



# Fractional Caputo and sensitivity heat map for a gonorrhea transmission model in a sex structured population

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## ABSTRACT

Gonorrhea is a disease that is spread by sexual contact, and it can potentially cause infections in the genital region, the rectum, and even the throat. Due to the shared history between infected individuals and their sexual partners, infected individuals will likely continue to have sexual relations with those same partners. As a result, this article aims to investigate how memory affects the transmission of gonorrhea in a structured population using the Caputo fractional derivative and sensitivity analysis. The model is shown to be positively invariant with a unique bound. The existence and uniqueness criteria of the fractional model are established using fixed-point theory. The stable nature of the model is obtained using the Ulam Hyers and Ulam Hyers Rassias ideas. To highlight the stability of the fractional model, the stability of solution trajectories to the disease-free and endemic steady states is graphically illustrated for the gonorrhea basic reproduction number,  $\mathcal{R}_0^{\alpha} < 1$  and  $\mathcal{R}_0^{\alpha} > 1$ , respectively. We showed the sensitivities linked to the proposed model using the Latin hypercube sampling, singular value analysis, box plots, scatter plots, contour plots, three-dimensional plots, and sensitivity heat maps. We noticed that the transmission rate from females to males,  $\beta_{fm}$ , is the most influential parameter in the spread of the disease. From the sensitivity heat maps, it is noticed that using the first four principal components analysis, the most sensitive state variables to the parameters in the model are symptomatic females, recovered males, susceptible females, and recovered females. In conjunction with the modified Adams–Bashforth method, the numerical trajectories of the fractional Caputo model are investigated. Finally, we noticed that memory changes impact the number of incubative females and incubative males.

## 1. Introduction

The bacteria *Neisseria gonorrhoea* is the source of the sexually transmitted infection (STI) known as gonorrhea. Both men and women can have gonorrhea, an infection of the urethra, rectum, and throat, which is highly prevalent, particularly among young adults aged 15 to 24. It may spread through vaginal, oral, or anal intercourse and even harm newborns after delivery. Babies born to infected mothers can also get the disease, with the eyes being the most commonly impacted organ. Infection with gonorrhea frequently has no signs. However, genital tract indications are widespread. However, it can also result in painful urine output, pus-like discharge, testicular swelling, increased secretions from the vagina, bleeds from the vagina, and abdominal or pelvic pain [1]. Carriers may continue propagating the disease, possibly much more so when asymptomatic. They could be more prone to infect their partner(s) during a sexual encounter without recognizing it [2]. Treatment for gonorrhea halts infection but does not reverse long-term effects; antimicrobial resistance concerns increase, making therapy more challenging to complete [3]. If left untreated, gonorrhea can result in severe, long-lasting health issues, including pelvic inflammatory disease (PID) in females and painful disorders in the tubes that connect the testicles, in rare circumstances, can make a man infertile [3]. Understanding the dynamics of gonorrhea using the mathematical model has provided many insightful views on the dynamics of the disease and how to prevent it. Lajmanovich and Yorke (1976) considered the mathematical model of gonorrhea by dividing the population into  $n$ -groups and studying the asymptotic stability characteristics [4]. The endemic scenario has a nontrivial periodic solution that is globally asymptotically stable, according to Aronsson and Mellander (1980), who adapted the Lajmanovich and Yorke model to add seasonal variation in the contact and removal rates [5]. A population of females to which the sailors were exposed was examined to evaluate the prevalence of *Neisseria gonorrhoea*. The number of sexual partners and the frequency of sex were statistically

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significantly correlated with the probability of gonorrhoea transmission. Further evidence reveals that most men are vulnerable to gonorrhoea if the amount of exposure is significant, as shown by the rising infection incidence with increasing numbers of exposures among men who had a single-sex partner [6]. Additionally, simulations of the whole HIV-gonorrhoea model revealed that a rise in gonorrhoea infections (singly or in combination with HIV) in the presence of therapy leads to a drop in gonorrhoea-only cases and an increase in HIV-only cases [7]. Alcoholism is now a major health concern for society and a menace on a worldwide scale. A deterministic alcohol model was developed and examined, and established its fundamental characteristics [8]. The work in [9] used sensitivity analysis to examine the effects of parameters on  $\mathcal{R}_0$ , including condom efficacy, effective contact rate, condom compliance, progression rate, and treatment rate. As the dynamics of the disease may take different paths, it is important also to understand the impact of memory on the spread of gonorrhoea, and to do this, is the use of fractional derivatives. In the past, there have been numerous studies on the relevance of fractional derivatives in epidemiology. For example, Ogunmiloro and Oluwatayo [10] presented a mathematical model with multiple population sections that describe the spread of fascioliasis in individuals, domestic animals, and the environment employing the Caputo fractional derivative. They demonstrated the effectiveness and convergence of the method via the numerically improved Euler method, and their selection of the Caputo derivative is due to the integration of conventional initial and boundary conditions. Using a modified Predictor–Corrector scheme, Kumar et al. [11] study examines the Lassa hemorrhagic outbreak, a disease that can harm pregnant women. Shah et al. [12] put out an epidemic model that employs the fractional-order derivative in the sense of Caputo–Fabrizio to explain the nature of the Ebola virus’s propagation. Aslam et al. [13] created a fractional-order COVID-19 model. They examined the domain’s convergence and nonnegative special solution. The model was solved using the Sumudu transformation method and the Caputo–Fabrizio methodology. Nwajeri et al. [14] studied a mathematical model for HPV and syphilis using the Caputo–Fabrizio fractional derivative, which was proven to be generalized Ulam Hyers–Rassias stable. Singh et al. [15] provided a mathematical model for stem cell development. The Atangana–Baleanu derivative in the Caputo (ABC) sense transformed the model from its non-fractional order formulation into a fractional sense. Farman et al. [16] investigated the dynamic propagation of computer viruses. They examined the consequences of corrupted portable storage devices and external computers. Adamu and Usman [17] created a mathematical model for the dynamics of Neisseria gonorrhoea infection transmission and investigated how the only known control interventions-natural immunity and treatment-affected the disease’s spread in a population. However, a conceptual model must take gonorrhoea’s specific epidemiologic characteristics into account. It is sufficient to consider only the community’s sexually active members who might spread the disease to their connections. The transmission dynamics of gonorrhoea in a sex-structured population can be modeled, which can assist in identifying factors that contribute to the disease’s spread and guide the creation of successful management techniques. Using a fractional Caputo differential equation to represent gonorrhoea transmission is one approach. This type of equation includes a non-integer order derivative, which allows for modeling long-range memory effects. It is possible to generate a sensitivity heat map to analyze how sensitive the model is to changes in the parameters. In this step, the values of the model’s parameters are changed within a predetermined range, and then the changes that occur in the equilibrium points of the model are calculated as a result. The equilibrium points are symbolic representations of the populations’ steady-state values under varying environmental conditions. Therefore, this study builds on the work of [17] by developing a novel Caputo fractional mathematical model for a crisscrossed population with the effects of memory. Crisscrossed models are a slight extension of the generalized population model to account for the fact that gonorrhoea can be transmitted through sexual activity between males and females, where the female infectives transmit the disease to a male susceptible and vice versa.

The remaining parts of the paper are organized as follows: The preliminaries of the research were provided in Section 2. In Section 3, we presented the fractional model. Section 4 contains the model’s qualitative characteristics are presented, that is, the model’s positivity and boundedness condition, the existence and uniqueness, basic reproduction number, the gonorrhoea-free equilibrium and gonorrhoea present equilibrium, and the Hyers–Ulam stability. Section 5 contains the numerical simulations. In Section 6, we presented the study’s conclusion.

**2. Preliminaries**

In this section, some basic definitions that will guide us through the formulation of the fractional gonorrhoea disease model are introduced based on the available information in [18–23].

**Definition 2.1.** The Caputo fractional derivative of the function  $\phi(p)$  of order  $\tau$  ( $\tau > 0$ ) is given by

$$\begin{cases} {}^C D_p^\tau \phi(p) = \frac{1}{\Gamma(m-\tau)} \int_0^p (p-\epsilon)^{m-\tau-1} \frac{d^m \phi(\epsilon)}{d\epsilon^m} d\epsilon, & m-1 < \tau < m, m \in \mathbb{N}, \\ \frac{d^m}{d p^m} \phi(p), & m = \tau, m \in \mathbb{N}. \end{cases}$$

with  $\Gamma(k) = \int_0^\infty e^{-p} p^{k-1} dp$ , being the Euler’s Gamma function.

The Caputo fractional integral of order  $\tau > 0$  is given by

$$I^\tau \left( {}^C D_p^\tau \phi(p) \right) = \frac{1}{\Gamma(\tau)} \int_0^p \frac{\phi(s)}{(p-s)^{1-\tau}} ds,$$

where  $p > 0$ , fulfills the following:

$$\begin{cases} (I^\tau \phi(p))^\tau = \phi(p), \\ I^\tau (\phi(p)^\tau) = \phi(p) - \phi(0). \end{cases}$$

**3. Formulation of the mathematical model**

We divided the model’s population into men and females. Each group is divided into four stages: the susceptible (vulnerable), the incubating, the symptomatic, and the recovered stages. The number of susceptible females is denoted by  $S_f(t)$ , the class of symptomatic females by  $I_{s,f}(t)$ , the class of incubating females by  $I_{c,f}(t)$ , and the class of recovered females by  $R_f^T(t)$ . For the male population,  $S_m(t)$  stands for those who are vulnerable,  $I_{c,m}(t)$  for those who are incubating,  $I_{s,m}(t)$  for those who are showing symptoms, and  $R_m^T(t)$  for those who have fully recovered. The male recruitment rate is represented by  $\theta_m$  and the female recruitment rate by  $\theta_f$ . Male recovery rate is denoted by the  $\alpha_m$ , whereas female recovery rate is written as  $\alpha_f$ . When a symptomatic man comes into touch with a susceptible female, the chance of infection is  $\eta_{sm}$ , but when an incubating male comes into

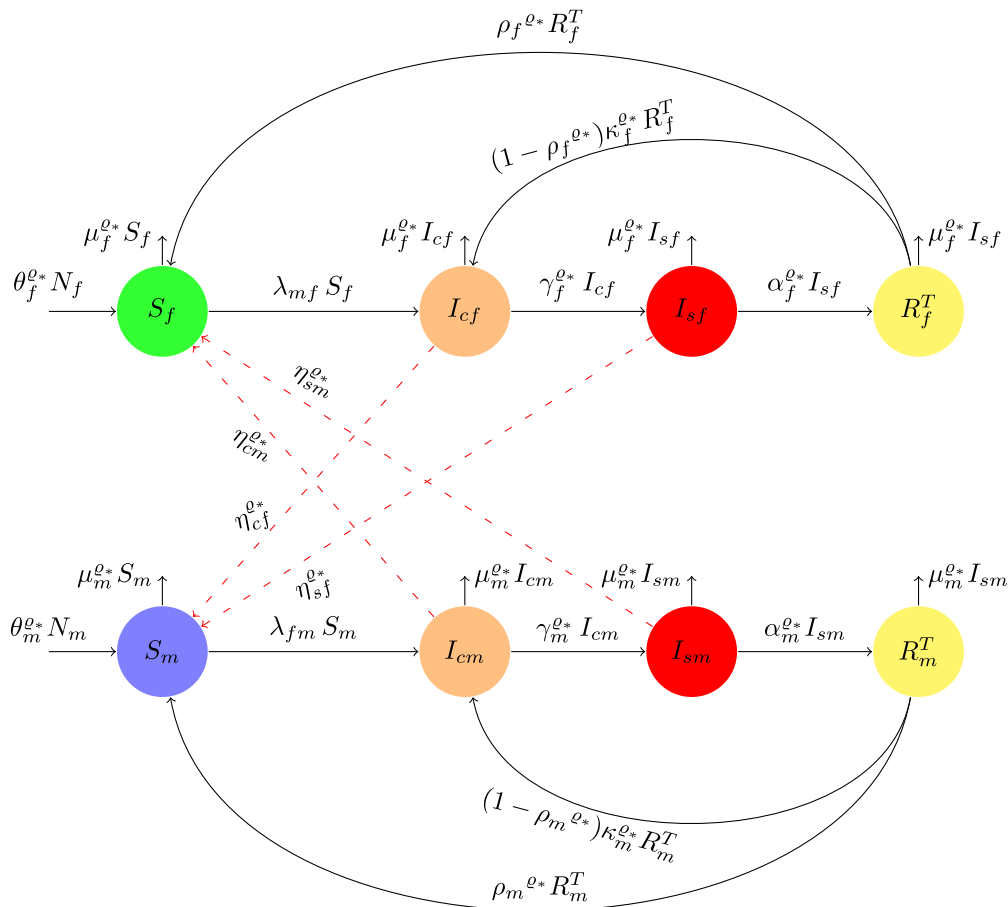


Fig. 1. Schematic diagram of the gonorrhea in a structured population, where  $\lambda_{mf} = \frac{\beta_{mf}^{e*}(\eta_{sm}^{e*} I_{sm} + \eta_{cm}^{e*} I_{cm})}{N_f}$ , and  $\lambda_{fm} = \frac{\beta_{fm}^{e*}(\eta_{sf}^{e*} I_{sf} + \eta_{cf}^{e*} I_{cf})}{N_m}$ .

contact with a susceptible female, the probability of infection is  $\eta_{cm}$ . Similarly,  $\eta_{sf}$  stands for the infectivity rate via contact between symptomatic females and susceptible men, and  $\eta_{cf}$  stands for the infectivity rate through contact between incubating females and susceptible males. The new transmission force between men and women is marked by the symbol  $\beta_{mf}$ , whereas the same force between women and men is represented by the symbol  $\beta_{fm}$ . Infection net force is indicated by  $\lambda_{mf}$ , and infection net force per unit mass is denoted by  $\lambda_{fm}$ . Females in recovery have a relapse rate of  $\kappa_f$ , while females in the incubative class have a transition rate of  $(1 - \rho_f)$ .  $(1 - \rho_m)$  is the percentage of cured men that revert to the incubative male susceptibility class. The  $\kappa_m$  denotes the rate at which men in the recovery class revert to the incubative class. The assumed disease-induced death rate in the female compartment is denoted by  $\epsilon_f$ , and that of the disease-induced death rate in the male compartment is represented by  $\epsilon_m$ . The natural death rate for women is denoted by  $\mu_f$ , whereas for men, it is denoted by  $\mu_m$ .  $\gamma_m$  and  $\gamma_f$  represent the rate at which men in the incubative class transition to the incubative male class, respectively. Male and female student populations are calculated as follows:  $N_f = S_f(t) + I_{cf}(t) + I_{sf}(t) + R_f^T(t)$ , and  $N_m = S_m(t) + I_{cm}(t) + I_{sm}(t) + R_m^T(t)$ . Below are the assumptions and table for all the parameters and their description.

**Assumption**

- i. Population size is constant.
- ii. All classes in each sub-compartmental block have the same natural death rate.
- iii. Death by disease is neglected because very few people die due to the contraction of the disease.
- iv. Temporal recovery (i.e. recovered individuals become susceptible again).
- v. There is the possibility of relapse; that is, an individual under treatment(recovered) does not complete treatment and relapses to the incubative classes.
- vi. Only the heterosexual class was considered (the disease transmission among individuals of the same gender is not considered).
- vii. It is assumed in this model that only individuals from the symptomatic class will recover because they seek medical help after seeing the symptoms of gonorrhea whereas the incubative class may not know that they have contracted the disease.
- viii. The incubative class is comprised of both exposed individuals and asymptomatic patients.

| Parameters and descriptions |  |
|-----------------------------|--|
| $\theta_f$                  | Recruitment rate in females  |
| $\theta_m$                  | Recruitment rate in males  |
| $\mu_f$                     | Natural death rate in females  |
| $\mu_m$                     | Natural death rate in males  |
| $\eta_{cf}$                 | Infectivity potential by contact between incubative females and susceptible males  |
| $\eta_{sf}$                 | Infectivity potential by contact between symptomatic females and susceptible males |
| $\eta_{cm}$                 | Infectivity potential by contact between incubative males and susceptible females  |
| $\eta_{sm}$                 | Infectivity potential by contact between symptomatic males and susceptible females |
| $\beta_{fm}$                | Transmission rate from females to males  |
| $\beta_{mf}$                | Transmission rate from males to a females  |
| $\alpha_f$                  | Recovery rate in females   |
| $\alpha_m$                  | Recovery rate in males   |
| $\rho_f$                    | The proportion at which recovered females become susceptible again                 |
| $\rho_m$                    | The proportion at which recovered males become susceptible again                   |
| $\kappa_f$                  | The rate at which females in the recovery class have a relapse                     |
| $\kappa_m$                  | The rate at which males in the recovery class have a relapse                       |
| $\gamma_f$                  | The rate at which females in the incubative class move to the symptomatic class    |
| $\gamma_m$                  | The rate at which males in the incubative class move to the symptomatic class      |

Our fractional Caputo model that describe the dynamics of gonorrhoea in the male and female population is represented in Fig. 1. Eq. (1) represents the nonlinear fractional Caputo mathematical model of gonorrhoea considering the two populations; for balance in the dimension in the model parameters, we raise the individual parameters to the fractional order  $\rho_*$ , where  $\rho_* \in (0, 1]$ . Making  $\rho_* = 1$ , turns the model into a classical integer model.

$$\begin{cases} {}^C D_t^{\rho_*} S_f &= \theta_f^{\rho_*} - \lambda_{mf} S_f + \rho_f^{\rho_*} R_f^T - \mu_f^{\rho_*} S_f, \\ {}^C D_t^{\rho_*} I_{cf} &= \lambda_{mf} S_f + (1 - \rho_f^{\rho_*}) \kappa_f^{\rho_*} R_f^T - (\gamma_f^{\rho_*} + \mu_f^{\rho_*}) I_{cf}, \\ {}^C D_t^{\rho_*} I_{sf} &= \gamma_f^{\rho_*} I_{cf} - (\alpha_f^{\rho_*} + \mu_f^{\rho_*} + \epsilon_f^{\rho_*}) I_{sf}, \\ {}^C D_t^{\rho_*} R_f^T &= \alpha_f^{\rho_*} I_{sf} - (\rho_f^{\rho_*} + (1 - \rho_f^{\rho_*}) \kappa_f^{\rho_*} + \mu_f^{\rho_*}) R_f^T, \\ {}^C D_t^{\rho_*} S_m &= \theta_m^{\rho_*} - \lambda_{fm} S_m + \rho_m^{\rho_*} R_m^T - \mu_m^{\rho_*} S_m, \\ {}^C D_t^{\rho_*} I_{cm} &= \lambda_{fm} S_m + (1 - \rho_m^{\rho_*}) \kappa_m^{\rho_*} R_m^T - (\gamma_m^{\rho_*} + \mu_m^{\rho_*}) I_{cm}, \\ {}^C D_t^{\rho_*} I_{sm} &= \gamma_m^{\rho_*} I_{cm} - (\alpha_m^{\rho_*} + \mu_m^{\rho_*} + \epsilon_m^{\rho_*}) I_{sm}, \\ {}^C D_t^{\rho_*} R_m^T &= \alpha_m^{\rho_*} I_{sm} - ((1 - \rho_m^{\rho_*}) \kappa_m^{\rho_*} + \rho_m^{\rho_*} + \mu_m^{\rho_*}) R_m^T, \end{cases} \tag{1}$$

where  $S_f(t) > 0, I_{cf}(t) \geq 0, I_{sf}(t) \geq 0, R_f^T(t) \geq 0, S_m(t) > 0, I_{cm}(t) \geq 0, I_{sm}(t) \geq 0, R_m^T(t) \geq 0$ ; with  $\lambda_{mf} = \frac{\beta_{mf}^{\rho_*} (\eta_{sm}^{\rho_*} I_{sm} + \eta_{cm}^{\rho_*} I_{cm})}{N_f}$ , and  $\lambda_{fm} = \frac{\beta_{fm}^{\rho_*} (\eta_{sf}^{\rho_*} I_{sf} + \eta_{cf}^{\rho_*} I_{cf})}{N_m}$ .

