KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI.

Valuation of Standard Option with Dividend Paying Stock Using Finite Difference Method.

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M.Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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"The fear of the Lord is the beginning of knowledge" (Proverbs 1:7).

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Dedication

I dedicate this dissertation to my dear sister, Mrs Naomi Asamoah.

Abstract

Numerical methods form a significant part of the pricing of financial derivatives, especially in cases where there is no closed form analytical solution. The evaluation of American options using the Black-Scholes Model where early exercise is possible and a general closed-form solution does not exist leads to a free boundary value problem. A common way to deal with this problem is to apply numerical methods. In this thesis we price American options with dividend paying stock on a single asset. We start from the Black-Scholes equation with a free boundary value, the free boundary value problem is then transformed into a Linear Complementarity Problem, and an Obstacle Problem. We solve the Linear Complementarity Problem by introducing the method of Finite Difference method. Finite difference methods is discussed quite extensively with a focus on the Crank-Nicolson scheme. This leads to a constraint linear system of equations which is solved on a discrete domain by applying the PSOR method. The simulation results showed that the price of the American option exceeds the analytical solution. The payoff function intersects the European option at lower prices relative to the American option; this gives us the early exercise value. We conclude that the American option with dividend paying stock is preferred to the European option.

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Chapter 1

Introduction

In this chapter, we present the background of the study to standard option pricing. The problem of the study is outlined clearly from which the objectives of the study are stated. The methods that are employed for this work and the significance of the study are also explained. The organization of the study is finally discussed.

1.1 Background of the Study

An option, one of the components of financial derivatives (contingent claims) has been considered to be the most dynamic segment in the financial markets traded for centuries. But it remained relatively obscure financial instruments until the inception of a listed options exchange in early 1970s Liu (2007). Since then, option trading has enjoyed an unprecedented expansion in American securities markets. Futures, Forwards and Swaps are also components of contingent claims. According to Cox et al. (1979), option pricing theory is a relevant field to almost every field of finance. It has a long and illustrious history, but it also underwent a completely new change in 1973.

In 1905, Albert Einstein expounded more on the work of botanist Robert Brown, who first described the motion of a pollen particle suspended in fluid in 1828 Klebaner (2005). Though, Brown observed the random movement of pollen particles immersed in water he argued that the movement is due to outpouring of the particle by the molecules of the fluid and obtained the equations for Brownian motion.

Before Einstein (1905), Louis Bachelier , a young French PhD student, was the first to analyse Brownian motion mathematically in order to develop a theory of option pricing in 1900 cited by Merton (1973). Louis further deduced Brownian motion as a model for speculative prices. However, the formula was based on unrealistic assumptions with zero interest rate (drift) and a process that allowed for a negative stock price Kishimoto (2008).

Moreover, Norbert Wiener proposed mathematical foundation for Brownian motion as a stochastic process in 1931 by publishing the first in a series of papers originating in 1918 cited by Ntwiga as in Klebaner (2005). The Brownian motion process, $(B(t))$ is also referred to as Wiener process, $(W(t))$.

Sprenkle (1961) enhanced Bachelier's expected-value theory by employing the assumption that the stock price follows a geometric Brownian motion. Further enhancement introduces different discount rates, namely, the expected return of the stock by Boness (1964) and the expected return of the option by Samuelson (1965). He further stated that Sprenkle, Boness and Samuelson improvement on Bachelier's formula in 1960s assumed that stock prices are log-normally distributed, which guarantees that the stock price is always positive and allowed for a non-zero interest rate.

Another assumption was that investors are risk-averse and demand a risk premium in addition to the risk-free interest rate. Additionally, an economist Samuelson propagated the exponential Brownian motion (Geometric Brownian motion) for modelling prices which are subject to uncertainty to his students at Massachusetts Institute of Technology (MIT) in 1960s.

According to Ralf and Elke, 1848 marked the foundation of the Chicago Board of Trade (CBOT), which later became the biggest future market in the world. In 1973, the CBOT opens the Chicago Board Options Exchange (CBOE) and became the first organized exchange to start trading option as well as other financial derivatives.

Before 1973, the valuation approach was, basically, to determine the expected value of an option at expiration and then to discount its value back to the time of the evaluation. The difficulty in this approach was determining the discount rate; in other words, assigning a risk premium was a tough problem, which was not successfully resolved at that time.

The most influential development in terms of impact on financial practice was the Black-Scholes model for option pricing. Black and Scholes published their first work in 1973, which contains the Black-Scholes formula for option pricing of European option. In the same year, Robert Merton extended their model in several important ways. Later 1977, Robert together with Scholes were awarded the nobel Prize for economics after the death of Fischer Black.

Option is a major financial derivative, which gives its holder the right, not the obligation to trade a fixed amount of underlying asset at an agreed-upon price on the maturity date(European option) or any time on or before the maturity date(American option). According to Hull (2009) a call option gives the holder the right, but not obligation to buy and a put option also gives the holder the right, but not obligation to sell its underlying asset at a certain time(t) for a fixed strike price(K).

Finally, options form the foundation of innovative financial instruments, which are extremely versatile securities that can be used in many different ways. Seydel (2002)

1.2 Statement of the Problem

Option pricing is widely used amongst academics, practitioners and professionals in the financial market. Over the last 30 years, option pricing on risky assets has long been an intriguing problem as valuation of American option is concerned. It is widely acknowledged that a general closed-form analytical solution does not exist for the American option valuation where early exercise is permitted at anytime during the life of the option, i e where early exercise may be optimal.

In contrast, the European option, which can only be exercised at its maturity date has been valued analytically by the celebrated Black-Scholes formula for the standard financial model stated by Black and Scholes (1973).

In real markets, many companies pay dividends to the stock holder. The celebrated Black-Scholes model cannot deal with dividend payments, therefore there is the need to extend (modify) the model to include the cash dividends. Since most traded options are of American type and where solution is based on an iterative method, there is considerable interest in searching for new valuation techniques. In view of that, the valuation of American option routinely resorts to numerical techniques, whose improvement is still an active field of research. Therefore efforts have been concentrated on approximate methods.

It is against this background that the study seeks to employ Finite difference methods (FDM) quite extensively with a focus on the Crank Nicolson method, using transformed Black-Scholes equation (PDE) in valuing standard Option with dividend and apply these numerical techniques to the pricing of standard (vanilla) options.

Finally, we compare the performance of the Crank Nicolson methods Finite difference methods to the analytical Black Scholes price of the standard option with dividend paying stock.

1.3 Objectives of the Study

The objectives of this research are as follows:

- To apply the Crank Nicolson method coupled with projected SOR in valuing standard option with dividend paying stock.
- To compare the performance of the Crank Nicolson methods to the analytical solution of basic Black Scholes model for pricing standard option.
- To determine the optimal value at early exercise.

1.4 Methodology

The Crank Nicolson method of FDM is applied in modelling standard option with dividend paying stock. A Matlab programming language is used to implement the method in generating the tables and graphs. The codes are listed in the appendix.

Finally, the results are discussed by comparing their performance to the solution of transformed Black-Scholes model.

1.5 Significance of the Study

Firstly, the tool could be used for the education purpose. It can also be used for academics to test accuracy of the option pricing models such as Black-Scholes against the market option price.

Secondly, for the market speculator to look for the best investment opportunity in market mispricing, and for the individual investor such as employee who holds employee options to find a best moment to sell the option.

1.6 Organization of the Thesis

The study consists of five chapters. Chapter 1 provides general background information of the study: the statement of problem, the objectives of the study, the methods to be used, the structure as well as the significance of the study. Chapter 2 reviews pertinent literature related to the dynamics of option pricing, the market strategies and the dynamics of derivative prices, specifically pricing standard options on stocks. Chapter 3 presents the methodology employed in valuing stock options.

The data analysis is presented by means of tables and graphs in Chapter 4. Chapter 5, which is the final chapter, summarizes the main findings of the study and provides suggestions and recommendations.

Chapter 2

Literature Review

2.1 Introduction

Option pricing, one of the financial derivatives, has been studied extensively in both academic and trade literature. There has been vast explosions of theoretical and empirical investigation on option pricing. The focus of this chapter highlights the overview of related work in the field of finance as assuming a basic Black-Scholes model, the Crank Nicolson finite difference methods and the option pricing.

2.2 Review of Black-Scholes Model

Options pricing is a very important problem encountered in financial domain. Options which are one of the financial derivatives are widely traded on financial markets. There are several method of determining the value (or price) of a given option. To determine the prices of the option, we require the formulation of a model for the way in which the asset price changes over time. One of the widely used models is the Black-Scholes model for pricing a European put and call option Brennan and Schwartz (1978).

A theoretical valuation formula was derived by Black and Scholes in 1973 for option pricing. This theoretical formula was based on the rationale that, if options are correctly priced in the market, it should not be possible to make profits by creating portfolios of long and short positions in options and their underlying stocks. The model is applicable to corporate liabilities namely, common stock, corporate bonds, and warrants since almost all corporate liabilities can be viewed as combination of options. In particular, the Black-Scholes formula can be used to derive the discount that should be applied to a corporate bond because of the possibility of default.

Rubinstein (1983), developed an option pricing formula that pushes the underlying source of risk back to the risk of individual assets of the firm. Relative to the Black-Scholes formula, the displaced diffusion formula has several desirable characteristics. The formula include differential riskiness of the assets of the firm, their relative weights in price determination of the firm, the effects of firms debt and finally the effects of a dividend policy with constant and random components.

Cox et al. (1979) introduced the binomial method for the valuation of American options, which is also flexible and requires time discretizations. The method is not stochastic in the sense that random numbers are used. The probabilities for the up and down movements are rather a consequence of the assumed market factors. The basic model was extended by additional factors to a multinomial method, with several possibilities for the up and down movements of the asset price. Boyle (1988). Geske and Shastri (1985) presented an analytic solution to American put option with or without dividends. However, their formula is an infinite series that must be approximated by numerical methods. Kim (1990) and Carr et al. (1992) provided an integral representation of the option price. These methods are compared by Broadie and Detemple (1996), who also derived the lower bound and upper bound for the value of American options.

The valuation of American options with dividends has also been studied by many researchers. Geske (1979), Roll. (1977) , and Whaley (1981) obtained analytical solutions for the case of known discrete dividends, while Brennan and Schwartz (1977) and Brennan and Schwartz (1978) introduced the finite difference approximation approach with log-transformation. This numerical method approximates differential terms of the value function by discretizing both time and state space. The finite difference method is one of the most popular methods because it is flexible and easy to implement, so that non-standard forms of options also may be solved.

Gallant et al. (1992), in their work investigated the joint dynamics of price changes and volume on the stock market making use of daily data on the S & P composite index and total NYSE trading volume from 1928 to 1987. Nonparametric methods were used to achieve the set objectives. Gallant et al. (1992) found out that the daily trading volume is positively and nonlinearly related to the magnitude of the daily price change and that price changes lead to volume movements.

Heston (1993) used a new technique, which was based on the Black-Scholes formula, to derive a closed-form solution for the price of a European call option on an asset with stochastic volatility. The model allows arbitrary correlation between volatility and spot asset returns. He introduced stochastic interest rate and showed how the model is applicable to bond options and foreign currency options. The result from his work showed that correlation between volatility and the spot asset price is important for explaining return skewness and strike price biases in the Black-Scholes model Black and Scholes (1973). Dempster and Hutton (1999) also studied American option pricing problem using linear programming approach.

Pastorello et al. (2000) dealt with the estimation of continuous-time stochastic volatility models of option pricing. They achieved this in a Monte Carlo experiment which compared two very simple strategies based on different information sets. An Ornstein-Uhlenbeck process for log of the volatility, a zero-volatility risk premium, and no leverage effect was assumed. Sticking to the framework with no over identifying restrictions, it was shown that, given the option pricing model, estimation based on option prices is much more precise in samples of typical size.

Kumar et al. (2012) obtained an analytic solution of the fractional Black-Scholes European option pricing equation. The Laplace homotopy perturbation method, a combined form of the Laplace transform and the homotopy perturbation method, was used with boundary condition for a European option pricing problem to obtain a quick and accurate solution to the fractional Black-Scholes equation. The analytic solution of the fractional Black-Scholes equation was calculated in the form of a convergent power series with easily computable components.

More recent studies on American option pricing are based on linear complementarity problems (LCPs). Huang and Pang (1998) provided discretized LCP formulations for various option problems including American options and suggested solution algorithms including projective successive over-relaxation (PSOR), Lemke's algorithm and a revised parametric principal pivoting (PPP) algorithm. Forsyth and Vetzal. (2002) considered a special penalty method for LCPs adequate to handle American option constraints, while Coleman et al. (2002) proposed a Newton type method for a nonlinear programming problem based on quadratic penalization of the complementarity conditions. Ikonen and Toivanen (2007) showed LU decomposition can improve the performance of several different algorithms for solving LCPs of American options.

Nevertheless, despite the different extensions that have been developed within the last three decades, the basic Black-Scholes-Model is still the most accepted and widely used framework in financial industry and research.

2.3 Finite Difference Methods

According to Hull (2009) the finite difference method is one of the most popular method under Numerical techniques for valuation of options in cases where a closed-form of analytical solutions are impossible. The finite difference method attempts to solve the Black-Scholes partial differential equation by approximating the differential equation over the area of integration by system of algebraic equations Tveito and Winther (1998). The most common finite difference methods for solving the partial differential equations are:

- Explicit scheme
- Implicit scheme
- Crank Nicolson scheme

According to them these schemes are closely related but differ in stability, accuracy and execution speed, The finite difference techniques was applied by Brennan and Schwartz to solve option valuation problems for which closed form solutions are unavailable by Brennan and Schwartz (1978). They considered the valuation of an American option on stock which pays discrete dividends. Brennan and Schwartz (1977) introduced the finite difference approximation approach with log-transformation. This numerical method approximates differential terms of the value function by discretizing both time and state space Kwon and Friesz (2008). Finite Difference scheme was also used by Courtadon (1982) to find the value of an American option.

Wilmott et al. (1996) defined Finite difference methods are a means of obtaining numerical solutions to partial differential equations and linear complementarity problems. They constitute a very powerful and flexible techniques and if applied correctly they are capable of of generating accurate solutions to all models. Shcherbakov and Szwaczkiewicz (2010) also stated that the main idea of the finite difference methods is to replace the partial derivatives which appear in the partial differential equations by difference quotients. In other words, it relies on replacing differential equations by Finite Differences equations.

Morton and Mayers (2005) of the view that Finite Difference Methods create a mathematical relationship which links together every point on the solution domain, like a chain. The first links in the chain are the boundary conditions and from these, we discover what every other point in the domain has to be. Perhaps the most popular FD methods used in computational finance are: Explicit Euler, Implicit Euler, and the Crank-Nicolson method. However, The main disadvantage to using Explicit Euler is that it is unstable for certain choices of domain discretisation. Though Implicit Euler and Crank-Nicolson involve solving linear systems of equations, they are each unconditionally stable with respect to the domain discretisation. But Crank-Nicolson exhibits the greatest accuracy of the three for a given domain discretisation.

Nwozo and Fadugba (2012) point out that among the three schemes considered, Crank Nicolson Scheme is unconditionally stable, more accurate and converges faster than binomial model and Monte Carlo Method when pricing standard options, while

Monte carlo simulation method is good for pricing path dependent options. Benbow (2005) used a Crank-Nicolson finite difference method formulated in a Lagrangian frame in Solving the Black-Scholes equation for the valuation of American options.

2.4 Financial Derivative

A financial derivative also known as contingent claim is a contract whose value is determined by the value of one or more underlying assets Hull (2009). Seydel (2002) of the view that derivatives are very essential in financial market as most firms use them to reduce or control risk (hedge). However, some also use them to speculate by buying or selling derivatives in hopes of earning a profit. He further stated that investors hold a great deal of risk. As a result, they incurred losses substantially, if speculations do not work out. For instance, in 1995, the United Kingdom's Barings', one of the world's oldest banks, collapsed when futures speculation by one of its traders in Singapore incurred losses of over $1 billion.

According to Shcherbakov and Szwaczkiewicz (2010), financial derivatives are instruments to assist and regulate agreements on transactions of the future. They can be traded on specialized exchanges throughout the world and its significance is evident.

Therefore, derivative is a type of contract that allows purchase or sale of an asset in the future on terms that are specified in the contract. The future value of a derivative is a stochastic process due to its uncertainty.

Futures, forwards, swaps and options are the main types of financial derivative products, but the most important underlying assets that can be traded under financial market are stocks(equities), bonds (Treasury Bills), foreign exchanges (currencies), commodities(such as, oil, cocoa, gold), interest rate, etc Wilmott et al. (1996).

