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Mathematical modeling for the transmission potential of Zika virus with optimal control strategies

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Abstract In this paper, we formulate a new Zika virus model in light of both mosquito and human transmission along with the human awareness in the host population. Initially, we assumed that the virus is transmitted to humans through a mosquito bite and then transmits to his or her sexual partner. Further, we investigated the mathematical results and stability analysis and proved that the model is asymptotically stable both locally and globally. We applied the Castillo-Chavez approach for establishing global stability. Similarly, we presented the existence of endemic equilibrium and demonstrate that the model is locally and globally asymptotically stable using a suitable Lyapunov function at endemic state, upon backward bifurcation analysis we proposed that no bifurcation exists for our model. The sensitivity analysis is carried out and verified that the probability per biting of the susceptible mosquito with the infected human is the most sensitive parameter. Furthermore, we developed a Zika control model and incorporated three controls. These controls are prevention through bed nets and mosquito repellents, treatment of Zika patients, and the spray of insecticides on mosquitoes. The graphical results of the model with control and without control are obtained through a numerical scheme. The infection caused by the Zika virus would be more efficiently eliminated using the new idea of human awareness and bilinear incidence presented in this paper.

1 Introduction

Zika Virus is a Flaviviridae virus that enters the body usually through the bite of female mosquitoes Aedes aegypti during the day. Sexual interaction, blood transfusion, and fetuses from their mothers are the other secondary transmission mechanisms. Conjunctivitis, rash, headache, fever, muscle, and joint discomfort are some of the symptoms of the Zika virus, which can last anywhere from 2 to 7 days. In some cases, the infected person may not show any symptoms at all. The Zika virus was initially discovered in rhesus monkeys in Uganda in 1947, and then in people in Uganda and Tanzania in 1952 [1]. The first major Zika virus outbreak occurred in Yap Island in 2007, and the second occurred in French Polynesia in 2013. Similarly, in 2015 and 2016, the world saw an outbreak of Zika virus in major European countries such as France, Italy, and Spain, among others. Colombia and Latin America have been severely affected by the Zika virus in the last ten years, with numbers of cases ranging from 31555 in Colombia to 150000 in Brazil, 11400 in Honduras, 4500 in Venezuela, and 6310 in El Salvador. In February 2016, about 735 new cases due to travel were reported from Florida state, according to [2]. In the last few years, Zika cases are reported from central and south American and the Caribbean. Many countries still face an alarming situation regarding the infection of Zika virus [3]. The complications associated with the Zika virus and its possible control are an important subject to public health authorities recognized by WHO [4]. The infection caused by the Zika virus is possibly the same as that of chikungunya, yellow, and dengue fever. The Aedes mosquitoes take the blood of an infected person through the bite and then passes the infection to a healthy individual, currently, no medicine, vaccine, proper treatment, or clinical test are available for Zika infection yet plenty of rest, protection measures against mosquitoes, and some WHO proposed medicine might be helpful to keep yourself safe from Zika virus. Besides this mosquitoes can be eliminated by larval devastation and insecticide spray. Mathematical modeling of infectious diseases is a significant tool in understanding the dynamics of disease and provides a base for the process of taking the necessary decisions to stop the disease transmission. To address the complications and key factors associated with Zika infection the mathematical models with control can be more efficient to predict and eradicate a future outbreak of the Zika virus. There are possibly three control interventions regarding Zika virus to reduce its spread in the host population, these strategies are prevention through wearing light-colored clothes, use of bed nets, window screens, and mosquito repellents, increasing autoimmunity by WHO proposed medicine, increasing death rate of mosquitoes through a spray of insecticide. The dynamics of Zika infection are modeled by many authors and each one explored a different perspective, for example, Zika infection studied by some mathematical models are discussed in [5–9]. The authors in

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[5] analyzed the actual data of two regions of the Island of Yap in 2007 and French Polynesia in 2013–2014. The coinfection of Zika virus with Chikungunya modeled in [6]. Zika virus with suggested control is considered in [9]. The spread of the Zika virus by sexual intercourse has been studied in [7]. Zika cases in Colombia are mathematically modeled in [8]. Zika infection with an optimal control is studied in [10]. The mathematical model that studies the Zika infection in the human compartment with logistic growth is considered in [11]. A recent temperature-dependent Zika model is investigated in [12]. An SEIR type model of Zika virus dynamics is studied in [13]. The issues faced by Zika patient in hospitals and the possible provision of resources is studied through a mathematical model [14]. In mathematical models of epidemiology, researchers have experimented with several forms of nonlinear incidence functions. In an SEIS model, Hethcote et al. [15] employed the nonlinear incidence function. Incident rates of the form $K(IS)^p$ with $p > 1$ may lead to periodic solutions, according to Cunningham [16]. The mathematical model for the dynamics of Zika virus with nonlinear incidence rate for both integer and non-integer order is discussed in [8,17] based on the Colombia data. This paper presents a Zika virus model with awareness including optimal control theory, the optimal control interventions are protection against mosquitoes, increasing autoimmunity, and increasing death rate of mosquitoes. In the proposed model, we consider the disease incidence function, which estimates the rate of new infection, is regarded as a critical parameter in the mathematical analysis of epidemiological problems. This rate is expected to be bilinear in both the infected and susceptible fractions. This hypothesis is based on the idea that people move around in a community, and infection spreads when a susceptible and an infective come into touch. Further, we considered the only human model and investigate the analysis. Similarly, we studied the full Zika carrier model with fundamental properties. Finally, we made the backward bifurcation analysis about bifurcation parameters at disease-free equilibrium and concluded that no bifurcation exists for our proposed model. In addition, sensitivity analysis is made to observe the effect of influential parameters on the basic reproduction number. The paper is organized as: Sects. 2 and 3 present formulation and dynamical properties of the proposed model. Section 4 includes the analysis of only the human model and Section 5 is composed of a mathematical analysis of the full Zika carrier model. Further Sect. 6 comprises optimal control and its numerical simulations while concluding remarks in Sect. 7.