#### 4. Model analytical study

Some necessary and sufficient criteria for the model are presented here, such as non-negativity, uniform boundedness, solutions' existence, uniqueness, local stability and Hyers–Ulam stability.

##### 4.1. Non-negativity and uniform boundedness

Given that

$$\mathcal{A} = \left\{ \mathcal{M}(t) : \mathcal{M}(t) = (S_f(t), I_{cf}(t), I_{sf}(t), R_f^T(t), S_m(t), I_{cm}(t), I_{sm}(t), R_m^T(t)) \in \mathcal{R}^8, \mathcal{M}(t) \geq 0 \right\}.$$

To prove that all solutions to the model under consideration are positive and fall within the set  $\mathcal{A}$ , we use the Theorem as discussed in [24,25].

**Theorem 4.1.** Let  $m(x) \in C[u, v]$  and  ${}^C D_x^{\rho_*} \in C(u, v)$  for  $0 < \rho_* \leq 1$ , then

$$m(x) = m(u) + \frac{1}{\gamma(\rho_*)} \left( {}^C D_x^{\rho_*} n \right) (\epsilon)(x - u)^{\rho_*},$$

where  $u \leq \epsilon \leq x$ , for all  $x \in (u, v)$ .

As a consequence from Theorem 4.1, we propose the following.

**Corollary 4.1.** Assume that  $n(x) \in [u, v]$  and  ${}^C D_x^{\rho_*} \in (0, v)$  for  $0 < \rho_* \leq 1$ . If  ${}^C D_x^{\rho_*} n(0) \geq 0, \forall t \in (0, v)$ , then the function  $n$  is not decreasing, and if  ${}^C D_x^{\rho_*} n(0) \leq 0, \forall x \in (0, v)$ , then function  $n$  is not increasing  $\forall x \in (0, v)$ .

**Theorem 4.2.** The solution to the gonorrhoea model (1) is oriented positively and belongs to  $\mathcal{A}$ .

**Proof.** Applying [Theorem 4.2](#), we show that the solution to the model (1) is positive, thus taking this

$${}^C D_t^{\rho_f} S_f \Big|_{S_f=0} = \theta_f^{\rho_f} N_f + \rho_f^{\rho_f} R_f^T > 0, \tag{2}$$

$${}^C D_t^{\rho_f} I_{cf} \Big|_{I_{cf}=0} = \frac{\beta_{mf}^{\rho_f} (\eta_{sm}^{\rho_f} I_{sm} + \eta_{cm}^{\rho_f} I_{cm})}{N_f} S_f + (1 - \rho_f^{\rho_f}) \kappa_f^{\rho_f} R_f^T \geq 0, \tag{3}$$

$${}^C D_t^{\rho_f} I_{sf} \Big|_{I_{sf}=0} = \gamma_f^{\rho_f} I_{cf} \geq 0, \tag{4}$$

$${}^C D_t^{\rho_f} R_f^T \Big|_{R_f^T=0} = \alpha_f^{\rho_f} I_{sf} \geq 0, \tag{5}$$

$${}^C D_t^{\rho_m} S_m \Big|_{S_m=0} = \theta_m^{\rho_m} N_m + \rho_m^{\rho_m} R_m^T > 0, \tag{6}$$

$${}^C D_t^{\rho_m} I_{cm} \Big|_{I_{cm}=0} = \lambda_{fm}^{\rho_m} S_m + (1 - \rho_m^{\rho_m}) \kappa_m^{\rho_m} R_m^T \geq 0, \tag{7}$$

$${}^C D_t^{\rho_m} I_{sm} \Big|_{I_{sm}=0} = \gamma_m^{\rho_m} i_{cm} \geq 0, \tag{8}$$

$${}^C D_t^{\rho_m} R_m^T \Big|_{R_m^T=0} = \alpha_m^{\rho_m} I_{sm} \geq 0. \tag{9}$$

Hence, in reference to [Corollary 4.1](#), it can be said that all the solutions associated with model (1) are nonnegative and is owned by  $\mathcal{A}$ .  $\square$

**Lemma 4.1.** The solution set  $\{S_f, I_{cf}, I_{sf}, R_f^T, S_m, I_{cm}, I_{sm}, R_m^T\}$  is bounded and positively invariant in

$$\Omega = \left\{ (S_f, I_{cf}, I_{sf}, R_f^T, S_m, I_{cm}, I_{sm}, R_m^T) \in \mathbb{R}^8 \mid N_f \leq \frac{\theta_f^{\rho_f}}{\mu_f^{\rho_f}} \text{ and } N_m \leq \frac{\theta_m^{\rho_m}}{\mu_m^{\rho_m}} \right\}. \tag{10}$$

**Proof.** Firstly, adding the female compartments, we have;

$${}^C D_t^{\rho_f} N_f = \theta_f^{\rho_f} - \mu_f^{\rho_f} (S_f + I_{cf} + I_{sf} + R_f^T) - \epsilon_f^{\rho_f}, \tag{11}$$

$$\leq \theta_f^{\rho_f} - \mu_f^{\rho_f} N_f, \tag{12}$$

Due to the fractional nature on the right-hand side, in solving (12), we apply the Laplace transform, and we obtain

$$N_f(t) \leq \left( N_f(0) - \frac{\theta_f^{\rho_f}}{\mu_f^{\rho_f}} \right) \mathbf{E} \left( -\mu_f^{\rho_f} t^{\rho_f} \right) + \frac{\theta_f^{\rho_f}}{\mu_f^{\rho_f}}, \tag{13}$$

where  $\mathbf{E} \left( -\mu_f^{\rho_f} t^{\rho_f} \right)$  represent the Mittag-Leffler function of fractional order  $\rho_f$ .

Therefore, considering that  $N_f(0) \leq \frac{\theta_f^{\rho_f}}{\mu_f^{\rho_f}}$ , then  $t > 0$ ,  $N_f(t) \leq \frac{\theta_f^{\rho_f}}{\mu_f^{\rho_f}}$ . Using a similar approach for the total male population, we have;

$$N_m(t) \leq \left( N_m(0) - \frac{\theta_m^{\rho_m}}{\mu_m^{\rho_m}} \right) \mathbf{E} \left( -\mu_m^{\rho_m} t^{\rho_m} \right) + \frac{\theta_m^{\rho_m}}{\mu_m^{\rho_m}}, \tag{14}$$

where  $\mathbf{E} \left( -\mu_m^{\rho_m} t^{\rho_m} \right)$  represent the Mittag-Leffler function of fractional order  $\rho_m$ .

Therefore, considering that  $N_m(0) \leq \frac{\theta_m^{\rho_m}}{\mu_m^{\rho_m}}$ , then  $t > 0$ ,  $N_m(t) \leq \frac{\theta_m^{\rho_m}}{\mu_m^{\rho_m}}$ . This ends the proof of [Lemma 4.1](#).  $\square$

### 4.2. Scaling

Now, before we identify the equilibrium points and the reproduction number, we use the scaling approach to lower the complexity of our system. To do this, we scale the equation in the original fractional model, which is denoted by (1), using the transformations;  $s_f = \frac{S_f}{N_f}$ ;  $i_{cf} = \frac{i_{cf}}{N_f}$ ;  $i_{sf} = \frac{I_{sf}}{N_f}$ ;  $r_f = \frac{R_f^T}{N_f}$  and  $\epsilon_f = \epsilon_m = 0$ , thus, assuming that disease-induced death rate is very low, gives:

$$\begin{aligned} {}^C D_t^{\rho_f} s_f &= \theta_f^{\rho_f} - \beta_{mf}^{\rho_f} (\eta_{sm}^{\rho_f} i_{sm} + \eta_{cm}^{\rho_f} i_{cm}) s_f + \rho_f^{\rho_f} r_f - \mu_f^{\rho_f} s_f, \\ {}^C D_t^{\rho_f} i_{cf} &= \beta_{mf}^{\rho_f} (\eta_{sm}^{\rho_f} i_{sm} + \eta_{cm}^{\rho_f} i_{cm}) s_f + (1 - \rho_f^{\rho_f}) \kappa_f^{\rho_f} r_f - (\gamma_f^{\rho_f} + \mu_f^{\rho_f}) i_{cf}, \\ {}^C D_t^{\rho_f} i_{sf} &= \gamma_f^{\rho_f} i_{cf} - (\alpha_f^{\rho_f} + \mu_f^{\rho_f}) i_{sf}, \\ {}^C D_t^{\rho_f} r_f &= \alpha_f^{\rho_f} i_{sf} - (\rho_f^{\rho_f} + (1 - \rho_f^{\rho_f}) \kappa_f^{\rho_f} + \mu_f^{\rho_f}) r_f, \\ {}^C D_t^{\rho_m} s_m &= \theta_m^{\rho_m} - \beta_{fm}^{\rho_m} (\eta_{sf}^{\rho_m} i_{sf} + \eta_{cf}^{\rho_m} i_{cf}) s_m + \rho_m^{\rho_m} r_m - \mu_m^{\rho_m} s_m, \\ {}^C D_t^{\rho_m} i_{cm} &= \beta_{fm}^{\rho_m} (\eta_{sf}^{\rho_m} i_{sf} + \eta_{cf}^{\rho_m} i_{cf}) s_m + (1 - \rho_m^{\rho_m}) \kappa_m^{\rho_m} r_m - (\gamma_m^{\rho_m} + \mu_m^{\rho_m}) i_{cm}, \\ {}^C D_t^{\rho_m} i_{sm} &= \gamma_m^{\rho_m} i_{cm} - (\alpha_m^{\rho_m} + \mu_m^{\rho_m}) i_{sm}, \\ {}^C D_t^{\rho_m} r_m &= \alpha_m^{\rho_m} i_{sm} - (\rho_m^{\rho_m} + (1 - \rho_m^{\rho_m}) \kappa_m^{\rho_m} + \mu_m^{\rho_m}) r_m. \end{aligned} \tag{15}$$

### 4.3. Existence and uniqueness

In this section, we investigate whether or not the model of fractional order, Eq. (1), has a solution. The existence argument has numerous practical uses and is a significant result of applied mathematics. If the model we are looking at exists, it tells us whether or not there is a

solution. Determine the possibility that there will be only one solution to the problem. It also helps us learn more about the solution’s qualitative characteristics. We use fixed-point theory to demonstrate the existence and uniqueness of the model [26,27]. Considering a Banach space  $\mathbb{B}(\mathcal{E})$  for a continuous real-valued function  $\mathcal{E} = [0, \theta]$ , we obtain the sub norm  $\mathbb{L} = \mathbb{B}(\mathcal{E}) \times \mathbb{B}(\mathcal{E}) \times \mathbb{B}(\mathcal{E}) \times \mathbb{B}(\mathcal{E}) \times \mathbb{B}(\mathcal{E}) \times \mathbb{B}(\mathcal{E}) \times \mathbb{B}(\mathcal{E}) \times \mathbb{B}(\mathcal{E})$  with the norm

$$\begin{aligned} \|(s_f, i_{cf}, i_{sf}, r_f, s_m, i_{cm}, i_{sm}, r_m)\| &= \|s_f\| + \|i_{cf}\| + \|i_{sf}\| + \|r_f\| + \|s_m\| + \|i_{cm}\| + \|i_{sm}\| + \|r_m\|, \\ \|s_f\| &= \sup_{t \in \mathcal{E}} |s_f|, \|i_{cf}\| = \sup_{t \in \mathcal{E}} |i_{cf}|, \|i_{sf}\| = \sup_{t \in \mathcal{E}} |i_{sf}|, \|r_f\| = \sup_{t \in \mathcal{E}} |r_f|, \|s_m\| = \sup_{t \in \mathcal{E}} |s_m| \\ \|i_{cm}\| &= \sup_{t \in \mathcal{E}} |i_{cm}|, \|i_{sm}\| = \sup_{t \in \mathcal{E}} |i_{sm}|, \|r_m\| = \sup_{t \in \mathcal{E}} |r_m|. \end{aligned}$$

Using the fundamental theorem of calculus and the Caputo fractional operator, we now obtain

$$\begin{aligned} s_f(t) - s_f(0) &= {}^C D_t^{\rho_f^*} \left[ \theta_f^{\rho_f^*} - \beta_{mf}^{\rho_f^*} (\eta_{sm}^{\rho_f^*} i_{sm}(t) + \eta_{cm}^{\rho_f^*} i_{cm}(t)) s_f(t) + \rho_f^{\rho_f^*} r_f(t) - \mu_f^{\rho_f^*} s_f(t) \right], \\ i_{cf}(t) - i_{cf}(0) &= {}^C D_t^{\rho_f^*} \left[ \beta_{mf}^{\rho_f^*} (\eta_{sm}^{\rho_f^*} i_{sm}(t) + \eta_{cm}^{\rho_f^*} i_{cm}(t)) s_f(t) + (1 - \rho_f^{\rho_f^*}) \kappa_f^{\rho_f^*} r_f(t) - (\gamma_f^{\rho_f^*} + \mu_f^{\rho_f^*}) i_{cf}(t) \right], \\ i_{sf}(t) - i_{sf}(0) &= {}^C D_t^{\rho_f^*} \left[ \gamma_f^{\rho_f^*} i_{cf}(t) - (\alpha_f^{\rho_f^*} + \mu_f^{\rho_f^*}) i_{sf}(t) \right], \\ r_f(t) - r_f(0) &= {}^C D_t^{\rho_f^*} \left[ \alpha_f^{\rho_f^*} i_{sf}(t) - (\rho_f^{\rho_f^*} + (1 - \rho_f^{\rho_f^*}) \kappa_f^{\rho_f^*} + \mu_f^{\rho_f^*}) r_f(t) \right], \\ s_m(t) - s_m(0) &= {}^C D_t^{\rho_m^*} \left[ \theta_m^{\rho_m^*} - \beta_{fm}^{\rho_m^*} (\eta_{sf}^{\rho_m^*} i_{sf}(t) + \eta_{cf}^{\rho_m^*} i_{cf}(t)) s_m(t) + \rho_m^{\rho_m^*} r_m(t) - \mu_m^{\rho_m^*} s_m(t) \right], \\ i_{cm}(t) - i_{cm}(0) &= {}^C D_t^{\rho_m^*} \left[ \beta_{fm}^{\rho_m^*} (\eta_{sf}^{\rho_m^*} i_{sf}(t) + \eta_{cf}^{\rho_m^*} i_{cf}(t)) s_m(t) + (1 - \rho_m^{\rho_m^*}) \kappa_m^{\rho_m^*} r_m(t) - (\gamma_m^{\rho_m^*} + \mu_m^{\rho_m^*}) i_{cm}(t) \right], \\ i_{sm}(t) - i_{sm}(0) &= {}^C D_t^{\rho_m^*} \left[ \gamma_m^{\rho_m^*} i_{cm}(t) - (\alpha_m^{\rho_m^*} + \mu_m^{\rho_m^*}) i_{sm}(t) \right], \\ r_m(t) - r_m(0) &= {}^C D_t^{\rho_m^*} \left[ \alpha_m^{\rho_m^*} i_{sm}(t) - (\rho_m^{\rho_m^*} + (1 - \rho_m^{\rho_m^*}) \kappa_m^{\rho_m^*} + \mu_m^{\rho_m^*}) r_m(t) \right]. \end{aligned} \tag{16}$$

Now, letting

$$\begin{aligned} \mathbb{H}_1 &= \left[ \theta_f^{\rho_f^*} - \beta_{mf}^{\rho_f^*} (\eta_{sm}^{\rho_f^*} i_{sm}(t) + \eta_{cm}^{\rho_f^*} i_{cm}(t)) s_f(t) + \rho_f^{\rho_f^*} r_f(t) - \mu_f^{\rho_f^*} s_f(t) \right], \\ \mathbb{H}_2 &= \left[ \beta_{mf}^{\rho_f^*} (\eta_{sm}^{\rho_f^*} i_{sm}(t) + \eta_{cm}^{\rho_f^*} i_{cm}(t)) s_f(t) + (1 - \rho_f^{\rho_f^*}) \kappa_f^{\rho_f^*} r_f(t) - (\gamma_f^{\rho_f^*} + \mu_f^{\rho_f^*}) i_{cf}(t) \right], \\ \mathbb{H}_3 &= \left[ \gamma_f^{\rho_f^*} i_{cf}(t) - (\alpha_f^{\rho_f^*} + \mu_f^{\rho_f^*}) i_{sf}(t) \right], \\ \mathbb{H}_4 &= \left[ \alpha_f^{\rho_f^*} i_{sf}(t) - (\rho_f^{\rho_f^*} + (1 - \rho_f^{\rho_f^*}) \kappa_f^{\rho_f^*} + \mu_f^{\rho_f^*}) r_f(t) \right], \\ \mathbb{H}_5 &= \left[ \theta_m^{\rho_m^*} - \beta_{fm}^{\rho_m^*} (\eta_{sf}^{\rho_m^*} i_{sf}(t) + \eta_{cf}^{\rho_m^*} i_{cf}(t)) s_m(t) + \rho_m^{\rho_m^*} r_m(t) - \mu_m^{\rho_m^*} s_m(t) \right], \\ \mathbb{H}_6 &= \left[ \beta_{fm}^{\rho_m^*} (\eta_{sf}^{\rho_m^*} i_{sf}(t) + \eta_{cf}^{\rho_m^*} i_{cf}(t)) s_m(t) + (1 - \rho_m^{\rho_m^*}) \kappa_m^{\rho_m^*} r_m(t) - (\gamma_m^{\rho_m^*} + \mu_m^{\rho_m^*}) i_{cm}(t) \right], \\ \mathbb{H}_7 &= \left[ \gamma_m^{\rho_m^*} i_{cm}(t) - (\alpha_m^{\rho_m^*} + \mu_m^{\rho_m^*}) i_{sm}(t) \right], \\ \mathbb{H}_8 &= \left[ \alpha_m^{\rho_m^*} i_{sm}(t) - (\rho_m^{\rho_m^*} + (1 - \rho_m^{\rho_m^*}) \kappa_m^{\rho_m^*} + \mu_m^{\rho_m^*}) r_m(t) \right]. \end{aligned} \tag{17}$$

