THE MATHEMATICS OF THE VIBRATING MEMBRANE: THE CASE OF THE "*GUNGON*".

By

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DECLARATION

I hereby declare that this submission is my own work towards the MPHIL and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University except where due acknowledgement has been made in the text.

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ABSTRACT

In this thesis, we developed a model of the "*Gungon"* drum based on the two dimensional wave equation to investigate the sound (the normal modes) made when hitting different points on the drumhead and the effect of the Fourier coefficient on the sound produced.

We used the mathematical software package, MATLAB to run all simulations to obtain the values of the deflections at different radii and time. The results churned out were then analysed and interpreted in the light of the thesis topic. It was observed that the highest deflection was always produced at the centre ($r=0$) at different time (t), this is what you hear most when you play the drum.

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DEDICATION

This thesis is dedicated to my husband

Abdulrahaman Ahmed Hafiz,

My son’s Chantiwun Hafiz and Wun-Nam Hafiz

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CHAPTER ONE

INTRODUCTION

1.1 History of Drums

The history of drums began with the emergence of human civilization. Drum beating as it is popularly known has been associated with the birth of humankind. Drum and bass is a sort of electronic dance music also known as jungle. It is also called a membranophone, means an instrument that makes sound by striking a prolonged membrane with some type of object, typically a curved stick.

Drums consist of a hollowed out piece (called the body), a covering extended over the end of the drum, and tuning keys or pegs which stiffens or loosens the membrane to achieve different tones. It is a common notion that the body of the drum generates the sound; it is in fact the membrane and its vibration that creates the sound when struck.

Following the history of drums, it is presumed origination of drums to be as early as in 6000 BC. Mesopotamian excavations discovered small cylindrical drums dated 3000 BC. Inside caves in Peru several wall markings were found which show the use of drums in various aspects of communal life. The American Indians used gourd and wooden constructed drums for their rituals and ceremonies. Drums are not always used for creating music only. It has also been used for communication purpose.

In African tribal cultures (also in the regional culture), drums bore an important part in the use of rituals and religious ceremonies. The people of different African tribes relied on the use of drums to express themselves and important messaging was done through series of drum beats along the length of the jungle. African drums and drummers provided an addictive and unique sound that deeply influenced Western Rock and Roll music. When it was discovered in the history of drums that one player could play two or more drums at the same time; people started placing groups of drums together for one musician to play. The player not only played drums of similar types but also imported from other cultures and around the globe. The early 1930s discovered a newly found trick. Musicians found out with the proper drum placement and a lot of practice, a single player could handle a set of drums, and an entire group of drum players was not necessary anymore. This leads to the origin of the Drum kit.

Double drumming is an important development in the history of drums. Double drumming is one drummer playing more than one drums. Cymbals and tom toms, invented in China were added to drum kits of the drummer, which was fast to accommodate different set of drums. Cowbells, wooden blocks, and chimes which were the percussion additions were incorporated as well. In the flow of the history of drums, by the 1930s the typical drum kit had taken shape with the various instruments in its armory. The kit consisted of a foot pedal and bass drum, snare, hi hat cymbal, tom toms, and large hanging cymbals.

1960s saw the rise of rock drummers, who began the development of drum kits that are the standard today. More toms and cymbals, as well as the accumulation of another bass drum to boost speed were added. Electronic drums then came into being with the intention of creating sounds that traditional drums were unable to generate. The history of drums thus echoed the variable sounds and rhythms of human development. In history, drumming and the use of percussive instruments have had a significant role in people’s lives. Not only do the people who play these instruments enjoy them, but it is said that "there is as much pleasure participating in, as listening to and admiring an expert drummer’s improvisations". The use of drums has been recognized as being able to put people into spiritual trances throughout history. The drum is a musical instrument with great power and presence that gives the "pulse" or backbone to the music it is incorporated with.

There are three rudimental rhythmic procedures that have been known in drumming for the use of communication, entertainment and both communication and entertainment together. These are the use of a drum as a speech surrogate or a "talking drum". These methods of playing were used for communicative purposes and often codes were used to be played over long distances for the sending and receiving of messages. The use of both iconic and symbolic dimensions of communication within music and dance. Throughout many festivals in Africa, depending on the event being celebrated, drumbeats are used to dictate the type of dance to be done by the listeners. For example, at the time of a birth of twins there is a different dance done than at a birth of a single child and the beat of the drum instructs the listeners to do the appropriate dance. This rhythmic procedure is most commonly used today and is the pure musical play of rhythms in dance. There are no communicative obligations within this type of music, which allows for free-form dance and unlimited use of improvisational strategies by the musician.

The drum is a member of the percussion group of musical instruments, technically classified as the membranous. Drums consist of at least one membrane, called a drumhead or drum skin, that is stretched over a shell and struck, either directly with the player's hands, or with a drumstick, to produce sound. There is usually a "resonance head" on the underside of the drum; these are usually tuned to a slightly lower pitch than the top drumhead. Other techniques have been used to cause drums to make sound, such as the thumb roll. Drums are the world's oldest and most ubiquitous musical instruments, and the basic design has remained virtually unchanged for thousands of years.

1.2 A BRIEF HISTORY OF AFRICAN MUSIC

"The history of nearly all musical instruments of Medieval Europe came from Asia, either from the South East through Byzantium, or from the Islamic Empire through North Africa, or from the northeast along the Baltic Coast. The direct correlation to Rome seems to be rather insignificant and the lyre is the only instrument that might possibly be European in origin." Curt Sachs

It is said that much of African music stems back to the European settlers. This of course is a rather large generalization considering the pride and innate musical abilities of Africans themselves. Being there are over 700 known languages within Africa, little can be said to generalize the entire music scene. One thing is for sure when speaking of Africa, music is the "ethnic bond" of the entire country.

There are many different "languages" of African music as well. Different styles of music come from the variations of environmental conditions within Africa. For example, cultures from the Savannah and Grassland tend to use different types of instruments than that of the cultures that occupy the country's Tropical Forest region.

Throughout history, mass population movements due to wars, famine and other crises forced villages to intermingle and combine their musical beliefs, methods and sounds. For example, in East Africa the people of the Luo culture are found in both Kenya and Tanzania, while members of the cattle culture are known throughout Uganda, Kenya, Sudan and Somalia. Given the variations of people throughout the country also implies the similar to same types of music are known by different names in Africa and likewise the instruments as well. An example of this is the Dahomean musical genres Kete, Ketehoun, Katanto and

Akofin are extremely similar to those in the Akan region of Ghana. This is not to say that their styles of music are the same, but it offers credibility when speaking of the similar bloodlines within all African music.

African music, before European settlement had strong relations with the Islamic style of music due to the close proximity of the two groups of people. Nomadic people such as Islamic pheasants and travelling merchants are known to have visited African villages in peace. Part of the welcoming ceremony would be the visitor's role of playing the drums of the natives in an action of proving self worth. Often this was the only common language between the two groups of people. After the initial drumming, both the visitors and the selected drummers of these villages would play together in harmony in a festival-like setting to show commonality and cooperation.

Although the Islamic style of music influenced its African neighbors, much of the underlying melodic and rhythmic modes never changed. This would have completely changed the sound and affect of the music to its listeners. The most influenced aspect of African music from the Islamic people was the role of the human voice and the use of melodic singing.

Later in African history, after the settlement of the Europeans, existed a time where African music intertwined with European styles and sounds. This was brought about by the need for musicians to entertain colonial officials and traders. Unfortunately for the African communities, the best drummers were recruited and taken from the towns in which they lived for governmental use. They were trained by colonial band conductors to learn the "new style" of drumming for these entertainment purposes. These drummers were considered the sole pride of the towns they were native to, and often times these towns were left in shambles without their presence. Through their playing, these men were viewed as interpreters of God’s words. An ancient African prophet said, "God is dumb, until the drum speaks." This alone tells of the great pride Africa has for the Drum and the drummer. There are many communities that still exist where their musical ties are still and always will be bound to the legacy of Europe in Africa.

