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Non-fractional and fractional mathematical analysis and simulations for Q fever

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ABSTRACT

The purpose of analysing the transmission dynamism of Q fever (Coxiellosis) in livestock and incorporating ticks is to outline some management practices to minimise the spread of the disease in livestock. Ticks pass coxiellosis from an infected to a susceptible animal through a bite. The faecal matter can also contain coxiellosis, thus contaminating the environment and spreading the disease. First, a nonlinear integer order mathematical model is developed to represent the spread of this infectious disease in livestock. The proposed integer model investigates the positivity and boundedness, disease equilibria, basic reproduction number, bifurcation, and sensitivity analysis. Through mathematical analysis and numerical simulations, it shows that if the environmental transmission and the effective shedding rate of *coxiella burnetii* into the environment by both asymptomatic and symptomatic livestock are zero, then the usual threshold hold and it produces forward bifurcation. It is noticed that an increase in the recruitment rate of ticks produces backward bifurcation. And also, it is seen that an increase in the natural decay rate of the bacterial in the environment reduces the backward bifurcation point. Furthermore, to take care of the memory aspect of ticks on their host, we modified the initially proposed integer order model by introducing Caputo, Caputo-Fabrizio, Atangana-Baleanu fractional differential operators. The existence and uniqueness of these three newly developed fractional-order differential models are shown using the Banach fixed point theorem. Numerical trajectories are obtained for each of the fractional-order mathematical models. The trajectory of some fractional orders converges to the same endemic equilibrium point as the integer order. Finally, it is seen that the Atangana-Baleanu fractional differential operator captures more susceptibilities and fewer infections than the other operators.

1. Introduction

Query (Q) fever is a zoonotic illness that affects both humans and animals and is caused by Coxiella burnetii, a Gram-negative obligate intracellular bacterium that has evolved to survive within the phagocyte's phagolysosome; the disease is also known as coxiellosis in animals [1]. In 1935, during an epidemic of a febrile fever among slaughterhouse employees in Brisbane, Australia, Edward Derrick identified this infection, which he named Q (query) fever [2,3]. Coxiella burnetii, the aetiological agent, is found in nearly all animal kingdoms, including arthropods, although it is mostly transferred to humans by infected agricultural animals such as goats, cattle, sheep, and pets [4]. The primary reservoirs of Coxiella burnetii are domestic ruminants. However, cats, dogs, rabbits, birds, and other animals have been linked to human sickness and infection [1,4,5]. Coxiella burnetii is extremely hardy, meaning it may live in the environment for months [1]. Chronic Q fever can cause endocarditis (inflammation of the heart valves) and endovascular infections. If a woman is infected while pregnant, she is more likely to have a bad pregnancy result unless she is treated. Premature birth, growth restriction, spontaneous abortion, or fetal death can all result from Coxiella burnetii infection in pregnant women. The acute infection in most cases of Q fever resolves

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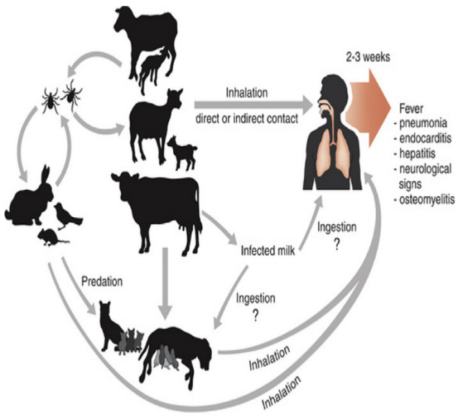
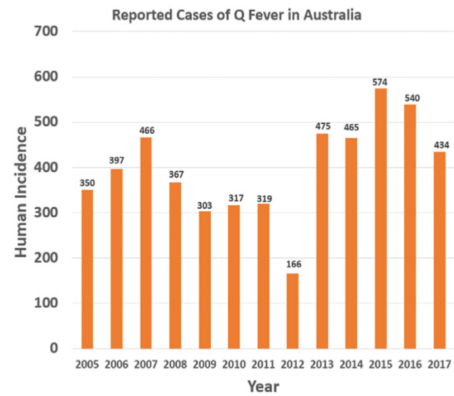
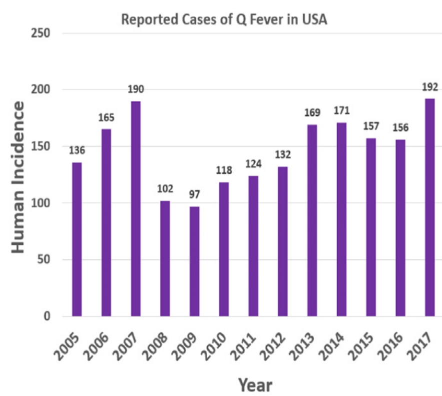


(a) Transmission of Coxiella burnetii. Image and caption was adopted from [8].

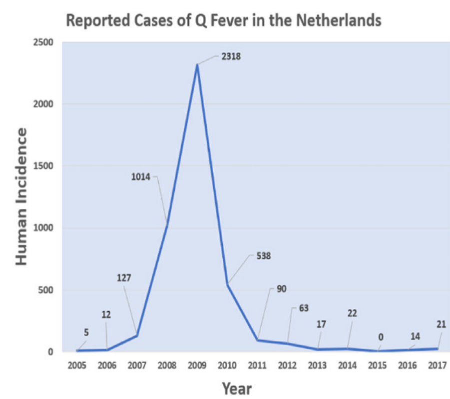


(b) Q fever incidence in Australia, data obtained from WAHID [9].

Fig. 1. Transmission cycle of Q fever and reported incidence of Q fever in humans in Australia.



(a) Q fever incidence in humans in the USA, data obtained from WAHID [9].



(b) Q fever incidence in humans in the Netherlands, data obtained from WAHID [9].

Fig. 2. A pictorial representation of Q fever in humans in USA and the Netherlands, data obtained from the World Animal Health Interface Database as at 23-04-2019 [9].

spontaneously or with proper antibiotic therapy, but the chronic infection requires long-term antibiotic therapy (at least 2 years) and serological surveillance [1,6]. Coxiella burnetii is a bacteria that causes difficulties in cattle reproductive. The illness can result in occasional miscarriages or epidemics, as well as dead or weak kids, all of which can be recovered without complications, although the infection is more often asymptomatic. With the exception of New Zealand, Iceland, French Polynesia, and Norway, Q fever (or Coxiellosis) is prevalent worldwide [7,5]. Ticks also carry the organism and are suspected to transmit it to animals, particularly rodents (such as tiny mice) and rabbits, as well as birds [8]. Coxiella burnetii can also be transmitted to livestock animals through ticks' bite [5,9]. Figures 1 and 2 gives a pictorial nature of the transmission routes, and incidence cases in some selected countries.

Mathematical modeling of infectious diseases and their constructive analytical study together with numerical simulations have greatly contributed to understanding diseases dynamics and their possible control measures [10–18]. Integer-order nonlinear differential equations have shown to be very useful in the construction of realistic epidemiological models, and they continue to gain popularity.

Fractional calculus which includes the calculus of derivatives and integrals of arbitrary order has been utilized to analyze the dynamics of real-world problems in numerous disciplines of science and engineering over the last few years [19–23]. Baleanu and Agarwal [24] have explored a variety of intriguing opinions and perspectives on fractional calculus and its application to real-world problems in science and engineering. In recent times, the application of fractional calculus in epidemiological modeling has broadened the mathematical analysis and numerical simulations perspectives of infectious disease models. It has been demonstrated in the literature that non-integer order differential equation models that are characterized by fractional-order derivative operators are very powerful and useful in formulating realistic models. One of the most widely used fractional derivative in fractional order models is the Caputo operator (see for example, [25–35]).

The authors in [36] and [37] have respectively developed Caputo–Fabrizio and Atangana–Baleanu fractional derivative operators with non-singular kernels. In [38], Khan and Atangana used three differential operators in fractional calculus namely Atangana–Baleanu, Caputo, and Caputo–Fabrizio to develop and examined epidemic models for Ebola virus disease (EVD). They performed numerical simulations for the three model problems and concluded that the EVD model with Atangana–Baleanu fractional derivative gives better results.

In [39], the author studied and analyzed an EVD compartmental model using Atangana-Baleanu fractional derivative operators. The author carried out the existence and uniqueness of solutions for the formulated EVD model and performed numerical experiments via the newly constructed and efficient iterative scheme developed by Toufik and Atangana [40]. Yavuz and Bonyah [41] in their study formulated and analyzed nonlinear mathematical models for schistosomiasis disease using Caputo–Fabrizio and Atangana–Baleanu fractional derivative operators. They obtained numerical solutions for the model problems using an iterative method through Laplace transform and Sumudu–Picard integration approach. The dynamical behavior of a two-strain Influenza virus infection is described using a fractional-order differential equation model based on the Mittag-Leffler kernel operator [42]. The transmission dynamics of diarrhea infection are investigated using real statistical data and deterministic differential equation models based on Mittag-Leffer kernel, exponential law kernel, and power law kernel operators [43]. A data-driven compartmental model for chickenpox has been developed and thoroughly studied using Mittag-Leffer kernel, exponential law kernel, and power law kernel operators in fractional calculus [44].

The literature on fractional calculus modelling techniques in the construction of realistic mathematical models connected to real-world situations, particularly compartmental modelling of infectious diseases, provided the motivation for this study. Furthermore, in the study of the transmission dynamics of Q fever in animals using deterministic models the following works can be cited [7,45], but none has consider the use of fractional calculus and the transmission dynamics of the disease between livestock and ticks. Even though numerous pioneering studies have been done on the relevance of ticks in Q fever epidemiology, the presence of Coxiella burnetii in ticks remains disputed [4,46,47]. Over 40 tick species contain Coxiella burnetii, according to early microscopic morphological observations [46]. Ticks with unusually high levels of Coxiella burnetii infection have been reported on occasion, raising the question of whether ticks play a critical role in Q fever transmission in livestock [4,48,49]. Thus, through non-fractional and fractional mathematical analysis and simulation, we seek to address this problem at the ecosystem (population) level using Caputo, Caputo-Fabrizio and Atangana-Baleanu derivatives and integrals.

The rest of the study comprises of eight sections, thus, Section 2 contain the model formulation and description of the state variables. Section 3 presents the model analysis in non-fractional form. In Section 4, we focus on some important definitions from factional derivatives in relation to the three fractional operators. In Section 5 the formulation of the three fractional operator models are given together with the prove of existence and uniqueness through fixed point theory. In Section 6, we provide the numerical schemes for our three proposed fractional operator models in livestock and ticks. Firstly, we start with the Caputo scheme, followed by the Caputo–Fabrizio scheme and lastly the Atangana–Baleanu scheme using the two step Lagrange interpolation. In Section 7, we apply the above numerical schemes to obtain numerical solutions for the three fractional derivatives. Section 8 contains the conclusion of the study.

2. Model formulation

We considered the dynamics of Q fever in livestock (cattle, sheep and goats) in a locality where the onset of a new infection is through environment, and contact with infected ticks. We assume a community where there are no vaccination and treatment dose for livestock. Let $S_L(t)$, $A_L(t)$ and $I_L(t)$ respectively denote the number of susceptible livestock, asymptomatic livestock and clinically ill livestock at time t. The susceptible tick population size and the number of the infected ticks at time t are respectively denoted by $S_T(t)$, and $I_T(t)$. Therefore, the total livestock population at time t is given as

$$N_L(t) = S_L(t) + A_L(t) + I_L(t), \tag{1}$$

and the total tick population is given as

$$N_T(t) = S_T(t) + I_T(t). \tag{2}$$

The total bacteria load at any time, t, in the environment is denoted by $B(t)$, Λ_L and Λ_T respectively denote the incremental rate of susceptible livestock and ticks. We assume that, the rate of transovarial transmission in ticks and livestock is zero, hence, all newborns are susceptible, μ_L and μ_T denote natural death rate in both compartments, ε is natural decay rate of *Coxiella burnetii* from the environment. The incidence of acquiring an infection in the livestock population is assumed to be

$$\lambda_L = (\beta_1 B + \beta_5 I_T)S_L, \tag{3}$$

this is due to the homogeneous mixing of livestock and a possible invasion of ticks on farms, the environmental transmission is denoted as β_1, β_5 is the per capita transmission rate of *coxiella burnetii* from infected ticks to a susceptible livestock, β_2 is the effective shedding rate of *coxiella burnetii* into the environment by both asymptomatic and symptomatic livestock, β_3 is the effective rate of transmission of *coxiella burnetii* to ticks after biting an infected livestock, thus $(A_L + I_L)$, and β_4 is the effective shedding rate of *coxiella burnetii* into the environment by infected ticks. The progression rate from A_L to I_L is denoted as α_L. The model flow diagram is shown in Fig. 3.

2.1. Model

From the above description we have the following system of equations for the proposed study.

$$\begin{cases} \frac{dS_L}{dt} = \Lambda_L - (\beta_1 B + \beta_5 I_T)S_L - \mu_L S_L, \\ \frac{dA_L}{dt} = (\beta_1 B + \beta_5 I_T)S_L - (\mu_L + \alpha_L)A_L, \\ \frac{dI_L}{dt} = \alpha_L A_L - \mu_L I_L, \\ \frac{dB}{dt} = \beta_2(A_L + I_L) + \beta_4 I_T - \varepsilon B, \\ \frac{dS_T}{dt} = \Lambda_T - \beta_3 S_T(A_L + I_L) - \mu_T S_T, \\ \frac{dI_T}{dt} = \beta_3 S_T(A_L + I_L) - \mu_T I_T, \end{cases} \tag{4}$$

with initial conditions

$$S_L(0) = S_{L_0} > 0, A_L(0) = A_{L_0} \geq 0,\ I_L(0) = I_{L_0} \geq 0,$$

$$B(0) = B_0 \geq 0, S_T(0) = S_{T_0} \geq 0,\ I_T(0) = I_{T_0} \geq 0.$$

3. Model analysis

This section focus on the feasible regions of the transmission model (4), thus, we need to show that the proposed model is epidemiologically well stated, and that, it leads to nonnegative state values as time, t, increases (thus, $t > 0$). If this is satisfied, then it is sufficient to obtain the disease equilibria, reproduction numbers, and further test the stability of the equilibria. The model (4) is basically divided into three invariant regions; thus, $\Omega = \Omega_L + \Omega_B + \Omega_T$, therefore with reference from Agusto et al. [50], we now proceed to analysis the positivity and boundedness of solutions; invariant regions.

3.1. Positivity and boundedness

Theorem 3.1. *Let the initial state values be* $S_L(0) > 0$, $A_L(0) > 0$, $I_L(0) > 0$, $S_T(0) > 0$, $I_T(0) > 0$, $B(0) > 0$. *Then, the solution set* $\Omega = \{S_L(t), A_L(t), I_L(t), S_T(t), I_T(t), B(t)\} \in \mathcal{R}_+^6$ *of the Q fever model (4) are nonnegative for all* $t > 0$.

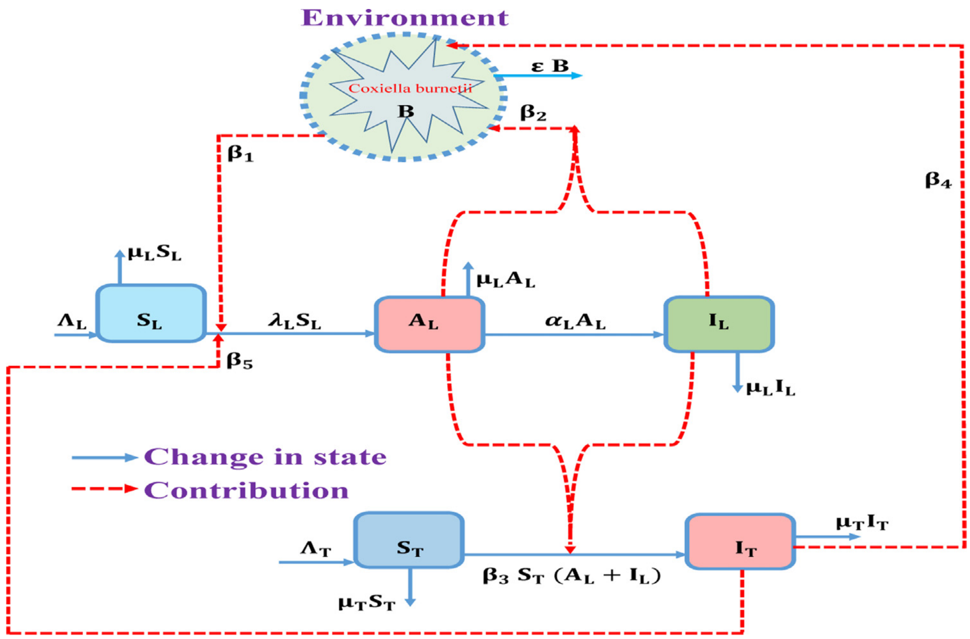


Fig. 3. Flow diagram for Q fever in a tick induced environment.

Proof. Let $Z^* = (S_L, A_L, I_L, B, S_T, I_T)^{T^*}, M_0^* = (\beta_1 B + \beta_5 I_T), M_1^* = \beta_2(A_L + I_L), M_2^* = \beta_3 S_T(A_L + I_L)$ where T^* represent transposition. Hence, model (4) can be written as

$$\begin{cases} \frac{dS_L}{dt} = \Lambda_L - M_0^* S_L - \mu_L S_L, \\ \frac{dA_L}{dt} = M_0^* S_L - (\mu_L + \alpha_L) A_L, \\ \frac{dI_L}{dt} = \alpha_L A_L - \mu_L I_L, \\ \frac{dB}{dt} = M_1^* + \beta_4 I_T - \varepsilon B, \\ \frac{dS_T}{dt} = \Lambda_T - M_2^* S_T - \mu_T S_T, \\ \frac{dI_T}{dt} = M_2^* S_T - \mu_T I_T, \end{cases} \tag{5}$$

Solving $\frac{dS_L}{dt}$ by applying the integration factor rule and the method of change of variables, we have

$$\frac{d}{dt}\left\{S_L(t)exp\left[\mu_L t + \int_0^t M_0^*(\tau)d\tau\right]\right\} = \Lambda_L\left\{exp\left[\mu_L t + \int_0^t M_0^*(\tau)d\tau\right]\right\}.$$

Thus,

$$S_L(t_1)exp\left[\mu_L t_1 + \int_0^{t_1} M_0^*(\tau)d\tau\right] - S_L(0) = \int_0^{t_1} \Lambda_L\left\{exp\left[\mu_L y^* + \int_0^{y^*} M_0^*(\tau)d\tau\right]\right\}dy^*.$$

Hence,

$$\begin{aligned} S_L(t_1) =& S_L(0)exp\left[-\mu_L t_1 - \int_0^{t_1} M_0^*(\tau)d\tau\right] \\ &+ exp\left[-\mu_L t_1 - \int_0^{t_1} M_0^*(\tau)d\tau\right] \times \int_0^{t_1} \Lambda_L\left\{exp\left[\mu_L y^* + \int_0^{y^*} M_0^*(\tau)d\tau\right]\right\}dy^* > 0. \end{aligned}$$

The same idea can be used to show that $A_L(t) > 0, I_L(t) > 0, B(t) > 0, S_T(t) > 0$ and $I_T(t) > 0$ for all $t > 0$. This guarantees that the compartments remain nonnegative and feasible during the entire scope of the study. In addition, model (5) can be written as $\frac{dZ^*}{dt} = D^*Z^* + Y^*$, where

$$D^* = \begin{pmatrix} -(M_0^* + \mu_L) & 0 & 0 & 0 & 0 & 0 \\ M_0^* & -(\mu_L + \alpha_L) & 0 & 0 & 0 & 0 \\ 0 & \alpha_L & -\mu_L & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon & 0 & \beta_4 \\ 0 & 0 & 0 & 0 & -(M_2^* + \mu_T) & 0 \\ 0 & 0 & 0 & 0 & M_2^* & -\mu_T \end{pmatrix}, Y^* = \begin{pmatrix} \Lambda_L \\ 0 \\ 0 \\ M_1^* \\ 0 \\ 0 \end{pmatrix}.$$

□

It is obvious that $Y^* \geq 0$ and the off-diagonal of D^* are positive for positive parameter values. This proves the property of Metzler matrix. Hence, this further support that all the equations in model (4) remains positive through out the scope of the study.

3.1.1. Boundedness

Using the epidemiologically feasible region, Ω, as defined below, we want to verify that a transformation on the Q fever model (4) will remain in a certain realistic limit for $t > 0$.

$$\Omega = \Omega_L + \Omega_B + \Omega_T \subset \mathcal{R}_+^3 \times \mathcal{R}_+ \times \mathcal{R}_+^2,$$

where

$$\Omega = \left\{ (S_L, A_L, I_L, S_T, I_T, B) \in \mathcal{R}_+^6 : N_L \leq \frac{\Lambda_L}{\mu_L}, N_T \leq \frac{\Lambda_T}{\mu_T},\ B \leq \frac{\beta_2 \Lambda_L \mu_T + \beta_4 \Lambda_T \mu_L}{\varepsilon \mu_L \mu_T} \right\}.$$

From model (4) the total population is given as

$$\frac{dN_L(t)}{dt} = \Lambda_L - \mu_L(S_L(t) + A_L(t) + I_L(t)), \tag{6}$$

$$\frac{dN_T(t)}{dt} = \Lambda_T - \mu_T(S_T(t) + I_T(t)), \tag{7}$$

$$\frac{dB(t)}{dt} = \beta_2(A_L(t) + I_L(t)) + \beta_4 I_T(t) - \varepsilon B(t). \tag{8}$$

Now to establish that $\mathcal{R}_+^6$ is positively invariant, we recall the following Eqs. (6), (7), and (8), here, it should be noted that, if $0 < A_L(t) + I_L(t) \leq N_L(t),\ 0 < I_T(t) \leq N_T(t)$, then it follows that

$$\frac{dN_L(t)}{dt} \leq \Lambda_L - \mu_L N_L(t),$$
$$\frac{dB(t)}{dt} \leq \beta_2 N_L(t) + \beta_4 N_T(t) - \varepsilon B,$$
$$\frac{dN_T(t)}{dt} \leq \Lambda_T - \mu_T N_L(t).$$

Now, solving the above inequalities, the following is established $N_L(t) \leq N_L(0)e^{-\mu_L t} + \frac{\Lambda_L}{\mu_L}\left(1 - e^{-\mu_L t}\right)$, $B(t) \leq \frac{\beta_2 N_L(t) + \beta_4 N_T(t)}{\varepsilon} + B(0)e^{-\varepsilon t} - \frac{\beta_2 N_L(0) + \beta_4 N_T(0)}{\varepsilon} e^{-\varepsilon t}$ and $N_T(t) \leq N_T(0)e^{-(\mu_T)t} + \frac{\Lambda_T}{\mu_T}\left(1 - e^{-(\mu_T)t}\right)$. In particular, $N_L(t) \leq \frac{\Lambda_L}{\mu_L}, B(t) \leq \frac{\beta_2 \Lambda_L}{\varepsilon \mu_L} + \frac{\beta_4 \Lambda_T}{\varepsilon \mu_T}$ and $N_T(t) \leq \frac{\Lambda_T}{\mu_T}$ as $t \to \infty$. Therefore, the region Ω is positively-invariant. Hence, the proposed model is considered to be epidemiological meaningful (mathematically well posed) and it is sufficient to say that all solution of the Q fever model, with initial conditions remains in the feasible region Ω.