Eq. (17) can be put in a Caputo integral form as

$$\begin{aligned} s_f(t) - s_f(0) &= \frac{1}{\Gamma(\rho_f^*)} \int_0^t \frac{\mathbb{H}_1(\rho_f^*, \mathfrak{s}, s_f(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_f^*}} d\mathfrak{s}, \\ i_{cf}(t) - i_{cf}(0) &= \frac{1}{\Gamma(\rho_f^*)} \int_0^t \frac{\mathbb{H}_2(\rho_f^*, \mathfrak{s}, i_{cf}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_f^*}} d\mathfrak{s}, \\ i_{sf}(t) - i_{sf}(0) &= \frac{1}{\Gamma(\rho_f^*)} \int_0^t \frac{\mathbb{H}_3(\rho_f^*, \mathfrak{s}, i_{sf}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_f^*}} d\mathfrak{s}, \\ r_f(t) - r_f(0) &= \frac{1}{\Gamma(\rho_f^*)} \int_0^t \frac{\mathbb{H}_4(\rho_f^*, \mathfrak{s}, r_f(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_f^*}} d\mathfrak{s}, \\ s_m(t) - s_m(0) &= \frac{1}{\Gamma(\rho_m^*)} \int_0^t \frac{\mathbb{H}_5(\rho_m^*, \mathfrak{s}, s_m(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_m^*}} d\mathfrak{s}, \\ i_{cm}(t) - i_{cm}(0) &= \frac{1}{\Gamma(\rho_m^*)} \int_0^t \frac{\mathbb{H}_6(\rho_m^*, \mathfrak{s}, i_{cm}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_m^*}} d\mathfrak{s}, \\ i_{sm}(t) - i_{sm}(0) &= \frac{1}{\Gamma(\rho_m^*)} \int_0^t \frac{\mathbb{H}_7(\rho_m^*, \mathfrak{s}, i_{sm}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_m^*}} d\mathfrak{s}, \\ r_m(t) - r_m(0) &= \frac{1}{\Gamma(\rho_m^*)} \int_0^t \frac{\mathbb{H}_8(\rho_m^*, \mathfrak{s}, r_m(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\rho_m^*}} d\mathfrak{s}. \end{aligned} \tag{18}$$

Notice that  $\mathbb{H}_1(s_f, \mathfrak{s}), \mathbb{H}_2(i_{cf}, \mathfrak{s}), \mathbb{H}_3(i_{sf}, \mathfrak{s}), \mathbb{H}_4(r_f, \mathfrak{s}), \mathbb{H}_5(s_m, \mathfrak{s}), \mathbb{H}_6(i_{cm}, \mathfrak{s}), \mathbb{H}_7(i_{sm}, \mathfrak{s}), \mathbb{H}_8(r_m, \mathfrak{s})$ , conforms to Lipschitz if and only if  $s_f(t), i_{cf}(t), i_{sf}(t), r_f(t), s_m(t), i_{cm}(t), i_{sm}(t), r_m(t)$  are limited from above. Now, let  $s_f(t)$  and  $s_f^*$  be dual-functions, thus

$$\|\mathbb{H}_1(\rho_f^*, t, s_f(t)) - \mathbb{H}_1(\rho_f^*, t, s_f^*(t))\| = \left\| \left( \beta_{mf}^{\rho_f^*} (\eta_{sm}^{\rho_f^*} i_{sm}(t) + \eta_{cm}^{\rho_f^*} i_{cm}(t)) + \mu_f^{\rho_f^*} \right) (s_f(t) - s_f^*(t)) \right\|. \tag{19}$$

Assuming  $\|i_{sm}(t)\| \leq \Delta_1$ , and  $\|i_{cm}(t)\| \leq \Delta_2$ . Now, supposing  $\hat{\Theta}_1 := \beta_{mf}^{\rho_f^*} (\eta_{sm}^{\rho_f^*} \Delta_1 + \eta_{cm}^{\rho_f^*} \Delta_2) + \mu_f^{\rho_f^*}$ , then the above expression can be written as

$$\|\mathbb{H}_1(\rho_f^*, t, s_f(t)) - \mathbb{H}_1(\rho_f^*, t, s_f^*(t))\| \leq \hat{\Theta}_1 \|(s_f(t) - s_f^*(t))\|. \tag{20}$$

Using the same method, the following can be extracted

$$\begin{aligned}
 \|\mathbb{H}_2(\varrho_*, t, i_{cf}(t)) - \mathbb{H}_2(\varrho_*, t, i_{cf}^*(t))\| &\leq \hat{\Theta}_2 \|(i_{cf}(t) - i_{cf}^*(t))\|, \\
 \|\mathbb{H}_3(\varrho_*, t, i_{sf}(t)) - \mathbb{H}_3(\varrho_*, t, i_{sf}^*(t))\| &\leq \hat{\Theta}_3 \|(i_{sf}(t) - i_{sf}^*(t))\|, \\
 \|\mathbb{H}_4(\varrho_*, t, r_f(t)) - \mathbb{H}_4(\varrho_*, t, r_f^*(t))\| &\leq \hat{\Theta}_4 \|(r_f(t) - r_f^*(t))\|, \\
 \|\mathbb{H}_5(\varrho_*, t, s_m(t)) - \mathbb{H}_5(\varrho_*, t, s_m^*(t))\| &\leq \hat{\Theta}_5 \|(s_m(t) - s_m^*(t))\|, \\
 \|\mathbb{H}_6(\varrho_*, t, i_{cm}(t)) - \mathbb{H}_6(\varrho_*, t, i_{cm}^*(t))\| &\leq \hat{\Theta}_6 \|(i_{cm}(t) - i_{cm}^*(t))\|, \\
 \|\mathbb{H}_7(\varrho_*, t, i_{sm}(t)) - \mathbb{H}_7(\varrho_*, t, i_{sm}^*(t))\| &\leq \hat{\Theta}_7 \|(i_{sm}(t) - i_{sm}^*(t))\|, \\
 \|\mathbb{H}_8(\varrho_*, t, r_m(t)) - \mathbb{H}_8(\varrho_*, t, r_m^*(t))\| &\leq \hat{\Theta}_8 \|(r_m(t) - r_m^*(t))\|,
 \end{aligned}
 \tag{21}$$

where

$$\begin{aligned}
 \hat{\Theta}_2 &= \rho_{mf}^{\varrho_*} (\Delta_1 + \Delta_2) + (\gamma_f^{\varrho_*} + \mu_f^{\varrho_*}), \hat{\Theta}_3 = (\alpha_f^{\varrho_*} + \mu_f^{\varrho_*}), \hat{\Theta}_4 = ((1 - \rho_f^{\varrho_*})\kappa_f^{\varrho_*} + \mu_f^{\varrho_*}), \\
 \hat{\Theta}_5 &= \rho_{fm}^{\varrho_*} (\eta_{sf}^{\varrho_*} \Delta_3 + \eta_{cf}^{\varrho_*} \Delta_4) + \mu_m^{\varrho_*} s_m(t), \|(i_{sf}(t))\| \leq \Delta_3, \|(i_{cf}(t))\| \leq \Delta_4, \hat{\Theta}_6 = (\gamma_m^{\varrho_*} + \mu_m^{\varrho_*}), \\
 \hat{\Theta}_7 &= (\alpha_m^{\varrho_*} + \mu_m^{\varrho_*}), \hat{\Theta}_8 = ((1 - \rho_m^{\varrho_*})\kappa_m^{\varrho_*} + \mu_m^{\varrho_*}),
 \end{aligned}$$

therefore, it proves that  $\mathbb{M}_i, i = 1, 2, 3, 4, 5, 6, 7, 8$  satisfies the Lipschitz condition. The Eq. (18) can be written recursively as

$$\begin{aligned}
 s_{fn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_1(\varrho_*, \mathfrak{s}, s_{fn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 i_{cfn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_2(\varrho_*, \mathfrak{s}, i_{cfn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 i_{sfn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_3(\varrho_*, \mathfrak{s}, i_{sfn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 r_{fn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_4(\varrho_*, \mathfrak{s}, r_{fn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 s_{mn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_5(\varrho_*, \mathfrak{s}, s_{mn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 i_{cmn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_6(\varrho_*, \mathfrak{s}, i_{cmn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 i_{smn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_7(\varrho_*, \mathfrak{s}, i_{smn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 r_{mn}(t) &= \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_8(\varrho_*, \mathfrak{s}, r_{mn-1}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s},
 \end{aligned}
 \tag{22}$$

connected to the following starting points

$$s_{f0}(t) = s_f(0), i_{cf0}(t) = i_{cf}(0), i_{sfn}(t) = i_{sf}(0), r_{f0}(t) = r_f(0), s_{m0}(t) = s_m(0),$$

$$i_{cm0}(t) = i_{cm}(0), i_{sm0}(t) = i_{sm}(0), r_{m0}(t) = r_m(0).$$

By deducting regular terms of succession, we get

$$\begin{aligned}
 \Sigma_{s_{fn}}(t) &= s_{fn}(t) - s_{fn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_1(\varrho_*, \mathfrak{s}, s_{fn-1}(\mathfrak{s})) - \mathbb{H}_1(\varrho_*, \mathfrak{s}, s_{fn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 \Sigma_{i_{cfn}}(t) &= i_{cfn}(t) - i_{cfn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_2(\varrho_*, \mathfrak{s}, i_{cfn-1}(\mathfrak{s})) - \mathbb{H}_2(\varrho_*, \mathfrak{s}, i_{cfn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 \Sigma_{i_{sfn}}(t) &= i_{sfn}(t) - i_{sfn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_3(\varrho_*, \mathfrak{s}, i_{sfn-1}(\mathfrak{s})) - \mathbb{H}_3(\varrho_*, \mathfrak{s}, i_{sfn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 \Sigma_{r_{fn}}(t) &= r_{fn}(t) - r_{fn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_4(\varrho_*, \mathfrak{s}, r_{fn-1}(\mathfrak{s})) - \mathbb{H}_4(\varrho_*, \mathfrak{s}, r_{fn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 \Sigma_{s_{mn}}(t) &= s_{mn}(t) - s_{mn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_5(\varrho_*, \mathfrak{s}, s_{mn-1}(\mathfrak{s})) - \mathbb{H}_5(\varrho_*, \mathfrak{s}, s_{mn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 \Sigma_{i_{cmn}}(t) &= i_{cmn}(t) - i_{cmn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_6(\varrho_*, \mathfrak{s}, i_{cmn-1}(\mathfrak{s})) - \mathbb{H}_6(\varrho_*, \mathfrak{s}, i_{cmn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 \Sigma_{i_{smn}}(t) &= i_{smn}(t) - i_{smn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_7(\varrho_*, \mathfrak{s}, i_{smn-1}(\mathfrak{s})) - \mathbb{H}_7(\varrho_*, \mathfrak{s}, i_{smn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}, \\
 \Sigma_{r_{mn}}(t) &= r_{mn}(t) - r_{mn-1}(t) = \frac{1}{\Gamma(\varrho_*)} \int_0^t \frac{\mathbb{H}_8(\varrho_*, \mathfrak{s}, r_{mn-1}(\mathfrak{s})) - \mathbb{H}_8(\varrho_*, \mathfrak{s}, r_{mn-2}(\mathfrak{s}))}{(t - \mathfrak{s})^{1-\varrho_*}} d\mathfrak{s}.
 \end{aligned}
 \tag{23}$$

After doing the computations, it becomes apparent that

$$\begin{aligned}
 s_{fn}(t) &= \sum_{k=0}^n Y_{s_{fn}}(t), \quad i_{cfn}(t) = \sum_{k=0}^n Y_{i_{cfn}}(t), \\
 i_{sfn}(t) &= \sum_{k=0}^n Y_{i_{sfn}}(t), \quad U_n(t) = \sum_{k=0}^n Y_{U_n}(t), \\
 s_{mn}(t) &= \sum_{k=0}^n Y_{s_{mn}}(t), \quad i_{cmn}(t) = \sum_{k=0}^n Y_{i_{cmn}}(t), \\
 i_{smn}(t) &= \sum_{k=0}^n Y_{i_{smn}}(t), \quad r_{mn}(t) = \sum_{k=0}^n Y_{r_{mn}}(t).
 \end{aligned}
 \tag{24}$$

In addition, by combining the ideas presented in Eqs. (20) and (21), we get

$$\begin{aligned}
 Y_{s_{fn-1}}(t) &= s_{fn-1}(t) - s_{fn-2}(t), \quad Y_{i_{cfn-1}}(t) = i_{cfn-1}(t) - i_{cfn-2}(t), \\
 Y_{i_{sfn-1}}(t) &= i_{sfn-1}(t) - i_{sfn-2}(t), \quad Y_{r_{fn-1}}(t) = r_{fn-1}(t) - r_{fn-2}(t), \\
 Y_{s_{mn-1}}(t) &= s_{mn-1}(t) - s_{mn-2}(t), \quad Y_{i_{cmn-1}}(t) = i_{cmn-1}(t) - i_{cmn-2}(t), \\
 Y_{i_{smn-1}}(t) &= i_{smn-1}(t) - i_{smn-2}(t), \quad Y_{r_{mn-1}}(t) = r_{mn-1}(t) - r_{mn-2}(t), \\
 \|Y_{s_{fn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_1 \int_0^t \frac{\|Y_{s_{fn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds, \\
 \|Y_{i_{cfn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_2 \int_0^t \frac{\|Y_{i_{cfn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds, \\
 \|Y_{i_{sfn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_3 \int_0^t \frac{\|Y_{i_{sfn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds, \\
 \|Y_{r_{fn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_4 \int_0^t \frac{\|Y_{r_{fn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds, \\
 \|Y_{s_{mn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_5 \int_0^t \frac{\|Y_{s_{mn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds, \\
 \|Y_{i_{cmn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_6 \int_0^t \frac{\|Y_{i_{cmn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds, \\
 \|Y_{i_{smn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_7 \int_0^t \frac{\|Y_{i_{smn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds, \\
 \|Y_{r_{mn}}(t)\| &= \frac{1}{\Gamma(\varrho_*)} \hat{\theta}_8 \int_0^t \frac{\|Y_{r_{mn-1}}(s)\|}{(t-s)^{1-\varrho_*}} ds.
 \end{aligned}
 \tag{25}$$

To conclude the existence and uniqueness of the gonorrhoea model in the Caputo sense, we then state and verify [Theorem 4.3](#).

**Theorem 4.3.** *A unique solution exists for the Caputo fractional gonorrhoea model (1) if and only if*

$$\frac{z^{\varrho_*}}{\Gamma(\varrho_* + 1)} \hat{\theta}_i < 1, \quad i = 1, 2, 3, 4, 5, 6, 7, 8,
 \tag{26}$$

when  $t \in [0, z]$ .

**Proof.** Since the preceding results hold, we know that the functions  $s_f(t), i_{cf}(t), i_{sf}(t), r_f(t), s_m(t), i_{cm}(t), i_{sm}(t), r_m(t)$  are limited and the functions  $\mathbb{H}_i, i = 1, 2, 3, 4, 5, 6, 7, 8$  meet the Lipschitz condition. As a result, we obtain the following results by applying the recursive principle and Eq. (25).

$$\begin{aligned}
 \|Y_{s_{fn}}(t)\| &\leq \|s_{f0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_1 \right)^n, \\
 \|Y_{i_{cfn}}(t)\| &\leq \|i_{cf0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_2 \right)^n, \\
 \|Y_{i_{sfn}}(t)\| &\leq \|i_{sf0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_3 \right)^n, \\
 \|Y_{U_n}(t)\| &\leq \|r_{f0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_4 \right)^n, \\
 \|Y_{s_{mn}}(t)\| &\leq \|s_{m0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_5 \right)^n, \\
 \|Y_{i_{cmn}}(t)\| &\leq \|i_{cm0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_6 \right)^n, \\
 \|Y_{i_{smn}}(t)\| &\leq \|i_{sm0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_7 \right)^n, \\
 \|Y_{r_{mn}}(t)\| &\leq \|r_{m0}(t)\| \left( \frac{z^{\varrho_*}}{\Gamma(\varrho_*)} \hat{\theta}_8 \right)^n.
 \end{aligned}
 \tag{27}$$

This means that

$$\begin{aligned} \|Y_{s_f,n}(t)\| &\rightarrow 0, \|Y_{i_{cf},n}(t)\| \rightarrow 0, \|Y_{i_{sf},n}(t)\| \rightarrow 0, \\ \|Y_{U,n}(t)\| &\rightarrow 0, \|Y_{r_f,n}(t)\| \rightarrow 0, \|Y_{s_m,n}(t)\| \rightarrow 0, \\ \|Y_{i_{cm},n}(t)\| &\rightarrow 0, \|Y_{i_{sm},n}(t)\| \rightarrow 0, \|Y_{r_m,n}(t)\| \rightarrow 0, \text{ provided } n \rightarrow \infty, \end{aligned}$$

Additionally, by utilizing the triangle inequality in conjunction with Eq. (27) for any value of  $\zeta$ , we can eventually arrive at

$$\begin{aligned} \|s_{f(n+\zeta)}(t) - s_{fn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_1^k = \frac{\Lambda_1^{n+1} - \Lambda_1^{n+\zeta+1}}{1 - \Lambda_1}, \\ \|i_{cf(n+\zeta)}(t) - i_{cfn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_2^k = \frac{\Lambda_2^{n+1} - \Lambda_2^{n+\zeta+1}}{1 - \Lambda_2}, \\ \|i_{sf(n+\zeta)}(t) - i_{sfn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_3^k = \frac{\Lambda_3^{n+1} - \Lambda_3^{n+\zeta+1}}{1 - \Lambda_3}, \\ \|r_{f(n+\zeta)}(t) - r_{fn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_4^k = \frac{\Lambda_4^{n+1} - \Lambda_4^{n+\zeta+1}}{1 - \Lambda_4}, \\ \|s_{m(n+\zeta)}(t) - s_{mn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_5^k = \frac{\Lambda_5^{n+1} - \Lambda_5^{n+\zeta+1}}{1 - \Lambda_5}, \\ \|i_{cm(n+\zeta)}(t) - i_{cmn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_6^k = \frac{\Lambda_6^{n+1} - \Lambda_6^{n+\zeta+1}}{1 - \Lambda_6}, \\ \|i_{sm(n+\zeta)}(t) - i_{smn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_7^k = \frac{\Lambda_7^{n+1} - \Lambda_7^{n+\zeta+1}}{1 - \Lambda_7}, \\ \|r_{m(n+\zeta)}(t) - r_{mn}(t)\| &\leq \sum_{k=n+1}^{n+\zeta} \Lambda_8^k = \frac{\Lambda_8^{n+1} - \Lambda_8^{n+\zeta+1}}{1 - \Lambda_8}. \end{aligned} \tag{28}$$

where  $\Lambda_i = \frac{z^{0*}}{I^{(0*)}} \hat{\Theta}_i$ ,  $i = 1, 2, 3, 4, 5, 6, 7, 8$ . As a consequence of this, the sequences  $s_{fn}, i_{cfn}, i_{sfn}, r_{fn}, s_{mn}, i_{cmn}, i_{smn}, r_{mn}$  are all examples of Cauchy sequences in the Banach space  $\mathbb{B}(\mathcal{E})$ . As a result, the state variables have a complete and uniform convergence. Therefore, using the limit theorem, we can claim that the limit of the sequence, denoted by the notation (22), is the only solution to the Caputo model.  $\square$