2.5 Financial Derivatives Tools

2.5.1 Risk

Risk is defined in a portfolio as the variance of the return. It can be almost completely eliminated by holding a well-diversified portfolio.

As stated by Hull (2009), the higher the risk of an investment, the higher the expected return demanded by an investor. So a highly volatile stock with a very uncertain return has a large variance and is a risky asset. We have two types of risk: specific and non-specific. Specific risk is the component of risk associated with a single asset or a sector of the market, whereas the one associated with factors affecting the whole market is considered as Non-specific risk Wilmott et al. (1996).

2.5.2 Arbitrage

Arbitrage is defined in finance as a strategy that allows to make a profit out of nothing without taking any risk. According to Ntwiga (2005) arbitrage is a trading strategy that involves two or more securities being mispriced relative to each other to realise a profit without taking a risk.

Hull (2009) stated that arbitrage is sometimes possible when the future price of an assets get out of line with its spot price. Further, arbitrage involves looking in a riskless profit by simultaneously entering into transaction in two or more markets.

However, Wilmott et al. (1996) of the view that 'there is no such thing as a free lunch', so in financial terms, there are never any opportunities to make instantaneous riskfree profits as in Neftci (2000). The main tool used to determine the fair price of a security or a derivative asset rely on the no-arbitrage principle.

Arbitrage-Free Market

According to Klebaner (2005) the main idea in pricing by no-arbitrage arguments is to replicate the payoff of the option at maturity by a portfolio consisting of stock and bond (cash). So, to avoid arbitrage, the price of the option at any other time must equal the value of the replicating portfolio, which is valued by the market.

Risk Neutral Valuation

Risk-Neutral Valuation states that any security dependent on other traded securities can be valued on the assumption that the world is risk neutral, according to Hull (2009). Further, a risk neutral world is a world where assets are valued solely in terms of their expected return but the expected return on all investment assets is the risk-free interest rate, r. So all investors are indifferent to risk. As a result, investors do not require a premium to urge them to take risks.

Moreover, the risk-free rate of interest is the appropriate discount rate to apply to any expected future cash flow. Consequently, derivative prices are determined by expected present value pay-off . Therefore, the risk-neutral valuation principle is important in option pricing, where the value of an option depends on the standard deviation of the asset price. In effect, the expected return (drift term), μ in the Stochastic differential equation for underlying asset is replaced by the risk-free rate of interest (r), whenever it appears Wilmott et al. (1996).

2.5.3 Log-normal Dynamics

Lognormal Transformation

The log transform method was suggested by Brennan and Schwartz (1978). According to them, when S is a stock price, it is efficient to use $\log(S)$ rather than S as the underlying variable. This is because, as indicated in Brennan and Schwartz (1978) by Hull and White, when σ is constant, the instantaneous standard deviation of $\log(S)$ is also constant.

Hull (2009) stated that a variable that has a log-normal distribution takes value between zero and infinity whereas variable that follows normal distribution can have values from negative infinity to positive infinity, which is not consistent with the real world stock price behaviour. He further argues that instead of a normal distribution of stock price, it is more appropriate to modelled stock that follow a log-normal distribution. According to Ntwiga (2005), the rate of return of a stock can be given as

$$\frac{(S_{t+\delta t} - S_t)}{S_t} = \mu\delta t + \sigma Z\sqrt{\delta t}$$

where $Z \sim N(0,1)$. He further stated that, as the time intervals become smaller and smaller and the limit as $t \to 0$, then

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

He denoted $\mu dt + \sigma dW_t$ by

$$dX_t = \mu dt + \sigma dW_t$$

and S_t by

$$S_t = S_0 e^{X_t}$$

This means that the logarithm of S_t is normally distributed. Hence, we say that the distribution of S_t is log-normal according to Wilmott et al. (1996).

Baz and Chacko (2004) state that the log-normal distribution has the following advantages over the normal distribution:

- It differs from the symmetric normal distribution in that it exhibits a skew with its mean and median all differing from that in a normal distribution. The stock dynamics will be treated as log-normally distributed with a specified mean and variance.

- It is mathematically tractable, and so we can obtain solutions for the value of the options if stock returns are log-normally distributed. The value of the option prices that we compute are very good approximations of actual market prices

2.5.4 Volatility

According to Chiu (2002), volatility (σ) is a measurement of change in price over a given period, commonly expressed in percentage terms. He further stated that volatility can be computed as the annualized standard deviation of the percentage change in daily price. Therefore, volatility is simply a measure of the degree of price movement in a stock, futures contract or any other market.

Volatility of a stock price is a measure of how uncertain we are about future stock price movements. The more volatile a stock market, the more an investor stands to gain or lose in a short time Hull (2009). As volatility increases, the chance of stock does very well or very poorly. For the owner of a stock, these two outcomes tend too offset each other. However, this is limited downside risk in the event of price decreases because the most the owner can lose is the price of the option. Similarly, the owner of a put benefits from prices decreases, but has limited downside risk in the event of price increases. The values of both calls and puts therefore increase as volatility increases. (Hull, 2002).

Historical and Implied volatility are the two major approaches for the estimation of volatility Jain (1997).

Historical Volatility

Following Jain (1997) historical volatility is the measure of a stock's price movement based on historical prices. He states that it is the measure of how active a stock price typically is over a certain period of time. Historic volatility also called realized volatility, is a measure of actual price changes during a specific time period in the past.

Mathematically, historic volatility is the annualized standard deviation of daily returns during a specific period. It estimates the volatility by calculating the standard deviation of the natural log of the price changes of a sample time series of historical data for the asset price. Therefore, the daily return is given as,

$$X_t = ln(\frac{S_t}{S_{t-1}})$$

The variance is estimated by the sample variance, which is normalised by $(n-1)$ to make it an unbiased statistic.

$$HV = \sigma = \sqrt{\frac{\sum_{t=1}^{n}(X_t^2 - X^2)}{(n-1)}}$$

The assumed uncertainty about the asset does not increase linearly. If the asset pays dividends, then the asset price sequence must be adjusted to reflect the non-homogeneous nature of the data series. The transition from cum-dividend to ex-dividend will affect the price of the asset. A dividend payment increases the return to be paid to the buyer. If the buyer has an asset that pays a dividend λ; then the daily price return is restated as

$$\ln[\frac{(S_t + \lambda)}{S_t - 1}].$$

Implied Volatility

Implied volatility is a volatility percentage that explains the current market price of an option by reflecting the volatility that options traders expect for the returns to the underlying stock during the life of the option Chiu (2002). As the common denominator of option prices, it enables comparison of options on different underlying instruments, and comparison of the same option at different times.

According to Jain (1997), it is the current volatility of a stock, as estimated by its option price. So, if the price of an option is known, which consists of several components such as the exercise price, maturity date, the spot stock price, dividends paid by the stock (if any), the implied volatility of the stock and interest rates except volatility, then you can modify the option-pricing model to calculate the implied volatility.

2.5.5 Dividend Paying Stock

Dividends are a share of profits paid to shareholders as cash or as additional shares of stock. They have influence on the stock price. Profits or earnings that are not distributed to shareholders stay with the firm and are called retained earnings. These earnings influence the value of the stock, because they increase the total asset value, or total amount of assets, of the firm . The stock price goes up in post-dividend date and it reduces the stock price on the ex-dividend date. According to Hull (2009) the value of call option is negatively related to the size of any anticipated dividends, and the value of a put option is positively related to the size of any anticipated dividends.

Discrete Dividend

Assumption

The underlying stock S is paying a discrete dividend at a fixed and known date $t_D \in (0,T)$. There will be either a known dividend payment of amount $\lambda > 0$ or a known dividend rate $\rho \in (0,1)$ at t_D. In case of a discrete payment, one has to take into consideration that at the instant of a discrete payment the price S_t of the asset instantaneously drops by the amount of the payment. Suppose that the stock pays N discrete dividends at known payments times (dates) $t_1, t_2, ..., t_N$ where $t_1 < t_2 < ... < t_N$ of amounts $D_1, D_2, ..., D_N$, respectively. Since ex-dividend dates and the actual amounts of dividends are known then we could assume that the asset price is composed of two components:

- The risk-free component that will be used to pay the known dividends during the life of the option. This is then taken to be the present value of all future dividends discounted at the risk free interest rate. The dividends will have been paid and the risk-less component will no longer exist by the time the option matures.
- Risky component which follows a stochastic process. The value of the risky component denoted as $\tilde{S}_t$ is

$$\tilde{S}_t = S_t - D_i e^{-rt_i} \qquad for \quad i = 1,2,...,N.$$

The new asset price $\tilde{S}_t$ is then used to compute the value of the option

Continuous Dividend yield

The continuous yield model is extremely useful to options on stocks. Let λ represent the constant continuous dividend yield which is known. The dividend yield is defined as the proportion of an underlying asset price that is paid out per unit time. Thus, in each time interval , δt, the underlying asset pays out a dividend $\lambda S \delta t$ to the holder, where $\lambda \geq 0$ is a constant. The share value is lowered after the dividend payout so the expected rate of return μ of a share becomes $(\mu - \lambda)$.

Chapter 3

Methodology

3.1 Introduction

The chapter discusses the methods used in achieving the objectives of the research work. The principal attention is about pricing American option, where early exercise is optimal. We derive the celebrated Black and Scholes model with a free boundary value. The problem is transformed equivalently into a Linear Complementarity Problem, an Obstacle Problem and a Variational Inequality. To solve the Linear Complementarity Problem, the chapter introduces the method of Finite Differences, which leads to a constraint linear system of equations. The latter can be solved by employing an iterative projection method.

3.2 Options

An option is the right but not the obligation for a transaction of a risky asset(stocks) at a fixed time for a given price in future. Dontwi et al. (2010) stated that options are used as valuable tools in numerous hedging strategies as they define the price at which underlying assets can be bought or sold in the future. It gives the holder the right to buy or sell an asset in the future at a price that is agreed upon today. The specified time and the prescribed amount in the contract are the expiration date or maturity and strike price (exercise price) respectively.

Option contract involves two parties; the writer(Bank) and the holder (investor) about trading underlying asset at a certain future time Seydel (2002). The holder of the option has the right, not an obligation, to exercise the option. He purchases the option by paying a premium, which is the price (V) of the option. The other party, the writer, who fixes the terms of the contract, has the potential obligation to sell the underlying in case the investor chooses to exercise. As a result, the writer of the option must be compensated for the obligation he assumed, if the investor fails to exercise. The holder is said to be in the long position (buy the option) while the other side of the investor takes the short position(sell the option) of the option contract Hull (2009). The party with the long position agrees to buy the underlying asset while the other party who assumes the short position agrees to sell the asset.

Moreover, the option on stock is said to be exercised when the holder chooses to buy or sell the underlying stock, S. As stated by Wilmott et al. (1996) there are two basic types of options; the call and put options. The call option allows the holder the right to buy the underlying for an agreed fixed strike price, K, by maturity date, T. The put option also gives the holder the right to sell the underlying at a certain time, T for an agreed fixed exercise price, K. The exercise rights under option are European and American option. They are not a geographical classification but refers to a technicality in the option contract. Both types are traded in each continent.

In general, there are two main groups of options: standard and non-standard options (exotic). The standard options, the American or European call or put options, with pay-off functions are based on a single underlying asset(single-factor). Exotic options also known as path-dependent options, with pay-off functions depend on several underlying assets (multi-factor options). Further, Options, whose pay-off functions only depend on the final value of the underlying asset, are called vanilla options. Options, whose payoffs depend on the path of the underlying asset, are called exotic or path-dependent options. In this thesis, we will be concerned with pricing plain-vanilla American options with dividend paying stock by means of finite differences.

European Options

According to Ntwiga (2005), European options can only be exercised at a maturity date $(t = T)$. An European call (put) option gives the holder the right but not the obligation to buy (sell) the underlying asset with an initial price S; at a given maturity date, $t = T$ and for a fixed strike price, K. Ntwiga then defined the price of the European call and put option as E_c and E_p respectively. He further stated that the pay-off of the European call at maturity time $t = T$ is

$$E_c = max(S_t - K, 0) := (S_t - K)^+ \tag{3.1}$$

If $S_t < K$, the call will be worthless and the holder will not exercise the right. The pay-off of an European put at maturity time $t = T$ is

$$E_p = max(K - S_t, 0) := (K - S_t)^+ \tag{3.2}$$

If $S_t > K$, the put will be worthless with no cash flows and the holder will not exercise the right . The pay-off of the European put at maturity can be obtained from the put- call parity. The put - call parity is the relationship between a European call and put, given by

$$E_c + Ke^{-rt} = E_p + S, \tag{3.3}$$

where r denotes the risk free interest rate, S the initial stock price and S_t represents the variation of the asset price(current price) S with time t.

American Options

American options can be executed(exercised) at anytime prior to their maturity date $(t \leq T$) Ntwiga (2005). He then stated that American call or put option gives to its holder the right (not the obligation) to buy (sell) the underlying asset, S at any time t $(0 < t < T)$; up to maturity date T; for a strike price K. Let denote A_c and A_p the price of American call and American put respectively. The pay-off of an American call at maturity time t is

$$A_c = max(S_t - K, 0) := (S_t - K)^+. \tag{3.4}$$

The pay-off of an American put at maturity time t is

$$A_p = max(K - S_t, 0) := (K - S_t)^+ \tag{3.5}$$

According to Ntwiga (2005) the price boundary and put-call parity for the American option is given by

$$S - K \leq A_c - A_p \leq S - Ke^{-rt} \tag{3.6}$$

The valuation of an American option can be shown to be uniquely specified by a set of constraints:

- the option value must be greater than or equal to the payoff function.
- the Black-Scholes equation is replaced by an inequality.
- the option value must be a continuous function of S.
- the option delta (slope) must be continuous.

American options are more flexible and more valuable than European ones. This is the reason why options on stocks are mostly of American style. For an American option, mostly there exist no explicit formulas and hence numerical solution techniques are required.

Option Value

The value of the option, $V = V(S,t)$ is a function of both the underlying asset (S) and the time (t) in option pricing. The calculation of the price of an option (premium) is our prime concern. The premium is the fair value of an option contract which the buyer pays to the writer. It is ascertained in the competitive market Seydel (2002).

Hull (2009) stated that the value of the stock option depends on six major factors: the current stock price (S_o), the strike price (K) and the maturity date (T). The market parameters affecting the price are the risk-free interest rate (r), the volatility (σ) of the price (S_t), and the dividend yield (λ), if the asset pays dividends.

Paully (2004) of the view that when the price S_T of the underlying asset is lower than the exercise price ($S_T < K$), the option is said to be ***in-the-money*** (ITM). The holder would exercise the put option, i.e sell the stock with value S_T for the higher price K, and earn the amount of $(K - S_T)$. If S_T, the spot price is equal to the strike price, K,$(S_T = K)$, the put is said to be ***at-the-money***(ATM) and the pay-off function is zero. Otherwise the option is ***out-of-the-money*** (OTM), if $(S_T > K)$.

In the second case, the holder would choose not to exercise the option, which would be worthless on maturity date with no cash flows. This is clear since the holder would not make any profit by selling the underlying asset for K, if its actual price were greater than, or equal to K. If a call is in the money, the holder would buy a stock worth S_T for a lower price K, yielding $(S_T - K)$. A call being at or out of the money would also be worthless on expiration date. The above considerations lead to the *payoff functions* for plain options:

1. for a call option

$$V_C(S_T, T) = max\{S_T - K, 0\} := (S_T - K)^+, \tag{3.7}$$

2. for a put option

$$V_P(S_T, T) = max\{K - S_T, 0\} := (K - S_T)^+, \tag{3.8}$$

All these factors are meant to be per year. Volatility describes the standard deviation, or uncertainty, in the movements of the value S of the underlying asset. The dividend is expressed as the continuous dividend yield λ. Last, but not least, it should be mentioned that the value of American options is always greater than, or equal to the value of their corresponding European, $V_Am \geq V_Eur$

3.3 Derivation of the Black-Scholes Model (PDE)

The famous Black-Scholes model has been used as the basis for pricing financial derivatives, particularly option pricing. The general idea of dynamic hedging was introduced by Black and Scholes (1973) and Merton (1973). This was to replicate the option payoff by a trading strategy in the underlying asset.

Prior to Black and Scholes famous model in 1973, various researchers work on the valuation of options has been presented in terms of warrants. This formula is useful because it relates the distribution of spot returns to the cross-sectional properties of option prices. Black and Scholes (1973) derived the theoretical valuation formula for options. The main conceptual idea of Black and Scholes lie in the construction of a riskless portfolio taking positions in bonds (cash), option and the underlying stock.