3.2. Disease-free equilibrium(DFE) and basic reproduction number

The disease-free equilibrium of our Q fever model is obtained by setting the right-hand sides of the equations in model (4) to zero, and further equating the *coxiella burnetii* induced livestock, ticks and bacteria load in the environment to zero, thus $A_L = I_L = I_T = B = 0$. Hence, the unique disease-free equilibrium is given by

$$\text{DEF} \equiv \mathcal{E}_0 := (S_{L0},\ A_{L0},\ I_{L0},\ B_0,\ S_{T0},\ I_{T0}) = \left(\frac{\Lambda_L}{\mu_L},\ 0,\ 0,\ 0,\ \frac{\Lambda_T}{\mu_T},\ 0 \right).$$

Using the DFE, we can obtain the basic reproduction number $\mathcal{R}_0$. Here, $\mathcal{R}_0$ is defined as the number of secondary infection(s) that an infected tick can generate in a completely susceptible livestock population during its entire period of infectiousness, and/or the number of secondary infection(s) produced when the bacteria is taken in from the environment by a susceptible livestock, $\mathcal{R}_0$ also contains the secondary infection(s) of ticks produced by both asymptomatic and symptomatic livestock. If $\mathcal{R}_0 < 1$ then the disease will eventually die out, whereas $\mathcal{R}_0 > 1$ means that the disease will persist in the population and $\mathcal{R}_0 = 1$ is a baseline which requires further investigation. The above account can be obtained mathematically, by using the next generation operator method $G = FV^{-1}$, thus we define two matrices F and V, where the element(s) in matrix F constitutes the new infection terms, while that of matrix V constitutes the new transfer of infection terms from one compartment to the another [51,52]. Now, considering the *Coxiella burnetii* induced state variables in model (4), we have the following equations

$$\frac{dA_L}{dt} = (\beta_1 B + \beta_5 I_T)S_L - (\mu_L + \alpha_L)A_L, \tag{9}$$

$$\frac{dI_L}{dt} = \alpha_L A_L - \mu_L I_L, \tag{10}$$

$$\frac{dB}{dt} = \beta_2(A_L + I_L) + \beta_4 I_T - \varepsilon B, \tag{11}$$

$$\frac{dI_T}{dt} = \beta_3 S_T(A_L + I_L) - \mu_T I_T. \tag{12}$$

Linearising Eqs. (9)–(12) about the DFE and assuming that shedding is not a new infection, then elements in B goes into matrix V(see [53], Sections 6 and 7, for similar approach), we obtain a Jacobian matrix J, given as, $\frac{d\vec{M}}{dt} := J\vec{M} = (F - V)\vec{M}$, where $\vec{M} = [A_L(t), I_L(t), B(t), I_T(t)]^{T^*}$ denotes vector of infectious variables.

Therefore,

$$F = \begin{pmatrix} 0 & 0 & \beta_1 \frac{\Lambda_L}{\mu_L} & \beta_5 \frac{\Lambda_L}{\mu_L} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_3 \frac{\Lambda_T}{\mu_T} & \beta_3 \frac{\Lambda_T}{\mu_T} & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\mu_L + \alpha_L) & 0 & 0 & 0 \\ -\alpha_L & \mu_L & 0 & 0 \\ -\beta_2 & -\beta_2 & \varepsilon & -\beta_4 \\ 0 & 0 & 0 & \mu_T \end{pmatrix}, \tag{13}$$

which leads to,

$$G := FV^{-1} = \begin{pmatrix} \frac{\Lambda_L \beta_1 \beta_2}{\varepsilon \mu_L{}^2} & \frac{\Lambda_L \beta_1 \beta_2}{\varepsilon \mu_L{}^2} & \frac{\Lambda_L \beta_1}{\varepsilon \mu_L} & \frac{\Lambda_L (\beta_1 \beta_4 + \beta_5 \varepsilon)}{\varepsilon \mu_L \mu_T} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\Lambda_T \beta_3}{\mu_L \mu_T} & \frac{\Lambda_T \beta_3}{\mu_L \mu_T} & 0 & 0 \end{pmatrix}. \tag{14}$$

Hence, the basic reproduction number $\mathcal{R}_0$ is the spectral radius (dominant eigenvalue) of Eq. (14). Therefore, $\mathcal{R}_0 = \rho(FV^{-1})$, is the positive root of the quadratic equation.

$$Q(\lambda) = \lambda^2 - \mathcal{R}_0^{01}\lambda - \mathcal{R}_0^{02}\mathcal{R}_0^{03} = 0, \tag{15}$$

where $\mathcal{R}_0^{01} = \frac{\beta_1\beta_2\Lambda_L}{\varepsilon\mu_L^2}$, $\mathcal{R}_0^{02} = \frac{\Lambda_L(\beta_1\beta_4+\beta_5\varepsilon)}{\mu_T\mu_L\varepsilon}$, $\mathcal{R}_0^{03} = \frac{\Lambda_T\beta_3}{\mu_T\mu_L}$.

Hence,

$$\mathcal{R}_0 = \mathcal{R}_0^{NG},$$

where

$$\mathcal{R}_0^{NG} = \frac{1}{2}\left(\mathcal{R}_0^{01} + \sqrt{(\mathcal{R}_0^{01})^2 + 4\mathcal{R}_0^{02}\mathcal{R}_0^{03}}\right).$$

Here, the reproduction number, $\mathcal{R}_0$, consist of the number of secondary infections generated per each stage of the disease transmission processes. It can be seen that

$$\mathcal{R}_0 < 1 \ \text{ iff } \ \mathcal{R}_0^{01} + \mathcal{R}_0^{02}\mathcal{R}_0^{03} < 1, \tag{16}$$

$$\mathcal{R}^0 = 1 \ \text{ iff } \ \mathcal{R}_0^{01} + \mathcal{R}_0^{02}\mathcal{R}_0^{03} = 1, \tag{17}$$

$$\mathcal{R}^0 > 1 \ \text{ iff } \ \mathcal{R}_0^{01} + \mathcal{R}_0^{02}\mathcal{R}_0^{03} > 1. \tag{18}$$

The above assertion can be verified by direct calculation,

$$\begin{aligned} \mathcal{R}_0^{NG} &< 1, \\ \frac{1}{2}\left(\mathcal{R}_0^{01} + \sqrt{(\mathcal{R}_0^{01})^2 + 4\mathcal{R}_0^{02}\mathcal{R}_0^{03}}\right) &< 1, \\ \sqrt{(\mathcal{R}_0^{01})^2 + 4\mathcal{R}_0^{02}\mathcal{R}_0^{03}} &< 2 - \mathcal{R}_0^{01}, \\ 4\mathcal{R}_0^{02}\mathcal{R}_0^{03} &< 4 - 4\mathcal{R}_0^{01}, \\ \mathcal{R}_0^{01} + \mathcal{R}_0^{02}\mathcal{R}_0^{03} &< 1. \end{aligned}$$

Analogously, (17)-(18) hold, which completes the proof of our assertion.

Also, the expressions in $\mathcal{R}_0^{NG}$ support some epidemiological facts: $\mathcal{R}_0^{01}$ indicates that the rate of shedding of the bacteria by both asymptomatic and symptomatic livestock into the environment, which may multiply the rate of intake of the bacteria by a susceptible animal, this may lead to an outbreak, but the term $\frac{1}{\varepsilon\mu_L^2}$, indicates that in such situation, any method aimed at increasing bacteria decay rate from the environment will eventually minimize the rate of new infection in a completely susceptible live animal population. The term $\frac{\beta_1\beta_4\Lambda_L}{\varepsilon\mu_L}$ in $\mathcal{R}_0^{02}$ indicates that shedding of *Coxiella burnetii* into the environment maintains the circle of new infection in a completely susceptible livestock population, where $\frac{1}{\mu_L}$, $\mu_L > 0$ and $\frac{1}{\varepsilon}$, $\varepsilon > 0$ is the mean time in $(A_L + I_L)$ and B respectively, hence any approach aimed at increasing μ_L and/or ε, will lead to a reduction of bacteria load shed by an infected livestock. The term $\frac{\beta_5\Lambda_L}{\mu_T}$ in $\mathcal{R}_0^{02}$ is the new infections that can be generated by an infected tick through a successful bite of susceptible livestock during its period of infectiousness. The expressions $\mathcal{R}_0^{03}$ is the secondary infections produced by both asymptomatic and symptomatic livestock in the susceptible tick population. In a tick free livestock farm, we will have $\mathcal{R}_0 = \mathcal{R}_0^{01}$.

3.3. Endemic equilibrium and existence of backward bifurcation

The endemic equilibrium is obtain by solving the following equation.

$$\begin{cases} \Lambda_L - (\beta_1 B + \beta_5 I_T)S_L - \mu_L S_L = 0, \\ (\beta_1 B + \beta_5 I_T)S_L - k_0 A_L = 0, \\ \alpha_L A_L - \mu_L I_L = 0, \\ \beta_2 (A_L + I_L) + \beta_4 I_T - \varepsilon B = 0, \\ \Lambda_T - \beta_3 S_T (A_L + I_L) - \mu_T S_T = 0, \\ \beta_3 S_T (A_L + I_L) - \mu_T I_T = 0, \end{cases} \tag{19}$$

where $k_0 = (\mu_L + \alpha_L)$. Now, let the force of infection be

$$\lambda_1^* = (\beta_1 B + \beta_5 I_T), \tag{20}$$

$$\lambda_2^* = \beta_2 (A_L + I_L), \tag{21}$$

$$\lambda_3^* = \beta_3 (A_L + I_L), \tag{22}$$

$$\lambda_4^* = \beta_4 I_T, \tag{23}$$

and let the impulsive endemic equilibrium point be denoted by

$\mathcal{E}^* = (S_L^*, A_L^*, I_L^*, B^*, S_T^*, I_T^*)$.

Therefore, the endemic equilibrium point from (19) expressed in terms of $\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*$ is given as

$$\begin{aligned} S_L^* &= \frac{\Lambda_L}{(\lambda_1^* + \mu_L)}, \\ A_L^* &= \frac{\lambda_1^* \Lambda_L}{(\lambda_1^* + \mu_L)k_0}, \\ I_L^* &= \frac{\alpha_L \lambda_1^* \Lambda_L}{(\lambda_1^* + \mu_L)k_0 \mu_L}, \\ B^* &= \frac{\lambda_2^* + \lambda_4^*}{\varepsilon}, \\ S_T^* &= \frac{\Lambda_T}{(\lambda_4^* + \mu_T)}, \\ I_T^* &= \frac{\lambda_3^* \Lambda_T}{(\lambda_3^* + \mu_T)\mu_T}, \end{aligned} \tag{24}$$

substituting (24) into (20)–(23), the force of infection now becomes

$$\lambda_1^* = \frac{\beta_1 \mu_T (\lambda_3^* + \mu_T)(\lambda_2^* + \lambda_4^*) + \beta_5 \lambda_3^* \varepsilon \Lambda_T}{\varepsilon \mu_T (\lambda_3^* + \mu_T)}, \tag{25}$$

$$\lambda_2^* = \frac{\beta_2 \Lambda_L \lambda_1^*}{\mu_L (\lambda_1^* + \mu_L)}, \tag{26}$$

$$\lambda_3^* = \frac{\beta_3 \Lambda_L \lambda_1^*}{\mu_L (\lambda_1^* + \mu_L)}, \tag{27}$$

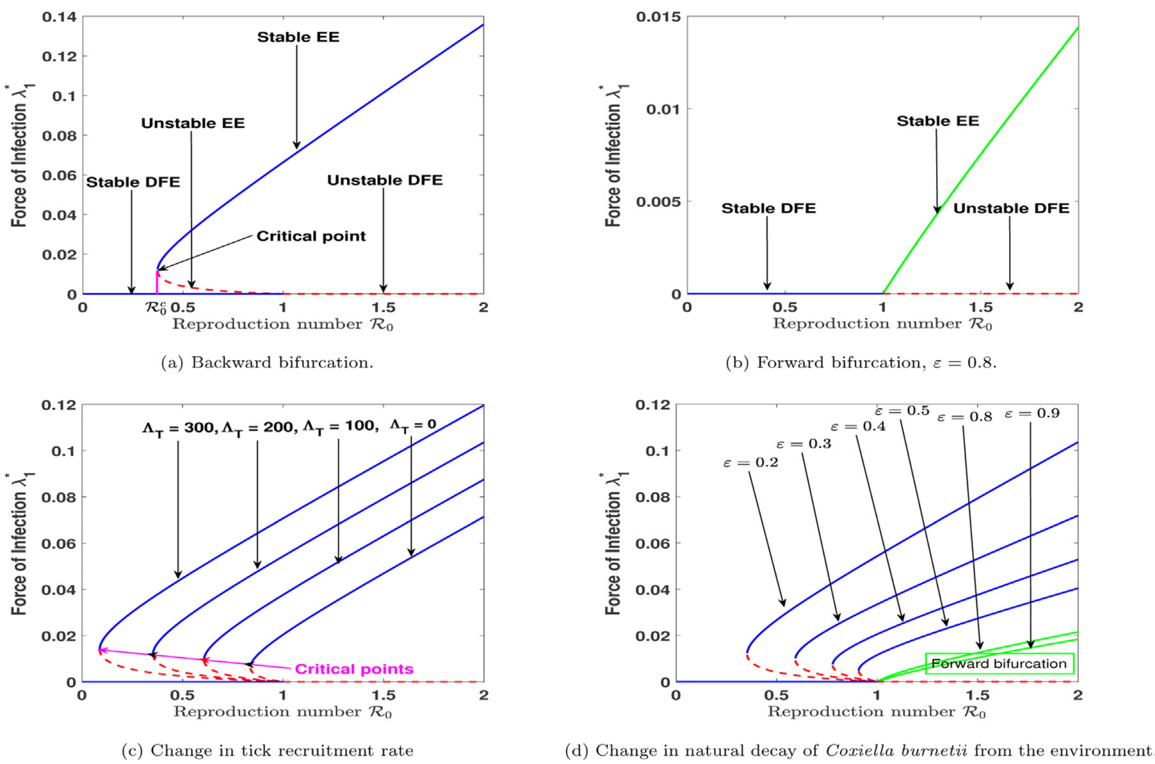
$$\lambda_4^* = \frac{\beta_4 \Lambda_T \beta_3 \Lambda_L \lambda_1^*}{\beta_3 \Lambda_L \lambda_1^* \mu_T + \mu_L \mu_T^2 (\lambda_1^* + \mu_L)}. \tag{28}$$

Thus, substituting (26)–(28) into (25), gives

$$\mathcal{A}_1 {\lambda_1^*}^3 + \mathcal{A}_2 {\lambda_1^*}^2 + \mathcal{A}_3 \lambda_1^* + \mathcal{A}_4 = 0, \tag{29}$$

where

$$\mathcal{A}_1 = 2\Lambda_L \beta_3 {\mu_L}^2 {\mu_T}^3 \varepsilon + {\mu_L}^3 {\mu_T}^4 \varepsilon + \beta_3^2 \Lambda_L^2 \mu_T^2 \mu_L \varepsilon,$$



(a) Backward bifurcation. (b) Forward bifurcation, $\varepsilon = 0.8$.

(c) Change in tick recruitment rate (d) Change in natural decay of *Coxiella burnetii* from the environment.

Fig. 4. Bifurcation diagrams as $\beta_1 \in \{0.001, 2.000\}$. Parameter values used are $\mu_L = 0.02, \Lambda_L = 9, \alpha_L = 0.0029, \beta_2 = 0.00003, \beta_3 = 0.0000001, \beta_4 = 0.000001, \beta_5 = 0.0001, \mu_T = 0.01$.

$$\mathcal{A}_2 = \beta_3\Lambda_L\mu_T^3\,{\mu_L}^3\varepsilon\left(4-\left[2\mathcal{R}_0^{01}+\mathcal{R}_{01}\right]\right)+\mu_T^4\mu_L^4\varepsilon\left(2-\left[\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03}\right]\right)+\beta_3^2\Lambda_L^2\mu_T^2\mu_L^2\varepsilon\left(1-\left[\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03}\right]\right)$$

$$\mathcal{A}_3 = \mu_T^4\mu_L^5\varepsilon\left(1-2\left[\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03}\right]\right)+\beta_3\Lambda_L\mu_L^4\mu_L^3\varepsilon\left(1-\mathcal{R}_0^{01}\right)+2\,\mu_L\mu_T^2\,{\mu_L}^4\,{\mu_T}^2\,\varepsilon+\beta_3\Lambda_L\mu_L^4\mu_T^3\varepsilon\left(1-\left[\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03}\right]\right)$$

$$\mathcal{A}_4 = \varepsilon{\mu_L}^6{\mu_T}^4\left(1-\left[\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03}\right]\right),$$

with $\mathcal{R}_0^{01} = \frac{\beta_1\beta_2\Lambda_L}{\varepsilon\mu_L^2}$, $\mathcal{R}_0^{02} = \frac{\Lambda_L(\beta_1\beta_4+\beta_5\varepsilon)}{\mu_T\mu_L\varepsilon}$, $\mathcal{R}_0^{03} = \frac{\Lambda_T\beta_3}{\mu_T\mu_L}$, $\mathcal{R}_{01} = \frac{\beta_2\Lambda_L^2\beta_3\beta_1}{\mu_L^3\varepsilon}$. It can be observed that $\mathcal{A}_1 > 0$, due to the fact that all the model parameters are nonnegative. Furthermore, $\mathcal{A}_4 > 0$ whenever $\mathcal{R}_0 < 1$, iff $\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03} < 1$. Therefore, the possible number of positive real roots for the cubic polynomial (29) is dependent on the sign of $\mathcal{A}_2$, and $\mathcal{A}_3$. Which can be analyzed using the Descartes' rule of signs on the cubic $p(x^*) = \mathcal{A}_1 x^{*3} + \mathcal{A}_2 x^{*2} + \mathcal{A}_3 x^* + \mathcal{A}_4$ (with $x^* = \lambda_1^*$). Therefore, the following are established.

Theorem 3.2. *The model (4)*

(a) *has a unique endemic equilibrium if* $\mathcal{R}_0 > 1$ iff $\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03} > 1$ *and either the following holds*
 (i) $\mathcal{A}_2 > 0, \mathcal{A}_3 > 0$;
 (ii) $\mathcal{A}_2 < 0, \mathcal{A}_3 < 0$;
 (iii) $\mathcal{A}_2 > 0, \mathcal{A}_3 < 0$;
(b) *could have more than one endemic equilibrium if* $\mathcal{R}_0 > 1$ iff $\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03} > 1$ *and this holds*
 (i) $\mathcal{A}_2 < 0, \mathcal{A}_3 > 0$,
(c) *could have 2 endemic equilibrium if* $\mathcal{R}_0 < 1$ iff $\mathcal{R}_0^{01}+\mathcal{R}_0^{02}\mathcal{R}_0^{03} < 1$ *and all of* $\mathcal{A}_2$, *and* $\mathcal{A}_3$ *are negative.*

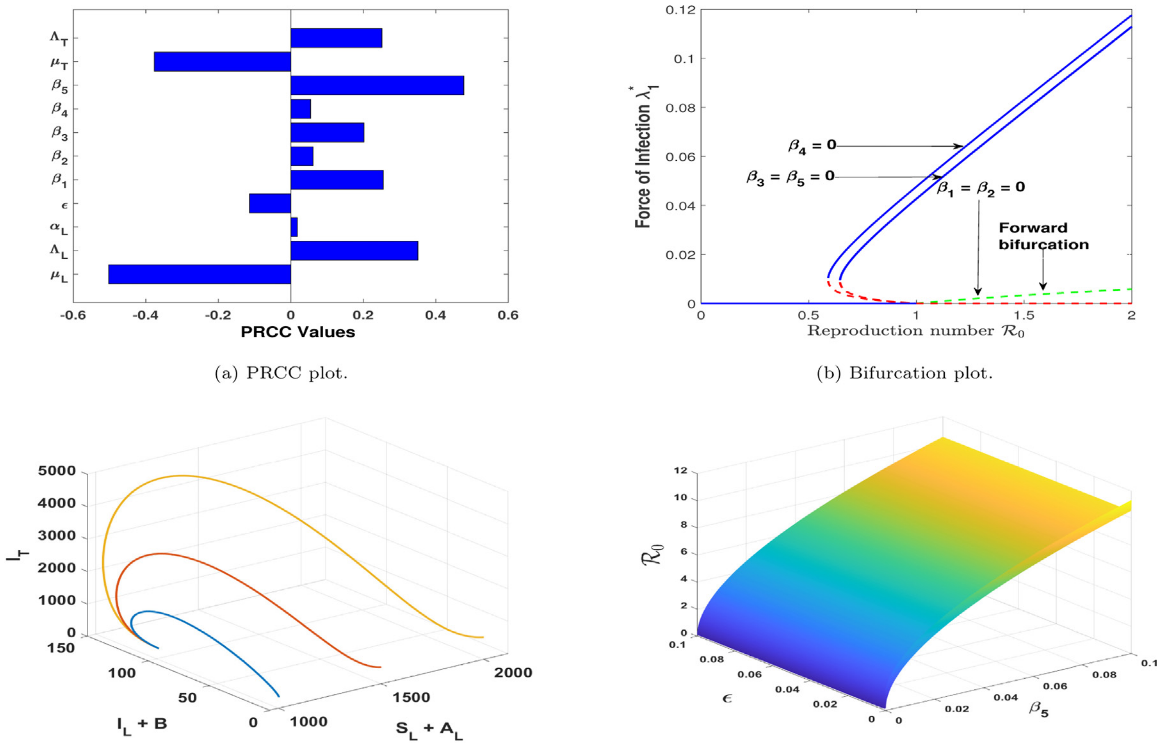
Item (c) of Theorem 3.2 indicates the existence of multiple endemic equilibrium when $\mathcal{R}_0 < 1$ (this normally suggests the phenomenon of backward bifurcation, see Fig. 4(a)). When the associated reproduction number of the model is less than unity, the phenomenon of backward bifurcation, which is characterized by the coexistence of a stable disease-free equilibrium (DFE) and a stable endemic equilibrium, has been observed in many disease transmission models (see for example [54–61]). Now focusing on three case scenarios of Eq. (29), we draw the following insights.

Case 1:

When $\beta_1 = \beta_2 = 0$, we have a unique endemic equilibrium, thus

$$\lambda_1^* = \frac{\varepsilon\mu_T^2 k_0\mu_L^2(\mathcal{R}^{0*}-1)}{\beta_3\Lambda_L(\mu_L+\alpha_L)+\mu_T k_0\mu_L}, \tag{30}$$

where $\mathcal{R}^{0*} = \frac{\beta_3\beta_5\Lambda_T\Lambda_L(\mu_L+\alpha_L)}{\varepsilon\mu_T^2 k_0\mu_L^0}$, which support the fact that, when $\beta_1 = \beta_2 = 0$, the usual threshold for the basic reproduction number holds, thus the disease can be eradicated when the basic reproduction number, $\mathcal{R}^{0*}$ is less than unity (no backward bifurcation), see Fig. 5(b).



(a) PRCC plot.

(b) Bifurcation plot.

(c) 3D bifurcation plot.

(d) 3D plot for ε, β_5, and $\mathcal{R}_0$.

Fig. 5. Numerical simulation of the non-fractional model.

