COMPARING KRASNOSELSKIJ AND MANN ITERATIVE METHODS FOR LIPSCHITZIAN GENERALIZED PSEUDO-CONTRACTIVE OPERATORS IN HILBERT SPACES

BY

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CERTIFICATION

I hereby declare that this submission is my own work towards the MSc. Mathematics degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgment has been made in the text.

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ABSTRACT

We introduce certain iterative methods (Krasnoselskij, Mann and Ishikawa) that ensure convergence to a fixed point for certain classes of operators that satisfy weak contractive type conditions, for which the Picard iteration guarantees no convergence. Some convergence theorems are stated and proved for these classes of operators.

We finally compare the convergence rate of Krasnoselskij and Mann iterative methods known to converge to a fixed point of Lipschitzian generalized pseudo-contractive operators in Hilbert spaces.

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DEDICATION

This work is gratefully dedicated to

Mr. and Mrs. Johnson Adzoro.

CHAPTER 1

1.0 INTRODUCTION

The topic under study falls under a branch of Mathematics called Functional Analysis. It is concerned with the study of *spaces of functions and operators acting on them*. It has its historical roots in the study of transformations, such as the Fourier transform, and in the study of differential and integral equations. This usage of the word functional goes back to the calculus of variations, implying a function whose argument is a function. Its use in general has been attributed to mathematician and physicist Vito Volterra and its founding is largely attributed to mathematician Stefan Banach (www.wikipedea.org, 15/02/10, 15:50GMT).

An important object of study in functional analysis is the *continuous linear operators* defined on Banach and Hilbert spaces. In the modern view, functional analysis is seen as the study of complete normed vector spaces over the real or complex numbers. Such studies are narrowed to the study of Banach spaces. An important example is a Hilbert space, where the norm arises from an inner product.

In this thesis, we consider some fixed point theorems – the existence of fixed points using well known iterative methods of (Picard, Krasnoselskij, Mann and Ishikawa iterative schemes). We also state a theorem to compare two iteration processes (Krasnoseskij and Mann iterative schemes) for Lipschitzian generalized pseudo-contractions in Hilbert spaces. This is intended to compare to know which of the schemes converges faster to the fixed point of the operator.

Definition 1.1 Let (X, d) be a metric space. A mapping $T : X \to X$ is called ***contractive*** if

$$d(Tx, Ty) < d(x, y) \quad \forall\, x, y \in X \,; \quad x \neq y \,;$$

Definition 1.2 Let *H* be a real Hilbert Space with norm $\| \cdot \|$ and an inner product $< \cdot, \cdot >$, and *K* be a non-empty subset of *H*.

An operator $T : K \to K$ is said to be a ***generalized pseudo-contraction*** if, for all *x*, *y* in *K*, there exists a constant $r > 0$ such that

$$\| Tx - Ty \|^2 \leq r^2 \| x - y \|^2 + \| Tx - Ty - r(x - y) \|^2$$

and ***Lipchitzian*** if there exist a constant $L > 0$ such that

$$\| Tx - Ty \| \leq L \cdot \| x - y \| \quad \text{for all } x, y \text{ in } K.$$

Generally, the ambient space *X*, say, considered in fixed point theorems cover a variety of spaces: lattice, metric space, normed linear space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator *T*, say, are generally metrical or compactness type conditions.

Judged from the perspective of its concrete applications, that is, from a numerical point of view, a fixed point theorem is valuable if, apart from the conclusion regarding the existence (and, possible, uniqueness) of the fixed point, it also satisfies some minimal numerical requirements, amongst which we mention:

(a) it provides a method (generally, iterative) for constructing fixed point(s);

(b) it is able to provide information on the error estimate (rate of convergence) of the iterative process used to approximate the fixed point, and

(c) it can give concrete information on the stability of this procedure, that is, on the data dependence of the fixed point(s).

Only a few fixed point theorems in literature are known to fulfill all three requirements above. Moreover the error estimate and the data dependence of fixed points appear to have been given for Picard iteration (sequence of successive approximation) in conjunction with various contraction conditions.

Example 1.1 If $T : X \to X$ is an *a−contraction* on a complete metric space, (X, d) that is, there exists a constant $0 \le a < 1$ such that

$$d(Tx, Ty) \le a\, d(x, y) \quad \forall\, x, y \in X$$

then by contraction mapping theorem (Banach) we know that

(a) *Fix (T)* = $\{x^*\}$;

(b) $x_n = T^n x_0$ Picard iteration converges to x^* for all $x_0 \in X$;

(c) Both the *a priori* and the *a posteriori* estimates

$$d(x_n, x^*) \le \frac{a^n}{1-a} \cdot d(x_0, x_1), \quad n = 0, 1, 2, \ldots, \tag{1}$$

$$d(x_n, x^*) \le \frac{a}{1-a} \cdot d(x_{n-1}, x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

respectively hold.

(d) The rate of convergence is given by

$$d(x_n, x^*) \le a \cdot d(x_{n-1}, x^*) \le a^n \cdot d(x_0, x^*) \quad n = 0, 1, 2, \ldots, \tag{3}$$

Remark.

The errors $d(x_n, x^*)$ are decreasing rapidly as the terms of geometric progression with ratio *a*, that is, $\{x_n\}_{n=0}^{\infty}$ converges to x^* at least as rapidly as the geometric series. The convergence is however *linear*, as shown by

$$d(x_n, x^*) \le a \cdot d(x_{n-1}, x^*), \quad n = 0, 1,2,...,$$

If T satisfies a weaker contractive condition, e.g., T is nonexpansive, then Picard iteration does not converge, generally, or even if it converges, its limit is not a fixed point of *T*. More general iterative procedures are needed. (Berinde, 2002).

Definition 1.3 Let (X, d) be a metric space. A mapping $T : X \to X$ is called ***nonexpansive*** if *T* is *1-Lipschitzian.*

Note. *T* is *L-Lipschitzian* if $d(Tx, Ty) \le L \cdot d(x, y) \quad \forall\, x, y \in X$;

Under weak contractive conditions, the general problem of studying the rate of convergence of a fixed point iterative method arises usually into two different contexts.

1) For certain fixed point iterative method, (Picard, Krasnoselskij, Mann, Ishikawa etc.) an analytical error estimate is not known. In this case an empirical study of the rate convergence is studied.

2) For large classes of operators (like quasi-contractions) two or more fixed point iteration procedures are known to be able to approximate the fixed points. In such situations, it is of theoretical importance to compare these methods with regard to the convergence rate, in order to establish if possible,

which one converges faster with respect to a certain concept of rate of convergence.

Definition 1.4 Let E be a normed space. An operator $T : E \to E$ is said to be *quasi-contractive* if there exists a number α, $0 \le \alpha < 1$ such that for all x, y in E

$$\| Tx - Ty \| \le \alpha \cdot M(x, y),$$

where

$$M(x, y) := \max\{\| x - y \|, \| x - Tx \|, \| y - Ty \|, \| x - Ty \|, \| y - Tx\| \}$$

In the absence of theoretical results, some authors have performed an empirical study of the rate of convergence of fixed point iterative methods, using the FIXPOINT software package, specially designed for that purpose (Berinde, 2007; Babu, 2006; Chatterjea, 1972; Ciric, 1974; Hardy and Rogers1973; Rhoades, 1976).

The empirical approach of the rate of convergence of fixed point iteration procedures is still of scientific interest and perspective because it also offers the possibility of inferring theoretical rate of convergence from empirical observations.

Fixed point iterative procedures are designed to be applied in solving concrete nonlinear operator equations, variational inequalities, etc. The classical importance of fixed point theory in functional analysis is due to its usefulness in the theory of ordinary and partial differential equations. The existence or construction of a solution to a differential equation is often reduced to the existence or location of a fixed point for an operator defined on a subset of a space of functions. Fixed point theorems have also been used to determine the existence of periodic solutions for functional equations when solutions are already known to exist.

The importance of metrical fixed point theory consists mainly in the fact that for most functional equations $F(x) = y$ we can equivalently transform them in a fixed point problem $x = T\,x$ and then apply a fixed point theorem to get information on the existence or existence and uniqueness of the fixed point, that is, of a solution for the original equation. Moreover, fixed point theorems usually provide a method for constructing such a solution.

Fixed point theorems are also used to obtain existence or existence and uniqueness theorems for various classes of operator equations (differential equations, integral equations, integro-differential equations, variational inequalities etc.)

Apart from this deep involvement in the theory of differential equations, fixed point theorems have been extremely useful in such problems as finding zeros of non-linear equations and proving surjectivity theorems. Partly as a consequence of the importance of its applications, fixed point theory has developed into an area of independent research.

Problems concerning the existence of fixed points for Lipschitz map have been given considerable interest in nonlinear Operator Theory. The study of nonlinear operators had its beginning about the start of the twentieth century with investigations into the existence property of solutions to certain boundary value problems in ordinary and partial differential equations. The earliest techniques, largely devised by E. Picard, involved the iteration of an integral operator to obtain solutions of such problems (Chidume, 1996). In 1922 these techniques of Picard were given precise abstract formulation by S. Banach and R. Cacciopoli in what is now generally referred to as the *Contraction Mapping Principle.* It is involved in many of the existence and uniqueness proofs of ordinary differential equations, and is probably the most useful fixed point theorem (Chidume, 1996).

An earlier fixed point theorem, called the *Brouwer Fixed Point Theorem*, concerns *continuous* mappings and has an advantage over The Banach Contraction Mapping Principle in that it applies to a much larger class of functions. It is, however, in a sense weaker than the Banach Contraction Mapping Principle because the sequence of iterates of the function at a given point need not converge to a fixed point. Furthermore it is confined to finite dimensional spaces.

1.1 Brouwer Fixed Point Theorem (1910)

Let B be the closed unit ball of any finite dimensional Euclidean space and $f : B \to B$ be continuous. Then f has a fixed point.

The first analytic attempt at generalizing Brouwer's Fixed Point Theorem to infinite dimensional spaces was made by Birkoff and Kellog. They were able to show that a continuous operator defined from a compact, convex subset of $C^n[0,1]$ into itself has a fixed point. This result was then applied in solving certain differential and integral equations. Further generalizations resulted in the following theorem:

1.2 Schauder–Tychonov Theorem

Let K be a compact convex subset of a Banach space E. If $T : K \to K$ is continuous, then T has a fixed point.

Despite the fact that there is no known constructive technique for determining a fixed point of *T*, the Schauder –Tychonov fixed point theorem is extremely important in the proofs of many existence theorems of differential equations. (Chidume, 1996)

1.3 Fixed Point Iterations

Iteration means to repeat a process integrally over and over again. To iterate a function, we begin with a *seed* for the iteration. This is a (real or complex) number x_0, say. Applying the function to x_0 yields the new number, x_1, say. Usually the iteration proceeds using the result of the previous computation as the input for the next. A sequence of numbers x_0, x_1, x_2,… is then generated. A very important question, then, is whether this sequence converges or diverges – and particularly for the purpose of this work, whether it converges to a ***fixed point*** or not.

This work focuses on fixed point theorems for maps defined on some ambient spaces (i.e. *Metric, Normed, Banach, Hilbert* spaces, etc.) and satisfying a variety of conditions. A lot of metrical fixed point theorems have been obtained, more or less important from a theoretical point of view, which establishes usually the existence, or the existence and the uniqueness of fixed points for a certain contractive operator. Even so only a small number of these fixed point theorems are important from a practical point of view, that is, they offer a constructive method for finding fixed points. Among the last ones, only a few gave information on the error estimate (the rate of convergence) of the method.

From a practical point of view it is important not to know that a fixed point exists (and, possible is unique), but also to be able to construct that fixed point(s). Since the constructive methods used in metrical fixed point theory are generally *iterative* procedures, it is also of crucial importance to have a priori or a posteriori error estimates (or alternatively, rate of convergence) for such a method.

In the last four decades, numerous papers were published on the iterative approximation of fixed points of self and nonself contractive type operators in metric spaces, Hilbert spaces or several classes of Banach spaces (Berinde, 2003).

In order to approximate fixed points of certain classes of operators which satisfy weak contractive type conditions that do not guarantee the convergence of Picard iterative process (or method of successive approximation) certain mean value fixed point iterations, namely Krasnoselskij, Mann and Ishikawa iteration methods are useful to approximate fixed points. Though these iterative procedures have been introduced mainly in order to approximate fixed points of those operators for which the Picard iteration does not converge, even so there are results on important classes of contractive mappings, that is, the class of quasi-contractions, for which all Picard, Krasnoselskij, Mann, and Ishikawa iterations converge.

The Krasnoselskij iteration [15], [5], [12], [13], the Mann iteration [16], [8], [17] and the Ishikawa iteration [10] are certainly the most studied of these fixed point iteration procedures, [1] (Berinde, 2003).

The classical Banach's contraction principle is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem 1.1. *Let* (X, d) *be a complete metric space and* $T: X \to X$ *a strict contraction, i.e. a map satisfying*

$$d(Tx, Ty) \leq a\, d(x, y), \quad \text{for all} \quad x, y \in X, \tag{1.1.1}$$

where $0 < a < 1$ *is a constant. Then T has a unique fixed point p and the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = T x_n \tag{1.1.2}$$

converges to p, for any $x_0 \in X$.

Theorem 1.1. has many applications in solving nonlinear equations, but suffers from one drawback – the contractive condition (1.1.1) forces *T* be continuous on *X*.