In developing the celebrated Black-Scholes model the following assumptions were made in the financial market under consideration, Black and Scholes (1973). It is assumed in the Black-Scholes model that

- the stock price follows a log normal random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal.
- The market is frictionless, thus there are no transaction costs (fees or taxes).
- There are no arbitrage possibilities exist, meaning that there are no opportunities of instantly making a risk-free profit.
- The underlying asset pays no dividends during the life of the options.
- The risk-free interest rate r and the variance of the return (volatility) σ are known functions of time over the life of the option.
- The underlying asset trading is continuous and the change of its price is continuous.

The stochastic differential equation (or SDE) can model the randomness of the underlying asset in financial derivatives. They are utilized in pricing derivative assets because they give a formal model of how an underlying asset's price changes over time. In pricing derivative assets, the randomness of the underlying instrument is essential. After all, it is the desire to eliminate or take risk that leads to the existence of derivative assets.

A trader continuously tries to forecast the price of an asset at any time interval, δt. These 'new events' recorded as time passes contain some parts that are unpredictable. After that, they become known and become part of the new information set $\{I_t\}$ the trader possesses. The formal derivation of SDE's is compatible with the way dealers behave in financial markets.

If S_t is the price of a security, then according to Ntwiga (2005) the dynamic behaviour of the asset price in a time interval dt can be represented by the SDE given by

$$dS_t = \alpha(S_t,t)dt + \sigma(S_t,t)dW_t \qquad for \quad t \in [0,\infty)$$

where dW_t is an innovation term representing unpredictable events that occur during the infinitesimal interval dt, $\alpha(S_t,t)$ is the drift parameter and $\sigma(S_t,t)$ the diffusion parameter which depends on the level of observed asset price S_t on time t Neftci (2000).

Following Sevcovic (2011) the stochastic process $X = \{X_t, t \geq 0\}$ that solves

$$X_t = X_0 + \int_0^t \alpha(X_s,t)ds + \int_0^t b(X_t,t)dW_s$$

is an Itô process. The corresponding stochastic differential equation is given by

$$dX_t = \alpha(X_t,t)dt + b(X_t,t)dW_t$$

where $\alpha(X_t,t)dt$ is the drift form, $b(X_t,t)dW_t$ is the diffusion form and W_s is a standard Wiener process.

According to Ntwiga (2005), if $V(S,t)$ be twice differentiable function of t and of the random process S_t, and S_t follows the Itô`s process

$$dX_t = \alpha_t dt + \sigma_t dW_t, \quad t \geq 0$$

with well behaved drift and diffusion parameters α_t and σ_t then,

$$dV_t = \frac{\partial V}{\partial S_t} dS_t + \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma_t^2 dt.$$

Now,the above conditions lead to an $It\hat{o}`s$ stochastic differential equation, describing the behaviour of the asset price which follows a geometric Brownian motion (GBM)

$$dS = \mu S dt + \sigma S dW, \tag{3.9}$$

where μ denotes the expected return of the underlying asset (drift), σ is the volatility and W follows a Wiener process (Brownian Motion).

We now look for a function $V(S,t)$ that gives the option value for any asset price $S \geq 0$ and at any time $0 \leq t \leq T$. In this setting, $V(S_0, 0)$ is the required time-zero option value. We further assume that such a function exists and is smooth in both variables. Therefore, $It\hat{o}`s$ Lemma provides us with a derivative chain rule for stochastic functions. Hence, by $It\hat{o}`s$ Lemma

$$df = \frac{df}{dS}(\mu S dt + \sigma S dW) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} dt \tag{3.10}$$

considering equation (3.10) We write as

$$dV = \sigma S \frac{\partial V}{\partial S} dW + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \tag{3.11}$$

This gives the random walk followed by V . We now construct a portfolio consisting of one option and a proportion $-\Delta$ of the underlying asset. The value of th portfolio is

$$\Pi = V - \Delta S \tag{3.12}$$

Then the change in the value of this portfolio in one time-step becomes

$$d\Pi = dV - \Delta dS \tag{3.13}$$

Combining equations (3.9), (3.11) and (3.12) we find that Π follows the random walk

$$d\Pi = \sigma S(\frac{\partial V}{\partial S} - \Delta)dW + (\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S)dt \tag{3.14}$$

We can eliminate the random component by choosing $\Delta = \dfrac{\partial V}{\partial S}$. This results in a portfolio whose increment is wholly deterministic

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt \tag{3.15}$$

The return on an amount Π invested in a riskless asset would see a growth of $r\Pi dt$ in a time dt. If the right hand side of equation (3.15) were greater than this amount, an arbitrageur could make a guaranteed risk less profit by borrowing an amount Π to invest in the portfolio. Conversely, if the right-hand side of equation (3.15) were less than $r\Pi dt$ then the arbitrageur would make a risk less, no cost, instantaneous profit. Thus we have

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right) dt \tag{3.16}$$

Substituting equation(3.12) into equation(3.16), where $\Delta = \dfrac{\partial V}{\partial S}$ and dividing by dt. Then we arrive at the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.17}$$

Any derivative security whose price depends only on the current value of S and on time, t, which is paid for up-front, must satisfy the Black-Scholes equation.

3.4 Modification to Black-Scholes Model

In this thesis, we attempt to model the price of American standard options based on dividend paying stock. In contrast, the Black-Scholes model discussed above is on assumption that no dividends are paid, where $\lambda = 0$. But when dividend payment is incorporated into the Black-Scholes model, the American options which can be exercised at any time t prior to the maturity date T, leads to the Black-Scholes-inequality. In modelling stock with dividends, the two important questions one needs to asked are:

- When and how often are dividend payments made?
- How large are the dividend payments?

The amounts paid as dividends may be modeled as either deterministic or stochastic. But the focus is on deterministic way only on those equities with dividends whose amount and timing is known at the start of the options life.
Using SDE, equation (3.9), which is the random walk of the asset price is modified to become

$$dS = (\mu - \lambda)Sdt + \sigma SdW \tag{3.18}$$

Considering the effect of the dividend payments on our hedged portfolio, we receive an amount $\lambda S\Delta dt$ for every asset held, and since we hold $-\Delta$ of the underlying, the portfolio changes by an amount

$$-\lambda S\Delta dt \tag{3.19}$$

Adding equations (3.13) and (3.19) we obtain

$$d\Pi = dV - \Delta dS - \lambda S\Delta dt \tag{3.20}$$

Following the same previous analysis, we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda)S\frac{\partial V}{\partial S} - rV = 0 \tag{3.21}$$

3.4.1 Solving Transformed Black-Scholes Equation

Solving the equation (3.21) for a dividend paying stock using European option, let $\tau = T - t$; where T denotes maturity time, t is current time and τ denotes the remaining life time. The value of European Call and Put options are respectively written as

$$E_c(S,\tau) = Se^{-\lambda\tau}N(d_1) - Ke^{-r\tau}N(d_2) \tag{3.22}$$

and

$$E_p(S,\tau) = Ke^{-r\tau}N(d_2) - Se^{-\lambda\tau}N(d_1) \tag{3.23}$$

where $\quad d_1 = \dfrac{\ln(S/K) + (r - \lambda + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad$ and $\quad d_2 = d_1 - \sigma\sqrt{\tau}$.

3.5 Transformed Black-Scholes To Diffusion Equation

It is useful to transform the Black-Scholes equation corresponding to (3.21) into the well known heat-conducting equation to simplify the computation of American options. So we obtain;

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial \tau} \tag{3.24}$$

for $y(x,\tau), where \quad x \in \mathbb{R}, and \quad \tau \geq 0.$

According to Seydel (2002), the equation (3.24) is a Partial Differential Equation of simplest parabolic type. It can also be written as $y_{xx} = y_\tau$, where y_{xx} is the diffusion term. Both equations (3.21) and (3.24) are linear in the dependent variables V or y. The transformation is obtained by applying:

$$\begin{gathered} S = Ke^x, \qquad t = T - \frac{2\tau}{\sigma^2}, \qquad q := \frac{2r}{\sigma^2}, \qquad q_\lambda := \frac{2(r-\lambda)}{\sigma^2} \\ v(x,\tau) := \mathrm{K}\exp\{-\tfrac{1}{2}(q_\lambda - 1)x - (\tfrac{1}{4}(q_\lambda - 1)^2 + q)\tau\}y(x,\tau) \\ V(S,t) = V(Ke^x, T - \frac{2r}{\sigma^2}) =: v(x,\tau) \end{gathered} \tag{3.25}$$

In view of the time transformation in equation (3.25), τ corresponds to the time variable t in the original Black-Scholes equation denotes the remaining life time of the option towards the assuming date: $t = T$ transforms to $\tau = 0$ and $t = 0$ is transformed to $\tau = \frac{1}{2}\sigma^2 T$. And the original domain of the half strip $S > 0$,$0 \leq t \leq T$ of equation (3.21) becomes the strip

$$-\infty < x < +\infty, \qquad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T$$

on which a solution $y(x,\tau)$ to equation (3.24) will be approximated. We now apply the transformations of equation (3.25) to derive out of $y(x,\tau)$ the value of the option $V(S,t)$ in the original variables, after the caculation.
Under the transformations of equation (3.25), the initial conditions will be

$$\text{call} : y(x,0) = \max\{e^{\frac{x}{2}(q_\lambda+1)} - e^{\frac{x}{2}(q_\lambda-1)}, 0\}$$

$$\text{put} : y(x,0) = \max\{e^{\frac{x}{2}(q_\lambda-1)} - e^{\frac{x}{2}(q_\lambda+1)}, 0\}$$

The payment of dividend lowers the stock price from S to $Se^{\tau-\lambda}$ and the risk-free interest rate which is the rate of return from r to $(r-\lambda)$ according to Hull (2009). Since American option may be exercised at any time prior to the maturity date, exercise under equation (3.21) is not optimal. The equality sign in equation (3.21) is replaced by an inequality sign to obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\lambda)S\frac{\partial V}{\partial S} - rV \leq 0 \qquad (3.26)$$

where V = V(S, t), $S > 0$, $0 \leq t \leq T$ and V does not depend on μ, but on the riskless interest rate r and the annual dividend yield $\lambda \geq 0$ of the asset Seydel (2002) .
In mathematical literature, $(r-\lambda)S\frac{\partial V}{\partial S}$ is called convection term, $\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}$ is a diffusion term and rV is a reaction term. In this sense, equation (3.26) is a convection-diffusion PDE. In finance, $\frac{\partial V}{\partial S}$ denotes the option delta (Δ), $\frac{\partial^2 V}{\partial S^2}$ is the option gamma (Γ), and $\frac{\partial V}{\partial t}$

is known as option theta (Θ). Hull (2009). The lower boundary condition for equation (3.26) is given as

$$\begin{aligned} V_c(S,t) &\geq (S-K)^+ \qquad \text{for all} \quad (S,t), \\ V_p(S,t) &\geq (K-S)^+ \qquad \text{for all} \quad (S,t), \end{aligned} \tag{3.27}$$

Therefore, the inequalities hold and hence, the value of American options can not be less than their payoff function.

3.6 Options as Free Boundary-Value Problems

3.6.1 Scheme for Dealing with American Options

The valuation of an American option is therefore more complicated than its European counterpart since we have to determine not only the option value but also, for each value of S, whether or not it should be exercised. This is what is known as a free boundary problem. However, the European options, put options, do not allow early exercise and can reach values $V_p(S,t) < K - S$ for S getting close to zero according to Paully (2004).

At each time t, there is a particular value of S, where $S > 0$ marks the boundary between two regions: to one side one should hold the option and to the other side one should exercise it. We denote this value, which varies with time, by $S_f(t)$, and refer to it as the optimal exercise price.

Initially, this high contact point $S_f(t)$ is unknown but very important to know as it tells whether its worth to hold or to execute the option. In the second case, the corresponding point t_S is called stopping time. If $S_f(t)$ which is $0 < S_f < K$ is calculated, one should obey the following scheme for American options.

call: $S < S_f(t)$: hold	$S \geq S_f(t)$: execute asap
put: $S \leq S_f(t)$: execute asap	$S > S_f(t)$: hold

If a put is executed, the earned amount K should be invested in a riskless asset at rate r for the remaining time τ. If a call is executed, the K purchased stock is sold for S and the profit $S - K$ should be invested in a riskless asset.

In figure 3.1, the curve $S_f(t)$ divides the $[0,\infty)\times[0,T]$ into the continuation region(grey) and the stopping region. Keeping the option longer than necessary reduces the profit of the alternative riskless investment, and that the calculation for this strategy does not yet include the cost of carry and the price of the option itself.

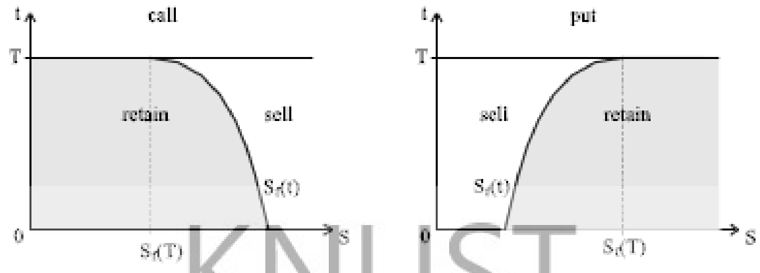

Figure 3.1: Scheme for Dealing with American Options

3.6.2 Free Boundary-Value Problem

The values of American call and put are given by equations (3.28) and (3.29) respectively. The visualized results is also shown in figure 3.1

$$V_c(S,t)\begin{cases} > (S-K)^+ & \text{for} \quad S < S_f(t) \\ = (S-K) & \text{for} \quad S \geq S_f(t) \end{cases} \tag{3.28}$$

$$V_p(S,t)\begin{cases} > (K-S) & \text{for} \quad S > S_f(t) \\ = (K-S) & \text{for} \quad S \leq S_f(t) \end{cases} \tag{3.29}$$

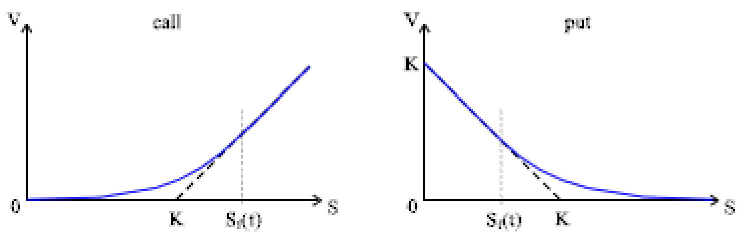

Figure 3.2: Value functions of American options at time $t < T$

Since S_f is not known theoretical, solving for $V(S,t)$ is called a free boundary-value problem. In order to calculate the unknown free boundary S_f , we need an

additional condition. Therefore, we consider the slope $\frac{\partial V}{\partial S}$ more closely with which $V_P(S,t)$ touches at $S_f(t)$ the straight line $K-S$ and has a slop -1 .
With similar arguments , the slope for the call which $V_C(S,t)$ coincides with the straight line $S-K$, is equals to 1. Finally, the two boundary conditions at the contact point for both the put and the call are known as smooth boundary condition:

$$\begin{aligned} V_C(S_f(t),t) &= S_f(t)-K, \qquad \frac{\partial V_C}{\partial S}(S_f(t),t) = 1 \\ V_P(S_f(t),t) &= K-S_f(t), \qquad \frac{\partial V_P}{\partial S}(S_f(t),t) = -1 \end{aligned} \tag{3.30}$$

This means that $V(S,t)$ touches the payoff function tangentially in $S_f(t)$. This tangent point also has an effect on the Black-Scholes-inequality. Here, the equality holds in case early exercise does not make sense (which is also valid for European options). For the other case, one has to deal with the inequality. Summarizing all those facts for American options, one obtains the following free boundary-value problems (FBVP):

American call option

$$\begin{array}{ll} \text{for } S < S_f(t): & V(S,t) > (S-K)^+ \text{ and} \\ & V_t + \frac{\sigma^2}{2}S^2V_{SS} + (r-\lambda)SV_S - rV = 0 \\ \text{for } S > S_f(t): & V(S,t) = S-K \text{ and} \\ & V_t + \frac{\sigma^2}{2}S^2V_{SS} + (r-\lambda)SV_S - rV < 0 \\ \text{boundary conditions:} & V(0,t) = 0 \\ & V(S_f(t),t) = S_f(t)-K \\ & V_S(S_f(t),t) = 1 \\ \text{terminal condition:} & V(S,T) = (S-K)^+ \end{array} \tag{3.31}$$

American put option

$$
\begin{array}{ll}
\text{for } S < S_f(t): & V(S,t) = K - S \text{ and} \\
& V_t + \frac{\sigma^2}{2} S^2 V_{SS} + (r-\lambda) S V_S - rV < 0 \\
\text{for } S > S_f(t): & V(S,t) > (K-S)^+ \text{ and} \\
& V_t + \frac{\sigma^2}{2} S^2 V_{SS} + (r-\lambda) S V_S - rV = 0 \\
\text{boundary conditions:} & \lim_{S\to\infty} V(S,t) = 0 \\
& V(S_f(t),t) = K - S_f(t) \\
& V_S(S_f(t),t) = -1 \\
\text{terminal condition:} & V(S,T) = (K-S)^+
\end{array}
\tag{3.32}
$$

Note that for American call a dividend yield $\lambda \neq 0$ is needed, because otherwise early exercise of the option is of no advantage to its holder, and the value of the American call equals the European-style call.