Case 2:

When $\beta_4 = 0$, Eq. (29) becomes

$$\mathcal{A}_1^*{\lambda_1^*}^3 + \mathcal{A}_2^*{\lambda_1^*}^2 + \mathcal{A}_3^*\lambda_1^* + \mathcal{A}_4^* = 0, \tag{31}$$

where

$$\mathcal{A}_1^* = 2\Lambda_L\beta_3{\mu_L}^2{\mu_T}^3\,\varepsilon + {\mu_L}^3\,{\mu_T}^4\,\varepsilon + \beta_3^2\Lambda_L^2\,\mu_T^2\mu_L\varepsilon,$$

$$\mathcal{A}_2^* = \beta_3\Lambda_L\mu_T^3\,{\mu_L}^3\varepsilon\big(4 - \big[2\mathcal{R}_0^{01} + \mathcal{R}_{01}\big]\big) + \mu_T^4\mu_L^4\varepsilon\big(2 - \big[\mathcal{R}_0^{01} + \mathcal{R}_0^{02*}\mathcal{R}_0^{03}\big]\big) + \beta_3^2\Lambda_L^2\mu_T^2\mu_L^2\varepsilon\big(1 - \big[\mathcal{R}_0^{01} + \mathcal{R}_0^{02*}\mathcal{R}_0^{03}\big]\big)$$

$$\mathcal{A}_3^* = \mu_T^4\mu_L^5\varepsilon\big(1 - 2\big[\mathcal{R}_0^{01} + \mathcal{R}_0^{02*}\mathcal{R}_0^{03}\big]\big) + \beta_3\Lambda_L\mu_L^4\mu_L^3\varepsilon\big(1 - \mathcal{R}_0^{01}\big) + 2\,\mu_L\mu_T^2\,{\mu_L}^4\,{\mu_T}^2\,\varepsilon + \beta_3\Lambda_L\mu_L^4\mu_T^3\varepsilon\big(1 - \big[\mathcal{R}_0^{01} + \mathcal{R}_0^{02*}\mathcal{R}_0^{03}\big]\big)$$

$$\mathcal{A}_4^* = \varepsilon{\mu_L}^6{\mu_T}^4\big(1 - \big[\mathcal{R}_0^{01} + \mathcal{R}_0^{02*}\mathcal{R}_0^{03}\big]\big),$$

with $\mathcal{R}_0^{01} = \frac{\beta_1\beta_2\Lambda_L}{\varepsilon\mu_L^2}$, $\mathcal{R}_0^{02*} = \frac{\Lambda_L\beta_5\varepsilon}{\mu_T\mu_L\varepsilon}$, $\mathcal{R}_0^{03} = \frac{\Lambda_T\beta_3}{\mu_T\mu_L}$, $\mathcal{R}_{01} = \frac{\beta_2\Lambda_L^2\beta_3\beta_1}{\mu_L^3\varepsilon}$. Thus, using the same assertion in Theorem 3.2, **case 2** indicates the existence of multiple endemic equilibrium when $\mathcal{R}_0 < 1$, iff $\mathcal{R}_0^{01} + \mathcal{R}_0^{02*}\mathcal{R}_0^{03} < 1$, see Fig. 5(b).

Case 3:

When $\beta_3 = \beta_5 = 0$, Eq. (29) becomes

$$\mathcal{A}_1^{**}{\lambda_1^*}^3 + \mathcal{A}_2^{**}{\lambda_1^*}^2 + \mathcal{A}_3^{**}\lambda_1^* + \mathcal{A}_4^{**} = 0, \tag{32}$$

where

$$\mathcal{A}_1^{**} = {\mu_L}^3\,{\mu_T}^4\,\varepsilon,\quad \mathcal{A}_2^{**} = \mu_T^4\mu_L^4\varepsilon\big(2 - \mathcal{R}_0^{01}\big),$$

$$\mathcal{A}_3^{**} = \mu_T^4\mu_L^5\varepsilon\big(3 - 2\mathcal{R}_0^{01}\big)$$

$$\mathcal{A}_4^{**} = \varepsilon{\mu_L}^6{\mu_T}^4\big(1 - \mathcal{R}_0^{01}\big),$$

with $\mathcal{R}_0^{01} = \frac{\beta_1\beta_2\Lambda_L}{\varepsilon\mu_L^2}$.

Thus, using the same assertion in Theorem 3.2, **case 3** also indicates the existence of multiple endemic equilibrium when $\mathcal{R}_0 < 1$, iff $\mathcal{R}_0^{01} < 1$, see Fig. 5(b). In what follows, we prove the existence of backward bifurcation using the center manifold theory [62].

Theorem 3.3. *The Q fever model (4) undergoes a backward bifurcation at $\mathcal{R}_0 = 1$.*

Proof. From model (4) we can define the following variables $y_1^* = S_L, y_2^* = A_L, y_3^* = I_L, y_4^* = B, y_5^* = S_T$, and $y_6^* = I_T$. Therefore, $N_L(t) = y_1^* + y_2^* + y_3^*$,$B(t) = y_4^*$,$N_T(t) = y_5^* + y_6^*$. Now, using the vector notation $y^* = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*, y_6^*)^{T^*}$, then model (4) can be written as $\frac{dy^*}{dt} := F =$

$(f_1, f_2, f_3, f_4, f_5, f_6)^{T^*}$ as iterated as follows:

$$\begin{cases} \frac{dy_1^*}{dt} := f_1(y^*) = \Lambda_L - \left(\beta_1 y_4^* + \beta_5 y_6^*\right) y_1^* - \mu_L y_1^*, \\ \frac{dy_2^*}{dt} := f_2(y^*) = \left(\beta_1 y_4^* + \beta_5 y_6^*\right) y_1^* - k_0 y_2^*, \\ \frac{dy_3^*}{dt} := f_3(y^*) = \alpha_L y_2^* - \mu_L y_3^*, \\ \frac{dy_4^*}{dt} := f_4(y^*) = \beta_2 (y_2^* + y_3^*) + \beta_4 y_6^* - \varepsilon y_4^*, \\ \frac{dy_5^*}{dt} := f_5(y^*) = \Lambda_T - \beta_3 y_5^* (y_2^* + y_3^*) - \mu_T y_5^*, \\ \frac{dy_6^*}{dt} := f_6(y^*) = \beta_3 y_5^* (y_2^* + y_3^*) - \mu_T y_6^*, \end{cases} \tag{33}$$

as defined before, $k_0 = (\mu_L + \alpha_L)$. The Jacobian of system (33), at the disease-free equilibrium, is given as

$$J(\mathcal{E}_0) = \begin{pmatrix} -\mu_L & 0 & 0 & -\frac{\beta_1 \Lambda_L}{\mu_L} & 0 & -\frac{\beta_5 \Lambda_L}{\mu_L} \\ 0 & -k_0 & 0 & \frac{\beta_1 \Lambda_L}{\mu_L} & 0 & \frac{\beta_5 \Lambda_L}{\mu_L} \\ 0 & \alpha_L & -\mu_L & 0 & 0 & 0 \\ 0 & \beta_2 & \beta_2 & -\varepsilon & 0 & \beta_4 \\ 0 & -\frac{\beta_3 \Lambda_T}{\mu_T} & -\frac{\beta_3 \Lambda_T}{\mu_T} & 0 & -\mu_T & 0 \\ 0 & \frac{\beta_3 \Lambda_T}{\mu_T} & \frac{\beta_3 \Lambda_T}{\mu_T} & 0 & 0 & -\mu_T \end{pmatrix}$$

Now, consider the case when $\mathcal{R}_0 = 1$, we can choose β_3^* as the bifurcation parameter and solving for β_3 from the expression for $\mathcal{R}_0 = 1$, we have

$$\beta_3 = \beta_3^* = \frac{\mu_T^2 \mu_L^2 \varepsilon}{\Lambda_L \Lambda_T (\beta_1 \beta_4 + \beta_5 \varepsilon)} \left[2 - \left(\frac{\beta_1^2 \beta_2^2 \Lambda_L}{\varepsilon^2 \mu_L^4} + \frac{\Lambda_L^2 (\beta_1 \beta_4 + \beta_5 \varepsilon)^2}{\mu_T^2 \mu_L^2 \varepsilon^2} \right) \right]. \tag{34}$$

It can be proven that the converted system (33), having $\beta_3 = \beta_3^*$, does have a nonhyperbolic equilibrium point with a simple eigenvalue with zero real part and all other eigenvalues with negative real parts in the linearized system [54]. As a result, the dynamics of the model (33) near $\beta_3 = \beta_3^*$ can be studied using center manifold theory [62].

Eigenvectors of $J(\mathcal{E}_0)|_{\beta_3=\beta_3^*}$

The right eigenvector in relation to the zero eigenvalue evaluated at $J(\mathcal{E}_0)|_{\beta_3=\beta_3^*}$ is denoted by

$\mathbf{W}^* = [w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^*]$,

where

$$\begin{cases} w_1^* = -\frac{\beta_1 \Lambda_L (w_4^* + w_6^*)}{\mu_L^2}, \\ w_2^* = w_2^* > 0, \\ w_3^* = \frac{\alpha_L}{\mu_L} w_2^*, \\ w_4^* = \frac{(\mu_L + \alpha_L)(\beta_2 \mu_T^2 + \beta_4 \beta_3^*)}{\mu_T^2 \mu_L} w_2^*, \\ w_5^* = -w_6^*, \\ w_6^* = \frac{\beta_3^* \Lambda_T (\mu_L + \alpha_L)}{\mu_T^2 \mu_L} w_2^*. \end{cases} \tag{35}$$

In addition, the left eigenvector in relation to the zero eigenvalue evaluated at $J(\mathcal{E}_0)|_{\beta_3=\beta_3^*}$ satisfying the property $\mathbf{W}^*.\mathbf{V}^* = 1$, is denoted by

$\mathbf{V}^* = [v_1^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*]$,

where

$$\begin{cases} v_1^* = 0, \\ v_2^* = v_2^* > 0, \\ v_3^* = \frac{\beta_2 \beta_1 \Lambda_L \mu_T^2 + \Lambda_T \Lambda_L \beta_3^* (\beta_1 + \beta_5 \varepsilon)}{\mu_T^2 \mu_L \varepsilon} v_2^*, \\ v_4^* = \frac{\beta_1 \Lambda_L}{\mu_L \varepsilon} v_2^*, \\ v_5^* = 0, \\ v_6^* = \frac{\Lambda_L (\beta_4 \beta_1 + \beta_5 \varepsilon)}{\mu_L \mu_T \varepsilon} v_2^*. \end{cases} \tag{36}$$

Computations of bifurcation coefficients a and b

Utilizing Castillo-Chavez and Song's notation, we have

$$\mathbf{a} = \sum_{k,i,j=1}^{6} v_k^* w_i^* w_j^* \frac{\partial^2 f_k}{\partial y_i^* \partial y_j^*}(\mathcal{E}_0), \tag{37}$$

$$\mathbf{b} = \sum_{k,i=1}^{6} v_k^* w_i^* \frac{\partial^2 f_k}{\partial y_i^* \partial \beta_3^*}(\mathcal{E}_0). \tag{38}$$

Thus, from Eqs. (37) and (38), we have the following:

$$\frac{\partial^2 f_2}{\partial y_1^* \partial y_4^*} = \beta_1, \ \frac{\partial^2 f_2}{\partial y_1^* \partial y_6^*} = \beta_5, \ \frac{\partial^2 f_6}{\partial y_2^* \partial y_5^*} = \beta_3^*, \ \frac{\partial^2 f_6}{\partial y_3^* \partial y_5^*} = \beta_3^*,$$
$$\frac{\partial^2 f_6}{\partial y_2^* \partial \beta_3^*} = \frac{\Lambda_T}{\mu_T}, \ \frac{\partial^2 f_6}{\partial y_3^* \partial \beta_3^*} = \frac{\Lambda_T}{\mu_T}.$$

Hence, it follows that

$$\mathbf{a} = v_2^* w_1^* w_5^* \beta_1 + v_2^* w_1^* w_6^* \beta_5 + v_6^* w_2^* w_5^* \beta_3^* + v_6^* w_3^* w_5^* \beta_3^* > 0, \tag{39}$$

$$\mathbf{b} = \frac{\Lambda_T}{\mu_T} v_6^* (w_2 + w_3) > 0. \tag{40}$$

Therefore, from Theorem 4.1 of Castillo-Chavez and Song [62], it indicates that the proposed model exhibits backward bifurcation at $\mathcal{R}_0 = 1$. □

Figure 4 (a) and (c) show the existence of backward bifurcation when the recruitment rate of susceptible ticks is varied or kept constant. This indicates that an increase in ticks in an environment where there is *coxiella burnetii* make the control of Q fever virtually impossible. Thus, the achievement of the usual threshold of $\mathcal{R}_0$ less than 1 is no longer feasible in the eradication of the disease. Figure 4(b) and (d) show that an increase in the natural death rate of the bacterial in the environment reduces the existence of backward bifurcation. Therefore to have effective control of the disease in an ecosystem, it is advised that the use of other methods in the decay rate of the *coxiella burnetii* in the environment can be employed to safeguard the livestock population and reproduction. Figure 5(b) show that making β_3 (transmission rate from $(A_L + I_L)$ to ticks), β_4 (shedding rate from infected ticks) and β_5 (transmission rate from I_T to S_L) zero do not remove the existence of backward bifurcation from the dynamics of Q fever between livestock and ticks. Nevertheless, when one focus on β_1 (environmental transmission) and β_2 (shedding rate from infected livestock) reduces the existence of backward bifurcation leading to a possible control of the disease in a tick induced farm. The partial rank correction coefficients (PRCC) values of the ten parameters are given in Fig. 5(a) to determine the dependency of parameters in $\mathcal{R}_0^{NG}$ or $\mathcal{R}_0$ using a sampling size of 2000. The longer the bar in Fig. 5(a) indicates that those parameters have a substantial statistical effect on changes in $\mathcal{R}_0^{NG}$ or $\mathcal{R}_0$. Figure 5(c) shows the 3D bifurcation for the proposed model; it can easily be seen that a lower number of infected livestock and bacterial load in the environment produces less infection in the infected ticks population. Figure 5(d) shows that an increase in β_5 (transmission rate from I_T to S_L) and a reduction in ε (natural decay of *Coxiella burnetii* from the environment) result in an increase in $\mathcal{R}_0^{NG}$ or $\mathcal{R}_0$. To take care of the memory aspect of ticks on their host, we introduce a Caputo, Caputo-Fabrizio, Atangana-Baleanu fractional differential operator for the proposed model in the next Section.

4. Basic definitions and preliminaries

In this section, we focus on some important definitions from factional derivatives in relation to the three fractional operators to be used for the Q fever model.

Definition 4.1. [38] Given a function $g : \mathbb{R}^+ \to \mathbb{R}$, then the fractional integral of order φ in Caputo sense is given as

$${}_0^C I_t^{\varphi}(g(t)) = \frac{1}{\Gamma(\varphi)} \int_0^t (t - \mathfrak{f})^{\varphi - 1} g(\mathfrak{f}) d\mathfrak{f}, \tag{41}$$

where Γ denote the Gamma function and φ denote the fractional order.

Definition 4.2. [38] Given a function $g \in C^n$, then the Caputo derivative with order φ is given as

$${}_0^C D_t^{\varphi}(g(t)) = I^{n-\varphi} D^n g(t) = \frac{1}{\Gamma(n - \varphi)} \int_0^t \frac{g^n(\mathfrak{f})}{(t - \mathfrak{f})^{\varphi - n + 1}} d\mathfrak{f}, \tag{42}$$

which is defined for the continuous functions in absolute terms and $n - 1 < \varphi < n \in N$. Clearly, ${}_0^C D_t^{\varphi}(g(t))$ approaches $g'(t)$ as φ approaches 1.

Definition 4.3. [36] Let $h \in H^1(a, b), b > a$ be a function. Then the Caputo-Fabrizio derivative of order $\varphi \in (0, 1)$ of a function h is defined as

$${}_a^{CF} D_t^{\varphi}[h(t)] = \frac{P(\varphi)}{1 - \varphi} \int_a^t h'(x) exp\Big[-\varphi \frac{t - x}{1 - \varphi} \Big] dx, \tag{43}$$

where $P(\varphi)$ is a normalization function with $P(0) = P(1) = 1$.

Definition 4.4. [63] The Caputo-Fabrizio fractional integral of order $\varphi \in (0,1)$ of a function h is defined as

$$I_t^{\varphi}[h(t)] = \frac{2(1-\varphi)}{2(1-\varphi)P(\varphi)}h(t) + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t h(\psi)d\psi, t \geq 0. \tag{44}$$

Definition 4.5. [37] Let $g \in H^1(a,b), a < b$ be a function and $\varphi \in (0,1)$. The Atangana-Baleanu derivative in Caputo sense of order φ of g is given as

$${}_a^{ABC}D_t^{\varphi}[g(t)] = \frac{G(\varphi)}{1-\varphi}\int_a^t g'(x)E_{\varphi}\Big[-\varphi\frac{(t-x)^{\varphi}}{1-\varphi}\Big]dx, \tag{45}$$

where $G(\varphi)$ is a normalization function with $G(0) = G(1) = 1$ and E_{φ} is the Mittag-Leffler function.

Definition 4.6. [37] Let $g \in H^1(a,b), a < b$ be a function and $\varphi \in (0,1)$. The Atangana-Baleanu derivative in Riemann-Liouville sense of order φ of g is given as

$${}_a^{ABR}D_t^{\varphi}[g(t)] = \frac{G(\varphi)}{1-\varphi}\frac{d}{dt}\int_a^t g(x)E_{\varphi}\Big[-\varphi\frac{(t-x)^{\varphi}}{1-\varphi}\Big]dx. \tag{46}$$

Definition 4.7. [37] The Atangana-Baleanu fractional integral is defined by:

$${}_a^{ABC}I_t^{\varphi}[g(t)] = \frac{1-\varphi}{G(\varphi)}g(t) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)}\int_a^t g(\psi)(t-\psi)^{\varphi-1}d\psi. \tag{47}$$

Theorem 4.1. *[37] For a continuous function g on $[a,b]$. The inequality given below holds on $[a,b]$:*

$$\left\|{}_0^{ABR}D_t^{\varphi}[g(t)]\right\| < \frac{G(\varphi)}{1-\varphi}\|g(t)\|, \tag{48}$$

where $\|g(t)\| = \max_{a\leq t\leq b}|g(t)|$.

Theorem 4.2. *[37] The Atangana–Baleanu derivative in Caputo and RL type satisfy Lipschitz condition:*

$\left\|{}_0^{ABC}D_t^{\varphi}[g(t)] - {}_0^{ABC}D_t^{\varphi}[h(t)]\right\| \leq \mathbf{H}\|g(t) - h(t)\|$, and

$\left\|{}_0^{ABR}D_t^{\varphi}[g(t)] - {}_0^{ABR}D_t^{\varphi}[h(t)]\right\| \leq \mathbf{H}\|g(t) - h(t)\|$.

5. Fractional models for Q fever in livestock together with the dynamism of ticks

Now, we introduce the fractional nature of our proposed model (4) in conformity of the fractional definitions above. For each of the fractional model we provide the existence and uniqueness of solutions.

5.1. Q fever model in Caputo operator

Based on standard literature on fractional order differential equation modeling, the Q fever model in (4) can be expressed in Caputo sense as

$$\begin{cases} {}_0^CD_t^{\varphi}S_L = \Lambda_L - (\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - \mu_L S_L(t), \\ {}_0^CD_t^{\varphi}A_L = (\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - (\mu_L + \alpha_L)A_L(t), \\ {}_0^CD_t^{\varphi}I_L = \alpha_L A_L(t) - \mu_L I_L(t), \\ {}_0^CD_t^{\varphi}B = \beta_2(A_L(t) + I_L(t)) + \beta_4 I_T(t) - \varepsilon B(t), \\ {}_0^CD_t^{\varphi}S_T = \Lambda_T - \beta_3 S_T(t)(A_L(t) + I_L(t)) - \mu_T S_T(t), \\ {}_0^CD_t^{\varphi}I_T = \beta_3 S_T(t)(A_L(t) + I_L(t)) - \mu_T I_T(t), \end{cases} \tag{49}$$

with initial conditions

$S_L(0) = S_{L_0} > 0, A_L(0) = A_{L_0} \geq 0,\ I_L(0) = I_{L_0} \geq 0,$

$B(0) = B_0 \geq 0, S_T(0) = S_{T_0} \geq 0,\ I_T(0) = I_{T_0} \geq 0.$

5.2. Existence and uniqueness analysis for Caputo fractional Q fever model

We shall prove the existence and uniqueness of our proposed model in Caputo operator using fixed-point theory. Assuming a Banach space $\mathcal{B}(\mathcal{C})$ for an interval-defined continuous real-valued function $\mathcal{C} = [0,m]$ with the sub norm and $\mathcal{L} = \mathcal{B}(\mathcal{C}) \times \mathcal{B}(\mathcal{C}) \times \mathcal{B}(\mathcal{C}) \times \mathcal{B}(\mathcal{C}) \times \mathcal{B}(\mathcal{C}) \times \mathcal{B}(\mathcal{C})$ with the norm

$\| (S_L, A_L, I_L, B, S_T, I_T) \| = \| S_L \| + \| A_L \| + \| I_L \| + \| B \| + \| S_T \| + \| I_T \|,$

$\| S_L \| = \sup_{t\in\mathcal{C}}|S_L|$, $\| A_L \| = \sup_{t\in\mathcal{C}}|A_L|$, $\| I_L \| = \sup_{t\in\mathcal{C}}|I_L|$, $\| B \| = \sup_{t\in\mathcal{C}}|B|$, $\| S_T \| = \sup_{t\in\mathcal{C}}|S_T|$ $\| I_T \| = \sup_{t\in\mathcal{C}}|I_T|$. Now, applying the Caputo fractional integral operator as defined in (41) on (49) and the fundamental theorem of calculus we have

$S_L(t) - S_L(0) = {}_0^CI_t^{\varphi}[\Lambda_L - (\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - \mu_L S_L(t)],$

$A_L(t) - A_L(0) = {}_0^CI_t^{\varphi}[(\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - k_0 A_L(t)],$

$$
\begin{aligned}
I_L(t)-I_L(0) &= {}_0^C I_t^{\varphi}[\alpha_L A_L(t)-\mu_L I_L(t)],\\
B(t)-B(0) &= {}_0^C I_t^{\varphi}[\beta_2(A_L(t)+I_L(t))+\beta_4 I_T(t)-\varepsilon B(t)],\\
S_T(t)-S_T(0) &= {}_0^C I_t^{\varphi}[\Lambda_T-\beta_3 S_T(t)(A_L(t)+I_L(t))-\mu_T S_T(t)],\\
I_T(t)-I_T(0) &= {}_0^C I_t^{\varphi}[\beta_3 S_T(t)(A_L(t)+I_L(t))-\mu_T I_T(t)].
\end{aligned}
\tag{50}
$$

Supposing that

$$
\begin{aligned}
\mathbf{M}_1 &= \Lambda_L-(\beta_1 B(t)+\beta_5 I_T(t))S_L(t)-\mu_L S_L(t),\\
\mathbf{M}_2 &= (\beta_1 B(t)+\beta_5 I_T(t))S_L(t)-k_0 A_L(t),\\
\mathbf{M}_3 &= \alpha_L A_L(t)-\mu_L I_L(t),\\
\mathbf{M}_4 &= \beta_2(A_L(t)+I_L(t))+\beta_4 I_T(t)-\varepsilon B(t),\\
\mathbf{M}_5 &= \Lambda_T-\beta_3 S_T(t)(A_L(t)+I_L(t))-\mu_T S_T(t),\\
\mathbf{M}_6 &= \beta_3 S_T(t)(A_L(t)+I_L(t))-\mu_T I_T(t).
\end{aligned}
\tag{51}
$$

Now (50) can further be written in Caputo integral fashion as

$$
\begin{aligned}
S_L(t)-S_L(0) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_1(\varphi,\mathfrak{f},S_L(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
A_L(t)-A_L(0) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_2(\varphi,\mathfrak{f},A_L(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
I_L(t)-I_L(0) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_3(\varphi,\mathfrak{f},I_L(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
B(t)-B(0) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_4(\varphi,\mathfrak{f},B(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
S_T(t)-S_T(0) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_5(\varphi,\mathfrak{f},S_T(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
I_T(t)-I_T(t) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_6(\varphi,\mathfrak{f},I_T(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},
\end{aligned}
\tag{52}
$$

We note that $\mathbf{M}_1(S_L,\mathfrak{f}),\mathbf{M}_2(A_L,\mathfrak{f}),\mathbf{M}_3(I_L,\mathfrak{f}),\mathbf{M}_4(B,\mathfrak{f}),\mathbf{M}_5(S_T,\mathfrak{f}),\mathbf{M}_6(I_T,\mathfrak{f})$ satisfies the Lipschitz condition if and only if $S_L(t),A_L(t),I_L(t),B(t),S_T(t)$, and $I_T(t)$, are bounded above. Now, let $S_L(t)$ and S_L^* be two functions, so that we have

$$
\| \mathbf{M}_1(\varphi,t,S_L(t))-\mathbf{M}_1(\varphi,t,S_L^*(t)) \| = ||(\beta_1 B(t)+\beta_5 I_T(t)+\mu_L)(S_L(t)-S_L^*(t))||. \tag{53}
$$

Now, let assume that $\|B(t)\|\le\theta_1^*$, and $\|I_T(t)\|\le\theta_2^*$.