The melodic and rhythmic modes in Africa are the essential part of their music. Much of the knowledge pertaining to much of the old customs of rhythm was lost with the deaths of the people that created them. They were considered the heartbeat to the society they belonged to and stayed a secret throughout all of their existence. What is known, is the essential importance of the drum to each of the societies within Africa and role it played for its inhabitants. The following is a quote from a person returning to Africa after years of exile:

Music provides a great way to vent frustrations and opinions of political and social disorder. This is seen throughout all types of music, worldwide and is often emphasized as such.

1.3 SOUND OF THE DRUM

Sound is a collection of vibrations, or pressure waves. The sound of a drum depends on several variables, including shape, size and thickness of its shell, materials from which the shell was made, counterhoop material, type of drumhead used and tension applied to it, position of the drum, location, and the velocity and angle in which it is struck.

Several factors determine the sound a drum produces, including the type, shape and construction of the drum shell, the type of drum heads it has, and the tension of these drumheads. Different drum sounds have different uses in music. Take, for example, the modern Tom-tom drum. A jazz drummer may want drums that are high pitched, resonant and quiet whereas a rock drummer may prefer drums that are loud, dry and low-pitched. Since these drummers want different sounds, their drums will be constructed a little differently.

The drum head has the most effect on how a drum sounds. Each type of drum head serves its own musical purpose and has its own unique sound. Double-ply drumheads dampen high frequency harmonics since they are heavier and they are suited to heavy playing. Drum heads with a white, textured coating on them muffle the overtones of the drum head slightly, producing a less diverse pitch. Drum heads with central silver or black dots tend to muffle the overtones even more. And drum heads with perimeter sound rings mostly eliminate overtones (Howie 2005). Some jazz drummers avoid using thick drum heads, preferring single ply drum heads or drum heads with no muffling. Rock drummers often prefer the thicker or coated drum heads.

The second biggest factor affecting the sound produced by a drum is the tension at which the drum head is held against the shell of the drum. When the hoop is placed around the drum head and shell tightened down with tension rods, the tension of the head can be adjusted. When the tension is increased, the amplitude of the sound is reduced and the frequency is increased, making the pitch higher and the volume lower.

The type of shell also affects the sound of a drum. Because the vibrations resonate in the shell of the drum, the shell can be used to increase the volume and to manipulate the type of sound produced. The larger the diameter of the shell, the lower the pitch and the larger the depth of the drum, the louder the volume. Shell thickness also determines the volume of drums. Thicker shells produce louder drums. Mahogany raises the frequency of low pitches and keeps higher frequencies at about the same speed. When choosing a set of shells, a jazz drummer may want smaller maple shells, while a rock drummer may want larger birch shells.

1.4 STATEMENT OF THE PROBLEM

This thesis will develop a model of the drum based on the wave equation to investigate the sound made (the normal modes) when hitting different points on the drumhead and the effect of the Fourier coefficient on the sound produced.

The study seeks to explain the mathematics involved in the vibration of a fixed edge circular membrane. The vibrations of an idealized circular drum, essentially an elastic membrane of uniform thickness attached to a rigid circular frame, are solutions of the wave equation with zero boundary conditions

1.5 OBJECTIVES OF THE STUDY

1. To transform a two dimensional wave equation into a differential equation and hence obtain a general solution
2. To calculate the vibrations (normal modes) of the drumhead at different radii and time.
3. To mathematically analyse the various deflections obtained using the software Mat lab.

1.6 METHODOLOGY

The Method of separation of variables is applied on the partial differential equation that describes the dynamics of the drum to obtain two second order ordinary differential equations with variable coefficients. These ordinary differential equations are solved and with the boundary and initial conditions imposed, the mathematical expression describing the motion of the "Gungon" is obtained.

1.7 ORGANISATION OF THE THESIS

In the chapters that follow, chapter one gives the introduction of the thesis. The researcher shall review the literatures on talking drums for the chapter two. The chapter three would see us make certain assumptions on a vibrating membrane in order to derive the two dimensional Cartesian wave equations. The Cartesian wave equation will be converted into Plane Polar equation in order to model the circular 'gongon' drum with a constant tension in the drum head. Chapter four, the researcher will impose some boundary conditions on the model derived from chapters three and apply mat lab to obtain the various deflections at different radius and time. Chapters five, the results chinned out from chapter four are then analyzed and interpreted in the light of the thesis topic. Other findings are also discussed and finally conclusions and recommendations are made.

CHAPTER TWO

LITERATURE REVIEW

2.1 TALKING DRUMS

In the past drums have been used not only for their musical qualities, but also as a means of communication, especially through signals. The talking drums of Africa can imitate the inflections and pitch variations of a spoken language and are used for communicating over great distances. Throughout Sri Lankan history drums have been used for communication between the state and the community, and Sri Lankan drums have a history stretching back over 2500 years.

Chinese troops used tàigǔ drums to motivate troops, to help set a marching pace, and to call out orders or announcements. For example, during a war between Qi and Lu in 684 BC, the effect of drum on soldier's morale is employed to change the result of a major battle. Fife-and-drum corps of Swiss mercenary foot soldiers also used drums. They used an early version of the snare drum carried over the player's right shoulder, suspended by a strap (typically played with one hand using traditional grip). It is to this instrument that the English word "drum" was first used. Similarly, during the English Civil War rope-tension drums would be carried by junior officers as a means to relay commands from senior officers over the noise of battle. These were also hung over the shoulder of the drummer and typically played with two drum sticks. Different regiments and companies would have distinctive and unique drum beats which only they would recognize. In the mid-19th century, the Scottish military started incorporating pipe bands into their Highland Regiments.

2.2 ATUMPAN DRUM

Atumpan are similar structurally to the “Lunga” and “Gungon” but Akan musicians use the tension drum heads primarily to create a descending or falling pitch on drum strokes rather to produce a wide tonal language.

The drum heads at either end of the drum's wooden body are made from hide, fish-skin or other membranes which are wrapped around a wooden hoop. Leather cords or thongs run the length of the drum's body and are wrapped around both hoops; when you squeeze these cords under your arm, the drum heads tighten, changing the instrument's pitch.

One of the unique features of the “Atumpan” is their ability to closely imitate the rhythms and intonations of spoken language. In the hands of skilled performers, they can reproduce the sounds of proverbs or praise songs through a specialized "drum language" - their dialogue can be easily understood by a knowledgeable audience. Whether accompanying dances or sending messages the sound of these instruments can carry many miles

Fig 1: The Atumpan

Fig 2: A drummer beating the Atumpan

2.3 DAGOMBA "LUNGA" AND "GUNGON"

The "*Lunga"* is played by members of a hereditary lineage. All members of the drumming family trace descent from one man, Bizung, who gave up the chance to be chief in favor of becoming a musician. If your father was born into this family line, then you also are a member. The word for drum and drummer are the same, "*Lunga"* (plural, Lunsi). Through speech, song and drumming the "*Lunsi"* recount the history of the Dagomba kingdom, called Dagbon, and the genealogy of the royal families. Among the "*Lunsi"* of Dagbon, highest prestige goes to those who chant the epic narrative of the nation in all-night performances called Sambanglunga. "*Lunsi"* also play music for dance at social events.

2.4 DRUMS OF THE DAGOMBA "*LUNSI*"

The Lunsi play drums of two kinds; "*Lunga*" and Gungon. The "*Lunga*" drum has two-heads that are connected by leather ropes over an hourglass-shaped wooden body. Held close into the player's armpit, the ropes are pressed and released to change the tension of the drumhead, thus changing its pitch. Using precisely intoned pitches, the "*Lunga*" drummer sets implicit texts, or drum language, to melodies that closely resemble the sound of speech. In music for dance, the "*Lunga*" drums have two roles--the lead "*Lunga*", which directs the drumming and presents many drum language texts, and the answer "*Lunga*", a part played by many drummers that provides a recurring melo-rhythmic theme in counterpoint to the lead "*Lunga*" part. Complementing melodic sound of the two "*Lunga*" roles is the gungon, a double-headed drum with a cylindrical wooden body. Dancers groove to the booming bass sound made by two gung-gong drummers. Depending on the particular piece of music, the two gungon drums may play in unison or may be divided into intertwined parts, lead and answer.

