fractional dynamics of the Isolated Human Compartment $J_w(t)$ at different fractional orders η_1 and a constant fractal $\eta_2 = 1$ value

Fig. 3. Fractal–fractional dynamics of the Susceptible Human ($S_w(t)$), the Exposed Human $E_w(t)$, the Infected Human ($I_w(t)$), and the Isolated Human ($J_w(t)$), at different fractional values with a constant fractal value $\eta_2 = 1$.



(a) Fractal-fractional dynamics of the Quarantined Human Compartment $Q_w(t)$ at different fractional orders η_1 with a constant fractal $\eta_2 = 1$ value

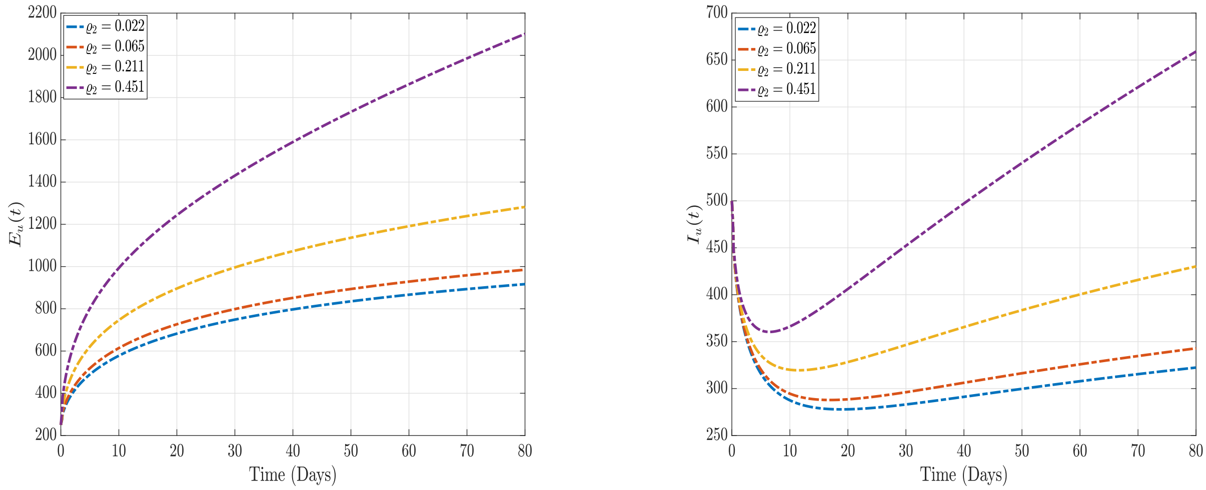
(b) Fractal-fractional dynamics of the Recovered Human Compartment $R_w(t)$ at different fractional orders η_1 with a constant fractal $\eta_2 = 1$ value



(c) Fractal-fractional dynamics of the Susceptible Rodents Compartment $S_v(t)$ at different fractional orders η_1 with a fractal $\eta_2 = 1$ value

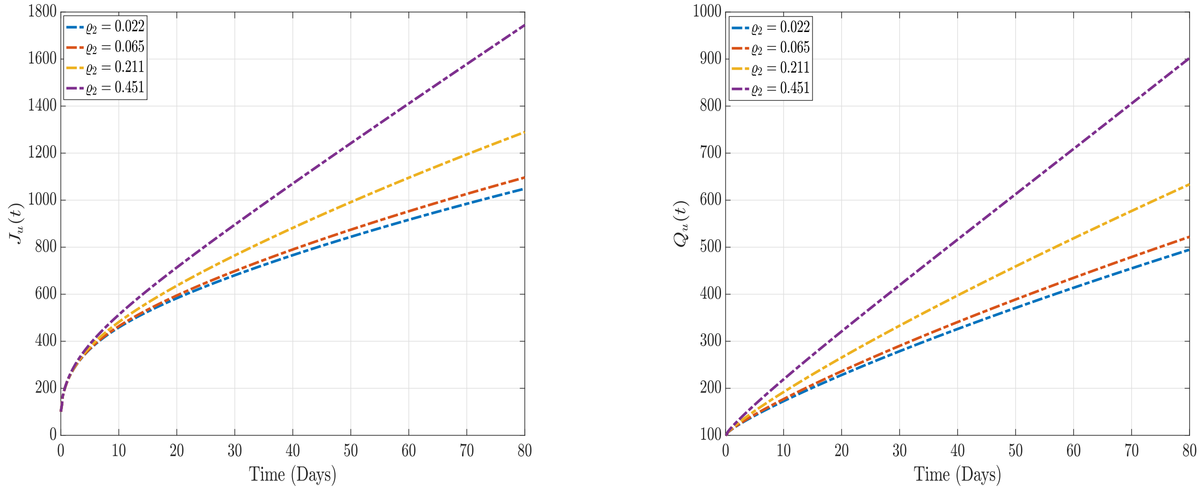
(d) Fractal-fractional dynamics of the Infected Rodents Compartment $I_v(t)$ at different fractional orders η_1 with a constant fractal $\eta_2 = 1$ value