Therefore, we let $\hat{a}_1 := \beta_1\theta_1^*+\beta_5\theta_2^*+\mu_L$ then we have

$$
\| \mathbf{M}_1(\varphi,t,S_L(t))-\mathbf{M}_1(\varphi,t,S_L^*(t)) \| \le \hat{a}_1||(S_L(t)-S_L^*(t))||, \tag{54}
$$

in similar manner, we obtain

$$
\begin{aligned}
\| \mathbf{M}_2(\varphi,t,A_L(t))-\mathbf{M}_2(\varphi,t,A_L^*(t)) \| &\le \hat{a}_2||(A_L(t)-A_L^*(t))||,\\
\| \mathbf{M}_3(\varphi,t,I_L(t))-\mathbf{M}_3(\varphi,t,I_L^*(t)) \| &\le \hat{a}_3||(I_L(t)-I_L^*(t))||,\\
\| \mathbf{M}_4(\varphi,t,B(t))-\mathbf{M}_4(\varphi,t,B^*(t)) \| &\le \hat{a}_4||(B(t)-B^*(t))||,\\
\| \mathbf{M}_5(\varphi,t,S_T(t))-\mathbf{M}_5(\varphi,t,S_T^*(t)) \| &\le \hat{a}_5||(S_T(t)-S_T^*(t))||,\\
\| \mathbf{M}_6(\varphi,t,I_T(t))-\mathbf{M}_6(\varphi,t,I_T^*(t)) \| &\le \hat{a}_6||(I_T(t)-I_T^*(t))||,
\end{aligned}
\tag{55}
$$

where $\hat{a}_2=\mu_L+\alpha_L, \hat{a}_3=\mu_L, \hat{a}_4=\varepsilon, \hat{a}_5=\mu_T, \hat{a}_6=\mu_T$, hence, it indicates that the Lipschitz condition is fulfilled for $\mathbf{M}_i, i=1,2,3.4.5,6$.

Recursively, Eq. (52) can be written as

$$
\begin{aligned}
S_{L_n}(t) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_1(\varphi,\mathfrak{f},S_{L_{n-1}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
A_{L_n}(t) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_2(\varphi,\mathfrak{f},A_{L_{n-1}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
I_{L_n}(t) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_3(\varphi,\mathfrak{f},I_{L_{n-1}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
B_n(t) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_4(\varphi,\mathfrak{f},B_{n-1}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},\\
S_{T_n}(t) &= \frac{1}{\Gamma(\varphi)}\int_0^t \frac{\mathbf{M}_5(\varphi,\mathfrak{f},S_{T_{n-1}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}}d\mathfrak{f},
\end{aligned}
\tag{56}
$$

$$I_{T_n}(t) = \frac{1}{\Gamma(\varphi)} \int_0^t \frac{\mathbf{M}_6(\varphi, \mathfrak{f}, I_{T_{n-1}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},$$

linked with the following initial conditions

$$S_{L_0}(t) = S_L(0), A_{L_0}(t) = A_L(0), I_{L_0}(t) = I_L(0), B_0(t) = B(0), S_{T_0}(t) = S_T(0).$$

Subtracting regular succession terms, we obtain

$$\begin{aligned}
\aleph_{S_L,n}(t) = S_{L_n}(t) - S_{L_{n-1}}(t) &= \frac{1}{\Gamma(\varphi)} \int_0^t \frac{\mathbf{M}_1(\varphi, \mathfrak{f}, S_{L_{n-1}}(\mathfrak{f})) - \mathbf{M}_1(\varphi, \mathfrak{f}, S_{L_{n-2}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
\aleph_{A_L,n}(t) = A_{L_n}(t) - A_{L_{n-1}}(t) &= \frac{1}{\Gamma(\varphi)} \int_0^t \frac{\mathbf{M}_2(\varphi, \mathfrak{f}, A_{L_{n-1}}(\mathfrak{f})) - \mathbf{M}_2(\varphi, \mathfrak{f}, A_{L_{n-2}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
\aleph_{I_L,n}(t) = I_{L_n}(t) - I_{L_{n-1}}(t) &= \frac{1}{\Gamma(\varphi)} \int_0^t \frac{\mathbf{M}_3(\varphi, \mathfrak{f}, I_{L_{n-1}}(\mathfrak{f})) - \mathbf{M}_3(\varphi, \mathfrak{f}, I_{L_{n-2}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
\aleph_{B,n}(t) = B_n(t) - B_{n-1}(t) &= \frac{1}{\Gamma(\varphi)} \int_0^t \frac{\mathbf{M}_4(\varphi, \mathfrak{f}, B_{n-1}(\mathfrak{f})) - \mathbf{M}_4(\varphi, \mathfrak{f}, B_{n-2}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
\aleph_{S_T,n}(t) = S_{T_n}(t) - S_{T_{n-1}}(t) &= \frac{1}{\Gamma(\varphi)} \int_0^t \frac{\mathbf{M}_5(\varphi, \mathfrak{f}, S_{T_{n-1}}(\mathfrak{f})) - \mathbf{M}_5(\varphi, \mathfrak{f}, S_{T_{n-2}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
\aleph_{I_T,n}(t) = I_{T_n}(t) - I_{T_{n-1}}(t) &= \frac{1}{\Gamma(\varphi)} \int_0^t \frac{\mathbf{M}_6(\varphi, \mathfrak{f}, I_{T_{n-1}}(\mathfrak{f})) - \mathbf{M}_6(\varphi, \mathfrak{f}, I_{T_{n-2}}(\mathfrak{f}))}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f}.
\end{aligned} \tag{57}$$

In the view of the above calculations, it is clear that

$$\begin{aligned}
&S_{Ln}(t) = \sum_{j=0}^{n} \aleph_{S_{L,n}}(t),\ A_{Ln}(t) = \sum_{j=0}^{n} \aleph_{A_{L,n}}(t),\\
&I_{Ln}(t) = \sum_{j=0}^{n} \aleph_{I_{L,n}}(t),\ B_n(t) = \sum_{j=0}^{n} \aleph_{B,n}(t),\\
&S_{Tn}(t) = \sum_{j=0}^{n} \aleph_{S_{T,n}}(t),\ I_{Tn}(t) = \sum_{j=0}^{n} \aleph_{I_{T,n}}(t).
\end{aligned} \tag{58}$$

Additionally, utilizing Eqs. (54) and (55) with the notion that

$$\aleph_{S_L,n-1}(t) = S_{L_{n-1}}(t) - S_{L_{n-2}}(t), \aleph_{A_L,n-1}(t) = A_{L_{n-1}}(t) - A_{L_{n-2}}(t),$$
$$\aleph_{I_L,n-1}(t) = I_{L_{n-1}}(t) - I_{L_{n-2}}(t), \aleph_{B,n-1}(t) = B_{n-1}(t) - B_{n-2}(t),$$
$$\aleph_{S_T,n-1}(t) = S_{T_{n-1}}(t) - S_{T_{n-2}}(t), \aleph_{I_T,n-1}(t) = I_{T_{n-1}}(t) - I_{T_{n-2}}(t),$$

we can arrive at

$$\begin{aligned}
||\aleph_{S_L,n}(t)|| &= \frac{1}{\Gamma(\varphi)} \hat{a}_1 \int_0^t \frac{||\aleph_{S_L,n-1}(\mathfrak{f})||}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
||\aleph_{A_L,n}(t)|| &= \frac{1}{\Gamma(\varphi)} \hat{a}_2 \int_0^t \frac{||\aleph_{A_L,n-1}(\mathfrak{f})||}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
||\aleph_{I_L,n}(t)|| &= \frac{1}{\Gamma(\varphi)} \hat{a}_3 \int_0^t \frac{||\aleph_{I_L,n-1}(\mathfrak{f})||}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
||\aleph_{B,n}(t)|| &= \frac{1}{\Gamma(\varphi)} \hat{a}_4 \int_0^t \frac{||\aleph_{B,n-1}(\mathfrak{f})||}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
||\aleph_{S_T,n}(t)|| &= \frac{1}{\Gamma(\varphi)} \hat{a}_5 \int_0^t \frac{||\aleph_{S_T,n-1}(\mathfrak{f})||}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f},\\
||\aleph_{I_T,n}(t)|| &= \frac{1}{\Gamma(\varphi)} \hat{a}_6 \int_0^t \frac{||\aleph_{I_T,n-1}(\mathfrak{f})||}{(t-\mathfrak{f})^{1-\varphi}} d\mathfrak{f}.
\end{aligned} \tag{59}$$

Next, we state and prove Theorem 5.1, so to complete the existence and uniqueness of the Q fever model in Caputo sense.

Theorem 5.1. *The Caputo fractional Q fever model (49) has a unique solution under the condition that*

$$\frac{\nu^{\varphi}}{\Gamma(\varphi)} \hat{a}_i < 1, \quad i = 1, 2, 3, 4, 5, 6. \tag{60}$$

when $t \in [0, \nu]$.

Proof. As it has been established above that, $S_L(t), A_L(t), I_L(t), B(t), S_T(t), I_T(t)$ are bounded and $\mathbf{M}_i, i = 1, 2, 3, 4, 5, 6$ satisfy the Lipschitz condition. Thus, through the recursive principle and Eq. (59), we get

$$\begin{aligned}
||\aleph_{S_L,n}(t)|| &\le ||S_{L0}(t)||\left(\frac{\upsilon^{\varphi}}{\Gamma(\varphi)}\hat{a}_1\right)^n,\\
||\aleph_{A_L,n}(t)|| &\le ||A_{L0}(t)||\left(\frac{\upsilon^{\varphi}}{\Gamma(\varphi)}\hat{a}_2\right)^n,\\
||\aleph_{I_L,n}(t)|| &\le ||I_{L0}(t)||\left(\frac{\upsilon^{\varphi}}{\Gamma(\varphi)}\hat{a}_3\right)^n,\\
||\aleph_{B,n}(t)|| &\le ||B_0(t)||\left(\frac{\upsilon^{\varphi}}{\Gamma(\varphi)}\hat{a}_4\right)^n,\\
||\aleph_{S_T,n}(t)|| &\le ||S_{T0}(t)||\left(\frac{\upsilon^{\varphi}}{\Gamma(\varphi)}\hat{a}_5\right)^n,\\
||\aleph_{I_T,n}(t)|| &\le ||I_{T0}(t)||\left(\frac{\upsilon^{\varphi}}{\Gamma(\varphi)}\hat{a}_6\right)^n.
\end{aligned} \tag{61}$$

As n tends to ∞ taking the limit on both sides, we have

$$||\aleph_{S_L,n}(t)|| \to 0,\ ||\aleph_{A_L,n}(t)|| \to 0,\ ||\aleph_{I_L,n}(t)|| \to 0,$$
$$||\aleph_{B,n}(t)|| \to 0,\ ||\aleph_{S_T,n}(t)|| \to 0,\ ||\aleph_{I_T,n}(t)|| \to 0.$$

Also, using the triangle inequality together with Eq. (61) for any q, we finally attain

$$\begin{aligned}
||S_{L_{n+q}}(t) - S_{L_n}(t)|| &\le \sum_{j=n+1}^{n+q} \omega_1^j = \frac{\omega_1^{n+1} - \omega_1^{n+q+1}}{1-\omega_1},\\
||A_{L_{n+q}}(t) - A_{L_n}(t)|| &\le \sum_{j=n+1}^{n+q} \omega_2^j = \frac{\omega_2^{n+1} - \omega_2^{n+q+1}}{1-\omega_2},\\
||I_{L_{n+q}}(t) - I_{L_n}(t)|| &\le \sum_{j=n+1}^{n+q} \omega_3^j = \frac{\omega_3^{n+1} - \omega_3^{n+q+1}}{1-\omega_3},\\
||B_{n+q}(t) - B_n(t)|| &\le \sum_{j=n+1}^{n+q} \omega_4^j = \frac{\omega_4^{n+1} - \omega_4^{n+q+1}}{1-\omega_4},\\
||S_{T_{n+q}}(t) - S_{T_n}(t)|| &\le \sum_{j=n+1}^{n+q} \omega_5^j = \frac{\omega_5^{n+1} - \omega_5^{n+q+1}}{1-\omega_5},\\
||I_{T_{n+q}}(t) - I_{L_n}(t)|| &\le \sum_{j=n+1}^{n+q} \omega_6^j = \frac{\omega_6^{n+1} - \omega_6^{n+q+1}}{1-\omega_6}.
\end{aligned} \tag{62}$$

where $\omega_i = \frac{\upsilon^{\varphi}}{\Gamma(\varphi)}\hat{a}_i < 1, \quad i = 1, 2, 3, 4, 5, 6$. Consequently, $S_{Ln}, A_{Ln}, I_{Ln}, B_n, S_{Tn}$ and I_{Tn} are Cauchy sequences in the Banach space $B(\mathcal{C})$. Therefore, the state variables converges uniformly. Hence via the limit theorem, we say the limit of the sequence (56) is the unique solution of the Caputo model. The end of the proof. □

5.3. Q fever model in Caputo-Fabrizio (CF) operator

The Q fever model in (4) is expressed in Caputo-Fabrizio sense as

$$\begin{aligned}
{}^{CF}_{0}D^{\varphi}_t\big(S_L(t)\big) &= Z_1\big(t, S_L\big),\\
{}^{CF}_{0}D^{\varphi}_t\big(A_L(t)\big) &= Z_2\big(t, A_L\big),\\
{}^{CF}_{0}D^{\varphi}_t\big(I_L(t)\big) &= Z_3\big(t, I_L\big),\\
{}^{CF}_{0}D^{\varphi}_t\big(B(t)\big) &= Z_4\big(t, B\big),\\
{}^{CF}_{0}D^{\varphi}_t\big(S_T(t)\big) &= Z_5\big(t, S_T\big),\\
{}^{CF}_{0}D^{\varphi}_t\big(I_T(t)\big) &= Z_6\big(t, I_T\big),
\end{aligned} \tag{63}$$

where

$$Z_1\big(t, S_L\big) = \Lambda_L - (\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - \mu_L S_L(t),\ \ Z_2\big(t, A_L\big) = (\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - (\mu_L + \alpha_L)A_L(t),$$

$$Z_3\big(t, I_L\big) = \alpha_L A_L(t) - \mu_L I_L(t),\ \ Z_4\big(t, B\big) = \beta_2(A_L(t) + I_L(t)) + \beta_4 I_T(t) - \varepsilon B(t),$$
$$Z_5\big(t, S_L\big) = \Lambda_T - \beta_3 S_T(t)(A_L(t) + I_L(t)) - \mu_T S_T(t),\ \ Z_5\big(t, I_T\big) = \beta_3 S_T(t)(A_L(t) + I_L(t)) - \mu_T I_T(t),$$

5.4. Existence and uniqueness analysis for Caputo-Fabrizio fractional Q fever model

Applying CF fractional integral to both sides of Eq. (63) we obtain

$$\begin{aligned}
S_L(t) - S_L(0) &= \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_1(t, S_L) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_1(\nu, S_L)d\nu,\\
A_L(t) - A_L(0) &= \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_2(t, A_L) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_2(\nu, A_L)d\nu,\\
I_L(t) - I_L(0) &= \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_3(t, I_L) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_3(\nu, I_L)d\nu,\\
B(t) - B(0) &= \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_4(t, B) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_4(\nu, B)d\nu,\\
S_T(t) - S_T(0) &= \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_5(t, S_T) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_5(\nu, S_T)d\nu,\\
I_T(t) - I_T(0) &= \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_6(t, I_T) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_6(\nu, I_T)d\nu.
\end{aligned} \tag{64}$$

Theorem 5.2. *If the following inequality holds*

$0 \le \eta_i < 1, \quad i = 1,\ 2,\ 3,\ 4,\ 5,\ 6,$

then the kernels $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$ satisfies the Lipschitz condition and are contractions.

Proof. Let assume that $S_L,\ A_L,\ I_L,\ B,\ S_T$ and I_T are bounded functions such that $\|S_L(t)\| \le k_1,\ \|A_L(t)\| \le k_2,\ \|I_L(t)\| \le k_3,\ \|B(t)\| \le k_4,\ \|S_T(t)\| \le k_5$ and $\|I_T(t)\| \le k_6$.

Let further assume that S_L and S_{L1} are two functions, then we have

$$\begin{aligned}
\|Z_1(t, S_L) - Z_1(t, S_{L1})\| &= \|\Lambda_L - (\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - \mu_L S_L(t) - (\Lambda_L - (\beta_1 B(t) + \beta_5 I_T(t))S_{L1}(t) - \mu_L S_{L1}(t))\|,\\
&= \|-(\beta_1 B(t) + \beta_5 I_T(t))(S_L(t) - S_{L1}(t)) - \mu_L(S_L(t) - S_{L1}(t))\|.
\end{aligned} \tag{65}$$

Using triangle inequality on (65), we obtain

$$\begin{aligned}
\|Z_1(t, S_L) - Z_1(t, S_{L1})\| &\le \|(\beta_1 B(t) + \beta_5 I_T(t))(S_L(t) - S_{L1}(t))\| + \|\mu_L(S_L(t) - S_{L1}(t))\|,\\
&\le (\beta_1\|B(t)\| + \beta_5\|I_T(t)\| + \mu_L)\|S_L(t) - S_{L1}(t))\|,\\
&\le \eta_1\|S_L(t) - S_{L1}(t)\|,
\end{aligned}$$

where $\eta_1 = \beta_1 k_4 + \beta_5 k_6 + \mu_L$, thus

$$\|Z_1(t, S_L) - Z_1(t, S_{L1})\| \le \eta_1\|S_L(t) - S_{L1}(t)\|. \tag{66}$$

Therefore, the Lipschitz condition is satisfied for Z_1 and if $0 \le \beta_1 k_4 + \beta_5 k_6 + \mu_L < 1$ then it is a contraction. Following similar analysis, it can be shown that the rest of the kernels satisfies Lipschitz conditions and also are contractions:

$$\begin{aligned}
\|Z_2(t, A_L) - Z_2(t, A_{L1})\| &\le \eta_2\|A_L(t) - A_{L1}(t)\|,\\
\|Z_3(t, I_L) - Z_3(t, I_{L1})\| &\le \eta_3\|I_L(t) - I_{L1}(t)\|,\\
\|Z_4(t, B) - Z_4(t, B_1)\| &\le \eta_4\|B(t) - B_1(t)\|,\\
\|Z_5(t, S_T) - Z_5(t, S_{T1})\| &\le \eta_5\|S_T(t) - S_{T1}(t)\|,\\
\|Z_6(t, I_T) - Z_6(t, I_{T1})\| &\le \eta_6\|I_T(t) - I_{T1}(t)\|,
\end{aligned}$$

where $\eta_2 = \mu_L + \alpha_L,\ \eta_3 = \mu_L,\ \eta_4 = \varepsilon,\ \eta_5 = \mu_T$ and $\eta_6 = \mu_T$. □

Recursively (64) can be we written as

$$\begin{aligned}
S_{Ln}(t) &= S_L(0) + \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_1(t, S_{L(n-1)}) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_1(\nu, S_{L(n-1)})d\nu,\\
A_{Ln}(t) &= A_L(0) + \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_2(t, A_{L(n-1)}) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_2(\nu, A_{L(n-1)})d\nu,\\
I_{Ln}(t) &= I_L(0) + \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_3(t, I_{L(n-1)}) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_3(\nu, I_{L(n-1)})d\nu,\\
B_n(t) &= B(0) + \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} Z_4(t, B_{n-1}) + \frac{2\varphi}{(2-\varphi)P(\varphi)} \int_0^t Z_4(\nu, B_{n-1})d\nu,
\end{aligned} \tag{67}$$

$$S_{Tn}(t) = S_T(0) + \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}Z_5(t, S_{T(n-1)}) + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t Z_5(\nu, S_{T(n-1)})d\nu,$$
$$I_{Tn}(t) = I_T(0) + \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}Z_6(t, I_{T(n-1)}) + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t Z_6(\nu, I_{T(n-1)})d\nu,$$

The difference between the successive terms of the above recursive expressions yields

$$\begin{aligned}
\Phi_{1n}(t) &= S_{Ln}(t) - S_{L(n-1)}(t) = \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left[Z_1(t, S_{L(n-1)}) - Z_1(t, S_{L(n-2)})\right] \\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left[Z_1(\nu, S_{L(n-1)}) - Z_1(\nu, S_{L(n-2)})\right]d\nu, \\
\Phi_{1n}(t) &= A_{Ln}(t) - A_{L(n-1)}(t) = \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left[Z_2(t, A_{L(n-1)}) - Z_2(t, A_{L(n-2)})\right] \\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left[Z_2(\nu, A_{L(n-1)}) - Z_2(\nu, A_{L(n-2)})\right]d\nu, \\
\Phi_{1n}(t) &= I_{Ln}(t) - I_{L(n-1)}(t) = \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left[Z_3(t, I_{L(n-1)}) - Z_3(t, I_{L(n-2)})\right] \\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left[Z_3(\nu, I_{L(n-1)}) - Z_3(\nu, I_{L(n-2)})\right]d\nu, \\
\Phi_{4n}(t) &= B_n(t) - B_{n-1}(t) = \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left[Z_4(t, B_{n-1}) - Z_4(t, B_{n-2})\right] \\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left[Z_4(\nu, B_{n-1}) - Z_4(\nu, B_{n-2})\right]d\nu, \\
\Phi_{5n}(t) &= S_{Tn}(t) - S_{T(n-1)}(t) = \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left[Z_5(t, S_{T(n-1)}) - Z_5(t, S_{T(n-2)})\right] \\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left[Z_5(\nu, S_{T(n-1)}) - Z_5(\nu, S_{T(n-2)})\right]d\nu, \\
\Phi_{6n}(t) &= I_{Tn}(t) - I_{T(n-1)}(t) = \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left[Z_6(t, I_{T(n-1)}) - Z_6(t, I_{T(n-2)})\right] \\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left[Z_6(\nu, I_{T(n-1)}) - Z_6(\nu, I_{T(n-2)})\right]d\nu.
\end{aligned} \tag{68}$$

In the view of above calculations, it is clear that

$$\begin{aligned}
S_{Ln}(t) &= \sum_{i=1}^{n}\Phi_{1i}(t), \quad A_{Ln}(t) = \sum_{i=1}^{n}\Phi_{2i}(t), \\
I_{Ln}(t) &= \sum_{i=1}^{n}\Phi_{3i}(t), \quad B_n(t) = \sum_{i=1}^{n}\Phi_{4k}(t), \\
S_{Tn}(t) &= \sum_{i=1}^{n}\Phi_{5i}(t), \quad I_{Tn}(t) = \sum_{i=1}^{n}\Phi_{6i}(t).
\end{aligned} \tag{69}$$

Implementing the norm to both sides of the equation and then triangular identity, we get

$$\begin{aligned}
&\|\Phi_{1n}(t)\| = \left\|S_{Ln}(t) - S_{L(n-1)}(t)\right\|, \\
&\leq \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left\|[Z_1(t, S_{L(n-1)}) - Z_1(t, S_{L(n-2)}]\right\| \\
&+ \frac{2\varphi}{(2-\varphi)P(\varphi)}\left\|\int_0^t \left[Z_1(t, S_{L(n-1)}) - Z_1(t, S_{L(n-2)}\right]d\nu\right\|.
\end{aligned} \tag{70}$$

Because the kernel Z_1 fulfills the Lipschitz condition proved in (70), we find

$$\begin{aligned}
\|\Phi_{1n}(t)\| &= \left\|S_{Ln}(t) - S_{L(n-1)}(t)\right\| \\
&\leq \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_1\left\|S_{L(n-1)}(t) - S_{L(n-2)}\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}\eta_1\int_0^t \left\|S_{L(n-1)}(t) - S_{L(n-2)}\right\|d\nu,
\end{aligned} \tag{71}$$

and we have

$$\|\Phi_{1n}(t)\| \leq \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_1\left\|\Phi_{1(n-1)}(t)\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}\eta_1\int_0^t \left\|\Phi_{1(n-1)}(\nu)\right\|d\nu. \tag{72}$$