2 Model description

2.1 Formulation of vector host population model

The total human population N_H is classified into susceptible S_H, exposed E_H, infected I_H, and recovered R_H. N_M represents the overall mosquito population, which is further separated into S_M-susceptible mosquitoes, E_M-exposed mosquitoes, and I_M-infected insects. As a result, we have the Zika model as follows:

$$\begin{cases} \frac{dS_H}{dt} = \Lambda_H - \beta_1 S_H I_H - \beta_2 S_H I_M - (k_1 + \omega_1) S_H, \\ \frac{dE_H}{dt} = \beta_1 S_H I_H + \beta_2 S_H I_M - (k_1 + \sigma_1) E_H, \\ \frac{dI_H}{dt} = \sigma_1 E_H - (\gamma + k_1) I_H, \\ \frac{dR_H}{dt} = \gamma I_H - k_1 R_H + \omega_1 S_H, \\ \frac{dS_M}{dt} = \Lambda_M - \mu S_M I_H - (k_2 + \omega_2) S_M, \\ \frac{dE_M}{dt} = \mu S_M I_H - (k_2 + \sigma_2 + \omega_2) E_M, \\ \frac{dI_M}{dt} = \sigma_2 E_M - (k_2 + \omega_2) I_M, \end{cases} \tag{1}$$

subjected to initial conditions

$$\begin{aligned} &S_H(0) = S_{H0} \geq 0,\ E_H(0) = E_{H0} \geq 0,\ I_H(0) = I_{H0} \geq 0,\ R_H(0) = R_{H0} \geq 0, \\ &S_M(0) = S_{M0} \geq 0,\ E_M(0) = E_{M0} \geq 0,\ I_M(0) = I_{M0} \geq 0. \end{aligned} \tag{2}$$

For simplicity, we choose

$$l_1 = (k_1 + \omega_1),\ \ l_2 = (k_1 + \sigma_1),\ \ l_3 = (\gamma + k_1),\ \ l_4 = (k_2 + \omega_2),\ \ l_5 = (k_2 + \sigma_2 + \omega_2).$$

The parameter Λ_H in the model (1) represents human growth rate, while Λ_M represents Zika mosquito birth rate. The factors β_1 and β_2 determine the rate of transmission between susceptible humans and infected humans, as well as between susceptible humans and infected mosquitoes. Humans and mosquitoes die naturally at the rate k_1 and k_2, respectively. The parameter γ shows the rate at which the infected become recovered. The parameter ω_1 shows the awareness rate of the host population, while ω_2 is the constant and effective control rate of mosquitoes. The exposed mosquitoes become infected at rate σ_2 and μ represents the

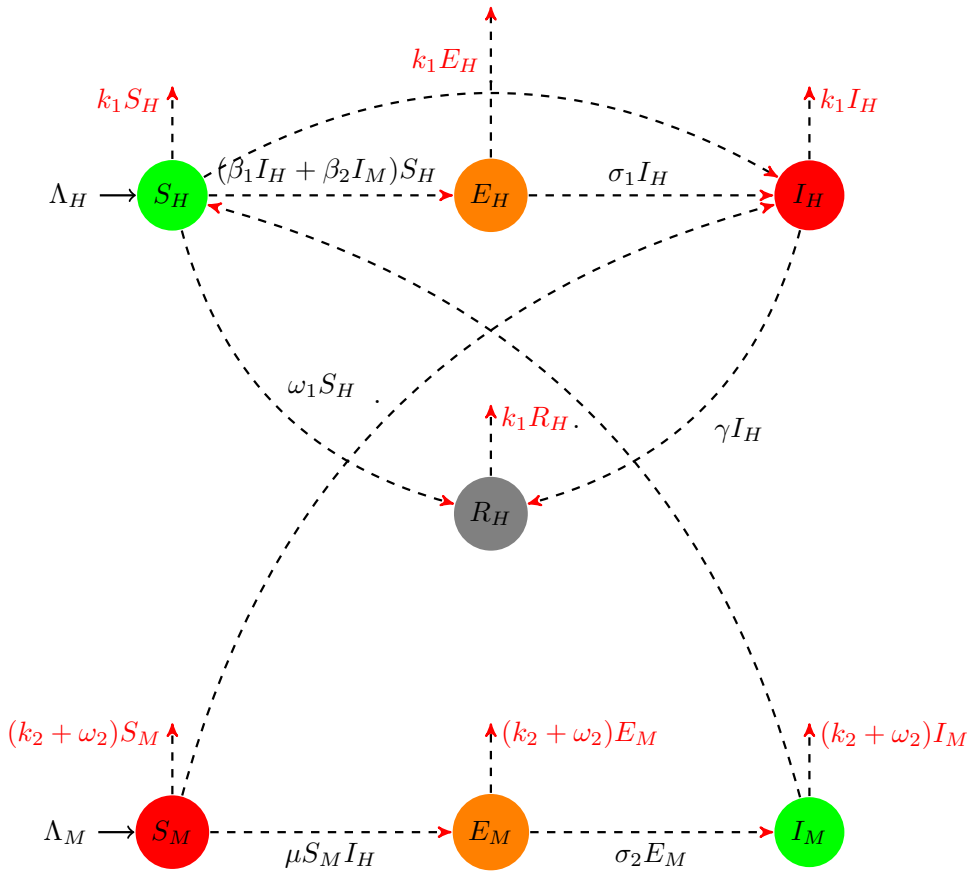


Fig. 1 Transition diagram of Zika virus model (1)

probability of susceptible mosquito biting-infected humans. The transmission pattern of the proposed model is depicted in Fig. 1. Since the parameters of the model are nonnegative, it is essential to show the positivity of all the state variables.

3 Model basic properties

3.1 Solution positivity

Lemma 3.1 *The solution* $G(t) = (S_H, E_H, I_H, R_H, S_M, E_M, I_M)$ *with the initial condition* $G(0) \geq 0$ *of the model (1) is nonnegative for* $t \geq 0$, *also* $\lim_{t\to\infty} sup\ N_H(t) = \frac{\Lambda_H}{l_1}$ *and* $\lim_{t\to\infty} sup\ N_M(t) = \frac{\Lambda_M}{l_4}$ *hold.*

Proof The first equation of the model (1) is

$$\frac{dS_H}{dt} = \Lambda_H - \beta_1 S_H I_H - \beta_2 S_H I_M - (k_1 + \omega_1)S_H. \tag{3}$$

Integrating (3) in the range $[0, T]$, we obtain

$$\frac{d}{dt}\Big[S_H(t)\ exp\Big\{\int_0^T \big(\beta_1 I_H(\eta) + \beta_2 I_M(\eta)\big)d\eta + (k_1+\omega_1)T\Big\}\Big]$$
$$= \Lambda_H\ exp\Big[\beta_1 I_H(\eta) + \beta_2 I_M(\eta)d\eta + (k_1+\omega_1)T\Big]dT,$$

or we can write it as,

$$S_H(T) = S_H(0)\ exp\Big[-\int_0^T \big(\beta_1 I_H(\eta) + \beta_2 I_M(\eta)\big)d\eta + (k_1+\omega_1)T\Big]$$
$$+exp\Big[-\int_0^T \big(\beta_1 I_H(\eta) + \beta_2 I_M(\eta)\big)d\eta + (k_1+\omega_1)T\Big]$$
$$\times \int_0^T \Lambda_H\ exp\Big\{\int_0^y \big(\beta_1 I_H(\eta) + \beta_2 I_M(\eta)\big)d\eta + (k_1+\omega_1)y\Big\}dy > 0.$$