Drumming energizes people to dance at social functions. Music may be for individuals to dance solo or for groups to do traditional choreography.

When a drummer first shoulders a drum, custom dictates that an invocation be played – Nawuni Mali Kpam Pam for "*Lunga*" and Jebo for "*Gungon*". Invocations align drummers with their history and with spiritual forces. Invocations also are a method of tuning up and alerting other musicians that you are ready to play.

The Praise Name Dances are for people to dance one-by-one to drumming that honors famous chiefs from the past. There is a strong association linking the music to the person who is the subject of the drum language. Often, the dancer is in the family lineage of the historical chief being saluted by the drummers.

Drummers play for people to dance in group formation. The dancers are united by a shared factor, for example, gender, social status, age or place of residence. Sometimes the dance group becomes a formal association that is hired to perform at social functions. Dances of this type also are in the repertory of performance arts organizations, such as the Ghana National Dance Ensemble. They are very appropriate for school groups or theater-style African ensembles. In Dagbon, the annual progression of the lunar months is marked by a cycle of festivals, each with its characteristic music and dance. Most well known is the Damba Festival, celebrated on the occasion of the birthday of the Prophet Mohammed. The music of the Damba Festival is closely associated with the ethnic identity of the Dagomba people.

The "*Lunga*" is a double-headed hourglass shaped tension drum carved from the Shea Butter tree (Tanga in Dagbani and Nkudua in Ashanti). Each end of the shell is covered with a goat skin head sewn onto a circular rim made of reed and grass. The two heads are connected by antelope skin tension cords from one end of the shell to the other end. The drum is played with a constructed curved wooden stick held in the right hand while the "*Lunga*" is suspended from the left shoulder by a scarf tied to the central cylinder shell and fits snugly into the player's armpit**.** By squeezing and pulling on the leather cords, the player can change the tension and thus the pitch of the drum. Pitches can vary over an octave and many intervals can be attained. In the hands of an expert, the drum becomes a way of communicating Dagbani, the spoken language of the Dagomba people. The "*Lunga*" expert mostly uses proverbs to praise people by their family name. Many styles of Ghanaian music are based upon speech, therefore the bond between language and music are so intimate that it is actually possible to tune and play an instrument so that the music it produces is linguistically comprehensible.

Fig 3: Various "Lunga"

All drums come with curved playing stick, and straps for the shoulder. The gungon is a medium sized cylindrical bass drum used to accompany the lunga. It is double headed drum with snares on each head. It produces a rich bass tone when struck in the middle and has a wide array of high overtones and buzz when played above the snares.

Fig 4: The "Gungon"

All shells for the drums are carved in Tamale, Northern Ghana. The drums are of the highest quality and satisfaction is guaranteed.

Fig 5: A drummer beating the “Lunga”

Fig 6: An expert manufacturing the “Lunga” and “Gungon”

2.5 THE DAMBA FESTIVAL

In Dagbon, the annual progression of the lunar months is marked by a cycle of festivals, each with its characteristic music and dance. Most well known is the Damba Festival, celebrated on the occasion of the birthday of the Prophet Mohammed. The music of the Damba Festival is closely associated with the ethnic identity of the Dagomba people. The festival provides a platform for the people to pay homage to the sub chiefs and the paramount chief who sits in state at his palace. The festival is headed by the paramount chief of the area. The Damba involves series of dances performed to commemorate the birth of the prophet Mohammed and are combined in a medley. The first two rhythms are Damba and the rest is Takai. The Damba dance can be segregated into two main parts. The naming of the prophet Mohammed is So Damba and the birthday of the prophet Mohammed is Na Damba. Damba is the signature dance of the Damba Festival which celebrates the Birth of the Holy Prophet Mohammed. Damba is almost an obligatory dance for the Dagomba chiefs, and is associated with festivals celebrating a large harvest on the farm.

Takai is a Dagomba area war dance. This war dance is performed for two reasons: To train warriors and to show what transpired on the field during the war. The dance is traditionally performed by men. The sticks they carry used to be swords in the past. In modern time the theater has put together Damba with Takai which is why you see women dancing it as well. The first rhythm is the traditional rhythm, and the other rhythms were incorporated by the Arts council of Ghana in the 1960's.

Fig 7: Drummers beating both the “Lunga” and “Gungon”

On “*Gungon”*: Iddrissu Wumbee, Rafik Wumbee

On answer *“Lunga”* : Amidu Wumbee, David Locke

On lead *“Lunga”* (seated): Alhaji Abubakari Lunna

Fig 8: A dancer dancing to the sound produced by the drummers

CHAPTER THREE

TWO - DIAMENSIONAL WAVE EQUATION

3.1 CIRCULAR MEMBRANE, BESSEL'S FUNCTION

Consider an open disk Ω of radius R centered at the origin, which will represent the "still" drum shape. At any time t, the height of the drum shape at a point (x,y) in Ω measured from the "still" drum shape will be denoted by $u(x,y,t)$, which can take both positive and negative values. Let $\partial\Omega$ denote the boundary of Ω, that is, the circle of radius R centered at the origin, which represents the rigid frame to which the drum is attached.

The vibrations of an idealized circular drum, essentially an elastic membrane of uniform thickness attached to a rigid circular frame, are solutions of the wave equation with zero boundary conditions.

There exist infinitely many ways in which a drum can vibrate, depending on the shape of the drum at some initial time and the rate of change of the shape of the drum at the initial time. Using separation of variables, it is possible to find a collection of "simple" vibration modes, and it can be proved that any arbitrarily complex vibration of a drum can be decomposed as a series of the simpler vibrations (analogously to the Fourier series).

Vibrating drums are surfaces which vibrate, like flat drum-heads in the shape of a circle or square.

The drum head is tied down at the boundary, i.e, the boundary conditions on the vibrating drum are "Dirichlet".

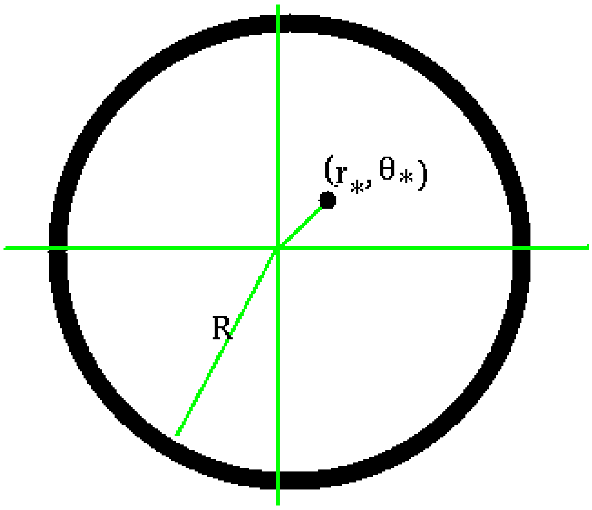

Fig 9: Polar Co-ordinates

The mathematical equation that governs the vibration of the drum is the two diamensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Due to the circular geometry of Ω, it will be convenient to use cylindrical coordinates, (r, θ, t).

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{dr}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}\right] \qquad 0 \le r \le R \quad 0 \le \phi \le 2\pi$$

where r and ϕ are polar coordinates of membrane and c is a positive constant, which gives the "speed" of vibration.

