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TRANSPORTATION WITH VOLUME DISCOUNT

A CASE STUDY OF MULTI-PLAN LIMITED

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DECLARATION

I hereby declare that this submission is my own work towards the Master of Science degree and that, to the best of my knowledge it contains no material previously published by another person nor material which has been accepted for award of any other degree of the university except where due acknowledgement has been made in the text.

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ABSTRACT

Network models and integer programming are well known variety of decision making problems. A very useful and widespread area of application is the management and efficient use of scarce resources to increase productivity. These applications include operational problems such as the distributions of goods, production scheduling and machine sequencing, and planning problems such as capital budgeting facility allocation, portfolio selection, and design problems such as telecommunication and transportation network design. The transportation problem, which is one of network integer programming problems is a problem that deals with distributing any commodity from any group of 'sources' to any group of destinations or 'sinks' in the most cost effective way with a given 'supply' and 'demand' constraints. Depending on the nature of the cost function, the transportation problem can be categorized into linear and nonlinear transportation problem. We applied Karush-Kuhn-Tucker (KKT) optimality algorithm to solve our problem.

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CHAPTER ONE

1.0 INTRODUCTION

To ensure the procurement function is aligned with the organization's long-term objectives such as savings and profitability, many organizations have transitioned to strategic sourcing to acquire their raw materials and service requirements. Strategic sourcing should be a systematic and comprehensive process to determine the procurement plan that minimizes linked costs in the supply chain, and maximizes the value of purchased goods and services. This is done by determining the best suppliers for needed goods or services and the conditions under which to use their services in a way that achieves the best value and contributes to the organization's long-term objectives. To fully achieve the alignment of the procurement process with the organizational objectives, transportation should be an essential element in the strategic sourcing plan. The reason is that strategic sourcing and transportation are interrelated processes and one of them cannot be optimized in isolation of the other. For example, if the delivery cost of the purchased items was not incorporated in the model, these costs can tremendously increase the overall costs.

When considering transportation, various considerations are apparent. This consideration includes port selection, inland movement, and port to port carrier selection and delivery movement. In addition to these transportation concerns, distribution-related considerations must be also given attention to such as packing/packaging, transit insurance, terms of sale, import duties, handling/loading and method of financing. Nevertheless, even freight companies projecting large volume movements can encounter serious transportation problem in organizing for distribution. Understanding these transportation problems especially that affects shipping costs is critical. Volume discount, more specifically, targets shipping costs and in minimizing the latter, volume discount must be acquired. In this study, the transportation problems encountered by freight companies will be explored especially those that affect the shipping cost.

In this chapter, we shall give a historical background of the transportation problem model as an integer programming problem; a brief description of the problem statement of the thesis is also presented as well as the objectives, the methodology, the justification and the organization of the thesis.

1.1 BACKGROUND OF STUDY

Contemporary research in logistics management relies on an increased recognition that an integrated plan requires coordinating different functional specialties within a system in keeping with this trend; we focus on the integration of production, inventory and transportation arising in a supplier- retailer logistic system. In the general inventory models, costs of such issues are usually accounted according to the following assumptions: the production cost is proportional to the quantity of products produced. The ordering cost, which refers to the charge for preparing of production, is independent of the quantity ordered. The inventory cost (shortage cost) is proportional to the quantity of products stored (out of order) as well as the duration for which these items are stored (stock out). When products are delivered from the supplier to the consumer, transportation costs are incurred. In the traditional economic order quantity (EOQ) model, the transportation cost is calculated together with the production cost, or with the ordering cost. However, in a practical logistic system, the transportation cost of a vehicle includes both of the fixed cost and the variable cost. The fixed cost, which is considered to be a constant sum in each period, refers to some necessary expenses such as parking fare and rewards to the driver. As to the variable cost, it depends mainly on the oil consumed, which is related directly to the distance travelled. In short, considering the real condition, it is unreasonable to assume that the transportation cost is proportional to the quantity delivered or is a constant sum.

Transportation models provide a powerful framework to meet this challenge. They ensure the efficient movement and timely availability of raw materials and finished goods. Transportation problem is a linear programming problem stemmed from a network structure consisting of a finite number of nodes and arcs attached to them.

When the transportation plan is made up, the volume discounts brought by large quantities of transportation should not be pursuited excessively. As this would bound to increase inventory costs throughout the system, also when the inventory strategy is determined, transportation costs cannot be dealt with as a fixed fee, but as a variable cost directly impacting on transportation frequency and inventory distribution. Under the prerequisite of comprehensively balancing the transportation costs and inventory costs, the objectives that Inventory-Transportation Integrated Optimization problem (ITIO) are to optimize the logistics system, reduce logistics costs, and determine the transportation program and inventory strategy of the system.

One of the earliest and most fruitful applications of linear programming techniques has been the formulation and solution of the transportation problems as a linear programming problem.

The basic transportation problem was originally developed by Hitchcock (1941). The objective of the transportation problem is to determine the optimal amounts of a commodity to be transported from various supply points to various demand points so that the total transportation cost is a minimum. The unit costs i.e. the cost of transporting one unit from a particular supply point to a particular demand point, the amounts available at the supply points and the amounts required at the demand points are the parameters of the transportation problem.

Kantorovich (1939) showed that a class of problems closely related to the classical transportation case has remarkable variety of applications concerned typically with the allotment of task to machines whose costs and rates of production vary by task and machine type, Dantzig (1963). He gave a useful but incomplete algorithm for solving such problems. In 1942, the author wrote a mathematical paper concerned with a continuous version of the transportation problem, and in 1948, he authored an implicational sturdy, jointly with Gavurin, on the capacitated transportation problem.

The standard form of the problem was first formulated, along with a constructive solution, by Hitchcock. His paper, "The Distribution of a Product from several Sources to Numerous Localities", (Hitchcock, 1941), sketched out the partial theory of a technique foreshadowing the simplex method; it did not exploit special properties of a transportation problem except in finding starting solutions. This failed to attract much interest.

Still another investigator, Koopmans (1949), as a member of the combined shipping board during world war II, became concerned with using solutions of the transportation problem to help reduce overall shipping times, for the shortage of cargo ships constituted a critical bottleneck, Flood (1954).

In 1947, Koopmans began to spearhead research on the potentialities of linear programs for the study of problems in economics. His historic paper, "Optimum Utilization of Transportation System", (Koopmans, 1949), was based on his wartime experience. Because of this and the work done earlier by Hitchcock, the classical case is often referred to as the Hitchcock-Koopmans transportation problem.

Another, whose work anticipated the recent era of development in linear programming, was Egervary (1931), a mathematician. His 1931 paper considered the problem of finding a permutation of ones in a matrix composed of zero and one elements, (Gauss, 1969). Based on this investigation, Kuhn developed an efficient algorithmic method for solving assignment problems (Kuhn and Tucker, 1956). Kuhn's approach, in its turn, underlies the Ford-Fulkerson Method for solution of the classical transportation problem, (Ford and Fulkerson, 1956).

Industrial development today depends on the efficiency of the transportation and logistics activities. Transportation can be described as a flow of materials between two organizations. The first formulation and discussion of a planar transportation model was introduced by Hitchcock (1941).

The objective was to find the way of transporting homogeneous product from several sources to several destinations so that the total cost can be minimized. The Transportation Problem (TP) is well known as one of the practical network problems and there are many investigations of evolutionary approaches to solve the varieties of transportation problem.

In the real-life applications, it is often that the problems to be solved have a large-scale and has to satisfy several other additional constraints. For example, Sun (1998) introduced the transportation problem with exclusionary side constraint. To solve this problem, he developed a Tabu search procedure. Another similar problem called the transportation problem with nonlinear side constraint was introduced by Cao (1995). In this model, the TP is extended satisfy additional constraints in which source centres cannot serve two different destinations which are given as the side constraint simultaneously. With this side constraint the difficulty of the problem increase drastically, while its applications to the real life also increase significantly.

Network models and integer programs are well known variety of decision problems. A very useful and widespread area of application is the management and efficient use of scarce resources to increase productivity. These applications include operational problems such as the distributions of goods, production scheduling and machine sequencing, and planning problems such as capital budgeting facility allocation, portfolio selection, and design problems such as telecommunication and transportation network design.

The transportation problem which is one of the network integer programming problems is a problem that deals with distributing any commodity from any group of sources to any group of destinations or sinks in the most cost effective way with a given supply and demand constraints.

Depending on the nature of the cost function, the transportation problem can be categorized into linear and nonlinear transportation problem.

In the linear transportation problem (ordinary transportation problem) the cost per unit commodity shipped from a given source to a given destination is constant, regardless of the amount shipped. Also it is always supposed that the mileage (distance) from every source to every destination is fixed. To solve such transportation problem we have the streamlined simplex algorithm which is very efficient. However, in actuality we can see at least two cases that the transportation problem fails to be linear.

First, the cost per unit commodity transported may not be fixed for volume discounts sometimes are available for large shipments. This would make the cost function either piecewise linear or just separable concave function. In this case the problem may be formulated as piecewise linear or concave programming problem with linear constraints.

In special conditions such as transporting emergency materials when natural calamity occurs or transporting military supplies during war time, where carrying network may be destroyed, mileage from some sources to some destinations are no longer definite. So the choice of different measures of distance leads to nonlinear (quadratic, convex, concave...) objective function.

In the above cases, solving the transportation problem is not as simple as that of the linear one.

In our work, solution procedures to the generalized transportation problem taking nonlinear cost function as a result of volume discounts are investigated. In particular, the nonlinear transportation problem considered in this research is stated as follows; we are given a set of n sources of commodity with known supply capacity and a set of m destinations with known demands.

The function of transportation cost, nonlinear, and differentiable for a unit of product from each source to each destination.

We are required to find the amount of product to be supplied from each source (may be market) to meet the demand of each destination in such a way as to minimize the total transportation cost.

1.2 PROBLEM STATEMENT

This thesis seeks to solve a transportation problem with volume discount. The costs of goods are determined by factors such as: the costs of raw materials, labour, and transport. When cost of raw materials rises, so does the cost of the goods. Transportation cost also affects the pricing system. It is assumed that the cost of goods per unit shipped from a give source to a given destination is fixed regardless of the volume shipped.

But in actuality the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal costs of shipping one unit might follow a particular pattern.

Our focus will be to develop a mathematical model using optimization techniques to close the demand and supply gap by discounting so as to minimize total transportation cost.

1.3 OBJECTIVE

The goal of this research is to minimize the total transportation cost with volume discount.

1.4 METHODOLOGY

This research seeks to apply the existing general nonlinear programming algorithms to solve our problem. The research strategy that the study will utilize is the descriptive method. A descriptive research intends to present facts concerning the nature and status of a situation, as it exists at the time of the study (Creswell, 1994). It is also concerned with relationships and practices that exist, beliefs and process that are ongoing, effects that are being felt, or trends that are developing (Best, 1970). In addition, such approach tries to describe present conditions, events or systems based on the impressions or reactions of the respondents of the research (Creswell, 1994).

In this study, primary and secondary research will be both incorporated. The reason for this is to be able to provide adequate discussion for the readers that will help them understand more about the issue and the different variables that involve with it. The primary data for the study will be represented by the survey results that will be acquired from the respondents. On the other hand, the literature reviews to be presented in the second chapter of the study will represent the secondary data of the study. The secondary sources of data will come from published articles from books, journals, theses and related literature.

Different algorithms to the various transportation problems will be presented.

1.5 JUSTIFICATION

Until recently, heavy trucks could load up to any capacity without limit. These trucks normally exceed the average loading capacity of the truck. This was partially due to high transportation cost. Drivers and transport owners together with transport users had to find a way of compensating for the high cost of transport by increasing the truck load so as to maximize profit. This had ripple effect on the state as a whole: increase road accidents, destruction of roads, pressure is also put on the vehicle, and longer time being spent on the road before getting destination. There is also the effect of increased cost of goods thereby increasing inflation. This has driven the attention of the stakeholders to find a lasting solution to the problems. There is therefore the need to determine the maximum loading capacity of trucks.

The purpose of this work is to find out whether given volume discounts on transportation costs could minimize total transportation cost thereby increasing total revenue of producers and retailers as well as solving some of the aforementioned problems associated with transportation.