#### 4.4. Gonorrhoea absent equilibrium point

By setting the right-hand side of the model equations to zero, it is possible to investigate the steady states in the absence of infection. Let  $D\mathcal{E}\mathcal{F}$  represent the disease-free equilibrium points. Now, solving the above equation with the right-hand equal to zero, and the states  $i_{cf}^* = i_{sf}^* = r_f^* = i_{cm}^* = i_{sm}^* = r_m^* = 0$ , gives,

$$D\mathcal{E}\mathcal{F} := \{s_f^*, i_{cf}^*, i_{sf}^*, r_f^*, s_m^*, i_{cm}^*, i_{sm}^*, r_m^*\} = \left\{ \frac{\theta_f^{0*}}{\mu_f^{0*}}, 0, 0, 0, \frac{\theta_m^{0*}}{\mu_m^{0*}}, 0, 0, 0 \right\}.$$

#### 4.5. Basic reproduction number, $\mathcal{R}_0^{0*}$

The basic reproduction number, typically written by  $\mathcal{R}_0$ , is a mathematical notion used in epidemiology to quantify an infectious disease’s contagiousness or transmissibility. Without interventions or control measures, it estimates the average number of secondary infections one infected person would generate in a vulnerable community. For example, if a disease’s  $\mathcal{R}_0$  is 3, each afflicted individual is predicted to infect three others. If the  $\mathcal{R}_0$  is less than unity, the disease may not be able to survive in a population and will eventually die out. If  $\mathcal{R}_0$  is greater than one, the disease will spread farther, potentially causing an epidemic or pandemic. Apart from the numerical interpretation, one can also explain the epidemiological effect of the parameters contained in  $\mathcal{R}_0$  on the spread of the disease. The  $\mathcal{R}_0$  is calculated by considering characteristics such as the route of transmission, the length of the infectious phase, population density, and population susceptibility. Public health experts and policymakers must consider when developing effective methods to restrict the spread of infectious illnesses. Here, we will use the new method proposed by [28], thus, the Jacobian-Determinant Method, to extract our basic reproduction number by grouping our disease state compartments. The fractional infected compartments are  $i_{cf}^*, i_{sf}^*, i_{cm}^*$  and  $i_{sm}^*$ , therefore, we reduce Eq. (29) to:

$$\begin{aligned} C D_t^{0*} i_{cf} &= \beta_{mf}^{0*} (\eta_{sm}^{0*} i_{sm} + \eta_{cm}^{0*} i_{cm}) s_f + (1 - \rho_f^{0*}) \kappa_f^{0*} r_f - (\gamma_f^{0*} + \mu_f^{0*}) i_{cf}, \\ C D_t^{0*} i_{sf} &= \gamma_f^{0*} i_{cf} - (\alpha_f^{0*} + \mu_f^{0*}) i_{sf}, \\ C D_t^{0*} i_{cm} &= \beta_{fm}^{0*} (\eta_{sf}^{0*} i_{sf} + \eta_{cf}^{0*} i_{cf}) s_m + (1 - \rho_m^{0*}) \kappa_m^{0*} r_m - (\gamma_m^{0*} + \mu_m^{0*}) i_{cm}, \\ C D_t^{0*} i_{sm} &= \gamma_m^{0*} i_{cm} - (\alpha_m^{0*} + \mu_m^{0*}) i_{sm}. \end{aligned} \tag{29}$$

Hence, the Jacobian of the affected subsystem, measured at the disease-free equilibrium point, is:

$$J(D\mathcal{EF}) = \begin{bmatrix} -(\gamma_f^{\theta_*} + \mu_f^{\theta_*}) & 0 & \frac{\beta_{mf}^{\theta_*} \eta_{cm}^{\theta_*} \theta_f^{\theta_*}}{\mu_f^{\theta_*}} & \frac{\beta_{mf}^{\theta_*} \eta_{sm}^{\theta_*} \theta_f^{\theta_*}}{\mu_f^{\theta_*}} \\ \gamma_f^{\theta_*} & -(\alpha_f^{\theta_*} + \mu_f^{\theta_*}) & 0 & 0 \\ \frac{\beta_{fm}^{\theta_*} \eta_{cf}^{\theta_*} \theta_m^{\theta_*}}{\mu_m^{\theta_*}} & \frac{\beta_{fm}^{\theta_*} \eta_{sf}^{\theta_*} \theta_m^{\theta_*}}{\mu_m^{\theta_*}} & -(\gamma_m^{\theta_*} + \mu_m^{\theta_*}) & 0 \\ 0 & 0 & \gamma_m^{\theta_*} & -(\alpha_m^{\theta_*} + \mu_m^{\theta_*}) \end{bmatrix},$$

whose determinant is given by

$$|J(D\mathcal{EF})| = (\gamma_f^{\theta_*} + \mu_f^{\theta_*})(\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\gamma_m^{\theta_*} + \mu_m^{\theta_*})(\alpha_m^{\theta_*} + \mu_m^{\theta_*}) - \frac{\beta_{mf}^{\theta_*} \eta_{cm}^{\theta_*} \theta_f^{\theta_*} (\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\alpha_m^{\theta_*} + \mu_m^{\theta_*})}{\mu_f^{\theta_*}} - \frac{\beta_{fm}^{\theta_*} \eta_{cf}^{\theta_*} \theta_m^{\theta_*}}{\mu_m^{\theta_*}} \\ - \frac{\gamma_f^{\theta_*} \gamma_m^{\theta_*} \beta_{mf}^{\theta_*} \eta_{sm}^{\theta_*} \theta_f^{\theta_*} \beta_{fm}^{\theta_*} \eta_{sf}^{\theta_*} \theta_m^{\theta_*}}{\mu_f^{\theta_*} \mu_m^{\theta_*}} - \frac{\beta_{mf}^{\theta_*} \eta_{sm}^{\theta_*} \theta_f^{\theta_*} (\alpha_f^{\theta_*} + \mu_f^{\theta_*}) \beta_{fm}^{\theta_*} \eta_{cf}^{\theta_*} \theta_m^{\theta_*} \gamma_m^{\theta_*}}{\mu_f^{\theta_*} \mu_m^{\theta_*}} \\ - \frac{\beta_{mf}^{\theta_*} \eta_{cm}^{\theta_*} \theta_f^{\theta_*} \beta_{fm}^{\theta_*} \eta_{sf}^{\theta_*} \theta_m^{\theta_*} (\alpha_m^{\theta_*} + \mu_m^{\theta_*})}{\mu_f^{\theta_*} \mu_m^{\theta_*}}. \tag{30}$$

Now Eq. (30) can be expressed as

$$\left| (\gamma_f^{\theta_*} + \mu_f^{\theta_*})(\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\gamma_m^{\theta_*} + \mu_m^{\theta_*})(\alpha_m^{\theta_*} + \mu_m^{\theta_*}) \left[ 1 - \frac{\omega}{(\gamma_f^{\theta_*} + \mu_f^{\theta_*})(\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\gamma_m^{\theta_*} + \mu_m^{\theta_*})(\alpha_m^{\theta_*} + \mu_m^{\theta_*})} \right] \right|, \tag{31}$$

where

$$\omega = \frac{\beta_{mf}^{\theta_*} \eta_{cm}^{\theta_*} \theta_f^{\theta_*} (\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\alpha_m^{\theta_*} + \mu_m^{\theta_*})}{\mu_f^{\theta_*}} + \frac{\beta_{fm}^{\theta_*} \eta_{cf}^{\theta_*} \theta_m^{\theta_*}}{\mu_m^{\theta_*}} \\ + \frac{\gamma_f^{\theta_*} \gamma_m^{\theta_*} \beta_{mf}^{\theta_*} \eta_{sm}^{\theta_*} \theta_f^{\theta_*} \beta_{fm}^{\theta_*} \eta_{sf}^{\theta_*} \theta_m^{\theta_*}}{\mu_f^{\theta_*} \mu_m^{\theta_*}} + \frac{\beta_{mf}^{\theta_*} \eta_{sm}^{\theta_*} \theta_f^{\theta_*} (\alpha_f^{\theta_*} + \mu_f^{\theta_*}) \beta_{fm}^{\theta_*} \eta_{cf}^{\theta_*} \theta_m^{\theta_*} \gamma_m^{\theta_*}}{\mu_f^{\theta_*} \mu_m^{\theta_*}} \\ + \frac{\beta_{mf}^{\theta_*} \eta_{cm}^{\theta_*} \theta_f^{\theta_*} \beta_{fm}^{\theta_*} \eta_{sf}^{\theta_*} \theta_m^{\theta_*} (\alpha_m^{\theta_*} + \mu_m^{\theta_*})}{\mu_f^{\theta_*} \mu_m^{\theta_*}}.$$

After some simplification, and from Lemma 2.3 of [28], we have

$$\mathcal{R}_0^{\theta_*} = \frac{\beta_{fm}^{\theta_*} \beta_{mf}^{\theta_*} \theta_f^{\theta_*} \theta_m^{\theta_*} \left( \alpha_f^{\theta_*} \eta_{cf}^{\theta_*} + \eta_{sf}^{\theta_*} \gamma_f^{\theta_*} + \eta_{cf}^{\theta_*} \mu_f^{\theta_*} \right) \left( \alpha_m^{\theta_*} \eta_{cm}^{\theta_*} + \eta_{sm}^{\theta_*} \gamma_m^{\theta_*} + \eta_{cm}^{\theta_*} \mu_m^{\theta_*} \right)}{\mu_f^{\theta_*} \mu_m^{\theta_*} (\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\alpha_m^{\theta_*} + \mu_m^{\theta_*}) (\gamma_f^{\theta_*} + \mu_f^{\theta_*})(\gamma_m^{\theta_*} + \mu_m^{\theta_*})}$$

Finally, from Remark 3.1 of [28], the basic reproduction number is given as

$$\mathcal{R}_0^{\theta_*} = \sqrt{\frac{\beta_{fm}^{\theta_*} \beta_{mf}^{\theta_*} \theta_f^{\theta_*} \theta_m^{\theta_*} \left( \alpha_f^{\theta_*} \eta_{cf}^{\theta_*} + \eta_{sf}^{\theta_*} \gamma_f^{\theta_*} + \eta_{cf}^{\theta_*} \mu_f^{\theta_*} \right) \left( \alpha_m^{\theta_*} \eta_{cm}^{\theta_*} + \eta_{sm}^{\theta_*} \gamma_m^{\theta_*} + \eta_{cm}^{\theta_*} \mu_m^{\theta_*} \right)}{\mu_f^{\theta_*} \mu_m^{\theta_*} (\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\alpha_m^{\theta_*} + \mu_m^{\theta_*}) (\gamma_f^{\theta_*} + \mu_f^{\theta_*})(\gamma_m^{\theta_*} + \mu_m^{\theta_*})}}. \tag{32}$$

Thus, the secondary infection of Gonorrhoea from male to female ( $\mathcal{R}_{mf}^{\theta_*}$ ) in the female population and female to male ( $\mathcal{R}_{fm}^{\theta_*}$ ) in the male population are respectively defined as

$$\mathcal{R}_{mf}^{\theta_*} = \sqrt{\frac{\beta_{mf}^{\theta_*} \theta_m^{\theta_*} \left( \alpha_m^{\theta_*} \eta_{cm}^{\theta_*} + \eta_{sm}^{\theta_*} \gamma_m^{\theta_*} + \eta_{cm}^{\theta_*} \mu_m^{\theta_*} \right)}{\mu_m^{\theta_*} (\alpha_m^{\theta_*} + \mu_m^{\theta_*})(\gamma_m^{\theta_*} + \mu_m^{\theta_*})}}, \quad \mathcal{R}_{fm}^{\theta_*} = \sqrt{\frac{\beta_{fm}^{\theta_*} \theta_f^{\theta_*} \left( \alpha_f^{\theta_*} \eta_{cf}^{\theta_*} + \eta_{sf}^{\theta_*} \gamma_f^{\theta_*} + \eta_{cf}^{\theta_*} \mu_f^{\theta_*} \right)}{\mu_f^{\theta_*} (\alpha_f^{\theta_*} + \mu_f^{\theta_*})(\gamma_f^{\theta_*} + \mu_f^{\theta_*})}}. \tag{33}$$

Now from (31), we see that, if  $\mathcal{R}_0^{\theta_*} \leq 1$ , the disease will die out in the population but if  $\mathcal{R}_0 > 1$ , the disease will spread. From Eq. (31), if all the parameters are held constant with fixed fractional order, and we vary  $\theta_f$ , and  $\theta_m$ , we can see that an increase in  $\theta_f$ , and  $\theta_m$ , will cause a decrease in  $\mathcal{R}_0^{\theta_*}$  and vice versa. The parameter,  $\gamma_f$ , which is the rate at which females move from the incubative class, indicates that if we increase  $\gamma_f$ , will, in turn, decrease the value of  $\mathcal{R}_0^{\theta_*}$ , which also holds for  $\gamma_m$ . This implies that early detection of the contraction of the disease will help reduce its spread in the female and male compartments. We also noticed that an increase in  $\beta_{mf}$  (transmission rate from a male to a female susceptible) increases the disease in the considered population. Similarly, an increase in  $\beta_{fm}$  (transmission rate from a female to a male susceptible) increases the disease in the considered population.

#### 4.6. Gonorrhoea present equilibrium point

By setting the right-hand side of the model equations to zero, it is possible to investigate the steady states in the presence of infection. Let  $\mathcal{EF} = (s_f^*, i_{cf}^*, i_{sf}^*, r_f^*, s_m^*, i_{cm}^*, i_{sm}^*, r_m^*)$  represent the endemic equilibrium points. Hence, model (29) becomes;

$$\begin{aligned} 0 &= \theta_f^{\theta_*} - \lambda_{mf} s_f^* + \rho_f^{\theta_*} i_{sf}^* - \mu_f^{\theta_*} s_f^*, \\ 0 &= \lambda_{mf} s_f^* + (1 - \rho_f^{\theta_*}) \kappa_f^{\theta_*} r_f^* - (\gamma_f^{\theta_*} + \mu_f^{\theta_*}) i_{cf}^*, \\ 0 &= \gamma_f^{\theta_*} i_{cf}^* - (\alpha_f^{\theta_*} + \mu_f^{\theta_*}) i_{sf}^*, \end{aligned} \tag{34}$$

$$\begin{aligned}
 0 &= \alpha_f^{0*} i_{sf}^* - (\rho_f^{0*} + (1 - \rho_f^{0*})\kappa_f^{0*} + \mu_f^{0*})r_f^*, \\
 0 &= \theta_m^{0*} - \lambda_{fm} s_m + \rho_m^{0*} i_m^* - \mu_m^{0*} s_m, \\
 0 &= \lambda_{fm} s_m^* + (1 - \rho_m^{0*})\kappa_m^{0*} r_m^* - (\gamma_m^{0*} + \mu_m^{0*})i_{cm}^*, \\
 0 &= \gamma_m^{0*} i_{cm}^* - (\alpha_m^{0*} + \mu_m^{0*})i_{sm}^*, \\
 0 &= \alpha_m^{0*} i_{sm}^* - (\rho_m^{0*} + (1 - \rho_m^{0*})\kappa_m^{0*} + \mu_m^{0*})r_m^*,
 \end{aligned}$$

$$\lambda_{mf}^* = \beta_{mf}^{0*} (\eta_{sm}^{0*} i_{sm}^* + \eta_{cm}^{0*} i_{cm}^*), \tag{35}$$

$$\lambda_{fm}^* = \beta_{fm}^{0*} (\eta_{sf}^{0*} i_{sf}^* + \eta_{cf}^{0*} i_{cf}^*). \tag{36}$$

Solving Eq. (34) in terms of Eqs. (35) and (36) we have the following

$$\begin{aligned}
 s_f^* &= \frac{\theta_f^{0*} (\alpha_f^{0*} \gamma_f^{0*} \rho_f^{0*} - k_1 k_3 k_4)}{\alpha_f^{0*} \gamma_f^{0*} \lambda_{mf}^* \rho_f^{0*} + \alpha_f^{0*} \gamma_f^{0*} \mu_f^{0*} \rho_f^{0*} - k_1 k_3 k_4 \lambda_{mf}^* - k_1 k_3 k_4 \mu_f^{0*} + \alpha_f^{0*} \gamma_f^{0*} k_2 \lambda_{mf}^*}, \\
 i_{cf}^* &= \frac{k_3 k_4 \lambda_{mf}^* \theta_f^{0*}}{k_1 k_3 k_4 \lambda_{mf}^* + k_1 k_3 k_4 \mu_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} \lambda_{mf}^* \rho_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} \mu_f^{0*} \rho_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} k_2 \lambda_{mf}^*}, \\
 i_{sf}^* &= \frac{\gamma_f^{0*} k_4 \lambda_{mf}^* \theta_f^{0*}}{k_1 k_3 k_4 \lambda_{mf}^* + k_1 k_3 k_4 \mu_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} \lambda_{mf}^* \rho_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} \mu_f^{0*} \rho_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} k_2 \lambda_{mf}^*}, \\
 r_f^* &= \frac{\alpha_f^{0*} \gamma_f^{0*} \lambda_{mf}^* \theta_f^{0*}}{k_1 k_3 k_4 \lambda_{mf}^* + k_1 k_3 k_4 \mu_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} \lambda_{mf}^* \rho_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} \mu_f^{0*} \rho_f^{0*} - \alpha_f^{0*} \gamma_f^{0*} k_2 \lambda_{mf}^*}, \\
 s_m^* &= \frac{\theta_m^{0*} (\alpha_m^{0*} \gamma_m^{0*} \rho_m^{0*} - k_5 k_7 k_8)}{\alpha_m^{0*} \gamma_m^{0*} \lambda_{fm}^* \rho_m^{0*} + \alpha_m^{0*} \gamma_m^{0*} \mu_m^{0*} \rho_m^{0*} - k_5 k_7 k_8 \lambda_{fm}^* - k_5 k_7 k_8 \mu_m^{0*} + \alpha_m^{0*} \gamma_m^{0*} k_6 \lambda_{fm}^*}, \\
 i_{cm}^* &= \frac{k_7 k_8 \lambda_{fm}^* \theta_m^{0*}}{k_5 k_7 k_8 \lambda_{fm}^* + k_5 k_7 k_8 \mu_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} \lambda_{fm}^* \rho_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} \mu_m^{0*} \rho_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} k_6 \lambda_{fm}^*}, \\
 i_{sm}^* &= \frac{\gamma_m^{0*} k_8 \lambda_{fm}^* \theta_m^{0*}}{k_5 k_7 k_8 \lambda_{fm}^* + k_5 k_7 k_8 \mu_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} \lambda_{fm}^* \rho_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} \mu_m^{0*} \rho_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} k_6 \lambda_{fm}^*}, \\
 r_m^* &= \frac{\alpha_m^{0*} \gamma_m^{0*} \lambda_{fm}^* \theta_m^{0*}}{k_5 k_7 k_8 \lambda_{fm}^* + k_5 k_7 k_8 \mu_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} \lambda_{fm}^* \rho_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} \mu_m^{0*} \rho_m^{0*} - \alpha_m^{0*} \gamma_m^{0*} k_6 \lambda_{fm}^*},
 \end{aligned} \tag{37}$$

where

$$\begin{aligned}
 k_1 &= (\gamma_f^{0*} + \mu_f^{0*}), k_2 = (1 - \rho_f^{0*})\kappa_f^{0*}, k_3 = (\alpha_f^{0*} + \mu_f^{0*}), k_4 = (\rho_f^{0*} + (1 - \rho_f^{0*})\kappa_f^{0*} + \mu_f^{0*}), \\
 k_5 &= (\gamma_m^{0*} + \mu_m^{0*}), k_6 = (1 - \rho_m^{0*})\kappa_m^{0*}, k_7 = (\alpha_m^{0*} + \mu_m^{0*}), k_8 = (\rho_m^{0*} + (1 - \rho_m^{0*})\kappa_m^{0*} + \mu_m^{0*}).
 \end{aligned} \tag{38}$$