Theorem 1.2 *Let* (X, d) *be a complete metric space and* $T: X \to X$ *a map for which there exist the real numbers* α, β *and* γ *satisfying* $0 < \alpha < 1,\ 0 < \beta,\ \gamma < 1/2$ *such that for each pair* $x, y \in X$ *, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq \alpha\, d(x, y)$;
(z_2) $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$;
(z_3) $d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration defined by

$$x_{n+1} = T x_n, \quad n = 0, 1, 2, \ldots$$

converges to p, for any $x_0 \in X$.

[The proof of the above Theorem 1.2 is clearly stated in chapter 2]

One of the most general contraction condition for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ciric [7] in 1974: there exists $0 < h < 1$ such that

$$d(Tx,Ty) \le h \cdot \max\{ d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx) \} \quad \forall\ x,y \in X \quad (1.2.1).$$

Theorem 1.3 *Let E be an arbitrary Banach space, K a closed convex subset of E, and* $T: X \to X$ *an operator satisfying condition of Theorem 1.2. Let* $\{x_n\}_{n=0}^{\infty}$ *be the Mann* iteration, *for* $x_0 \in X$ *with* $\alpha_n \in [0,1]$ *satisfying*

$$\sum_{n=0}^{\infty} \alpha_n = \infty . \qquad (ii)$$

Then the sequence converges strongly to the fixed point of T.

In 1968 R. Kannan [11], obtained a fixed point theorem which extends Theorem 1.3 to mappings that need not be continuous, by considering instead of (1.1.2) the next condition: there exists $b \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le b\left[d(x, Tx) + d(y, Ty) \right] \quad \forall\, x,y \in X. \qquad (1.3.1)$$

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of *T*, see for example, Rus [22], and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [6], is based on a condition similar to (1.3.1): there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le c\left[d(x, Ty) + d(y, Tx) \right], \quad \forall\, x,y \in X. \qquad (1.3.2)$$

It is known, from Rhoades [19] that (1.1.1) and (1.3.1), (1.1.1) and (1.3.2), respectively, are independent contractive conditions.

In 1972, Zamfirescu [24] obtained a very interesting fixed point theorem, by combining (1.1.1), (1.3.1) and (1.3.2). The theorem is stated above (Theorem 1.2) without proof.

Remarks.

A mapping satisfying (1.2.1) is commonly called ***quasi contraction*.** It is obvious that each of the conditions (1.1.1), (1.3.1), (1.3.2) and (z_1)– (z_3) implies (1.2.1). An operator *T* which satisfies the contractive conditions in Theorem 1.2 will be called a Zamfirescu.

One of the most studied class of quasi-contractive type operators is that of Zamfirescu operators, for which all important fixed point iteration procedures, i.e., the Picard [24], Mann [17] and Ishikawa [18] iterations, are known to converge to the unique fixed point of *T*. Zamfirescu showed in [24] that an operator which satisfies the contractive conditions in Theorem 1.2 has a unique fixed point that can be approximated using the Picard iteration. Later, Rhoades [17], [18] proved that the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu operators.

The class of operators satisfying contractive conditions in Theorem 1.2 is independent, see Rhoades [17], of the class of strictly (strongly) pseudocontractive operators, extensively studied by several authors in the last years. For the case of pseudocontractive type operators, the pioneering convergence theorems, due to Browder [4] and Browder and Petryshyn [5], established in Hilbert spaces, were successively extended to more general Banach spaces and to weaker conditions on the parameters that define the fixed point iteration procedures, as well as to several classes of weaker contractive type operators.

It is shown by Rhoades ([18], Theorem 8), that in a uniformly Banach space *E*, the Ishikawa iteration, for $x_0 \in K$ converges (strongly) to the fixed point of *T*, where $T: K \to K$ is a mapping satisfying conditions of Theorem 1.2, *K* is a closed convex subset of *E*, and $\{\alpha_n\}$, a sequence of numbers in [0, 1] such that

$$\sum_{n=0}^{\infty} \alpha_n (1-\alpha_n) = \infty . \tag{i}$$

[Note: The various iterative schemes are more fully defined in the chapter 2].

Verma [7] approximated fixed points of Lipschitzian and generalized pseudo-contractive operators in Hilbert spaces by both Krasnoselskij and Mann type iterative methods. When for a certain class of mappings, two or more fixed points iteration procedures can be used to approximate their fixed points, it is of theoretical and practical importance to compare the rate of convergence of these methods and to find out, if possible which of them converges faster.

This work focuses on some Fixed Point Theorems – thus we major on the existence of fixed points of four iterative procedures for certain operators, and it also compares the convergence rate of the Krasnoselskij and Mann iterative methods, both known to converge to a fixed point of Lipschitzian generalized pseudo-contractive operators. Finally, we shall obtain a result on the fastest iteration in the family of the Krasnoselskij iterative scheme.

CHAPTER 2

2.0 BASIC DEFINITIONS AND CONVERGENCE RESULTS

We consider some basic definitions of maps, spaces and convergence results.

2.1 Fixed Point: Let X be a non-empty set and $T : X \to X$ a self map. We say $x \in X$ is a ***fixed point*** of T if $T(x) = x$.

We denote the set of all fixed points of T by $F_T = \{x \in X \,/\, T(x) = x\}$ or by *Fix T*.

2.2 Linear Map: Let X and Y *be* linear spaces over a scalar field *K*. A mapping $T : X \to Y$ is said to be a *linear map* if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

For arbitrary $x, y \in X$ and arbitrary scalars $\alpha, \beta \in K$. Sometimes the terms ***linear operator*** or ***linear transformation*** are used instead of ***linear map*****.** The above condition is equivalent to the following two conditions:

i) $T(x+y) = T(x) + T(y) \quad \forall\, x, y \in X$; and

ii) $T(\alpha x) = \alpha Tx \quad \forall\, x \in X$ and for each scalar, α.

2.3 Strong Contractive Map: Let $(X, \|\,,\|)$ be a normed vector space over Real or Complex numbers. An operator T acting from a closed convex set $\Omega \subset X$ into itself is strongly contractive if there exists a constant $0 < q < 1$ such that for all x and y in Ω,

$$\| Tx - Ty \| \le q \| x - y \|.$$

Then the well known Banach contraction mapping principle asserts that there exists a fixed point x^* of the map T in Ω (i.e. $Tx^* = x^*$) and it is unique. The approximating sequence defined by $x_{n+1} = Tx_n$, $n = 1, 2, \ldots.$ converges strongly to x^*.

2.4 Weak Contraction: Let (X, d) be a metric space. A mapping $T : X \to X$ is called ***weak contraction*** if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad \forall\, x, y \in X \tag{2.4.1}$$

Remark.

Due to the symmetry of the distance, the weak contraction condition (2.4.1) implicitly includes the following dual one

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(x, Ty), \quad \forall\, x, y \in X \tag{2.4.2}$$

obtained from (2.4.1) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$ respectively, and then interchanging *x* and *y*. Consequently, in order to check the weak contractiveness of *T*, it is necessary to check both (2.4.1) and (2.4.2).

Obviously, any strict contraction satisfies (2.4.1), with $\delta = a$ and $L = 0$, and hence is a weak contraction (that possesses a unique fixed point).

2.5 Let (X, d) be a metric space. A mapping $T : X \to X$ is called

2.5.1 Lipschitzian (or ***L* – *Lipschitzian***) if there exists $L > 0$ such that

$$d(Tx, Ty) \leq L \cdot d(x, y) \qquad \forall\, x, y \in X\ ;$$

2.5.2 (Strict) contraction (or a-contraction) if there exists a constant $a \in (0,1]$ such that T is a-Lipschitzian;

2.5.3 Nonexpansive if T is 1-*Lipschitzian*;

2.5.4 Contractive if $d(Tx, Ty) < d(x, y) \quad \forall\, x, y \in X\,; \quad x \neq y\,;$

2.5.5 Isometry if $d(Tx, Ty) = d(x, y) \quad \forall\, x, y \in X\,;$

2.6 Metric Space: Let X be a non empty set. A mapping $d : X \times X \to R$ is called metric or distance on X provided that

(d_1) $d(x, y) \geq 0$ for every pair $x, y \in X$

(d_2) $d(x, y) = 0 \Leftrightarrow x = y$

(d_3) $d(y, x) = d(x, y)$, for all $x, y \in X$;

(d_4) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$ (triangular inequality).

A set X endowed with metric d is called ***metric space*** and is denoted by (X, d).

2.7 Norm Vector Space: Let X be a linear space over K (field of all Real or Complex nnumbers). A norm on X is a real-valued function $\| \cdot \|$,

$$\| \ , \ \| : X \to [0, \infty)$$

such that the following conditions are satisfied:

N1 $\| x \| \geq 0$ for every $x \in X$

N2 $\| x \| = 0$ if and only if $x = 0$,

N3 $\| kx \| = |k| \| x \|$ for all $k \in K$ and $x \in X$,

N4 $\| x + y \| \leq \| x \| + \| y \|$ (triangle inequality)

The pair $(X, \| \ , \ \|)$ is called a ***normed (linear) space.***

2.8 Banach Contraction Mapping Principle: Let (X,ρ) be a complete metric space and $T : X \to X$ be a contraction map. Then T has a unique fixed point in X. moreover, the sequence $\{T^n(x_0)\}_{n=0}^{\infty}$ converges to the fixed point.

Various generalizations of the contraction mapping principle abound, and are usually obtained in two ways:

1) By weakening the contractive properties of the map and, possibly, by simultaneously giving the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness;
2) By extending the structure of the ambient space.

2.9 Inner Product: Let X be a linear space. An inner product on X is a function $\langle ; \rangle : X \times X \to C$ (the set of complex numbers) such that the following three conditions are satisfied: for $x, y, z \in X$, $\alpha, \beta \in C$,

$I_1 : \langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only $x = 0$

$I_2 : \langle x, y \rangle = \overline{\langle x, y \rangle}$ where the "bar" indicates complex conjugation

$I_3 : \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$. The pair $(X, \langle ; \rangle)$ is called an inner product space.

2.10 Hilbert Space: A sequence $\{x_n\}_{n=0}^{\infty}$ in X is called Cauchy if and only if

$\langle x_n - x_m, x_n - x_m \rangle^{1/2} := \| x_n - x_m \| \to 0$ as $n, m \to 0$

An inner product space X is said to be complete if every Cauchy sequence in X converges to a point of X. A complete inner product space is called a ***Hilbert space***.

2.11 Convex Set: The subset C of a real vector space X is called convex if, for any pair of points x, y in C, the closed segment with extremities x, y, that is, the set $\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ is contained in C. A subset C of a real normed space is called *bounded* if there exists $M > 0$ such that $\| x \| \leq M$, for all $x \in C$.

2.12 Let H be a real Hilbert Space with norm $\| \cdot \|$ and an inner product $< \cdot, \cdot >$, and K be a non-empty subset of H. An operator $T : K \to K$ is said to be a ***generalized pseudo-contraction*** if, for all x, y in K, there exists a constant $r > 0$ such that

$$\| Tx - Ty \|^2 \leq r^2 \| x - y \|^2 + \| Tx - Ty - r(x-y) \|^2 \tag{2.1}$$

condition (2.1) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq r \| x - y \|^2 \tag{2.2}$$

or to

$$\langle (I-T)x - (I-T)y, x - y \rangle \geq (1-r) \| x - y \|^2,$$

where I is the identity map. Clearly, if T is ***generalized pseudo-contraction*** with $r < 1$, then $I - T$ is strongly monotone.

For $r = 1$ in (2.1), T is called ***pseudo-contraction*.**

The operator T is called ***Lipschitzian* (or Lipschitz continuous)** if there exist a constant $L > 0$ such that

$$\| Tx - Ty \| \leq L \cdot \| x - y \| \quad \text{for all } x, y \text{ in } K. \tag{2.3}$$

By the Cauchy-Schwarz inequality,

$$|\langle Tx - Ty, x - y \rangle| \leq \| Tx - Ty \| \cdot \| x - y \|,$$

It is clear that any Lipschitzian operator T, that is, for which there exists $L > 0$ such that

$$\| Tx - Ty \| \leq L \cdot \| x - y \| \qquad x, y \in K$$

is also a generalized pseudo-contractive operator with $r = L$. Consequently, for a Lipschitzian operator with $L > 0$, the only reason to consider also a *generalized pseudo-contractive* condition of the form (2.1) is that r could be smaller than L.

The following example shows that an operator T can be simultaneously Lipschitzian with constant L and generalized pseudo-contractive with constant r, and $r < L$. It also shows the limitation of the Picard iteration in approximating fixed points of certain operators.

Example 2.1 Let H a real line with usual norm, $K = \left[\frac{1}{2}, 2\right]$ and $T : K \to K$ a self map defined by $Tx = \frac{1}{x}$, for all x in K. Then L is Lipschitzian with constant $L = 4$ and also

generalized pseudo-contractive with constant $r = 4$. Moreover, T is also generalized pseudo-contractive with any constant $r > 0$ arbitrary.

The Picard iteration, $x_{n+1} = Tx_n$, $n > 0$, does not converge, for any initial guess $x_o \neq 1$(which is the unique fixed point of T). The Picared iteration yields an oscillatory sequence

$$\frac{1}{x_0}, x_0, \frac{1}{x_0}, \ldots$$

In order to approximate fixed points of the operators considered in this work we shall make use of other well known iterative methods.