For circular membranes, the radial part of the vibrations is a Bessel's function while the angular part is an ordinary trigonometric function

The only boundary condition we have is $u(R,0,t) = 0 \quad t \succ 0$

and the initial conditions are the standard ones

$u(r,\phi,0) = f(r,0)$

$\frac{\partial u}{\partial t}(r,\phi,0) = g(r,0)$

We begin with the separation of variables

$u(r,\phi,t) = \varphi(r,\phi)T(t)$

but $\varphi(r,\phi) = W(r)\Theta(\phi)$

$u(r,\phi,t) = W(r)\Theta(\phi)T(t)$ so that we get the ODE

$$T''(t) = -\lambda c^2 T(t)$$

And the Helmholtz PDE (in polar coordinates)

$$\nabla^2\varphi + \lambda\varphi = 0$$

With boundary condition $\varphi(R,0) = 0$

We can write this PDE eigenvalue problem as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{dr}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \lambda\varphi = 0$$

$\varphi(R,0) = 0$

Now we again apply separation of variables for this polar coordinates problem using

$\varphi(r,0) = W(r)\Theta(\phi)$

This gives us

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(W(r)\Theta(\phi))\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2}(W(r)\Theta(\phi)) + \lambda(W(r)\Theta(\phi)) = 0$$

Or

$$\frac{\Theta(\phi)}{r}\frac{\partial}{\partial r}\left(rW'(r)\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}\left(\Theta''(\phi)\right)+\lambda\left(W(r)\Theta(\phi)\right)=0$$

Multiply by $\dfrac{r^2}{W(r)\Theta(\phi)}$ and a little rearranging gives

$$\frac{r}{W(r)}\frac{\partial}{\partial r}\left(rW'(r)\right)+\lambda r^2=-\frac{\Theta''(\phi)}{\Theta(\phi)}=\mu$$

This results in two additional SL ODE eigenvalue problems

Altogether, we now have three ODEs

- The time dependent

We also know that since $\lambda \succ 0$

$$T''(t)=-\lambda c^2 T(t)$$

has the general solution

$$T(t)=c_1\cos\left(\sqrt{\lambda}ct\right)+c_2\sin\left(\sqrt{\lambda}ct\right)$$

- And from $\dfrac{r}{W(r)}\dfrac{\partial}{\partial r}\left(rW'(r)\right)+\lambda r^2=-\dfrac{\Theta''(\phi)}{\Theta(\phi)}=\mu$

We get the two singular sturm liouville problems

$$\Theta''(\phi)=-\mu\Theta(\phi)$$

with periodic boundary conditions $\Theta(2\pi)=\Theta(0)$, $\Theta'(2\pi)=\Theta'(0)$

$$r\frac{\partial}{\partial r}\left(rW'(r)\right)+\left(\lambda r^2-\mu\right)W(r)=0$$

With singularity boundary conditions

$$W(R)=0 \qquad \left|W(0)\right|\prec\infty$$

The first problem

$$\Theta''(\phi)=-\mu\Theta(\phi)$$

With periodic boundary conditions has eigenvalues and eigenfunctions

$$\mu_n=n^2$$

$$\Theta(\phi)=c_1\cos n\phi+c_2\sin n\phi \qquad n=0,\ 1,\ 2\ldots$$

The second problem is more easily investigated if we first re write it using the product rule we have

$$0=r\frac{\partial}{\partial r}\left(rW'(r)\right)+\left(\lambda r^2-\mu\right)=rW''(r)+rW'(r)+\left(\lambda r^2-\mu\right)W(r)$$

If we apply the substitution $z=\sqrt{\lambda}r$ an eigenvalue $\mu_n=n^2$ to the equation

$$rW''(r)+rW'(r)+\left(\lambda r^2-\mu\right)W(r)=0$$

We get

$$\frac{z^2}{\lambda}\lambda W''(z)+\frac{z}{\sqrt{\lambda}}\sqrt{\lambda}W'(z)+\left(z^2-n^2\right)W(z)=0$$

$$z^2W''(z)+zW'(z)+\left(z^2-n^2\right)W(z)=0$$

The above equation is known as the Bessel's equation of order *n*

$$W_n(z) = J_n\left(\sqrt{\lambda_{n,m}}\,r\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{\sqrt{\lambda_{n,m}}}{2}\right)^{2k} \qquad n=0,\ 1,\ 2\ \ldots$$

Now that the boundary conditions tell us that the eigenvalues $\lambda_{n,m}$ are such that

$$W_n(R) = J_n\left(\sqrt{\lambda_{n,m}}\,R\right) = 0$$

$$\lambda_{n,m} = \left(\frac{z_{n,m}}{R}\right)^2$$

Where is the $z_{n,m}$ m-th zero of the Bessel function of order n

$$J_n(z_{n,m}) = 0$$

Boundary conditions

The boundary condition is that the edge of the drum head is fixed so that it cannot be displaced. This means that

$$W(R) = J_0\left(\sqrt{\lambda}R\right) = 0$$

where R is the radius of the drum.

Initial conditions

The coefficients a_n and b_n are determined form the initial conditions. The initial conditions are specified as

$$u(r,0) = f(r)$$

$$\frac{\partial u}{\partial t}(r,0) = 0$$

We substitute the general solution above into these expressions to give

$$\sum_{n=1}^{\infty} a_n J_0\left(\sqrt{\lambda_n} r\right) = f(r)$$

$$\sum_{n=1}^{\infty} \sqrt{\lambda_n} c b_n J_0\left(\sqrt{\lambda_n} r\right) = g(r)$$

These expressions give the initial functions $f(r)$ and $g(r)$ in terms of the coefficients a_n and b_n. What we want to do is to invert this result to give the coefficients a_n and b_n in terms of the initial functions $f(r)$ and $g(r)$. And when these coefficients are found in this way then the complete solution for the (circularly symmetric) behavior of the drum head is given by

$$u(r,\phi,t) = W(r)\Theta(\phi)T(t)$$

The solution of the initial value problem can be found as a superposition of normal modes.

$$u(r,\phi,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[a_{n,m} J_n\left(\sqrt{\lambda_{n,m}} r\right)\cos n\phi \cos\left(\sqrt{\lambda_{n,m}} ct\right) + b_{n,m} J_n\left(\sqrt{\lambda_{n,m}} r\right)\cos n\phi \sin\left(\sqrt{\lambda_{n,m}} ct\right)$$

$$c_{n,m} J_n\left(\sqrt{\lambda_{n,m}} r\right)\sin n\phi \cos\left(\sqrt{\lambda_{n,m}} ct\right) + d_{n,m} J_n\left(\sqrt{\lambda_{n,m}} r\right)\sin n\phi \sin\left(\sqrt{\lambda_{n,m}} ct\right)\Big]$$

So when you bang a drum, the sound produced is a combination of the normal modes.

3.2 A CIRCULAR SYMMETRIC DRUM WITH ZERO INITIAL DISPLACEMENT

This physical problem can be represented by the following boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{dr}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} \qquad 0 \le r \le R$$

Because of circular symmetry there is no change in the angular variable and the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{dr}\right)$$

$u(R,t) = 0$ because the sides of the drum are nailed down

$u(r,0) = f(r) = 0$ initial displacement

$\frac{\partial u}{\partial t}(r,0) = g(r)$ initial velocity

We have

$$u(r,t) = W(r)T(t)$$

$$T''(t) = -\lambda c^2 T(t)$$

$$T(t) = c_1 \cos\left(\sqrt{\lambda}ct\right) + c_2 \sin\left(\sqrt{\lambda}ct\right)$$

$$\frac{d}{dr}\left(rW'(r) + \lambda rW(r)\right) = 0$$

Using the product rule we can re write the radial ODE as

$$rW''(z) + W'(r) + \lambda rW(z) = 0$$

And multiply by r and do the substitution $z = \sqrt{\lambda} r$

$$z^2 W''(z) + zW'(z) + z^2 W(z) = 0$$

As Bessel's equation for the case n=0, i.e J_0

Where is the λ_n m-th zero of the Bessel function of order n

$$\mu = \lambda^2 \succ 0$$

The solutions of the radial part is the equation below

$$W(r) = AJ_0(r) + BY_0(r)$$

J_0 and Y_0 are called ordinary Bessel Function of the First and second kinds, of order zero, respectively.