3.7 Formulation as Linear Complementarity Problem

3.7.1 Obstacle Problem

The free boundary conditions do not show up explicitly, so there is a need to reformulate the obstacle problem. This may anticipate computational advantages.

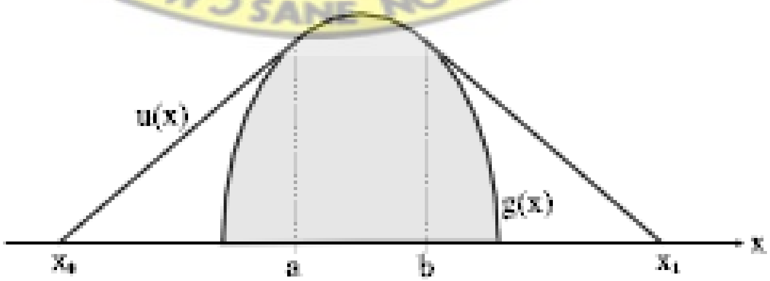

Figure 3.3: Setup for the obstacle problem

Let a function $g(x), x \in R, g \in C^2, g''(x) < 0$ be given, it represents an obstacle as shown in the figure above. Let $u(x), u \in C^1[x_0, x_1]$ be a function that is stretched over $g(x)$ like a string, and for simplicity let $u(x0) = u(x1) = 0$. On $[a, b]$ both functions coincide, everywhere else one has $u > g$. Initially, a and b are unknown.

Now we can formulate the example as free boundary-value problem:

$$\begin{aligned} &\text{for } x_0 < x < a: && u'' = 0 \text{ , then } u > g \\ &\text{for } a < x < b: && u'' = g'' < 0 \text{, then } u = g \\ &\text{for } b < x < x_1: && u'' = 0 \text{, then } u > g \end{aligned} \tag{3.33}$$

This situation attests a complementarity in the sense that $u(x)$ is a straight line.

$$\text{If} \quad u > g \quad \text{meaning,} \quad u'' = 0;$$

and

$$\text{If} \quad u = g, \text{then} \quad u'' < 0$$

.

So on $[x_0, x_1]$ one has either $u'' = 0$ or $u - g = 0$, but not both at the same time. Hence this obstacle problem can be equivalently reformulated as a linear complementarity problem:

$$(LCP) = \begin{cases} \text{find a function} \quad u(x) \quad \text{such that :} \\ u''(u-g) = 0, & -u'' \geq 0, \quad u - g \geq 0, \\ u(x_0) = u(x_1) = 0, & u \in C^1[x_0, x_1] \end{cases} \tag{3.34}$$

The reverse equivalence is clear, since when a suitable $u(x)$ is found, for a given $g''(x) < 0$ one gets the original obstacle problem. Note that in the LCP the boundary values a and b are not mentioned explicitly. However, if one knows a solution for it, the boundaries will also be known.

Recall the free boundary-value problems 3.31, they are of a similar form as 3.33.

So they can also be seen as obstacle problems, that is with $u := V(S,t), g := (K-S)+$ and $b := S_f$ for the put. Therefore it is obvious that we can also formulate the evaluation of American options as LCP, where the free boundary S_f is not mentioned explicitly, but will be known when the problem can be solved.

3.7.2 Linear Complementarity Problem (LCP)

It is useful to transform the Black-Scholes Inequality corresponding to equation (3.26) into the well known heat-conducting equation. This will simplify and enhance easy computation of American options.

Technically, it is easier to solve heat-conducting equation as compared with Black-Scholes Inequality for several reasons: The most obvious one is that the diffusion Equation contains only two terms, i.e. the diffusion and the time derivative, as compared to the four terms in (3.26), which results in a simpler algorithm for the computation.

The convection term in equation (3.26) might lead to numerical spurious oscillations in our application. The problem of numerical instabilities were not caused by the PDE itself, but by the particular numerical algorithms we will use to solve it. However, these difficulties do not occur for the heat-conducting equation.

As originally one has to deal with a Black-Scholes inequality when evaluating American options, the direct transformation yields:

$$\frac{\partial^2 y}{\partial x^2} \leq \frac{\partial y}{\partial \tau} \tag{3.35}$$

for $y(x,\tau)$ with $0 \leq \tau \leq \tau_{max}$, $x \in \Re$.

For this inequality, one can construct a linear complementarity problem (LCP) similar to (3.34). Specifically, the constraints of the FBVP for an American put in equation (3.31) can also be transformed as LCP. Applying the transformation to them lead to

$$V_P(S,t) \geq (K-S)^+ = \text{K} \max\{1-e^x, 0\} \tag{3.36}$$

Inserting this into (3.25) yields

$$\begin{aligned} y(x,\tau) &\geq \max\{1-e^x,0\}\exp\left\{\frac{1}{2}(q_\lambda-1)x-(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\} \\ &= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{(1-e^x)e^{\frac{1}{2}(q_\lambda-1)x},0\right\} \\ &= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{e^{\frac{1}{2}(q_\lambda-1)x}-e^{\frac{1}{2}(q_\lambda+1)x},0\right\} =: g(x,\tau) \end{aligned} \tag{3.37}$$

The terminal condition of the American put, $V(S,T)=(K-S)^+$ implies equality in the above equation, so the initial condition for y is $y(x,0)=g(x,0)$. For $x\to\pm\infty$, we also have $y(x,\tau)=g(x,\tau)$.

With an adjusted function g, it also works for American call with $0<\lambda<r$.

So formulating the linear complementarity problem

$$\begin{aligned} \text{call}: g(x,\tau) &:= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{e^{\frac{1}{2}(q_\lambda+1)x}-e^{\frac{1}{2}(q_\lambda-1)x},0\right\} \\ \text{put}: g(x,\tau) &:= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{e^{\frac{1}{2}(q_\lambda-1)x}-e^{\frac{1}{2}(q_\lambda+1)x},0\right\} \end{aligned}$$

$$(LCP)\begin{cases} \text{find a y}(x,\tau)\text{such that}: & \\ \left(\frac{\partial y}{\partial\tau}-\frac{\partial^2 y}{\partial x^2}\right)(y-g)=0 & \\ \frac{\partial y}{\partial\tau}-\frac{\partial^2 y}{\partial x^2}\geq 0, & y-g\geq 0 \\ y(x,0)=g(x,0), & 0\leq\tau\leq\frac{1}{2}\sigma^2 T, \\ y(x,\tau)=g(x,\tau) & \text{for x}\to\pm\infty \end{cases} \tag{3.38}$$

In financial terms, the heat-conducting inequality (3.35) means that the expected return from the riskless delta-hedged portfolio is less than the riskless interest rate Wilmott et al. (1996).

3.8 Finite Differences Method

Introduction

This section presents the method of Finite Differences(FD), one of the first approaches used to compute the value of options on a single underlying asset. However, the original FD approach, which was adapted to the Black-Scholes partial differential equations (B-S PDEs) directly, suffered spurious oscillations.

In order to avoid such disorders, we applied the transformation in equation (3.25), yielding equation (3.24) , and proceed to formulate the LCP. The latter can be solved by employing an iterative projection method. Before we start, the domain for our computations will be considered first.

3.8.1 Types of Finite Differences

The main idea of the method of Finite Differences is to replace differentials by differential quotients. Therefore, we consider the Taylor expansion of an arbitrary function $f : D \mapsto \Re, D \subset \Re^n$ open and convex, $f \in C^4$: The most common finite difference methods for solving the partial differential equations are:

- Explicit scheme.
- Implicit scheme.
- Crank Nicolson scheme.

We discuss the Finite difference method quite extensively with a focus on the Crank-Nicolson schemes, and apply these techniques to the pricing of vanilla options.

3.8.2 Foundations of Finite-Difference Methods

This section describes the basic ideas of finite differences as they are applied to the equation (3.24) . Each two times continuously differentiable function f satisfies

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(\xi)$$

where ξ is an intermediate number between x and $x+h$. The accurate position of ξ is usually unknown. Such expressions are derived by Taylor expansions.
We discretize $x \in \mathbb{R}$ by introducing a one-dimensional grid of discrete points x_i with

$$\cdots < x_{i-1} < x_i < x_{i+1} < \ldots$$

For example, choose an equidistant grid with mesh size $h := x_{i+1} - x_i$. The x is discretized, but the function values $f_i := f(x_i)$ are not discrete, $f_i \in \mathbb{R}$. For $f \in C^2$ the derivative f'' is bounded, and the term $-\frac{h}{2} f''(\zeta)$ can be conveniently written as $O(h)$. This leads to the ***forward difference*** with $f \in C^2$

$$f'(x_i) = \frac{f(i+1) - f(i)}{h} + O(h), \tag{3.39}$$

and the ***backward difference*** yields

$$f'(x_i) = \frac{f(i) - f(i-1)}{h} + O(h), \tag{3.40}$$

Analogous expressions hold for the partial derivatives of $y(x, \tau)$, which includes a discretization in τ . This suggests to replace the neutral notation h by either Δx or $\Delta\tau$, respectively. The fraction in equation (3.39) is the difference quotient that approximates the differential quotient f' of the left-hand side(LHS); the $O(h^p)$ term is the error. The one-sided (i.e. non-symmetric) difference quotient of equation (3.39) is of the order $p = 1$. Error orders of $p = 2$ are obtained by ***central differences***

$$f'(x_i) = \frac{f(i+1) - f(i-1)}{2h} + O(h^2) \qquad (\text{for } f \in c^3)$$

$$f''(x_i) = \frac{f(i+1) - 2f(i) + f(i-1)}{h^2} + O(h^2) \qquad (\text{for } f \in c^4) \tag{3.41}$$

or by one-sided differences that involve more terms, such as

$$f'(x_i) = \frac{-f(i+2) + 4f(i+1) - 3f(i)}{2h} + O(h^2) \qquad (\text{for } f \in c^3)$$

Rearranging terms and indices provides the approximation formula

$$f_i \approx \frac{4}{3} f_{i-1} - \frac{1}{3} f_{i-2} + \frac{2}{3} h f' x_i \tag{3.42}$$

which is of second order.

3.8.3 Domain Discretization

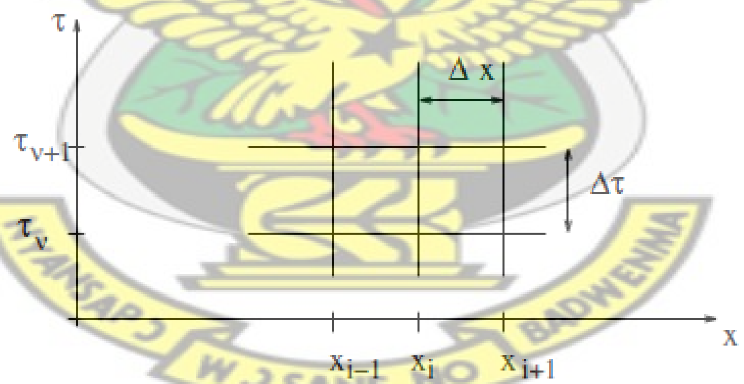

Figure 3.4: Detail and notations of the grid

Actually, an American option works on the $S-t$ half strip $[0,\infty) \times [0,T]$. But it became an $x-\tau$ strip $(-\infty,\infty) \times [0,\tau_{max}]$ after applying the transformation, where $\tau_{max} := \frac{1}{2}\sigma^2 T$.

Under this section, the domain needs to be discretized to a finite lattice i.e. $[x_{min}, x_{max}] \times [0,\tau_{max}]$. Let Δx and $\Delta\tau$ be the equidistant mesh sizes of the discretizations of x and τ. The choice of the x-discretization is more complicated. So, the infinite interval

$-\infty < \times < \infty$ must be replaced by $[x_{min} \leq x \leq x_{max}]$. We chose x_{min} and x_{max} such that the solution on the interval $[x_{min}, x_{max}]$ is in line with the solution on $-\infty < x < \infty$.

For m and ν_{max} be a suitable integers, we define the mesh density by $\Delta x := \frac{x_{max} - x_{min}}{m}$ and the step in τ as $\Delta\tau := \frac{\tau_{max}}{\nu_{max}}$. Since the equidistant of the grid simplifies the implementation and the estimation of the error terms, the work stands better side of it.

The transformation $S = S_i = Ke^{x_i}$, which makes it appropriate to choose $x_{min} < 0$,and $x_{max} > 0$ fit the original limits of the S-interval correctly. The grid is then based on the knots;

$\tau_v := v.\Delta\tau, \quad \text{for v} = 0, 1, \ldots, \nu_{max}$

$x_i := x_{min} + i\Delta\text{x} \quad \text{for i} = 0, 1, \ldots, m$

Furthermore, w_i^v denotes the approximation for y_i^v, where $y_i^v := y(xi, \tau_v)$. This is only defined on the discrete nodes and the nodes are the intersection of the points x_i and τ_v. In contrast to the theoretical solution $y(x, \tau)$, y_i^v is defined on a continuum.

The error $\| w_i^v - y_i^v \|$ depends on the prior choice of parameters m, x_{min}, x_{max} and v_{max}. A priori we do not know whose choice of parameters matches a prespecified error tolerance. For instance, if the order of magnitude of these parameters is given by $x_{min} = -5, x_{max} = 5, v_{max} = 100, m = 100$. This choice of x_{min}, x_{max} has shown to be reasonable for a wide range of r, σ-values and accuracies. The actual error is then controlled via the numbers v_{max} and m of grid lines.

With the reference to equation (3.24), the RHS and LHS of it can be written as equations (3.39) and 3.41 respectively to obtained;

$$\frac{\partial}{\partial\tau} y_i^v = \frac{y_i^{v+1} - y_i^v}{\Delta\tau} + O(\Delta\tau), \qquad \text{the forward difference} \qquad (3.43)$$

and

$$\frac{\partial^2}{\partial x^2} y_i^v = \frac{y_{i+1}^v - 2y_i^v + y_{i-1}^v}{\Delta x^2} + O(\Delta x^2), \qquad \text{the central difference} \qquad (3.44)$$

Then the backward difference of the RHS of equation (3.24) also yields

$$\frac{\partial}{\partial\tau}y_i^v = \frac{y_i^v - y_i^{v-1}}{\Delta\tau} + O(\Delta\tau), \tag{3.45}$$

The Explicit Scheme

With w being the approximation for y, where Δx and $\Delta\tau$ denoted the introduced mesh sizes, we replace equations (3.43) and (3.44) into equation (3.24) and discarding the 0-error terms leads to

$$\frac{w_i^{v+1} - w_i^v}{\Delta\tau} = \frac{w_{i+1}^v - 2w_i^v + w_{i-1}^v}{\Delta x^2} \tag{3.46}$$

Solving for w_i^{v+1} under the idea of explicit scheme, where all values w are calculated for the time level v, then the values of the time level $(v+1)$ are given by

$$w_i^{v+1} = w_i^v + \frac{\Delta\tau}{\Delta x^2}(w_{i+1}^v - 2w_i^v + w_{i-1}^v)$$

.

Further, starting at $v = 0$, as all $w_i^0 := y(x_i, 0)$, $i = 0, \ldots, m$ are known, each w_i^1 can explicitly be calculated (hence the name of the method). Then, successively the next levels of time can be proceeded, for $1 \leq v \leq v_{max}$.
With the notation $\zeta := \frac{\Delta\tau}{\Delta x^2}$, the result is written compactly in time-iteration form as

$$w_i^{v+1} = \zeta w_{i+1}^v + (1 - 2\zeta)w_i^v + \zeta w_{i-1}^v. \tag{3.47}$$

The total error is $O(\Delta\tau + \Delta x^2)$ for $y \in C^{4,2}(\bar{D}_w)$, for $D_w := (x_{min}, x_{max}) \times (0, \tau_{max})$.