Fig. 4. Fractal–fractional dynamics of the Quarantined Human ($Q_w(t)$), the Recovered Human $R_w(t)$, the Susceptible Rodents ($S_v(t)$), and the Infected Rodents ($I_v(t)$), at different fractional values with a constant fractal value $\eta_2 = 1$.



(a) Dynamics of the Exposed Human Compartment $E_w(t)$ at different ϱ_2 values with $\eta_1 = \eta_2 = 0.8$

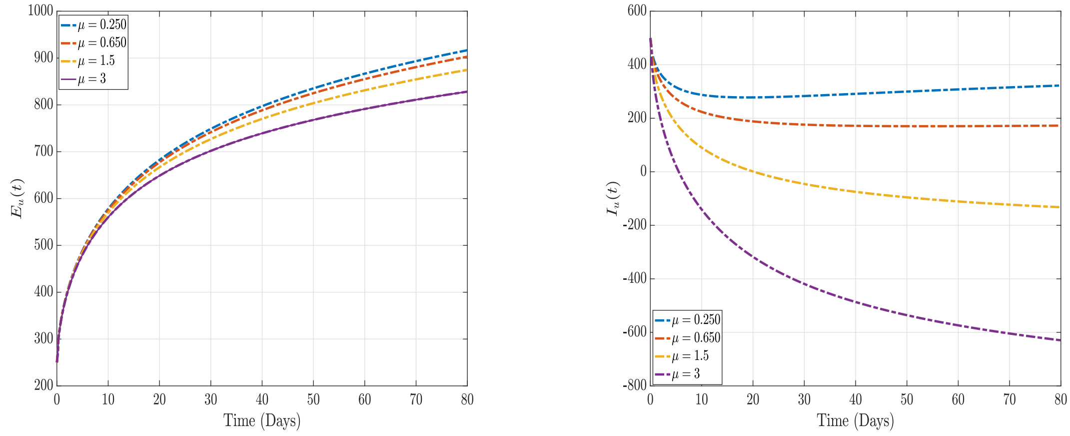
(b) Dynamics of the Infected Human Compartment $I_w(t)$ at different ϱ_2 values with $\eta_1 = \eta_2 = 0.8$



(c) Dynamics of the Isolated Human Compartment $J_w(t)$ at different ϱ_2 values with $\eta_1 = \eta_2 = 0.8$

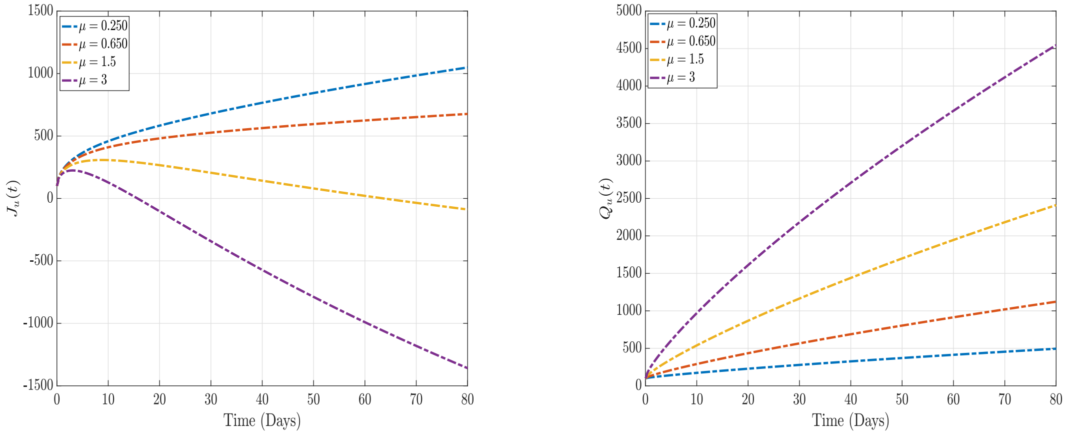
(d) Dynamics of the Exposed Human Compartment $Q_w(t)$ at different ϱ_2 values with $\eta_1 = \eta_2 = 0.8$

Fig. 5. Dynamics of the Exposed Human ($E_w(t)$), the Infected Human $I_w(t)$, the Isolated Human ($J_w(t)$), and the Quarantined Human ($I_v(t)$), at different σ_2 values with a fractional and fractal value $\eta_1 = \eta_2 = 0.8$.