1.6 LIMITATION OF THE STUDY

Our study is limited to a nonlinear transportation problem with concave shape which is as a result of discount given on volume of goods transported. Unlike the linear transportation problems, maximization of profit is realized with discounts on large volumes, which means the determination of the best transportation route that would lead to low transportation cost and the effective transportation of these goods.

1.7 ORGANIZATION OF THE THESIS

In chapter one, we presented a background study of transportation problem, objectives, methodology, justification and limitations of the study.

In chapter two, related works in the field of transportation problems will be discussed.

Chapter three presents various existing algorithms for solving the various transportation problems.

Chapter four presents data collection and analysis of the study.

Chapter five is devoted for the conclusion and recommendations of the study.

CHAPTER TWO

LITERATURE REVIEW

It is known to be real that the per unit transportation cost from a specific supply source to a given demand sink is dependent on the quantity shipped, so that there exist finite intervals for quantities where price breaks are offered to customers. Thus, such a quantity discount results in a non-convex, piecewise linear functional. Balachandran and Avinoam (2006) presented a model with an algorithm to solve this problem. This algorithm, with minor modifications, is shown to encompass the "incremental" quantity discount and the "fixed charge" transportation problems as well. It is based upon a branch-and-bound solution procedure. The branches lead to ordinary transportation problems, the results of which are obtained by utilizing the "cost operator" for one branch and "rim operator" for another branch. Suitable illustrations and extensions were also provided.

Goossens and Maas (2007) studied the procurement problem faced by a buyer who needs to purchase a variety of goods from suppliers applying a so-called total quantity discount policy. This policy implies that every supplier announces a number of volume intervals and that the volume interval in which the total amount ordered lies determines the discount. Moreover, the discounted prices apply to all goods bought from the supplier, not only to those goods exceeding the volume threshold. The author's referred to this cost-minimization problem as the TQD problem. The authors give a mathematical formulation for this problem and argue that not only it is NP-hard, but also that there exists no polynomial-time approximation algorithm with a constant ratio (unless P = NP). Apart from the basic form of the TQD problem, the authors described three variants. In a first variant, the market share that one or more suppliers can obtain is constrained. Another variant allows the buyer to procure more goods than strictly needed, in order to reach a lower total cost. In a third variant, the number of winning suppliers is limited. The authors showed that the TQD problem and its variants can be solved by solving a series of min-cost flow problems. Finally, they investigated the performance of three exact algorithms (min-cost flow based branch-and-bound, linear programming based branch-and-bound, and branch-and-cut) on randomly generated instances involving fifty (50) suppliers and hundred (100) goods. It turns out that even the large instances of the basic problem are solved to optimality within a limited amount of time. However, the authors found that different algorithms perform best in terms of computation time for different variants.

Discount in transportation cost on the basis of transported amount is extended to a solid transportation problem. In a transportation model, the available discount is normally offered on items/criteria, etc., in the form AUD (all unit discounts) or IQD (incremental quantity discount) or combination of these two.

Ojha et al., (2009) considered a transportation model with fixed charges and vehicle costs where AUD, IQD or combination of AUD and IQD on the price depending upon the amount is offered and varies on the choice of origin, destination and conveyance. To solve the problem, Genetic Algorithm (GA) based on Roulette wheel selection, arithmetic crossover and uniform mutation has been suitably developed and applied. To illustrate the models, numerical examples have been presented. Here, different types of constraints are introduced and the corresponding results are obtained. To have better customer service, the entropy function is considered and it is displayed by a numerical example. To exhibit the efficiency of GA, another method-weighted average method for multi-objective is presented, executed on a multi-objective problem and the results of these two methods are compared.

Logistics managers often encounter incremental quantity discounts when choosing the best transportation mode to use. This could occur when there is a choice of road, rail, or water modes to move freight from a set of supply points to various destinations. The selection of mode depends upon the amount to be moved and the costs, both continuous and fixed, associated with each mode. This can be modelled as a transportation problem with a piecewise-linear objective function. Henig et al., (1997) presented a vertex ranking algorithm to solve the incremental quantity discounted transportation problem. Computational results for various test problems are presented and discussed.

Transportation and production contracts often specify the frequency and volume reserved by the supplier for a particular customer's deliveries. This practice motivated Henig et al., (1997) presented an inventory model embedded in designing a supply contract to study a periodic-review inventory-control model where ordering cost is zero if the order quantity does not exceed a given contract volume and is linear in the excess quantity otherwise.

Xiuli and Paul (2002) studied the same problem but with a fixed cost if the order quantity is above the contract volume. The fixed cost may represent the cost of disruption for the supplier (finding more trucks, arranging extra processing capacity, persuading other customers to wait, etc.) as well as additional administrative costs.

Suppliers may impose such costs simply to induce desired behaviour by buyers. This order-cost function is neither convex nor concave. The classical inventory models with fixed costs are special cases with contract volume zero. The authors partially characterized the optimal policy for this system and develop a simple, effective heuristic policy. The authors also applied the model to a production-control problem in which an incentive is provided for not ordering over a certain quota.

Dries (2003) studied the procurement problem faced by a buyer who needs to purchase a variety of goods from suppliers applying a so-called total quantity discount policy. This policy implies that every supplier announces a number; of volume intervals and that the volume interval in which the total amount ordered lies determines the discount. Moreover, the discounted prices apply to all goods bought from the supplier, not only to those goods exceeding the volume, threshold. The authors referred to this cost-minimization problem as the total quantity discount (TQD) problem. The authors gave a mathematical, formulation for this problem and, argued that not only it is NP-hard, but also that there exists no polynomial time approximation algorithm with a constant ratio (unless P = NP). Apart from the basic form of the TQD problem, we describe four variants. In a first variant, the market share that one or more suppliers can obtain is constrained. Another variant allows the buyer to procure more goods than strictly needed, in order to reach a lower total cost. We also consider a setting where the buyer needs to pay a disposal cost for the extra goods bought. In a third variant, the number of winning suppliers is limited, both in general and per product. Finally, we investigate a multi-period variant, where the buyer not only needs to decide what goods to buy from what supplier, but also when to do this, while considering the inventory costs. The authors showed that the

TQD problem and its variants can be solved by solving a series of min-cost flow problems. Finally, the authors investigated the performance of three exact algorithms on randomly generated instances involving 50 suppliers and 100 goods.

Yves and Tores (2004) described the purchasing decisions faced by a multi-plant company. The suppliers of this company offer complex discount schedules based on the total quantity (rather than cost) of ingredients purchased. The schedules simultaneously account both for corporate purchases and for purchases at the individual plant level. The complexity of the purchasing decisions is further increased due to the existence of alternative production recipes for each final product. We formulate the corresponding cost-minimization problem as a nonlinear mixed 0-1 programming problem. We propose various ways to linearize this formulation, and the authors evaluated the quality of the resulting models on real-world data.

Boris et al., (2009) investigated a model for pricing the demand for a set of goods when suppliers operate discount schedules based on total business value. The authors formulated the buyers' decision problem as a mixed binary integer problem (MIP) which is a generalization of the capacitated facility location problem (CFLP). A branch and bound procedure using lagrangean relaxation and sub gradient optimization is developed for solving large-scale problems that can arise when suppliers' discount schedules contain multiple price breaks. Results of computer trials on specially adapted large benchmark instances of the CFLP, conform that a sub gradient optimization procedure based on Shor and Zhurbenko's r-algorithm, which employs a space dilation strategy in the direction of the difference between two successive sub gradients, can solve such instances efficiently.

Keane et al., (2007) considered a problem, of optimal, order allocation faced for example, by an internet trading agent who seeks to fulfill an order for specified amounts, of several products from a pre-arranged list of suppliers, taking into account availability and price. The authors presented a mixed, integer programming, (MILP) formulation, for the case that suppliers impose, a fixed charge which, is waived, or discounted, on orders above, a certain threshold value. This formulation is extended, to cases where, suppliers operate, a discount, schedule, with multiple price breaks. We show that a modified, capacitated facility location (CFLP) model is appropriate for the general case and outline a solution approach, by Lagrangean, relaxation.

MacKinnon (1975)described a new computational technique for solving spatial economic equilibrium problems which are generalizations of the classical transportation problem. Existing algorithms employ quadratic programming, and they therefore require that demand and supply functions are linear. By contrast, the algorithm of the author can handle nonlinear or even semi-continuous demand and supply relationships. It can also handle non-constant transport costs and other complications. The technique is capable of yielding highly accurate solutions, and it appears to be computationally efficient on problems of reasonable size.

The solid transportation problem is a generalization of the traditional transportation problem in which three kinds of constraint are taken into account instead of two. In general, the three kinds of constraint are understood as source, destination, and transport mode. The fixed charge transportation problem is an extension of the traditional transportation problem in which two kinds of costs, says direct cost and fixed charge, are taken into consideration.

Linzhong and Liang (2007) modelled the fixed charge solid transportation problem with fuzzy data as a chance-constrained programming by using the credibility measure.

The classical transportation problem can be applied in a more general way in practice. Related problems as Multi-commodity transportation problem, Transportation problems with different kind of vehicles, Multi-stage transportation problems, Transportation problem with capacity limit is an extension of the classical transportation problem considering the additional special condition. For solving such problems many optimization techniques (dynamic programming, linear programming, special algorithms for transportation problem etc.) and heuristics approaches (e.g. evolutionary techniques) were developed. Brezina et al., (2007) considered a Multi-stage transportation problem with capacity limit that reflects limits of transported materials (commodity) quantity. Discussed issues are: theoretical base, problem formulation as way as new proposed algorithm for that problem.

Andrew et al., (2003) studied a transportation problem with the minimum quantity commitment (MQC), which is faced by a famous international company. The company has a large number of cargos for carriers to ship to the United States. However, the U.S. Marine Federal Commission stipulates that when shipping cargos to the United States, shippers must engage their carriers with an MQC. With such a constraint of MQC, the transportation problem becomes intractable. To solve it practically, the authors provided a mixed-integer programming model defined by a number of strong facets. Based on this model, a branch-and-cut search scheme is applied to solve small-size instances and a linear programming rounding heuristic for large ones. The authors also devised a greedy approximation method, whose solution quality depends on the scale of the minimum quantity if the transportation cost forms a distance metric. Extensive experiments have been conducted to measure the performance of the formulations and the algorithms and have shown that the linear rounding heuristic behaves best.

Paolo and Daniele (1997) examined the problem of determining an optimal schedule for a fleet of vehicles used to transport handicapped persons in an urban area. The problem is a generalization of the well-known advance-request Pickup and Delivery Problem with Time Windows. Due to the high level of service required by this kind of transport, several additional operational constraints must be considered. The problem is NP-hard in the strong sense, and exact approaches for the solution of real-life problems (typically with hundreds of users to be transported) are not practicable. The authors described a fast and effective parallel insertion heuristic algorithm which is able to determine good solutions for real-world instances of the problem in a few seconds on a personal computer. The authors also presented a Tabu Thresholding procedure which can be used to improve the starting solution obtained by the insertion algorithm. The application of the proposed procedures to a set of real-life instances for the city of Bologna, involving about 300 trips each, is also discussed. The heuristic algorithms obtain very good results compared with the hand-made schedules, both in terms of service quality (all the service requirements are met) and overall cost.

The generalised transportation problem (GTP) is an extension of the linear Hitchcock transportation problem. However, it does not have the unimodularity property, which means the linear programming solution (like the simplex method) cannot guarantee to be integer. This is a major difference between the GTP and the Hitchcock transportation problem. Although some special algorithms, such as the generalized stepping-stone method, have been developed, they are based on the linear programming model and the integer solution requirement of the GTP is relaxed. Ho and Ji (2005) proposed a genetic algorithm (GA) to solve the GTP and a numerical example is presented to show the algorithm and its efficiency.

The classical transportation problem is actually well known both in theory and numerical resolution.

Zitouni and Keraghel (2003) studied the multi-subscripts capacitated transportation problem of axial sum launched by specialists some years ago. The authors work dealt with the capacitated problem with four subscripts for which we have established an existence criterion, an optimality condition and an algorithm of resolution.

Transportation problems (TP) are one of the most prominent fields of application of the mathematical disciplines to optimization and operations research. In general, there are three starting basic feasible solution methods: Northwest Corner, Least Cost Method, and VAM – Vogel's Approximation Method. The three methods differ in the quality of the starting basic solution. Çakmak and Ersöz (2007) studied a problem which showed a new method for starting basic feasible solution to one-criterion-transportation problems: Çakmak Method. This method can be used for balanced or unbalanced one-criterion transportation problems, and gives the basic feasible optimum solution accordingly.