Similarly it can be shown that $E_H(t)$, $I_H(t)$, $R_H(t)$, $S_M(t)$, $E_M(t)$, and $I_M(t)$ are all positive for $T > 0$. Thus $G > 0$ for all $t > 0$, for the next part of lemma, we consider the human compartments in model (1) is,

$$\frac{dN_H}{dt} = \Lambda_H - k_1 N_H. \tag{4}$$

Similarly, the mosquitoes compartments in model (1), we have

$$\frac{dN_M}{dt} = \Lambda_M - k_2 N_M. \tag{5}$$

By integrating and taking limit for $t \to \infty$ [18], both (4) and (5) follows that,

$$\frac{\Lambda_H}{k_1} \leq \lim_{t\to\infty} inf\ N_H(t) \leq \lim_{t\to\infty} sup\ N_H(t) \leq \frac{\Lambda_H}{k_1},$$

$$\lim_{t\to\infty} sup\ N_H(t) = \frac{\Lambda_H}{k_1},$$

and

$$\frac{\Lambda_M}{k_2} \leq \lim_{t\to\infty} inf\ N_M(t) \leq \lim_{t\to\infty} sup\ N_M(t) \leq \frac{\Lambda_M}{k_2},$$

$$\lim_{t\to\infty} sup\ N_M(t) = \frac{\Lambda_M}{k_1}.$$

Hence, the biologically feasible region is

$$\Omega = \left\{(S_H, E_H, I_H, R_H, S_M, E_M, E_M) \in \mathbb{R}^7_+ : N_H \leq \frac{\Lambda_H}{k_1}, N_M \leq \frac{\Lambda_M}{k_2}\right\}.$$

□

3.2 Invariant region

Proposition 3.1 *For nonnegative initial conditions, the closed set Ω is bounded and positively invariant in $\mathbb{R}^7_+$.*

Proof The rates of change of whole human and mosquito populations are

$$N_H(t) \leq N_H(0)e^{-k_1(t)} + \frac{\Lambda_H}{k_1}(1 - e^{-k_1 t}),$$

$$N_M(t) \leq N_M(0)e^{-k_1(t)} + \frac{\Lambda_H}{k_1}(1 - e^{-k_2 t}).$$

Since $e^{-k_1 t} \to 0$ and $e^{-k_2 t} \to 0$ as $t \to \infty$, the solutions are

$$N_H(t) = \frac{\Lambda_H}{k_1},$$

$$N_M(t) = \frac{\Lambda_M}{k_2}.$$

Thus the closed set Ω is bounded and positively invariant. □

4 Dynamics of only human model

In this section we investigate the mathematical results of only human model that comprises of first four equations of model (1) followed as,

$$\begin{cases} \frac{dS_H}{dt} = \Lambda_H - \beta_1 S_H I_H - \beta_2 S_H I_M - (k_1 + \omega_1) S_H, \\ \frac{dE_H}{dt} = \beta_1 S_H I_H + \beta_2 S_H I_M - (k_1 + \sigma_1) E_H), \\ \frac{dI_H}{dt} = \sigma_1 E_H - (\gamma + k_1) I_H, \\ \frac{dR_H}{dt} = \gamma I_H - k_1 R_H + \omega_1 S_H. \end{cases} \tag{6}$$

Disease-free equilibrium
The infection-free equilibrium of only human model is given as

$$\mathcal{E}_0^H = (S_H^0, E_H^0, I_H^0, R_H^0) = \left(\frac{\Lambda_H}{l_1}, 0, 0, \frac{\omega\Lambda_H}{k_1 l_1}\right). \tag{7}$$

Reproduction number
The basic reproduction number is an important parameter in mathematical epidemiology, it determines the existing situation and future outlook of an infectious disease. We follow the approach [19], for the computation of basic reproduction number ($\mathcal{R}_0^H$) of the only human model.

$$\mathbf{F} = \begin{pmatrix} \frac{\beta_1\Lambda_H}{l_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} l_2 & 0 \\ -\sigma_1 & l_3 \end{pmatrix}.$$

$$\mathcal{R}_0^H = \rho(\mathbf{F}\mathbf{V}^{-1}) = \frac{\sigma_1\beta_1\Lambda_H}{l_1 l_2 l_3}. \tag{8}$$

Endemic equilibria
The endemic equilibria of only human model (6) are denoted by $\mathcal{E}_H^* = (S_H^*, E_H^*, I_H^*, R_H^*)$.

$$\begin{cases} S_H^* = \frac{\Lambda_H l_4 l_5(\mu I_H^* + l_4)}{I_H^*(\beta_1 l_4 l_5(\mu I_H^* + l_4) + \beta_2\sigma_2\mu\Lambda_M) + l_1 l_4 l_5(\mu I_H^* + l_4)}, \\ E_H^* = \frac{l_3 I_h^*}{\sigma_1}, \\ R_H^* = \frac{l_4 l_5(\mu I_H^* + l_4)(\gamma I_H^*(\beta_1 I_H^* + l_1) + \omega_1\Lambda_H + \gamma I_H^*\beta_2\sigma_2\mu\Lambda_M)}{k_1(l_4 l_5(\mu I_H^* + l_4)(\beta_1 I_H^* + l_1) + beta_2\sigma_2\mu\Lambda_M I_H^*)}. \end{cases} \tag{9}$$

4.1 Asymptotic stability of disease-free equilibrium

Theorem 4.1 *The only human model (6) is locally asymptotically stable (LAS) at* $\mathcal{E}_0^H$ *if* $\mathcal{R}_0^H < 1$.

Proof The Jacobian of only human model at $\mathcal{E}_0^H$ is,

$$J(\mathcal{E}_0^H) = \begin{pmatrix} l_1 & 0 & \frac{-\beta_1\Lambda_H}{l_1} & 0 \\ 0 & l_2 & \frac{-\beta_1\Lambda_H}{l_1} & 0 \\ 0 & \sigma_1 & -l_3 & 0 \\ \omega_1 & 0 & \gamma & -k_1 \end{pmatrix}.$$

The two eigenvalues are $\lambda_{1,2} = -l_1, -k_1$, having negative real parts while the other eigenvalues can be obtained from characteristic equation followed as,

$$\lambda^2 + d_1\lambda + d_2 = 0. \tag{10}$$

The coefficients of (10) are,

$$d_1 = l_2 + l_3,$$

$$d_2 = l_2 l_3(1 - \mathcal{R}_0^H).$$

Clearly, d_1 is positive and d_2 is positive only when $\mathcal{R}_0^H < 1$. So it is concluded that only the human model at $\mathcal{E}_0^H$ yields four eigenvalues having negative real parts. So, the only human model (6) at $\mathcal{E}_0^H$ is locally asymptotically stable whenever $\mathcal{R}_0^H < 1$. □

4.2 Asymptotic stability of endemic equilibria

Theorem 4.2 *The only human model is LAS at* $\mathcal{E}_H^*$ *if* $\mathcal{R}_0^H > 1$.