Now replacing (37) and (38) into Eqs. (35) and (36), and simplifying yields;

$$\lambda_{fm}^* = \frac{\beta_{fm}^{0*} k_4 \lambda_{mf}^* \theta_f^{0*} (\eta_{sf} \gamma_f^{0*} + \eta_{cf} k_3)}{(k_1 k_3 k_4 \lambda_{mf}^* + k_1 k_3 k_4 \mu_f^{0*}) - (\alpha_f^{0*} \gamma_f^{0*} \lambda_{mf}^* \rho_f^{0*} + \alpha_f^{0*} \gamma_f^{0*} \mu_f^{0*} \rho_f^{0*} + \alpha_f^{0*} \gamma_f^{0*} k_2 \lambda_{mf}^*)}, \tag{39}$$

$$\lambda_{mf}^* = \frac{\beta_{mf}^{0*} k_8 \lambda_{fm}^* \theta_m^{0*} (\eta_{sm} \gamma_m^{0*} + \eta_{cm} k_7)}{(k_5 k_7 k_8 \lambda_{fm}^* + k_5 k_7 k_8 \mu_m^{0*}) - (\alpha_m^{0*} \gamma_m^{0*} \lambda_{fm}^* \rho_m^{0*} + \alpha_m^{0*} \gamma_m^{0*} \mu_m^{0*} \rho_m^{0*} + \alpha_m^{0*} \gamma_m^{0*} k_6 \lambda_{fm}^*)}. \tag{40}$$

Substituting (39) into (40), the Gonorrhoea present (nonzero Gonorrhoea) equilibria of the model (29) fulfill

$$\lambda_{mf}^* = \frac{\mathcal{R}_1^{0*} - 1}{\mathbf{J}_1 + \mathbf{J}_2}, \tag{41}$$

where

$$\begin{aligned}
 \mathcal{R}_1^{0*} &= \sigma (\mathcal{R}_0^{0*})^2, \\
 \sigma &= \frac{\mu_f^{0*} \mu_m^{0*} (\alpha_f^{0*} + \mu_f^{0*}) (\alpha_m^{0*} + \mu_m^{0*}) (\gamma_f^{0*} + \mu_f^{0*}) (\gamma_m^{0*} + \mu_m^{0*}) k_4 k_8}{\mu_f^{0*} \mu_m^{0*} (\alpha_f^{0*} \gamma_f^{0*} \rho_f^{0*} - k_1 k_3 k_4) (\alpha_m^{0*} \gamma_m^{0*} \rho_m^{0*} - k_5 k_7 k_8)}, \\
 \mathbf{J}_2 &= \mu_m^{0*} (\alpha_m^{0*} \gamma_m^{0*} \rho_m^{0*} - k_5 k_7 k_8) (\alpha_f^{0*} \gamma_f^{0*} k_2 + \alpha_f^{0*} \gamma_f^{0*} \rho_f^{0*} - k_1 k_3 k_4), \\
 \mathbf{J}_2 &= \mathcal{R}_{fm}^{0*} \mu_f^{0*} (\gamma_f^{0*} + \mu_f^{0*}) (k_5 k_7 k_8 \gamma_f^{0*} - \mu_f^{0*}) k_4 (\alpha_m^{0*} \gamma_m^{0*} k_6 - \alpha_m^{0*} \gamma_m^{0*} \rho_m^{0*}).
 \end{aligned}$$

When,  $\sigma = 1, \mathcal{R}_1^{0*} = (\mathcal{R}_0^{0*})^2$ , we have a unique gonorrhoea present equilibrium point (endemic point) and the  $\mathcal{R}_0^{0*} > 1$  will lead to endemicity.

#### 4.7. Stability analysis

Stability theory is a branch of mathematics that examines the degree to which solutions of differential equations (fractional and non-fractional) and how the trajectories of dynamical systems are resilient in the face of minor changes to their initial conditions. The most used form of stability

analysis when using fractional derivatives is the Ulam–Hyers and Ulam–Hyers–Rassias stability analysis. First described in [29], Ulam–Hyers stability was generalized by Rassias in a later paper [30]. Ulam–Hyers and Ulam–Hyers–Rassias stability is used to effectively govern the dynamics of a suggested model where precise solutions might be difficult to discover. Here, comparable notions from [31,32] will be presented in the context of discussing the stable solutions to the Caputo fractional gonorrhoea model (1) employing Ulam–Hyers and Ulam–Hyers–Rassias stability conditions.

**Definition 4.1.** The fractional Caputo gonorrhoea model (1) is Ulam–Hyers stable if there exist  $0 < D_{Q_i} \in \mathbb{R}, i = 1, 2, 3, 4, 5, 6, 7, 8$  such that  $\forall \delta_i > 0$  and for all  $(s_f^{**}, i_{cf}^{**}, i_{sf}^{**}, r_f^{**}, s_m^{**}, i_{cm}^{**}, i_{sm}^{**}, r_m^{**}) \in \mathbb{Y}$  satisfying

$$\left\{ \begin{aligned} |{}^C D_{0,t}^{\alpha} s_f^{**}(t) - \mathcal{N}_1(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_1, \\ |{}^C D_{0,t}^{\alpha} i_{cf}^{**}(t) - \mathcal{N}_2(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_2, \\ |{}^C D_{0,t}^{\alpha} i_{sf}^{**}(t) - \mathcal{N}_3(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_3, \\ |{}^C D_{0,t}^{\alpha} r_f^{**}(t) - \mathcal{N}_4(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_4, \\ |{}^C D_{0,t}^{\alpha} s_m^{**}(t) - \mathcal{N}_5(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_5, \\ |{}^C D_{0,t}^{\alpha} i_{cm}^{**}(t) - \mathcal{N}_6(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_6, \\ |{}^C D_{0,t}^{\alpha} i_{sm}^{**}(t) - \mathcal{N}_7(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_7, \\ |{}^C D_{0,t}^{\alpha} r_m^{**}(t) - \mathcal{N}_8(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_8, \end{aligned} \right. \tag{42}$$

$\exists (s_f, i_{cf}, i_{sf}, r_f, s_m, i_{cm}, i_{sm}, r_m) \in \mathbb{Y}$  satisfying the fractional gonorrhoea model (1) with

$$\left\{ \begin{aligned} |s_f^{**}(t) - s_f(t)| &\leq D_{Q_1} \delta_1, \\ |i_{cf}^{**}(t) - i_{cf}(t)| &\leq D_{Q_2} \delta_2, \\ |i_{sf}^{**}(t) - i_{sf}(t)| &\leq D_{Q_3} \delta_3, \\ |r_f^{**}(t) - r_f(t)| &\leq D_{Q_4} \delta_4, \\ |s_m^{**}(t) - s_m(t)| &\leq D_{Q_5} \delta_5, \\ |i_{cm}^{**}(t) - i_{cm}(t)| &\leq D_{Q_6} \delta_6, \\ |i_{sm}^{**}(t) - i_{sm}(t)| &\leq D_{Q_7} \delta_7, \\ |r_m^{**}(t) - r_m(t)| &\leq D_{Q_8} \delta_8. \end{aligned} \right. \tag{43}$$

**Remark 4.1.**  $(s_f^{**}, i_{cf}^{**}, i_{sf}^{**}, r_f^{**}, s_m^{**}, i_{cm}^{**}, i_{sm}^{**}, r_m^{**}) \in \mathbb{Y}$  is a solution of (42) iff

$$\exists \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7, \mathbf{p}_8 \in C([0, T], \mathbb{R})$$

(based on  $s_f^{**}, i_{cf}^{**}, i_{sf}^{**}, r_f^{**}, s_m^{**}, i_{cm}^{**}, i_{sm}^{**}, r_m^{**}$ , respectively) such that  $\forall t \in \mathbb{T}$ ,

(i).  $|\mathbf{p}_i(t)| < \delta_i, (i = 1, 2, 3, 4, 5, 6, 7, 8)$ , and

$$(ii) \left\{ \begin{aligned} |{}^C D_{0,t}^{\alpha} s_f^{**}(t) - \mathcal{N}_1(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_1(t), \\ |{}^C D_{0,t}^{\alpha} i_{cf}^{**}(t) - \mathcal{N}_2(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_2(t), \\ |{}^C D_{0,t}^{\alpha} i_{sf}^{**}(t) - \mathcal{N}_3(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_3(t), \\ |{}^C D_{0,t}^{\alpha} r_f^{**}(t) - \mathcal{N}_4(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_4(t), \\ |{}^C D_{0,t}^{\alpha} s_m^{**}(t) - \mathcal{N}_5(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_5(t), \\ |{}^C D_{0,t}^{\alpha} i_{cm}^{**}(t) - \mathcal{N}_6(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_6(t), \\ |{}^C D_{0,t}^{\alpha} i_{sm}^{**}(t) - \mathcal{N}_7(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_7(t), \\ |{}^C D_{0,t}^{\alpha} r_m^{**}(t) - \mathcal{N}_8(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &+ \mathbf{p}_8(t). \end{aligned} \right. \tag{44}$$

**Definition 4.2.** The fractional Caputo gonorrhoea (1) is Ulam–Hyers–Rassias stable with respect to the functions  $\chi_i, i = 1, 2, 3, 4, 5, 6, 7, 8$  whenever there exists  $0 < D_{Q_i, \chi_i} \in \mathbb{R}, i = 1, 2, 3, 4, 5, 6, 7, 8$  such that  $\forall \delta_i > 0$  and  $\forall (s_f^{**}, i_{cf}^{**}, i_{sf}^{**}, r_f^{**}, s_m^{**}, i_{cm}^{**}, i_{sm}^{**}, r_m^{**}) \in \mathbb{Y}$  satisfying

$$\left\{ \begin{aligned} |{}^C D_{0,t}^{\alpha} s_f^{**}(t) - \mathcal{N}_1(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_1 \chi_1(t), \\ |{}^C D_{0,t}^{\alpha} i_{cf}^{**}(t) - \mathcal{N}_2(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_2 \chi_2(t), \\ |{}^C D_{0,t}^{\alpha} i_{sf}^{**}(t) - \mathcal{N}_3(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_3 \chi_3(t), \\ |{}^C D_{0,t}^{\alpha} r_f^{**}(t) - \mathcal{N}_4(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_4 \chi_4(t), \\ |{}^C D_{0,t}^{\alpha} s_m^{**}(t) - \mathcal{N}_5(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_5 \chi_5(t), \\ |{}^C D_{0,t}^{\alpha} i_{cm}^{**}(t) - \mathcal{N}_6(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_6 \chi_6(t), \\ |{}^C D_{0,t}^{\alpha} i_{sm}^{**}(t) - \mathcal{N}_7(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_7 \chi_7(t), \\ |{}^C D_{0,t}^{\alpha} r_m^{**}(t) - \mathcal{N}_8(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))| &< \delta_8 \chi_8(t), \end{aligned} \right. \tag{45}$$

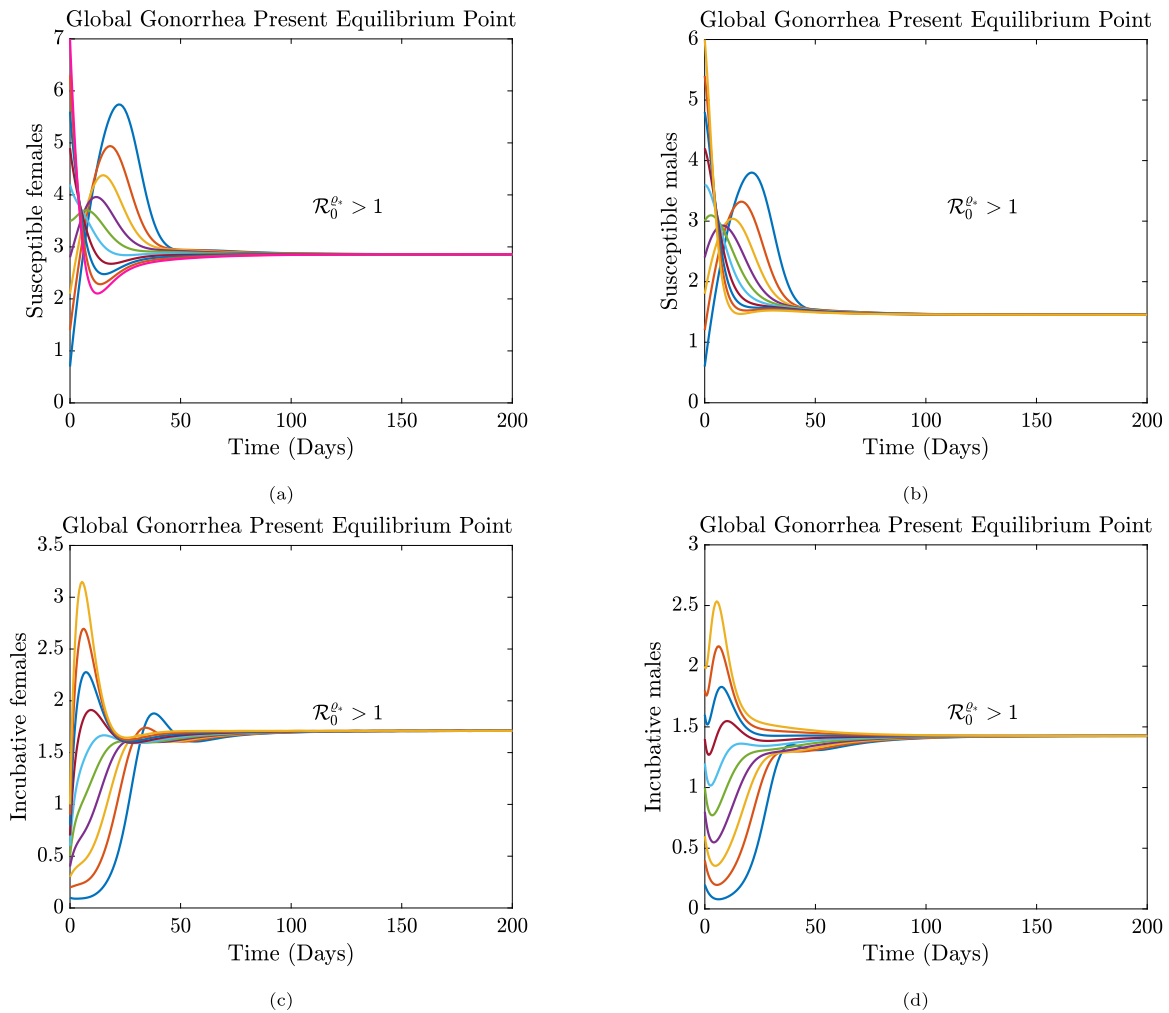


Fig. 2. Stability plot when  $\mathcal{R}_0^e > 1$ , for the susceptible and incubative classes.

there exists  $(s_f, i_{cf}, i_{sf}, r_f, s_m, i_{cm}, i_{sm}, r_m) \in \mathbb{Y}$  fulfilling the fractional Caputo model (1) together with

$$\begin{cases} |s_f^{**}(t) - s_f(t)| \leq D_{Q_1, \chi_1} \delta_1 \chi_1(t), \\ |i_{cf}^{**}(t) - i_{cf}(t)| \leq D_{Q_2, \chi_2} \delta_2 \chi_2(t), \\ |i_{sf}^{**}(t) - i_{sf}(t)| \leq D_{Q_3, \chi_3} \delta_3 \chi_3(t), \\ |r_f^{**}(t) - r_f(t)| \leq D_{Q_4, \chi_4} \delta_4 \chi_4(t), \\ |s_m^{**}(t) - s_m(t)| \leq D_{Q_5, \chi_5} \delta_5 \chi_5(t), \\ |i_{cm}^{**}(t) - i_{cm}(t)| \leq D_{Q_6, \chi_6} \delta_6 \chi_6(t), \\ |i_{sm}^{**}(t) - i_{sm}(t)| \leq D_{Q_7, \chi_7} \delta_7 \chi_7(t), \\ |r_m^{**}(t) - r_m(t)| \leq D_{Q_8, \chi_8} \delta_8 \chi_8(t). \end{cases} \tag{46}$$

**Remark 4.2.**  $(s_f^{**}, i_{cf}^{**}, i_{sf}^{**}, r_f^{**}, s_m^{**}, i_{cm}^{**}, i_{sm}^{**}, r_m^{**}) \in \mathbb{Y}$  is a solution of (45) iff there exists

$$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7, \mathbf{p}_8 \in C([0, T], \mathbb{R})$$

(based on  $s_f^{**}, i_{cf}^{**}, i_{sf}^{**}, r_f^{**}, s_m^{**}, i_{cm}^{**}, i_{sm}^{**}, r_m^{**}$ , respectively) such that  $\forall t \in \mathbb{T}$ ,