(a) Dynamics of the Exposed Human Compartment $E_w(t)$ at different μ values with $\eta_1 = \eta_2 = 0.8$

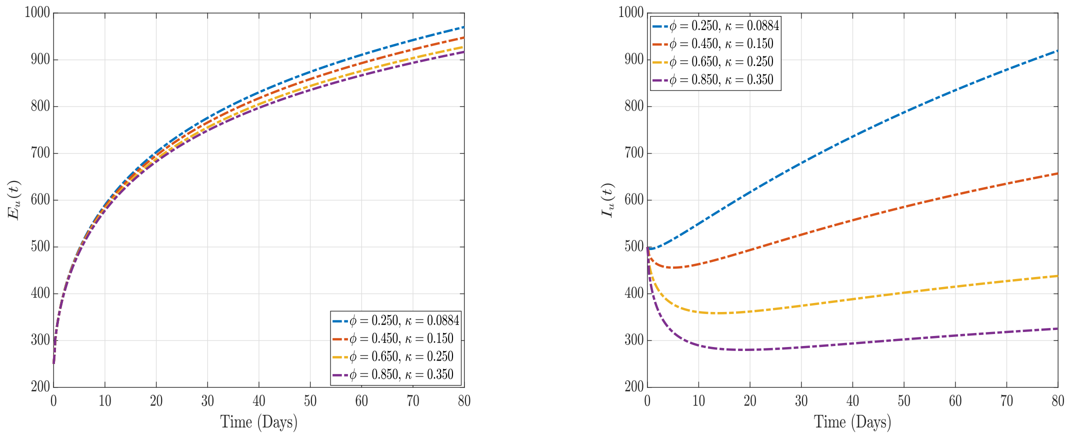
(b) Dynamics of the Infected Human Compartment $I_w(t)$ at different μ values with $\eta_1 = \eta_2 = 0.8$



(c) Dynamics of the Isolated Human Compartment $J_w(t)$ at different μ values with $\eta_1 = \eta_2 = 0.8$

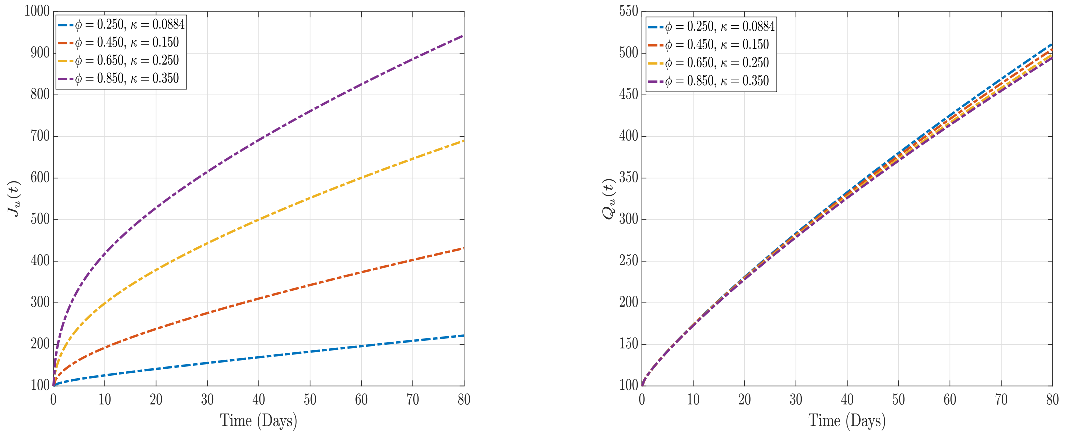
(d) Dynamics of the Quarantined Human Compartment $Q_w(t)$ at different μ values with $\eta_1 = \eta_2 = 0.8$

Fig. 6. Dynamics of the Exposed Human ($E_w(t)$), the Infected Human $I_w(t)$, the Isolated Human ($J_w(t)$), and the Quarantined Human ($I_v(t)$), at different σ_2 values with a fractional and fractal value $\eta_1 = \eta_2 = 0.8$.