Proof The Jacobian of model (6) at $\mathcal{E}_H^*$ is followed as,

$$J(\mathcal{E}_H^*) = \begin{pmatrix} -N_1 - l_1 & 0 & -N_2 & 0 \\ N_1 & -l_2 & N_2 & 0 \\ 0 & \sigma_1 & -l_3 & 0 \\ \omega_1 & 0 & \gamma & -k_1 \end{pmatrix}, \tag{11}$$

where $N_1 = \beta_1 I_H^* + \beta_2 I_M^*,\ N_2 = \beta_1 S_H^*$. We get the negative eigenvalue $-k_1$, while the remaining eigenvalues can be obtained from the characteristic equation that yields,

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{12}$$

where the coefficients are of the form,

$$c_1 = l_1 + l_2 + l_3 + N_1,$$

$$c_2 = (N_1 + l_1)(l_2 + l_3) + (l_2 l_3 - N_2\sigma_1),$$

$$c_3 = l_2 l_3 N_1 + l_1(l_2 l_3 - N_2\sigma_1).$$

Clearly c_1 is positive and $c_2,\ c_3$, becomes positive if $(l_2 l_3 - N_2\sigma_1) > 0$. Thus the Routh–Hurwitz conditions [20], $c_1 > 0,\ c_2 > 0,\ c_3 > 0$ and $c_1 c_2 > c_3$ can be easily verified. If $c_i > 0$ for $i = 1, 2, 3$ then the characteristic polynomial (12) has negative real parts and the model (6) at $\mathcal{E}_H^*$ is asymptotically stable. □

4.3 Global stability of $\mathcal{E}_H^*$ using geometric approach

In this section, we use the geometric technique [21] to examine the global stability of only a human model at $\mathcal{E}_H^*$.

Theorem 4.3 *The only human model is globally asymptotically stable (GAS) at $\mathcal{E}_H^*$ if $\mathcal{R}_0^H > 1$.*

Proof The model (6) evaluated at $\mathcal{E}_H^*$ is given as

$$J(\mathcal{E}_H^*) = \begin{pmatrix} -\beta_1 I_H - \beta_2 I_M - l_1 & 0 & -\beta_1 S_H \\ \beta_1 I_H + \beta_2 I_M & -l_2 & \beta_1 S_H \\ 0 & \sigma_1 & -l_3 \end{pmatrix}.$$

Also the second additive compound matrix is

$$J^2 = \begin{pmatrix} M_1 & \beta_1 S_H & \beta_1 S_H \\ \sigma_1 & M_2 & 0 \\ 0 & M_3 & -l_2 - l_3 \end{pmatrix}, \tag{13}$$

where

$$M_1 = -\beta_1 S_H - \beta_2 I_M - l_1 - l_2,$$
$$M_2 = -\beta_1 I_H - \beta_2 I_M - l_1 - l_3,$$
$$M_3 = \beta_1 I_H + \beta_2 I_M.$$

Consider a matrix,

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{E_H}{I_H} & 0 \\ 0 & 0 & \frac{E_H}{I_H} \end{pmatrix}, \quad then \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{I_H}{E_H} & 0 \\ 0 & 0 & \frac{I_H}{E_H} \end{pmatrix}.$$

The matrix P_f shows the derivative of P in the direction of vector field f.

$$P_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{I_H E'_H - I'_H E_H}{I_H^2} & 0 \\ 0 & 0 & \frac{I_H E'_H - I'_H E_H}{I_H^2} \end{pmatrix}, \quad and \quad P_f P^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{E'_H}{I_H} - \frac{I'_H}{I_H} & 0 \\ 0 & 0 & \frac{E'_H}{I_H} - \frac{I'_H}{I_H} \end{pmatrix}.$$

Moreover, we obtained the matrix,

$$P_f J^2 P^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ M_4\sigma_1 & M_4 M_2 \frac{I_H}{E_H} & 0 \\ 0 & M_4 M_3 \frac{I_H}{E_H} & -M_4(l_2 + l_3)\frac{I_H}{E_H} \end{pmatrix}, \tag{14}$$

where $M_4 = \frac{I_H E'_H - I'_H E_H}{I_H^2}$.

$$A = P_f P^{-1} + P_f J^2 P^{-1} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \tag{15}$$

where

$$G_{11} = 0, \quad G_{12} = (0,0), \quad G_{21} = (M_4\sigma_1, 0)^T,$$

$$G_{22} = \begin{pmatrix} \frac{E'_H}{E_H} - \frac{I'_H}{I_H} + M_2M_4\frac{I_H}{E_H} & 0 \\ M_3M_4\frac{I_H}{E_H} & \frac{E'_H}{E_H} - \frac{I'_H}{I_H} - M_4(l_2+l_3)\frac{I_H}{E_H} \end{pmatrix}.$$

Let the vector $(u, v, w) \in \mathbb{R}^3$ and its norm defined by $\|.\|$.

$$\|(u,v,w)\| = max\{|u|, |v|, |w|\},$$

Consider $\nu(L)$ gives the Lozinski measure with norm already defined,

$$\nu(L) \leq sup(\phi_1, \phi_2),$$

where

$$\phi_1 = \nu(G_{11}) + |G_{12}|, \quad \phi_2 = |G_{21}| + \nu(G_{22}).$$

$|G_{21}|$ and $|G_{12}|$ shows the vector norm related to the vector l and ν, denote the Lozinski measure with respect to l norm. Then, we have

$$\nu(G_{11}) = 0, \quad |G_{12}| = sup\{0,0\} = 0.$$

Therefore, $\phi_1 = \nu(G_{11}) + |G_{12}| = 0$. Also for ϕ_2, we have

$$\begin{aligned} G_{21} &= \frac{I_H E'_H - I'_H E_H}{I_H^2}\sigma_1, \\ \nu G_{22} &= sup\Big\{M_4M_2\frac{I_H}{E_H} + \frac{E'_H}{E_H} - \frac{I'_H}{I_H} + M_4M_3\frac{I_H}{E_H},\ -M_4\frac{I_H}{E_H}(l_2+l_3) + \frac{E'_H}{E_H} - \frac{I'_H}{I_H}\Big\}, \\ &= \frac{E'_H}{E_H} - \frac{I'_H}{I_H} + max\Big\{-M_4\frac{I_H}{E_H}(l_1+l_3),\ -M_4\frac{I_H}{E_H}(l_2+l_3)\Big\}. \end{aligned}$$

$$\begin{aligned} \phi_2 &= |G_{21}| + \nu G_{22}, \\ &\leq \frac{E'_H}{I_H}\sigma_1 - \frac{I'_H E_H}{I_H^2}\sigma_1 + \frac{E'_H}{E_H} - \frac{I'_H}{I_H} + max\Big\{-M_4\frac{I_H}{E_H}(l_1+l_3),\ -M_4\frac{I_H}{E_H}(l_2+l_3)\Big\}; \end{aligned}$$

furthermore, the third equation of the model (6) gives,

$$\frac{I'_H}{I_H} = \sigma_1\frac{E_H}{I_H} - l_3, \quad \frac{E'_H}{E_H} = \frac{S_H(\beta_1 I_H + \beta_2 I_M)}{E_H} - (k_1 + \sigma_1),$$