The Solid Transportation Problem arises when bounds are given on three item properties. Usually, these properties are source, destination and mode of transport (conveyance), and may be given in an interval way. Jose and Fernando (1999) studied the solid transportation problems in which the data in the constraint set are expressed in an interval form, i.e. when sources, destinations and conveyances have interval values instead of point values. An arbitrary linear or nonlinear objective function is also considered. To solve the problem, an Evolutionary Algorithm which extends and generalizes other approaches considering only point values, is proposed.

Shu et al., (2010) studied a multistage production–transportation problem for a make-to-order company with outsourcing options at each stage of production. The authors formulated the problem as a multi-commodity network flow problem with piecewise linear cost structures by assuming the less-than-truckload transportation mode and non-linear production cost structure. The authors used polymatroid inequalities to strengthen its linear programming relaxation and present a cutting plane approach to tackle it. The computational results showed that the strong formulation gives a tight lower bound and the cutting plane approach can solve the problem efficiently.

Nonlinear transportation problems may be successfully used to model problems in economics, due to the nonlinear relationship between quantities and the real cost for their transportation. Dangalchev (1997) considered transportation problems with two-piece linear functions on arcs (which may be nonconvex), where the nonlinearity is represented by absolute value functions. A necessary and sufficient condition for local optimality is given and an algorithm for solving these problems was suggested.

Hussein (1998) presented a model that dealt with the complete solutions of multiple objective transportation problems with possibilistic coefficients. The author considered the problem by incorporating possibilistic data into the coefficients of objective functions. A solution concept that is attractive from the standpoint of efficiency is specified. A necessary and sufficient condition for such a solution is established. A relationship between solutions of possibilistic levels is constructed. The parametric analysis is used to decompose the parametric space of the equivalent problem. A numerical example was given to illustrate the aspects of the developed results.

The Solid Transportation Problem arises when bounds are given on three item properties. Usually, these properties are source, destination and type of product or mode of transport, and often are given in a uncertain way. Fernando et al., (1975) studied a problem which dealt with two of the ways in which uncertainty can appear in the problem: Interval Solid Transportation Problem and Fuzzy Solid Transportation Problem. The first arises when data problem are expressed as intervals instead of point values, and the second when the nature of the information is vague. Both models are treated in the case in which the uncertainty affects only the constraint set. For interval case, an auxiliary problem is obtained in order to find a solution. This auxiliary problem is a standard solid transportation problem which can be solved with the efficient methods existing. For fuzzy case, a parametric approach which makes it possible to find a fuzzy solution to the former problem is used.

As a kind of particular programming problem, transportation problem attracts much attention in many fields, such as energy development, materials management, etc.

Fachao and Wang (2010) presented a model which, after analyzing the essence of stochastic programming and the deficiencies of existing methods, proposed a quasi-linear pattern based on expectation and variance for the satisfaction of the random constraints. Give a stochastic programming model (compound quantification model) with good operability, and establish its corresponding model in stochastic transportation problem. Its performance is discussed through an example. All these indicate the compound quantification model generalizes existing methods, and can solve the stochastic transportation problems with unknown distribution of the random variable under random environment. It is worthy to point out the solution reflects the consciousness of the decision maker, so it enriches methods of stochastic programming.

The conventional transportation problems model requires the parameters to be known as constants. In the real-life, however, the parameters are seldom known exactly and have to be estimated. Interval transportation problems are models in which some or all of its parameters including variables are in the interval forms. It is known that interval programming is one of the tools to tackle the uncertainty in the mathematical programming models. Ismail and Herry (2009) presented a modified simplex method for solving transportation problems with interval numbers as coefficients and values of its variables are also in the form of intervals.

Multi-modal transportation is a logistics problem in which a set of goods have to be transported to different places, with the combination of at least two modes of transport, without a change of container for the goods. Jose et al., (2003) presented TIMIPLAN, a system that solves multi-modal transportation problems in the context of a project for a big company. The authors combined Linear Programming (LP) with automated planning techniques in order to obtain good quality solutions. The direct use of classical LP techniques is difficult in this domain, because of the non-linearity of the optimization function and constraints; and planning algorithms cannot deal with the entire problem due to the large number of resources involved. The authors proposed a new hybrid algorithm, combining LP and planning to tackle the multi-modal transportation problem, exploiting the benefits of both kinds of techniques. The system also integrates an execution component that monitors the execution, keeping track of failures and replans if necessary, maintaining most of the plan in execution. We also present some experimental results that show the performance of the system.

Adams et al., (2006) studied a transportation problem to discover a minimum cost transportation system for the banana industry of the PaulistaLitoral. The PaulistaLitoral is the most important banana producing region in Brazil and the banana industry is the most important element in the agricultural economy of the region. Most of the bananas grown in this area are sold in either Sao Paulo or Buenos Aires. The location pattern of production and the nature of the markets are described, and the transportation system linking production and markets is examined.

The present transportation system is described; minimum cost solutions are calculated using linear programming; and then, the actual and optimal systems were compared. It was concluded that: (i) the costs of the present system are close to those of the optimal solution, (ii) the industry is in a chaotic state and in need of regulation, (iii) the presently available statistics are of such dubious reliability that the descriptions and solutions presented can only be considered tentative estimates subject to revision and, (iv) the approach used for calculating the minimum cost solutions could be useful to a future regulating agency if more accurate statistics become available.

Many transportation problems are such that, when origins and destinations are suitably indexed, the cost matrix contains elements along the main diagonal, a band above it, and a band below it, while the other elements of the cost matrix are infinite. Lev (1970) developed a procedure which yields optimal solution to such tri-diagonal problems in n steps for a n-origin, n-destination problem. A second model has been solved for a tri-diagonal and a coupling column of the cost matrix. A third model, a four-diagonal one, has been partially solved. The author suggested and showed a method to solve any other model which is close to a tri-diagonal one, by Benders' Algorithm. The algorithm presented here works by eliminating all of diagonal variables in terms of the diagonal ones, and sub sequentially models and small linear programming problems.

The dynamic transportation problem is a transportation problem over time. That is, a problem of selecting at each instant of time, the optimal flow of commodities from various sources to various sinks in a given network so as to minimize the total cost of transportation subject to some supply and demand constraints. While the earliest formulation of the problem dates back to 1958 as a problem of finding the maximal flow through a dynamic network in a given time, the problem has received wider attention only in the last ten years. During these years, the problem has been tackled by network techniques, linear programming, dynamic programming, combinational methods, nonlinear programming and finally, the optimal control theory. James and Suresh (1980) presented a survey of the various analyses of the problem along with a critical discussion, comparison, and extensions of various formulations and techniques used. The survey concluded with a number of important suggestions for future work.

Sivri et al., (2010) studied the transportation problem of minimizing the ratio of two linear functions subject to a set of linear equations and non-negativity conditions on the variables (or constraints of the classical transportation problem). The authors extended the transportation problem with the linear objective function to the transportation problem with the linear fractional objective function and we propose a new algorithm in order to obtain an initial solution for this problem which is similar to Vogel's approximation method in the classical transportation problem and then we construct the optimality conditions for the transportation problem with the linear fractional objective functions.

Peerayuth and Saeree (2007) studied two classes of the bottleneck transportation problem with an additional budget constraint. An exact approach was proposed to solve both problem classes with proofs of correctness and complexity. Moreover, the approach was extended to solve a class of multi-commodity transportation network with a special case of the multi-period constrained bottleneck assignment problem.

In the real-life applications, frequently one may be faced with transportation problems that quantities may not be known in precise manner. The supplies and demands may be uncertain due to some uncontrollable factors. Nuran GÜZEL (2007)presented an algorithm for solving fuzzy transportation problem using membership functions of these fuzzy numbers when the unit shipping costs, the supply quantities and the demand quantities are fuzzy numbers. The proposed solution algorithm to fuzzy transportation problem yields optimal compromise solutions. To show the ability the proposed solution, the numerical example was presented. The given example is solved using optimization software W INQSB.

Minghao et al., (2007) considered fuzzy transportation problems with satisfaction degrees of routes since except of transportation costs about routes, its safety or transportation time etc should be taken into account. Further flexibility of demand and supply quantity should also be taken into account. Moreover the fuzzy goal about total transportation cost is considered in place of minimizing the total transportation cost directly. The authors considered two criteria. One is to maximize the minimal satisfaction degree with respect to the flexibility and fuzzy goal .The other is to maximize the minimal satisfaction degree among routes used in transportation. But usually there exists no solution that optimizes both objectives at a time. So we seek some non-dominated solutions after defining non-domination.

In the late 1940s, George Dantzig and his contemporaries were faced with monumental problems that arose in the areas of military logistics, management, shipping, and economics. In 1947 Dantzig invented the simplex method—a way to reduce the number of calculations involved in optimization problems. This was the advent of linear programming. Chea (2007) presented a cost minimization transportation problem associated with Royal Dutch Shell's distribution system in the Chicago area. The solution is made easier by using a program called SIMPMETH, which was developed by the author. This software was designed as a TI-83 calculator application.

The Simplex algorithm was the forerunner of many computer programs that are used to solve complex optimization problems (Baynton, 2006). These applications are used extensively in a variety of situations. One of the most important applications of the simplex method is the transportation method (Zitarelli and Coughlin). The transportation method has been employed to develop many different types of processes. From machine shop scheduling (Mohaghegh, 2006) to optimizing operating room schedules in hospitals (Calichman, 2005). The wood products industry has used this method to maximize raw material component value (Baynton, 2006). The transportation method can also be used to reduce the impact of using fossil fuels to transport materials (Case, 2007).

Zierer, Mitchell and White (1976) studied the practical applications of linear programming to Royal Dutch Shell's distribution system. In 1976 Shell marketed over a dozen grades of liquid petroleum products. Their East of the Rockies (EOR) region included three refineries and over 100 terminal demand points. Shell's other distribution system, West of the Rockies Region (WOR) comprised the rest of the U.S. The study was restricted to the EOR Region. The task of making Shell's products available to customers was considerably complex but the computations were essential since from 10 to 20 percent of Shell's revenues were allocated to transportation costs.

Zitarelli and Coughlin (1992) presented the Shell oil study but concentrated on the Chicago area sub region to reduce the number of variables. The present study used their problem to illustrate how transportation problems can be solved using a simplex tableau.

Chea (2004) created a graphing calculator program for the TI-83 calculator.

His program significantly reduced the computational steps involved in solving a simplex matrix. The use of this software is described in conjunction with a description of the solution to the Zitarelli and Coughlin problem.

In 1976 the Chicago area sub region had two primary Shell oil refineries where oil was refined into various grades of petroleum products. These refineries were located in East Chicago, Indiana and Hammond, Indiana. The two major storage and shipment terminals were located in Des Plains, Illinois, and Niles, Michigan. In actual practice the problem was much more complex than the one presented by Zitarelli and Coughlin (1992). It involved over 1,200 variables and 800 constraints because there were more complex decisions to be made such as which mode of transportation to use (including pipelines, barges, trucks and tankers). In 1976 the typical problem faced each day could be solved on a computer in about one-half hour at a cost of about $100. Such reports generated about ten optional reports because there were various goals and managers with different responsibilities using the same data (Zitarelli and Coughlin).

Denardo et al., (1988) studied a problem, which uses supplied item travel time averages to determine the 'cost' of satisfying the demand at a particular location. Items that arrive first receive the greatest weight, and decreasing weights are given to each succeeding item. An equivalent transportation problem is used for problems with a known demand. If the demand is stochastic a transportation problem whose aim is to minimize the sum of a linear function is used. The function is linearized by substituting the product and a linear term for the convex function.

The transportation problem has been formulated by various investigators and solved to various degrees. The systematic method of solution was first given by Dantzig. In general, the computational procedures are adaptation of the simplex method. However, almost all of the techniques either take too long to be solved by a digital computer or are not readily adaptable for use on digital computers. The northwest corner rule has been presented for solving the transportation problem. The essentials of the stepping stone method are then reviewed. This technique does not consider costs for determining the initial basic feasible solution. Ramesh Gupta (1972) presented a modified technique for solving transportation problem by digital computer, which is presented along with the unique features of the method which give its high speed in solving problems. This method according to the author was implemented and was found to reduce the solution time by 2.6 times as compared to the well known matrix minima method of solution.