(a) Dynamics of the Exposed Human Compartment $E_w(t)$ at different ϕ and κ values with $\eta_1 = \eta_2 = 1$

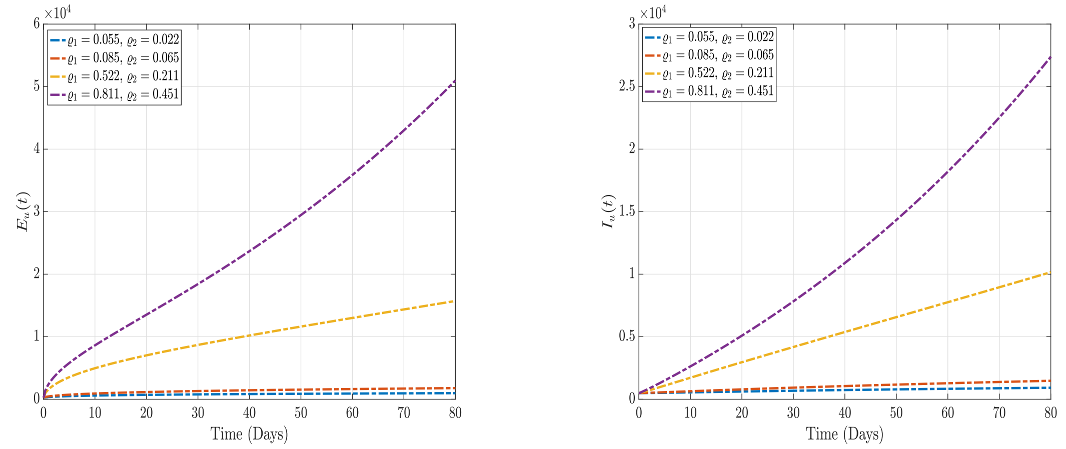
(b) Dynamics of the Infected Human Compartment $I_w(t)$ at different ϕ and κ values with $\eta_1 = \eta_2 = 1$



(c) Dynamics of the Isolated Human Compartment $J_w(t)$ at different ϕ and κ values with $\eta_1 = \eta_2 = 1$

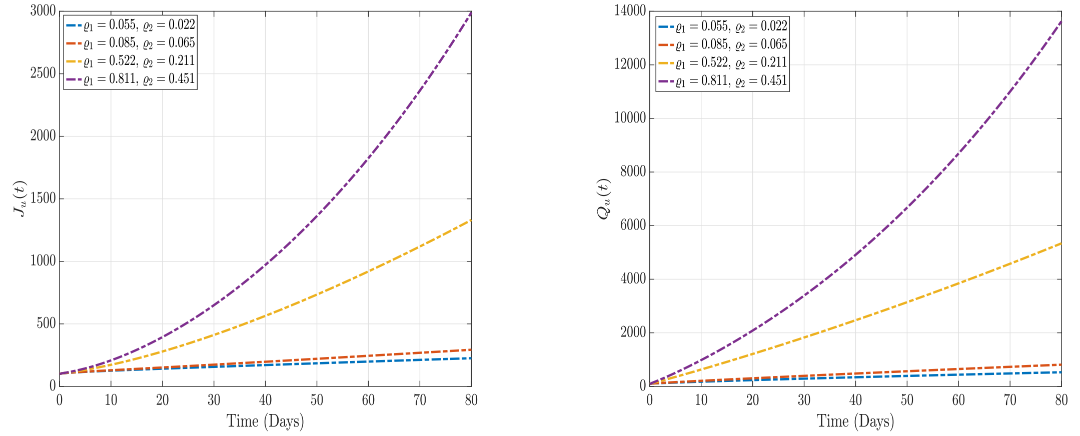
(d) Dynamics of the Quarantined Human Compartment $Q_w(t)$ at different ϕ and κ values with $\eta_1 = \eta_2 = 1$

Fig. 7. Dynamics of the Exposed Human ($E_w(t)$), the Infected Human $I_w(t)$, the Isolated Humans ($J_w(t)$), and the Quarantined Humans ($Q_w(t)$) with $\eta_1 = \eta_2 = 0.8$.



(a) Dynamics of the Exposed Human Compartment $E_w(t)$ at different ϱ_1 and ϱ_2 values with $\eta_1 = \eta_2 = 1$

(b) Dynamics of the Infected Human Compartment $I_w(t)$ at different ϱ_1 and ϱ_2 values with $\eta_1 = \eta_2 = 1$



(c) Dynamics of the Isolated Human Compartment $J_w(t)$ at different ϱ_1 and ϱ_2 values with $\eta_1 = \eta_2 = 1$

(d) Dynamics of the Quarantined Human Compartment $Q_w(t)$ at different ϱ_1 and ϱ_2 values with $\eta_1 = \eta_2 = 1$

Fig. 8. Dynamics of the Exposed Human ($E_w(t)$), the Infected Human $I_w(t)$, the Isolated Humans ($J_w(t)$), and the Quarantined Humans ($Q_w(t)$) with varied values of ϱ_1 and ϱ_2 at $\eta_1 = \eta_2 = 0.8$.

Declaration of competing interest

I wish to declare that the work has no competing interest.

Data availability

No data was used for the research described in the article

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