2.13 Let E be an arbitrary real Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is called

(a) ***Strongly Pseudo-contraction*** if there exists $k>0$ such that for all $x, y \in D(T)$ there exists $j(x, y) \in J(x-y)$ such that

$$\langle (I-T)x-(I-T)y,\ j(x-y)\rangle \geq k \cdot \| x-y \|^2;$$

(b) ***Pseudo-contractive*** if for each $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-T)x-(I-T)y,\ j(x-y)\rangle \geq 0$$

where J is normalized duality mapping.

2.14 A mapping U with domain and range in E is called

(a) ***Strongly accretive*** if there exists a positive number k such that for each $x, y \in D(U)$

there exists a $j(x-y) \in J(x-y)$ such that

$$\langle Ux-Uy,\ j(x-y)\rangle \geq k\| x-y \|^2;$$

(b) ***Accretive*** if for each $x, y \in D(U)$ we have

$$\langle Ux-Uy,\ j(x-y)\rangle \geq 0$$

Remarks.

1) By comparing Definitions 2.11 and 2.12, ,we remark that an operator T is (strongly) pseudo-contractive if and only if $(I-T)$ is (strongly) accretive;

2) As a consequence of a result of Kato [1], the concepts of pseudo-contractive and accretive operators can be equivalently defined as follows:

(i) T is strongly pseudo-contractive if there exists $t > 1$ such that, for all $x, y \in D(T)$ and $k > 0$, the following inequality holds

$$\| x - y \| \leq \| (1 + r)(x + y) - rt\,(Tx - Ty) \|;$$

(ii) T is **pseudocontractive** if $t = 1$ in the previous inequality;

(iii) T is strongly accretive if there exists $k > 0$ such that the inequality

$$\| x - y \| \leq \| (x - y) + r\,[\,(T - kI)x - (T - kI)y\,] \|$$

holds for all $x, y \in D(U)$ and $r > 0$;

(iv) T is accretive if $k = 0$ in previous the inequality.

2.15 The Picard iteration method: Let X be any set and $T : X \to X$ a self map. For any $x_0 \in X$, the sequence $\{ x_n \}_{n \geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \qquad n = 1, 2, \ldots$$

is called the ***sequence of successive approximation*s** with the initial value x_0. It is also known as the ***Picard iteration***.

2.16 The Krasnoselskij iteration method. For $x_o \in K$ and $\lambda \in [0, 1]$ the sequence $\{ x_n \}_{n=0}^{\infty}$, defined by

$$x_{n+1} = (1 - \lambda)\, x_n + \lambda\, Tx_n, \qquad n = 0, 1, 2, \ldots \qquad (2.4)$$

is called *Krasnoselskij iterative method* or *Kranoselskij iteration* and is denoted by $K_n(x_0, \lambda, T)$.

2.17 The Mann iteration method. For $x_o \in K$ and $\{\alpha\}_{n=0}^{\infty}$ a sequence in $[0,1]$, the sequence $\{y_n\}_{n=0}^{\infty}$ defined by

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \qquad n = 0, 1, 2, \ldots \tag{2.5}$$

is called *Mann iterative method* or *Mann iteration* and will be denoted by $M_n(y_0, \alpha_n, T)$

2.18 The Ishikawa iteration method: For $x_o \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-a_n)x_n + a_n T\left[(1-b_n)x_n + b_n \cdot Tx_n\right] \qquad n = 0, 1, 2, \ldots, \tag{2.6}$$

where $\{a\}_{n=0}^{\infty}$ $\{b\}_{n=0}^{\infty}$ are sequences of reals satisfying $0 \le a_n,\ b_n < 1$ is called the *Ishikawa iteration*, and is denoted by $I(x_0, \alpha_n, \beta_n, T)$.

The above equation (2.6) can be rewritten in a system form

$$\begin{cases} y_n = (1-b_n)x_n + b_n T x_n & n = 0, 1, 2, \ldots \\ x_{n+1} = (1-a_n)\cdot x_n + a_n T y_n & n = 0, 1, 2, \ldots \end{cases} \tag{2.7}$$

Remarks

It's obvious that, for $\lambda = 1$, the Krasnoselskij iteration reduces to the Picard iteration (the method of successive approximations), while for $\alpha_n = \lambda$ (const), the Mann iteration reduces to the Krasnoselskij method.

Again equation (2.7) can be regarded as a sort of double Mann iteration, with two different parameters sequences.

Despite this apparent similarity and the fact that, for $b_n = 0$, Ishikawa iteration reduces to the Mann iteration, there is not a general dependence between convergence results for the Mann iteration and Ishikawa iteration.

As mentioned in the introduction the special software package (the FIXEDPOINT software) was designed by Andrei BOŽ̂ANTAN (as a MSc Dissertation thesis). The execution of the program FIXPOINT for some input data had lead to the following observations:

1) The Krasnoselskij iteration converges to $p = 1$ for any $\lambda \in (0,1)$ and any initial guess x_0 (recall that the Picard iteration does not converge for any initial value $x_0 \in [1/2, 2]$ different from the fixed point). The convergence is slow for λ close enough to 0 (that is, for Krasnoselskij iterations close enough to the Picard iteration) or close enough to 1. The closer to $1/2$, the middle point of the interval $(0,1)$, λ is, the faster it converges.

For $\lambda = 0.5$ the Krasnoselskij iteration converges very fast to $p = 1$, the unique fixed point of T. For example, starting with $x_0 = 1.5$, only 4 iterations are needed in order to obtain p with 6 exact digits: $x_1 = 1.08335$, $x_2 = 1.00325$, $x_3 = 1.000053$, and $x_4 = 1$,

For the same value of λ and $x_0 = 2$, again only 4 iterations are needed to obtain p with the same precision, even though the initial guess is far away from the fixed point: $x_1 = 1.25$, $x_2 = 1.025$, $x_0 = 1.0003$, and $x_4 = 1$;

2) The speed of Mann and Ishikawa iterations also depends on the position of $\{\alpha_n\}$ and $\{\beta_n\}$ in the interval $(0,1)$. If we take $x_0 = 1.5$, $\alpha_n = 1/(n+1)$, $\beta_n = 1/(n+2)$, then the Mann and Ishikawa iterations converge (slowly) to $p=1$: after $n=35$ iterations we get: $x_{35} = 1.000155$ for both Mann and Ishikawa iterations.

For $\alpha_n = 1/\sqrt[3]{n+1}$, $\beta_n = 1/\sqrt[4]{n+2}$, we obtain the fixed point with 6 exact digits performing 8 iterations (the Mann scheme) and, respectively, 9 iterations (the Ishikawa iteration). Notice that in this case both Mann and Ishikawa iterations converge not monotonically to $p=1$.

Conditions like $\alpha_n \to 0$ (as $n \to \infty$) or /and $\beta_n \to 0$ (as $n \to \infty$) are usually involved in convergence theorems. The next results show that these conditions are in general not necessary for the convergence of Mann and Ishikawa iterations.

Indeed, taking

$$x_0 = 2,\ \alpha_n = \frac{n}{2n+1} \to \frac{1}{2}, \qquad \beta_n = \frac{n+1}{2n} \to \frac{1}{2},$$

we obtain the following results.

For the Mann iteration: $x_1 = 2$, $x_2 = 1.5$, $x_3 = 1.166$, $x_4 = 1.034$, $x_5 = 1.0042$ $x_6 = 1.00397$, $x_7 = 1.000031$, $x_8 = 1.000002$, and $x_9 = 1$,

For the Ishikawa iteration: $x_1 = x_2 = 2$, $x_3 = 1.357$, $x_4 = 1.120$, $x_5 = 1.0289$ $x_6 = 1.0047$, $x_7 = 1.0057$, $x_8 = 1.000054$, $x_9 = 1.00004$, and $x_{10} = 1$,

For all combinations of x_0, λ, α_n and β_n, we notice the following decreasing (with respect to their speed of convergence) chain of iterative methods: Krasnoselskij, Mann, Ishikawa. Consequently, if for a certain operator in the same class, all these methods converge, then we shall use the fastest one (empirically deduced).

The next example presents a function with two repulsive fixed points with respect to the Picard iteration

Example. Let $K = [0,1]$ and $T : K \to K$ given by $Tx = (1-x)^6$.

Then T has $p_1 \approx 0.2219$ and $p_1 \approx 2.1347$ as fixed points (obtained with Maple).

Here there are some numerical results obtained by running the new version of the program FIXPOINT, to support the previous assertions.

Krasnoselskij iteration: for $x_0 = 2,$ and $\lambda = 0.5$, we obtain $x_1 = 1.5,$ $x_2 = 0.757,$ $x_3 = 0.379,$ $x_4 = 0.2181$ $x_5 = 0.2322$ and $x_6 = 0.2214;$

Mann iteration: for $x_0 = 2,$ and $\alpha_n = 1/(n+1)$, we obtain $x_1 = 1.0,$ $x_2 = 0.5,$ $x_3 = 0.338,$ $x_4 = 0.2748$ $x_5 = 0.2489$ and $x_6 = 0.2378;$

Ishikawa iteration: for $x_0 = 2,$ $\alpha_n = 1/(n+1)$, and $\beta_n = 1/(n+1)$ we obtain and $x_1 = 0.01,$ $x_2 = 0.55,$ $x_3 = 0.346,$ $x_4 = 0.2851,$ $x_5 = 0.2527$ and $x_6 = 0.2392.$

The previous numerical results suggest that Krasnoselskij iteration converges faster than both Mann and Ishikawa iterations. This fact is more clearer illustrated if we choose $x_0 = p_2$, the repulsive fixed point of *T*: after 20 iterations, Krasnoselskij method gives $x_{20} = 0.2219,$ while Mann and Ishikawa iteration procedures give $x_{20} = 0.6346,$ and $x_{20} = 0.6347,$ respectively. The convergence of Mann and Ishikawa iteration procedures is indeed very slow in this case: after 500 iterations we get $x_{500} = 0.222,$ for both methods.

Note that for $x \in \{-2, 3, 4\}$ and the previous values of the parameters λ, α_n and β_n all the three iteration procedures: Krasnoselskij, Mann and Ishikawa, converge to 1, which is not a fixed point of *T*.

The convergence Theorems below, stated for the Krasnoselskij and Mann iterative methods respectively, by Verma [7] will be used in the proof of the main results.

Theorem 2.2 *Let K be a non-empty closed convex subset of a real Hilbert space H, and $T : K \to K$ be a Lipschitzian and generalized pseudo-contractive, with the corresponding constants $L > 0$ and $r > 0$ satisfying*

$$0 < r < 1 \text{ and } r \le L \tag{2.2.1}$$

Then:

(i) T has a unique fixed point p in K;

(ii) The Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \lambda, T)$ converges strongly to p, for any $x_0 \in K$ and all $\lambda \in (0, a) \cap (0, 1)$, where

$$a = 2(1-r)/(1-2r+L^2) \tag{2.2.2}$$

Theorem 2.3 *Let H be a real Hilbert space and K be a non-empty closed convex subset of H. Let $T : K \to K$* be *Lipschitzian and generalized pseudo-contractive operator with the corresponding constants L>1 and $r > 0$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be an increasing sequence in $[0,1]$ such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \tag{2.3.1}$$

Then

(i) T *has a unique fixed point p in K;*

(ii) *The Mann iteration* $\{y_n\}_{n=0}^{\infty} = M_n(y_0, t\alpha_n, T)$ *converges strongly to p for any* $y_0 \in K$ *and all t in* $(0, a)$ *that satisfy*

$$0 \le (1-t)^2 - 2t(1-t)r + t^2 L^2 < 1$$

where a is given by (2.2.2)

Remarks

Theorem 2.2 was obtained under the assumptions $r < 1$ and $L \ge 1$. Thus in the following we shall assume that the Lipschitzian constant r and the generalized pseudo-contractivity constant L fulfill the conditions.

$$0 < r < 1 \quad \text{and} \quad r < L. \tag{β}$$

2.19 Rate of Convergence

Now in order to compare two fixed point iteration procedures, we shall make use of the following concept of rate of convergence, introduced and studied by Berinde [1, 2, 3, 4].

Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume there exists

$$l = \lim_{n \to \infty} \frac{|a_n - a|}{|b_n - b|}$$

a) If $l = 0$, then it is said that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b;

b) If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

c) If $l = \infty$, $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{a_n\}_{n=0}^{\infty}$.

Suppose that for two fixed points iteration, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point *p*, the following error estimates

$$\| u_n - p \| \le a_n, \qquad n = 0, 1, 2, \ldots \qquad (*)$$

and

$$\| v_n - p \| \le b_n \qquad n = 0, 1, 2, \ldots$$

$(**)$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences of positive numbers (both converging to 0).

Definition 2.1. Let and $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iterations procedures that converge to the same fixed point *p*, such that $(*)$ and $(**)$ are satisfied.

If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then we say that $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ or simply, that $\{u_n\}_{n=0}^{\infty}$ is better than $\{v_n\}_{n=0}^{\infty}$.

Remarks:

Rhoades [6] considered that $\{u_n\}_{n=0}^{\infty}$ is better than $\{v_n\}_{n=0}^{\infty}$ if

$$\| u_n - p \| \le \| v_n - p \|, \quad \text{for all } n,$$

Example 2.2. Consider p= 0, $u_n = \frac{1}{n+1}$ and $v_n = \frac{1}{n}$, $n \geq 1$.