If we look at the two kinds of Bessel's functions, Y_0 cannot be a solution due to the fact that it diverges when r approaches 0. Therefore $B = 0$, if we look at J_0 it has to vanish on the boundary of the membrane meaning that $J_0\left(\sqrt{\lambda_n} r\right) = 0$, where r is the radius of the drum. As we see on the plots in figure below there is more than one point where J_0 is equal zero.

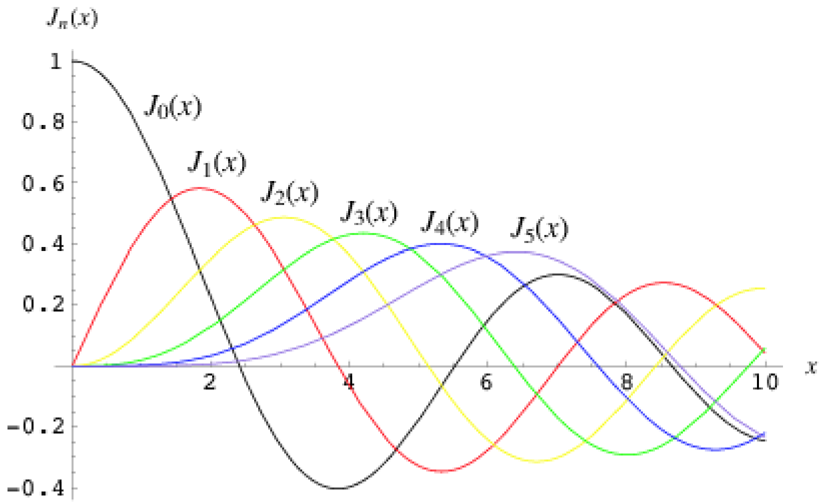

Fig 10: Bessel Function of the First Kind

$$W(r) = AJ_0\left(\sqrt{\lambda}r\right)$$

Let $A = 1$

$$W(r) = J_0\left(\sqrt{\lambda}r\right)$$

$$u(r,t) = W(r)T(t)$$

$$u(r,t) = J_0\left(\sqrt{\lambda}r\right)\left(c_1 \cos\left(\sqrt{\lambda}ct\right) + c_2 \sin\left(\sqrt{\lambda}ct\right)\right)$$

$$W(R) = 0 \qquad |W(0)| \prec \infty$$

$$W(r) = J_0(\alpha r) = 0$$

$$\sqrt{\lambda}R = z_{m,n} \qquad \sqrt{\lambda} = \frac{z_{m,n}}{R}$$

$$W(R) = J_o\left(\frac{z_{m,n}}{R}\right)$$

Hence the frequency of vibration are given by

$$\sqrt{\lambda} = \frac{z_{m,n}}{R}$$

$$u(r,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\sqrt{\lambda_n} ct\right) + b_n \sin\left(\sqrt{\lambda_n} ct\right)\right) J_0\left(\sqrt{\lambda_n} r\right)$$

The first initial condition gives us

$$u(r,0) = \sum_{n=1}^{\infty} a_n J_0\left(\sqrt{\lambda_n} r\right) = f(r) = 0$$

By orthogonality property, the Fourier Bessel's coefficient a_n is

$$a_n = \frac{\int_0^R r f(r) J_0\left(\sqrt{\lambda} r\right) dr}{\int_0^R r J_0^{\,2}\left(\sqrt{\lambda} r\right) dr}$$

$$a_n = 0$$

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} c b_n J_0\left(\sqrt{\lambda_n} r\right) = b_n$$

By orthogonality property

$$b_n = \frac{1}{c\sqrt{\lambda_n}} \frac{\int_0^R r g(r) J_0\left(\sqrt{\lambda} r\right) dr}{\int_0^R r J_0^{\,2}\left(\sqrt{\lambda} r\right) dr}$$

$$u(r,t) = \sum_{n=1}^{\infty} \left(b_n \sin\left(\sqrt{\lambda_n} ct\right)\right) J_0\left(\sqrt{\lambda_n} r\right)$$

CHAPTER FOUR

APPLICATION OF THE TWO-DIMENSIONAL WAVE EQUATION: THE CASE OF THE "*GUNGON*".

We shall be concerned with the two dimensional geometry of the "*Gungon*" in a plane with a little but significant modification in the assumptions of a vibrating membrane. Thus in preparing a model for this problem of how the "Gungon" (the single talking drum) sounds the way it does.

Physical description of the "*Gungon"* drum

1. Radially symmetric
2. Radius r=25cm
3. Edges fixed
4. Beginning at rest
5. The tension of the drum is fixed
6. The deflection $u(x, y, t)$ the membrane while in motion is relatively small in comparison to the size of the membrane; the angles of inclination of the deflections are small.

We are aiming at a solution of the form $u(x, y, t)$ which is radially symmetric due to the circular physical orientation of the drum and assumed a periodic function for the tension.

4.2 THE CASE WHERE THE DRUM INITIAL DEFLECTION IS ZERO AND INITIAL VELOCITY IS NOT ZERO

In this section an attempt is made to calculate the vibrational modes for the "*Gungon"* drum with constant tension as a function of time, discussed in chapter four using Mat Lab.

The initial velocity used is dependent on the radius and had only one trough with no crest was a quadratic function of r i.e. of the form

$$g(r) = \left(1 - \frac{r}{25}\right)^2$$

The choice of an initial velocity functions with repeated roots of the form:

$$\left(a - \frac{r}{b}\right)^2$$

where the constants a and b are chosen so that the centre would have the largest amplitude and least amplitude near the highest value of r, is adapted to ensure that at least a nodal point exist on the drum head because the entire drumhead do not vibrate at the same time.

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{dr}\right) \qquad 0 \le r \le 25 \qquad t \ge 0$$

$u(r,0)$, $\frac{\partial u}{\partial t}(r,0)$ finite

$$u(25,t) = 0$$

$$u(r,0) = 0$$

$$\frac{\partial u}{\partial t}(r,0)=\left(1-\frac{r}{25}\right)^2$$

This represents a circular drum, of radius 25, with a tension and density such that $c^2 = 1$

There is no initial displacement, but an initial velocity. According to the above development,

the eigenfunctions are

$$W(r)=J_0\left(\sqrt{\lambda}r\right) \qquad let \qquad \sqrt{\lambda}==\frac{z_n}{R}=k$$

$$W(r)=J_o\left(\frac{z_n}{R}r\right)$$

$$W(r)=J_o(kr)$$

The Bessel's function J_0 has (infinitely) many real zeros. Let us denote the positive zeros of $J_0(s)$ by $s=\alpha_1,\alpha_{2,}\alpha_{3,.........}$

The numerical values of the zeros of J_0 (exact to 4 decimal places) are

$\alpha_1 = 2.4048$ $\alpha_2 = 5.5200$ $\alpha_3 = 8.6537$ $\alpha_4 = 11.7915$ $\alpha_5 = 14.9309$

$\alpha_6 = 18.0711$ $\alpha_7 = 21.2116$ $\alpha_8 = 24.3525$ $\alpha_9 = 27.4935$ $\alpha_{10} = 30.6346$

The general solution is then

$$u(r,t)\approx\sum_{n=1}^{10}\left(a_n\cos\left(\sqrt{\lambda_n}t\right)+b_n\sin\left(\sqrt{\lambda_n}t\right)\right)J_0\left(\sqrt{\lambda_n}r\right)$$

The first initial condition gives us

$$\frac{\partial u}{\partial t}(r,0)=\sum_{n=1}^{10}\sqrt{\lambda_n}\,b_n J_0\left(\sqrt{\lambda_n}\,r\right)=\left(1-\frac{r}{25}\right)^2$$

$$b_n=\frac{1}{\sqrt{\lambda_n}}\frac{\int_0^{25} r\left(1-\frac{r}{25}\right)^2 J_0\left(\sqrt{\lambda_n}\,r\right)dr}{\int_0^{25} rJ_0^{\,2}\left(\sqrt{\lambda_n}\,r\right)dr}$$

$$u(r,0)=\sum_{n=1}^{10}a_n J_0\left(\sqrt{\lambda_n}\,r\right)=f(r)=0$$

$$a_n=\frac{\int_0^{25} r\bullet 0\bullet\left(J_0\left(\sqrt{\lambda_n}\,r\right)\right)dr}{\int_0^{25} rJ_0^{\,2}\left(\sqrt{\lambda_n}\,r\right)dr}$$

$$a_n=0$$

We used the mathematical software package, MATLAB to run all simulations to obtain the values of b_n