The Implicit Scheme

This method is sometimes called implicit method. But to distinguish it from other implicit methods, we call it fully implicit, or backward-difference method, or more accurately, backward time centered space scheme (BTCS).

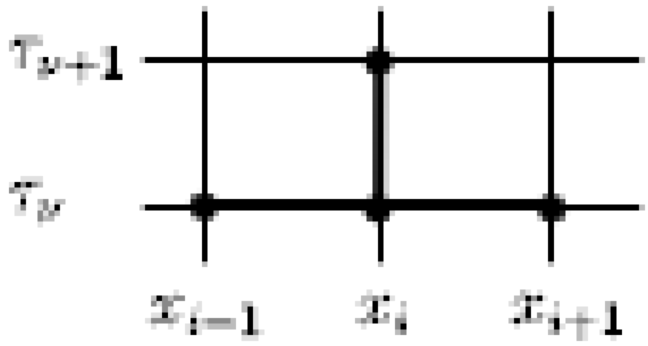

Figure 3.5: Connection scheme of the explicit method

Using the backward difference, equation (3.45) and equation (3.44) to discretize the heat-conducting equation,(3.24), yields

$$\frac{w_i^\nu - w_i^{\nu-1}}{\Delta\tau} = \frac{w_{i+1}^\nu - 2w_i^\nu + w_{i-1}^\nu}{\Delta x^2}$$

This is rewritten as

$$\frac{w_i^{\nu+1} - w_i^\nu}{\Delta\tau} = \frac{w_{i+1}^{\nu+1} - 2w_i^{\nu+1} + w_{i-1}^{\nu+1}}{\Delta x^2}, \tag{3.48}$$

with the same 0-error terms as in the explicit scheme. Sorted by time-levels, we obtain the iteration form

$$-\zeta w_{i+1}^{\nu+1} + (2\zeta + 1)w_i^{\nu+1} - \zeta w_{i-1}^{\nu+1} = w_i^\nu \tag{3.49}$$

The equation (3.49) couples three unknowns. Therefore, only the value w_i^ν of the RHS of the equation (3.49) is known, whereas on the LHS of the same equation in each step, one has to compute three unknown variables. The corresponding molecule is shown in Figure 4.3

Eventually this leads to a linear system of equations (LSE) that includes all time stages. This system can then be solved. The method is unconditionally stable for all $\Delta\tau > 0$

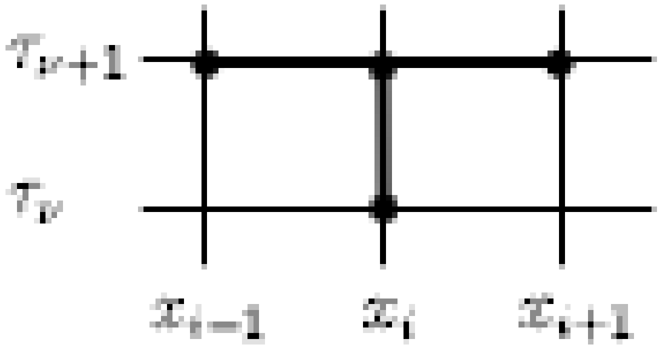

Figure 3.6: Molecule of the backward-difference method

The Crank-Nicholson Scheme

This last scheme was proposed by Crank and Nicholson (1947). They applied the forward scheme in time step v of equation (3.46), the backward scheme at step $(v+1)$ of equation (3.48) and to average them. The summation of equations (3.46) and (3.48), and truncating the error terms yields

$$\frac{w_i^{v+1}-w_i^v}{\Delta\tau}=\frac{w_{i+1}^v-2w_i^v+w_{i-1}^v+w_{i+1}^{v+1}-2w_i^{v+1}+w_{i-1}^{v+1}}{2\Delta x^2} \tag{3.50}$$

With $\zeta := \frac{\Delta\tau}{\Delta x^2}$ the equation (3.50) can be rewritten as

$$-\frac{\zeta}{2}w_{i-1}^{v+1}+(1-\zeta)w_i^{v+1}-\frac{\zeta}{2}w_{i+1}^{v+1}=\frac{\zeta}{2}w_{i-1}^v+(1-\zeta)w_i^v+\frac{\zeta}{2}w_{i+1}^v. \tag{3.51}$$

To get the error for the CN scheme for a $y\in C^{4,3}(\bar{D}_w)$, D_w defined as before, first consider the L.H.S. of 3.50 by using the first three terms of the Taylor expansion, it can be approximated by

$$\frac{(w(v-\Delta\tau)-w(v))}{\Delta\tau}=w_\tau+\frac{1}{2}w_{\tau\tau}\Delta\tau+O((\Delta\tau)^2)$$

From the R.H.S. of equation (3.50) it follows

$$\frac{1}{2}(w_{xx}(x,\tau)+w_{xx}(x,\tau+\Delta\tau))=\frac{1}{2}(2w_{xx}+w_{xx\tau}\Delta\tau+O((\Delta x)^2+(\Delta\tau)^2))$$

Eventually, we get the total consistency error

$$c_{err} = w_\tau - w_{xx} + \frac{1}{2}\Delta\tau(w_{\tau\tau}) - w_{xx\tau} + O((\Delta x)^2 + (\Delta\tau)^2)$$

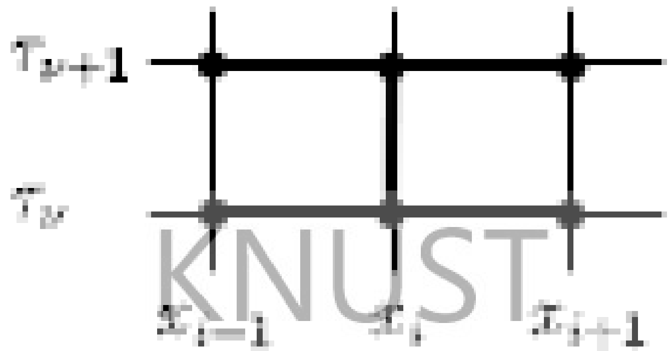

Figure 3.7: Molecule of the Crank-Nicolson method

So finally the Crank-Nicholson approach has got a better order than the former two methods. But similar to the one before, there is no explicit way to solve (3.51). Again, one needs to set up an LSE, which then can be evaluated. Thus, the CN scheme is also of implicit type.

3.9 FDM in a Linear System of Equations

The iteration of the three FD schemes i. e. equations ((3.51), (3.49) and (3.47)) can be written as a sequence of LSEs, that is

$$Aw^{\nu+1} = Bw^{\nu} + c^{\nu}, \qquad \nu = 0, \ldots, \nu_{max} - 1 \tag{3.52}$$

with the matrices

$$A = \begin{pmatrix} 1+2\zeta\theta & -\zeta\theta & & 0 \\ -\zeta\theta & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}, \; B = \begin{pmatrix} 1-2\zeta\tilde{\theta} & \zeta\tilde{\theta} & & 0 \\ \zeta\tilde{\theta} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

The two matrices are tridiagonal and of the dimension $(m-1)\times(m-1)$. The free paremeters θ and $\tilde{\theta} := 1-\theta$ denote the particular FD scheme, where for;

- $\theta = 0,$ one has the explicit,
- $\theta = \frac{1}{2},$ the Crank-Nicolson
- $\theta = 1,$ the implicit one .

$\zeta = \frac{\Delta\tau}{\Delta x^2}$, as defined before. The vectors are $w^i := (w_1^i, \ldots, w_{m-1}^i)^T, i = \{v, v+1\}$ and $c^v := (c_0^v, 0, \ldots, 0, c_m^v)^T$. The elements c_0^v and c_m^v contain the terms that were discarded when setting up matrix A and B. In particular, they are defined by the boundary conditions of the PDE. Note that the actual setup of the matrices A and B depends both on m and v_{max}, where the former parameter influences the size, and both of them affect the eigenvalues.

3.10 Existence of a Uniform Solution

A uniform solution of the system of equations (3.52) exists, if matrix A has an inverse, which is actually true for $\zeta > 0$ and $\theta \in [0,1]$. To prove this statement, we need to show that no eigenvalue λ of A equals zero:

Let $x := (x_1, \ldots, x_n)^T$ be an arbitrary eigenvector of A with the corresponding eigenvalue λ. Let $x_i := max\{|x_j| : x_j \text{ is element of} x\}, i, j = \{1, \ldots, n\}$. Then, from $\lambda x_i = (Ax)_i = \sum_{j=1,\ldots,n} a_{ij} x_j$, after division by $x_i \neq 0$ follows:

$$|\lambda - a_{ii}| = \left| \sum_{j \neq i} a_{ij} \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{ij}|$$

So by claim of Gerschgorin's Theorem, for the eigenvalues of A as in (3.52), one has

$$1 \leq \lambda \leq 1 + 4\zeta\theta,$$

with ζ and θ defined as before, and in particular,$\lambda \leq 0$, q.e.d.

3.11 Finite Differences Applied to American Options

Following Seydel (2002) Finite Differences are an efficient tool to solve the parabolic equation. Recall from (3.38), that in order to evaluate American options, we actually need to solve a LCP, containing a heat-inequality. This means, that the iteration (3.52) needs to be adjusted to

$$Aw^{(v+1)} \geq Bw^{(v)} + c^{(v)},\ v = 0, \ldots, v_{max} - 1 \tag{3.53}$$

Additionally, the LCP claims $y - g \geq 0$, which in terms of the FD discretization leads to $w^{(v)} \geq g^{(v)}$. Note that inequalities in vectors are meant to be component-wise. The last things missing are the initial conditions (w^0), and the structure of vector c, which is defined by the boundary conditions. From the LCP (3.38) we get $w_i^0 = g_i^0$ for $i = 1, \ldots, m-1$, i.e. $w^0 = g^0$. The boundary conditions are $w_0^{(v)} = g_0^{(v)}$ and $w_m^{(v)} = g_m^{(v)}$ for all $v \geq 1$, yielding

$$c^{(v)} := \begin{pmatrix} \zeta\theta g_0^{(v+1)} + \zeta\tilde{\theta} g_0^{(v)} \\ 0 \\ \vdots \\ 0 \\ \zeta\theta g_m^{(v+1)} + \zeta\tilde{\theta} g_m^{(v)} \end{pmatrix} \tag{3.54}$$

Since in step (v+1) the R.H.S. of (3.53) is completely known, we can set

$$b := Bw^{(v)} + c^{(v)}, \tag{3.55}$$

and rewrite the LCP (3.38):

$$\begin{cases} \text{find } w := w^{(v+1)}, \text{ such that} \\ Aw \geq b, \ w \geq g, \ (w-g)^T(Aw-b) = 0 \end{cases} \tag{3.56}$$

3.12 Implementation of the Methods

This section adds on to the LCPs derived by the finite difference method (3.56). The problem involves the three equations

$$Aw \geq b, \ \ w \geq g, \ \ (w-g)^T(Aw-b) = 0. \tag{3.57}$$

This section will introduce a solution scheme for these equations and provide a closed form algorithm for the actual computation.

3.12.1 Iterative Solution of a Linear System of Equations

We consider the linear system of equations

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \ \ b, x \in \mathbb{R}^n, \tag{3.58}$$

and assume that A is symmetric and positive definite. There are many ways to solve a linear system of equations with numerical methods. However, in our context, when dealing with large sparse matrices (A), iteration methods are of advantage, compared to ordinary elimination schemes, since they require less memory and arithmetic cost.

The main idea to solve a linear system of equations by fixed point iteration is to choose a suitable regular matrix $Q \in \mathbb{R}^{n\times n}$, such that we rewrite equation (3.58)

$$Qx = (Q-A)x + b \tag{3.59}$$

$$x = (I - Q^{-1}A)x + Q^{-1}b = \Phi(x) \tag{3.60}$$

Now let $M = (I - Q^{-1}A)$ and $c = Q^{-1}b$, then

$$x^{k+1} = \Phi(x^{(k)}) = Mx^{(k)} + c. \tag{3.61}$$

Equation (3.61) converges if and only if $\rho(M) < 1$, where $\rho(M)$ is the spectral radius of M. In order to optimize convergence, we require the matrix M in an appropriate manner. Therefore, we decompose A into $A = D - L - U$, where D contains the diagonal, L and U are the lower and the upper elements of A, respectively. We now focus on Relaxation Methods. Let us choose

$$Q := \frac{1}{\omega}D - L \quad \Longrightarrow M_\omega = I - (\frac{1}{\omega}D - L)^{-1}A,$$

with a relaxation parameter $\omega > 0$. This leads to the iteration

$$x^{(k+1)} = \left(I - \left(\frac{1}{\omega}D - L\right)^{-1} A\right) x^{(k)} + \left(\frac{1}{\omega}D - L\right)^{-1} b. \tag{3.62}$$

When $\omega = 1$ then equation (3.62) equals the Gauss-Seidel method. For $0 < \omega < 1$, the iteration is called damped, and for $1 < \omega < 2$ the scheme is called Successive Over-relaxation (SOR). In practice, we can approach the iteration by rewriting (3.62) equivalently as

$$\frac{1}{\omega}(D - L)x^{(k+1)} = \left(\left(\frac{1}{\omega}D - L\right) - A\right)x^{(k)} + b \quad \Longleftrightarrow \tag{3.63}$$

$$\frac{1}{\omega}Dx^{(k+1)} = Lx^{(k+1)} + \left(\left(\frac{1}{\omega}D - L\right) - D + L + U\right)x^{(k)} + b \quad \Longleftrightarrow \tag{3.64}$$

$$x^{(k+1)} = x^{(k)} + \omega D^{-1}(Lx^{(k+1)} - Dx^{(k)} + Ux^{(k)} + b), \tag{3.65}$$

and assuming that for step $x^{(k+1)}$ the components $x_i^{(k+1)}$, $1 \leq i \leq j-1$, are already known.

3.12.2 The Projected SOR Method

We were dealing with problem (3.57), which contained a linear system of inequalities. To be able to use an iterative scheme for linear system of equations on this problem, we need to make some modifications. In particular, we will focus on an extension of the SOR method. We rewrite (3.57) as

$$w - A^{-1}b \geq b, \;\; w - g \geq g, \;\; (w-g)^T(Aw-b) = 0 \iff \tag{3.66}$$

$$\min_w \{w - A^{-1}b, \; w - g\} = 0 \iff \tag{3.67}$$

$$w = max\{A^{-1}b, \; g\} \tag{3.68}$$

Chapter 4

Analysis and Results

4.1 Introduction

This chapter presents an analysis on the implementation of the black Scholes model with and without dividend. The thesis sake of the analysis the various benchmark parameter are chosen. The tolerance parameter (ε) is chosen as 10^{-6}. This is due to the fact that PSOR iteration, the accuracy of the convergence test depends on the tolerance parameter. Usually tolerance parameter value less than 10^{-3} leads to quite inaccurate results. The iteration speed also depends on the relaxation parameter(ω). For PSOR iteration, the relation parameter is usually between one and two inclusive, $1 \leq w \leq 2$. Par the nature of our iteration, we choose the relaxation parameter as 1.15. We discuss the impact of dividend on both American option and European option. The numerical computations are implemented in MatLab.

4.2 A Put Option with Non-dividend Paying Stock

Analysing the American and European put option with non-dividend paying stock, the risk free interest rate is taken as 0.25. The price of the underlying (S) and the strike price (K) are both chosen as 50. An annual volatility (σ) is chosen as 0.4. Because higher volatility sometimes deters investors from investing. The maturity time of underlying (T) is $\frac{5}{12}$, ie that fifth month in the year and dividend λ is chosen as zero because we are considering a non-divided paying stock.