Analogously, for the rest of the equations of the model, we get the followings:

$$\begin{aligned}
\|\Phi_{2n}(t)\| &\le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_2\left\|\Phi_{2(n-1)}(t)\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}\eta_2\int_0^t \left\|\Phi_{2(n-1)}(\nu)\right\|d\nu,\\
\|\Phi_{3n}(t)\| &\le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_3\left\|\Phi_{3(n-1)}(t)\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}\eta_3\int_0^t \left\|\Phi_{3(n-1)}(\nu)\right\|d\nu,\\
\|\Phi_{4n}(t)\| &\le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_4\left\|\Phi_{4(n-1)}(t)\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}\eta_4\int_0^t \left\|\Phi_{4(n-1)}(\nu)\right\|d\nu,\\
\|\Phi_{5n}(t)\| &\le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_5\left\|\Phi_{5(n-1)}(t)\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}\eta_5\int_0^t \left\|\Phi_{5(n-1)}(\nu)\right\|d\nu,\\
\|\Phi_{6n}(t)\| &\le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_6\left\|\Phi_{6(n-1)}(t)\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}\eta_6\int_0^t \left\|\Phi_{6(n-1)}(\nu)\right\|d\nu.
\end{aligned} \tag{73}$$

Theorem 5.3. *The fractional model (63) has a solution, if we can find t_0 satisfying the inequality*

$$\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_i + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_i < 1 \qquad for \qquad i = 1,2,3,4,5,6. \tag{74}$$

Proof. We know that $S_L(t), A_L(t), I_L(t), B(t), S_T(t), I_T(t)$ are bounded functions and carry out Lipschitz condition. Having regard the Eqs. (72) and (73), we get the succeeding relations:

$$\begin{aligned}
\|\Phi_{1n}(t)\| &\le \|S_{Ln}(0)\|\left[\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_1 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_1\right]^n,\\
\|\Phi_{2n}(t)\| &\le \|A_{Ln}(0)\|\left[\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_2 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_2\right]^n,\\
\|\Phi_{3n}(t)\| &\le \|I_{Ln}(0)\|\left[\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_3 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_3\right]^n,\\
\|\Phi_{4n}(t)\| &\le \|B_n(0)\|\left[\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_4 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_4\right]^n,\\
\|\Phi_{5n}(t)\| &\le \|S_{Tn}(0)\|\left[\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_5 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_5\right]^n,\\
\|\Phi_{6n}(t)\| &\le \|I_{Tn}(0)\|\left[\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_5 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_6\right]^n.
\end{aligned} \tag{75}$$

Thus, the existence and continuity of the aforementioned solutions are proved. Now, our goal is to show that the above functions are solutions of (63), suppose that

$$\begin{aligned}
S_L(t) - S_L(0) &= S_{Ln}(t) - \rho_{1n}(t),\\
A_L(t) - A_L(0) &= A_{Ln}(t) - \rho_{2n}(t),\\
I_L(t) - I_L(0) &= I_{Ln}(t) - \rho_{3n}(t),\\
B(t) - B(0) &= B_n(t) - \rho_{4n}(t),\\
S_T(t) - S_T(0) &= S_{Tn}(t) - \rho_{5n}(t),\\
I_T - I_T(0) &= I_{Tn}(t) - \rho_{6n}(t).
\end{aligned}$$

Next, we have

$$\begin{aligned}
\|\rho_{1n}(t)\| &= \|\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left[Z_1(t,S_L) - Z_1(t,S_{L(n-1)})\right]\\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left[Z_1(\nu,S_L) - Z_1(\nu,S_{L(n-1)})\right]d\nu\|,\\
&\le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\left\|\left[Z_1(t,S_L) - Z_1(t,S_{L(n-1)})\right]\right\|\\
&\quad + \frac{2\varphi}{(2-\varphi)P(\varphi)}\int_0^t \left\|\left[Z_1(\nu,S_L) - Z_1(\nu,S_{L(n-1)})\right]\right\|d\nu,\\
&\le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_1\left\|S_L - S_{L(n-1)}\right\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_1\left\|S_L - S_{L(n-1)}\right\|.
\end{aligned} \tag{76}$$

By continuing this method recursively, it yields at t_0

$$\|\rho_{1n}(t)\| \le \left(\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)} + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\right)^{n+1}\eta_1^{n+1}a. \tag{77}$$

As n tends to ∞ taking the limit on both sides, we have $\|\rho_{1n}(t)\| \to 0$. In an analogous way, it can be shown $\|\rho_{2n}(t)\| \to 0$, $\|\rho_{3n}(t)\| \to 0$, $\|\rho_{4n}(t)\| \to 0$, $\|\rho_{5n}(t)\| \to 0$. □

Theorem 5.4. *The fractional order Caputo-Fabrizo mathematical model (63) has a unique provided the inequality below holds*

$$\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_i + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_i < 1, \ \textit{for } i = 1,2,3,4,5,6. \tag{78}$$

Proof. Suppose that $S_{L1}(t), A_{L1}(t), I_{L1}(t), B_1(t), S_{T1}(t)$ and $I_{T1}(t)$ are also solutions of our constructed model problem (63) such that

$$\begin{aligned} S_L(t) - S_{L1}(t) = {} & \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\big[Z_1(t,S_L) - Z_1(t,S_{L1})\big] + \frac{2\varphi}{(2-\varphi)P(\varphi)} \\ & \times \int_0^t \big[Z_1(\nu,S_L) - Z_1(\nu,S_{L1})\big]d\nu, \end{aligned} \tag{79}$$

$$\begin{aligned} A_L(t) - A_{L1}(t) = {} & \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\big[Z_2(t,A_L) - Z_2(t,A_{L1})\big] + \frac{2\varphi}{(2-\varphi)P(\varphi)} \\ & \times \int_0^t \big[Z_2(\nu,A_L) - Z_2(\nu,A_{L1})\big]d\nu, \end{aligned} \tag{80}$$

$$\begin{aligned} I_L(t) - I_{L1}(t) = {} & \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\big[Z_3(t,I_L) - Z_3(t,I_{L1})\big] + \frac{2\varphi}{(2-\varphi)P(\varphi)} \\ & \times \int_0^t \big[Z_3(\nu,I_L) - Z_3(\nu,I_{L1})\big]d\nu, \end{aligned} \tag{81}$$

$$\begin{aligned} B(t) - B_1(t) = {} & \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\big[Z_4(t,B) - Z_4(t,B_1)\big] + \frac{2\varphi}{(2-\varphi)P(\varphi)} \\ & \times \int_0^t \big[Z_4(\nu,B) - Z_4(\nu,B_1)\big]d\nu, \end{aligned} \tag{82}$$

$$\begin{aligned} S_T(t) - S_{T1}(t) = {} & \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\big[Z_5(t,S_T) - Z_5(t,S_{T1})\big] + \frac{2\varphi}{(2-\varphi)P(\varphi)} \\ & \times \int_0^t \big[Z_5(\nu,S_T) - Z_5(\nu,S_{T1})\big]d\nu, \end{aligned} \tag{83}$$

$$\begin{aligned} I_T(t) - I_{T1}(t) = {} & \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\big[Z_6(t,I_T) - Z_6(t,I_{T1})\big] + \frac{2\varphi}{(2-\varphi)P(\varphi)} \\ & \times \int_0^t \big[Z_6(\nu,I_T) - Z_6(\nu,I_{T1})\big]d\nu. \end{aligned} \tag{84}$$

Knowing that the kernel satisfies the Lipschitz condition and taking the norm on both sides of Eq. (79) we obtain

$$\|S_L(t) - S_{L1}(t)\| \le \frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\eta_1\|S_L(t) - S_{L1}(t)\| + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\eta_1\|S_L(t) - S_{L1}(t)\|. \tag{85}$$

The inequality above can further be simplify to give

$$\|S_L(t) - S_{L1}(t)\|\left[1 - \left(\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\gamma_1 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\gamma_1\right)\right] \le 0. \tag{86}$$

Since

$$\left[1 - \left(\frac{2(1-\varphi)}{(2-\varphi)P(\varphi)}\gamma_1 + \frac{2\varphi}{(2-\varphi)P(\varphi)}t\gamma_1\right)\right] \le 0,$$

it then follows that $\|S_L(t) - S_{L1}(t)\| = 0$. This further implies that $S_L(t) = S_{L1}(t)$. Following similar steps for Eqs. (80)–(84), we can deduce that

$$A_L(t) = A_{L1}(t), I_L(t) = I_{L1}(t), B(t) = B_1(t), S_T(t) = S_{T1}(t), I_T(t) = I_{T1}(t).$$

This completes the proof. □

5.5. Model description

Let us modify this model by replacing the integer order time derivative by the fractional time derivative:

$${}_0^{ABC}D_t^{\varphi}\big(S_L(t)\big) = \Lambda_L - (\beta_1 B + \beta_5 I_T)S_L - \mu_L S_L,$$

$${}_0^{ABC}D_t^{\varphi}\big(A_L(t)\big) = (\beta_1 B + \beta_5 I_T)S_L - (\mu_L + \alpha_L)A_L,$$

$$
\begin{aligned}
&{}_{0}^{ABC}D_t^{\varphi}\Big(I_L(t)\Big) = \alpha_L A_L - \mu_L I_L,\\
&{}_{0}^{ABC}D_t^{\varphi}\Big(B(t)\Big) = \beta_2(A_L + I_L) + \beta_4 I_T - \varepsilon B,\\
&{}_{0}^{ABC}D_t^{\varphi}\Big(S_T(t)\Big) = \Lambda_T - \beta_3 S_T(A_L + I_L) - \mu_T S_T,\\
&{}_{0}^{ABC}D_t^{\varphi}\Big(I_T(t)\Big) = \beta_3 S_T(A_L + I_L) - \mu_T I_T.
\end{aligned}
\tag{87}
$$

5.6. Existence and uniqueness analysis

Let $A = C(N) \times C(N) \times C(N) \times C(N) \times C(N)$ and $C(N)$ be a Banach space of continuous $\mathbb{R} \to \mathbb{R}$ valued functions on the interval N with the norm

$$\|(S_L, A_L, I_L, B, S_T, I_T)\| = \|S_L\| + \|A_L\| + \|I_L\| + \|B\| + \|S_T\| + \|I_T\|.$$

where $\|S\| = \sup\{|S(t)| : t \in N\}$, $\|A_L\| = \sup\{|A_L(t)| : t \in N\}$, $\|I_L\| = \sup\{|I_L(t)| : t \in N\}$, $\|B\| = \sup\{|B(t)| : t \in N\}$, $\|S_T\| = \sup\{|S_T(t)| : t \in N\}$, $\|I_T\| = \sup\{|I_T(t)| : t \in N\}$. Now, we redecorate the model (87), thus we have:

$$
\begin{aligned}
&{}_{a}^{ABC}D_t^{\varphi}\Big(S_L(t)\Big) = Z_1\Big(t, S_L\Big),\\
&{}_{a}^{ABC}D_t^{\varphi}\Big(A_L(t)\Big) = Z_2\Big(t, A_L\Big),\\
&{}_{a}^{ABC}D_t^{\varphi}\Big(I_L(t)\Big) = Z_3\Big(t, I_L\Big),\\
&{}_{a}^{ABC}D_t^{\varphi}\Big(B(t)\Big) = Z_4\Big(t, B\Big),\\
&{}_{a}^{ABC}D_t^{\varphi}\Big(S_T(t)\Big) = Z_5\Big(t, S_T\Big),\\
&{}_{a}^{ABC}D_t^{\varphi}\Big(I_T(t)\Big) = Z_5\Big(t, I_T\Big).
\end{aligned}
\tag{88}
$$

Applying Atangana-Baleanu fractional integral to both sides of Eq. (88) we obtain

$$
\begin{aligned}
S_L(t) - S_L(0) &= \frac{1-\varphi}{G(\varphi)} Z_1(t, S_L) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_1(\psi, S_L) d\psi,\\
A_L(t) - A_L(0) &= \frac{1-\varphi}{G(\varphi)} Z_2(t, A_L) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_2(\psi, A_L) d\psi,\\
I_L(t) - I_L(0) &= \frac{1-\varphi}{G(\varphi)} Z_3(t, I_L) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_3(\psi, I_L) d\psi,\\
B(t) - B(0) &= \frac{1-\varphi}{G(\varphi)} Z_4(t, B) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_4(\psi, B) d\psi,\\
S_T(t) - S_T(0) &= \frac{1-\varphi}{G(\varphi)} Z_5(t, S_T) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_5(\psi, S_T) d\psi,\\
I_T - I_T(0) &= \frac{1-\varphi}{G(\varphi)} Z_6(t, I_T) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_6(\psi, I_T) d\psi,
\end{aligned}
\tag{89}
$$

Theorem 5.5. *If the following inequality holds*

$$0 \le \gamma_i < 1, \quad i = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

then the kernel Z_i satisfies the Lipschitz condition and contraction.

Proof. Let assume that $S_L,\ A_L,\ I_L,\ B,\ S_T$ and I_T are bounded functions such that
$\|S_L(t)\| \le k_1,\ \|A_L(t)\| \le k_2,\ \|I_L(t)\| \le k_3,\ \|B\|(t) \le k_4,\ \|S_T(t)\| \le k_5$ and $\|I_T(t)\| \le k_6$.
Let further assume that S_L and S_{L1} are two functions, then we have

$$
\begin{aligned}
\|Z_1(t, S_L) - Z_1(t, S_{L1})\| &= \|\Lambda_L - (\beta_1 B(t) + \beta_5 I_T(t))S_L(t) - \mu_L S_L(t) - (\Lambda_L - (\beta_1 B(t) + \beta_5 I_T(t))S_{L1}(t) - \mu_L S_{L1}(t))\|,\\
&= \|-(\beta_1 B(t) + \beta_5 I_T(t))(S_L(t) - S_{L1}(t)) - \mu_L(S_L(t) - S_{L1}(t))\|.
\end{aligned}
\tag{90}
$$

Using triangle inequality on (90), we obtain

$$
\begin{aligned}
\|Z_1(t, S_L) - Z_1(t, S_{L1})\| &\le \|(\beta_1 B(t) + \beta_5 I_T(t))(S_L(t) - S_{L1}(t))\| + \|\mu_L(S_L(t) - S_{L1}(t))\|,\\
&\le (\beta_1\|B(t)\| + \beta_5\|I_T(t)\| + \mu_L)\|S_L(t) - S_{L1}(t))\|,\\
&\le \gamma_1\|S_L(t) - S_{L1}(t)\|,
\end{aligned}
$$

where $\gamma_1 = \beta_1 k_4 + \beta_5 k_6 + \mu_L$, thus

$$\|Z_1(t, S_L) - Z_1(t, S_{L1})\| \le \gamma_1\|S_L(t) - S_{L1}(t)\|. \tag{91}$$

Therefore, the Lipschitz condition is satisfied for Z_1 and if $0 \leq \beta_1 k_4 + \beta_5 k_6 + \mu_L < 1$ then it is a contraction. Following similar analysis, it can be shown that the rest of the kernels satisfies Lipschitz conditions and also are contractions:

$$\begin{aligned}
\|Z_2(t, A_L) - Z_2(t, A_{L1})\| &\leq \gamma_2 \|A_L(t) - A_{L1}(t)\|,\\
\|Z_3(t, I_L) - Z_3(t, I_{L1})\| &\leq \gamma_3 \|I_L(t) - I_{L1}(t)\|,\\
\|Z_4(t, B) - Z_4(t, B_1)\| &\leq \gamma_4 \|B(t) - B_1(t)\|,\\
\|Z_5(t, S_T) - Z_5(t, S_{T1})\| &\leq \gamma_5 \|S_T(t) - S_{T1}(t)\|,\\
\|Z_6(t, I_T) - Z_6(t, I_{T1})\| &\leq \gamma_6 \|I_T(t) - I_{T1}(t)\|,
\end{aligned}$$

where $\gamma_2 = \mu_L + \alpha_L,\ \gamma_3 = \mu_L,\ \gamma_4 = \varepsilon,\ \gamma_5 = \mu_T$ and $\gamma_6 = \mu_T$.

Recursively (89) can be we written as

$$\begin{aligned}
S_{Ln}(t) &= S_L(0) + \frac{1-\varphi}{G(\varphi)} Z_1(t, S_{L(n-1)}) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_1(\psi, S_{L(n-1)}) d\psi,\\
A_{Ln}(t) &= A_L(0) + \frac{1-\varphi}{G(\varphi)} Z_2(t, A_{L(n-1)}) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_2(\psi, A_{L(n-1)}) d\psi,\\
I_{Ln}(t) &= I_L(0) + \frac{1-\varphi}{G(\varphi)} Z_3(t, I_{L(n-1)}) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_3(\psi, I_{L(n-1)}) d\psi,\\
B_n(t) &= B(0) + \frac{1-\varphi}{G(\varphi)} Z_4(t, B_{n-1}) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_4(\psi, B_{n-1}) d\psi,\\
S_{Tn}(t) &= S_T(0) + \frac{1-\varphi}{G(\varphi)} Z_5(t, S_{T(n-1)}) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_5(\psi, S_{T(n-1)}) d\psi,\\
I_{Tn}(t) &= I_T(0) + \frac{1-\varphi}{G(\varphi)} Z_6(t, I_{T(n-1)}) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} Z_6(\psi, I_{T(n-1)}) d\psi.
\end{aligned} \tag{92}$$

The difference between the successive terms of the above recursive expressions yields

$$\begin{aligned}
\vartheta_{1n}(t) = S_{Ln}(t) - S_{L(n-1)}(t) &= \frac{1-\varphi}{G(\varphi)} \left[Z_1(t, S_{L(n-1)}) - Z_1(t, S_{L(n-2)}) \right]\\
&+ \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} \left[Z_1(\psi, S_{L(n-1)}) - Z_1(\psi, S_{L(n-2)}) \right] d\psi,\\
\vartheta_{1n}(t) = A_{Ln}(t) - A_{L(n-1)}(t) &= \frac{1-\varphi}{G(\varphi)} \left[Z_2(t, A_{L(n-1)}) - Z_2(t, A_{L(n-2)}) \right]\\
&+ \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} \left[Z_2(\psi, A_{L(n-1)}) - Z_2(\psi, A_{L(n-2)}) \right] d\psi,\\
\vartheta_{1n}(t) = I_{Ln}(t) - I_{L(n-1)}(t) &= \frac{1-\varphi}{G(\varphi)} \left[Z_3(t, I_{L(n-1)}) - Z_3(t, I_{L(n-2)}) \right]\\
&+ \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} \left[Z_3(\psi, I_{L(n-1)}) - Z_3(\psi, I_{L(n-2)}) \right] d\psi,\\
\vartheta_{4n}(t) = B_n(t) - B_{n-1}(t) &= \frac{1-\varphi}{G(\varphi)} \left[Z_4(t, B_{n-1}) - Z_4(t, B_{n-2}) \right]\\
&+ \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} \left[Z_4(\psi, B_{n-1}) - Z_4(\psi, B_{n-2}) \right] d\psi,\\
\vartheta_{5n}(t) = S_{Tn}(t) - S_{T(n-1)}(t) &= \frac{1-\varphi}{G(\varphi)} \left[Z_5(t, S_{T(n-1)}) - Z_5(t, S_{T(n-2)}) \right]\\
&+ \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} \left[Z_5(\psi, S_{T(n-1)}) - Z_5(\psi, S_{T(n-2)}) \right] d\psi,\\
\vartheta_{6n}(t) = I_{Tn}(t) - I_{T(n-1)}(t) &= \frac{1-\varphi}{G(\varphi)} \left[Z_6(t, I_{T(n-1)}) - Z_6(t, I_{T(n-2)}) \right]\\
&+ \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^t (t-\psi)^{\varphi-1} \left[Z_6(\psi, I_{T(n-1)}) - Z_6(\psi, I_{T(n-2)}) \right] d\psi.
\end{aligned} \tag{93}$$

In the view of above calculations, it is clear that

$$S_{Ln}(t) = \sum_{i=1}^{n} \vartheta_{1i}(t), \quad A_{Ln}(t) = \sum_{i=1}^{n} \vartheta_{2i}(t),$$

$$I_{Ln}(t)=\sum_{i=1}^{n}\vartheta_{3i}(t),\quad B_n(t)=\sum_{i=1}^{n}\vartheta_{4k}(t),$$
$$S_{Tn}(t)=\sum_{i=1}^{n}\vartheta_{5i}(t),\quad I_{Tn}(t)=\sum_{i=1}^{n}\vartheta_{6i}(t). \tag{94}$$

Implementing the norm to both sides of the equation and then triangular identity, we get

$$\begin{aligned}\|\vartheta_{1n}(t)\| &= \left\|S_{Ln}(t)-S_{L(n-1)}(t)\right\|,\\ &\leq \frac{1-\varphi}{G(\varphi)}\left\|[Z_1(t,S_{L(n-1)})-Z_1(t,S_{L(n-2)}]\right\|,\\ &+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\left\|\int_0^t (t-\psi)^{\varphi-1}\big[Z_1(t,S_{L(n-1)})-Z_1(t,S_{L(n-2)}\big]d\psi\right\|.\end{aligned} \tag{95}$$

Because the kernel Z_1 fulfills the Lipschitz condition proved in (91), we find

$$\begin{aligned}\|\vartheta_{1n}(t)\| &= \left\|S_{Ln}(t)-S_{L(n-1)}(t)\right\|,\\ &\leq \frac{1-\varphi}{G(\varphi)}\gamma_1\left\|S_{L(n-1)}(t)-S_{L(n-2)}\right\|+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_1\int_0^t (t-\psi)^{\varphi-1}\left\|S_{L(n-1)}(t)-S_{L(n-2)}\right\|d\psi,\end{aligned} \tag{96}$$

and we have

$$\|\vartheta_{1n}(t)\| \leq \frac{1-\varphi}{G(\varphi)}\gamma_1\left\|\vartheta_{1(n-1)}(t)\right\|+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_1\int_0^t (t-\psi)^{\varphi-1}\left\|\vartheta_{1(n-1)}(\psi)\right\|d\psi. \tag{97}$$

Analogously, we get the followings:

$$\begin{aligned}\|\vartheta_{2n}(t)\| &\leq \frac{1-\varphi}{G(\varphi)}\gamma_2\left\|\vartheta_{2(n-1)}(t)\right\|+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_2\int_0^t (t-\psi)^{\varphi-1}\left\|\vartheta_{2(n-1)}(\psi)\right\|d\psi,\\ \|\vartheta_{3n}(t)\| &\leq \frac{1-\varphi}{G(\varphi)}\gamma_3\left\|\vartheta_{3(n-1)}(t)\right\|+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_3\int_0^t (t-\psi)^{\varphi-1}\left\|\vartheta_{3(n-1)}(\psi)\right\|d\psi,\\ \|\vartheta_{4n}(t)\| &\leq \frac{1-\varphi}{G(\varphi)}\gamma_4\left\|\vartheta_{4(n-1)}(t)\right\|+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_4\int_0^t (t-\psi)^{\varphi-1}\left\|\vartheta_{4(n-1)}(\psi)\right\|d\psi,\\ \|\vartheta_{5n}(t)\| &\leq \frac{1-\varphi}{G(\varphi)}\gamma_5\left\|\vartheta_{5(n-1)}(t)\right\|+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_5\int_0^t (t-\psi)^{\varphi-1}\left\|\vartheta_{5(n-1)}(\psi)\right\|d\psi,\\ \|\vartheta_{6n}(t)\| &\leq \frac{1-\varphi}{G(\varphi)}\gamma_6\left\|\vartheta_{5(n-1)}(t)\right\|+\frac{\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_6\int_0^t (t-\psi)^{\varphi-1}\left\|\vartheta_{6(n-1)}(\psi)\right\|d\psi.\end{aligned} \tag{98}$$

□

Theorem 5.6. *The fractional model (87) has a solution, if we can find t_0 satisfying the inequality*

$$\frac{1-\varphi}{G(\varphi)}\gamma_i+\frac{t_0^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_i<1 \qquad for \qquad i=1,2,3,4,5,6. \tag{99}$$