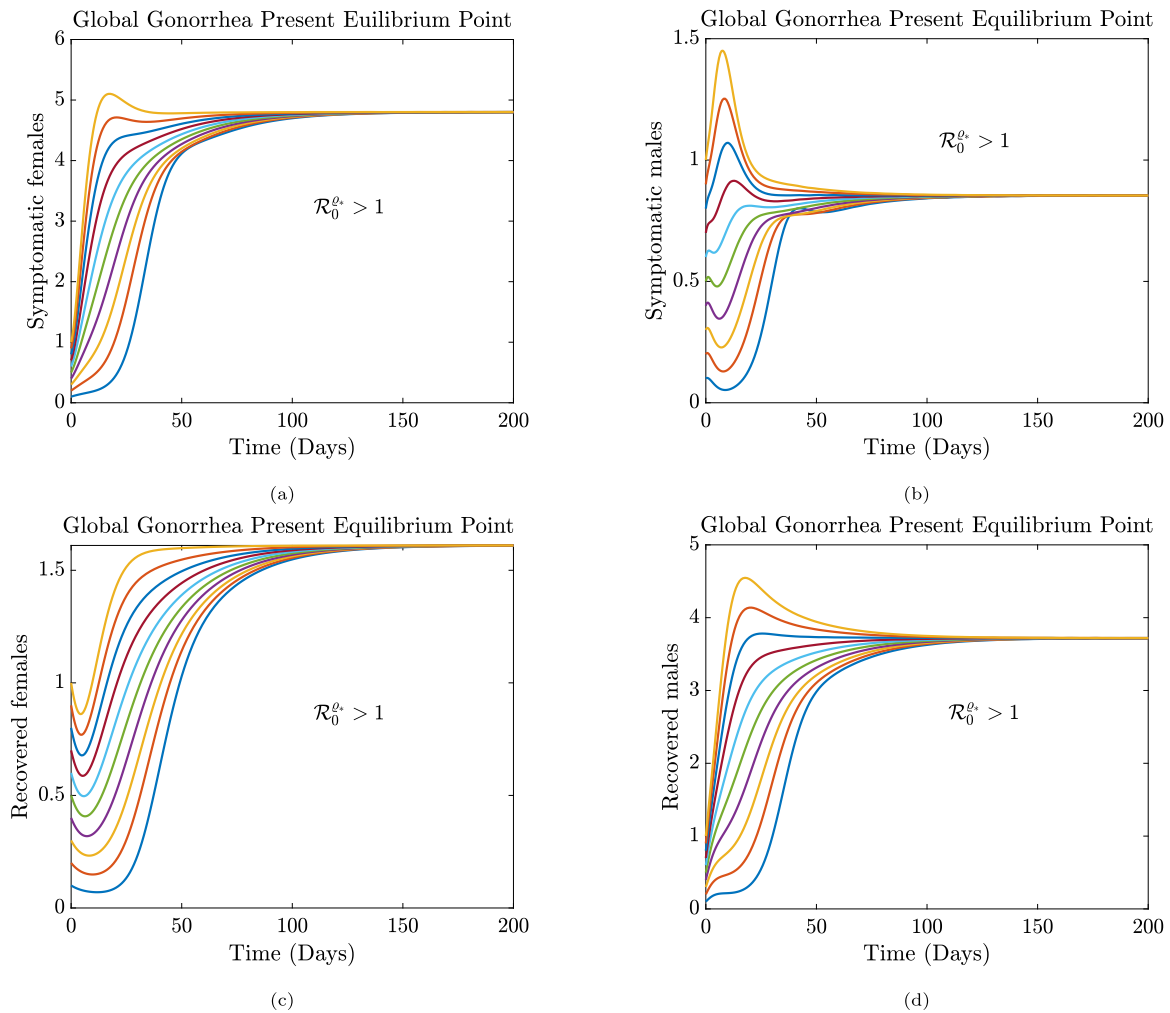


Fig. 3. Stability plot when  $\mathcal{R}_0^e > 1$  for the symptomatic and recovered classes.

(i).  $|\mathbf{p}_i(t)| < \chi_i(t)\delta_i$ , ( $i = 1, 2, 3, 4, 5, 6, 7, 8$ ), and

$$(ii) \begin{cases} {}^C D_{0,t}^{\alpha_s} s_f^{**}(t) = \mathcal{N}_1(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_1(t), \\ {}^C D_{0,t}^{\alpha_s} i_{cf}^{**}(t) = \mathcal{N}_2(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_2(t), \\ {}^C D_{0,t}^{\alpha_s} i_{sf}^{**}(t) = \mathcal{N}_3(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_3(t), \\ {}^C D_{0,t}^{\alpha_s} r_f^{**}(t) = \mathcal{N}_4(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_4(t), \\ {}^C D_{0,t}^{\alpha_s} s_m^{**}(t) = \mathcal{N}_5(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_5(t), \\ {}^C D_{0,t}^{\alpha_s} i_{cm}^{**}(t) = \mathcal{N}_6(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_6(t), \\ {}^C D_{0,t}^{\alpha_s} i_{sm}^{**}(t) = \mathcal{N}_7(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_7(t), \\ {}^C D_{0,t}^{\alpha_s} r_m^{**}(t) = \mathcal{N}_8(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t))) + \mathbf{p}_8(t). \end{cases} \tag{47}$$

**Theorem 4.4.** The fractional Caputo gonorrhea model (1) is Ulam–Hyers stable on  $\mathbb{T} := [0, T]$  such that

$$\hat{\Delta}\hat{\theta}_i < 1, \quad i \in \{1, 2, 3, 4, 5, 6, 7, 8\},$$

where  $\hat{\theta}_i$  and  $\hat{\Delta}$  are given as

$$\begin{aligned} \hat{\theta}_1 : \hat{\theta}_1 &:= \beta_{mf}^{\alpha_s}(\eta_{sm}^{\alpha_s}A_1 + \eta_{cm}^{\alpha_s}A_2) + \mu_f^{\alpha_s}, \quad \hat{\theta}_2 = \beta_{mf}^{\alpha_s}(A_1 + A_2) + (\gamma_f^{\alpha_s} + \mu_f^{\alpha_s}), \quad \hat{\theta}_3 = (\alpha_f^{\alpha_s} + \mu_f^{\alpha_s}), \\ \hat{\theta}_4 &= (1 - \rho_f^{\alpha_s})\kappa_f^{\alpha_s} + \mu_f^{\alpha_s}, \quad \hat{\theta}_5 = \beta_{fm}^{\alpha_s}(\eta_{sf}^{\alpha_s}A_3 + \eta_{cf}^{\alpha_s}A_4) + \mu_m^{\alpha_s}s_m(t), \quad \hat{\theta}_6 = (\gamma_m^{\alpha_s} + \mu_m^{\alpha_s}), \\ \hat{\theta}_7 &= (\alpha_m^{\alpha_s} + \mu_m^{\alpha_s}), \quad \hat{\theta}_8 = (1 - \rho_m^{\alpha_s})\kappa_m^{\alpha_s} + \mu_m^{\alpha_s}, \end{aligned}$$

if the assumption (D1) is true.  $D(1) : \|i_{sm}(t)\| \leq \Delta_1, \|i_{cm}(t)\| \leq \Delta_2, \|i_{sf}(t)\| \leq \Delta_3, \|i_{cf}(t)\| \leq \Delta_4.$

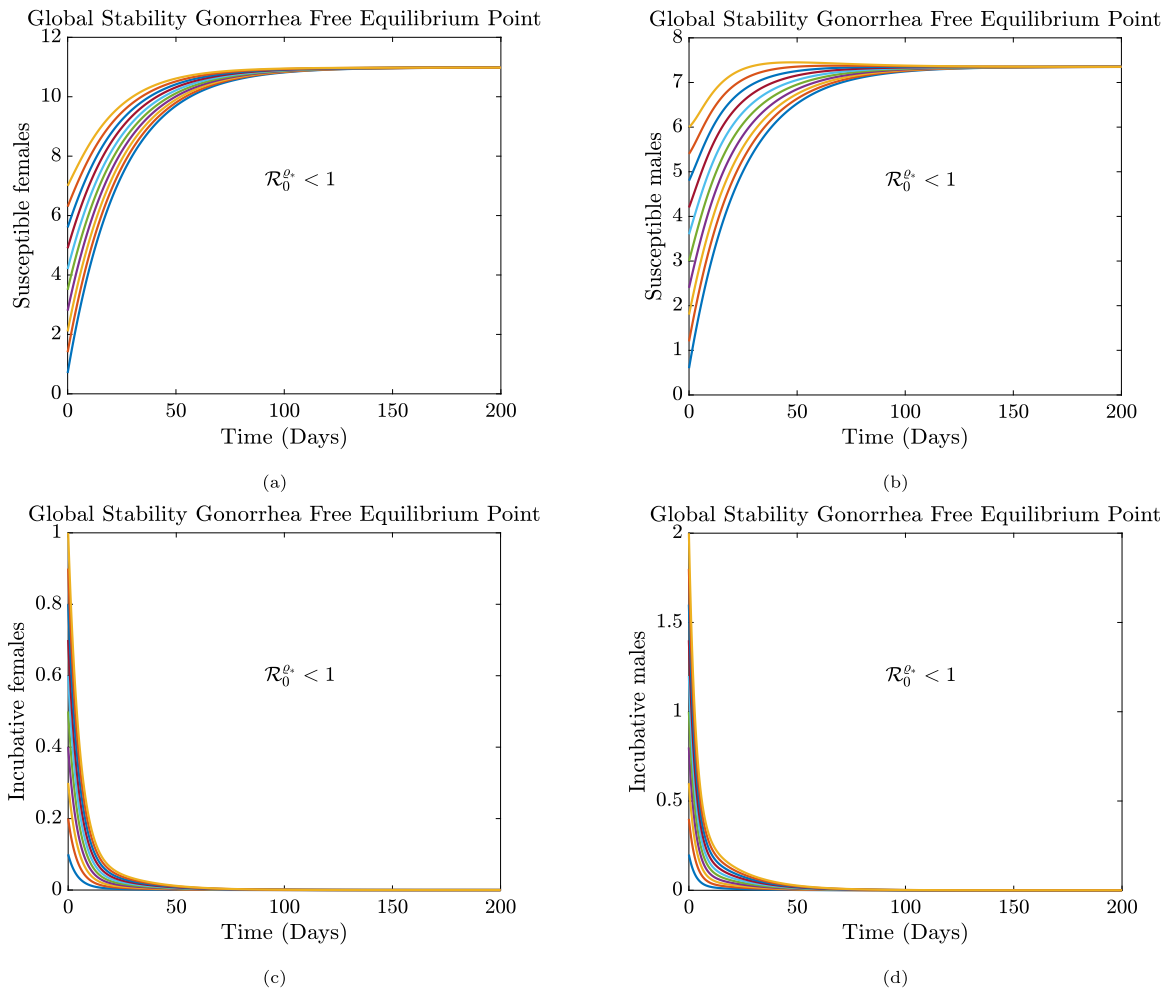


Fig. 4. Stability plot when  $\mathcal{R}_0^{e_s} < 1$ , for the susceptible and incubative classes.

**Proof.** Let  $\delta_1 > 0$  and  $s_f^{**} \in \mathbb{Y}$  such that

$$\left| {}^C D_{0,t}^{\varrho_s} s_f^{**}(t) - \mathcal{N}_1(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t)) \right| < \delta_1.$$

Then, in line with Remark 4.1, there exists,  $\mathbf{p}_1(t)$  in order to

$${}^C D_{0,t}^{\varrho_s} s_f^{**}(t) = \mathcal{N}_1(s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t)) + \mathbf{p}_1(t),$$

and  $|\mathbf{p}_1(t)| \leq \delta_1$ . Therefore,

$$s_f^{**}(t) = s_{f_0} + \frac{1}{\Gamma(\varrho_s)} \int_0^t (t-\tau)^{\varrho_s-1} \mathcal{N}_1(s_f^{**}(\tau), i_{cf}^{**}(\tau), i_{sf}^{**}(\tau), r_f^{**}(\tau), s_m^{**}(\tau), i_{cm}^{**}(\tau), i_{sm}^{**}(\tau), r_m^{**}(\tau)) d\tau + \frac{1}{\Gamma(\varrho_s)} \int_0^t (t-\tau)^{\varrho_s-1} \mathbf{p}_1(\tau) d\tau.$$

From Theorem 4.3, we let  $s_f \in \mathbb{Y}$  as the unique solution of the fractional Caputo model. Then  $s_f(t)$  is formulated by

$$s_f(t) = s_{f_0} + \frac{1}{\Gamma(\varrho_s)} \int_0^t (t-\tau)^{\varrho_s-1} \mathcal{N}_1(s_f(\tau), i_{cf}(\tau), i_{sf}(\tau), r_f(\tau), s_m(\tau), i_{cm}(\tau), i_{sm}(\tau), r_m(\tau)) d\tau.$$

Then

$$\begin{aligned} |s_f^{**}(t) - s_f(t)| &\leq \frac{1}{\Gamma(\varrho_s)} \int_0^t (t-\tau)^{\varrho_s-1} |\mathbf{p}_1(\tau)| d\tau \\ &+ \frac{1}{\Gamma(\varrho_s)} \int_0^t (t-\tau)^{\varrho_s-1} \\ &\times \left| \mathcal{N}_1(s_f^{**}(\tau), i_{cf}^{**}(\tau), i_{sf}^{**}(\tau), r_f^{**}(\tau), s_m^{**}(\tau), i_{cm}^{**}(\tau), i_{sm}^{**}(\tau), r_m^{**}(\tau)) \right. \\ &- \left. \mathcal{N}_1(s_f(\tau), i_{cf}(\tau), i_{sf}(\tau), r_f(\tau), s_m(\tau), i_{cm}(\tau), i_{sm}(\tau), r_m(\tau)) \right| d\tau \\ &\leq \hat{\Delta} \delta_1 + \hat{\Delta} \hat{\Theta}_1 \|s_f^{**} - s_f\|. \end{aligned}$$

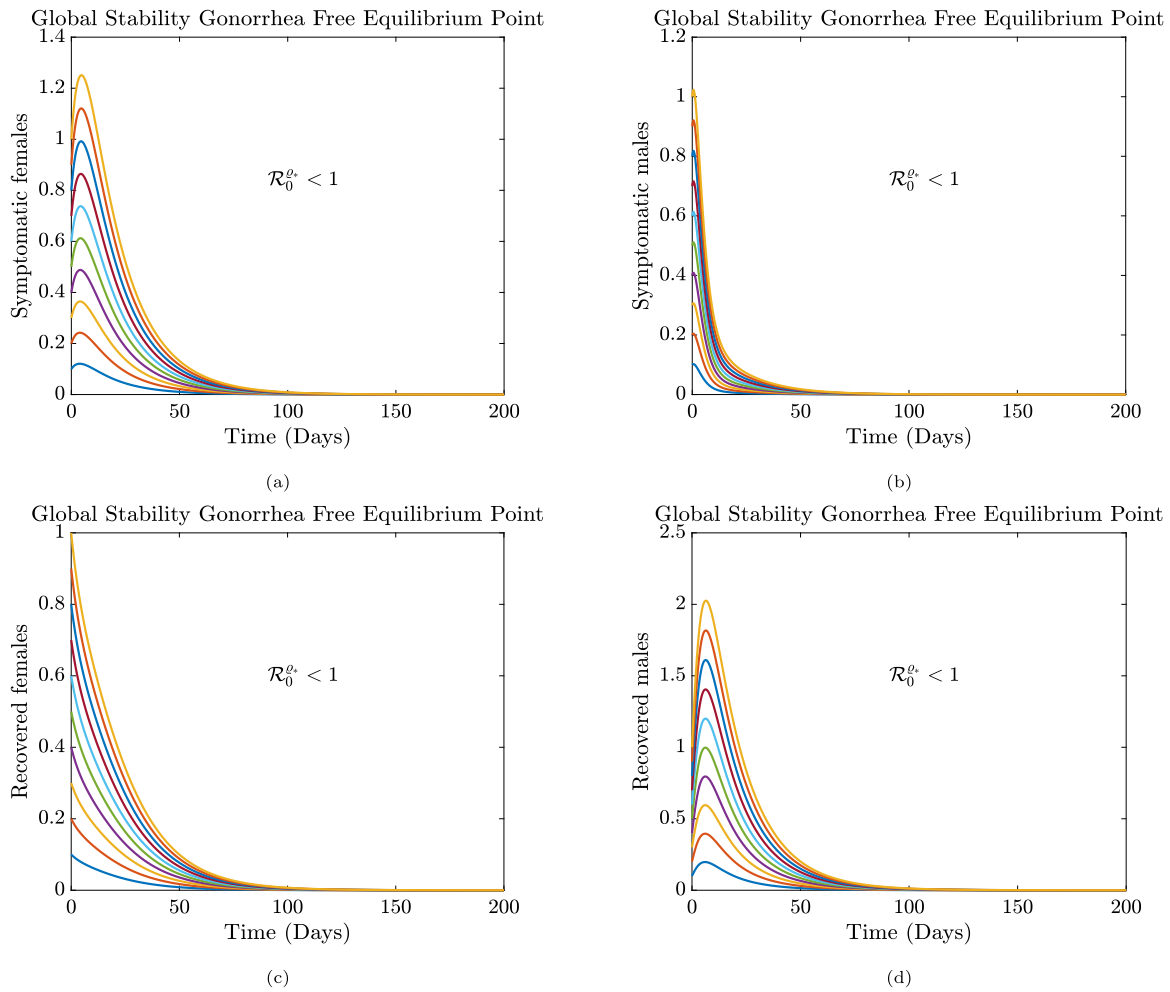


Fig. 5. Stability plot when  $\mathcal{R}_0^e < 1$ , for the symptomatic and recovered classes.

Thus, by using sup norm on the left and right of the inequality, we get

$$\|s_f^{**} - s_f\| - \hat{\Delta}\hat{\Theta}_1\|s_f^{**} - s_f\| \leq \hat{\Delta}\delta_1.$$

Therefore,

$$\|s_f^{**} - s_f\| \leq \frac{\hat{\Delta}\delta_1}{1 - \hat{\Delta}\hat{\Theta}_1}.$$

If  $D_{Q_1} = \frac{\hat{\Delta}}{1 - \hat{\Delta}\hat{\Theta}_1}$ , then  $\|s_f^{**} - s_f\| \leq D_{Q_1}\delta_1$ . In the same way,

$$\|i_{cf}^{**} - i_{cf}\| \leq D_{Q_2}\delta_2, \quad \|i_{sf}^{**} - i_{sf}\| \leq D_{Q_3}\delta_3, \quad \|r_f^{**} - r_f\| \leq D_{Q_4}\delta_4,$$

$$\|s_m^{**} - s_m\| \leq D_{Q_5}\delta_5, \quad \|i_{cm}^{**} - i_{cm}\| \leq D_{Q_6}\delta_6, \quad \|i_{sm}^{**} - i_{sm}\| \leq D_{Q_7}\delta_7, \quad \|r_m^{**} - r_m\| \leq D_{Q_8}\delta_8,$$

where

$$D_{Q_i} = \frac{\hat{\Delta}}{1 - \hat{\Delta}\hat{\Theta}_i}, \quad (i \in \{2, 3, 4, 5, 6, 7, 8\}).$$

Hence, the Ulam–Hyers stability of the fractional Caputo gonorrhoea (1) is accomplished.  $\square$

**Theorem 4.5.** Suppose (D'), such that there exists a non decreasing mappings  $\Omega_i \in \mathbf{C}([0, T], \mathbb{R}^+)$ ,  $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$  and there exists  $\Xi_{\Omega_i} > 0$  such that for all  $t \in \mathbb{T}$ ,

$${}^C I_{0,t}^{\alpha} \Omega_i(t) < \Xi_{\Omega_i} \Omega_i(t), \quad i \in \{1, 2, 3, 4, 5, 6, 7, 8\}. \tag{48}$$

If (H1) holds, then the Ulam–Hyers–Rassias stability of the fractional Caputo gonorrhoea model (1) holds.