Efroymson and Ray (1966) presented an algorithm to find an optimal solution for the uncapacitated transportation problem. The authors assumed that each of the unit production cost functions has a fixed charge form. The authors also remarked that their branch-and-bound method can be extended to the case in which each of these functions is concave and consists of several linear segments, and each unit transportation cost function is linear.

Spielberg (1968) studied the uncapacitated transportation problem and proposed an algorithm which included some features that added speed computation time. The method can also accommodate such side conditions as budget constraints on plant expense and mutually exclusive alternatives.

Algorithms for the capacitated transportation problem have been presented by Davis (1969), Ellwein (1970), and Gray (1970) Marks (1969) and SA (1969). In all of these the cost functions were assumed to be linear and the production cost is linear where ever the production is and zero where not. Ellwein's technique allows the easy incorporation of configuration constraints that restrict the allowable combinations of open plants and generalization of the production.

Frank et al., (1970) developed an algorithm for reaching an optimal solution to the production-transportation problem for the convex case. The algorithm utilizes the decomposition approach it iterates between a linear programming transportation problem which allocates previously set plant production quantities to various markets and a routine which optimally sets plant production quantities to equate total marginal production costs, including a shadow price representing a relative location cost determined from the transportation problem.

Williams (1962) applied the decomposition principle of Dantzing and Wolf to the solution of the Hitchcock transportation problem and to several generalizations of it. In this generalizations, the case in which the costs are piecewise linear convex functions is included. Theauthor decomposed the problem and reduced to a strictly linear program. In addition, he argued that the two problems are the same by a theorem that he called the reduction theorem. The algorithm given by him, to solve the problem, is a variation of the simplex method with generalized pricing operation. It ignores the integer solution property of the transportation problem so that some problems of not strictly transportation type, and for which the integer solution property may not hold be solved.

Shetty in 1959 also formulated an algorithm to solve transportation problems taking nonlinear costs. The author considered the case when a convex production cost is included at each supply centre besides the linear transportation cost.

Feldman et al., (1976) assessed the concavity of the cost curve brought about by economies of scale leads to multiple-optima, and thus problems like these are not susceptible to conventional mathematical techniques. The power of the simplex method in solving linear programs is based on the general theorem which states that the number of variables-including slack variables, whose values are positive in an optimal solution, is at most equal to the number of constraints in the problem. For this reason, nearsighted computational techniques are used to examine the corners of the feasible region (basic solution). Unfortunately, these myopic computational and optimality testing techniques can be employed only when the problem involves a convex feasible region and increasing marginal cost.

Soland (1971) presented a branch and bound algorithm to solve concave separable transportation problem, which called it the Simplified algorithm in comparison with similar algorithm given by Falk and himself in 1969. The algorithm reduces the problem to a sequence of linear transportation problem with the same constraint set as the original problem.

Caputo et al., (2006) presented a methodology for optimally planning long-haul road transport activities through proper aggregation of customer orders in separate full-truckload or less-than-truckload shipments in order to minimize total transportation costs. The authors have demonstrated that evolutionary computation techniques may be effective in tactical planning of transportation activities. The model shows that substantial savings on overall transportation cost may be achieved adopting the methodology in a real-life scenario.

Crainic and Laporte (1997) reviewed the optimization models for freight transportation. A main distinction can be established between strategic-tactical and operational models that respectively consider a national or an international multimodal network, such as in the service network design problem, and the unimodal distribution management models that are variants of the vehicle routing problem.

Macharis and Bontekoning (2004) presented a freight logistics literature review focused on intermodal transportation. The authors proposed a classification based on two criteria: the type of operator and the length of the problem's time horizon. Four types of operators are distinguished: drayage operators, terminal managers, network planners, and intermodal operators. The time horizon criterion resulted in the classical differentiation of strategic, tactical, and operational levels.

Benton (1991) considered quantity discount procedures under conditions of multiple items, resource limitations and multiple suppliers. The author offered an efficient heuristic programming procedure for evaluating alternative discount schedules which provided encouraging findings for the managers.

CHAPTER THREE

METHODOLOGY

3.0 INTRODUCTION

In most transportation problem cases it was assumed that the cost per unit shipped from a given source to a given destination is fixed, regardless of the amount shipped. In actuality, this cost may not be fixed. Volume discounts sometimes are available for large shipments, so that the marginal cost of shipping one more unit might follow a nonlinear pattern. The resulting cost of shipping *x* units then is given by a nonlinear function $C(x)$, which is a piecewise linear function with slope equal to the marginal cost. Consequently, if each combination of source and destination has a similar shipping cost function, so that the cost of shipping x_{ij} units from source $i(i = 1,2, \dots m)$ to destination $j(j = 1,2, \dots n)$ is given by a nonlinear function $C_{ij}(x_{ij})$, then the overall objective function to be minimized is $f(x) = C_{ij}(x_{ij})$.

Even with this nonlinear objective function, the constraints normally are still the special linear constraints that fit the general transportation problem model.

In this chapter we shall provide an in depth explanation of the solution procedures to the generalized transportation problem taking nonlinear cost function. In particular, the nonlinear transportation problem considered in this paper as a result of volume discount on shipping cost is stated as follows; we are given: (i) a set of n sources of commodity with known supply capacity and a set of m destinations with known demands, (ii) the function of transportation cost, nonlinear, and differentiable for a unit of product from each source to each destination. We are required to find the amount of product to be supplied from each source to meet the demand of each destination in such a way as to minimize the total transportation cost.

Our approach to solve this problem is applying the existing general nonlinear programming algorithms to it making suitable modifications in order to use the special structure of the problem.

In order to understand our approach, it is necessary to have a good understanding of some of the background polyhedral theory for both the general linear and nonlinear programming problems.

The general transportation problem is modelled as;

Minimize Z = $\sum_{i,j} C_{ij}X_{ij}$

Subject to the constraints

$$\sum_{j=1}^{n} X_{ij} = S_i i = 1,2,.,.,m$$

$$\sum_{i=1}^{m} X_{ij} = D_j \quad j = 1,2, \quad ., \quad .,n$$

$$X_{ij} \geq 0$$

3.1 Polyhedral Sets

A set P in an n dimensional normed vector space E^n is called polyhedral set if it is the intersection of a finite number of closed-half spaces, i.e.$P = \{X: P_i^t X \leq \alpha_i, i = 1, \ldots, m\}$, where P_i is a non zero vector in E^n and α_i is a scalar.

A polyhedral set is a closed convex set and can be represented by a finite number of inequalities and/or equations.

We consider the polyhedral set $P = \{x: Ax = b, x \geq 0\}$, where A is an m x n matrix and b is an m-vector, assume also that the rank of A is m. If not, assuming that $Ax = b$ is consistent, we can leave aside any redundant equations.

3.1.0 Extreme Points

Let P be non empty convex set in E^n. A vector $x \in P$ is called an extreme point of P if $x = \beta x_1 + (1 - \beta)x_2$ with x_1 and x_2 elements of P and $\beta \in (0, 1)$.

The following are basic theorems concerning extreme points: for their proofs one can refer to (2).

Theorem 3.1.1

Let P = {x: Ax = b, x ≥ 0}, where A is m x n matrix of rank m, and b is an m vector. A point x is an extreme point of P if and only if a can be decomposed into [B, N] such that

$$x = \binom{xB}{xN} = \binom{B-1b}{0}$$

Where B is an m x n invertible matrix satisfying $B^{-1}b \geq 0$. Any such solution is called a basic feasible solution for (BFS) P.

The number of extreme points of P is finite.

Theorem 3.1.2 (Existence of extreme points)

Let P = {x: Ax = b, x ≥ 0} be non empty; where A is an m x n matrix of rank m and b is an m vector. Then P has at least one extreme point.

3.2 Extreme Direction

Let P be a non empty polyhedral set in E^n. A none zero vector d in E^n is called direction or recession direction of P if $x + \beta d \in P$ for each $x \in P$ for all $\beta \geq 0$.

It follows that, d is a direction of P if and only if Ad = 0 and d ≥ 0.

Theorem 3.1.3 Characterization of Extreme Directions

Let P = {x: Ax = b, x ≥ 0} ≠ ∅, where A is an m x n matrix of rank m, and b is an m vector. A vector $\bar{d}$ is an extreme direction of P if and only if A can be decomposed into [B, N] such that $B^{-1}a_j \leq 0$ for some column a_j of N, and $\bar{d}$ is a positive multiple of d = $\begin{pmatrix} B-1aj \\ ej \end{pmatrix}$, where e_j is an n-m vector of zero except for in position j which is 1.

Theorem 3.1.4 Representation theorem

Let P = {x: Ax = b, x ≥ 0} ≠ ∅. Let x_1, .,. x_k be the extreme points of P and d_1, d_2, .., d_l be the extreme direction of P. Then $x \in P$ if and only if x can be written as:

$$x = \sum_{j=1}^{k} \beta j x_j + \sum_{i=1}^{l} \gamma \ i d_i$$

$\sum_{j=1}^{k} \beta j = 1$

$\beta_j \geq 0$, and $\gamma_i \geq 0$.

Theorem 3.1.5 Existence of extreme directions

P = {x: Ax = b, x ≥ 0} where A is an m x n matrix with rank m. Then, P has at least one extreme direction if and only if it is unbounded.

3.3 The Karush-Kuhn-Tucker (KKT) optimality condition for nonlinear programming problem (NPP)

Given the nonlinear programming problem:

$$\text{(NPP)} \ min \ f(x)$$

$$\text{s. t.} \qquad g_i(x) \leq 0 \quad i = 1,2, \ldots, k$$

$$h_i(x) = 0 \quad j = 1,2, \ldots, l$$

3.3.1 KKT Necessary optimality conditions

3.3.1.1 Theorem

Given the objective function f : $R^n \rightarrow R$ and the constraint functions are g_i : $R^n \rightarrow R$ and hj : $R^n \rightarrow R$ and I = { i : $g_i(x^*) = 0$}. In addition, suppose they are continuously differentiable at a feasible point x* and $\nabla g_i(x^*)$ for i ∈ I and $\nabla h_j(x^*)$ for j = 1, .., l be linearly independent. If x* is minimizer of the problem (NPP), then there exist scalars $\bar{\beta}_i$ i = 1, ..., k and $\bar{\gamma}_j$ j = 1, ., ., l, called Lagrange multiplier, such that

$$\nabla f(x^*) + \sum_{j=1}^{k} \beta j \, \nabla g_i(x^*) + \sum_{i=1}^{l} \gamma_i \nabla h_i(x^*) = 0$$

$$\bar{\beta}_j g_j(x^*) = 0, \quad \bar{\beta} j \geq 0, \text{ and } \bar{\gamma}_j \in R$$

3.3.2 KKT Necessary optimality conditions for convex NPP

Further, if f and $g_i$are convex, each h_j as affine, then the above necessary optimality conditions will also be sufficient.

3.4The Linear Transportation Problem

The linear transportation problem is concerned with distributing any commodity from any group of supply centres, called sources, to any group of receiving centres, called destinations in such a way as to minimize the total distribution cost, where the cost per commodity is constant regardless of the amount transported. By letting z to be the total distribution cost and x_{ij} the number of units to be distributed from source $i(s_i)$ to destination $j(d_j)$ the linear programming formulation of this problem become:

$$\min z = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

$$\text{s. t} \quad \sum_{j=1}^{m} x_{ij} = s_i \qquad \text{for } i = 1, 2, \ldots n$$

$$\sum_{i=1}^{n} x_{ij} = d_j \text{ for } j = 1, 2, \ldots m$$

$$x_{ij} \geq 0 \quad \forall \ i, j$$

3.5 Methods for Finding Initial Basic Feasible Solutions

The first phase of the solving a transportation problem for optimal solution involves finding the initial basic feasible solution. An initial feasible solution is a set of arc flows that satisfies each demand requirement without supplying more from any origin node than the supply available. Heuristic, a common – sense procedure for quickly finding a solution to a problem is a producer most employed to find an initial feasible solution to a transportation problem. This project examines three of the more popular heuristics for developing an initial solution to transportation problem.

(i). The Northwest Corner Method

(ii). The Least Cost Method

(iii). The Vogel's Approximation Method

(i) The Northwest Corner Method

This method is the simplest of the three methods used to develop an initial basic feasible solution. This notwithstanding, it is the least likely to give a good "low cost" initial solution because it ignores the relative magnitude of the costs c_{ij} in making allocations The procedure of this method is as follows.