Then, we can get

$$\begin{aligned} \phi_2 &\leq \frac{E'_H}{I_H}\sigma_1 - \frac{I'_H E_H}{I_H^2}\sigma_1 + \frac{E'_H}{E_H} - \frac{I'_H}{I_H} + max\Big\{-M_4\frac{I_H}{E_H}(l_1+l_3),\ -M_4\frac{I_H}{E_H}(l_2+l_3)\Big\}, \\ &< \frac{E'_H}{E_H} - min\Big\{M_4\frac{I_H}{E_H}(l_1+l_3),\ M_4\frac{I_H}{E_H}(l_2+l_3)\Big\}. \end{aligned}$$

As a result, for $R_0^H > 1$, the method discussed in [21] concluded that the model (6) is globally asymptotically stable at $\mathcal{E}_H^*$. □

5 Analysis of the full model

We analyzed the dynamics of the full Zika virus model (1). The disease-free equilibrium DFE of full model is $\mathcal{E}_0$ and is given as,

$$\mathcal{E}_0 = (S_H^0, E_H^0, I_H^0, R_H^0, S_M^0, E_M^0, I_M^0) = \Big(\frac{\Lambda_H}{l_1}, 0, 0, \frac{\omega\Lambda_H}{k_1 l_1}, \frac{\Lambda_M}{l_4}, 0, 0\Big). \tag{16}$$

The next generation technique [19] is used to obtain the reproduction number, such as

$$\mathbf{F} = \begin{pmatrix} 0 & \beta_1 S_H^0 & 0 & \beta_2 S_H^0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu S_M^0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} l_2 & 0 & 0 & 0 \\ -\sigma_1 & l_3 & 0 & 0 \\ 0 & 0 & l_5 & 0 \\ 0 & 0 & -\sigma_2 & l_4 \end{pmatrix}.$$

The $\rho(\mathbf{FV}^{-1})$ gives,

$$\mathcal{R}_0 = \frac{\sigma_1\beta_1\Lambda_H}{2l_1l_2l_3} + \sqrt{\frac{\sigma_1^2\beta_1^2\Lambda_H^2}{4l_1^2l_2^2l_3^2} + \frac{\mu\sigma_1\sigma_2\beta_2\Lambda_H\Lambda_M}{l_1l_2l_3l_4^2l_5}} = \frac{\mathcal{R}_0^H}{2} + \mathcal{R}_0^M, \tag{17}$$

where

$$\mathcal{R}_0^M = \sqrt{\frac{\sigma_1^2\beta_1^2\Lambda_H^2}{4l_1^2l_2^2l_3^2} + \frac{\mu\sigma_1\sigma_2\beta_2\Lambda_H\Lambda_M}{l_1l_2l_3l_4^2l_5}}, \quad \mathcal{R}_0^H = \frac{\sigma_1\beta_1\Lambda_H}{l_1l_2l_3}.$$

5.1 Local stability of $\mathcal{E}_0$ (DFE)

The Jacobian of the model (1) at $\mathcal{E}_0$ is

$$J(\mathcal{E}_0) = \begin{pmatrix} -l_1 & 0 & -\frac{\beta_1\Lambda_H}{l_1} & 0 & 0 & 0 & -\frac{\beta_2\Lambda_H}{l_1} \\ 0 & -l_2 & \frac{\beta_1\Lambda_H}{l_1} & 0 & 0 & 0 & \frac{\beta_2\Lambda_H}{l_1} \\ 0 & \sigma_1 & -l_3 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \gamma & -k_1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu\Lambda_M}{l_4} & 0 & -l_4 & 0 & 0 \\ 0 & 0 & \frac{\mu\Lambda_M}{l_4} & 0 & 0 & -l_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2 & -l_4 \end{pmatrix}.$$

The eigenvalues are $\lambda = -k_1, -l_1, -l_4$, which are clearly negative, and the remaining can be obtained from the following characteristic equation

$$\lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + m_4 = 0. \tag{18}$$

The coefficients are

$$\begin{aligned} m_1 &= l_2 + l_3 + l_4 + l_5, \\ m_2 &= l_4l_5 + (l_2 + l_3)(l_4 + l_5) + l_2l_3(1 - \mathcal{R}_0^H), \\ m_3 &= l_4l_5(l_2 + l_3) + l_2l_3(l_4 + l_5)(1 - \mathcal{R}_0^H), \\ m_4 &= l_2l_3l_4l_5(1 - \mathcal{R}_0^*), \end{aligned}$$

where

$$\mathcal{R}_0^* = \frac{\sigma_1\beta_1\Lambda_H}{l_1l_2l_3} + \frac{\mu\sigma_1\sigma_2\beta_2\Lambda_H\Lambda_M}{l_1l_2l_3l_4^2l_5} = \mathcal{R}_0^H(1 - \mathcal{R}_0) + \mathcal{R}_0^2. \tag{19}$$

The characteristic equation (18) yields four negative values if $\mathcal{R}_0 < 1$, while from Routh–Hurwitz criteria [20] $m_i > 0$, for $i = 1, 2, 3, 4$ and $m_1m_2m_3 > m_1^2m_4 + m_3^2$ satisfy. Obviously the coefficients are positive if $\mathcal{R}_0 < 1$. Hence, the system (1) at $\mathcal{E}_0$ is locally asymptotically stable.

5.2 Global stability of $\mathcal{E}_0$

The global stability of DFE is calculated by the Castillo-Chavez approach [22], the model is rewritten as,

$$\begin{aligned} \frac{dX}{dt} &= F(X, Z), \\ \frac{dZ}{dt} &= G(X, Z), \quad G(X, 0) = 0, \end{aligned} \tag{20}$$

we choose $X = (S_H, S_M) \in \mathbb{R}_+^2$ and $Z = (E_H, I_H, E_M, I_M) \in \mathbb{R}_+^4$, the DFE is $\mathcal{E}_0 = (X^0, 0)$, where $X^0 = (\frac{\Lambda_H}{l_1}, \frac{\Lambda_M}{l_4})$.

The existence of global asymptotic stability of DFE depends on two conditions described by Castillo-Chavez, needs to be satisfied.

(i). For $\frac{dX}{dt} = F(X, Z)$, X^0 is global asymptotic stable.

(ii). $G(X, Z) = MZ - \hat{G}(X, Z)$, where $\hat{G}(X, Z) \geq 0$, for $(X, Z) \in \Omega$.

where $M = D_zG(X^0, Z)$ is an G-matrix and Ω is a feasible region.