**Proof.** For every  $\delta_1 > 0$  and for all  $s_f^{**} \in \mathbb{Y}$  satisfying

$$\left| {}^C D_{0,t}^{\alpha,\nu} s_f^{**}(t) - \mathcal{N}_1((t), s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t)) \right| < \delta_1 \Omega_1(t),$$

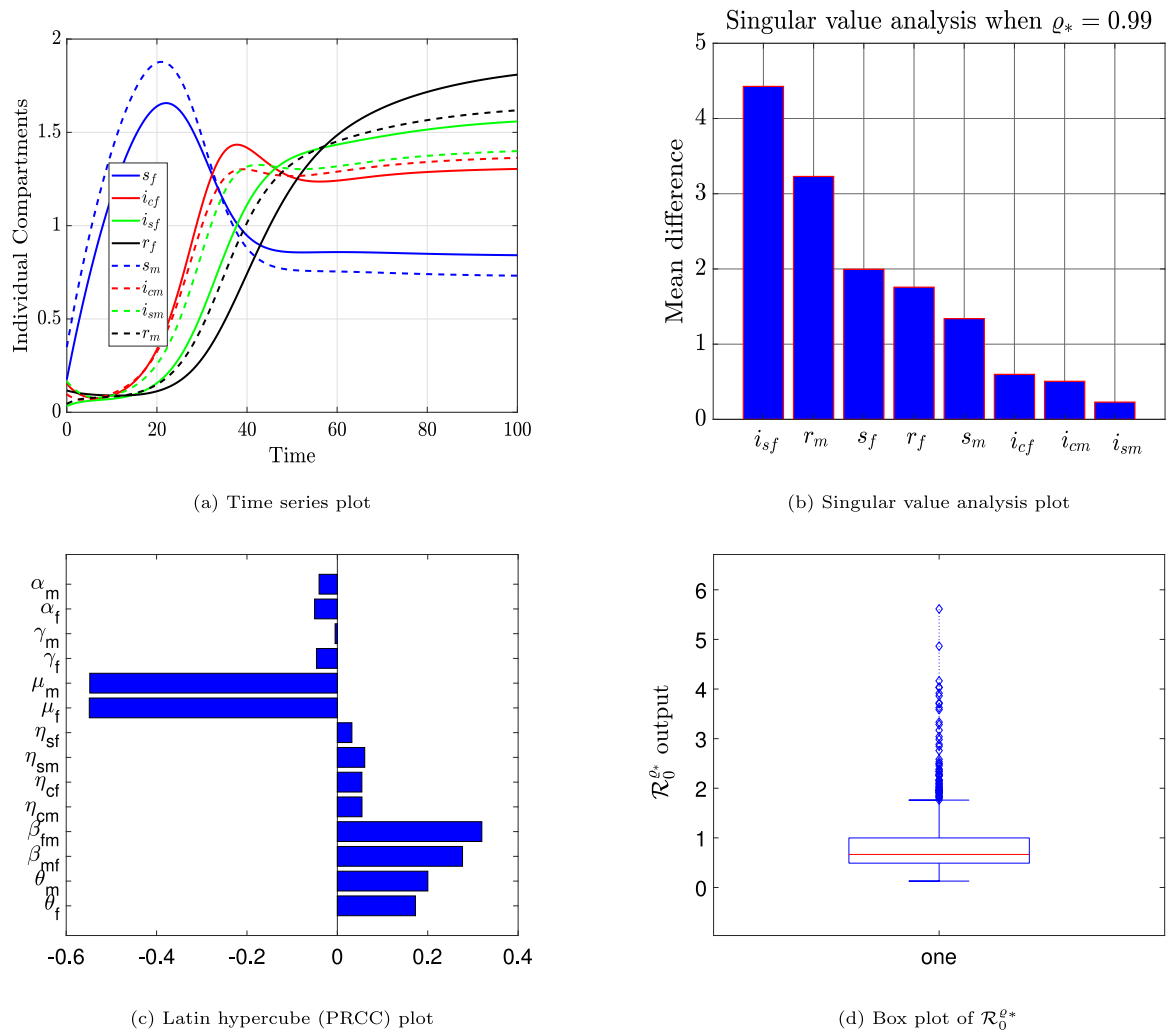


Fig. 6. Fractional time series plot and sensitivity dynamics of Gonorrhoea with  $\varrho_* = 0.99$ .

there exists  $\mathbf{p}_1(t)$  such that

$${}^C D_{0,t}^{\varrho_*} s_f^{**}(t) = \mathcal{N}_1(t, s_f^{**}(t), i_{cf}^{**}(t), i_{sf}^{**}(t), r_f^{**}(t), s_m^{**}(t), i_{cm}^{**}(t), i_{sm}^{**}(t), r_m^{**}(t)),$$

and  $|\mathbf{p}_1(t)| \leq \delta_1 \Omega_1(t)$ . Hence

$$s_f^{**}(t) = s_{f0} + \frac{1}{\Gamma(\varrho_*)} \int_0^t (t-\tau)^{\varrho_*-1} \mathcal{N}_1(\tau, s_f^{**}(\tau), i_{cf}^{**}(\tau), i_{sf}^{**}(\tau), r_f^{**}(\tau), s_m^{**}(\tau), i_{cm}^{**}(\tau), i_{sm}^{**}(\tau), r_m^{**}(\tau)) d\tau + \frac{1}{\Gamma(\varrho_*)} \int_0^t (t-\tau)^{\varrho_*-1} \mathbf{p}_1(\tau) d\tau.$$

From Theorem 4.3, we presume that  $s_f \in \mathbb{Y}$  has a single solution for the fractional Caputo gonorrhoea model(1). Therefore  $s_f(t)$  is

$$s_f(t) = s_{f0} + \frac{1}{\Gamma(\varrho_*)} \int_0^t (t-\tau)^{\varrho_*-1} \mathcal{N}_1(\tau, s_f^{**}(\tau), i_{cf}^{**}(\tau), i_{sf}^{**}(\tau), r_f^{**}(\tau), s_m^{**}(\tau), i_{cm}^{**}(\tau), i_{sm}^{**}(\tau), r_m^{**}(\tau)) d\tau$$

We get

$$\begin{aligned} |s_f^{**}(t) - s_f(t)| &\leq \frac{1}{\Gamma(\varrho_*)} \int_0^t (t-\tau)^{\varrho_*-1} |\mathbf{p}_1(\tau)| d\tau \\ &+ \frac{1}{\Gamma(\varrho_*)} \int_0^t (t-\tau)^{\varrho_*-1} \times \left| \mathcal{N}_1(\tau, s_f^{**}(\tau), i_{cf}^{**}(\tau), i_{sf}^{**}(\tau), r_f^{**}(\tau), s_m^{**}(\tau), i_{cm}^{**}(\tau), i_{sm}^{**}(\tau), r_m^{**}(\tau)) \right. \\ &\quad \left. - \mathcal{N}_1(\tau, s_f(\tau), i_{cf}(\tau), i_{sf}(\tau), r_f(\tau), s_m(\tau), i_{cm}(\tau), i_{sm}(\tau), r_m(\tau)) d\tau \right| \\ &\leq \frac{\delta_1}{\Gamma(\varrho_*)} \int_0^t (t-\tau)^{\varrho_*-1} \Omega_1(\tau) d\tau + \hat{\Delta} \hat{\Theta}_1 \|s_f^{**} - s_f\| \\ &\leq \delta_1 \Xi_{\Omega_1} \Omega_1(t) + \hat{\Delta} \hat{\Theta}_1 \|s_f^{**} - s_f\|. \end{aligned}$$

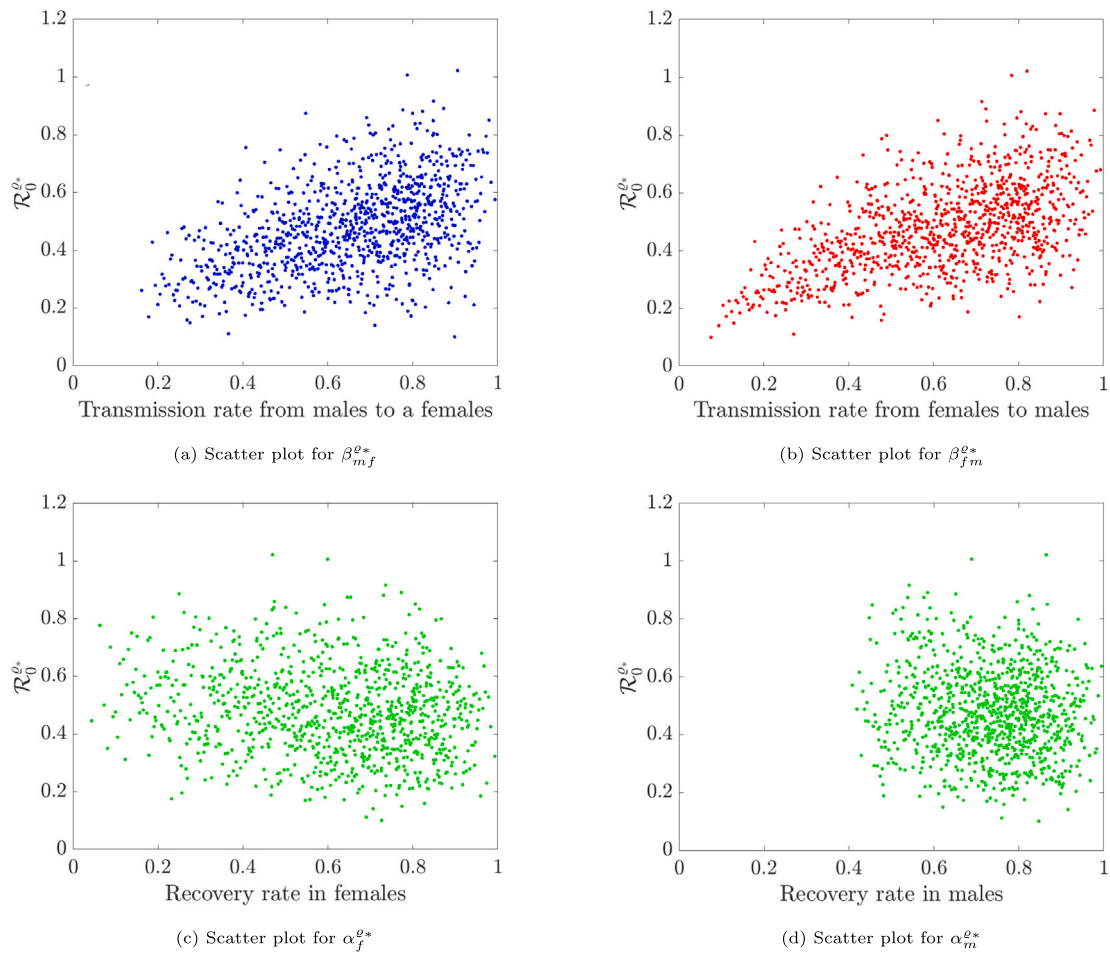


Fig. 7. Scatter plot of the fractional Gonorrhea reproduction number when  $\rho_* = 0.99$ .

Therefore

$$\|s_f^{**} - s_f\| \leq \frac{\delta_1 \Xi_{\Omega_1} \Omega_1(t)}{1 - \Delta \hat{\Theta}_1}.$$

If

$$D_{Q_1, \Omega_1} = \frac{\Xi_{\Omega_1}}{1 - \Delta \hat{\Theta}_1},$$

then  $\|s_f^{**} - s_f\| \leq \delta_1 D_{Q_1, \Omega_1} \Omega_1(t)$ . In a way analogous to this, we have

$$\|i_{cf}^{**} - i_{cf}\| \leq \delta_2 D_{Q_2, \Omega_2} \Omega_2(t), \quad \|i_{sf}^{**} - i_{sf}\| \leq \delta_3 D_{Q_3, \Omega_3} \Omega_3(t), \quad \|r_f^{**} - r_f\| \leq \delta_4 D_{Q_4, \Omega_4} \Omega_4(t),$$

$$\|s_m^{**} - s_m\| \leq \delta_5 D_{Q_5, \Omega_5} \Omega_5(t), \quad \|i_{cm}^{**} - i_{cm}\| \leq \delta_6 D_{Q_6, \Omega_6} \Omega_6(t), \quad \|i_{sm}^{**} - i_{sm}\| \leq \delta_7 D_{Q_7, \Omega_7} \Omega_7(t),$$

$$\|r_m^{**} - r_m\| \leq \delta_8 D_{Q_8, \Omega_8} \Omega_8(t),$$

where

$$D_{Q_i, \Omega_i} = \frac{\Xi_{\Omega_i}}{1 - \Delta \hat{\Theta}_i}, (i \in \{2, 3, 4, 5, 6, 7, 8\}).$$

Therefore, we state that the fractional Caputo gonorrhea model of (1) is Ulam–Hyers–Rassias stable. Hence, this completes the stability result of the proposed Caputo gonorrhea model (1).  $\square$

### 5. Numerical simulations

In this section, we show the numerical simulation concerning the stability of the proposed model, the sensitivity analysis using singular value analysis, Latin hypercube sampling, box-plot, scatter plot and sensitivity heat map. At the latter, we showed the time series plots when the fractional order is changed from 0.50 to the integer order 1.00. Using the parameter values in Table 1. We used the well-known fractional Adams–Bashforth iterative method by [33] for all the fractional Caputo simulations with initial conditions  $s_f = 0.6, i_{cf} = 0.2, i_{sf} = 0.1, r_f = 0.1, s_m = 0.7, i_{cm} = 0.1, i_{sm} = 0.1, r_m = 0.1$ .

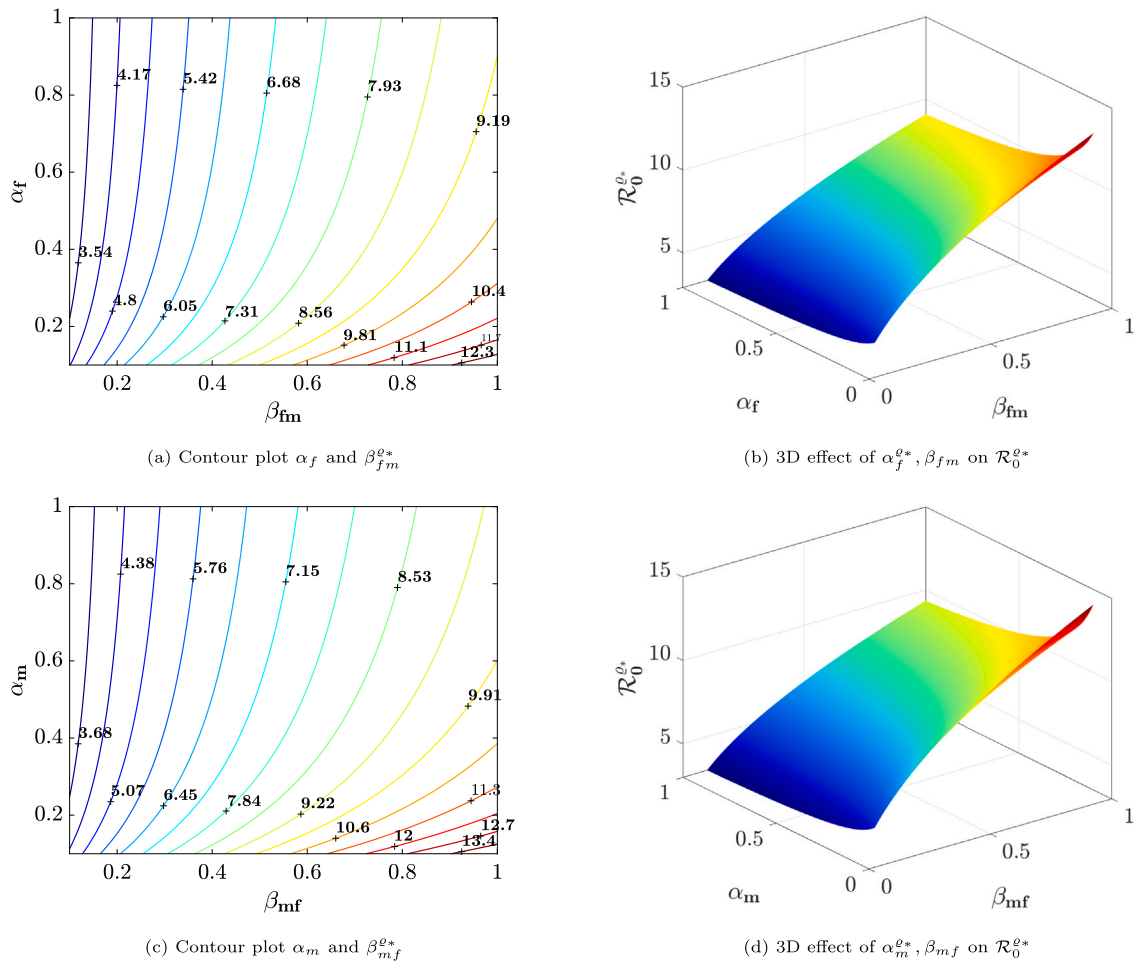


Fig. 8. Contour and surface plot of the fractional Gonorrhoea reproduction number when  $\rho_* = 0.99$ .