Proof. We know that $S_L(t), A_L(t), I_L(t), B(t), S_T(t), I_L(t)$ are bounded functions and carry out Lipschitz condition. Having regard the Eqs. (97) and (98), we get the succeeding relations:

$$\begin{aligned}\|\vartheta_{1n}(t)\| &\leq \|S_{Ln}(0)\|\left[\frac{1-\varphi}{G(\varphi)}\gamma_1+\frac{t^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_1\right]^n,\\ \|\vartheta_{2n}(t)\| &\leq \|A_{Ln}(0)\|\left[\frac{1-\varphi}{G(\varphi)}\gamma_2+\frac{t^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_2\right]^n,\\ \|\vartheta_{3n}(t)\| &\leq \|I_{Ln}(0)\|\left[\frac{1-\varphi}{G(\varphi)}\gamma_3+\frac{t^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_3\right]^n,\\ \|\vartheta_{4n}(t)\| &\leq \|B_{n}(0)\|\left[\frac{1-\varphi}{G(\varphi)}\gamma_4+\frac{t^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_4\right]^n,\\ \|\vartheta_{5n}(t)\| &\leq \|S_{Tn}(0)\|\left[\frac{1-\varphi}{G(\varphi)}\gamma_5+\frac{t^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_5\right]^n,\\ \|\vartheta_{6n}(t)\| &\leq \|I_{Tn}(0)\|\left[\frac{1-\varphi}{G(\varphi)}\gamma_5+\frac{t^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_6\right]^n.\end{aligned} \tag{100}$$

Thus, the existence and continuity of the aforementioned solutions are proved. Now, our goal is to show that the above functions are solutions of (87), suppose that

$$S_L(t) - S_L(0) = S_{Ln}(t) - \bar{\varepsilon}_{1n}(t),$$
$$A_L(t) - A_L(0) = A_{Ln}(t) - \bar{\varepsilon}_{2n}(t),$$
$$I_L(t) - I_L(0) = I_{Ln}(t) - \bar{\varepsilon}_{3n}(t),$$
$$B(t) - B(0) = B_n(t) - \bar{\varepsilon}_{4n}(t),$$
$$S_T(t) - S_T(0) = S_{Tn}(t) - \bar{\varepsilon}_{5n}(t),$$
$$I_T - I_T(0) = I_{Tn}(t) - \bar{\varepsilon}_{6n}(t).$$

Next, we have

$$\begin{aligned}\|\bar{\varepsilon}_{1n}(t)\| &= ||\frac{1-\varphi}{G(\varphi)}\big[Z_1(t,S_L) - Z_1(t,S_{L(n-1)})\big] \\ &\quad + \frac{\varphi}{G(\varphi)\Gamma(\varphi)}\int_0^t (t-\psi)^{\varphi-1}\big[Z_1(\psi,S_L) - Z_1(\psi,S_{L(n-1)})\big]d\psi||, \\ &\le \frac{1-\varphi}{G(\varphi)}\big\|\big[Z_1(t,S_L) - Z_1(t,S_{L(n-1)})\big]\big\| \\ &\quad + \frac{\varphi}{G(\varphi)\Gamma(\varphi)}\int_0^t (t-\psi)^{\varphi-1}\big\|\big[Z_1(\psi,S_L) - Z_1(\psi,S_{L(n-1)})\big]\big\|d\psi, \\ &\le \frac{1-\varphi}{G(\varphi)}\gamma_1\big\|S_L - S_{L(n-1)}\big\| + \frac{t^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_1\big\|S_L - S_{L(n-1)}\big\|.\end{aligned} \tag{101}$$

By continuing this method recursively, it yields at t_0

$$\|\bar{\varepsilon}_{1n}(t)\| \le \left(\frac{1-\varphi}{G(\varphi)} + \frac{t_0^{\varphi}}{G(\varphi)\Gamma(\varphi)}\right)^{n+1}\gamma_1^{n+1}a^*. \tag{102}$$

As n tends to ∞ taking the limit both sides, we have $\|\bar{\varepsilon}_{1n}(t)\| \to 0$. In an analogous way, it can be shown $\|\bar{\varepsilon}_{2n}(t)\| \to 0$, $\|\bar{\varepsilon}_{3n}(t)\| \to 0$, $\|\bar{\varepsilon}_{4n}(t)\| \to 0$, $\|\bar{\varepsilon}_{5n}(t)\| \to 0$. □

Theorem 5.7. *The fractional order Atangana-Baleanu mahematical model (87) has a unique provided the inequality below holds*

$$\frac{1-\varphi}{G(\varphi)}\gamma_i + \frac{t_0^{\varphi}}{G(\varphi)\Gamma(\varphi)}\gamma_i < 1 \qquad for \qquad i = 1,2,3,4,5,6. \tag{103}$$

Proof. Suppose that $S_{L1}(t), A_{L1}(t), I_{L1}(t), B_1(t), S_{T1}(t)$ and $I_{T1}(t)$ are also solutions our constructed model problem (87) such that

$$\begin{aligned}S_L(t) - S_{L1}(t) &= \frac{1-\varphi}{G(\varphi)}\big[Z_1(t,S_L) - Z_1(t,S_{L1})\big] + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \\ &\quad \times \int_0^t (t-\psi)^{\varphi-1}\big[Z_1(\psi,S_L) - Z_1(\psi,S_{L1})\big]d\psi,\end{aligned} \tag{104}$$

$$\begin{aligned}A_L(t) - A_{L1}(t) &= \frac{1-\varphi}{G(\varphi)}\big[Z_2(t,A_L) - Z_2(t,A_{L1})\big] + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \\ &\quad \times \int_0^t (t-\psi)^{\varphi-1}\big[Z_2(\psi,A_L) - Z_2(\psi,A_{L1})\big]d\psi,\end{aligned} \tag{105}$$

$$\begin{aligned}I_L(t) - I_{L1}(t) &= \frac{1-\varphi}{G(\varphi)}\big[Z_3(t,I_L) - Z_3(t,I_{L1})\big] + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \\ &\quad \times \int_0^t (t-\psi)^{\varphi-1}\big[Z_3(\psi,I_L) - Z_3(\psi,I_{L1})\big]d\psi,\end{aligned} \tag{106}$$

$$\begin{aligned}B(t) - B_1(t) &= \frac{1-\varphi}{G(\varphi)}\big[Z_4(t,B) - Z_4(t,B_1)\big] + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \\ &\quad \times \int_0^t (t-\psi)^{\varphi-1}\big[Z_4(\psi,B) - Z_4(\psi,B_1)\big]d\psi,\end{aligned} \tag{107}$$

$$\begin{aligned}S_T(t) - S_{T1}(t) &= \frac{1-\varphi}{G(\varphi)}\big[Z_5(t,S_T) - Z_5(t,S_{T1})\big] + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \\ &\quad \times \int_0^t (t-\psi)^{\varphi-1}\big[Z_5(\psi,S_T) - Z_5(\psi,S_{T1})\big]d\psi,\end{aligned} \tag{108}$$

$$I_T(t) - I_{T1}(t) = \frac{1-\varphi}{G(\varphi)}\left[Z_6(t, I_T) - Z_6(t, I_{T1})\right] + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \times \int_0^t (t-\psi)^{\varphi-1}\left[Z_6(\psi, I_T) - Z_6(\psi, I_{T1})\right]d\psi. \tag{109}$$

Knowing that the kernel satisfies the Lipschitz condition and taking the norm on both sides of Eq. (104) we obtain

$$\|S_L(t) - S_{L1}(t)\| \le \frac{1-\varphi}{G(\varphi)}\gamma_1\|S_L(t) - S_{L1}(t)\| + \frac{t^\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_1\|S_L(t) - S_{L1}(t)\|. \tag{110}$$

The inequality above can further be simplify to give

$$\|S_L(t) - S_{L1}(t)\|\left[1 - \left(\frac{1-\varphi}{G(\varphi)}\gamma_1 + \frac{t^\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_1\right)\right] \le 0. \tag{111}$$

Since

$$\left[1 - \left(\frac{1-\varphi}{G(\varphi)}\gamma_1 + \frac{t^\varphi}{G(\varphi)\Gamma(\varphi)}\gamma_1\right)\right] \le 0,$$

it then follows that $\|S_L(t) - S_{L1}(t)\| = 0$. This further implies that $S_L(t) = S_{L1}(t)$. Following similar steps for Eqs. (105)–(109), we can deduce that

$A_L(t) = A_{L1}(t), I_L(t) = I_{L1}(t), B(t) = B_1(t), S_T(t) = S_{T1}(t), I_T(t) = I_{T1}(t).$

This completes the proof. □

6. Numerical schemes

Here, we provide the numerical schemes for our three proposed fractional operator model for the Q fever dynamics in livestock and ticks. Firstly, we start with the Caputo scheme [64], then followed by the Caputo–Fabrizio scheme [65] and lastly the Atangana–Baleanu scheme [40] using the two step Lagrange interpolation.

6.1. Numerical scheme for the Caputo fractional Q fever disease model

The Cauchy problem of the Caputo derivative can be written as

$${}_0^C D_t^\varphi g(t) = Z^*(t, g(t)), \quad g(0) = g_0. \tag{112}$$

Using the Caputo integral, the Cauchy problem (112) can be converted into

$$g(t) - g(0) = \frac{1}{\Gamma(\varphi)}\int_0^t Z^*(\Upsilon, g(\Upsilon))(t-\Upsilon)^{\varphi-1}d\Upsilon. \tag{113}$$

At the point $t_{\upsilon^*+1} = (\upsilon^*+1)h$ and $t_{\upsilon^*} = \upsilon^* h$, $\upsilon^* = 0, 1, 2, 3, 4, \ldots$ with h being the time step, Eq. (113) can be formulated as:

$$g(t_{\upsilon^*+1}) - g(0) = \frac{1}{\Gamma(\varphi)}\int_0^{t_{\upsilon^*+1}} Z^*(\Upsilon, g(\Upsilon))(t_{\upsilon^*+1}-\Upsilon)^{\varphi-1}d\Upsilon. \tag{114}$$

Which can be written as

$$g(t_{\upsilon^*+1}) = g(0) + \frac{1}{\Gamma(\varphi)}\sum_{m^*=0}^{\upsilon^*}\int_{t_{m^*}}^{t_{m^*+1}} Z^*(\Upsilon, g(\Upsilon))(t_{\upsilon^*+1}-\Upsilon)^{\varphi-1}d\Upsilon. \tag{115}$$

Thus, simplifying the integral on the right hand in Eq. (115) and using the Lagrange polynomial, Eq. (115) can now be written as

$$g_{\upsilon^*+1} = g_0 + \frac{1}{\Gamma(\varphi)}\sum_{m^*=0}^{\upsilon^*}\left\{\begin{array}{l} \frac{Z^*(t_{m^*}, g_{m^*})}{h}\int_{t_{m^*}}^{t_{m^*+1}}(\Upsilon - t_{m^*-1})(t_{\upsilon^*+1}-\Upsilon)^{\varphi-1}d\Upsilon \\ -\frac{Z^*(t_{m^*-1}, g_{m^*-1})}{h}\int_{t_{m^*}}^{t_{m^*+1}}(\Upsilon - t_{m^*})(t_{\upsilon^*+1}-\Upsilon)^{\varphi-1}d\Upsilon \end{array}\right\}. \tag{116}$$

Where Eq. (116) can be written as

$$g_{\upsilon^*+1} = g_0 + \frac{1}{\Gamma(\varphi)}\sum_{m^*=0}^{\upsilon^*}\frac{Z^*(t_{m^*}, g_{m^*})}{h}\int_{t_{m^*}}^{t_{m^*+1}}(\Upsilon - t_{m^*-1})(t_{\upsilon^*+1}-\Upsilon)^{\varphi-1}d\Upsilon - \frac{1}{\Gamma(\varphi)}\sum_{m^*=0}^{\upsilon^*}\frac{Z^*(t_{m^*-1}, g_{m^*-1})}{h}\int_{t_{m^*}}^{t_{m^*+1}}(\Upsilon - t_{m^*})(t_{\upsilon^*+1}-\Upsilon)^{\varphi-1}d\Upsilon. \tag{117}$$

Solving the integral in (117) we arrive at the following:

$$\int_{t_{m^*}}^{t_{m^*+1}}(\Upsilon - t_{m^*-1})(t_{\upsilon^*+1}-\Upsilon)^{\varphi-1}d\Upsilon = \frac{h^{\varphi-1}}{\varphi(\varphi+1)}\left[\begin{array}{l}(\upsilon^* - m^* + 1)^\varphi(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^\varphi(\upsilon^* - m^* + 2 + 2\varphi)\end{array}\right], \tag{118}$$

$$\int_{t_{m^*}}^{t_{m^*+1}} (\Upsilon - t_{m^*})(t_{\upsilon^*+1} - \Upsilon)^{\varphi-1} d\Upsilon = \frac{h^{\varphi-1}}{\varphi(\varphi+1)} \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}. \tag{119}$$

Replacing Eqs. (118) and (119) into Eq. (117), the numerical scheme for the Caputo derivative is given as

$$\begin{aligned} g_{\upsilon^*+1} = g_0 &+ \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*}, g_{m^*}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi}(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 2 + 2\varphi) \end{bmatrix} \\ &- \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*-1}, g_{m^*-1}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}. \end{aligned} \tag{120}$$

Thus, in terms of our Caputo model we get:

$$\begin{aligned}
S_{L_{\upsilon^*+1}} = S_{L_0} &+ \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*}, S_{L_{m^*}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi}(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 2 + 2\varphi) \end{bmatrix} \\
&- \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*-1}, S_{L_{m^*-1}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}, \\
A_{L_{\upsilon^*+1}} = A_{L_0} &+ \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*}, A_{L_{m^*}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi}(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 2 + 2\varphi) \end{bmatrix} \\
&- \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*-1}, A_{L_{m^*-1}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}, \\
I_{L_{\upsilon^*+1}} = S_{L_0} &+ \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*}, I_{L_{m^*}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi}(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 2 + 2\varphi) \end{bmatrix} \\
&- \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*-1}, I_{L_{m^*-1}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}, \\
B_{\upsilon^*+1} = B_0 &+ \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*}, B_{m^*}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi}(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 2 + 2\varphi) \end{bmatrix} \\
&- \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*-1}, B_{m^*-1}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}, \\
S_{T_{\upsilon^*+1}} = S_{L_0} &+ \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*}, S_{T_{m^*}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi}(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 2 + 2\varphi) \end{bmatrix} \\
&- \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*-1}, S_{T_{m^*-1}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}, \\
I_{T_{\upsilon^*+1}} = I_{T_0} &+ \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*}, I_{T_{m^*}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi}(\upsilon^* - m^* + 2 + \varphi) \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 2 + 2\varphi) \end{bmatrix} \\
&- \frac{h^{\varphi}}{\Gamma(\varphi+2)} \sum_{m^*=0}^{\upsilon^*} Z^*(t_{m^*-1}, I_{T_{m^*-1}}) \begin{bmatrix} (\upsilon^* - m^* + 1)^{\varphi+1} \\ -(\upsilon^* - m^*)^{\varphi}(\upsilon^* - m^* + 1 + \varphi) \end{bmatrix}.
\end{aligned} \tag{121}$$

6.2. Numerical scheme for the Caputo–Fabrizio (FC) fractional Q fever disease model

The Cauchy problem of the Caputo–Fabrizio derivative can be written as

$${}_0^{\mathrm{CF}}D_t^{\varphi} w(t) = Z(t, w(t)). \tag{122}$$

Using the Caputo–Fabrizio integral, the Cauchy problem of FC can be restated as

$$w(t) - w(0) = \frac{(1-\varphi)}{P(\varphi)} Z(t, w(t)) + \frac{\varphi}{P(\varphi)} \int_0^t Z(\Upsilon, w(\Upsilon)) d\Upsilon. \tag{123}$$

Writing (123) at the point $t_{\upsilon+1} = (\upsilon+1)h$ and $t_\upsilon = \upsilon h$, results in

$$w(t_{\upsilon+1}) - w(0) = \frac{(1-\varphi)}{P(\varphi)} Z(t_\upsilon, w(t_\upsilon)) + \frac{\varphi}{M(\varphi)} \int_0^{t_{\upsilon+1}} Z(\Upsilon, w(\Upsilon)) d\Upsilon, \tag{124}$$

$$w(t_\upsilon) - w(0) = \frac{(1-\varphi)}{P(\varphi)} Z(t_{\upsilon-1}, w(t_{\upsilon-1})) + \frac{\varphi}{P(\varphi)} \int_0^{t_\upsilon} Z(\Upsilon, w(\Upsilon)) d\Upsilon. \tag{125}$$

Taking (124)-(125) results in

$$w(t_{\upsilon+1}) - w(t_\upsilon) = \frac{(1-\varphi)}{P(\varphi)} (Z(t_\upsilon, w(t_\upsilon)) - Z(t_{\upsilon-1}, w(t_{\upsilon-1}))) + \frac{\varphi}{P(\varphi)} \int_{t_\upsilon}^{t_{\upsilon+1}} Z(\Upsilon, w(\Upsilon)) d\Upsilon. \tag{126}$$

Equation (126) in two-step Lagrange polynomial gives

$$\begin{aligned} w_{\upsilon+1} - w_\upsilon =& \frac{(1-\varphi)}{P(\varphi)} (Z(t_\upsilon, w(t_\upsilon)) - Z(t_{\upsilon-1}, w(t_{\upsilon-1}))) \\ &+ \frac{\varphi}{P(\varphi)} \int_{t_\upsilon}^{t_{\upsilon+1}} \left(\frac{Z(t_\upsilon, w(t_\upsilon))}{h} (\Upsilon - t_{\upsilon-1}) - \frac{Z(t_{\upsilon-1}, w(t_{\upsilon-1}))}{h} (\Upsilon - t_\upsilon) \right) d\Upsilon. \end{aligned} \tag{127}$$

The above Eq. (127) leads to

$$\begin{aligned} w_{\upsilon+1} - w_\upsilon =& \frac{(1-\varphi)}{P(\varphi)} (Z(t_\upsilon, w(t_\upsilon)) - Z(t_{\upsilon-1}, w(t_{\upsilon-1}))) \\ &+ \frac{\varphi}{P(\varphi)} \left(\frac{Z(t_\upsilon, w(t_\upsilon))}{h} \int_{t_\upsilon}^{t_{\upsilon+1}} (\Upsilon - t_{\upsilon-1}) d\Upsilon - \frac{Z(t_{\upsilon-1}, w(t_{\upsilon-1}))}{h} \int_{t_\upsilon}^{t_{\upsilon+1}} (\Upsilon - t_\upsilon) d\Upsilon \right). \end{aligned} \tag{128}$$

Solving the integrals in Eq. (128) yields

$$\begin{aligned} \int_{t_\upsilon}^{t_{\upsilon+1}} (\Upsilon - t_{\upsilon-1}) d\Upsilon =& \frac{3}{2} h^2, \\ \int_{t_\upsilon}^{t_{\upsilon+1}} (\Upsilon - t_\upsilon) d\Upsilon =& \frac{1}{2} h^2. \end{aligned} \tag{129}$$

Substituting Eq. (129) into Eq. (128), the numerical scheme of FC is given as:

$$w_{\upsilon+1} = w_\upsilon + \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{3h\varphi}{2P(\varphi)} \right] Z(t_\upsilon, w_\upsilon) - \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{h\varphi}{2P(\varphi)} \right] Z(t_{\upsilon-1}, w_{\upsilon-1}). \tag{130}$$

Thus, in terms of our Caputo–Fabrizio model we get:

$$\begin{aligned} S_{L_{\upsilon+1}} &= S_{L_\upsilon} + \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{3h\varphi}{2P(\varphi)} \right] Z(t_\upsilon, S_{L_\upsilon}) - \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{h\varphi}{2P(\varphi)} \right] Z(t_{\upsilon-1}, S_{L_{\upsilon-1}}), \\ A_{L_{\upsilon+1}} &= A_{L_\upsilon} + \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{3h\varphi}{2P(\varphi)} \right] Z(t_\upsilon, A_{L_\upsilon}) - \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{h\varphi}{2P(\varphi)} \right] Z(t_{\upsilon-1}, A_{L_{\upsilon-1}}), \\ I_{L_{\upsilon+1}} &= I_{L_\upsilon} + \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{3h\varphi}{2P(\varphi)} \right] Z(t_\upsilon, I_{L_\upsilon}) - \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{h\varphi}{2P(\varphi)} \right] Z(t_{\upsilon-1}, I_{L_{\upsilon-1}}), \\ B_{\upsilon+1} &= B_\upsilon + \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{3h\varphi}{2P(\varphi)} \right] Z(t_\upsilon, B_\upsilon) - \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{h\varphi}{2P(\varphi)} \right] Z(t_{\upsilon-1}, B_{\upsilon-1}), \\ S_{T_{\upsilon+1}} &= S_{T_\upsilon} + \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{3h\varphi}{2P(\varphi)} \right] Z(t_\upsilon, S_{T_\upsilon}) - \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{h\varphi}{2P(\varphi)} \right] Z(t_{\upsilon-1}, S_{T_{\upsilon-1}}), \\ I_{T_{\upsilon+1}} &= I_{T_\upsilon} + \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{3h\varphi}{2P(\varphi)} \right] Z(t_\upsilon, I_{T_\upsilon}) - \left[\frac{(1-\varphi)}{P(\varphi)} + \frac{h\varphi}{2P(\varphi)} \right] Z(t_{\upsilon-1}, I_{T_{\upsilon-1}}). \end{aligned} \tag{131}$$

6.3. Numerical scheme for the ABC fractional Q fever disease model

The Cauchy problem of the ABC derivative can also be written as

$${}_0^{ABC}D_t^{\varphi} w(t) = Z(t, w(t)). \tag{132}$$

The ABC integral, the Cauchy problem of ABC can be restated using the fundamental theory of calculus

$$w(t) - w(0) = \frac{(1-\varphi)}{G(\varphi)} Z(t, w(\Upsilon)) + \frac{\varphi}{\Gamma(\varphi) G(\varphi)} \int_0^t (t - \Upsilon)^{\varphi-1} Z(\Upsilon, w(\Upsilon)) d\Upsilon. \tag{133}$$

Writing (133) at the point $t_{n+1} = (n+1)h$ and $t_n = nh$, results in

$$w(t_{n+1}) - w(0) = \frac{(1-\varphi)}{G(\varphi)} Z(t_n, w(t_n)) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \int_0^{t_{n+1}} (t_{n+1} - \Upsilon)^{\varphi-1} Z(\Upsilon, w(\Upsilon)) d\Upsilon,$$

$$= \frac{(1-\varphi)}{G(\varphi)} Z(t_n, w(t_n)) + \frac{\varphi}{G(\varphi)\Gamma(\varphi)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{n+1} - \Upsilon)^{\varphi-1} Z(\Upsilon, w(\Upsilon)) d\Upsilon. \tag{134}$$

The two-step Lagrange polynomial in the interval $[t_n, t_{n+1}]$ can be written as

$$\begin{aligned} \gamma_k(\Upsilon) &= \frac{\Upsilon - t_{k-1}}{t_k - t_{k-1}} Z(t_k, w(t_k)) - \frac{\Upsilon - t_k}{t_k - t_{k-1}} Z(t_{k-1}, w(t_{k-1})), \\ &= \frac{Z(t_k, w(t_k))}{h}(\Upsilon - t_{k-1}) - \frac{Z(t_{k-1}, w(t_{k-1}))}{h}(\Upsilon - t_k), \end{aligned} \tag{135}$$

$$\cong \frac{Z(t_k,\ w_k)}{h}(\Upsilon - t_{k-1}) - \frac{Z(t_{k-1},\ \ w_{k-1})}{h}(\Upsilon - t_k). \tag{136}$$