Then $\{u_n\}$ is better than $\{v_n\}$ in this sense;

i.e.

$$\| u_n - p \| \leq \| v_n - p \|, \quad \text{for all } n,$$

although $\{u_n\}$ and $\{v_n\}$ have the same rate of convergence, in the sense of Definition 2.1, since $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$.

The previous example also shows that the concept used by Rhoades [6] is independent by that given by Definition 2.1.

2.20 Some Results Obtained from the Iterative Methods

Results have shown that, if an operator T is continuous and the Mann iterative process converges, then it converges to a fixed point of T. But if T is not continuous, then there is no guarantee that, even if the Mann process converges, it will converge to a fixed point of T, as shown by the following example.

Example 1.1. Let $T:[0,1] \to [0,1]$ be given by $T(0) = T(1) = 0$ and Tx = 1, $0 < x < 1$.

Then $F_T = \{0\}$ and the Mann iteration $M(x_1, \alpha_n, T)$ with $0 < x_1 < 1$ and $\alpha_n = \frac{1}{n}$, $n \geq 1$, converges to 1, which is not a fixed point of T. (F_T denotes the set of fixed points of T).

CHAPTER 3

3.0 THE PICARD AND KASNOSELSKIJ ITERATION

Let (X,d) be a metric space, $D \subset X$ a closed subset of X (we often have $D = X$) and $T: D \to D$ a self map possessing at least one fixed point $p \in F_T$. For a given $x_0 \in X$, we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method

$$x_n = T(x_{n-1}) = T^n(x_0), \qquad n = 1, 2, \ldots \tag{3.1}$$

We are interested in obtaining (additional) conditions on *T*, *D*, and *X*, as general as possible, and which should guarantee the (strong) convergence of the iterates $\{x_n\}_{n=0}^{\infty}$ to the fixed point of *T* in *D*.

The sequence defined by (3.1) is known as ***successive approximations with the initial value*** x_0. It is also known as the ***Picard iteration***.

Usually, if the Picard iteration converges to a fixed point of *T*, we will be interested in evaluating the error estimate (or, alternatively, the rate of convergence) of the method – that is in obtaining a *stopping criterion* for the sequence of successive approximation

When the contractive conditions are slightly weaker, then the Picard iteration need not converge to a fixed point of the operator *T*, and some other iterative procedures must be considered.

The next fixed point iteration scheme considered is the Krasnoselskij iteration. We define it in the real normed spaced $(E, \|\cdot\|)$. Let $T : E \to E$ be a self-map, $x_0 \in E$, and $\lambda \in [0,1]$. The sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, \ldots \tag{3.2}$$

is called the *Krasnoselskij iteration.*

From (3.2) when $\lambda = 1$, the Krasnoselskij iteration reduces to the Picard iteration.

In this work the Picard iteration is studied in connection with strict contractiveness type while the Krasnoselskij is associated with Lipschitzian and pseudocontractive type conditions. The theorems stated in this chapter are done to show the existence of fixed points for the Picard and Krasnoselskij iterative schemes.

In chapter 2 we stated the *Contraction Mapping Principle* (Theorem 2.1). This is reformulated here in an extended form. This fundamental result in metrical fixed point theory is usually called *theorem of Banach* or *theorem of Picard-Cacccioppoli* or *Contraction mapping theorem (principle).*

3.1 The Contraction Mapping Principle

Theorem 3.1 *Let (X, d) be a complete metric space and $T : X \to X$ be an a a–contraction , that is an operator satisfying*

$$d(Tx, Ty) \le a d(x, y), \text{ for any } x, y \in X \tag{3.1.1}$$

with $a \in [0,1)$ fixed. Then

(i) T has a unique fixed point, that is $F_T = \{x^*\}$;

(ii) The Picard iteration associated to T; i.e., the sequence $\{x_n\}_{n=0}^{\infty}$, defined by

$$x_n = T(x_{n-1}) = T^n(x_0), \qquad n = 1, 2, \ldots, \tag{3.1.2}$$

Converges to x^*, *for any initial guess* $x_0 \in X$

(iii) *The a priori and a posteriori error estimates*

$$d(x_n, x^*) \le \frac{a^n}{1-a} \cdot d(x_0, x_1), \qquad n = 0, 1, 2, \ldots \tag{3.1.3}$$

$$d(x_n, x^*) \le \frac{a}{1-a} \cdot d(x_{n-1}, x_n), \qquad n = 0, 1, 2, \ldots \tag{3.1.4}$$

respectively hold.

(iv) The rate of convergence is given by

$$d(x_n, x^*) \le a \cdot d(x_{n-1}, x_n), \qquad n = 0, 1, 2, \ldots \tag{3.1.5}$$

Proof. There is at most one fixed point, i.e., $F_T \le 1$. Indeed, assuming $x^*, y^* \in F_T$ $x^* \ne y^*$ we get the contradiction

$$d(x^*, y^*) = d(Tx^*, Ty^*) \le a \cdot d(x^*, y^*) < d(x^*, y^*),$$

since $0 \le a < 1$.

To prove the existence of a fixed point, we will show that, for any given $x_0 \in X$, the Picard iteration $\{x\}_{n=0}^{\infty}$ is a Cauchy sequence. Notice that by (3.1) we have

$$d(x_2, x_1) = d(Tx_1, Tx_0) \le a\, d(x_1, x_0),$$

and by induction,

$$d(x_{n+1}, x_n) \le a^n\, d(x_1, x_0), \qquad n = 0, 1, 2, \ldots \tag{3.1.6}$$

Thus, for any numbers $n, p \in N$, $p > 0$, we have

$$d\left(x_{n+p}, x_n\right) \le \sum_{k=n}^{n+p-1} d\left(x_{k+1}, x_k\right) \le \sum_{k=n}^{n+p-1} a^k \; d\left(x_1, x_0\right) \le \frac{a^n}{1-a} \cdot d\left(x_1, x_0\right). \tag{3.1.7}$$

Since $0 \le a < 1$, it results that $a_n \to 0$ (as $n \to \infty$), which together with (3.1.7) shows that $\{x_n\}_{n=0}^{\infty}$ is Cauchy sequence. But (X, d) is a complete metric space, therefore $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in X$.

On the other hand, any Lipschitzian mapping is continuous. So denoting

$$\lim_{n \to \infty} = x^*,$$

we find

$$x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} = T\left(x_n\right) = T\left(\lim_{n \to \infty} = x_n\right) = T\ x^*,$$

which gives $x^* = Tx^*$, i.e. x^* is a fixed point of *T*.

This shows that for any $x_0 \in X$ the Picard iteration converges in *X* and it's limit is a fixed point of *T*. Since *T* has at most one fixed point, we deduce that, for every choice of $x_0 \in X$, the Picard iteration converges to the same value x^*, that is, the unique fixed point of *T*. So we proved (i) and (ii).

To prove (iii) we use (3.1.7),

$$d\left(x_{n+p}, x_n\right) \le \frac{a^n}{1-a} \cdot d\left(x_0, x_1\right) \qquad \text{for all} \quad p \in \mathrm{N}^*,$$

and the continuity of the metric and so, by letting $p \to \infty$, we find

$$d\left(x_n, x^*\right) = d\left(x^*, x_n\right) = \lim_{p \to \infty} d\left(x_{n+p}, x_n\right) \le \frac{a^n}{1-a} \cdot d\left(x_0, x_1\right) \qquad \mathrm{n} \ge 0$$

And so (3.1.7) is proved.

To obtain the *posteriori* estimation (4), let us notice that by (1) we have

$$d(x_{n+1}, x_n) \le a\, d\,(x_n, x_{n-1})$$

and by induction,

$$d(x_{n+k}, x_{n+k-1}) \le a^k\, d(x_n, x_{n-1}), \qquad k \in N^*$$

so

$$d(x_{n+p}, x_n) \le (a + a^2 + \ldots + a^p\ )d(x_n, x_{n-1}) \le \frac{a}{1-a}\ d(x_n, x_{n-1}).$$

By letting $p \to \infty$ in the last inequality we get exactly (4).

Remarks.

1) The *a priori* estimate (3.1.7) shows that, when starting from an initial guess $x_0 \in X$, the approximation error of the n^{th} iterate is completely determined by the contraction constant *a* and the initial displacement $d(x_1, x_0)$

2) Similarly, the *a posteriori* estimate shows that, in order to obtain the desired error approximation of the fixed point by means of Picard iteration, that is, to have $d(x_n, x^*) < \varepsilon$, we need to stop the iterative process at the step *n* for which the displacement between two consecutive iterates is at most $\frac{(1-a)\varepsilon}{a}$.

So, the *a posteriori* estimation offers a direct stopping criterion for the iterative approximation of fixed points by Picard iteration, while the *a priori* estimation indirectly gives a stopping criterion.

3) It is easy to see that the *a posteriori* estimation is better than the *a priori* one, in the sense that from (3.1.4) we can obtain (3.1.5), by means of (3.1.6).

4) Each of the three estimations given in Theorem 3.1 shows that the rate of convergence of the Picard iteration is at least as quick as that of the geometric series $\sum a^n$.

When the contractive conditions are slightly weaker, then the Picard iteration need not converge to a fixed point of the operator *T*, and some other iteration procedures can be used. The next fixed point iteration scheme (the Krasnoselskij iteration) is defined in a real normed space $(E, \| \ \|)$. Let $T: E \to E$ be a self-map, $x_0 \in E$ and $\lambda \in [0,1]$.

The sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \qquad n = 0, 1, 2, \ldots \tag{4.1}$$

is called is called the Krasnoslskij iteration procedure or, simply, Krasnoselskij iteration. It is easy to see that the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by is exactly the Picard iteration corresponding to associated operator

$$T_\lambda = (1-\lambda)I + \lambda \cdot T, \qquad I = \text{the identity operator}$$

and that for $\lambda = 1$ the Krasnoselskij iteration reduces to the Picard iteration.

Moreover we have $Fix(T) = Fix(T_\lambda)$, for all $\lambda \in (0, 1]$.

It is known that if *T* is assumed to be only a nonexpansive map, then the Picard iteration $\{T^n x_0\}_{x \geq 0}$ need no longer converge (to a fixed point of *T*). In fact, in general, *T* need not have a fixed point. For our work in this thesis the Krasnoselskij iteration will be mainly associated with the approximation of fixed points for nonexpansive operators. This kind of result is gotten by imposing certain additional conditions on the operator *T* and/ or on the ambient space itself, and to consider a convex combination of two successive terms of the Picard iteration, that defined the Krasnoselskij iteration.

The following proof is called the Browder–Gohde–Kirk fixed point theorem. It is known to be a basic fixed point existence result for nonexpansive operators. Unlike the result gotten for Picard iteration (Theorem 3.1), this proof is given in a Hilbert space setting, suitable for many convergence theorems on the Krasnoselskij iteration.

3.2 Browder–Gohde–Kirk fixed point theorem

Theorem 3.2. *Let C be a closed bounded convex subset of a Hilbert space H and* $T : C \to C$ *be a nonexpansive operator. Then T has at least one fixed point.*

Proof. For fixed point element v_0 in C and a number s with $0 < s < 1$, we denote

$$U_s(x) = (1-s)v_0 + s\,Tx\,, \qquad x \in C\,.$$

Since C is convex and closed, we deduce that $U_s : C \to C$ is and s–contraction and it has a unique fixed point u_s (from the contraction mapping principle). On the other hand, since C is closed , convex and bounded in a Hilbert space H, is weakly compact. Hence we may find a sequence $\{ s_j \}$ in (0, 1) such that $s_j \to 1$ ($j \to \infty$) and $u_j \to u_{sj}$ converges weakly to an element p of H.

Since C is weakly closed, p lies in C. We shall prove that p is a fixed point of T. If u any arbitrary point in H, we have

$$\| u_j - u \|^2 = \| (u_j - p) + (p - u) \|^2 = \| u_j - p \|^2 + \| p - u \|^2 + 2 \langle u_j - p, p - u \rangle,$$

where

$$2\left\langle u_j - p,\, p-u \right\rangle \to 0 \qquad (\text{as } j\to\infty),$$

since $u_j - p$ converges weakly to zero in H. Moreover since $s_j \to 1$ and $U_{s_j} u_j = u_j$, we have

$$Tu_j - u_j = \lfloor s_j Tu_j + (1-s_j)v_0 \rfloor - u_j + (1-s_j)\lfloor Tu_j - v_0 \rfloor =$$

$$= (U_{s_j} u_j - u_j) + (1-s_j)(Tu_j - v_0) = 0 + (1-s_j)(Tu_j - v_0) \to 0, \text{ as } j\to\infty$$

Setting $u = Tp$ above, we obtain

$$\lim_{j\to\infty}\left(\| u_j - Tp \|^2 - \| u_j - p \|^2\right) = \| p - Tp \|^2 .$$

On the other hand, since T is nonexpansive, we have

$$\| Tu_j - Tp \| \le \| u_j - p \|$$

and hence

$$\| u_j - Tp \| \le \| u_j - Tu_j \| + \| Tu_j - Tp \| \le \| u_j - Tu_j \| + \| u_j - Tp \| .$$

Thus

$$\limsup \left(\| u_j - Tp \| - \| u_j - p \| \right) \le \lim_{j\to\infty} \| u_j - Tu_j \| = 0$$

and, due to boundedness of C, we have also

$$\limsup \left(\| u_j - Tp \|^2 - \| u_j - p \|^2 \right) =$$

$$\limsup \left(\| u_j - Tp \| - \| u_j - p \| \right)\left(\| u_j - Tp \| - \| u_j - p \| \right) \le 0$$

which yields

$$\| p - Tp \|^2 = 0 ,$$

that is, p is a fixed point of T.