TABLE 1: THE ASSOCIATED FOURIER COEFFICIENTS FOR DRUMHEAD

n	1	2	3	4	5	6	7	8	9	10
b_n	1.0957	0.0832	0.0734	0.0148	0.0053	0.0024	0.0019	0.0017	0.0015	0.0004

$$u(r,t)\approx\sum_{n=1}^{10}\left(b_n\sin\left(\sqrt{\lambda_n}\,t\right)\right)J_0\left(\sqrt{\lambda_n}\,r\right)$$

The solution to the vibrating drumhead can now be written as

$$u(r,t) \approx 1.0957 J_0\left(\sqrt{2.4048} r\right)\sin(2.4048t) + 0.0832 J_0\left(\sqrt{5.5200} r\right)\sin(5.5200t) + 0.0734 J_0\left(\sqrt{8.6537} r\right)\sin(8.6537t)$$
$$+ 0.0148 J_0\left(\sqrt{11.7915} r\right)\sin(11.7915t) + 0.0053 J_0\left(\sqrt{14.9309} r\right)\sin(14.9309t)$$

$$+ 0.0024 J_0\left(\sqrt{18.0711} r\right)\sin(18.0711t) + 0.0019 J_0\left(\sqrt{21.2116} r\right)\sin(21.2116t) + 0.0017 J_0\left(\sqrt{24.3525} r\right)\sin(24.3525t)$$
$$+ 0.0015 J_0\left(\sqrt{27.4935} r\right)\sin(27.4935t) + 0.0004 J_0\left(\sqrt{30.6346} r\right)\sin(30.6346t)$$

4.3 THE CASE WHERE BOTH THE DRUM INITIAL DEFLECTION AND INITIAL VELOCITY IS NOT ZERO

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{dr}\right) \qquad 0 \le r \le 25 \qquad t \ge 0$$

$u(r,0)$, $\frac{\partial u}{\partial t}(r,0)$ finite

$$u(25,t) = 0$$

$$u(r,0) = \left(5 - \frac{r}{5}\right)^2$$

$$\frac{\partial u}{\partial t}(r,0) = \left(1 - \frac{r}{25}\right)^2$$

The first initial condition gives us

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{10}\sqrt{\lambda_n}\, c b_n J_0\left(\sqrt{\lambda_n} r\right) = \left(1 - \frac{r}{25}\right)^2$$

$$u(r,0) = \sum_{n=1}^{10} a_n J_0\left(\sqrt{\lambda_n} r\right) = \left(5 - \frac{r}{5}\right)^2$$

This is a Fourier Bessel's series with coefficients

$$a_n = \frac{\int_0^{25} rf(r)J_0\left(\sqrt{\lambda}r\right)dr}{\int_0^{25} rJ_0{}^2\left(\sqrt{\lambda}r\right)dr}$$

We use the mathematical software package, MATLAB to run all simulations to obtain the values of a_n *and* b_n

TABLE 2: THE ASSOCIATED FOURIER COEFFICIENTS FOR DRUMHEAD

n	1	2	3	4	5	6	7	8	9	10
a_n	0.0741	0.0613	0.0044	0.0017	0.0512	0.0055	0.0235	0.0135	0.0055	0.0057
b_n	1.0957	0.0832	0.0734	0.0148	0.0053	0.0024	0.0019	0.0017	0.0015	0.0004

The general solution is then

$$u(r,t) \approx \sum_{n=1}^{10} \left(a_n \cos\left(\sqrt{\lambda_n}t\right) + b_n \sin\left(\sqrt{\lambda_n t}\right)\right) J_0\left(\sqrt{\lambda_n} r\right)$$

We showed that the vibrations of a circular membrane, fixed at the outer edge and with a non-zero initial displacement and initial velocity, is given by

$$\begin{aligned}
u(r,t) \approx\ & 0.0741J_0(2.4048r)\cos(2.4048t) + 0.0613J_0(5.5200r)\cos(5.5200t) + 0.0044J_0(8.6537r)\cos(8.6537t) \\
& + 0.0017J_0(11.7915r)\cos(11.7915t) + 0.0512J_0(14.9309r)\cos(14.9309t) + 0.0055J_0(18.0711r)\cos(18.0711t) \\
& + 0.0235J_0(21.2116r)\cos(21.2116t) + 0.0135J_0(24.3525r)\cos(24.3525t) + 0.0055J_0(27.4935r)\cos(27.4935t) \\
& + 0.0057J_0(30.6346r)\cos(30.6346t) + 1.0957J_0(2.4048r)\sin(2.4048t) + 0.0832J_0(5.5200r)\sin(5.5200t) \\
& + 0.0734J_0(8.6537r)\sin(8.6537t) + 0.0148J_0(11.7915r)\sin(11.7915t) + 0.0053J_0(14.9309r)\sin(14.9309t) \\
& + 0.0024J_0(18.0711r)\sin(18.0711t) + 0.0019J_0(21.2116r)\sin(21.2116t) + 0.0017J_0(24.3525r)\sin(24.3525t) \\
& + 0.0015J_0(27.4935r)\sin(27.4935t) + 0.0004J_0(30.6346r)\sin(3t)0.6346
\end{aligned}$$

4.4 The Case Where the Drum Initial Deflection and Initial velocity is Zero

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{dr}\right) \qquad 0 \le r \le 25 \qquad t \ge 0$$

$u(r,0)$, $\frac{\partial u}{\partial t}(r,0)$ finite

$$u(25,t) = 0$$

$$u(r,0) = 0$$

$$\frac{\partial u}{\partial t}(r,0) = 0$$

The first initial condition gives us

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}cb_n J_0\left(\sqrt{\lambda_n r}\right) = 0$$

And the second initial condition yields

$$u(r,0) = \sum_{n=1}^{\infty} a_n J_0\left(\sqrt{\lambda_n r}\right) = 0$$

This is a Fourier Bessel's series with coefficients

$$a_n = 0 \qquad b_n = 0$$

The general solution is then

$$u(r,t) \approx \sum_{n=1}^{10}\left(a_n \cos\left(\sqrt{\lambda_n}t\right) + b_n \sin\left(\sqrt{\lambda_n}t\right)\right)J_0\left(\sqrt{\lambda_n}r\right)$$

We showed that the vibrations of a circular membrane, fixed at the outer edge and with a zero initial displacement and initial velocity, is given by

$$u(r,t) = 0$$

No sound is therefore produced with a zero initial displacement and initial velocity. It simple means the drum is not struck or hit.

We will consider generating values for a practical case, when there is zero initial deflection and non zero initial velocity, since the "Gungon" is not played with the hand on the drum head to produce an initial displacement.

Taking the time (t) values from 0 to 9secs and the radius r takes on values

r = 0, 5, 10, 15, 20 and 25cm

The results in using the initial velocity functions are tabulated below.

TABLE 3: VALUES OF U(r,t) FOR VARIOUS TIME AND RADII

t/s \ r/cm	0	5	10	15	20	25
0	0	0	0	0	0	0
1	3.2297	2.5954	1.7302	1.0004	0.0916	0
2	2.9906	2.0017	1.4628	0.8123	0.0625	0
3	2.3713	1.9841	0.9001	0.7042	0.0126	0
4	1.8419	1.0072	0.7355	0.0393	0.0086	0
5	1.3191	0.7312	0.6789	0.0154	0.0073	0
6	1.2159	0.6703	0.3061	0.0094	0.0041	0
7	0.9603	0.5117	0.1013	0.0076	0.0015	0
8	0.7503	0.3228	0.0842	0.0071	0.0009	0
9	0.5712	0.0802	0.0671	0.0052	0.0004	0

Note: Where r/cm is the radius in centimeters and t/s is the time in seconds.