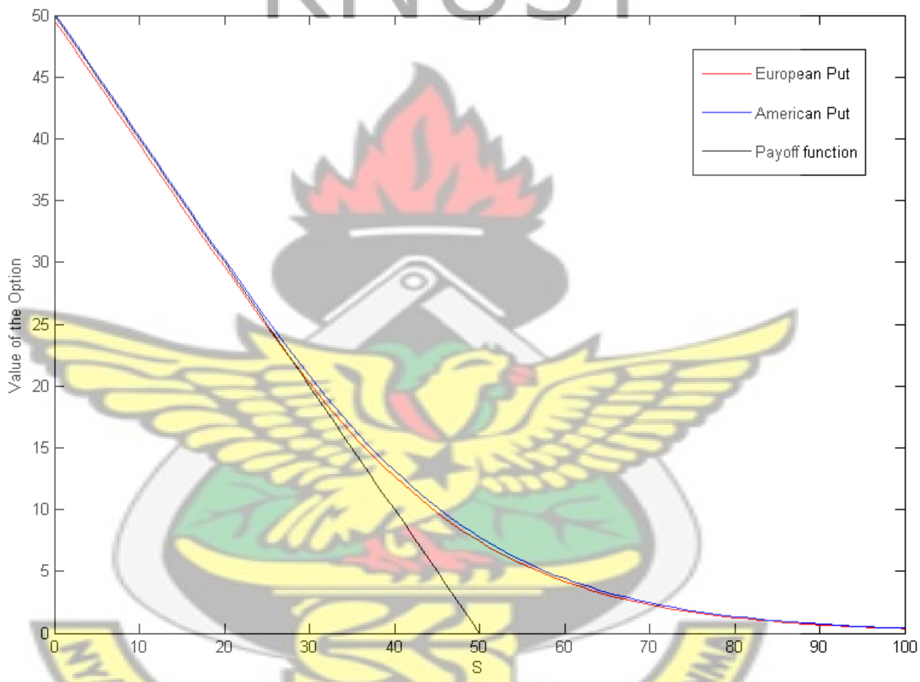

Figure 4.1: The value function $V(S,0)$ of the European and American put when $K = 50$, $\sigma = 0.6$, $r = 0.25$, $T = \frac{5}{12}$, $S = 50$.

Figure 4.1 shows the comparison between American put option, American Call option and the payoff function. The payoff function is zero when the price of the underlying asset is greater than the strike price. The American put is in parallel shape to the European put. With the American put lying on top of the European implies the American put with dividend equal to zero have similar prices to the European put. The price of the American put is also the same as the payoff function at the initial stages of the iteration but moves away from each other at the certain stages of the calculation.

4.3 A Call Option with different Dividend values

An American put option is evaluated with the following financial values. The strike price (K) is chosen as 50. An annual volatility (σ) equal 0.6. The risk free interest rate (r) and the maturity time of underlying(T) are both 0.25 and 1 respectively. We compute this based on different dividend values.

From Figure 4.2 the red graph represents a dividend level of 0.6, while as the blue line represents a dividend of 0.3. As dividend is increased the value of the call option also increases. At stock prices less then 30, there are no distinct differences between the call option with dividend equal 0.3 and call option with dividend equal 0.6.

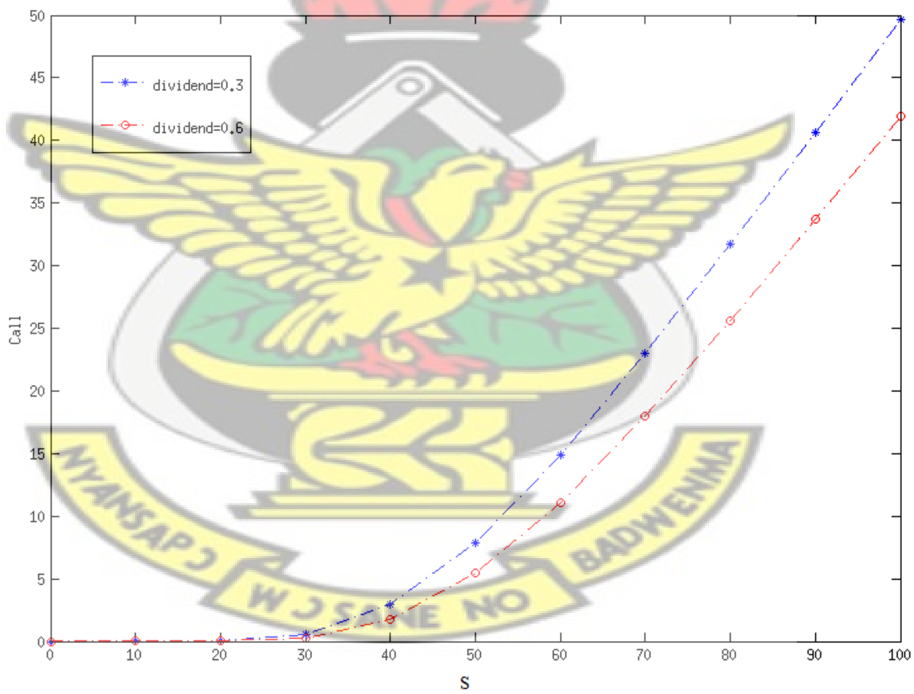

Figure 4.2: The value function $V(S,0)$ of the American call when $K=50, \sigma=0.6,\ r=0.25,\ T=\frac{5}{12}$

These are evaluated with the projected over-successive relaxation algorithm. The table below gives an elaboration of the Figure 4.2.

Table 4.1: A Call Option with different Dividend

Stock Price	Call Option $\lambda = 0.3$	Call Option $\lambda = 0.6$
5	1.5721e-09	5.4211e-23
10	0.0001	3.8319e-07
15	0.0069	0.0030
20	0.0756	0.0380
25	0.3538	0.1967
30	1.0313	0.6192
35	2.2495	1.4349
40	4.0593	2.7179
45	6.4324	4.4799
50	9.2944	6.6852
55	12.5545	9.2734
60	16.1247	12.1757
65	19.9288	15.3269
70	23.9056	18.6701
75	28.0077	22.1587
80	32.1999	25.7558
85	36.4563	29.4330
90	40.7583	33.1690
95	45.0925	36.9480
100	49.4494	40.7583

As the spot price of the stock increases the value of the call option at these different dividend values also increases. All call values at $\lambda = 0.3$ are greater then call values at $\lambda = 0.6$.

4.4 A Call Option with Dividend Paying Stock

Now examining the differences between the European call, the payoff function and the American call option with dividend paying stock, we considered dividend λ value of 0.2 is used for the computation. The price of the underlying (S) and the strike price (K) are both chosen as 80. An annual volatility (σ) is chosen as 0.6. The risk free interest rate (r) and the maturity time of underlying(T) are chosen as 0.25 and $\frac{5}{12}$ respectively.

Though the pictorial view depict same prices for the American call, the European call and the payoff function when the strike price is less than thirty. There are slight differences between the corresponding numerical values. The payoff function is zero when the price of the underlying asset is less than the strike price. The payoff function rises immediately the price of the underlying asset becomes greater than the strike price.

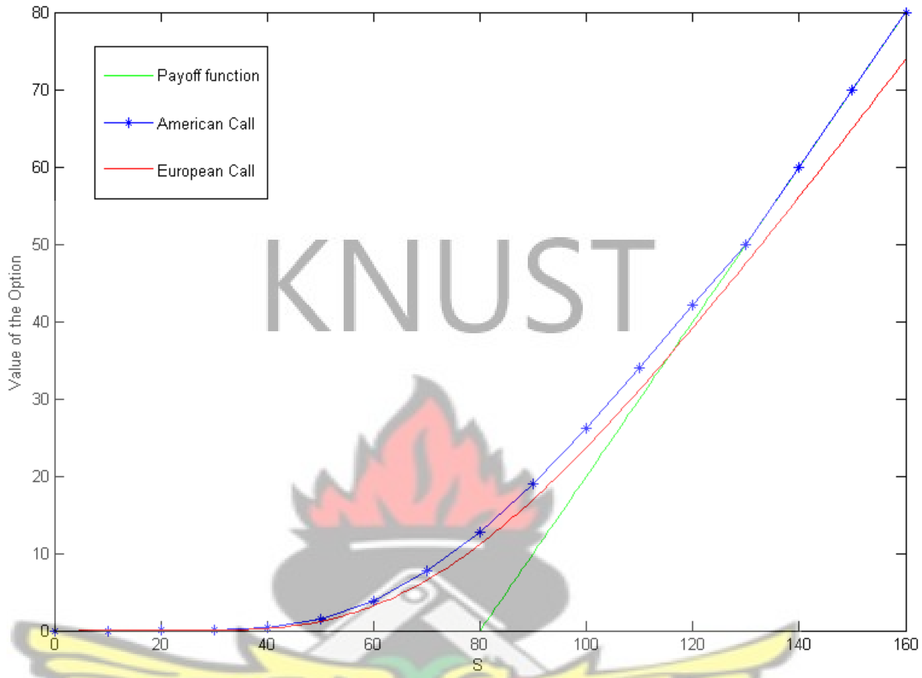

Figure 4.3: The value function $V(S,0)$ of the European and American call when $K = 80$, $\sigma = 0.6$, $\lambda = 0.2$, $r = 0.25$, $T = \frac{5}{12}$, $S = 80$.

The payoff function intersect both the European call and the American call functions. But, it intersects the European call at lower call price with it relativity to the American option. The PSOR simulation of the American call with dividend paying stock more desirable than the European call. A similar analysis of a put option with dividend paying stock is shown in Figure 4.4.

4.5 A put option with dividend paying stock

An American put option with dividend paying stock is evaluated with the following financial values. The strike price $(K) = 80$, an annual volatility (σ) equal 0.6. The risk free interest rate $(r) = 0.25$, the maturity time of underlying, measured in years $(T) = 1$. $S = 80$, that is, the asset value price at issuing date and dividend $\lambda = 0.2$.

Below shows a clear cut distinction between American put dividend paying stock, the pay-off function and the European put. The blue line represents for the American put option, the green line for the European put, and the pay-off function by the red line. The pay-off function has an intersection with both the American and European put. Figure 4.4 presents a put option on which dividend are paid.

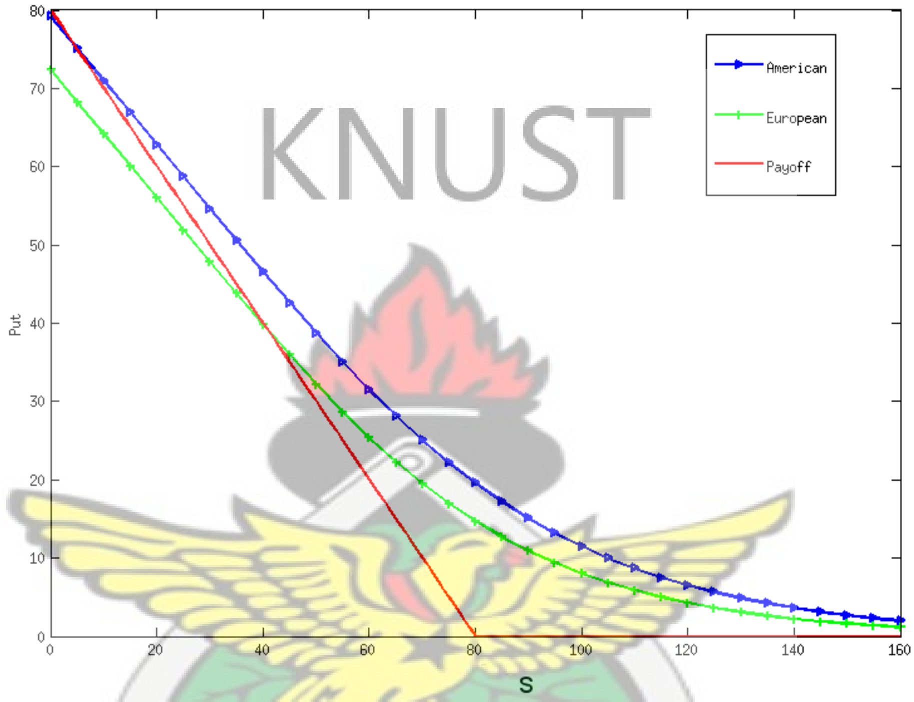

Figure 4.4: The value function $V(S,0)$ of the European and American put when $K = 80,\ \sigma = 0.6,\ r = 0.25,\ T = 1,\ S = 80.$

4.6 Convergence to the Exact Solution

The following financial values were employed for the Numerical Simulation in Table 4.2. The strike price $(K) = 10$, an annual volatility (σ) equal 0.6, dividend $\lambda = \frac{2}{10}$. The risk free interest rate $(r) = 0.1$, the maturity time of underlying, measured in years $(T) = \frac{5}{12}$. $S = 10$, that is, the asset value price at issuing date.

$V_{max} = M$	Time (sec)	Value of option
100	0.3899	26.9890
200	1.5302	27.0059
300	4.0597	27.0090
400	7.9792	27.0101
500	14.0717	27.0106
600	22.3674	27.0109
700	32.2696	27.0111
800	45.3769	27.0112
900	61.1237	27.0112
1000	78.5957	27.0112
2000	154.0145	27.0112
3000	434.9952	27.0112
4000	912.5345	27.0112
5000	1684.2001	27.0112

Table 4.2: Value of the option as the mesh size M is varied

As the mesh size for the PSOR iteration $V_{max} = M$ increases, the price of the option converges to the exact solution. There are no changes in the price of the option as the value of mesh size goes beyond 800. All higher values of mesh size has an option value of 27.0115. Hence, the exact solution of the option is 27.0115.

4.7 Option at Early Exercise

An American call option with dividend paying stock, where early exercise exist is evaluated with the strike price $(K) = 80$, an annual volatility (σ) equal 0.6. The risk free interest rate $(r) = 0.25$, the maturity time of underlying, measured in years $(T) = 1$. The current price was issued at $S = 80$ attracting dividend (λ) of 0.2.

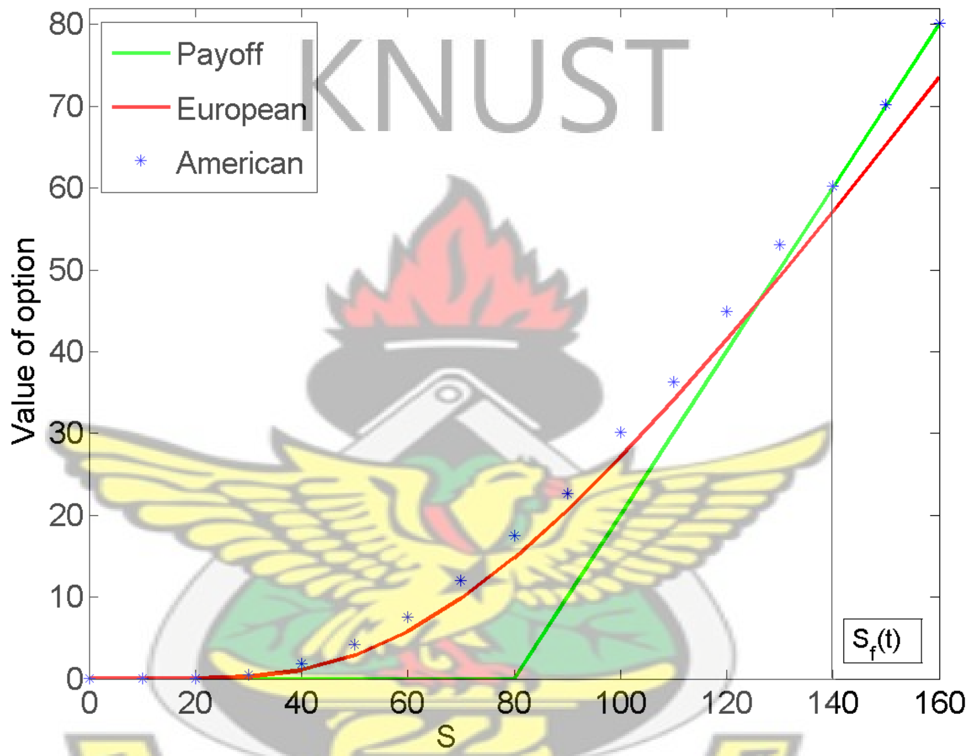

Figure 4.5: The value function $V(S,0)$ of the European and American call when $K = 80,\ \sigma = 0.6,\ r = 0.25,\ T = 1,\ \lambda = 0.2,\ S = 80.$

At a price $S_f(t)$, the American call option behaves almost identically to the pay-off function which implies that early exercise is possible. This means that the option value tangentially touches the pay-off function in $S_f(t)$. As long as the the option value, $V(S,t)$ coincides with the pay-off function, a financial investor executes the option as early as possible to maximize profit.

Furthermore, at the point when S is less than $S_f(t)$ i.e. ($S < S_f(t)$), the holder will retain or hold the option and allow it goes worthless. Conversely, the holder will exercise the call option and make profit if the spot price S is greater than or equal to

$S_f(t)$. This means that if a call is executed, K purchased stock is sold for S and the profit $S-K$ should be invested in a risk-less asset.

Therefore, early exercise becomes possible at a stock price of 139.0154. This is shown by the position of the free boundary point $S_f(t)$ in Figure 4.5 above.

4.8 Selceted PSOR tolerance values

The numerical results for the adjusted tolerances 10^{-6}, 10^{-7} , 10^{-8} and 10^{-9} gives the same price of option as the number of step size, (M) keeps on changing. However, the time which is measured in seconds vary as M keeps changing.