Definition 3.1. Let H be a Hilbert space and C a subset of H. A mapping $T : C \to H$ is called ***demicompact*** if it has the property that whenever $\{u_n\}$ is a bounded sequence in H and $\{Tu_n - u_n\}$ is strongly convergent, the there exists a subsequence $\{u_{nk}\}$ of $\{u_n\}$ which is strongly convergent.

We now consider a result on approximating fixed points of nonexpansive and demicompact mappings by means of Krasnoselskij iteration.

3.3 Nonexpansive And Demicompact Operator

Theorem 3.3. *Let C be a bounded closed convex subset of a Hilbert space H and sequence $T : C \to C$ be a nonexpansive and demicompact operator. Then the set F_T of fixed points of T is a nonempty convex set and for any given x_0 in C and any fixed number λ* with $0 < \lambda < 1$, *the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n \qquad n = 0, 1, 2, ... \tag{3.3.1}$$

converges (strongly) to a fixed point of T.

Proof. Since T is nonexpansive, by Theorem 3.2, T has fixed points in C, that is, $F_T \neq \phi$. Furthermore, F_T is convex, i.e., when $x, y \in F_T$ and $\lambda \in [0,1]$ we have

$$u_\lambda = (1-\lambda)x + \lambda_y \in F_T$$

Indeed, we have

$$\| Tu_\lambda - x \| = \| Tu_\lambda - Tx \| \leq \| u_\lambda - x \| \text{ and } \| Tu_\lambda - y \| \leq \| u_\lambda - y \|,$$

which imply that

$$\| x - y \| \leq \| x - Tu_\lambda \| + \| Tu_\lambda - y \| \leq \| x - y \|$$

This shows that for some *a*, *b* with $0 \leq a, b \leq 1$, we have

$$x - Tu_{\lambda} = a\left(x - u_{\lambda}\right) \text{ and } y - Tu_{\lambda} = b\left(y - u_{\lambda}\right)$$

from which it follows that $Tu_{\lambda} = u_{\lambda} \in F_T$.

For any $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by (3.3.1) lies in *C* and is bounded. Let *p* be a fixed point of *T*, and, so of U_{λ} given by

$$U_{\lambda} = (1-\lambda)I + \lambda T \quad (I = \text{the identity map}). \tag{3.3.2}$$

We first prove that the sequence $\{Tu_n - u_n\}_{n \in N}$ converges strongly to zero. Indeed

$$x_{n+1} - p = (1-\lambda)\,x_n + \lambda\,Tx_n - p = (1-\lambda)(x_n - p) + \lambda(Tx_n - p)\,.$$

On the other hand, for any constant *a*,

$$a\left(x_n - Tx_n\right) = a\left(x_n - p\right) - a\left(Tx_n - p\right).$$

Then

$$\| x_n - p \|^2 = (1-\lambda)^2 \| x_n - p \|^2 + \lambda^2 \| Tx_n - p\|^2 +$$

$$+ 2\lambda\left(1-\lambda\right) < Tx_n - p, x_n - p >$$

and

$$a^2 \| x_n - Tx_n \|^2 = a^2 \| x_n - p \|^2 + a^2 \| Tx_n - p \|^2 - 2a^2 < Tx_n - p, x_n - p > .$$

Hence adding the corresponding sides of the preceding two inequalities and using the fact that *T* is nonexpansive and $Tp = p$, we get

$$\| x_{n+1} - p \|^2 + a^2 \| x_n - Tx_n \|^2 \leq \left[2a^2 + \lambda^2 + (1-\lambda)^2\right] \cdot \| x_n - p \|^2 +$$

$$+ 2\left[\lambda\left(1-\lambda\right) - a^2\right] \cdot < Tx_n - p, x_n - p >$$

If we choose now an *a* such that $a^2 \leq \lambda\left(1-\lambda\right)$ then from the last inequality we obtain

$$\| x_{n+1} - p \|^2 + a^2 \| x_n - Tx_n \|^2 \leq$$

$$\leq \left(2a^2 + \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) - 1a^2 \right) \| x_n - p \|^2 = \| x_n - p \|^2$$

$$\left(\begin{array}{l} \text{we use the Cauchy - Schwarz inequality} \\ \qquad\qquad < Tx_n - p, x_n - p > \leq \| Tx_n - P \| \cdot \| x_n - p \| \leq \| x_n - p \|^2 \end{array} \right).$$

Letting $a^2 = \lambda(1-\lambda) > 0$ and summing up the obtained inequality

$$a^2 \| x_n - Tx_n \|^2 \leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2$$

for $n = 0$ to $n = N$ we get

$$\lambda(1-\lambda) \sum_{n=0}^{N} \| x_n - Tx_n \|^2 \leq \sum_{n=0}^{N} \left[\| x_n - p \|^2 - \| x_{n+1} - p \|^2 \right] =$$

$$= \| x_0 - p \|^2 - \| x_{N+1} - p \|^2 \leq \| x_0 - p \|^2$$

which shows that $\sum_{n=0}^{\infty} \| x_n - Tx_n \|^2 < \infty$ and hence $\| x_n - Tx_n \| \to 0$, as $n \to \infty$.

As T is demicompact, it results that there exists a strong convergent subsequence $\{ x_{ni} \}$ such that $x_{ni} \to p \in F_T$; since T is nonexpansive, $Tx_{ni} \to Tp$ and $Tp \to p$.

The convergence of the entire sequence $\{ x_n \}_{n=0}^{\infty}$ to p follows from the inequality $= \| x_{n+1} - p \| \leq \| x_n - p \|$, which is deduced from the nonexpansiveness of T and is valid for each n.

Remarks.

1) It has been shown that if in Theorem 3.3, we remove the assumption that T is demicompact, the Krasnoseskij iteration does not longer converge strongly, in general, but it converges (at least) weakly to a fixed point.

CHAPTER 4

4.0 THE MANN ITERATION

The Mann iteration was chronologically introduced two years earlier than the Krasnoselskij iteration; even so it is a generalization of the latter and in its normal form is obtained by replacing the parameter λ in the Krasnoselskij iteration formula by a sequence $\{a_n\}$.

The normal *Mann iteration procedure* or *Mann iteration*, starting from

$x_0 \in E$, is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n, \qquad n = 0, 1, 2, \ldots,$$

where $\{a_n\}_{n=0}^{\infty} \subset [0, 1]$.

If we consider

$$T_n = (1 - a_n) I + a_n \cdot T$$

then we have $Fix(T) = Fix(T_n)$, for all $a_n \in (0, 1]$.

If the sequence $a_n = \lambda\,(const)$, then the Mann iterative procedure obviously reduces to the Krasnoselskij iteration.

There is a lot of literature on the convergence of Mann iteration for different classes of operators considered on various spaces.

4.1 Strongly Pseudocontractive Operators

Let E be a Banach space, K a subset of E, and $T : K \to K$ a strongly pseudcontractive operator, then there exist a number $t > 1$ such that the inequality

$$\| x - y \| \leq \| (1 + r)(x - y) - rt\,(Tx - Ty)\| \tag{4.1}$$

holds for all $x, y \in K$ and $r > 0$.

In chapter 2, it was stated that a mapping is strongly pseudo contractive if and if $I - T$ is a strongly accretive mapping, i.e. there exist $j(x - y) \in J(x - y)$ and a positive number k such that

$$\langle (I - T)x - (I - T)y\,,\, j(x - y) \rangle \geq k \| x - y \|^2 \tag{4.2}$$

that, in

$$\| x - y \| \leq \| x - y + r\,[\,(I - T - kI)x - (I - T - kI)y\,]\|$$

turn, is equivalent to the fact that the next inequality

(4.3)

holds for any $x, y \in K$ and any $r > 0$ (where $k = \frac{t-1}{t}$).

Based on the form (4.3) of the strong pseudo-contractiveness property, it can be proved that the Mann iteration process converges strongly to the unique fixed point of a Lipschitzian and strongly pseudocontractive operator.

Theorem 4.1 *Let E be a Banach space and K a nonempty closed convex and bounded subset of E. If* $T : K \to K$ *is a Lipchitzian strongly pseudocontractive operator such that the fixed point set of T,* F_T *is nonempty, then the Mann iteration*

$\{x_n\} \subset K$ generated by $x_1 \in K$ and the sequence $\{\alpha_n\} \subset (0, 1]$, with $\{\alpha_n\}$ satisfying

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; *(ii)* $\alpha_n \to 0$ (as $n \to \infty$),

converges strongly to the unique fixed point of T.

Proof. Let p be a fixed point of T. Since is a strongly pseudocontractive operator, $I - T$ is strongly accretive, i.e., the inequality (4.3) holds for any $x, y \in K$ and $r > 0$. Let $L > 0$ be the Lipschitz constant. Then from the definition of $\{x_n\}$,

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \quad n = 1, 2, \ldots \tag{4.1.1}$$

and therefore we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T x_n = (1 + \alpha_n) x_{n+1} + \\ &+ \alpha_n (1 - T - kI) x_{n+1} - (2 - k) \alpha_n x_{n+1} + \alpha_n x_n + \alpha_n (T x_{n+1} - T x_n) = \\ &= (1 - \alpha_n) x_{n+1} + \alpha_n (1 - T - kI) x_{n+1} - (2 - k) \alpha_n [(1 - \alpha_n) x_n + \alpha_n T x_n] + \\ &+ \alpha_n x_n + \alpha_n (T x_{n+1} - T x_n) = (1 + \alpha_n) x_{n+1} + \alpha_n (1 - T - kI) x_{n+1} - \\ &- (1 - k) \alpha_n x_n + (2 - k) \cdot \alpha_n^2 (x_n - T x_n) + \alpha_n (T x_{n+1} - T x_n). \end{aligned}$$

As $Tp = p$, we have

$$\begin{aligned} x_n - p &= (1 + \alpha_n)(x_{n+1} - p) + \alpha_n (1 - T - kI)(x_{n+1} - p) - (1 - k) \alpha_n (x_n - p) + \\ &+ (2 - k) \cdot \alpha_n^2 (x_n - T x_n) + \alpha_n (T x_{n+1} - T x_n) \end{aligned}$$

Now, using the inequality (4.3), we get

$$\begin{aligned} \| x_n - p \| &\geq (1 + \alpha_n) \| x_{n+1} - p \| - (1 - k) \alpha_n \| x_n - p \| - \\ &- (2 - k) \cdot \alpha_n^2 \| x_n - T x_n \| - \alpha \|_n T x_{n+1} - T x_n \| \end{aligned}$$

Since T is Lipschitzian, it follows that

$$\| T x_{n+1} - T x_n \| \leq L \| x_{n+1} - x_n \| \leq L(L+1)\alpha_n \| x_n - p \| ,$$

and then

$$\| x_n - p \| \geq (1+\alpha_{\mathrm{n}}) \| x_{n+1} - p \| - (1-k)\alpha_n \| x_{n+1} - p \| -$$

$$-(2-k)\alpha_n^2 \| x_n - T x_n \| - L(L+1)\alpha_n^2 \| x_n - p \|$$

Hence

$$\| x_{n+1} - p \| \leq [1 + (1-k)\alpha_n] (1+\alpha_n)^{-1} \| x_n - p \| + (2-k)\alpha_n^2 (1+\alpha_n)^{-1}$$

$$\| x_n - T x_n \| + L(L+1)\alpha_n^2 (1+\alpha_n)^{-1} \| x_n - p \| \leq$$

$$\leq [1 + (1-k)\alpha_n] (1-\alpha_n+\alpha_n^2) \| x_n$$

$$+ L(L+1)\alpha_n^2 \| x_n - p \| \tag{4.1.2}$$

and so we obtain

$$\| x_{n+1} - p \| \leq (1-k\alpha_n) \| x_n - p \| + M \alpha_n^2 ,$$

for some constant $M > 0$, in view of the fact that K is bounded.

Now using Lemma 2.1, part (ii), it follows that the sequence $\{\| x_n - p \|\}$ converges to 0, that is, $\{x_n\}$ converges strongly to the (unique) fixed point p of T.

4.2 Quasi–Contractive type Operators

An important class of quasi-contractive mappings, which is independent of the class of strictly pseudo-contractive mappings, is the class of *Zamfirescu mappings*.

In chapter 2 we have proven (Theorem 1.4 of chapter 1) that for any Zamfirescu mappings *T* considered on a complete metric space, the Picard iteration converges to the unique fixed point of *T*.

We now show that in a more particular ambient space, the Mann iteration converges as well.

Theorem 4.2 *Let E be a uniformly convex Banach space, K a closed convex subset of E, and* $T : K \to K$ *be a Zamfirescu mapping. Then the Mann iteration* $\{x_n\}$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 1, 2, \ldots \tag{4.2.1}$$

with $\{\alpha_n\}$ *satisfying the conditions*

(i) $\alpha_n = 1$; *(ii)* $0 \le \alpha_n < 1$, *for* $n > 1$ *and* *(iii)* $\sum \alpha_n (1-\alpha_n) = \infty$,

converges to the fixed point of T.