CHAPTER FIVE

ANALYSIS OF RESULTS, CONCLUSION AND RECOMMENDATIONS

5.1 DISCUSSION

In the thesis, we applied mathematics of the vibrating membrane to the drum based on the wave equation to investigate the sound (the normal modes) made when hitting different points on the drumhead and the effect of the Fourier coefficient on the sound produced.

Several qualitative properties of Bessel functions may be directly inferred from these graphs. The Bessel's functions oscillate and appear to resemble a decaying trigonometric function for large values of *x*. The amplitude of the Bessel's function decreases as the value of the positive (*n*) increases. Thus all Bessel's functions with $n \succ 1$ replicate the behavior of the sine graph although $J_o(x)$ has certain common features with the graph of cosine. Unlike sine graph, the oscillations of Bessel's functions are not periodic and their amplitudes fall off slowly with increasing values of *x*.

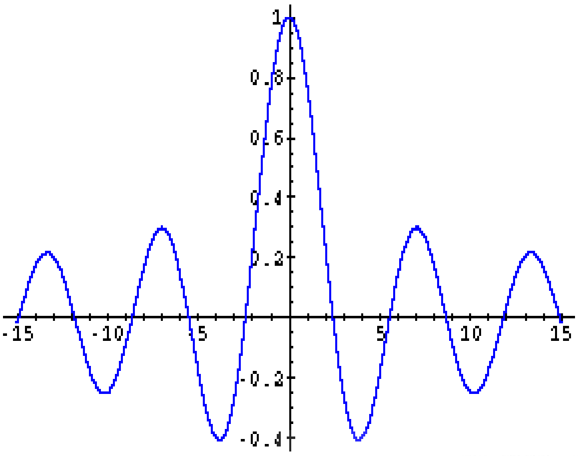

Fig 11: Graph of Bessel function of order zero

5.2 ANALYSIS OF RESULTS

The most common Gungon have a diameter of 38 to 50 cm. The vibration of the membrane corresponding to $u_m(r,t)$ is called the normal mode. The first normal mode, that is for *m=1*, all the points of the membrane move upward (or downward) at the same time. Here there is no concentric circle or nodal line. When the drum is hit at *m=2* the situation is as follows. The function

$$W_2(r) = J_0\left(\frac{\alpha_2}{R} r\right)$$

is zero for $\frac{\alpha_2}{R} r = \alpha_1$ or $r = \frac{\alpha_1 R}{\alpha_2}$.

The circle $r = \frac{\alpha_1 R}{\alpha_2}$ is, therefore, a nodal line, and when at some instant the central part of the membrane moves upwards, the outer part $\left(r \succ \frac{\alpha_1 R}{\alpha_2} \right)$ moves downwards, and conversely.

From table 3, the calculations revealed that the overtones and vibrations of the drumhead are not integral multiples of their fundamental frequencies and normal modes. The highest deflection was always produced at the centre (*r=0*) at different time (*t*), this is what you hear most when you play the drum. It was also observed that the sound seems to die off gradually when the time and radius was increasing, that is vibration breaks down close to the edge of the drum and no vibrations at the edge. The reason that the drum expert gives is that when you beat the drum closer to the edge it gradually wears off and the sound produced is not rich enough. The mathematical findings above attest to the fact that the "Gungon" has scientific basis in addition to the notion that it is just an artistic creation of our culture. The first fundamental vibration obtained with different radius and time vibrates up and down throughout the entire membrane. The varying amplitude due to the damping nature of the wave function prevents the sound produced by the drums to last long, as a results of the presence of the Bessel's function.

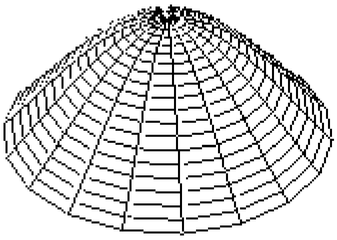

Fig 12: Fundamental mode

The animated figure above shows the fundamental modes shape of a vibrating circular membrane. The mode number is designated as $m = 1$ since there are no nodal diameters, but one circular node (the outside edge). (Remember that a node is a point (or line) on a structure that does not move while the rest of the structure is vibrating). The $m = 1$ mode of a drum, such as a 'Gungon', is excited when the drum head is struck at its center. When vibrating in this mode the membrane acts much like a monopole source, which radiates sound very effectively. Since it radiates sound so well when vibrating in this manner, the membrane quickly transfers its vibrational energy into radiated sound energy and the vibration dies away. The short duration (fraction of a second) of the $m = 1$ mode means that this mode does not contribute greatly to the musical tone quality of a drum. In fact, when struck at the center of the 'Gungon', it produces a "thump" which decays quickly and with no definite pitch.

The Fourier coefficients a_n, b_n also determines the character of the sound. If a_1, b_1 is much larger than all the $n > 1$ coefficients, then the note sounds smooth. But if a_n, b_n does not decrease rapidly with *n* (for example, if the *n-th* coefficient behaves like *1/n*) then the note sound quite sharp in character and perhaps even harsh. Thus you can hear something about the Fourier coefficients in a musical sound (University of Bristol Journal 2011). Since in our examples above, the a_1 and b_1 was much larger than all the $n > 1$ coefficients, then the note sound quite smooth. Therefore the sound produced by the drum (a two-dimensional instrument) in when there is no deflection and when deflection and velocity are present the sound produced is much pure and easily pleasing to the ear.

The frequencies with which a drum vibrates radially are the zeros λ_n of the Bessel function J_0. A vibrating string has frequencies $n\omega$ where ω^2 is the bass note, i.e. the frequencies are integer multiples of one bass frequency. The higher harmonics are in harmony with each other. We hear 'music' to our ears. For a drum, the frequencies λ_n are far from being an arithmetic progression. The radial frequencies λ_n, i.e. the zeros of J_0 are not an arithmetic progression. They are not integer multiples of the fundamental frequency. The totality of frequencies behaves like 'random numbers', i.e. numbers put out by a random number generator. We hear the bass note of a drum; the drum's muffled sound (Muffling is often referred to as muting, which can also refer to playing the drums with mutes on them) is due to the randomness of the higher frequencies. This fact makes such a drum a rhythmic instrument and not a harmonic instrument. The Drum with constant tension provides the rhythm of any musical piece, rather than add to the harmony due to its inharmonic nature since it overtones are not integral multiples of their fundamental frequency.

5.3 CONCLUSION

The "*Gungon"* drum used in this thesis with constant tension provides the rhythm, rather than harmony due to its inharmonic nature since its overtones and modes of vibrations are not integral multiples of their fundamental frequency and mode. The "*Gungon"* is therefore a bass drum, since it produces a rich bass tone (highest deflection) when struck in the middle as was evident from the values generated in chapter four.

The numerical results presented in this thesis showed that our local drums can be considered as having scientific basis.

5.4 RECOMMENDATIONS

1. We recommend that future research on this thesis topic should include the development of techniques to determine whether the results and analysis will be different from our work, when initial velocity and initial displacement are applied at the same time.