Table 1  
Parameter table.

| Parameters   | Values, day <sup>-1</sup> | Reference | Parameters   | Values, day <sup>-1</sup> | Reference |
|--------------|---------------------------|-----------|--------------|---------------------------|-----------|
| $\theta_f$   | 0.45                      | [34]      | $\beta_{mf}$ | 0.15                      | [35]      |
| $\theta_m$   | 0.3                       | [34]      | $\eta_{cf}$  | 0.65–0.99                 | Variable  |
| $\eta_{sm}$  | 0.4–0.8                   | Variable  | $\eta_{sf}$  | 0.65–0.99                 | Variable  |
| $\mu_f$      | 0.04                      | [8]       | $\eta_{cm}$  | 0.4                       | Variable  |
| $\mu_m$      | 0.04                      | [8]       | $\rho_f$     | 0.04                      | [8]       |
| $\alpha_f$   | 0.03                      | [35]      | $\rho_m$     | 0.04                      | [8]       |
| $\kappa_f$   | 0.01                      | Assumed   | $\kappa_m$   | 0.01                      | Assumed   |
| $\alpha_m$   | 0.4                       | [35]      | $\gamma_f$   | 0.2                       | Assumed   |
| $\beta_{fm}$ | 0.0625                    | [35]      | $\gamma_m$   | 0.26                      | Assumed   |

5.1. Results and discussion

5.1.1. Global stability plot

Figs. 2 and 3 shows that the gonorrhoea present equilibrium point is globally asymptotically stable when  $\mathcal{R}_0^{\rho_*} > 1$ . That is, for Figs. 2 and 3 we used the parameter values in Table 1, and  $\mathcal{R}_0^{\rho_*} = 4.1224$  with fractional order  $\rho_* = 0.99$ . Figs. 4 and 5 show that the Gonorrhoea-free equilibrium is globally asymptotically stable when  $\mathcal{R}_0^{\rho_*} < 1$ , thus, using the parameter values in Table 1,  $\mathcal{R}_0^{\rho_*} = 0.2735$  with  $\beta_{mf} = 0.0625$ ,  $\eta_{cm} = 0.04$ ,  $\eta_{cf} = 0.065$ ,  $\eta_{sm} = 0.04$ ,  $\eta_{sf} = 0.065$  and fractional order  $\rho_* = 0.99$ . Therefore, we state that the proposed model has unique gonorrhoea present and gonorrhoea-free equilibrium point whenever  $\mathcal{R}_0^{\rho_*} > 1$  and  $\mathcal{R}_0^{\rho_*} < 1$ , respectively.

5.1.2. Sensitivity analysis

In this subsection, we show various sensitivity analyses and other plots. Figs. 6–9, shows the various sensitivity plots. Fig. 6(a) shows the time series plot for all state variables when one normalized the date with a fractional order of 0.99. It indicates that, when one considers the spread of

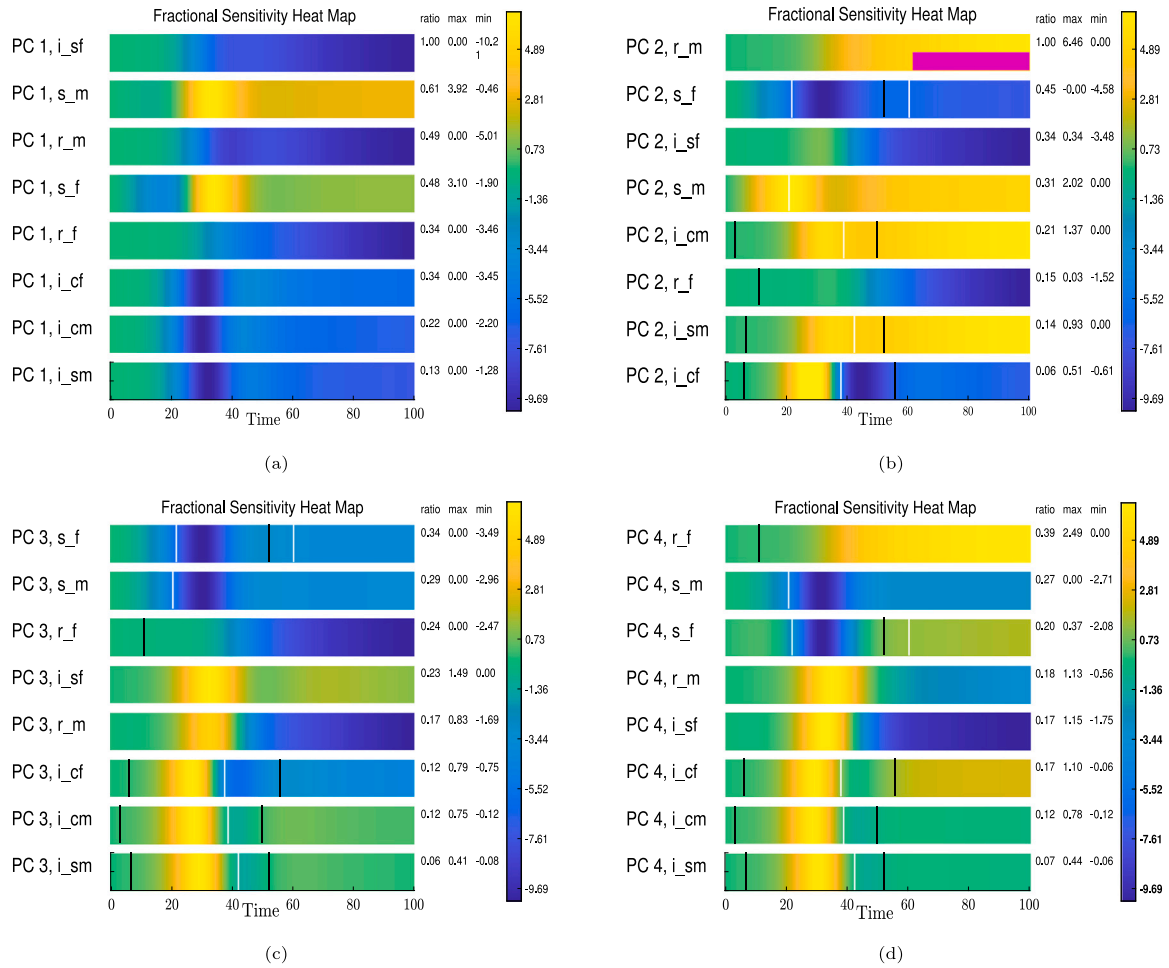


Fig. 9. Sensitivity heat map considering four principal components. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the disease for 100 days, there will be more recovered females than recovered males, more symptomatic females than symptomatic males, more incubative males than females and more susceptible females than susceptible males at the end of the simulation. Fig. 6(b) shows that the most sensitive state variable is the symptomatic females, followed by the recovered males. This indicates that the disease can be controlled if more females show symptoms than are in the incubative class. It also indicates that the spread of the disease can be managed with more recovered males.

Fig. 6(c) shows that global sensitivity analysis of the parameters in the basic reproduction number. The plot indicates that the infectivity potential by contact between incubative females and susceptible males,  $\eta_{cf}$ , the infectivity potential by contact between symptomatic females and susceptible males,  $\eta_{sf}$ , the infectivity potential by contact between incubative males and susceptible females,  $\eta_{cm}$ , the infectivity potential by contact between symptomatic males and susceptible females,  $\eta_{sm}$ , the transmission rate from females to males,  $\beta_{fm}$ , the transmission rate from males to females,  $\beta_{mf}$ , the recruitment rate of females,  $\theta_f$ , and the recruitment rate of males,  $\theta_m$ , contribute positively to the spread of the disease in a community. Also, the PRCC plot indicates that  $\alpha_f$ , recovery rate in females,  $\alpha_m$ , recovery rate in males,  $\rho_f$ , the proportion at which recovered females become susceptible again,  $\rho_m$ , the proportion at which recovered males become susceptible again,  $\gamma_f$ , the rate at which females in the incubative class moves to the symptomatic class, and,  $\gamma_m$  the rate at which males in the incubative class moves to the symptomatic class, reduces the spread of the disease in a community. The plot indicates that reducing the disease in females will go a long way in reducing the overall gonorrhoea endemicity in the population. The box plot in Fig. 6(d) shows that the threshold value ranges between 0.5 to 5.5 using 2000 uniformly distributed parameter values.

Figs. 7(a) and 7(b) show that, as the transmission rate from males to females and females to males increases from 0.1 to 1. The basic reproduction number linearly increases. Controlling these two transmission rates could lead to a significant drop in secondary infections. Figs. 7(c) and 7(d) show that, as the recovery of females and males increases from 0.1 to 1. The basic reproduction number linearly reduces. Any control measure to increase the recovery of these two classes will lead to a significant increase in healthy people.

Figs. 8(a) and 8(b) shows the contour and surface plot for female class. It indicates that the rate of recovery and transmission rate in the female population has an inverse relationship. That is, an increase in  $\alpha_f$  distorts the dynamical effect of the transmission rate  $\beta_{fm}$ . This similar thought holds for the male population in Figs. 8(c) and 8(d), respectively. The fractional model's sensitivity analysis utilizes the various compartments, visualized

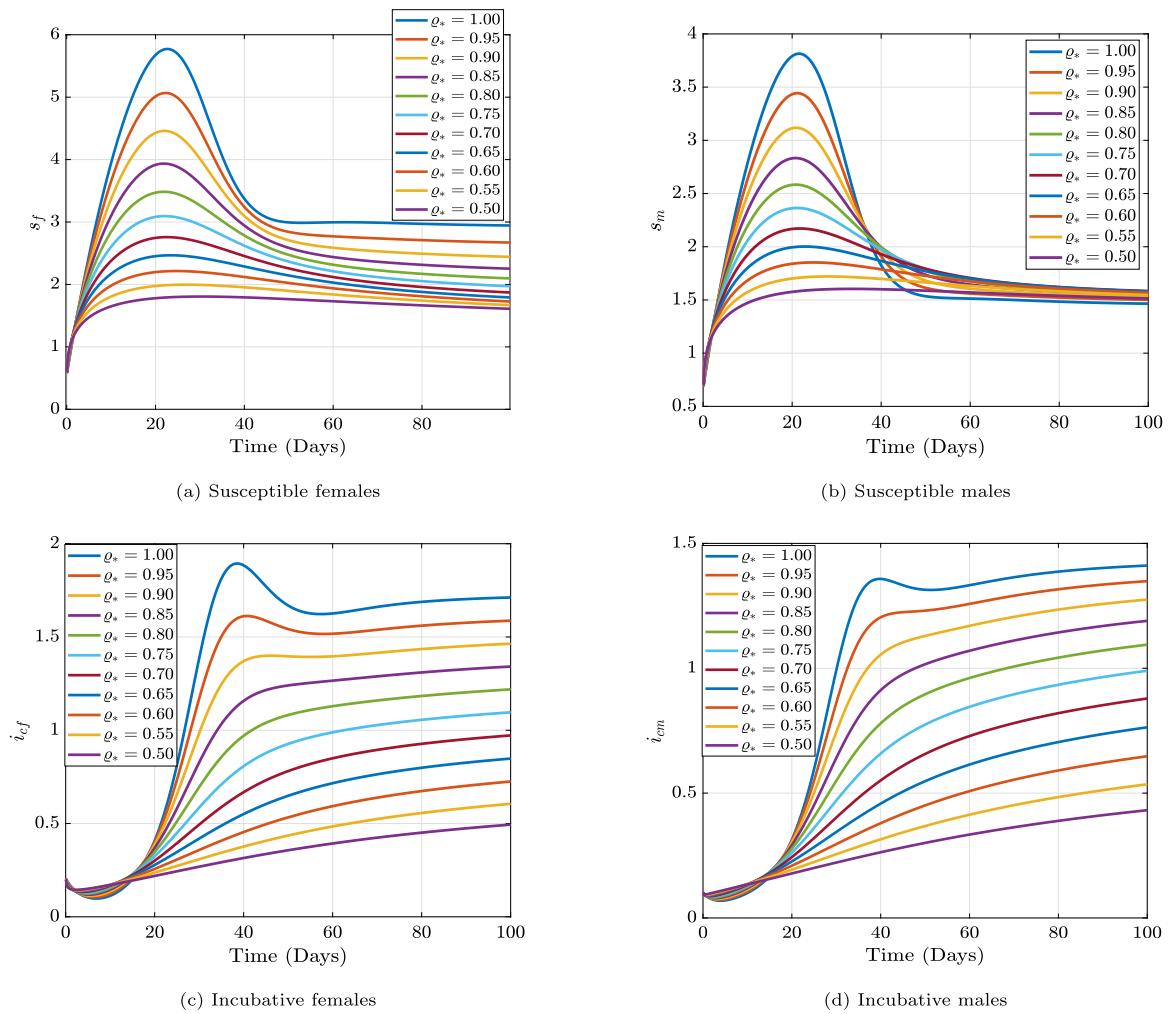


Fig. 10. Fractional dynamics of Gonorrhea considering susceptible and incubative subpopulation.

as a heat map for four different principal components (PC) in Fig. 9. Principal component analysis (PCA) is a widely employed technique for reducing the dimensionality of extensive data sets. It accomplishes this by transforming a multitude of variables into a more compact representation that retains the majority of the information present in the original data set. The sensitivity indices of gonorrhea compartments for each model parameter are displayed. The sensitivity index measures how much each partition shifts due to a change in one or more parameters. Each panel's color intensity corresponds to a different level of parameter sensitivity, with light yellow indicating a strong positive correlation between the parameters and the model compartments, deep yellow indicating a moderate positive correlation, green indicating a weak positive correlation, light blue showing a strong negative correlation, and deep blue indicating a robust negative correlation. Figs. 9(a)–9(d) shows that symptomatic females, recovered males, susceptible females and recovered females are the compartments that are most sensitive to the parameter values.

### 5.1.3. Caputo fractional simulations

Figs. 10 and 11 show the dynamical importance of fractional derivatives in epidemic studies. Fig. 10(a) shows that memory changes impact the number of susceptible females. For a structured gonorrhea study, each fractional order corresponds to a different number of susceptibility with no crisscrossing behavior. Fig. 10(b) shows that memory changes impact the number of susceptible males. For a structured gonorrhea study, each fractional order corresponds to a different number of susceptibility for males with crisscrossing behavior after 40 days. Figs. 10(c) and 10(d) show that memory changes impact the number of incubative females and males. That is, for structured gonorrhea there exists a crisscrossing behavior for the early stages of the disease, and afterwards, no crisscrossing behaviors occur. Figs. 11(a) and 11(b) show that memory changes impact the number of symptomatic females and males. For a structured gonorrhea model, there exists a close trajectory for 20 days in the symptomatic female class and no crisscrossing behavior for the female symptomatic class. However a crisscrossing behavior occurs for 25 days in the symptomatic males class, and afterwards, no crisscrossing behaviors show up.

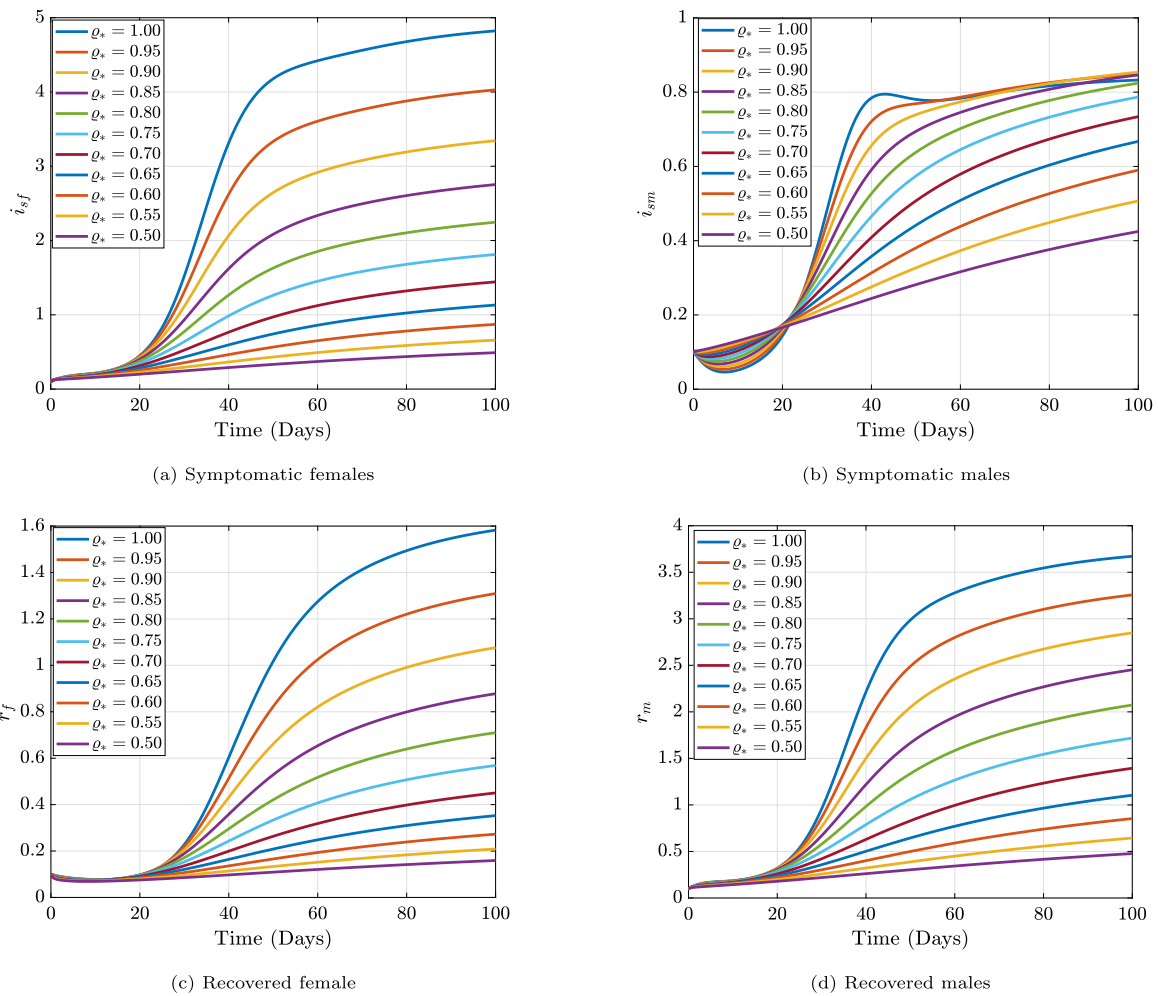


Fig. 11. Fractional dynamics of Gonorrhoea considering symptomatic and recovered subpopulation.

### 6. Conclusion

This paper studied the dynamics of gonorrhoea spread between females and males. The proposed model is shown to be positively invariant and bounded. The conditions for existence and uniqueness are obtained using the Lipschitz condition. The stability of the model was also studied using the Ulam Hyers and Ulam Hyers Rassias stability conditions. Compared to previous mathematical studies, this work showed the importance of memory in a structured population. We showed numerically that the gonorrhoea present equilibrium point is globally asymptotically stable when  $\mathcal{R}_0^{\rho_*} > 1$ , that the gonorrhoea-free equilibrium is globally asymptotically stable when  $\mathcal{R}_0^{\rho_*} < 1$ . The numerical simulation indicates that a change in memory affects the population sizes in each compartment. The sensitivity analysis shows that the most sensitive state variable is the symptomatic females, followed by the recovered males. This indicates that the disease can be controlled if more females show symptoms than are in the incubative class. It also indicates that the spread of the disease can be managed with more recovered males. Using the sensitivity heat map, we showed that the symptomatic females, recovered males, susceptible females and recovered females are the compartments most sensitive to the parameter values. The limitation of this work is that we did not compare this study to real data. The future outlook is to compare this model with gonorrhoea data; however, this study can serve as a guide since most parameters are related to gonorrhoea studies.

### CRediT authorship contribution statement

**Joshua Kiddy K. Asamoah:** Conceptualization, Methodology, Formal analysis, Writing – review & editing. **Gui-Quan Sun:** Formal analysis, Funding, Writing – review & editing.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Data availability

No data was used for the research described in the article.

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