Proof. Theorem 1.4 shows that *T* has a unique fixed point in *K*. Let us denote it by *p*. For any $x_1 \in K$, we have

$$\| x_{n+1} - p \| \le (1-\alpha_n)\| x_n - p \| + \alpha_n \| Tx_n - p \|.$$

Since any Zamfirescu mapping is quasi-contractive we deduce that

$$\| Tx_n - p \| \le \| x_n - p \|,$$

which shows that the sequence $\{\| x_n - p \|\}$ is decreasing. We also have

$$\| x_n - Tx_n \| = \| (x_n - p) - (Tx_n - p) \| \le 2\| (x_n - p) \|$$

Now let us assume that there exist a number $a > 0$ such that $\| x_n - p \| \ge a$ for all n.

Suppose $\{\| x_n - Tx_n \|\}_{n\geq 1}$ does not converge to zero. Then there are two possibilities: either there exists an $\varepsilon > 0$ such that $\| x_n - Tx_n \| \geq \varepsilon$ for all n or

$$\liminf \| x_n - Tx_n \| = 0$$

In the first case, using Lemma of Groetsch [1] with $b = 2\delta \left(\varepsilon / \| x_0 - p \| \right)$ we get

$$\| x_{n+1} - p \| \leq \left(\left(1 - \alpha_n (1 - \alpha_n) b\right) \right) \| (x_n - p) \| \leq$$

$$\leq \| x_{n-1} - p \| - \alpha_{n-1}(1 - \alpha_{n-1}) b \| x_n - p \| - b\alpha_n (1 - \alpha_n) \| (x_n - p) \| \leq$$

$$\leq \| x_{n-1} - p \| - b\left[\alpha_{n-1}(1 - \alpha_{n-1}) + \alpha_n (1 - \alpha_n)\right] \cdot \| (x_n - p) \|.$$

By induction one obtains

$$a \leq \| x_{n+1} - p \| \leq \| (x_0 - p) \| - b\sum_{k=1}^{n} \alpha_k (1 - \alpha_k) \cdot \| (x_n - p) \|.$$

Therefore

$$a \left[1 + b\sum_{k=1}^{n} \alpha_k (1 - \alpha_k) \right] \leq \| (x_n - p) \|, \text{ which contradicts } \textit{(iii)}$$

In the second case, there exists a subsequence $\{ x_{nk} \}$ such that

$$\lim_{k} \| x_{nk} - Tx_{nk} \| = 0 \tag{4.2.2}$$

If x_{nk}, x_{nl} satisfy (z_1) (see Theorem 1.4), that is

$$\| Tx_{nk} - Tx_{nl} \| \leq \alpha \| x_{nk} - x_{nl} \|,$$

then

$$\| Tx_{nk} - Tx_{nl} \| \leq \alpha \left[\| x_{nk} - Tx_{nk} \| + \| Tx_{nk} - Tx_{nl} \| + \| Tx_{nl} - x_{nl} \| \right],$$

and hence

$$\| Tx_{nk} - Tx_{nl} \| \leq \alpha (1 - \alpha)^{-1} \left[\| x_{nk} - Tx_{nk} \| + \| Tx_{nl} - x_{nl} \| \right],$$

and if x_{nk}, x_{nl} satisfy (z_2), then

$$\| Tx_{nk} - Tx_{nl} \| \leq \beta \left[\| x_{nk} - Tx_{nk} \| + \| x_{nl} - Tx_{nl} \| \right],$$

and if x_{nk}, x_{nl} satisfy (z_3), then

$$\| Tx_{nk} - Tx_{nl} \| \leq \gamma \left[\| x_{nk} - Tx_{nl} \| + \| x_{nl} - Tx_{nk} \| \right],$$

which yields

$$\| Tx_{nk} - Tx_{nl} \| \leq \gamma (1 - 2\gamma)^{-1} \left[\| x_{nk} - Tx_{nk} \| + \| x_{nl} - Tx_{nl} \| \right],$$

Therefore in all situations $\{Tx_{nk}\}$ is a Cauchy sequence and hence convergent.

Let u be it's limit. From (4.2.2) it results that

$$\lim_{k \to \infty} x_{nk} = \lim_{k \to \infty} Tx_{nk} = u$$

Moreover,

$$\| u - Tu \| \leq \| u - x_{nk} \| + \| x_{nk} - Tx_{nk} \| + \| Tx_{nk} - Tu \|.$$

We will show that $u = Tu$, that is, u is a fixed point of T. indeed, if x_{nk}, u satisfy (z_1), then

$$\| Tx_{nk} - Tu \| \leq \alpha \| x_{nk} - u \|.$$

If x_{nk}, u satisfy (z_2), then

$$\| Tx_{nk} - Tu \| \leq \beta \left[\| x_{nk} - Tx_{nk} \| + \| u - Tu \| \right]$$

which leads to

$$\| u - Tu \| \leq \left[\| u - x_{nk} \| + (1 + \beta) \| x_{nk} - Tx_{nk} \| \right] / (1 - \beta)$$

and, finally, if x_{nk}, u satisfy (z_3), then

$$\| Tx_{nk} - Tu \| \leq \gamma \left[\| x_{nk} - Tu \| + \| u - Tx_{nk} \| \right] \leq$$

$$\leq \gamma \left[\| x_{nk} - Tx_{nk} \| + \| Tx_{nk} - Tu \| + \| u - Tx_{nk} \| \right],$$

or

$$\| Tx_{nk} - Tu \| \leq \gamma (1 - \gamma)^{-1} \left[\| x_{nk} - Tx_{nk} \| + \| u - Tx_{nk} \| \right].$$

Hence $u = Tu$.

Now, since p is the unique fixed point of T, it results that $p = u$ and so the two conditions $\lim_k x_{nk} = u\,(= p)$ and $\{\| x_n - p \|\}$ decreasing with respect to n yield $\lim_n x_n = p$.

Remarks.

1) Having in view that any Kannan mapping is a Zamfirescu mapping, from Theorem 4.2 we obtain the convergence of the Mann iteration, in the class of Kannan mappings;

2) If $\alpha_n = \frac{1}{2}$ for all n, from Theorem 4.2 we obtain two theorems (Theorem 2 and Theorem 3) of Kannan [2], while if $\alpha_n = \lambda$ for all n, we obtain Theorem 3 of Kannan [3].

3) As both Picard iteration and Krasnoselskij iteration converge in the class of Zamfirescu mappings, it is natural to try to compare these methods in order to know which one converges faster to the (unique) fixed point of T. However, such results have not been made available in this work.

CHAPTER 5

5.0 THE ISHIKAWA ITERATION

The Ishikawa iteration method: Let X be any set and $T : X \to X$ a self map. For $x_o \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-a_n)\,x_n + a_n T\,[(1-b_n)\,x_n + b_n \cdot Tx_n] \qquad n = 0, 1, 2, ..., \tag{5.1}$$

where $\{a\}_{n=0}^{\infty}$ $\{b\}_{n=0}^{\infty}$ are sequences of reals satisfying $0 \le a_n,\ b_n < 1$ is called the *Ishikawa iteration*, and is denoted by $I(x_0, \alpha_n, \beta_n, T)$.

The above equation (5.1) can be rewritten in a system form

$$\begin{cases} y_n = (1-b_n)x_n + b_n Tx_n & n = 0, 1, 2, ... \\ x_{n+1} = (1-a_n)\cdot x_n + a_n Ty_n & n = 0, 1, 2, ... \end{cases} \tag{5.2}$$

Then we can regard the ***Ishikawa iteration*** as a sort of double Mann iteration, with two different parameter sequences.

It's obvious that, for $\lambda = 1$, the Krasnoselskij iteration reduces to the Picard iteration (the method of successive approximations), while for $\alpha_n = \lambda$ (constant), the Mann iteration reduces to the Krasnoselskij method.

Despite this apparent similarity and the fact that, for $b_n = 0$, Ishikawa iteration reduces to Mann iteration, there is not a general dependence between convergence results for Mann and Ishikawa iterations. Recently, some authors considered the so called modified Mann iteration, respectively modified Ishikawa iteration, by replacing the operator T by its *n-th* iterate T^n.

It is mentioned in chapter 2, if an operator *T* is continuous and the Mann iterative procedure converges, then it converges to a fixed point of *T*. But if *T* is not continuous, then there is no guarantee that, even if the Mann process converges, it will converge to a fixed point of *T*.

If instead of the Mann iteration, we consider another iterative process, which is in some sense a double Mann iterative process, then it is possible to approximate the fixed point of some other classes of contractive mappings. This new iterative process, is called the *Ishikawa iteration*, and was introduced for the class of Lipchitzian pseudo-contractive operators. Thus it first used to establish the strong convergence to a fixed point of a Lipschitzian and pseudo-contractive selfmap of a convex compact subset of a Hilbert space.

As shown in chapter 4, the Mann iteration process converges in a special case of Lipschitzian and strongly pseudocontractive operators**.** However, if *T* is only a pseudocontractive mapping, then generally the Mann iterative process does not converge to fixed point.

Interest in pseudocontractive maps stems mainly from their firm connection with the class of non-linear accretive operators. It is a classical result, that if *T* is an accretive operator, then the solution of the equations

$$T(x) = 0$$

correspond to the equilibrium points of some evolution systems.

This explains why a considerable research effort has been devoted to iterative methods for approximating solutions of the equation above, when T is accretive or, correspondingly, to the iterative approximation of fixed points of pseudocontractions. Results of this kind have been obtained firstly in Hilbert spaces, but only for Lipschitz operators, and then, they have been extended to more general Banach spaces and to more general classes of operators.

There are still no results for the case of arbitrary Lipschitzian and pseudocontractive operators, even when the domain of the operator is a compact convex subset of a Hilbert space. This explains the importance, from this point of view, of the improvement brought by the Ishikawa iteration.

It is shown that, under certain assumptions of the sequences $\{\alpha_n\}$, $\{\beta_n\}$, the Ishikawa iterative process associated to a Lipschitzian pseudocontractive operator converges strongly to a fixed point of T. The original result of Ishikawa is stated in the following.

5.1 Lipschitzian Pseudocontractive Operators

Theorem 5.1. *Let K be a convex compact subset of a Hilbert space H,* $T : K \to K$ *a Lipschitzian pseudocontractive map and* $x_1 \in K$. *Then the Ishikawa iteration* $\{x_n\}$, $x_n = I\{x_1\ \alpha_n, \beta_n, T\}$, *i.e., the sequence defined by*

$$x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n T\left[(1-\beta_n)\,x_n + \beta_n T x_n\right], \qquad (5.1.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *are sequences of positive numbers satisfying*

$$(i)\ \ 0 \le \alpha_n \le \beta_n \le 1,\ n \ge 1; \quad (ii)\ \lim_{n\to\infty} \beta_n = 0; \quad (iii)\ \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty,$$

converges strongly to a fixed point of T.

Proof. Since T is pseudo-contractive, for any $x, y \in K$ we have

$$\| Tx - Ty \|^2 \le \| x - y \|^2 + \| (I - T)x - (I - T)y \|^2, \tag{5.1.2}$$

where I is the identity map.

From the assumption that T is Lipschitzian, we state that there exists a positive number L such that

$$\| Tx - Ty \| \le L \| x - y \|, \qquad \text{for any } x, y \in K. \tag{5.1.3}$$

Since K is a convex compact set and T is continuous (being Lipschitzian), from Schauder's fixed point theorem we obtain that the set of fixed points of T, $F(T)$, is non-empty. Let p denote any point of $F(T)$.