(i). Start at the northwest corner (upper-left-hand corner) cell of the tableau and allocate as much as possible to x_{11} without violating the supply or demand constraints(i.e. x_{11} is equal to the minimum of the values of S_i or d_j.)

(ii). This will exhaust the supply at source **i** and or the demand for destination **j**. As a result, no more units can be allocated to the exhausted row or column, and it is eliminated. Next, allocate as much as possible to the adjacent cell in the row or column that has not been eliminated. If both row and column are exhausted, move diagonally to the next cell.

(iii).Continue the process in the same manner until all supply has been exhausted and demand requirements have been met. Following is an example to illustrate the use of the Northwest Corner Method of finding an initial basic feasible solution to transportation problems.

(ii) Least – Cost Method

The Least– Cost Method tries to reflect the objective of cost minimization by systematically allocating to cells according to the magnitude of their unit costs.

Following is the general procedure for the Least –Cost Method.

(i). Select the x_{ij} variable (cell) with the minimum c_{ij} transportation cost and allocate as much as possible thus, for minimum c_{ij}.

$x_{ij} = minimum\ (s_i, d_j)$This will exhaust either row i or column j.

(ii). From the remaining cells that are feasible (i.e. have not been filled or their row or column eliminated), select the minimum c_{ij}value and allocate as much a possible

(iii). Continue the process until all supply and demand requirements are satisfied

(iv). In case of ties between the min c_{ij}values select between the tied cells arbitrarily and apply the procedure.

(iii) VOGEL'S APPROXIMATION METHOD

The Vogel's Approximation Method (VAM) is by far the best method (better than the Northwest Corner Method and the Last-Cost Method) of developing an initial basic feasible solution to transportation problems. In many cases the initial solution obtained by the VAM will be optimal.

It consists of making allocations in a manner that will minimize the penalty (regret or opportunity cost) for selecting the wrong cell for an allocation. The procedure for the use of the VAM is as follows;

(i). Calculate the penalty cost for each row and column. The penalty costs for each row i are computed by subtracting the smallest c_{ij} values in the row from the next smallest c_{ij} values in the same row.

(ii). Column penalty costs are similarly obtained, by subtracting the smallest c_{ij} value in each column from the next smallest column c_{ij} value. These costs are the penalty for mot selecting the minimum cell cost.

(iii). Select the row or column with the greatest penalty cost (breaking any ties arbitrarily) and allocate as much as possible to the cell with the minimum c_{ij} value in the selected row or column, that is for minimum $c_{ij}, x_{ij} = minimum\ (s_i, d_j)$. This will avoid the greatest penalties.

(iv). Adjust the supply and demand requirements to reflect the allocations already made. Eliminate any rows and columns in which supply and demand have been exhausted.

(v). If all supply and demand requirements have not been satisfied, go to the first step and recalculate new penalty costs. If all row and column values have been satisfied the initial solution has been obtained.

3.6 Optimality-Test Algorithm for Transportation Problems

These are methods of determining the optimal solutions for transportation problems following the determination of the initial basic feasible solution. Two methods,

(i) The stepping stone method

(ii) The Modified Distribution Method shall be the focus of this project.

(1) The Stepping Stone Method: This optimality test begins, once an initial basic feasible solution is obtained for the transportation problem, by determining if the total transportation cost can be further reduced by entering a nonbasic variable (i.e. allocating units to an empty cell) into the solution. Thus each empty cell is evaluated to determine if the cost of shifting a unit to that cell from a cell containing a positive unit will decrease. A closed loop of occupied cells is used to evaluate each nonbasic valuable. An initial basic feasible solution is considered optimal if the total transportation cost cannot be lowered/ decreased by reallocating units between cells.

The following three steps are involved in the stepping-stone method

(i). Determine an initial feasible solution by using any of the afore-discussed initial feasible solution determination methods

(ii). Compute a cell evaluator for each empty cell, determined by computing the next cost of shifting one unit from a cell containing a positive unit to the empty cells. The sign of cell evaluators are then checked for optimality

(iii). If a cell evaluator fails the sign test, if the solution is not optimal, determine a new lower total cost solution, accomplished by shifting the maximum amount to that empty cell so that the supply or demand constraints are not violated.

(2) The Modified Distribution Method (MODI)

The modified distribution method of solution is a variation of the steeping-stone method based on the dual formulation. The difference between the two is that with the MODI, unlike the stepping-stone method, it is not necessary to determine all closed paths for nonbasic variable. The C^*_{ij} values are instead determined simultaneously and the closed path is identified only for the entering nonbasic variable. In the MODI method, a value u_i is defined for each row (i) and a value v_j is defined for each column (j) in the transportation tableau. For each basic variable, (occupied cell), x_{ij} the following relationship exists.

$C_{ij} = u_i + v_j$, where C_{ij} is the unit cost of transportation.

The steps employed in the MODI method are;

(i). Determine u_i value for each row and v_j value for each column by using the relationship $C_{ij} = u_i + v_j$ for all basic variables beginning with an assignment of zero to u_i.

(ii). Compute the net cost change C^*_{ij}, for each nonbasic variable using the formula $C^*_{ij} = C_{ij} - u_i - v_j$.

(iii). If a negative C^*_{ij} value exists, the solutions is not optimal. Select the x_{ij} variable with the greatest negative C^*_{ij} value as the entering nonbasic variable.

(iv). Allocate units to the entering C^*_{ij} value as the entering nonbasic variable, x_{ij}, according as the stepping-stone procedure. Return to step 1.

3.7 Solution procedures to nonlinear transportation Problems (NTP)

This section considers the solution to the transportation problem with nonlinear cost function arising from volume discount. We shall consider different solution procedures depending on the nature of the objective cost function.Before considering the different special cases, let us first formulate the KKT condition and general algorithm for the problem. Given a differentiable function $C: R^{nm} \rightarrow R$.

We consider a nonlinear transportation problem (NTP),

min $\quad C(x)$

s.t $\quad Ax = b, \quad x \geq 0$

The KKT Optimality Condition for the NTP

Given the transportation table as below:

$\frac{\partial C(\bar{x})}{\partial x_{11}}$	.....	......	$\frac{\partial C(\bar{x})}{\partial x_{1m}}$	S_1	U_1	
....						
.. ..	$\frac{\partial C(\bar{x})}{\partial x_{ij}}$	....	S_i U_i			
$\frac{\partial C(\bar{x})}{\partial x_{n1}}$	....	.... $\frac{\partial C(\bar{x})}{\partial x_{nm}}$	S_n	U_n		
d_1....	.... d_m					
v_1	v_j....	v_m				

Where $\bar{x}$ is the current basic solution.

The Lagrange function for the NTP is formulated as:

$z(x,\beta, w) = C(x) + w\,(b- Ax) - \beta x$

Where β and w are Lagrange multipliers and

$\beta \in R^{nm}$

The optimal point $\bar{x}$ should satisfy the KKT conditions:

$\nabla z = \nabla C(\bar{x}) - w^T A - \beta = 0$

$\beta\bar{x} = 0$

$\beta \geq 0$

$\bar{x} \geq 0$

Specifically for each cell (i,j) we have

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{x})}{\partial x_{ij}} - (u,v)\,(e_i, e_{n+j}) - \beta_{ij} = 0 \qquad (3.1)$$

$\beta_{ij} x_{ij} = 0$

$x_{ij} \geq 0$

$\beta_{ij} \geq 0$

where k = 1 ... mn and w = (u, v) = (u_1, u_2, ..., u_n, v_1, v_2, ..., v_m), $e_k \in R^{m+n}$ is a vector of zeros except at position k which is 1.

From the conditions (3.1) and $\beta k \geq 0$, we get,

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0 \qquad (3.2)$$

$$x_{ij}\frac{\partial z}{\partial x_{ij}} = x_{ij}\frac{\partial C(\bar{x})}{\partial x_{ij}} - (u_i + v_j) = 0 \qquad (3.3)$$

$x_{ij} \geq 0$

General Solution Procedure for the NTP.

- Initialization

Find an initial basic feasible solution $\bar{x}$

- Iteration

Step 1 If $\bar{x}$ is KKT point, stop. Otherwise go to the next step.

Step 2 Find the new feasible solution that improves the cost function and go to step 1.

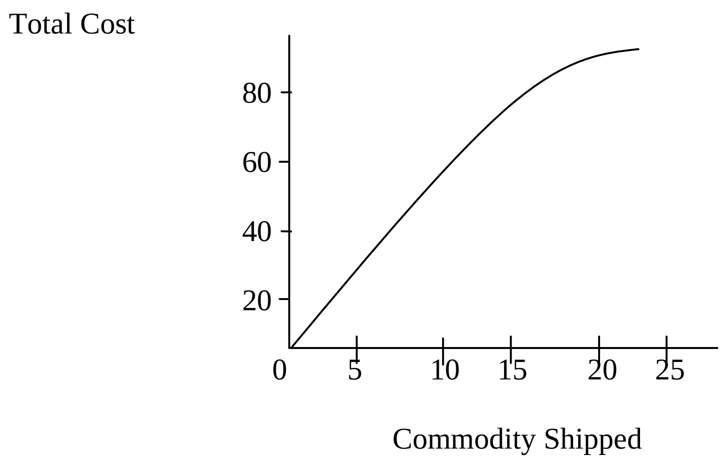
3.8 Transportation Problem with Concave Cost Functions

For large shipments, volume discount may be available sometimes. In this case the cost function of the transportation problem generally takes concave structure for it is separable and the marginal cost (cost per unit commodity shipped) decreases with increase of the amount of shipment; and increasing, because of the total cost increase per addition of unit commodity shipped.

The discount (1) may be either directly related to the unit commodity: (2) or have the same rate for some amount.

Case 1: If the discount is directly related to the unit commodity the resulting cost function will be continues and have continues first order partial derivatives.

The graph of C_{ij} (x_{ij}) will look like,



Nonlinear programming formulation of such a problem is given by

$$\text{Minimize } Z = \sum_{i,j} C_{ij} X_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} X_{ij} = S_i \qquad I = 1, 2, ., ., m$$

$$\sum_{i=1}^{m} X_{ij} = D_j \qquad j = 1, 2, ., ., n$$

$$X_{ij} \geq 0$$

Where

$C_{ij} : R \rightarrow R$

Now before we go to look for an optimal solution let us state an important theorem:

Theorem 3.3.1.2 Let f be concave and continues function and P be a non empty compact polyhedral set. The optimal solution to the problem min f(x), x ∈ P exists and can be found at an extreme point of P.

Proof

Let E = {x_1, x_2, ..., x_k, ..., x_n} be the set of extreme points of P, and $x_k \in E$ such that $f(x_k) = \min\{f(x_i) : i = 1, ..., n\}$. Now since P is compact and f is continuous, f attains its minimum in P, call it $\bar{x}$.

If $\bar{x}$ is extreme point, we are done. Otherwise, we have that

$$\bar{x} = \sum_{i=1}^{n} \beta_i X_i , \qquad \sum_{i=1}^{n} \beta_i = 1 \quad ,\beta_i > 0$$

where x_1, x_2, ..., x_n are extreme points of P.

The by concavity of it follows that,

$$f(\bar{x}) = f\left(\sum_{i=1}^{n} \beta_i X_i\right) \geq \sum_{i=1}^{n} \beta_i f(x_i) \geq f(x_k) \sum_{i=1}^{n} \beta_i$$

$$\Rightarrow f(\bar{x}) \geq f(x_k) \qquad \text{(Since for each I = 1, …, n; } f(x_k) \leq f(x_i) \text{ and } \sum_{i=1}^{n} \beta_i = 1\text{)}$$

Since $\bar{x}$ is minimizer, in addition we have,

$$f(\bar{x}) \leq f(x_k)$$

The above two relations imply

$$f(\bar{x}) \leq f(x_k)$$

This completes the proof.

Solution Procedure

Because of the above theorem, it suffice to consider only the extreme points to find the minimum; the following is the procedure.

After we find the initial basic feasible solution (which are n + m – 1 in number), let $\bar{x}$ be the basic solution we have in the current iteration.