Lemma 5.1 *The fixed point $\mathcal{E}_0$ of the model (1) is GAS if $\mathcal{R}_0 < 1$ and above conditions are satisfied.*

Theorem 5.1 *The DFE of the presented model is globally asymptotic stable if $R_0 < 1$ and both above conditions are satisfied.*

Proof

$$\frac{dX}{dt} = F(X, Z) = \begin{pmatrix} \Lambda_H - \beta_1 S_H I_H - \beta_2 S_H I_M - (k_1 + \omega_1) S_H \\ \\ \Lambda_M - \mu S_M I_H - (k_2 + \omega_2) S_M \end{pmatrix},$$

after evaluating $\frac{dX}{dt}$ at $\mathcal{E}_0$, so $S^0 = (S_H^0, S_M^0)$ and $\frac{dX}{dt} = F(X, 0)$, we obtain

$$\frac{dX}{dt} = F(X, 0) = \begin{pmatrix} \Lambda_H - l_1 S_H^0 \\ \Lambda_M - l_4 S_M^0 \end{pmatrix},$$

As $t \to \infty$ and $X \to X^0$ so $X = X^0 = (S_H^0, S_M^0)$. So,

$$MZ - \hat{G}(X, Z) = \begin{pmatrix} -l_2 & \beta_1 S_H^0 & 0 & \beta_2 S_H^0 \\ -\sigma_1 & -l_3 & 0 & 0 \\ 0 & \mu S_M^0 & -l_5 & 0 \\ 0 & 0 & \sigma_2 & -l_4 \end{pmatrix} \begin{pmatrix} E_H \\ I_H \\ E_M \\ I_M \end{pmatrix} - \begin{pmatrix} \beta_1 I_H (S_H^0 - S_H) + \beta_2 I_M (S_H^0 - S_H) \\ 0 \\ \mu I_H (S_M^0 - S_M) \\ 0 \end{pmatrix},$$

where

$$M = \begin{pmatrix} -l_2 & \beta_1 S_H^0 & 0 & \beta_2 S_H^0 \\ -\sigma_1 & -l_3 & 0 & 0 \\ 0 & \mu S_M^0 & -l_5 & 0 \\ 0 & 0 & \sigma_2 & -l_4 \end{pmatrix}, \quad Z = \begin{pmatrix} E_H \\ I_H \\ E_M \\ I_M \end{pmatrix},$$

$$\hat{G}(X, Z) = \begin{pmatrix} \beta_1 I_H (S_H^0 - S_H) + \beta_2 I_M (S_H^0 - S_H) \\ 0 \\ \mu I_H (S_M^0 - S_M) \\ 0 \end{pmatrix}.$$

Clearly, $S^0 = (S_H^0, S_M^0) = (\frac{\Lambda_H}{l_1}, \frac{\Lambda_M}{l_4})$ is a bound for the entire population. While $G(X, Z)$ fulfills the condition that is $G(X, Z) = 0$ and $G(X, Z) = MZ - \hat{G}(X, Z) \geq 0$, the $\mathcal{E}_0$ is GAS. □

5.3 Endemic equilibrium of full model

The endemic equilibrium $\mathcal{E}^*$ of full model (1) is denoted by,

$$\mathcal{E}^* = (S_H^*, E_H^*, I_H^*, R_H^*, E_M^*, I_M^*), \tag{21}$$

and its expressions are,

$$\begin{cases} S_H^* = \frac{\Lambda_H l_4 l_5 (\mu I_H^* + l_4)}{I_H^* (\beta_1 l_4 l_5 (\mu I_H^* + l_4) + \beta_2 \sigma_2 \mu \Lambda_M) + l_1 l_4 l_5 (\mu I_H^* + l_4)}, \\ E_H^* = \frac{l_3 I_h^*}{\sigma_1}, \\ R_H^* = \frac{l_4 l_5 (\mu I_H^* + l_4)(\gamma I_H^* (\beta_1 I_H^* + l_1) + \omega_1 \Lambda_H + \gamma I_H^* \beta_2 \sigma_2 \mu \Lambda_M)}{k_1 (l_4 l_5 (\mu I_H^* + l_4)(\beta_1 I_H^* + l_1) + \beta_2 \sigma_2 \mu \Lambda_M I_H^*)}, \\ E_M^* = \frac{\mu \Lambda_M I_H^*}{l_5 (\mu I_H^* + l_4)}, \\ I_M^* = \frac{\sigma_2 \mu \Lambda_M I_H^*}{l_4 l_5 (\mu I_H^* + l_4)}. \end{cases} \tag{22}$$

Using (22) in the second equation of model (1), we have

$$q_1 {I^*}^2_H + q_2 I_H^* + q_3 = 0, \tag{23}$$

where

$$q_1 = \mu \beta_1 l_2 l_3 l_4 l_5,$$
$$q_2 = l_2 l_3 l_4 l_5 (\beta_1 l_4 + \mu l_1) + \mu (\beta_2 \sigma_2 \Lambda_M l_2 l_3 - \beta_1 \sigma_1 \Lambda_H l_4 l_5),$$

$$q_3 = l_1 l_2 l_3 l_4^2 l_5 (1 - \mathcal{R}_0^*),$$

In (23), q_1 is positive and q_3 become positive, when $\mathcal{R}_0 < 1$ and negative when $\mathcal{R}_0 > 1$. Therefore, $q_3 = 0$ if $\mathcal{R}_0^* = 1$ and thus we acquire a unique nonzero solution $I_H^* = -\frac{q_2}{q_1}$ for $q_2 < 0$. The above discussion concludes that our equilibria continuously depend on $\mathcal{R}_0$.

$$I_{H_1}^* = \frac{-q_2 - \sqrt{q_2^2 - 4q_1q_3}}{2q_1}, \quad I_{H_2}^* = \frac{-q_2 + \sqrt{q_2^2 - 4q_1q_3}}{2q_1}.$$

No positive solution exists for (23) if $q_3 > 0$ and either $q_2 \geq 0$ or $q_2^2 < 4q_1q_3$ and hence there exists no endemic equilibria. The following is an analysis of the preceding discussion:

Theorem 5.2 *The Zika model has:*

(i) if $q_3 < 0$ iff $\mathcal{R}_0 > 1$, then there exists a unique EE.
(ii) if $q_2 < 0 \wedge\ q_3 = 0 \vee q_2^2 - 4q_1q_3 = 0$, a unique EE exist.
(iii) if $q_3 > 0$, $q_2 < 0 \wedge Disc > 0$, then two equilibria exists.
(iv) No EE exists otherwise.

Case (iii) of above theorem incorporates the possibility of backward bifurcation for the model (1) when $\mathcal{R}_0^* < 1$. The backward bifurcation is acquired by putting $q_2^2 - 4q_1q_3 = 0$. Then the critical value $\mathcal{R}_c$ of $\mathcal{R}_0^*$ is,

$$\mathcal{R}_c = 1 - \frac{q_2^2}{4q_1 l_1 l_2 l_3 l_4^2 l_5},$$

Thus $\mathcal{R}_c < \mathcal{R}_0^*$ is equivalent to $q_2^2 - 4q_1q_3 > 0$; therefore, backward bifurcation might happen for $\mathcal{R}_0^*$ provided that $\mathcal{R}_c < \mathcal{R}_0^* < 1$. The set of considered values of Zika model (1) parameters are $k_1 = 0.19204$, $k_2 = 0.020531$, $\sigma_1 = 0.35809$, $\sigma_2 = 0.020706$, $\gamma = 0.07098$, $\omega_1 = 0.35809, \omega_2 = 0.00071429$, $\beta_1 = 0.01599$, $\beta_2 = 0.00014874$, $\Lambda_H = 0.014957$, $\Lambda_M = 119.96$, and $\mu = 0.0008134$ and the bifurcation plot is shown in Figure 2.