Now, for any x, y, z in a Hilbert space H and a real number λ, we have

$$\| \lambda x + (1-\lambda) y - z \|^2 = \lambda \| x - z \|^2 + (1-\lambda) \| y - z \|^2 - \lambda(1-\lambda) \| x - y \|^2 \tag{5.1.4}$$

Using (5.1.4) we obtain the following three equalities

$$\|x_{n+1} - p\|^2 = \|\alpha_n T[\beta_n T x_n + (1-\beta_n)x] + (1-\alpha_n)x_n - p\|^2 =$$

$$= \alpha_n \|T[\beta_n T x_n + (1-\beta_n)x_n] - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 -$$

$$- \alpha_n (1-\alpha_n) \| T[\beta_n T x_n + (1-\beta_n)x_n] - x_n \|^2;$$

(5.1.5)

$$\| \beta_n Tx_n + (1-\beta_n)x_n - p \|^2 = \beta_n \| Tx_n - p \|^2 + (1-\beta_n) \| x_n - p \|^2 - (1-\alpha_n)$$

$$- \beta_n (1-\beta_n) \| Tx_n - x_n \|^2, \tag{5.1.6}$$

and, respectively,

$$\| \beta_n Tx_n + (1-\beta_n) x_n - T [\beta_n Tx_n + (1-\beta_n)x_n] \|^2 =$$

$$\beta_n \| Tx_n - T [\beta_n Tx_n + (1-\beta_n) x_n] \|^2 + (1-\beta_n) =$$

$$\| x_n - T [\beta_n Tx_n + (1-\beta_n)x_n] \|^2 - \beta_n (1-\beta_n) \| Tx_n - x_n \|^2. \tag{5.1.7}$$

Applying (5.1.2) we deduce the following two inequalities

$$\| T [\beta_n Tx_n + (1-\beta_n) x_n] - p \|^2 = \| T [\beta_n Tx_n + (1-\beta_n) x_n] - Tp \|^2 \leq$$

$$\leq \| \beta_n Tx_n + (1-\beta_n) x_n - p \|^2 +$$

$$+ \| \beta_n Tx_n + (1-\beta_n) x_n - T [\beta_n Tx_n + (1-\beta_n) x_n] \|^2, \tag{5.1.8}$$

and

$$\| Tx_n - p \|^2 = \| Tx_n - Tp \|^2 \leq \| x_n - p \|^2 + \| x_n - Tx_n \|^2 \tag{5.1.9}$$

Now performing the computations in $(5.5) + \alpha_n \left[(5.6) + (5.7) + (5.8) + \beta_n (5.9) \right]$, we get

$$\| x_{n+1} - p \|^2 \leq \| x_n - p \|^2 - \alpha_n \beta_n (1 - 2\beta_n) \| Tx_n - x_n \|^2 +$$

$$+\alpha_n\beta_n \| Tx_n - T[\beta_n Tx_n + (1-\beta_n)x_n]\|^2 -$$

$$-\alpha_n(\beta_n - \alpha_n)\| x_n - T[\beta_n Tx_n + (1-\beta_n)x_n]\|^2,$$

and so, in view of *(i)*, it follows that

$$\| x_{n+1} - p\|^2 \leq \| x_n - p\|^2 - \alpha_n\beta_n(1-2\beta_n)\| Tx_n - x_n\|^2 +$$

$$+\alpha_n\beta_n \| Tx_n - T[\beta_n Tx_n + (1-\beta_n)Tx_n]\|^2 \tag{5.1.10}$$

Since T is Lipschitzian, we have

$$\| Tx_n - T[\beta_n Tx_n + (1-\beta_n)x_n]\| < L\beta_n \| Tx_n - x_n\| \tag{5.1.11}$$

and hence, from (5.1.10) and (5.1.11) we deduce

$$\| x_{n+1} - p\|^2 \leq \| x_n - p\|^2 - \alpha_n\beta_n\left(1-2\beta_n - L^2\beta_n^{\ 2}\right)\| Tx_n - x_n\|^2 \tag{5.1.12}$$

By summing (5.1.12) for $n \in \{m, m+1, ..., n\}$ we obtain

$$\| x_{n+1} - p\|^2 \leq \| x_m - p\|^2 - \sum_{k=m}^{n}\alpha_k\beta_n\left(1-2\beta_k - L^2\beta_k^{\ 2}\right)\| Tx_k - x_k\|^2,$$

which can be written as

$$\sum\alpha_k\beta_k\left(1-2\beta_k - L^2\beta_k^{\ 2}\right)\| Tx_k - x_k\|^2 \leq \| x_m - p\|^2 - \| x_{n+1} - p\|^2$$

Now by exploiting the assumption (*ii*), we deduce that there exists a positive integer N such that

$$2\beta_k + L^2\beta_k{}^2 \leq \frac{1}{2} \text{ for all integers } \quad k \geq N\,.$$

Then, for $\frac{1}{2}m > N$ we obtain

$$\frac{1}{2}\sum_{k=m}^{n} \alpha_k\beta_k \; \| Tx_k - x_k \|^2 \leq \| Tx_m - p \|^2 - \| Tx_{n+1} - p \|^2 \qquad (5.1.13)$$

Since K is bounded, the right-hand side quantity in (5.1.13) is bounded. This means that the series in the left hand side is convergent and therefore, by (*iii*),

it results that

$$\liminf_n \; \| Tx_n - x \| = 0\,,$$

which in turn implies (K is compact) that there is a subsequence $\left\{ x_{n_k} \right\}_{k=1}^{\infty}$ that converges to certain point q of $F(T)$.

Now, since q is a fixed point of T, from (5.12) we obtain for $n \geq N$

$$\| x_{n+1} - q \| \leq \| x_n - q \|,$$

that is, the sequence $\left\{ \| x_n - q \| \right\}$ is decreasing.

Having in view that there is a subsequence $\left\{ \| x_{n_k} - q \| \right\}$ converging to zero, it finally results that $\{x_n\}$ converges to q.

Remarks.

1) In its original form, the Ishikawa iteration does not include the Mann iteration, because of the assumption (*i*) in Theorem 5.1. Indeed, if one had $\beta_n = 0\,(n \geq 1),$ then it would result $\alpha_n = 0,$ as well.

2) In the effort to obtain an Ishikawa iteration which should include a Mann iteration as a special case, some authors, amongst them Naimpally and Singh, K.L. [1] and Liu, Q. [1], have modified (*i*) to a weaker condition of the form $0 \le \alpha_n,\ \beta_n \le 1$.

3) Liu, Q. [1] extended Theorem 5.1 to the class of Lipchitzian hemicontractive maps. A hemicontractive is a pseudocontractive map with respect to a fixed point, i.e., if *p* is a fixed point of *T*, and *x* is a point in the space, then *T* satisfies

$$\| Tx - p \|^2 \le \| x - p \|^2 + \| x - Tx \|^2 .$$

4) However neither the proof of Q. Liu nor that of Ishikawa can be used to establish a similar result for the Mann iterative process.

5.2 Quasicontractive type operators

As mentioned earlier in chapter 1, the Picard, Krasnoselskij, Mann and Ishikawa iterative methods all converge for quasi-contractive operators. It was again mentioned (as shown by Rhoades ([18], Theorem 8),) that in a uniformly Banach space *E*, the Ishikawa iteration $\{x_n\}_{n=0}^{\infty}$ given by (5.1.1) and $x_0 \in K$ converges (strongly) to the fixed point of *T*, where $T : X \to X$ is a mapping satisfying conditions $(z_1), (z_2)$, and (z_3) of Theorem 1.2, *K* is a closed convex subset of *E*, and $\{\alpha_n\}$, a sequence of numbers in [0, 1] such that

$$\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty . \qquad (i)$$

It was also stated that in [3] the Berinde proved the following convergence theorem in arbitrary Banach spaces, for the Mann iteration associated to operators satisfying conditions $(z_1),(z_2)$, and (z_3), extending in this way another result of Rhoades ([17], Theorem 4). We briefly state it here also.

Theorem 5.2. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : X \to X$ an operator satisfying condition Z. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration defined by (1.1) and $x_0 \in X$ with $\alpha_n \in [0,1]$ satisfying*

$$\sum_{n=0}^{\infty} \alpha_n = \infty . \tag{ii}$$

converges strongly to the fixed point of T.

We now present a convergence theorem for the Ishikawa iteration, corresponding to a typical representative of the class of quasicontractive operators, i.e., the class of Zamfirescu operators.

5.2.1 The Zamfirescu operator

Theorem 5.3 *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ a Zamfirescu operator. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by (5.1.1) and $x_0 \in K$, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (ii). Then, the Ishikawa iteration, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

Proof. By Theorem 1.4 , we know that T has a unique fixed point in K, say *p*. consider $x, y \in K$. Since T is a Zamfirescu operator, at least one of the conditions *(z₁)*, *(z₂)* and *(z₃)* is satisfied. If *(z₂)* holds, then

$$\| Tx - Ty \| \leq b\left[\| x - Tx \| + \| y - Ty \|\right]$$

$$\leq b\left\{\| x - Tx \| + \left[\| y - x \| + \| x - Tx \| + \| Tx - Ty \|\right]\right\}$$

So

$$(1-b)\| Tx - Ty \| \leq b \cdot \left[\| x - y \| + 2b\| x - Tx \|\right],$$

which yields (using the fact that $0 \leq b < 1$)

$$\| Tx - Ty \| \leq \frac{b}{1-b}\| x - y \| + \frac{2b}{1-b}\| x - Tx \| \tag{5.3.1}$$

If *(z₃)* holds, then similarly we obtain

$$\| Tx - Ty \| \leq \frac{c}{1-c}\| x - y \| + \frac{2c}{1-c}\| x - Tx \| \tag{5.3.2}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{5.3.3}$$

Then we have $0 \leq \delta \leq 1$ and, in view of *(z₁)*, (5.1) and (5.2) it results that the inequality

$$\| Tx - Ty\| \leq \delta\| x - y\| + 2\delta\| x - Tx\| \tag{5.3.4}$$

holds for all $x, y \in K$.

Now let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration and $x_0 \in K$ arbitrary.

Then

$$\| x_{n+1} - p \| = \| (1-\alpha_n)x_n + \alpha_n Ty_n - (1-\alpha_n + \alpha_n)p\| =$$

$$= \| (1-\alpha_n)(x_n - p) + \alpha_n (Ty_n - p) \| \leq$$

$$\leq (1-\alpha_n) \| (x_n - p) \| + \alpha_n \| Ty_n - p \| . \qquad (5.3.5)$$

With $x := p$ and $y := y_n$, from (5.3.4) we obtain

$$\| Ty_n - p \| \leq \delta \cdot \| y_n - p \| , \qquad (5.3.6)$$

Where δ is given by (5.3.6). Further we have

$$\| y_n - p \| = \| (1-\beta_n) x_n + \beta_n Tx_n - (1 - \beta_n + \beta_n) p \|$$

$$= \| (1-\beta_n)(x_n - p) + \beta_n (Tx_n - p) \|$$

$$\leq (1-\beta_n) \| (x_n - p) \| + \beta_n \| Tx_n - p \| . \qquad (5.3.7)$$

Again by (5.3.4), this time with $x := p$; $y := x_n$, we find that

$$\| Tx_n - p \| \leq \delta \cdot \| x_n - p \| \qquad (5.3.8)$$

and hence, by (5.3.5) – (5.3.8) we obtain

$$\| x_{n+1} - p \| \leq [1 - (1-\delta)\alpha_n (1 - \delta\beta_n)] \cdot \| x_n - p \| ,$$

Which, by the inequality

$$1 - (1-\delta)\alpha_n (1 - \delta\beta_n) \leq 1 - (1-\delta)^2 \alpha_n ,$$

Implies

$$\| x_{n+1} - p \| \leq \left[1 - (1-\delta)^2 \alpha_n \right] \cdot \| x_n - p \| , \qquad n = 0, 1, 2, \ldots \qquad (5.3.9)$$

By (5.3.9) we inductively obtain

$$\| x_{n+1} - p \| \leq \prod_{k=0}^{n} \left[1 - (1-\delta)^2 \alpha_k \right] \cdot \| x_0 - p \| \qquad n = 0, 1, 2, \ldots \qquad (5.3.10)$$

Using the fact that $0 \leq \delta < 1$, $\alpha_k , \beta_n \in [0, 1]$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, by *(ii)* it results that

$$\lim_{n=\infty} \prod_{k=0}^{n}\left[1-\left(1-\delta\right)^2 \alpha_k\right] = 0,$$

which by (5.3.10) implies

$$\lim_{n=\infty} \| x_{n+1} - p \| = 0,$$

i.e. $\{x_n\}_{n=0}^{\infty}$ converges strongly to *p*.

Remarks.

1) Condition *(i)* in Theorem 1 is slightly more restrictive than condition *(ii)* in our Theorem 2, known as a necessary condition for the convergence of Mann and Ishikawa iterations. Indeed, in virtue of *(i)* we cannot have $\alpha_n \equiv 1$ or $\alpha_n \equiv 0$ and hence

$$0 < \alpha_n(1-\alpha_n) < \alpha_n, \quad n = 0, 1, 2, \ldots,$$

which shows that *(i)* always implies *(ii)*. But values of $\{\alpha_n\}$ exist that satisfy *(ii)*, e.g., $\alpha_n \equiv 1$, such that *(i)* is not true.

2) Since the Kannan's and Chattejea's contractive conditions are both included in the class Zamfirescu operator, by Theorem 5.3 we obtain corresponding convergence theorems for the Ishikawa iteration in these classes of operators.

Corollary 5.1. *Let E be an arbitrary Banach space, K a closed convex subset of E, and* $T: K \to K$ *a Kannan operator, i.e., an operator satisfying* (1.5). *Let* $\{x_n\}_{n=0}^{\infty}$ *be the Ishikawa iteration defined by* (5.1) *and* $x_0 \in K$, *with* $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *satisfying (ii). Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of T.*

Corollary 5.2. *Let E be an arbitrary Banach space, K a closed convex subset of E, and* $T : K \to K$ *a Chatterjea operator, i.e., an operator satisfying* (1.6). *Then the Ishikawa iteration* $\{x_n\}_{n=0}^{\infty}$ *be the defined by* (5.1) *and* $x_0 \in K$*, with* $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ *satisfying (ii). Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of T.*

Conclusions.

1) The convergence theorems of two mean value fixed point iteration procedures for Kannan operators [12], [13] are extended to the larger class of Zamfirescu operators and simultaneously from uniformly convex Banach spaces to arbitrary Banach spaces and to the Ishikawa iteration;

2) The fixed point theorem of Chatterjea is extended from the Picard iteration to the Ishikawa iteration. This also contains, as a particular case, the corresponding convergence theorem for Mann and Krasnoselskij iterations;

3) While the convergence of Picard iteration in the class of Zamfirescu operators cannot be deduced by Theorem 8 of Rhoades [18], our main result also include, as a particular case, the convergence of both Picard and Krasnoselskij iterations.