Next let us decompose our $\bar{x}$ to ($\bar{x}_B$, $\bar{x}_N$) where $\bar{x}_B$ and x_N are the basic and nonbasic variables respectively. Since $\bar{x}_B > 0$, the complementary slackness condition given in equation (3.3) above gives as m + n – 1 equations;

$$\frac{\partial z}{\partial XBij} = \frac{\partial C(\bar{x})}{\partial XBij} - (u_i + v_j) = 0 \qquad (3.4)$$

From the above relation we can determine the values of u_i and v_j by assigning one of u'$_i$s the value zero for we have m + n variables, u_i and v_j.

Then we calculate $\frac{\partial z}{\partial Xij}$ for the non basic variable x_{ij}. Since all x_{ij} are zero at the extreme, the complementary slackness condition is satisfied. Therefore if equation (3.2) is satisfied for all no basic variables x_{ij}, $\bar{x}$ is a KKT point.

Otherwise, if

$$\frac{\partial z}{\partial Xij} - (u_i + v_j) < 0,$$

We shall move to look for better basic solution such that all the constraints (feasibility conditions) are satisfied. We do this by using the same procedure as the transportation simplex algorithm as stated below.

3.8.1 The Transportation Concave Simplex Algorithm (TCSA)

Initialization

Find the initial basic feasible solution using some rule like the north west corner rule.

Iteration

Step1 : determine the values of u_i and v_j from the equation,

$$\frac{\partial C(\bar{x})}{\partial XBij} - (u_i + v_j) = 0 ,$$

where x_{Bij} are the basic variables.

Step 2 : If

$$\frac{\partial C(\bar{x})}{\partial XBij} - (u_i + v_j) \geq 0 ,$$

for all x_{ij} non basic, stop, $\bar{x}$ is KKT point. Otherwise go to step 3.

Step 3: Calculate

$$\frac{\partial z}{\partial Xrl} = \min \{\frac{\partial C(\bar{x})}{\partial Xij} - (u_i + v_j) \}$$

x_{rl} will enter the basis. Allocate $x_{rl} = \theta$ where θ is found as in the linear transportation case.

Adjust the allocation so that the constraints are satisfied.

Determine the leaving variable say x_{Brk}, where x_{Brk} is the basic variable comes to zero first while making the adjustment. Then find the new basic variable and go to step 1.

Finite Convergence of the Algorithm

The feasible set of our problem is a non empty polyhedral set. And by definition, a polyhedral set P is a set bounded with a finite number of hyperplanes from which it follows that it possesses finite number of extreme points.

In each step of the algorithm, we jump from one extreme point to another looking for a better feasible solution implying that the algorithm will terminate after a finite iteration. In addition since for all i and j, $0 \leq xij \leq \max \{ s_i, d_j \}$, P is bounded that guarantees the existence of minimum value.

Case 2: In the case when the volume discount is fixed for some amount of commodity, rather than varying with unit amount shipped, the transportation cost function will be piecewise linear concave yet increasing.

The graph is like;

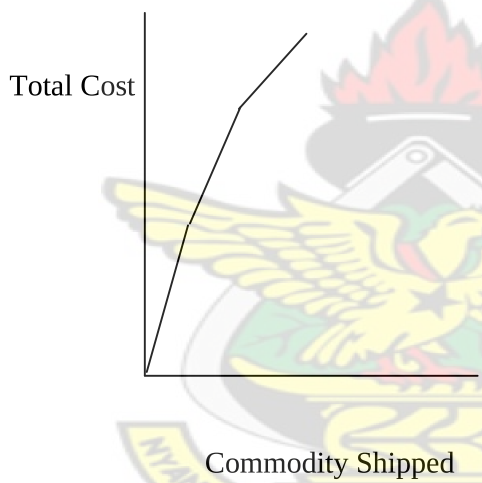


Figure 3.2: Transportation problem with piecewise linear concave cost

To avoid complication, assuming that to each combination of source and destination, the interval in which the marginal cost (cost per unit commodity) changes is the same, the cost of shipping x_{ij} units from source *i* to destination *j* is given by $C_{ij}(x_{ij})$,then the nonlinear programming formulation of the problem is given by

$$\text{Minimize } Z = \sum_{i,j} CijXij$$

Subject to the constraints

$\sum_{j=1}^{n} Xij$ = S_i I = 1, 2, ., ., m

$\sum_{i=1}^{m} Xij$ = D_j j = 1, 2, ., ., n

$X_{ij} \geq 0$

Where,

$$\begin{cases} C_{ij}^{0}(x_{ij}), & 0 \leq x_{ij} \leq a_1 \\ C_{ij}^{1}(x_{ij}), & a_1 \leq x_{ij} \leq a_{l+1} \\ C_{ij}^{l}(x_{ij}), & a_l \leq x_{ij} \leq a_2 \\ C_{ij}^{k-1}(x_{ij}), & a_{k-1} \leq x_{ij} \leq a_k \\ C_{ij}^{k}(x_{ij}), & a_k \leq x_{ij} \leq b = \max\{s_i, d_j\} \end{cases}$$

And

1. $\{0, a_1, ..., a_l, ..., a_{k-1}, a_k, b\}$is the partition of the interval $[0, b]$ into k+ 1 sub intervals

2. Each C^{l}_{ij} is linear in the sub interval $[a_l, a_{l+1}]$

To solve this problem, as we can see from the structure of the cost function, it's impossible to directly apply the algorithm of the previous section for non differentiability of the total cost function hinders as to do so.

But, since the function, also, has a simple structure and differentiability fails at discrete points, it can be easily approximated using differentiable functions like Chebshev, trigonometric or Legendre polynomials.

We choose to approximate it by the so called shifted Legendre polynomials.

These set of Legendre polynomials say $\{p_0, p_1 \ldots, p_r,\}$is orthogonal in [0,1]with respect to weight function $w(x) = 1$, where the inner product on C[0,1]is defined by

$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, for all f, g ∈ C[0, 1],

Where $C[0; 1]$ is the space of continuous functions on [0,1] .

The first four of them are,

$$p_0(x) = 1$$

$$p_1(x) = 2x - 1$$

$$p2(x) = 6x2 \text{ ¡ } 6x + 1$$

$$p3(x) = 20x3 \text{ ¡ } 30x2 + 12x \text{ ¡ } 1$$

and the others can be obtained from

$p_r(x) = \frac{1}{2^r!}\frac{d^r}{dx^2}[(x^2 - 1)^r]$

Then, the space spanned by $\{p_0, p_1 \ldots, p_r\}$is a subspace of C [0,1]. Hence, given any $f(x) \in C$ [0,1], we can find a unique least square approximation of f in the subspace. Note that every element of the subspace spanned by $\{p_0, p_1, \ldots, p_r\}$is at least twice differentiable.

The least square approximation of any function $f(x)$ with r of these polynomials in [0, 1] is given by,

$f(x) = a_0p_0(x) + a_1p_1(x) + \ldots + a_ip_i(x) + \ldots + a_rp_r(x)$

where

$ai = \frac{\int_0^1 PiF(x)dx}{\int_0^1 [Pi(x)]2dx}$,   i = 0, 1, ..., r.

To approximate our functions $C_{ij}(x_{ij})$, in the same manner, we define a one to one correspondence between [0,b] to [0,1] by

g: [0, b] →[0, 1]

$g(x_{ij}) = \frac{1}{b}x_{ij}$

That is, we substitute x_{ij} by $\frac{1}{b}x_{ij}$ so that it's domain will be [0,1] then we have,

$$
C_{ij}(x_{ij}) \rightarrow \hat{C}_{ij}(x_{ij}) = C_{ij}(\tfrac{1}{b}x_{ij}) = \begin{cases} C^{o}_{ij}(\frac{1}{b}x_{ij}), & 0 \leq x_{ij} \leq \frac{a1}{b} \\ C^{1}_{ij}(\frac{1}{b}x_{ij}), & \frac{a1}{b} \leq x_{ij} \leq \frac{a2}{b} \\ \cdot & \\ \cdot & \\ C^{k}_{ij}(\frac{1}{b}x_{ij}), & \frac{ak}{b} \leq x_{ij} \leq 1 \end{cases}
$$

Now, after approximating $\hat{C}_{ij}x_{ij}$ by the shifted Legendre polynomials on [0, 1], assume we have found it's best approximation $\hat{C}_{ij}(x_{ij})$.

Then, substituting back the x_{ij} in C_{ij} by bx_{ij} gives us the approximation to $Cij(x_{ij})$ over [0,b]. Therefore the best approximation of $C_{ij}(x_{ij})$ over [0,b]will be

$\overline{C}_{ij}(x_{ij}) = \hat{C}_{ij}(bx_{ij})$,

Which has continuous derivatives.

Consequently, we solve the problem

$\min\sum_{i=1}^{n}\sum_{j=1}^{m}\bar{C}(x_{ij}) = \sum_{l=0}^{2}\sum_{i=1}^{n}\sum_{j=1}^{m} a_l p_l(x_{ij})$

s. t $\sum_{j=1}^{n} xij = \frac{si}{b}$

$\sum_{i=1}^{n} xij = \frac{dj}{b}$

i = 1, 2, ..., n and j = 1, 2, ..., m

Using exactly the same procedure as the previous case

3.9 Convex Transportation Problem

This case may arise when the objective function is composed of not only the unit transportation cost but also of production cost related to each commodity, or in the case when the distance from each source to each destination is not fixed.

The problem can be formulated as :

min $C(\mathrm{x})$

s.t $A\mathrm{x} = b$

$\mathrm{x} \geq 0$

Where C(x) is convex, continuous and has continuous first order partial derivatives.

The Convex Simplex solution procedure for Transportation Problem.

In the case when the cost function is convex, the minimum point may not be attained necessarily at an extreme; it may be found before reaching a boundary of the feasible set.

What precisely happens is that there may be non basic variable with positive allocation while non of the basis is driven to zero.

To solve this problem, we use the idea of the convex simplex algorithm of Zangwill (1967) which was originally designed to take care of convex and pseudoconvex problem with linear constraints. Actually the original procedure is used to look for a local optimal solution for any other linearly constrained programming problem. We use the special structure of transportation problem in the procedure so as to make it efficient for our particular problem. The method reduces to the ordinary transportation simplex algorithm whenever the objective is linear, to the method of Beal when it is quadratic and to the above concave simplex procedure when the function is concave.

We partition the variable $x = (x_{11}, ..., x_{nm})$ to (x_B, x_N), where x_B is n+m -1 component vector of basic variables and x_N is nm - (n + m -1)) component vector of non basic variables, corresponding to the (n +m -1)X(n +m -1)basic sub matrix and (n + m - 1)X(nm - (n + m - 1)) non basic sub matrix of A.

Suppose we have the initial basic feasible solution $\bar{x}_0$.In the procedure what we do is to find a mechanism in which non optimal basic solution $\bar{x}$ at a given iteration is improved until it satisfies the KKT conditions which are also sufficient conditions for convex transportation problem, i. e, until for each cell we have;

$$x_{ij} \left(\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j)\right) = 0$$

and

$$\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j) \geq 0$$

Since we have each basic variable $x_{Bij}>0$, the above complementary slackness condition implies that for each basic cell, we must have

$$\frac{\partial f(\bar{x})}{\partial X Bij} - (u_i + v_j)) = 0$$

x_{Bij}- basic variable.

Since we have $n + m$ -1 of such equations, by letting $u_1 = 0$ we obtain all the values of u_i and v_j as we have done exactly for the concave and linear cases.

Now for a non basic cell, at a feasible iterate point $\bar{x}$; we may have:

$$\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j) > 0 , \quad x_{ij} \left(\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j)\right) > 0 ,$$

$$\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j) < 0 , \quad x_{ij} \left(\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j)\right) < 0 ,$$

$$\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j) = 0 , \quad x_{ij} \left(\frac{\partial f(\bar{x})}{\partial X ij} - (u_i + v_j)\right) = 0$$

Or for non basic x_{ij}, we may have;

$\frac{\partial f(\bar{x})}{\partial Xij}$ - $(u_i + v_j)$ ≥ 0, x_{ij} ($\frac{\partial f(\bar{x})}{\partial Xij}$ - $(u_i + v_j)$) = 0

From the KKT conditions given earlier, the last case occurs when $\bar{x}$ is optimal.

But if the solution $\bar{x}$ falls on either of the other three, it must be improved as follows.

Let IJ = {ij:x_{ij} is non basic variable} and suppose that we are in the k^{th} iteration.