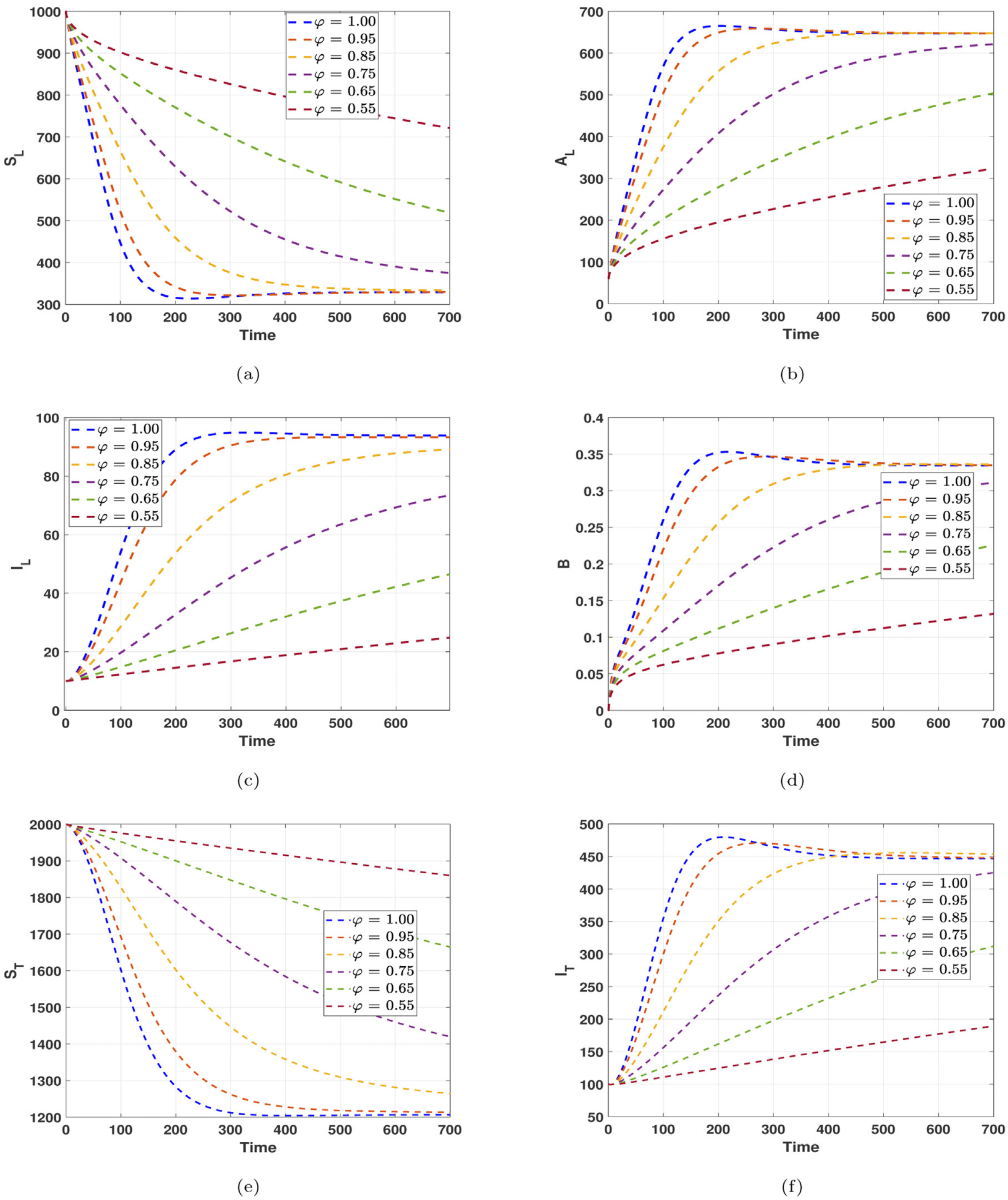


Fig. 6. Numerical simulation of Caputo operator on the model with different fractional order, φ.

Now, the above approximation (136) into Eq. (134) gives:

$$
\begin{aligned}
w_{n+1} =& w(0) + \frac{(1-\varphi)}{G(\varphi)} Z(t_n, w(t_n)) + \frac{\varphi}{G(\varphi) \times \Gamma(\varphi)} \\
& \times \sum_{k=0}^{n} \left(\frac{Z(t_k,\ w_k)}{h} \int_{t_k}^{t_{k+1}} (\Upsilon - t_{k-1})(t_{n+1} - t)^{\varphi-1} d\Upsilon \right. \\
& \left. - \frac{Z(t_{k-1}, w_{k-1})}{h} \int_{t_k}^{t_{k+1}} (\Upsilon - t_k)(t_{n+1} - t)^{\varphi-1} d\Upsilon \right).
\end{aligned}
\tag{137}
$$

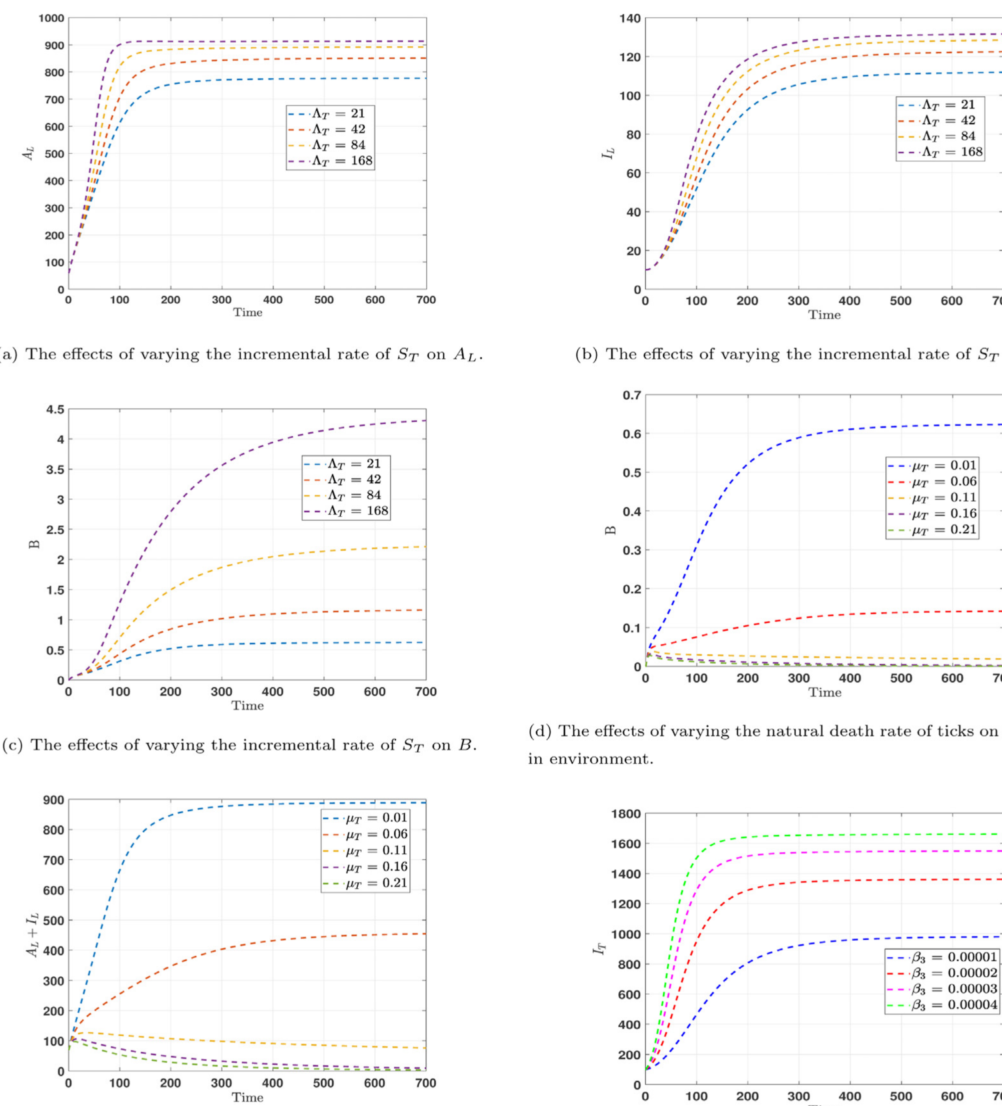
The integrals of Eq. (137) can be denoted as:

$$
U_{\varphi,k,1} = \int_{t_k}^{t_{k+1}} (\Upsilon - t_{k-1})(t_{n+1} - t)^{\varphi-1} d\Upsilon, \tag{138}
$$

$$
U_{\varphi,k,2} = \int_{t_k}^{t_{k+1}} (\Upsilon - t_k)(t_{n+1} - t)^{\varphi-1} d\Upsilon. \tag{139}
$$

Solving Eqs. (138) and (139) using integration by substitution, it then follows that

$$
U_{\varphi,k,1} = h^{\varphi+1} \frac{(n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi)}{\varphi(\varphi+1)}, \tag{140}
$$



(a) The effects of varying the incremental rate of S_T on A_L.

(b) The effects of varying the incremental rate of S_T on I_L.

(c) The effects of varying the incremental rate of S_T on B.

(d) The effects of varying the natural death rate of ticks on the bacterial in environment.

(e) The effects of varying the natural death rate of ticks on infected livestock.

(f) Varying the transmission rate from $(A_L + I_L)$ to ticks.

Fig. 7. Numerical simulation under Caputo operator with fractional order, $\varphi = 0.95$.

$$U_{\varphi,k,2} = h^{\varphi+1}\frac{(n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi)}{\varphi(\varphi+1)}. \tag{141}$$

Thus, replacing the solution of $U_{\varphi,k,1}$ and $U_{\varphi,k,2}$ into (137), gives the following numerical scheme:

$$w_{n+1} = w(0) + \frac{(1-\varphi)}{G(\varphi)}Z(t_n, w(t_n)) + \frac{\varphi}{G(\varphi)}\sum_{k=0}^{n} \times \left(\frac{h^{\varphi}Z(t_k,\ w_k)}{\Gamma(\varphi+2)}((n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi))\right)$$

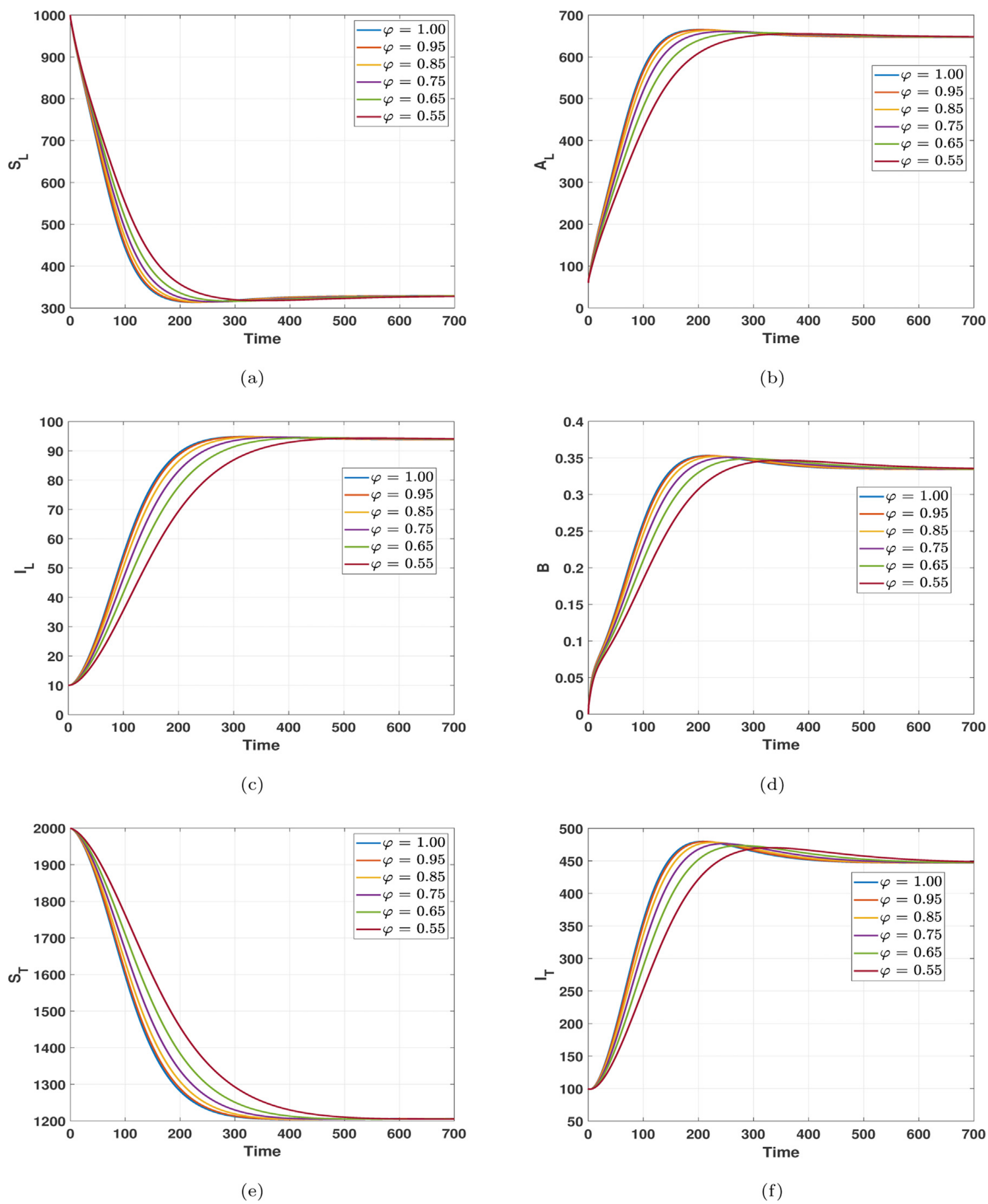


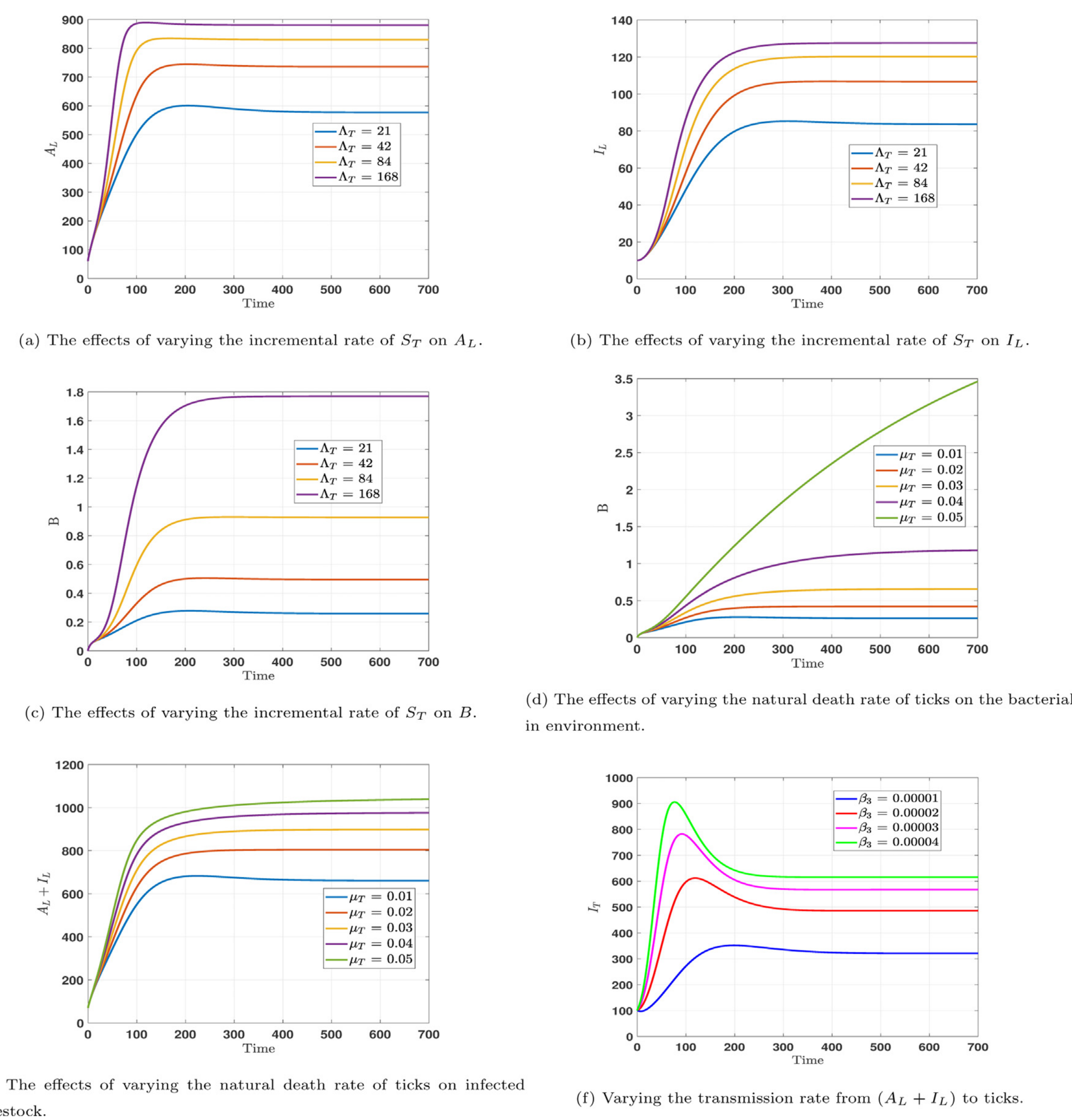
Fig. 8. Numerical simulation under Caputo-Fabrizio differential operator with different fractional order, φ.

$$- \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \left(\frac{h^{\varphi} Z(t_{k-1}, w_{k-1})}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi) \right) \right).$$

Therefore, in terms of the Q fever ABC model we get:

$$S_{Ln+1} = S_L(0) + \frac{(1-\varphi)}{G(\varphi)} Z(t_n, S_L(t_n)) + \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n}$$
$$\times \left(\frac{h^{\varphi} Z(t_k,\ S_{Lk})}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi) \right) \right)$$
$$- \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \left(\frac{h^{\varphi} Z\left(t_{k-1}, S_{L_{k-1}}\right)}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi) \right) \right),$$

$$A_{L_{n+1}} = A_L(0) + \frac{(1-\varphi)}{G(\varphi)} Z(t_n, A_L(t_n)) + \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n}$$
$$\times \left(\frac{h^{\varphi} Z(t_k, A_{Lk})}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi) \right) \right)$$



(a) The effects of varying the incremental rate of S_T on A_L.

(b) The effects of varying the incremental rate of S_T on I_L.

(c) The effects of varying the incremental rate of S_T on B.

(d) The effects of varying the natural death rate of ticks on the bacterial in environment.

(e) The effects of varying the natural death rate of ticks on infected livestock.

(f) Varying the transmission rate from $(A_L + I_L)$ to ticks.

Fig. 9. Numerical simulation under Caputo-Fabrizio operator with fractional order, $\varphi = 0.95$.

$$- \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \left(\frac{h^{\varphi} Z\left(t_{k-1}, A_{L_{k-1}}\right)}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi)\right) \right),$$

$$I_{L_{n+1}} = I_L(0) + \frac{(1-\varphi)}{G(\varphi)} Z(t_n, I_L(t_n)) + \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n}$$

$$\times \left(\frac{h^{\varphi} Z(t_k,\ I_{Lk})}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi)\right) \right)$$

$$- \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \left(\frac{h^{\varphi} Z\left(t_{k-1}, I_{L_{k-1}}\right)}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi)\right) \right),$$

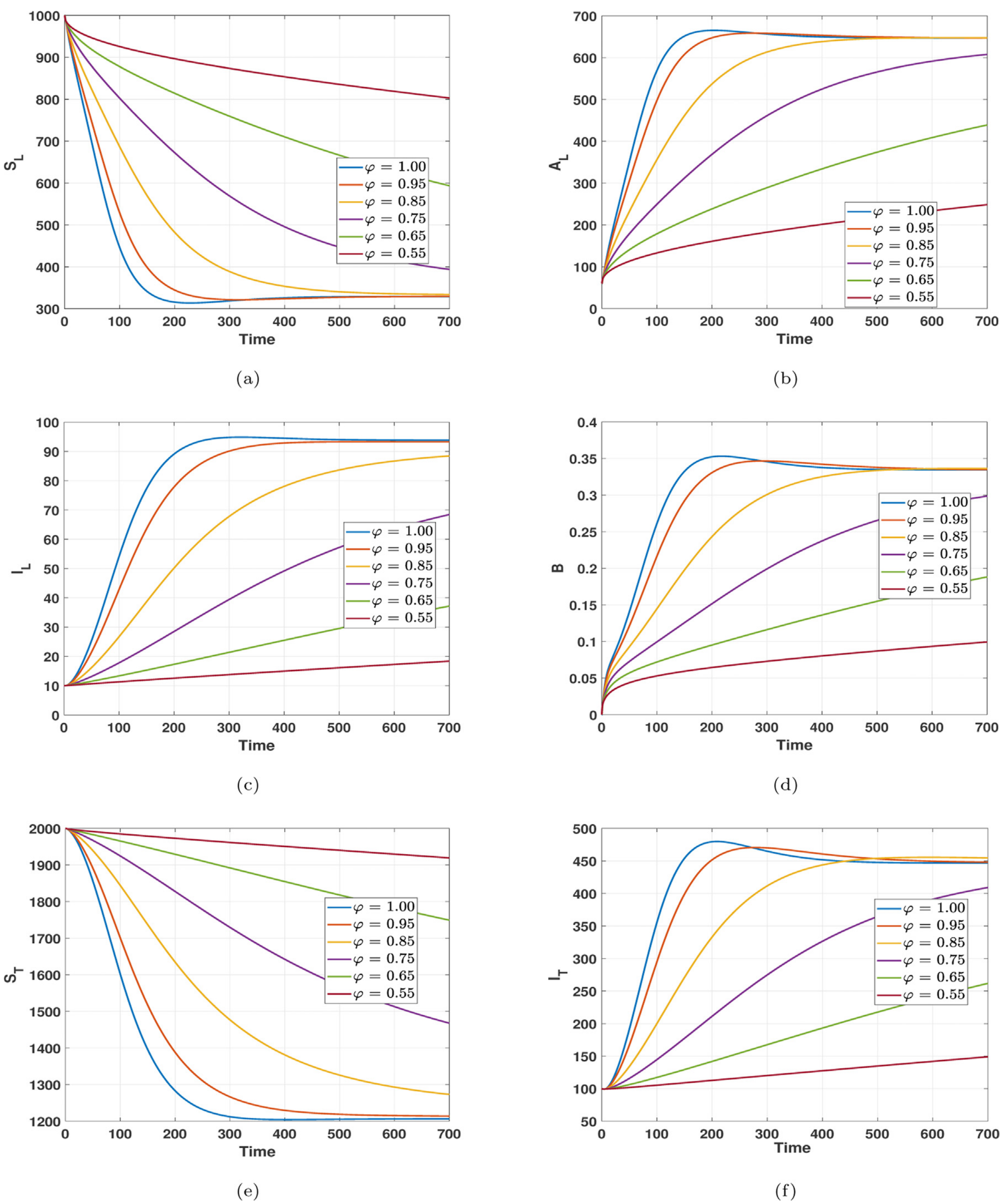
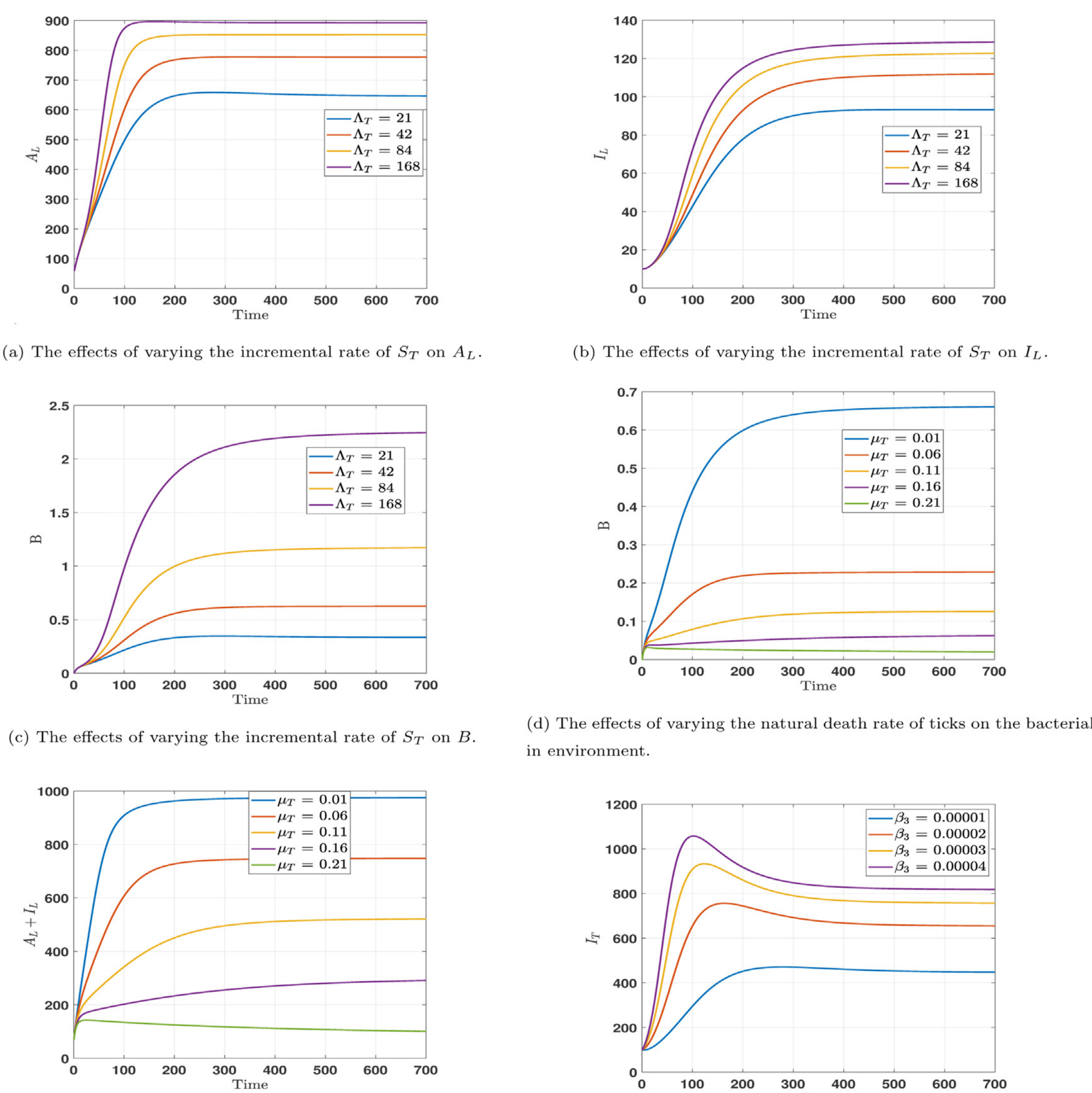