5.4 Existence of bifurcation

We applied the center manifold theorem [23,24] to prove the phenomenon of backward bifurcation. Let β_1 and β_2 be the bifurcation parameter, such that $\mathcal{R}_0^* = 1$ iff,

$$\beta_1 = \beta_1^* = \frac{l_1 l_2 l_3}{\sigma_1 \Lambda_H}, \tag{24}$$

$$\beta_2 = \beta_2^* = \frac{l_2 l_3 l_4^2 l_5}{\sigma_1 \Lambda_H (\beta_1 l_4^2 l_5 + \mu \sigma_2 \Lambda_M)}. \tag{25}$$

The state variables in model (1) are replaced by $S_H = x_1$, $E_H = x_2$, $I_H = x_3$, $R_H = x_4$, $S_M = x_5$, $E_M = x_6$ and $I_M = x_7$ and the vector notation is $[x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T$. Then the Zika carrier model can be expressed as $\frac{dx}{dt} = F(x)$ with $F = [f_1, f_2, f_3, f_4, f_5, f_6, f_7]^T$ and written as,

$$\begin{cases} \frac{dx_1}{dt} = f_1 = \Lambda_H - x_1(\beta_1 x_3 + \beta_2 x_7) - l_1 x_1, \\ \frac{dx_2}{dt} = f_2 = x_1(\beta_1 x_3 + \beta_2 x_7) - l_2 x_2), \\ \frac{dx_3}{dt} = f_3 = \sigma_1 x_2 - l_3 x_3, \\ \frac{dx_4}{dt} = f_4 = \gamma x_3 - k_1 x_4 + \omega_1 x_1, \\ \frac{dx_5}{dt} = f_5 = \Lambda_M - \mu x_3 x_5 - l_4 x_5, \\ \frac{dx_6}{dt} = f_6 = \mu x_3 x_5 - l_5 x_6, \\ \frac{dx_7}{dt} = f_7 = \sigma_2 x_6 - l_4 x_7. \end{cases} \tag{26}$$

After evaluating the system (26) at $\mathcal{E}_0$ with $\beta_1 = \beta_1^*$ and $\beta_2 = \beta_2^*$. We obtain,

$$J(\mathcal{E}_0) = \begin{pmatrix} -l_1 & 0 & \frac{l_2 l_3}{\sigma_1} & 0 & 0 & 0 & -\frac{l_2 l_3 l_4^2 l_5}{\sigma_1(\beta_1 l_4^2 l_5 + \mu\sigma_2\Lambda_M)} \\ 0 & -l_2 & \frac{l_2 l_3}{\sigma_1} & 0 & 0 & 0 & \frac{l_2 l_3 l_4^2 l_5}{\sigma_1(\beta_1 l_4^2 l_5 + \mu\sigma_2\Lambda_M)} \\ 0 & \sigma_1 & -l_3 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \gamma & -k_1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu\Lambda_M}{l_4} & 0 & -l_4 & 0 & 0 \\ 0 & 0 & \frac{\mu\Lambda_M}{l_4} & 0 & 0 & -l_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2 & -l_4 \end{pmatrix}.$$

First, we calculate the left and right eigenvalues denoted by $W = [w_1, w_2, w_3, w_4, w_5, w_6, w_7]$ and $V = [v_1, v_2, v_3, v_4, v_5, v_6, v_7]$, respectively.

$$w_1 = -\frac{l_2 w_2}{l_1}, \quad w_2 = w_2 > 0, \quad w_3 = \frac{\sigma_1 w_2}{l_3}, \quad w_4 = \frac{(\gamma\sigma_1 l_1 - \omega_1 l_2 l_3) w_2}{k_1 l_1 l_3},$$
$$w_5 = -\frac{\mu\Lambda_M\sigma_1 w_2}{l_3 l_4^2}, \quad w_6 = -\frac{\mu\Lambda_M\sigma_1 w_2}{l_3 l_4 l_5}, \quad w_7 = \frac{\mu\Lambda_M\sigma_1\sigma_2 w_2}{l_3 l_4^2 l_5},$$

and

$$v_1 = v_4 = v_5 = 0, \quad v_2 = v_2 > 0, \quad v_3 = \frac{l_2 v_2}{\sigma_1},$$
$$v_6 = \frac{l_2 l_3 l_4 \sigma_2 v_2}{\sigma_1(\beta_1 l_4^2 l_5 + \mu\sigma_2\Lambda_M)}, \quad v_7 = \frac{l_2 l_3 l_4 l_5 v_2}{\sigma_1(\beta_1 l_4^2 l_5 + \mu\sigma_2\Lambda_M)}.$$

The bifurcation coefficients a and b are calculated as,

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f}{\partial x_i \partial x_j}(0,0), \quad b = \sum_{k,i=1}^{n} v_k w_i \Big(\frac{\partial^2 f}{\partial x_i \partial \beta_1}(0,0) + \frac{\partial^2 f}{\partial x_i \partial \beta_2}(0,0)\Big),$$

so we have,

$$a = -\frac{l_2 v_2 \sigma_1 w_2^2(\pi_1\pi_2 + \mu\sigma_2\Lambda_M l_1 l_4 l_5)}{l_1 l_3 l_4^2 l_5(\beta_1 l_4^2 l_5 + \mu\sigma_2\Lambda_M)} < 0,$$
$$b = \frac{(l_4^2 l_5 + \mu\sigma_2\Lambda_M)\sigma_1\Lambda_H v_2 w_2}{l_1 l_3 l_4^2 l_5} > 0,$$

where

$$\pi_1 = (\beta_1 l_4^2 l_5 + \beta_2\mu\sigma_2\Lambda_M), \quad \pi_2 = (\beta_1 l_4^2 l_5 + \mu\sigma_2\Lambda_M).$$

As the bifurcation coefficient, b, is positive, the existence of backward bifurcation for a proposed model can be predicted from the sign of a [24], if $a > 0, b > 0$ then the system will undergo backward bifurcation. Here in this case $a < 0$ and no backward bifurcation exists for model (1). In backward bifurcation, the endemic equilibrium coexists with stable disease-free equilibrium between which one is stable and the other is unstable if $\mathcal{R}_0 < 1$. The biological importance of the phenomenon of backward bifurcation for Zika model (1), the condition $\mathcal{R}_0 < 1$ is necessary but not sufficient for the eradication of Zika virus from society and depends on the initial size of the population.