CHAPTER 6

6.0 COMPARISON OF FASTNESS OF THE CONVERGENCE AMONG KRASNOSELSKIJ AND MANN ITERATIONS IN HILBERT SPACE

The interest of this chapter is to compare the fastness of the convergence to the fixed point among the Krasnoselskij and Mann iterations for the class of Lipschitzian and generalized pseudocontractive operators in Hilbert spaces. Thus it is shown that to each Mann iteration there is a Krasnosleskij iteration which converges faster than the Mann iteration.

Theorem 6.1 in this section shows that the Krasnoselskij iteration is more suitable than the Mann iteration for approximating fixed points of a Lipschitzian and generalized pseudo-contractive operators.

We shall also show that amongst all the Krasnoselskij there exists one which is fastest with respect to the concept of rate of convergence given by Definition 2.1.

6.1 Lipschitzian generalized pseudo-contractive operator

Theorem 6.1. *Let H be a real Hilbert space and K a non-empty closed convex subset of H. Let* $T : K \to K$ *be a Lipschitzian generalized pseudo-contractive operator with corresponding constants* $L \geq 1$ *and* $0 < r < 1$.

Then:

1) T *has a unique fixed point p in K;*

2) *For any* $x_o \in K$ *and* $\lambda \in (0, a)$ *with a given by* (2.2.2) *the Krasnoselskij iteration* $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \lambda, T)$ *converges strongly to p;*

3) *For any* $y_o \in K$ *and* $\{\alpha_n\}_{n=0}^{\infty}$ *in* [0, 1] *satisfying* (2.2.3), *the Mann iteration* $\{y_n\}_{n=0}^{\infty} = M_n(y_0, \alpha_n, T)$ *converges strongly to p.*

4) *For any Mann iteration converging to p, with* $0 \leq \alpha_n \leq b < 1$, *there exists a Krasnoselskij iteration that converges faster to p.*

Proof.

1)- 2) For all $\lambda \in [0,1]$, consider the operator T_λ on *K* given by

$$T_\lambda x = (1-\lambda)x + \lambda Tx, \qquad x \in K \tag{6.1.1}$$

Since *K* is convex, we have $T_\lambda(K) \subset K$, for all $\lambda \in [0,1]$.

From the generalized pseudo-contractive and Lipschitzian conditions on *T* and

$$\| T_\lambda x - T_\lambda y \|^2 = \| (1-\lambda)(x-y) + \lambda(Tx - Ty) \|^2 =$$

$$= (1-\lambda)^2 \| x-y \|^2 + 2\lambda(1-\lambda) \cdot \langle Tx - Ty, x-y \rangle + \lambda^2 \cdot \| Tx - Ty \|^2$$

we find that

$$\| T_\lambda x - T_\lambda y \|^2 \le \left[(1-\lambda)^2 + 2\lambda(1-\lambda) r + \lambda^2 L^2 \right] \| x-y \|^2,$$

so

$$\| T_\lambda x - T_\lambda y \| \le \theta \cdot \| x-y \|, \quad \text{for all } x, y \text{ in } K, \tag{6.1.2}$$

where $0 < \theta = \left[(1-\lambda)^2 + 2\lambda\ (1-\lambda) r + \lambda^2 L^2 \right]^{1/2} < 1$, as $\lambda < a$.

Since K is a closed subset of a Hilbert space, K is a complete metric space. Then by Banach contraction mapping principle, T_λ has a unique fixed point q in K and the Picard iteration associated to T_λ,

$$x_{n+1} = T_\lambda x_n, \quad n \ge 0, \tag{6.1.3}$$

converges strongly to q, for any $x_0 \in K$.

Now using the fact that $\{x_n\}_{n=0}^{\infty}$ given by (3.3) is exactly the Krasnoselskij iteration $K_n(x_0, \lambda, T)$ associated to T, on the one hand, and the fact that $F(T) = F(T_\lambda)$, for all $\lambda \in (0, 1)$, that is $p = q$ is the unique fixed point of T, on the other hand we obtain 1) and 2).

3) Let $\{y_n\}_{n=0}^{\infty}$ be the Mann iteration with $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ satisfying (2.3.1). Consider t, $0 < t < 1$, and denote $a_n = \frac{1}{t}\alpha_n$, $n = 0, 1, 2, \ldots$.

Then the Mann iteration will be given by

$$y_{n+1} = (1 - t\,a_n) y_n + t a_n T\, y_n, \quad n = 0, 1, 2, \ldots.$$

we have

$$\| y_{n+1} - p \| = \| (1-a_n) y_n + a_n \left[(1-t)\, y_n \; + \; tTy_n \right] - p \| \leq$$

$$\leq (1-a_n) \| y_n - p \| + a_n \| (1-t)(y_n - p) + t\,(Ty_n - Tp) \| \qquad (6.1.4)$$

Using the properties of T we find that

$$\| t\,(Ty_n - Tp) + (1-t)(y_n - p) \|^2 = (1-t)^2 \| y_n - p \|^2 +$$

$$+ 2t\,(1-t) \langle Ty_n - p, y_n - p \rangle + t^2 \| Ty - p \|^2 \leq \qquad = \left[(1-t)^2 \right.$$

By (6.1.4) and (6.1.5) we get

$$\| y_{n+1} - p \| \leq \left\{ 1 - a_n + a_n \left[(1-t)^2 \; + \; 2t \; (1-t)\, r + t^2 L^2 \right]^{\frac{1}{2}} \right\} \cdot \| y_n - p \|$$

$$= (1 - (1-\theta)\, a_n) \cdot \| y_n - p \|$$

$$\leq \prod_{k=1}^{n} (1 - (1-\theta)\, a_k) \| y_1 - p \| \qquad 6.1.6$$

where

$$0 \leq \theta = \left[(1-t)^2 + 2t\,(1-t)\, r + t^2 L^2 \right]^{1/2} < 1,$$

for all t such that $0 < t < 2(1-r)/(1-2r+L^2)$.

Since by (2.3.1) $\sum_{n=0}^{\infty} \alpha_n$ diverges, follows that $\sum_{n=0}^{\infty} a_n$ diverges, too, and in view of $\theta < 1$ we get that

$$\lim_{n \to \infty} \prod_{k=1}^{n} (1 - (1-\theta)\, a_k) = 0,$$

which by (6.1.6) shows that $\{y_n\}$ converges strongly to p.

4) Take $x := x_n$, $y := x_{n-1}$ in (6.1.2) to obtain

$$\| x_{n+1} - x_n \| \leq \theta \| x_n - x_{n-1} \|$$

which inductively yields

$$\| x_{n+1} - x_n \| \leq \theta^n \| x_1 - x_0 \|$$

and hence by triangle rule we obtain

$$\| x_{n+p} - x_n \| \leq \theta^n \left(1 + \theta + \cdots + \theta^{p-1}\right) \| x_1 - x_0 \|, \qquad (6.1.7)$$

valid for all $n, p \in N^*$

Now letting $p \to \infty$ in (6.1.7) and using part 2), we get

$$\| x_n - x^* \| = \frac{\theta^n}{1 - \theta} \| x_1 - x_0 \| \qquad (6.1.8)$$

Therefore in view of (6.1.6) and (6.1.8), in order to compare the Krasnoselskij and Mann iterations, we have to compare

$$\theta^n \text{ and } \prod_{k=1}^{n} \left[1 - (1 - \theta) a_k\right]$$

Let $\{ y_n \}_0^\infty$ be a certain Mann iteration converging to p, with $\{ \alpha_n \}_n^\infty$ satisfying $0 \leq \alpha_n \leq b < 1$. Then, $a_k \leq b/t$ (denote b/t by b) and for any m, $0 < m < 1$, we find $\theta \in (0, 1)$ such that

$$b < \frac{1 - \left(\theta/m\right)}{1 - \theta}.$$

Indeed, to this end it is enough to take $\theta < \dfrac{m(1 - b)}{1 - bm}$.

Using the fact $a_k \leq b$, it results

$$\frac{\theta}{1 - (1-\theta) a_k} \leq m < 1,$$

which shows that

$$\lim \frac{\theta}{\prod_{k=1}^{n} [1-(1-\theta) a_k]} \leq \lim_{n\to\infty} m^n = 0,$$

So the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \theta, T)$ converges faster than the considered Mann iteration, $\{y_n\}_{n=0}^{\infty} = M_n(y_0, \alpha_n, T)$.

To end the proof we still need to show that the intervals $(0, a)$ with a given by (2.2.2) and $\left(0, \frac{m(1-b)}{1-m}\right)$ have a nonempty intercession. But this is immediate, since $\frac{m(1-b)}{1-m} > 0$ and $0 < a = \frac{2(1-r)}{1-2r+L^2} \leq 1$, under the hypothesis of Theorem 6.1.

Remarks

1) Part 4) in Theorem 6.1 shows that, in order to approximate the fixed points of a Lipschitzian and pseudo-contractive operator T, it is always more convenient to use a certain Krasnoselskij iteration in the family (2.4) with $\lambda \in (0, a)$ and a is given by (2.2.2).

2) Moreover, amongst the Krasnoselskij iterations $\{x_n\}_{n=0}^{\infty}$ there exists one which is the fastest in that family in the sense of Definition 2.1.

6.2 Fastest iteration in the family of Krasnoselskij

Theorem 6.2 [1] *Let all assumptions in Theorem 2.2 be satisfied. Then the fastest iteration* $\{x_n\}_{n=0}^{\infty}$ *in the family (2.4) with* $\lambda \in (0,\ a)$ *is that obtained for*

$$\lambda_{\min} = (1-r)/(1-2r+L) \tag{6.2.1}$$

Proof

We have to find the minimum of the quadratic function

$$f(x) = (1-x)^2 + 2x(1-x)r + x^2L^2$$

with respect to *x*, that is to minimize the function

$$f(x) = (1-r+L^2)x^2 - 2(1-r)x + 1, \qquad x \in (0,\ a)$$

with a given by

$$a = 2(1-r)/(1-2r+L^2)$$

From (β) we have that

$$1-2r+L^2 \geq (1-r)^2 > 0,$$

and hence *f* does admit a minimum, which is attained for

$$x = \lambda_{\min},$$

with $\lambda_{\min}$ given by (6.2.1). The minimum value of $f(x)$ is then

$$f_{\min} = (L^2-r^2)/(1-2r+L^2),$$

which shows that the minimum value of θ given (δ) is then

$$\theta = [(L^2-r^2)/(1-2r+L^2)], \qquad \text{that completes the proof.}$$

Remarks

1) Theorem 4 shows that the fastest iteration is commonly obtained for λ situated in the middle of the interval to which the parameter belongs.

2) Observe that in view of the condition $\lambda < 1$, the convergence of the Picard iteration cannot be obtained from Theorems 2.2 – 6.2. Actually, as shown by Example 1, the Picard iteration does not generally converge and this is the reason we need to consider other fixed point iteration procedures, like Krasnoselskij and Mann, in order to approximate fixed point of Lipchitzian and generalized pseudo-contractions.

CHAPTER 7

7.0 CONCLUSION AND DISCUSSION

This topic was chosen basically to underline some fixed point theorems and to study the strong convergence to a fixed point of four iterative procedures (Picard, Krasnoselskij, Mann and Ishikawa iterative methods). Finally, in chapter 6 we compared the rate of convergence of the Krasnoselskij and Mann iterative schemes for Lipschitzian and generalized pseudocontractive operators to determine which of the scheme is better, i.e., which one converges faster to the fixed point of the operator? It's quite significant because of the several applications of fixed points iterative procedures – some of which have been listed in the introduction.

It is clear from results displayed in chapters 3 to 5 that each of the four iterative procedures converge strongly to a fixed points of the various operators considered. Theorem 6.1 clearly shows that in comparing the rate of convergence of the Krasnoselskij and Mann iterative schemes for Lipschitzian and generalized Pseudo-contractive operator, the Krasnoselskij iteration method converges faster than the Mann iteration method.

In summary, each of these topics was introduced by the basic definitions of the iterative schemes, it's relationship to the preceding one, and then we state one or two theorems for which they converge to a fixed point of the operator.

Under the Picard iteration we considered the convergence to a fixed point in a ***complete metric space*** for an ***a–contraction,*** a priori and posteriori estimates. Results for the Krasnoselskij iteration was obtained for ***a closed bounded convex subset of a Hilbert space H,*** and the operators considered were ***Nonexpansive and Demicompact.*** Under the Mann iteration the spaces and operators considered were a ***Banach Space*** (***Complete Normed Linear Space***), ***Uniformly Convex Banach Space*** and ***Strongly Pseudcontractive*** and the class of ***Zamfirescu Mappings*** respectively. Finally for the Ishikawa iteration: ***Banach Space***, ***Convex Compact subset of a Hilbert space*** and ***Lipschitzian Pseudcontractive*** and ***Zamfirescu Map***.

This work was limited only to the strong convergence of iteration methods to fixed points. Weak convergence was not our focus.

For exciting new problems for research, we may consider the following questions:

- How do we compare the convergence to a fixed point for three or four different iterative schemes in order to measure which of them is the fastest?
- If that were possible, do we restrict this study to one space and for the same operators, or varied spaces and operators? What criteria do we use to determine the fastest scheme for the latter condition?

- Thirdly, given that we consider spaces and maps different from those considered in this work will the iterative methods yield a fixed point for maps. Is it possible to impose some conditions on the space or map to obtain a fixed point?

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