	10^{-6}		10^{-7}		10^{-8}		10^{-9}	
$V_{max} = M$	Time	Value	Time	Value	Time	Value	Time	Value
100	0.3899	26.9890	0.4234	26.9890	0.4430	26.9890	0.4376	26.9896
200	1.5302	27.0059	1.7624	27.0059	1.9834	27.0059	2.1346	27.0059
300	4.0597	27.0090	4.5112	27.0090	5.1234	27.0090	5.5700	27.0090
400	7.9792	27.0101	9.0976	27.0101	10.3388	27.0101	11.3361	27.0101
500	14.0717	27.0106	15.6140	27.0106	17.4682	27.0106	195427	27.0106
600	22.3674	27.0109	24.9805	27.0109	27.9952	27.0109	30.9696	27.0109
700	32.2696	27.0110	36.5399	27.0111	416835	27.0111	45.4040	27.0111
800	45.3769	27.0112	51.8811	27.0112	58.3954	27.0112	64.0929	27.0112
900	61.1237	27.0112	69.0228	27.0112	79.2385	27.0112	88.4214	27.0112
1000	78.5957	27.0112	90.3653	27.0112	105.2096	27.0112	117.6885	27.0112

Table 4.3: Numerical simulation times, with selected PSOR tolerances 10^{-6}, 10^{-7} , 10^{-8} and 10^{-9} , measured in seconds

The corresponding graph for the numerical results for the adjusted tolerances 10^{-6}, 10^{-7} , 10^{-8} and 10^{-9} is shown in Figure 4.6

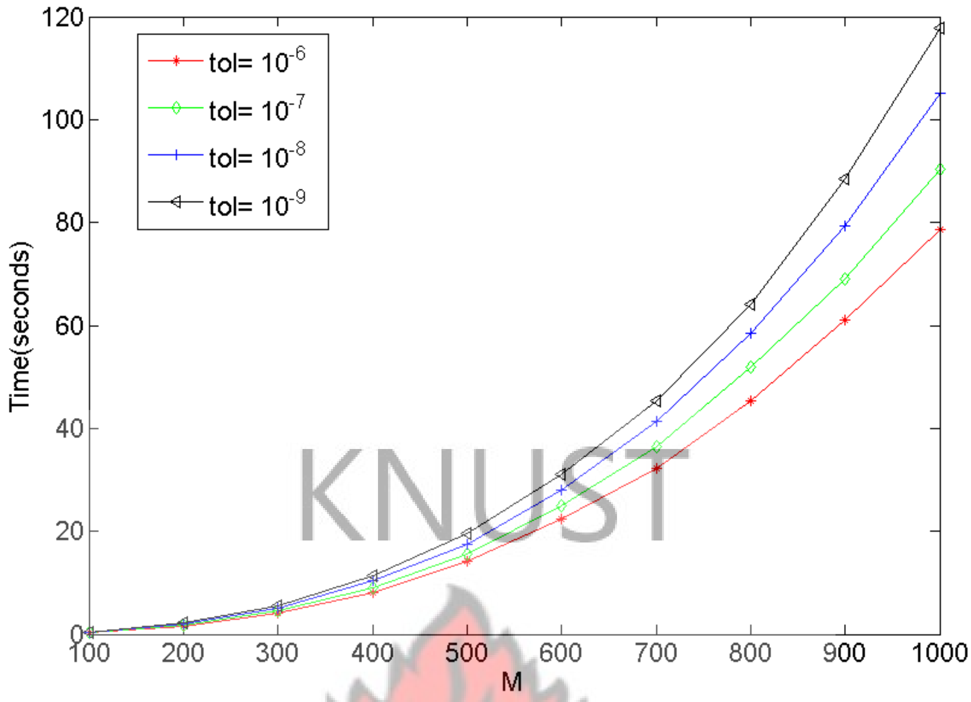

Figure 4.6: Error analysis when $\lambda = 0.2K = 10$, $\sigma = 0.6$, $r = 0.1$, $T = \frac{5}{12}$, $\lambda = 0.2$, $S = 10$.

As the tolerance value is increased, the time needed to complete a single iteration also increase. This iteration was done with HP ProBook 4530s laptop with Windows 7 operating system. Intel Core i3 2350M CPU @ 2.3GHz, 4GB DDR3 RAM Memory, 500GB Hard Disk Drive, 15.6" HD LED.

4.9 Effect of varied Interest rate

With the dividend rate, λ given as 0.2. The price of the underlying (S) and the strike price (K) are both chosen as 80. An annual volatility (σ) is chosen as 0.6. The maturity time of underlying (T) is 1. The asset matures at the end of the years.

The price of the option at 10% interest rate for all values of the strike price are much higher than at 30%, 60% and 80% interest rate. As the rate of interest rises, the price of American put option falls. Figure 4.7 demonstrated this distinct values of interest rate.

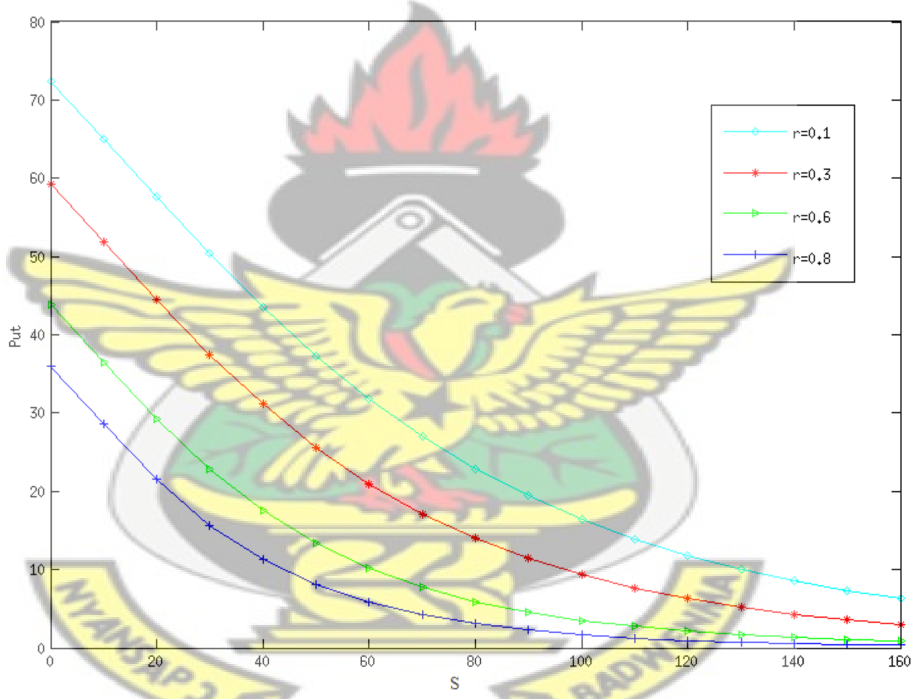

Figure 4.7: Value of American put option with interest rate values of 10%, 30%, 60%, and 80%

4.10 Value of the Option with varying annual volatility

American put option are evaluated with the following financial values. The risk free interest rate (r) and the maturity time of underlying(T) are both 0.25 and $\frac{5}{12}$ respectively. The asset price at issuing date, $S = 80$. The strike price (K) is chosen as 80 and dividend $\lambda = 0.2$. We compute this based on different annual volatility values.

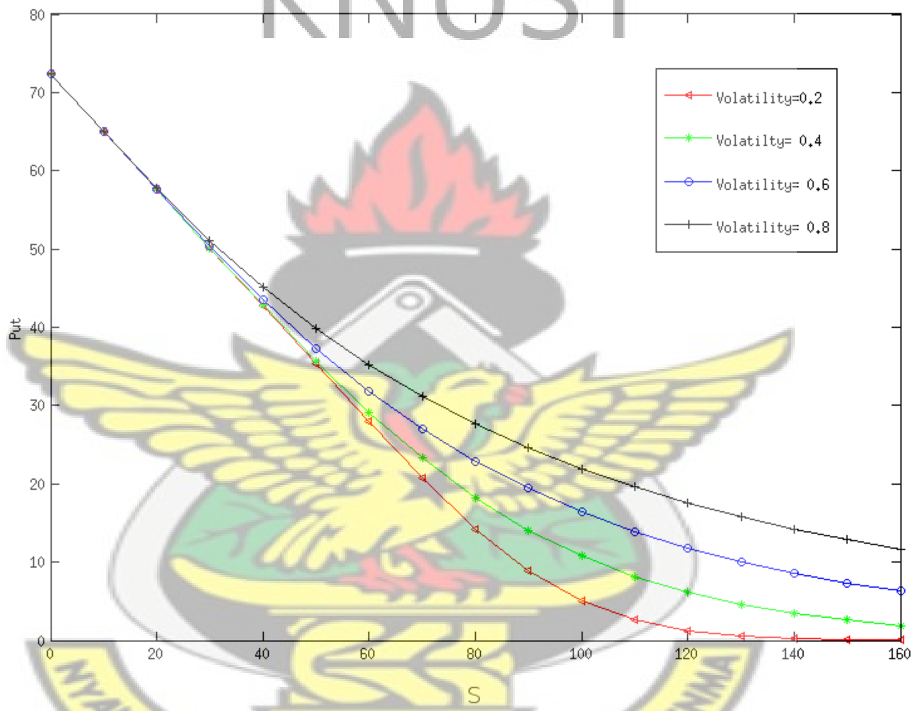

Figure 4.8: An American put option with selected annual values for volatility

The volatility rates for this simulation are 0.2, 0.4, 0.6, and 0.8. This is indicated in Figure 4.8. Within a particular range of the strike price, the prices of the American put remains same for all values of volatility selected for the simulation. At the latter part, the disparity between the prices of the option is shown clearly.

4.11 Different Maturity Times Measured in Years

Quarterly time intervals in years are chosen for computing the value of the on American call option. The following financial values were employed for the calculation in Figure 4.9. The strike price $(K) = 50$, an annual volatility (σ) equal 0.6, dividend $\lambda = \frac{2}{10}$. The risk free interest rate $(r) = 0.25$, $S = 10$, that is, the asset value price at issuing date.

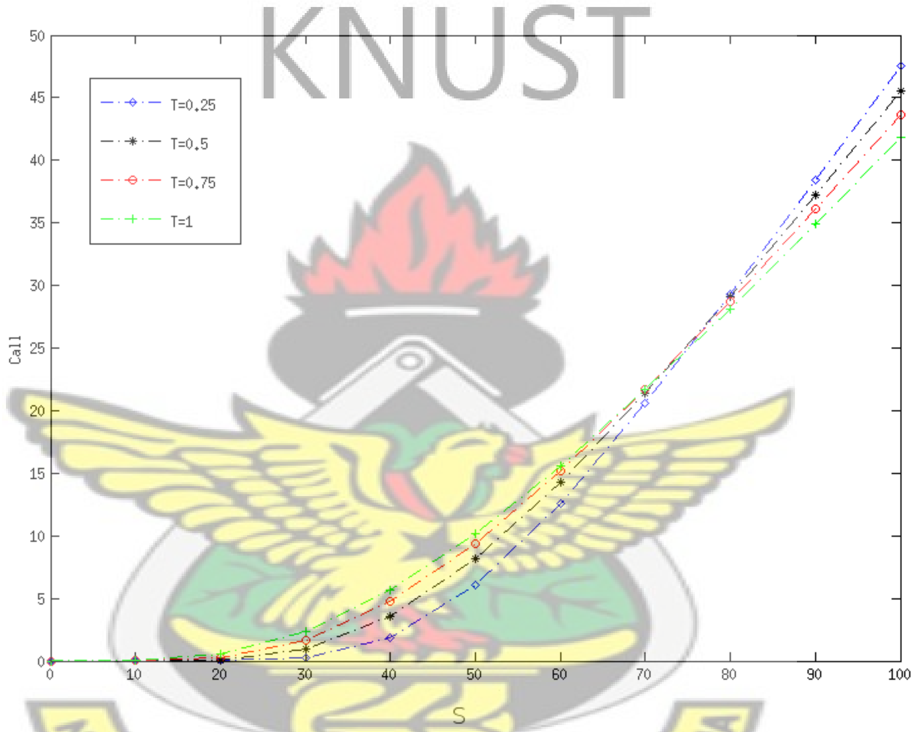

Figure 4.9: American call option when $\lambda = 0.2$, $K = 10$, $\sigma = 0.6$, $r = 0.1$, $T = \frac{5}{12}$, $\lambda = 0.2$, $S = 10$.

For American call with dividend paying stock, the early exercise value remains the same irrespective of the time of maturity. This is shown by the point of intersections in Figure 4.9. Maturity time is an important element to consider when dealing with American option.

Chapter 5

Conclusion and Recommendations

5.1 Introduction

In this chapter, we conclude based on the discussed results obtained from the simulations in chapter four. Based on these same results we make possible recommendations to researchers and businessmen who are into options.

5.2 Conclusion

In this study, we made a review on financial derivatives. The main types of financial derivatives are futures, forwards, swaps and options. The underlying assets that are traded under financial markets are stocks, bonds, interest rate and commodities such as cocoa, gold and oil. The various types of options that were considered are the American options- call and put; and European options - call and put.

The study looked at the Black Scholes model. We transformed the Black Scholes PDE to heat conduction PDE for easy computation. We formulated a linear complementarity problem (LCP) to solve the heat equation in a discrete domain by applying the projected successive over-relaxation (PSOR) method and showed that the details of discritization and assembling of matrices for the LCP depend on the choice of the finite difference method. We considered all the finite difference approaches; the explicit method, the implicit method and the Crank Nicholson method. The Crank

Nicholson Scheme was chosen because it was found to be an unconditionally stable compared to the other two finite difference scheme.

We applied the Crank Nicolson method in valuing standard option with dividend paying stock. Different financial values were chosen for the computation. We evaluated for a call option at different dividends values. It was realized that the price of the option has negative relationship with the dividend value.

The result revealed that the American option and European options does not intersect their pay-off function on the non-dividend paying stock ($\lambda = 0$). As a result, the price of American call equals to the European style call where early exercise of the option is of no important to its holder. So it is important to note that for American calls a dividend yield $\lambda \neq 0$ is needed as early exercise will be observed.

For the dividends paying stocks, the computational results for the prices of the American option exceed the analytical solution. The pay-off function intersects both the European call and the American call functions. But, it intersects the European call at lower call price compared to the American option. The PSOR simulation of the American call with dividend paying stock is more desirable than the European call. For the put option, the American option is also favourite.

For early exercise, the holder has to exercise the call option and make profit when the spot price S is greater than or equal to $S_f(t)$. This means that, if a call is executed, K purchased stock is sold for S and the profit $S - K$ should be invested in a risk-less asset.

5.3 Recommendations

The PSOR method is recommended for solving the transformed Black Scholes equation because of it high speed of convergence.

The result of the study showed that American option with non-dividend paying stock has a close solution to the European option. But for dividend paying stock the American option is preferred to the European option. We would advise the financial institutions to adapt to the American option when dividends are paid.

In the area of mathematical finance, there is still the need for more advanced mathematical models which make simulation very efficient and accurate. The thesis used the finite difference scheme (Crank Nicolson Method) at a discretization scheme for the transformed Black-Scholes equation. The finite element method (FEM) may minimize the error function and may produce a stable, because it connects many tiny straight lines to form a curve over many small sub-domains. Researchers can consider an extension of these work by discretizing the Black-Scholes equation with either iterative techniques or meshfree methods.

Extension can also be done on pricing American option with two or more underlying assets since the study looked at the valuation of a single underlying asset.

References

Bachelier, L. (1900). Theory of Speculation. *Paris*.

Baz, J. and Chacko, G. (2004). *Financial Derivatives: Pricing, Applications and Mathematics*. Cambridge University Press, Cambridge.

Benbow, S. C. (2005). Numerical methods for american options. Master's thesis, University of Reading.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659.

Boyle, P. P. (1977). Options: A monte carlo approach. *Journal of Financial Economics*, 23:323–338.

Boyle, P. P. (1988). A lattice framework for option pricing with two state variables. *Journal of Financial and Quantitative Analysis*, 23:1–12.

Brennan and Schwartz (1978). Finite difference methods and jump processes arising in the pricing of contingent claims. *Journal of Financial and Quantitative Analysis*, 5(4):461–474.

Brennan, M. J. and Schwartz, E. S. (1977). The valuation of american put options. *Journal of Finance*, 32:449–462.

Broadie, M. and Detemple, J. (1996). American option valuation: New bounds, approximations, and a comparison of existing methods. *The Review of Financial Studies*, 9(4):1211–1250.