We first begin by computing;

$$\frac{\partial z}{\partial Xrl} = \min \{\frac{\partial f(\bar{x})}{\partial Xij} - u_i - v_j\}\ ij \in IJ$$

$$xs_t \frac{\partial z}{\partial Xrl} = \max \{x_{ij}(\frac{\partial f(\bar{x})}{\partial Xij} - u_i - v_j)\}\ ij \in IJ$$

Here we don't want to improve (decrease) a positive - valued non basic variable x_{ij} unless its partial derivative is positive. Therefore we only focus on positive values of the product $\frac{\partial z}{\partial Xij} x_{ij}$.

Now the variables to be adjusted are selected as;

Case 1 If

$$\frac{\partial z}{\partial Xrl} \geq 0 \text{ and } xst\ (\frac{\partial z}{\partial Xst}) > 0$$

Decrease x_{st} by the value θ using the transportation table as in the linear and concave cases.

Let y^k= (y^k_{11}, y^k_{12} , ..., y^k_{nm}) be the value of $\bar{x}^k$= ($\bar{x}^k_{11}$, ..., $\bar{x}^k_{nm}$) after making the necessary adjustment by adding and subtracting θ in the loop containing x_{st} so that all the constraints are satisfied.

By doing so, either x_{st} itself or a basic variable say x_{Bst} will be driven to zero.

Now y^k may not be the next iterate point; since the function is convex, a better point could be found before reaching y^k to check this, we solve problem;

$$f(\bar{x}^{k+1}) = \min \{ f(\beta\bar{x}^k + (1-\beta)\, y^k : 0 \leq \beta \leq 1 \} \qquad (3.5)$$

and get $\bar{x}^{k+1} = \bar{\beta}\bar{x}^k + (1 - \bar{\beta})\, y^k$ where $\bar{\beta}$ is the optimal solution of equation 3.5.

Before the next iteration,

If $\bar{x}^{k+1} = y^k$ and if a basic variable became zero during the adjustment made, we change the basis.

If $\bar{x}^{k+1} \neq y^k$ or if $\bar{x}^{k+1} = y^k$ and x_{st} is driven to zero, we don't change the basis by substituting the leaving basic variable by x_{st}.

case 2 If

$\frac{\partial z}{\partial Xrl} < 0$ and x_{st} ($\frac{\partial z}{\partial Xst}$) ≤ 0

In this case the value of x_{rl} should be increased by θ and then we find y^k, where θ and y^k are defined as in the case 1.

Note that: as we increase the value of x_{rl} one of the basic variables, say, x_{Bt} will be driven to zero, and this is the exit criteria of the linear and concave transportation simplex algorithm and y^k would have been the next iterate point of the procedure. But now after solving for $\bar{x}^{k+1}$ from 3.5, before going to the next iteration, we will have the following possibilities.

If $\bar{x}^{k+1} = y^k$, we change the former basis, substitute x_{Bt} by x_{rl}

If $\bar{x}^{k+1} \neq y^k$, we do not change the basis.

All the basic variables outside of the loop will remain unchanged.

case 3 If

$\frac{\partial z}{\partial Xrl} < 0$ and x_{st} ($\frac{\partial z}{\partial Xst}$) > 0

In this case either we decrease x_{st} as in the case 1 or increase x_{rl} according to case 2.

3.10 The Transportation Convex Simplex Algorithm

Now we write the formal algorithm¤ for solving the convex transportation problem.

Initialization

Find the initial basic feasible solution.

Iteration

Step 1: Determine all u_i and v_j from

$\frac{\partial f(\bar{x})}{\partial XBij}$ - u_i -v_j = 0 for each basic cell.

Step 2: For each non basic cell, calculate;

$\frac{\partial z}{\partial Xrl} = \min \{\frac{\partial f(\bar{x})}{\partial Xij} - u_i - v_j\}$

$xs_t\frac{\partial z}{\partial Xrl} = \max \{x_{ij}(\frac{\partial f(\bar{x})}{\partial Xij} - u_i - v_j)\}$

If

$\frac{\partial z}{\partial Xrl} \geq 0$ and x_{st} ($\frac{\partial z}{\partial Xst}$) = 0

Stop. Otherwise go to step 3.

Step 3: Determine the non basic variable to change.

Decrease x_{st} according to case 1 if $\frac{\partial z}{\partial Xrl} \geq 0$ and x_{st} ($\frac{\partial z}{\partial Xst}$) > 0

Increase x_{rl} according to case 2 if

$\frac{\partial z}{\partial Xrl} < 0$ and x_{st} ($\frac{\partial z}{\partial Xst}$) ≤ 0

Either increase x_{rl} or decrease x_{st} if $\frac{\partial z}{\partial Xrl} < 0$ and x_{st} ($\frac{\partial z}{\partial Xst}$) > 0

Step 4 : Find the values of y^k, by means of θ, and $\bar{x}^{k+1}$, from 3.5

If $y^k = \bar{x}^{k+1}$ and a basic variable is driven to zero, change the basis.

Otherwise do not change the basis.

$\bar{x}^k = \bar{x}^{k+1}$

go to step 1.

CHAPTER FOUR

DATA COLLECTION AND ANALYSIS

4.0 INTRODUCTION

In this chapter, we shall consider a computational study of the above solution procedures. Emphasis will be given to a transportation problem where discounts are given to volume on quantity of goods transported which is concave in nature. Data from the Multi-Plan Limited shall be examined.

4.1 Data Collection and Analysis

The Multi-Plan Limited, a distributor of various kinds of drinks located in Accra, purchase from three manufacturing companies in different places and sell the same to four market segments in Ghana. The cost of purchasing and transporting the drinks from the traders place to the market centres is given in Table 4.1 below.

Table 4.1: Cost of transporting the drinks to the various market zones

	AVAILABILITY	LOADING AND PACKAGING	MARKET SEGMENTS				SUPPLY
			A	B	C	D	
P. RED	15,000	3,000	12,000	7,000	1,000	17,000	15,000
OVIDIO	25,000	2,000	5,000	4,000	6,000	1,000	25,000
MERLOT	10,000	600	400	8,400	4,400	2,400	10,000
REQUIREMENT OF DRINKS			20,000	10,000	8,000	12,000	

All values in Table 4.1 apart from requirements and supply are in cedi monetary value. The policy of the company allows discounts on each box transported from source to destination and it is directly related to the unit commodity purchased and transported, and the percentage discounts are shown in Table 4.2 below.

	A	B	C	D
P. RED	0.02	0.01	0.04	0.07
OVIDIO	0.01	0.04	0.03	0.02
MERLOT	0.005	0.03	0.015	0.01

The problem is to determine how many boxes of each product to be transported from the source to each destination on a monthly basis in order to minimize the total transportation cost.

<u>Forming the transportation tableau</u> (Table 4.3)

	A	B	C	D	SUPPLY
P. RED	15	10	4	20	15
OVIDIO	7	6	8	3	25
MERLOT	1	9	5	3	10
DEMAND	20	10	8	12	**50**

To form transportation tableau, let

i = product to be shipped.

j = destination of each product.

s_i = the capacity of source node *i*,

d_j = the demand of destination *j*,

x_{ij}= the total capacity from source *i* to destination *j*

C_{ij}= the per unit cost of transporting commodity from *i* to destination *j*.

If we suppose that discount is given on each box transported from *i* to *j* then the non linear transportation problem can be formulated as:

The problem can be modeled as:

Minimize $15x_{11} + 10\,x_{12} + 4\,x_{13} + 20\,x_{14}$

$7x_{21} + 6\,x_{22} + 8\,x_{23} + 3\,x_{24}$

$x_{31} + 9\,x_{32} + 5\,x_{33} + 3\,x_{34}$

Subject to

$x_{11} + x_{12} + x_{13} + x_{14} = 15$

$x_{21} + x_{22} + x_{23} + x_{24} = 25$

$x_{31} + x_{32} + x_{33} + x_{34} = 10$

$x_{11} + x_{21} + x_{31} = 20$

$x_{12} + x_{22} + x_{32} = 10$

$x_{13} + x_{23} + x_{33} = 8$

$x_{14} + x_{24} + x_{34} = 12$

where

$C_{11}x_{11} = 15x_{11} – p_{11}x^2_{11}$ $C_{22}x_{22} = 6x_{22} – p_{22}x^2_{22}$

$C_{12}x_{12} = 10x_{12} – p_{12}x^2_{12}$ $C_{23}x_{23} = 8x_{23} – p_{23}x^2_{23}$

$C_{13}x_{13} = 4x_{13} – p_{13}x^2_{13}$ $C_{24}x_{24} = 3x_{24} – p_{24}x^2_{24}$

$C_{14}x_{14} = 20x_{14} – p_{14}x^2_{14}$ $C_{31}x_{31} = x_{31} – p_{31}x^2_{31}$

$C_{21}x_{21} = 7x_{21} – p_{21}x^2_{21}$ $C_{32}x_{32} = 9x_{32} – p_{32}x^2_{32}$

$C_{33}x_{33} = 5x_{33} – p_{33}x^2_{33}$ $C_{34}x_{34} = 3x_{34} – p_{34}x^2_{34}$

If we allow the discounts on each transported product i from the source to each of the destinations j as given in table 4.2, the cost function become:

$C_{11}x_{11} = 15x_{11} – 0.02x^2_{11}$ $C_{22}x_{22} = 6x_{22} – 0.04x^2_{22}$

$C_{12}x_{12} = 10x_{12} – 0.01x^2_{12}$ $C_{23}x_{23} = 8x_{23} – 0.03x^2_{23}$

$C_{13}x_{13} = 4x_{13} – 0.04x^2_{13}$ $C_{24}x_{24} = 3x_{24} – 0.02x^2_{24}$

$C_{14}x_{14} = 20x_{14} – 0.07x^2_{14}$ $C_{31}x_{31} = x_{31} – 0.005x^2_{31}$

$C_{21}x_{21} = 7x_{21} –0.01x^2_{21}$ $C_{32}x_{32} = 9x_{32} – 0.03x^2_{32}$

$C_{33}x_{33} = 5x_{33} – 0.04x^2_{33}$ $C_{34}x_{34} = 3x_{34} – 0.01x^2_{34}$

Using the West Corner rule we get the initial basic solution.

The solution tableau is as shown below,

	A	B	C	D	SUPPLY
P. RED	15 $\boxed{15}$	$\boxed{10}$	$\boxed{4}$	$\boxed{20}$	15
OVIDIO	5 $\boxed{7}$	10 $\boxed{6}$	8 $\boxed{8}$	2 $\boxed{3}$	25
MERLOT	$\boxed{1}$	$\boxed{9}$	$\boxed{5}$	10 $\boxed{3}$	10
DEMAND	20	10	8	12	**50**

The initial basic feasible solution is;

$\bar{x} = (x_{B11}, x_{12}, x_{13}, x_{14}, x_{B21}, x_{B22}, x_{B23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{B34})$

This from the table is given as;

$\bar{x} = (15, 0,0, 0, 5, 10, 8, 2, 0, 0, 0, 10)$ in thousands

with the total transportation cost of

Cost = (1,500*15) + (5000*5) + (10,000*6) + (8,000*8) + (2,000*3) + (10,000*2)

Total Cost = GH¢400,000.00

Now, we use the KKT optimality conditions to improve upon our solution.

The partial derivatives at $\bar{x}$ for the cost function are given as:

$\frac{\partial f(x)}{\partial x_{11}} = 14.4$ $\quad \frac{\partial f(x)}{\partial x_{12}} = 10$ $\quad \frac{\partial f(x)}{\partial x_{13}} = 4$ $\quad \frac{\partial f(x)}{\partial x_{14}} = 20$

$\frac{\partial f(x)}{\partial x_{21}} = 6.9$ $\quad \frac{\partial f(x)}{\partial x_{22}} = 5.2$ $\quad \frac{\partial f(x)}{\partial x_{23}} = 7.52$ $\quad \frac{\partial f(x)}{\partial x_{24}} = 2.92$

$\frac{\partial f(x)}{\partial x_{31}} = 1$ $\frac{\partial f(x)}{\partial x_{32}} = 9$ $\frac{\partial f(x)}{\partial x_{33}} = 5$ $\frac{\partial f(x)}{\partial x_{34}} = 1.8$

Now we find from the cost equation of the occupied cell;

$$\frac{\partial Z}{\partial x_{Bij}} = \frac{\partial f(x)}{\partial x_{Bij}} - u_i - v_j = 0$$

Thus,

$$\frac{\partial f(x)}{\partial x_{Bij}} = u_i + v_j$$

$u_1 + v_1 = 14.4$ $u_1 + v_2 = 10$ $u_2 + v_2 = 5.2$

$u_2 + v_4 = 2.92$ $u_2 + v_1 = 6.9$ $u_2 + v_3 = 7.52$ $u_3 + v_4 = 1.8$

Letting $u_1 = 0$, from the equations we have;

$u_1 = 0$, $u_2 = -7.5$, $u_3 = -8.62$, $v_1 = 14.4$, $v_2 = 12.7$, $v_3 = 15.02$, and $v_4 = 10.42$

We find the net evaluation factor or the reduced costs for the non-basic variables.