2. We suggest that from the conclusion, we may be able to reconstruct the shape of the drumhead by way of modifying the boundary conditions. For instance, what prevents us from presenting a drumhead of the "*Gungon"* in rectangular form

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APPENDIX A

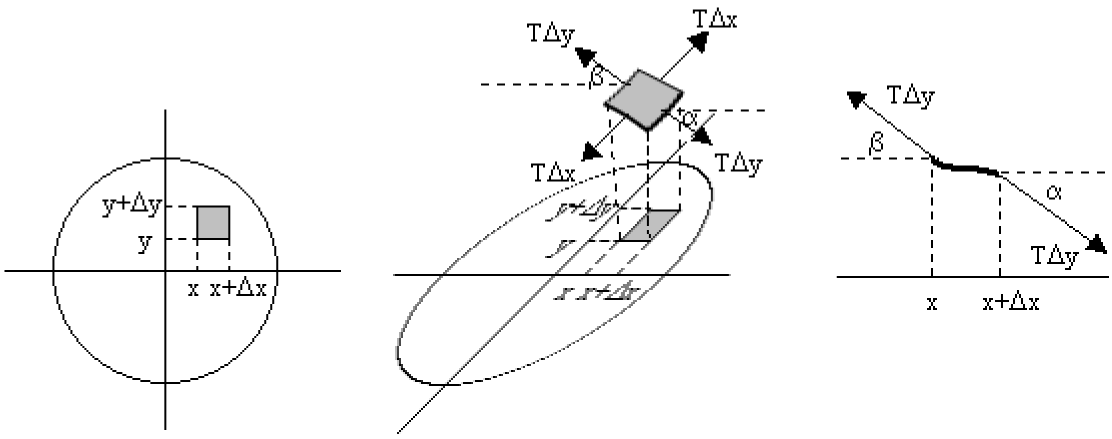
DERIVATION OF THE TWO - DIMENSIONAL WAVE EQUATION

We study the motion of a stretched circular membrane, such as the vibrating surface of a drum, subject to the following physical assumptions:

- The membrane has uniform mass per unit area, is perfectly flexible and so thin that it does not offer any resistance to bending.
- The membrane is stretched and then fixed along its circular boundary in the xy-plane, the tension T per unit length being the same at all points and along all directions, not changing during the motion.
- The vertical deflection $u = u(x, y, t)$ of the point (x, y) of the membrane at time t is small compared to the diameter.

The aim is to find an explicit formula for $u(x, y, t)$, given the initial deflection, so that we may model the motion of the vibrating drum

Consider a small portion of the membrane:



The sides of the portion are of small lengths *Δx and Δy*. The tension T is the force per unit length, hence the forces acting on the edges are *TΔx and TΔy*, tangent to the membrane since it is perfectly flexible. First consider the components of the forces parallel to the *xy*-plane. Along the *x*-axis, the resultant $T\Delta y \cos(\beta) - T\Delta y \cos(\alpha)$ is (almost) zero, since α and β are small. Similarly, the resultant force is (almost) zero along the y-axis. Thus, the motion of the portion is completely governed by the vertical components of the forces parallel to the *u-axis*. The vertical resultant force for the edges parallel to the x-axis is

$T\Delta y \sin(\beta) - T\Delta y \sin(\alpha)$

$= T\Delta y(\sin(\beta) - \sin(\alpha))$

$= T\Delta y(\tan(\beta) - \tan(\alpha))$ (since α and β are small)

$= T\Delta y(u_x(x + \Delta x, y_1) - u_x(x, y_2))$ (by Rolle's Mean Value Theorem)

where u_x denotes the partial derivative of u with respect to x and y_1 and y_2 are values between y and y+Δy. Similarly, the vertical resultant force for the edges parallel to the y-axis is

$T\Delta x(u_y(x_1, y+\Delta y) - u_y(x_2, y))$

where u_y denotes the partial derivative of u with respect to y and x_1 and x_2 are values between x and x+Δx. Thus the total resultant force on the portion is given by

$F = T\Delta y(u_x(x+\Delta x, y_1) - u_x(x, y_2)) + T\Delta x(u_y(x_1, y+\Delta y) - u_y(x_2, y))$.

Let ρ denote the mass per unit area of the membrane. The mass of the portion is $m = \rho\Delta x\Delta y$ and its acceleration is $a = \partial^2 u/\partial t^2$. By Newton's second law of motion F = ma we have

$\rho\Delta x\Delta y\partial^2 u/\partial t^2 = T\Delta y(u_x(x+\Delta x, y_1) - u_x(x, y_2)) + T\Delta x(u_y(x_1, y+\Delta y) - u_y(x_2, y))$.

Dividing both sides by ρΔxΔy we get

$\partial^2 u/\partial t^2 = (T/\rho)((u_x(x+\Delta x, y_1) - u_x(x, y_2))/\Delta x) + (u_y(x_1, y+\Delta y) - u_y(x_2, y))/\Delta y)$.

Now let Δx and Δy approach zero to obtain the differential equation

$\partial^2u/\partial t^2 = c^2(\partial^2u/\partial x^2+\partial^2u/\partial y^2)$ where $c^2 = T/\rho$.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

This differential equation is called the 2-dimensional wave equation.

APPENDIX B

TRANSFORMATION OF 2-D WAVE EQUATION INTO PLANE POLAR CO-ORDINATES

Changing from Cartesian to Polar Coordinates in 2-D

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

We ask what the form in polar coordinates with $x = r\cos\phi$, $y = r\sin\phi$ *and* $z = z$. To find out we use the chain rule. This gives use the results

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial r}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

To work out these partial derivatives we need $r = r(x, y)$ and $\phi = \phi(x, y)$. These functions are

$$r = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

So that

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\phi$$

$$\frac{\partial r}{\partial r} = \frac{y}{\sqrt{x^2 + y^2}} = \sin\phi$$

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin\phi}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\cos\phi}{r}$$

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = 0$$

These imply

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right)\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right) + \left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\right)\left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\right)$$

$$= \cos\phi\left(\cos\phi\frac{\partial^2}{\partial r^2} + \frac{\sin\phi}{r^2}\frac{\partial}{\partial \phi} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right) - \frac{\sin\phi}{r}\left(-\sin\phi\frac{\partial}{\partial r} + \cos\phi\frac{\partial^2}{\partial \phi \partial r} - \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} - \frac{\sin\phi}{r}\frac{\partial^2}{\partial \phi^2}\right)$$

$$\sin\phi\left(\sin\phi\frac{\partial^2}{\partial r^2} - \frac{\cos\phi}{r^2}\frac{\partial}{\partial \phi} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\right) + \frac{\cos\phi}{r}\left(\cos\phi\frac{\partial}{\partial r} + \sin\phi\frac{\partial^2}{\partial \phi \partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} + \frac{\cos s\phi}{r}\frac{\partial^2}{\partial \phi^2}\right)$$

$$\left(\cos^2\phi + \sin^2\phi\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)$$

$$\frac{1}{c^2}u_{tt} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)$$

So the wave equation for $u(r,\phi)$ in polar coordinates reads

$$\frac{1}{c^2}u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi}$$

APPENDIX C

CODE FOR DETERMINING THE ZEROS OF THE BESSEL'S FUNCTION $J_0(x)$ *and* $Y_0(x)$

```
x = linspace (-15:15);

jbess = besselj(0,x);

ybess = bessely(0,x);

plot(x,jbess,x,ybess);
```


APPENDIX D

MATLAB CODE FOR EVALUATING THE NORMAL MODES

% the code is made to evaluate the normal mode of a vibrating membrane

% by keeping the tension in the drumhead constant

% the code is prepared for the talking drum with radius 25cm.

% in order to make comparism from the output of the code.

% the following initial and boundary conditions $u(r,t)=0$ $u_t = g(r)$ were used

% with velocity $g(r)=\left(1-\frac{r}{25}\right)^2$ this functions was chosen so that nearer the origin the normal

%Mode would be greatest and further from the origin the normal mode would be least.

disp('DRUMHEAD WITH FIXED TENSION')

```
c=1; R=25;

disp([R'])

disp('where  _1  is the associated velocity functions used in computing the normal mode')

disp('j_x are the bessel function of order zero')

disp('A_n are the co-efficients associated with the drum')

disp('B_n are the co-efficients associated with the drum')

% k the first ten zeros of Bessel function of order zero

k=[2.4048,5.5201,8.6537,11.7915,14.9309,18.0711,21.2116,24.3525,27.4935,30.6346];

j_x=besselj(0,x)

%intj is the integration in the model

intJ=double(int(w*besselj(0,w.*u),w,0,R));

% t the values for the varying time

t1= [0 1 2 3 4 5 6 7 8 9];

u=k/R;

%p the values for the varying r's

p= [0, 5,10,15,20, 25];

p(r);

x(r,:)=p(r).*u;

end
```