Broadie, M. and Glasserman, P. (1997). Pricing american-style securities using simulation. *Journal of Economic Dynamics and Control*, 21:1323–1352.

Carr, P., Jarrow, R., and Myneni, R. (1992.). Alternative characterizations of american put options. *Mathematical Finance*, 2:87–106.

Chiu, C. (2002). Analysis of historical and implied volatility of the s&p 100 and nasdaq 100 indices. Master's thesis, School of Business, New York University.

Coleman, T., Li, Y., and Verma, A. (2002). A newton method for american option pricing. *Journal of Computational Finance*, 5(3):51–78.

Courtadon, G. (1982). A more accurate finite difference approximation for the valuation of options. *Journal of Financial and Quantitative Analysis,*, pages 697–705.

Cox, J., Ross, S., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, 7:229–263.

Dempster, M. A. H. and Hutton, J. P. (1999). Pricing american stock options by linear programming. *Mathematical Finance*, 9(3):229–254.

Dontwi, I., Dedu, V., and Biney, F. (2010). Pricing of foreign currency options in a developing financial market. *European Journal of Scientific Research*, 45:608–619.

Forsyth, P. A. and Vetzal., K. R. (2002). Quadratic convergence for valuing american options using a penalty method. *SIAM Journal of Science Computution*, 23(6):2095–2122.

Gallant, A. R., Rossi, P. E., and Tauchen, G. (1992). Stock prices and volume. *The Review of Financial Studies*, 5:199–242.

Geske, R. (1979). A note on an analytical valuation formula for unprotected american options on stocks with known dividends. *Journal of Financial Economics*, 7:375–380.

Geske, R. and Shastri, K. (1985). Valuation by approximation: A comparison of alternative option valuation techniques. *Journal of Financial and Quantitative Analysis*, 20(1):45–71.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6.

Huang, J. and Pang, J. (1998). Option pricing and linear complementarity. *Journal of Computational Finance*, 2(3).

Hull, J. (2009). *Options, Futures and other Derivatives*. Pearson Education Inc, Fifth Edition, New Jersey., 7 edition.

Ikonen, S. and Toivanen, J. (2007). Pricing american options using lu decomposition. *Applied Mathematical Sciences*, 1(51):2529–2551.

Jain, R. K. (1997). Putting volatility to work. unpublished work.

Jandacka, M. and Sevcovic, D. (2005). On the risk adjusted pricing methodology based valuation of vanilla options and explanation of the volatility smile. *Journal of Applied Mathematics*, 3:235–258.

Kim, I. J. (1990). The analytic valuation of american options. *Review of Financial Studies*, 3:547–572.

Kishimoto, M. (2008). On the black-scholes equation: Various derivations. Master's thesis.

Klebaner, F. C. (2005). *Introduction To Stochastic Calculus With Applications*. Imperial College Press.

Kumar, S., Yildirim, A., Khan, Y., Jafari, H., Sayevand, K., and Wei., L. (2012). Analytical solution of fractional black-scholes european option pricing equation by using laplace transform. *Journal of Fractional Calculus and Applications*, 2(8):1–9.

Kwon, C. and Friesz, T. L. (2008). Valuation of american options by the gradient projection method.

Liu, S. (2007). Review of Option Pricing Literature and an Online Real-Time Option Pricing Application Development. Master's thesis, Nottingham University Business School.

Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics Management Science*, 4:141–183.

Morton, K. W. and Mayers, D. F. (2005). *Numerical Solution of Partial Differential Equations: An Introduction*. Cambridge University Press.

Neftci, S. N. (2000). *An Introduction to the Mathematics of Financial Derivatives.* Academic Press, New York.

Ntwiga, D. B. (2005). Numerical methods for the valuation of financial derivatives. Master's thesis, University of Western Cape, South Africa.

Nwozo, C. and Fadugba, S. (2012). Some numerical methods for options valuation. *Communications in Mathematical Finance*, 1:51–74.

Pastorello, S., Renault, E., and Touzi., N. (2000). Statistical inference for random-variance option pricing. *Journal of Business & Economic Statistics*, 18:358–367.

Paully, O. (2004). *Numerical Simulation of American Options*. PhD thesis.

Roll., R. (1977). An analytic valuation formula for unprotected american call options on stocks with known dividends. *Journal of Financial Economics*, 5:251–258.

Rubinstein, M. (1983). Displayed diffusion option pricing. *Journal of Finance*, pages 213–217.

Sevcovic, D. (2011). Analytical and numerical methods for pricing financial derivatives.

Seydel, R. (2002). *Tools for Computational Finance*. Springer.

Shcherbakov, D. and Szwaczkiewicz, S. (2010). Exponential fitting, finite volume and box methods in option pricing. Master's thesis, School of Information Science, Computer and Electrical Engineering, Halmstad University.

Tveito, A. and Winther, R. (1998). *Introduction to Partial Differential Equation Instruments: The Finite Difference Method.* John Wiley and Son, New York.

Whaley, R. E. (1981). On the Valuation of American Call Options on Stocks with known Dividends. *Journal of Financial Economics*, 9:207–211.

Wilmott, P., Dewynne, J., and Howison, S. (1996). *Option Pricing: Mathematical Models and Computation,.* Oxford Financial Press.

Wu, L. (1999). The black-Scholes Model. Option Market.

Appendix A

Matlab code for PSOR


```
function [x,iter]  = PSOR(A, b, f, x, omega, maxIter, tol)
%PSOR(A, b, f, x, omega, maxIter, tol)  solve the LCP
%A*x >= b, x >= f, (x - f)'*(A*x - b) = 0, for x
%omega - relaxation parameter (0,2)
%maxIter - maximum interations
%tol - tolerance for stopping critia
%Note if A is not symmetric positive definite and consistently
% orders this may not converge
%see also LCPforPDE

%The don't run signal
if(tol>=1.0)
    iter = 0;
    return
end
%check the LCP conditions to see if we already have the solution
r = A*x - b;
if(all(r >= 0) && all(x >= f) && (x-f)'*(r)==0.0)
    iter = 0;
    return
```

```
end
omegaOD = omega./diag(A);
At = transpose(A);
N = length(x);
lim = 8*floor(N/8);
for iter = 1:maxIter
    xOld = xt;
    %loop unwind
    for i = 1:8:lim
        xt(i) = max(f(i),xt(i) + omegaOD(i)*(b(i)-xt*At(:,i)));
        xt(i+1) = max(f(i+1),xt(i+1) + omegaOD(i+1)*(b(i+1)-xt*At(:,i+1)));
        xt(i+2) = max(f(i+2),xt(i+2) + omegaOD(i+2)*(b(i+2)-xt*At(:,i+2)));
        xt(i+3) = max(f(i+3),xt(i+3) + omegaOD(i+3)*(b(i+3)-xt*At(:,i+3)));
        xt(i+4) = max(f(i+4),xt(i+4) + omegaOD(i+4)*(b(i+4)-xt*At(:,i+4)));
        xt(i+5) = max(f(i+5),xt(i+5) + omegaOD(i+5)*(b(i+5)-xt*At(:,i+5)));
        xt(i+6) = max(f(i+6),xt(i+6) + omegaOD(i+6)*(b(i+6)-xt*At(:,i+6)));
        xt(i+7) = max(f(i+7),xt(i+7) + omegaOD(i+7)*(b(i+7)-xt*At(:,i+7)));
    end
    for i = lim+1:N
        xt(i) = max(f(i),xt(i) + omegaOD(i)*(b(i)-xt*At(:,i)));
    end
    error = norm( xt - xOld ) / norm( xt );
    if ( error <= tol ), break, end
end
if ( error > tol )
    errMsg= MException('OpenGamma:PSOR','no convergence');
    throw (errMsg);
end x = transpose(xt); end
```

Appendix B

Matlab Code for LCP


```
function x = LCP(M,q,parameters)
%LCP(M,q,parameters) Solve the Linear Complementarity Problem.
% USAGE
%   x = LCP(M,q) solves the LCP
%        x >= 0
%      Mx + q >= 0
%   x'(Mx + q) = 0
%   x = LCP(M,q,l,u) solves the generalized LCP (a.k.a MCP)
%   l < x < u   =>   Mx + q = 0
%       x = u   =>   Mx + q < 0
%   l = x       =>   Mx + q > 0
%x = LCP(M,q,l,u,x0,display) allows the optional initial value 'x0' and
%a binary flag 'display' which controls the display of iteration data.
% Parameters:
%tol-   Termination criterion. Exit when 0.5*phi(x)'*phi(x) < tol.
%mu -   Initial value of Levenberg-Marquardt mu coefficient.
% mu_step   -   Coefficient by which mu is multiplied / divided.
%mu_min    -   Value below which mu is set to zero (pure Gauss-Newton).
%max_iter  -   Maximum number of (successful) Levenberg-Marquardt steps.
%b_tol-  Tolerance of degenerate complementarity: Dimensions where
```

```
%   max( min(abs(x-l),abs(u-x)) , abs(phi(x)) ) < b_tol
%   are clamped to the nearest constraint and removed from
%      the linear system.
%    ALGORITHM
%This function implements the semismooth algorithm as described in [1],
%with a least-squares minimization of the Fischer-Burmeister function using
%a Levenberg-Marquardt trust-region scheme with mu-control as in [2].

n = size(M,1);
%defaults
tol       = 1.0e-12;
mu        = 1e-3;
mu_step   = 5;
mu_min    = 1e-5;
max_iter = 10;
b_tol     = 1e-6;
l         = zeros(n,1);
u         = inf(n,1);
x0        = ones(n,1);
display   = false;
if(nargin>2)
if(isfield(parameters,'tol')), tol = parameters.tol; end
if(isfield(parameters,'mu')), mu = parameters.mu; end
if(isfield(parameters,'mu_step')), mu_step = parameters.mu_step; end
if(isfield(parameters,'mu_min')), mu_min = parameters.mu_min; end
if(isfield(parameters,'max_iter')), max_iter = parameters.max_iter; end
if(isfield(parameters,'b_tol')), b_tol = parameters.b_tol; end
if(isfield(parameters,'l')), l = parameters.l; end
if(isfield(parameters,'u')), u = parameters.u; end
```

```
if(isfield(parameters,'x0')), x0 = parameters.x0; end
if(isfield(parameters,'display')), display = parameters.display; end
end

lu              = [l u];
x               = x0;
[psi,phi,J]     = FB(x,q,M,l,u);
new_x           = true;
warning off MATLAB:nearlySingularMatrix
for iter = 1:max_iter
   if new_x
      [mlu,ilu]       = min([abs(x-l),abs(u-x)],[],2);
      bad             = max(abs(phi),mlu) < b_tol;
      psi             = psi - 0.5*phi(bad)'*phi(bad);
      J               = J(~bad,~bad);
      phi             = phi(~bad);
      new_x           = false;
      nx              = x;
      nx(bad)         = lu(find(bad)+(ilu(bad)-1)*n);
   end
   H               = J'*J + mu*speye(sum(~bad));
   Jphi            = J'*phi;

   d               = -H\Jphi;

   nx(~bad)        = x(~bad) + d;
   [npsi,nphi,nJ] = FB(nx,q,M,l,u);
   r               = (psi - npsi)  / -(Jphi'*d + 0.5*d'*H*d);
   % actual reduction / expected reduction
```

```
   if r < 0.3           % small reduction, increase mu
      mu = max(mu*mu_step,mu_min);
   end
   if r > 0             % some reduction, accept nx
      x     = nx;
      psi   = npsi;
      phi   = nphi;
      J     = nJ;
      new_x = true;
      if r > 0.8       % large reduction, decrease mu
         mu = mu/mu_step * (mu > mu_min);
      end
   end
   if display
fprintf('iter = %2d, psi = %3.0e, r = %3.1f, mu = %3.0e\n',iter,psi,r,mu);
   end
   if psi < tol
      break;
   end
end
warning on MATLAB:nearlySingularMatrix
x = min(max(x,l),u);

function [psi,phi,J] = FB(x,q,M,l,u)
n     = length(x);
Zl    = l >-inf & u==inf;
Zf    = l==-inf & u==inf;

a     = x;
```

```
b      = M*x+q;

a(Zl) = x(Zl)-l(Zl);

a(Zu) = u(Zu)-x(Zu);
b(Zu) = -b(Zu);

if any(Zlu)
   nt     = sum(Zlu);
   at     = u(Zlu)-x(Zlu);
   bt     = -b(Zlu);
   st     = sqrt(at.^2 + bt.^2);
   a(Zlu) = x(Zlu)-l(Zlu);
   b(Zlu) = st -at -bt;
end

s         = sqrt(a.^2 + b.^2);
phi       = s - a - b;
phi(Zu)   = -phi(Zu);
phi(Zf)   = -b(Zf);
psi       = 0.5*(phi'*phi);
if nargout == 3
   if any(Zlu)
      M(Zlu,:) = -sparse(1:nt,find(Zlu),at./st-ones(nt,1),nt,n)
       - sparse(1:nt,1:nt,bt./st-ones(nt,1))*M(Zlu,:);
   end
   da        = a./s-ones(n,1);
   db        = b./s-ones(n,1);
   da(Zf)    = 0;
```

```
    db(Zf)    = -1;
    J         = sparse(1:n,1:n,da) + sparse(1:n,1:n,db)*M;
end
```

Appendix C

Matlab Code for the Value of the option

```
function opt = roo05_06_2013(status)
% input parameters
% american options
%clc, close all hidden
global  r vol lambda qLambda q

K = 80; % strike
T = 1; % time to expiration
r = 0.25; % risk free interest rate
vol = 0.6; % volatility
lambda = 0; % dividend
% status = 'call';
xmax = 0;
xmin = -5;
% LCP
theta = 1/2;
% if isequal(status,'put')
```

```
m = 100;

h = (xmax - xmin)/m;

vmax = 100;

tauMax = ((vol^2)*T)/2;

dtau = tauMax/vmax;

zeta = dtau/(h^2);


x = xmin:h:xmax;

S0 = K*exp(x(1)); % spot price

tau = 0:dtau:tauMax;

% creating the matrices

A = zeros(m,m);

B = zeros(m,m);

for i = 1:m

    A(i,i) = 1 + 2*zeta + theta;

    B(i,i) = 1 - 2*zeta + (1 - theta);

end

for i = 1:m-1

    A(i,i+1) = -zeta*theta;

    A(i+1) = A(i);

    B(i+1,i) = B(i,i+1);

end


c = zeros(m,1);

c(1) = zeta*theta*gfunction(x(1),0,status)

+ zeta*(1 - theta)*gfunction(x(1),0,status);

c(m) = zeta*theta*gfunction(x(m),0,status)

+ zeta*(1 - theta)*gfunction(x(m),0,status);

w = zeros(m,1);
```

```
f = zeros(m,1);
for i = 1:m
    w(i) = gfunction(x(i),0,status);
    f(i) = gfunction(x(i),tau(i),status);
end

b = B*w + c;
% w = LCP(A,b);
tol = 10^(-6);
omega = 1.5;
maxIter = 100;
[w,iter] = PSOR(A,b,f,w,omega,maxIter,tol);
for i = 2:m+1
    V(i-1) = K*w(i-1)*exp(-(x(i)/2)*(qLambda - 1))
    *exp(-tauMax*((1/4)*(qLambda - 1)^2 + q));
end

% temp = [];
if isequal(status,’put’)
    err = K*10^(-5);
    f = abs(V + S -K);
    idx = find(f < err);
    Sif = max(f(idx));
    plot(fliplr(S));hold on;plot(V(end),S(round(V(end))),’r.’)
    if S0 < Sif
        return
%          temp = [temp f];
    end
```

```
elseif isequal(status,'call')
    err = K*10^(-5);
    f = abs(K - S + V);
    idx = find(f < err);
    Sif = min(f(idx));
    plot(S);hold on;plot(V(end),S(round(V(end))),'r.');
    if S0 > Sif
        return
%         temp = [temp f];
    end
end
opt = V(end);

% the prototype core algorithm
function g = gfunction(x,tau,status)
% required parameters
global r vol lambda qLambda q

q = (2*r)/(vol^2);
qLambda = (2*(r - lambda))/(vol^2);
if isequal(status,'put')
    g1 = exp(tau/4*((qLambda - 1)^2 + 4*q));
    g2 = max(exp((x/2)*(qLambda - 1)) - exp((x/2)*(qLambda + 1)),0);
    g = g1 + g2;
elseif isequal(status,'call')
    g1 = exp(tau/4*((qLambda - 1)^2 + 4*q));
    g2 = max(exp((x/2)*(qLambda + 1)) - exp((x/2)*(qLambda - 1)),0);
    g = g1 + g2;
end
```