Fig. 10. Numerical simulation under Atangana-Baleanu differential operator with different fractional order, φ.

$$
\begin{aligned}
B_{n+1} = B(0) + \frac{(1-\varphi)}{G(\varphi)} Z(t_n, B(t_n)) + \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \\
\times \left(\frac{h^{\varphi} Z(t_k, B_k)}{\Gamma(\varphi+2)} ((n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi)) \right) \\
- \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \left(\frac{h^{\varphi} Z(t_{k-1}, B_{k-1})}{\Gamma(\varphi+2)} ((n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi)) \right),
\end{aligned}
$$

$$
\begin{aligned}
S_{T_{n+1}} = S_T(0) + \frac{(1-\varphi)}{G(\varphi)} Z(t_n, S_T(t_n)) + \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \\
\times \left(\frac{h^{\varphi} Z(t_k, S_{Tk})}{\Gamma(\varphi+2)} ((n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi)) \right) \\
- \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \left(\frac{h^{\varphi} Z\big(t_{k-1}, S_{T_{k-1}}\big)}{\Gamma(\varphi+2)} ((n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi)) \right),
\end{aligned}
$$



(a) The effects of varying the incremental rate of S_T on A_L.

(b) The effects of varying the incremental rate of S_T on I_L.

(c) The effects of varying the incremental rate of S_T on B.

(d) The effects of varying the natural death rate of ticks on the bacterial in environment.

(e) The effects of varying the natural death rate of ticks on infected livestock.

(f) Varying the transmission rate from $(A_L + I_L)$ to ticks.

Fig. 11. Numerical simulation under Atangana-Baleanu differential operator with fractional order, $\varphi = 0.95$.

$$
\begin{aligned}
I_{T_{n+1}} &= I_T(0) + \frac{(1-\varphi)}{G(\varphi)} Z(t_n, I_T(t_n)) + \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \\
&\quad \times \left(\frac{h^{\varphi} Z(t_k, I_{Tk})}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi}(n-k+2+\varphi) - (n-k)^{\varphi}(n-k+2+2\varphi) \right) \right) \\
&\quad - \frac{\varphi}{G(\varphi)} \sum_{k=0}^{n} \left(\frac{h^{\varphi} Z\big(t_{k-1}, I_{T_{k-1}}\big)}{\Gamma(\varphi+2)} \left((n+1-k)^{\varphi+1} - (n-k)^{\varphi}(n-k+1+\varphi) \right) \right).
\end{aligned} \tag{142}
$$

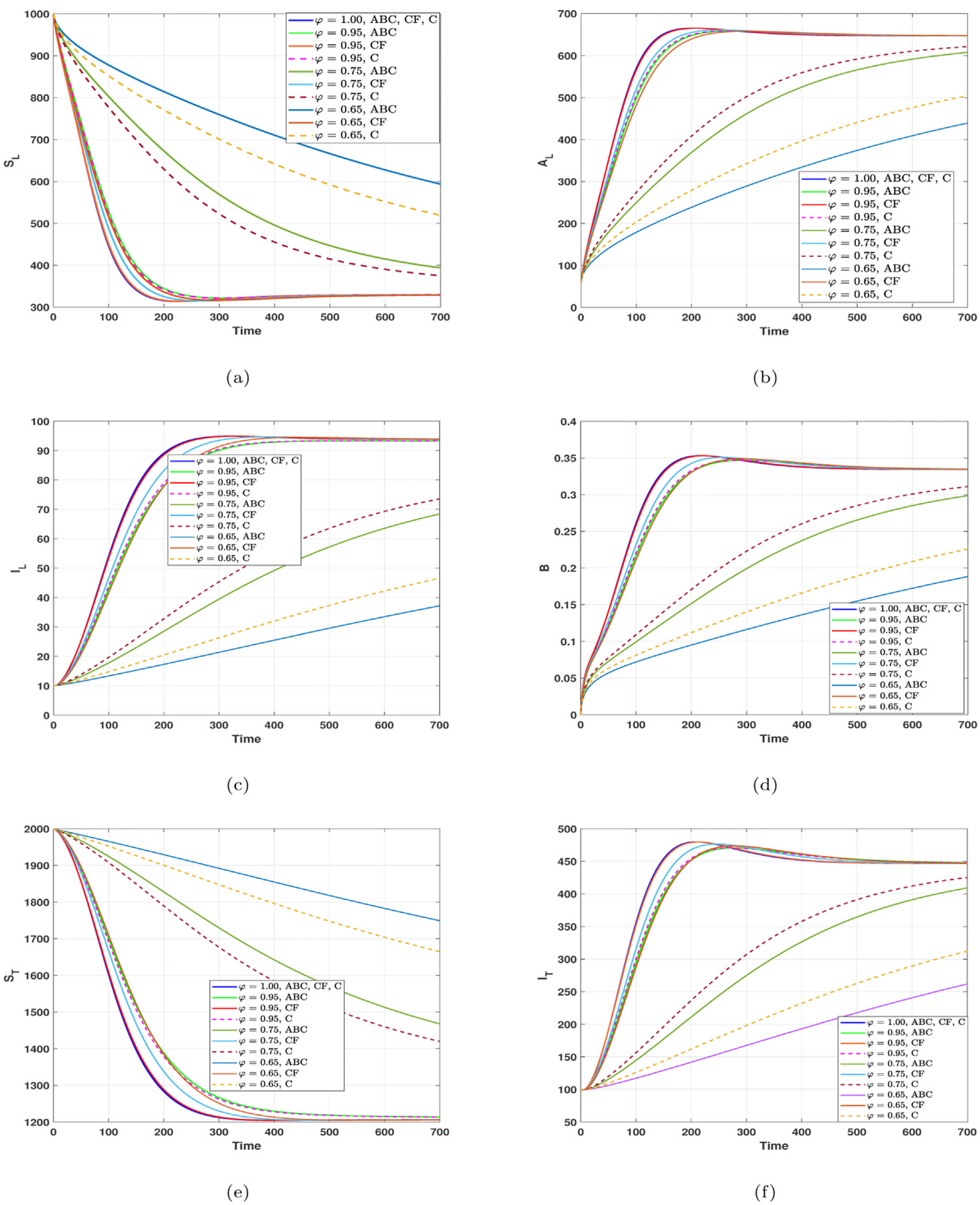


Fig. 12. Numerical simulation under Caputo, Caputo-Fabrizio and Atangana-Baleanu operator with integer order and different fractional order, $\varphi = 1, 0.95, 0.75, 0.65$.

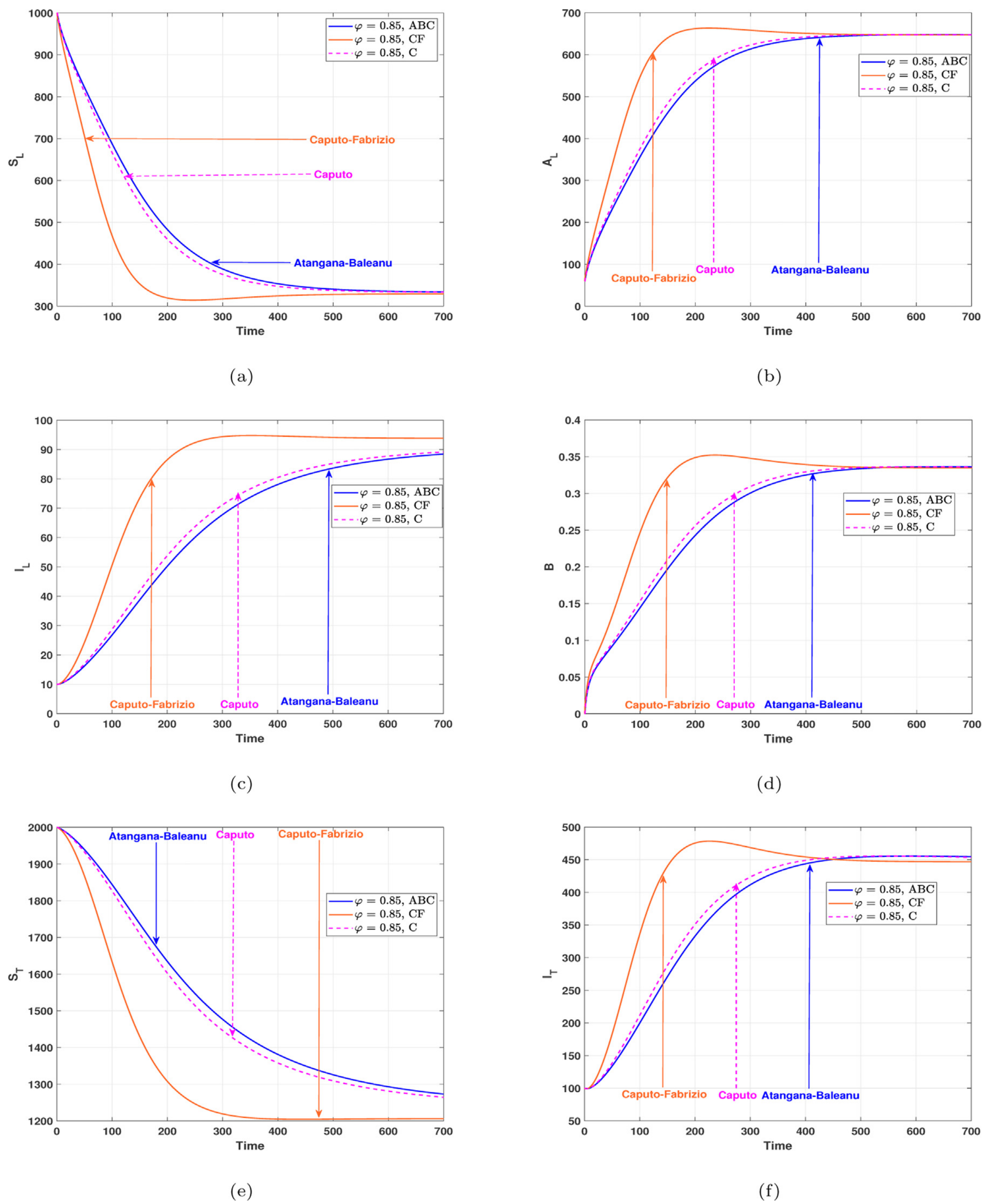


Fig. 13. Numerical simulation under Caputo, Caputo-Fabrizio and Atangana-Baleanu operator with fractional order, $\varphi = 0.85$.

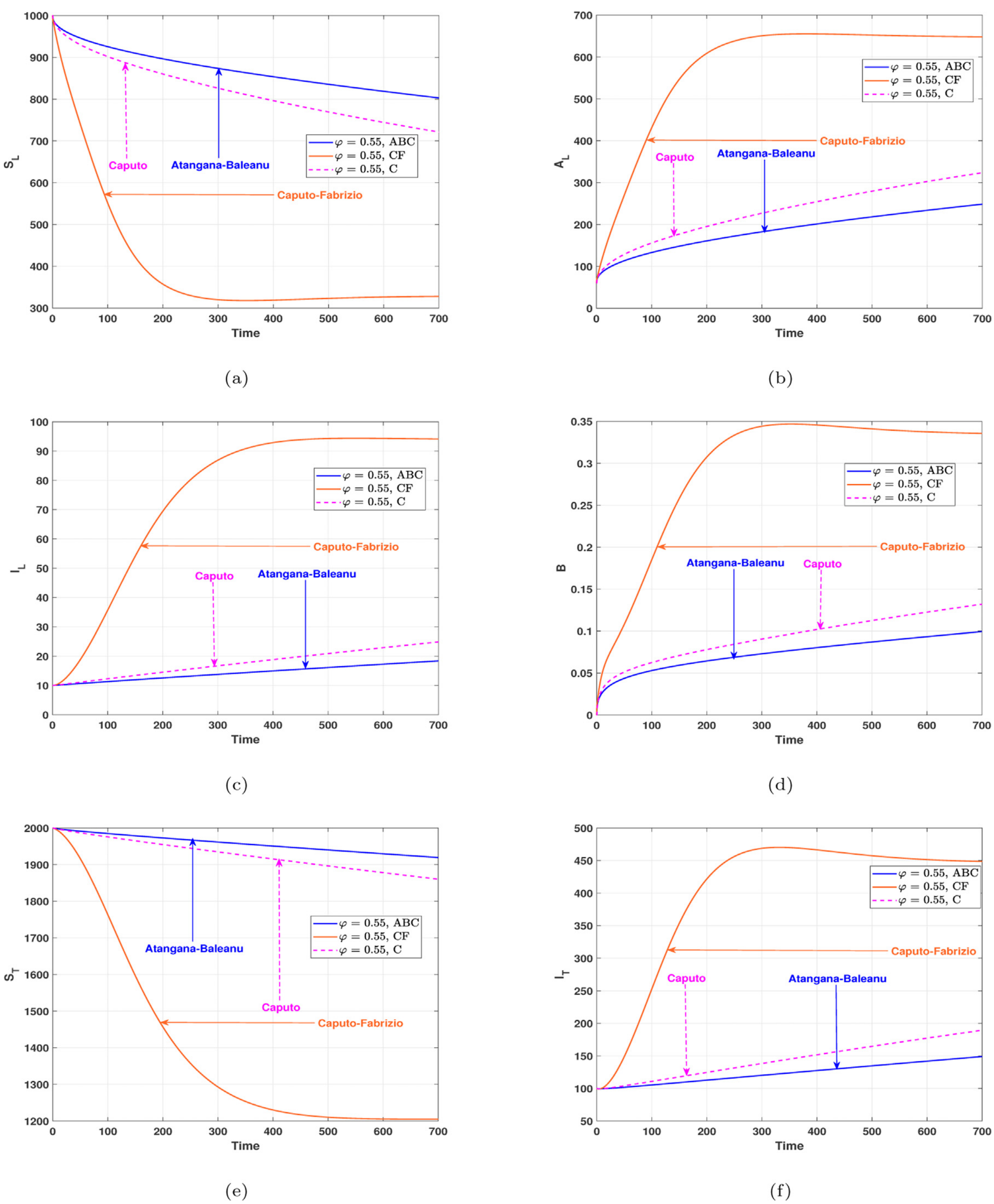


Fig. 14. Numerical simulation under Caputo, Caputo-Fabrizio and Atangana-Baleanu operator with fractional order, $\varphi = 0.55$.

Table 1
Interpretation of model parameters with values used in numerical simulations, which are subject to time in days.

Parameter	Interpretation	Range	Source	Values used
Λ_L	Incremental rate of S_L	$\mu_L \times N_L$	Assumed	22
β_1	Environmental transmission	0.001–0.0943	[66]	0.001
β_2	Shedding rate from infected livestock	0.00003–0.0400	[66,67]	0.00003
β_3	Transmission rate from $(A_L + I_L)$ to ticks	0.00001	Assumed	0.00001
β_4	Shedding rate from infected ticks	0.0001	Assumed	0.0001
β_5	Transmission rate from I_T to S_L	0.0001	Assumed	0.0001
Λ_T	Incremental rate of S_T	$\mu_T \times N_T$	Assumed	21
μ_L	Natural death rate of livestock	0.0002-0.02	[7]	0.02
μ_T	Natural death rate of ticks	0.01	Variable	0.01
ε	Natural decay of *Coxiella burnetii* from the environment	0.0083 – 0.2000	[7]	0.2000
α_L	Progression rate from A_L to I_L	0.0029	[7]	0.0029

7. Numerical simulations

Here, we apply the above numerical schemes to obtain numerical solutions for the three fractional derivatives, using the fractional order $\varphi \in (0,1]$. For illustration purposes we used the following initial conditions: $S_L(0) = 1000,\ A_L(0) = 60,\ I_L(0) = 10,\ B = 0,\ S_T(t) = 2000,\ I_T(0) = 100$. Together with the parameter values in Table 1. We employ the Adams–Bashforth method for our numerical simulations. The following fractional values [0.95,0.85,0.75,0.65,0.55] were used to study the effect of the fractional-order on our model. In Figs. 6, 8 and 10, we represent the numerical solutions of the Q fever model using Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional models, respectively. We noticed in Fig. 6(a) and (c) that the trajectory of the fractional-order 0.95 converges to the endemic equilibrium point as same as the integer-order, whiles the trajectories of the fractional-orders 0.85, 0.75, 0.65 and 0.55 takes different paths respectively. In Fig. 6(b) and (d), we noticed that the trajectory of the fractional-order 0.95 and 0.85 converges to the endemic equilibrium point as same as the integer-order, whiles the trajectories of the fractional-orders 0.75, 0.65 and 0.55 takes different paths respectively. In Fig. 6(e) we noticed that the trajectories of the fractional-orders take different directions and do not converge to the endemic equilibrium point when the fractional-order is 1. In Fig. 6(f) we noticed that trajectory of the fractional-order 0.95 links to the endemic equilibrium point as same as the integer-order, but the fractional-order 0.85 has a higher endemic equilibrium point than the integer-order. Also, in Fig. 6(f) we see that the fractional-order 0.75, 0.65 and 0.55 take different paths with different converging points. The Caputo-Fabrizio numerical simulations in Fig. 9 indicate that, in all the subplots Fig. 9(a)–(f), the fractional-orders converge to the same actual endemic equilibrium point at the end of the simulation time.

The Atangana–Baleanu numerical simulations in Fig. 10, we notice that in Fig. 10(a), (c) and (f) that the trajectories of the fractional-order 0.95 converges to the endemic equilibrium point as same as the integer-order, whiles the trajectories of the fractional-orders 0.85, 0.75, 0.65 and 0.55 converge respectively to different points at the end of the simulation time. In Fig. 10(b) and (d), we notice that the trajectory of the fractional-order 0.95 and 0.85 converges to the endemic equilibrium point as same as the integer-order, whiles the trajectories of the fractional-orders 0.75, 0.65 and 0.55 takes different paths respectively. In Fig. 10(e) we notice that the trajectories of the fractional-orders take different paths and do not converge to the same endemic equilibrium point when the fractional-order is 1. Figures 7, 9 and 11 shows the trajectory effects of varying Λ_T, μ_T and β_3 on the infected classes A_L, I_L, I_T and the bacterial load in the environment, B. It is obvious to see in Fig. 7 that each trajectory has a special rate of convergence as the parameters Λ_T, μ_T and β_3 are varied with a fractional-order of 0.95. We also notice that the bacterial get extinct from the environment and the infected livestock population at the end simulation time, when $\mu_T = 0.16$ and $\mu_T = 0.21$ in the Caputo sense, but not in Atangana–Baleanu sense as shown in Fig. 11. In Fig. 12, we compared the three different operators when the fractional-orders are $\varphi = 1, 0.95, 0.75, 0.65$; it is evident that the Caputo and Atangana-Baleanu operators converge to the integer endemic equilibrium point at the end of the simulation time when $\varphi = 0.95$. At the same time, the trajectories for the Caputo-Fabrizio converge to the same endemic equilibrium point when the fractional-order is perturbed.

In Fig. 12 we further notice that the Atangana-Baleanu operator captures more susceptibilities than the other operators when the fractional-order is 0.75 and 0.65, respectively. In Figs. 13 and 14 we show the comparison of the three operators when the fractional-order is 0.85 and 0.55 respectively. It is obvious to see that the three operators converge at the end of the simulation time when $\varphi = 0.85$ for Fig. 13(a), (b), and (d), and distinct path for Fig. 13(c), (e) and (f). In Fig. 14 it is easy to see that all the three operators have distinct paths when the fractional-order is 0.55.

8. Conclusion

In this study, we have developed nonlinear dynamical models for Q fever to represent the disease dynamics in livestock and ticks together with the effect of the bacterial in the environment using integer and non-integer order models. Authors in previous mathematical studies on Q fever did not consider the impact of ticks (see [7,45]). Hence, our study is a novel contribution to the dynamical analysis of Q fever. From the study, we obtain the positivity and boundedness of our model equations. We obtained the disease-free equilibrium and further used it to get the expression of the basic reproduction number. We noticed a multiplicity of endemic equilibrium when the basic reproduction number is less than unity, thus resulting in the existence of backward bifurcation. We showed instances where the backward bifurcation can be removed both theoretically and numerically. We proposed three fractional order differential models to capture the memory aspect of disease spread using three fractional derivative operators namely Caputo, Caputo-Fabrizio, and Atangana-Baleanu in Caputo sense. The existence and uniqueness of these three newly developed epidemiological models are demonstrated using the Banach fixed point theorem. The numerical solutions showed that each fractional operator captures the disease spread in nature differently but sometimes converge to the integer-order, $\varphi = 1$.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Joshua Kiddy K. Asamoah: Conceptualization, Methodology, Formal analysis, Writing – review & editing. **Eric Okyere:** Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing. **Ernest Yankson:** Formal analysis, Writing – original draft, Writing – review & editing. **Alex Akwasi Opoku:** Formal analysis, Writing – original draft, Writing – review & editing. **Agnes Adom-Konadu:** Formal analysis, Writing – original draft, Writing – review & editing. **Edward Acheampong:** Formal analysis, Writing – original draft, Writing – review & editing. **Yarhands Dissou Arthur:** Writing – review & editing.

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