$\frac{\partial Z}{\partial x_{12}} = \frac{\partial f(x)}{\partial x_{12}} - u_1 - v_2 = -2.7$ $\frac{\partial Z}{\partial x_{13}} = \frac{\partial f(x)}{\partial x_{13}} - u_1 - v_3 = -11.02$

$\frac{\partial Z}{\partial x_{14}} = \frac{\partial f(x)}{\partial x_{14}} - u_1 - v_4 = 9.58$ $\frac{\partial Z}{\partial x_{31}} = \frac{\partial f(x)}{\partial x_{31}} - u_3 - v_1 = -4.78$

$\frac{\partial Z}{\partial x_{32}} = \frac{\partial f(x)}{\partial x_{32}} - u_3 - v_2 = 4.92$ $\frac{\partial Z}{\partial x_{33}} = \frac{\partial f(x)}{\partial x_{33}} - u_3 - v_3 = -1.4$

The presence of negative values for the reduced cost signifies non optimality; hence we readjust. From the above, the minimum reduced costs for the non-basic variable is x_{13}. Therefore x_{13} should enter the basis since it is the most negative reduced cost.

We then move on to next iteration.

At the end of this stage of iteration, the basic feasible solution is:

$$\bar{x}^1 = (15, 0, 0, 0, 5, 10, 8, 2, 0, 0, 0, 10)$$

After adjusting the values x_{23} entered the solution.

Next we find the cost equation for the occupy cell.

$$\frac{\partial Z}{\partial x_{Bij}} = \frac{\partial f(x)}{\partial x_{Bij}} - u_i - v_j = 0$$

Thus,

$$\frac{\partial f(x)}{\partial x_{Bij}} = u_i + v_j$$

$u_1 + v_1 = 14.4$ $\quad$ $u_1 + v_3 = 4$ $\quad$ $u_2 + v_1 = 6.9$

$u_2 + v_2 = 5.2$ $\quad$ $u_2 + v_4 = 2.92$ $\quad$ $u_3 + v_4 = 1.8$

Letting $u_1 = 0$, from the equations we have;

$u_1 = 0$, $u_2 = -7.5$, $u_3 = -8.62$, $v_1 = 14.4$, $v_2 = 12.7$, $v_3 = 4$, and $v_4 = 10.42$

The net evaluation factor or the reduced costs for the non-basic variables is;

$\frac{\partial Z}{\partial x_{12}} = \frac{\partial f(x)}{\partial x_{12}} - u_1 - v_2 = -2.7$ $\quad$ $\frac{\partial Z}{\partial x_{23}} = \frac{\partial f(x)}{\partial x_{23}} - u_2 - v_3 = 11.02$

$\frac{\partial Z}{\partial x_{14}} = \frac{\partial f(x)}{\partial x_{14}} - u_1 - v_4 = 9.58$ $\quad$ $\frac{\partial Z}{\partial x_{31}} = \frac{\partial f(x)}{\partial x_{31}} - u_3 - v_1 = -4.78$

$\frac{\partial Z}{\partial x_{32}} = \frac{\partial f(x)}{\partial x_{32}} - u_3 - v_2 = 4.92$ $\quad$ $\frac{\partial Z}{\partial x_{33}} = \frac{\partial f(x)}{\partial x_{33}} - u_3 - v_3 = 9.62$

The presence of negative values for the reduced cost signifies non optimality; hence we readjust. From the above, the minimum reduced costs for the non-basic variable is x_{31}. Therefore x_{31} should enter the basis since it is the most negative reduced cost.

We then move on to next iteration.

At the end of this stage of iteration, the basic feasible solution is:

$$\bar{x}^2 = (7, 0, 8, 0, 13, 10, 0, 2, 0, 0, 0, 10)$$

Next we find the cost equation for the occupy cell.

$$\frac{\partial Z}{\partial x_{Bij}} = \frac{\partial f(x)}{\partial x_{Bij}} - u_i - v_j = 0$$

Thus,

$$\frac{\partial f(x)}{\partial x_{Bij}} = u_i + v_j$$

$u_1 + v_1 = 14.4$ $\quad$ $u_1 + v_3 = 4$ $\quad$ $u_2 + v_1 = 6.9$

$u_2 + v_2 = 5.2$ $\quad$ $u_2 + v_4 = 2.92$ $\quad$ $u_3 + v_4 = 1.8$

Letting $u_1 = 0$, from the equations we have;

$u_1 = 0$, $u_2 = -7.5$, $u_3 = -8.62$, $v_1 = 14.4$, $v_2 = 12.7$, $v_3 = 4$, and $v_4 = 10.42$

The net evaluation factor or the reduced costs for the non-basic variables is;

$$\frac{\partial Z}{\partial x_{12}} = \frac{\partial f(x)}{\partial x_{12}} - u_1 - v_2 = -2.7 \qquad \frac{\partial Z}{\partial x_{23}} = \frac{\partial f(x)}{\partial x_{23}} - u_2 - v_3 = 11.02$$

$$\frac{\partial Z}{\partial x_{14}} = \frac{\partial f(x)}{\partial x_{14}} - u_1 - v_4 = 9.58 \qquad \frac{\partial Z}{\partial x_{31}} = \frac{\partial f(x)}{\partial x_{31}} - u_3 - v_1 = -4.78$$

$$\frac{\partial Z}{\partial x_{32}} = \frac{\partial f(x)}{\partial x_{32}} - u_3 - v_2 = 4.92 \qquad \frac{\partial Z}{\partial x_{33}} = \frac{\partial f(x)}{\partial x_{33}} - u_3 - v_3 = 9.62$$

The presence of negative values for the reduced cost signifies non optimality; hence we readjust. From the above, the minimum reduced costs for the non-basic variable is x_{31}. Therefore x_{31} should enter the basis since it is the most negative reduced cost.

We then move on to next iteration.

At the end of this stage of iteration, the basic feasible solution is:

$$\bar{x}^3 = (7, 0, 8, 0, 3, 10, 0, 12, 10, 0, 0, 0)$$

Next we find the cost equation for the occupy cell.

$$\frac{\partial Z}{\partial x_{Bij}} = \frac{\partial f(x)}{\partial x_{Bij}} - u_i - v_j = 0$$

Thus,

$$\frac{\partial f(x)}{\partial x_{Bij}} = u_i + v_j$$

$u_1 + v_1 = 14.4$ $\quad$ $u_1 + v_3 = 4$ $\quad$ $u_2 + v_1 = 6.9$

$u_2 + v_2 = 5.2$ $\quad$ $u_2 + v_4 = 2.92$ $\quad$ $u_3 + v_1 = 1$

Letting $u_1 = 0$, from the equations we have;

$u_1 = 0$, $u_2 = -7.5$, $u_3 = -13.4$, $v_1 = 14.4$, $v_2 = 12.7$, $v_3 = 4$, and $v_4 = 10.42$

The net evaluation factor or the reduced costs for the non-basic variables is;

$$\frac{\partial Z}{\partial x_{12}} = \frac{\partial f(x)}{\partial x_{12}} - u_1 - v_2 = -2.7 \qquad \frac{\partial Z}{\partial x_{23}} = \frac{\partial f(x)}{\partial x_{23}} - u_2 - v_3 = 28.42$$

$$\frac{\partial Z}{\partial x_{34}} = \frac{\partial f(x)}{\partial x_{34}} - u_3 - v_4 = 4.78$$

$$\frac{\partial Z}{\partial x_{32}} = \frac{\partial f(x)}{\partial x_{32}} - u_3 - v_2 = 9.7 \qquad \frac{\partial Z}{\partial x_{33}} = \frac{\partial f(x)}{\partial x_{33}} - u_3 - v_3 = 14.4$$

The presence of negative values for the reduced cost signifies non optimality; hence we readjust. From the above, the minimum reduced costs for the non-basic variable is x_{12}. Therefore x_{12} should enter the basis since it is the most negative reduced cost.

We then move on to next iteration.

At the end of this stage of iteration, the basic feasible solution is:

$$\bar{x}^4 = (0, 7, 8, 0, 10, 3, 0, 12, 10, 0, 0, 0)$$

Next we find the cost equation for the occupy cell.

$$\frac{\partial Z}{\partial x_{Bij}} = \frac{\partial f(x)}{\partial x_{Bij}} - u_i - v_j = 0$$

Thus,

$$\frac{\partial f(x)}{\partial x_{Bij}} = u_i + v_j$$

$u_1 + v_2 = 10$ $u_1 + v_3 = 4$ $u_2 + v_1 = 6.9$

$u_2 + v_2 = 5.2$ $u_2 + v_4 = 2.92$ $u_3 + v_1 = 1$

Letting $u_1 = 0$, from the equations we have;

$u_1 = 0$, $u_2 = -4.8$, $u_3 = -10.7$, $v_1 = 11.7$, $v_2 = 10$, $v_3 = 4$, and $v_4 = 7.09$

The net evaluation factor or the reduced costs for the non-basic variables is;

$$\frac{\partial Z}{\partial x_{11}} = \frac{\partial f(x)}{\partial x_{11}} - u_1 - v_1 = 2.7 \qquad \frac{\partial Z}{\partial x_{14}} = \frac{\partial f(x)}{\partial x_{14}} - u_1 - v_4 = 12.91$$

$$\frac{\partial Z}{\partial x_{23}} = \frac{\partial f(x)}{\partial x_{23}} - u_2 - v_3 = 8.32 \qquad \frac{\partial Z}{\partial x_{34}} = \frac{\partial f(x)}{\partial x_{34}} - u_3 - v_4 = 2.5$$

$$\frac{\partial Z}{\partial x_{32}} = \frac{\partial f(x)}{\partial x_{32}} - u_3 - v_2 = 9.7 \qquad \frac{\partial Z}{\partial x_{33}} = \frac{\partial f(x)}{\partial x_{33}} - u_3 - v_3 = 11.7$$

Since all the reduced costs for the non-basic variables are all positive, it implies $\bar{x}^4$ is the KKT optimality point. Because optimal solution is our goal, we then proceed to make our allocation and calculate our total optimal cost of transportation.

From our feasible solution, 7000 boxes of P.Red should be supplied to market zone B, 8000 boxes to market zone C, 10000 boxes of Ovidio to market zone A, 3000 to market zone B, 12000 to market zone D, and 10000 boxes of Merlot be supplied to market zone A.

Total Cost = (10*7) + (8*4) + (10*7) + (3*6)+ (12*3)+ (10*1) thousand

Total Cost = GH¢236,000

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.0 INTRODUCTION

We have described the transportation problem of a company as a non-linear transportation problem. We applied KKT optimality algorithm to solve the company's problem. Our research focused on the model of the non-linear transportation problem for a particular company in Ghana. It can however be applied to any situation that can be modelled as such.

5.1 CONCLUSIONS

This thesis seeks to solve transportation problem with volume discount on quantity of goods shipped which is a non-linear transportation problem. Using KKT optimality algorithm, with a data from a Ghanaian company, it was observed that the optimal solution that gave minimum achievable cost of supply was the supply of 7000 boxes of P.Red to market zone B, 8000 boxes to market zone C, 10000 boxes of Ovidio to market zone A, 3000 to market zone B, 12000 to market zone D, and 10000 boxes of Merlot be supplied to market zone A at a cost of GH¢236,000.

5.2 RECOMMENDATIONS

Using the more scientific transportation problem model for the company's transportation problem gave a better result. Management may benefit from the proposed approach for their transportation problem purposes. We therefore recommend that the transportation problem model should be adopted by the company for their transportation problem planning.

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