5.5 Local stability of EE

Theorem 5.3 *The Zika model (1) is LAS at endemic equilibrium (EE) if* $\mathcal{R}_0 > 1$.

Proof The model (1) has Jacobian matrix at endemic equilibrium in the form,

$$J(\mathcal{E}^*) = \begin{pmatrix} -N_1 - l_1 & 0 & -N_2 & 0 & 0 & 0 & -N_3 \\ N_1 & -l_2 & N_2 & 0 & 0 & 0 & N_3 \\ 0 & \sigma_1 & -l_3 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \gamma & -k_1 & 0 & 0 & 0 \\ 0 & 0 & -N_4 & 0 & -N_5 - l_4 & 0 & 0 \\ 0 & 0 & N_4 & 0 & N_5 & -l_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2 & -l_4 \end{pmatrix}.$$

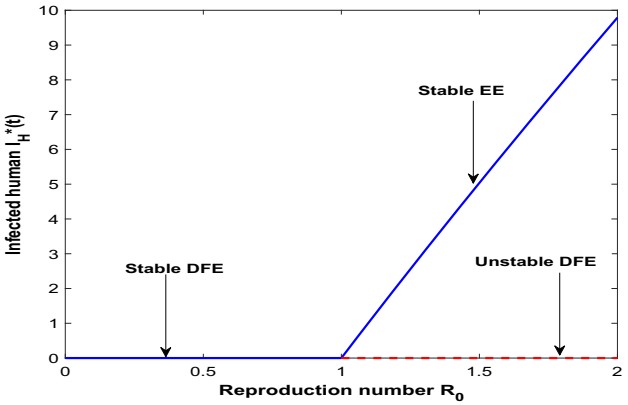


Fig. 2 Backward bifurcation plot for the Zika model

where

$$N_1 = \beta_1 I_H^* + \beta_2 I_M^*, \quad N_2 = \beta_1 S_H*, \quad N_3 = \beta_2 S_H^*, \quad N_4 = \mu S_M^*, \quad N_5 = \mu I_H^*.$$

The eigenvalue $-k_1$ is negative while the characteristic equation gives,

$$\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0. \tag{27}$$

The constants are,

$$\begin{aligned}
a_1 &= l_1 + l_2 + l_3 + 2l_4 + l_5 + N_1 + N_5,\\
a_2 &= (l_2 + l_3 + l_5 + N_5 + 2l_4)l_1 - N_2\sigma_1 + \big(2(l_2 + l_3 + l_5 + N_1) + N_5 + l_4\big)l_4\\
&\quad + l_2(l_3 + l_5 + N_1 + N_5) + l_3(l_5 + N_1 + N_5) + l_5(N_1 + N_5) + N_1N_5,\\
a_3 &= -N_2N_6\sigma_1 + \big((2l_5 + N_5)l_4 + l_5N_5\big)l_1 + l_3(N_5N_{10} + l_4N_7 + l_5N_8) + (N_9 + l_3)l_4^2\\
&\quad + l_2(N_5N_{10} + l_4N_7 + l_5N_8 + l_3(l_5 + N_8) + l_4^2) + l_4l_5(2N_1 + N_5) + N_1N_5(l_4 + l_5),\\
a_4 &= -\sigma_1\Big(N_2\big(N_5(l_1 + l_4) + l_5(l_1 + 2l_4 + N_5) + l_4(2l_1 + l_4) + N_3N_4\sigma_2\big)\Big)\\
&\quad + l_2\Big(l_3\big(l_4(2(l_1 + N_1) + N_5) + l_5N_8 + l_5N_{10} + l_4^2\big) + N_{10}N_{11}l_4\Big) + l_4l_5N_{10}N_{11}\\
&\quad + l_2l_5\big((N_7 + l_4)l_4 + N_5N_{10}\big) + l_3\Big(N_{10}N_{11}l_4 + l_5\big((N_7 + l_4)l_4 + N_5N_{10}\big)\Big),\\
a_5 &= -\sigma_1\Big(N_3N_4\sigma_2(l_1 + l_4) + l_1l_4N_2N_{11} + l_5N_2\big(N_5(l_1 + l_4) + l_4(l_4 + 2l_5)\big)\Big)\\
&\quad + l_2\Big(l_3\big(l_5(2N_{10} + N_5) + N_5N_{10}l_4^2\big) + N_{10}N_{11}l_4\Big) + l_4l_5N_{10}N_{11} + N_{10}N_{11}l_3l_4),\\
a_6 &= l_4\big(l_2l_3l_5N_{10}N_{11} - l_1\sigma_1(N_3N_4\sigma_2 + l_5N_2N_{11})\big),
\end{aligned}$$

where

$$\begin{aligned}
&N_6 = (l_1 + 2l_4 + l_5 + N_5), \quad N_7 = \big(2(N_1 + l_1) + N_5\big), \quad N_8 = (l_1 + 2l_4 + N_1 + N_5),\\
&N_9 = (l_1 + l_5 + N_1), \quad N_{10} = (l_1 + N_1), \quad N_{11} = (l_4 + N_5).
\end{aligned}$$

Now the Hurwitz matrices are,

$$H_1 = a_1, \quad H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \quad H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, \quad H_4 = \begin{pmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & 1 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{pmatrix},$$

$$H_5 = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix}, \quad H_6 = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\ 0 & 0 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & 0 & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{pmatrix}.$$

The polynomial given by (27) gives six eigenvalues having negative real parts if $a_i > 0$ for $i = 1, 2, ...6$, and the corresponding Routh–Hurwitz criteria satisfied. So that, the model (1) is LAS at $\mathcal{E}^*$ if $\mathcal{R}_0 > 1$. □

5.6 Global stability of EEP

In this section, we prove that the Zika model is globally asymptotically stable (GAS) at the endemic equilibrium point.

$$\begin{cases} \Lambda_H = (\beta_1 I_H^* + \beta_2 I_M^*)S_H^* + l_1 S_H^*, \\ l_2 E^* = (\beta_1 I_H^* + \beta_2 I_M^*)S_H^* + l_1 S_H^*, \\ \sigma_1 E_H^* = l_3 I_H^*, \\ \frac{l_2 l_3}{\sigma_1} I_H^* = (\beta_1 I_H^* + \beta_2 I_M^*)S_H^*, \\ \Lambda_M = \mu S_M^* I_H^* + l_4 S_M^*, \\ \mu S_M^* I_H^* = l_5 E_M^*, \\ \sigma_2 E_M^* = l_4 I_M^*, \\ \mu S_M^* I_H^* = \frac{l_4 l_5}{\sigma_2} I_H^*. \end{cases} \tag{28}$$

Theorem 5.4 *If* $\mathcal{R}_0 > 1$ *and*

$$\left[6 - \frac{S_H}{S_H^*} + \left(1 - \frac{S_H}{S_H^*}\frac{E_H^*}{E_H}\right)\frac{(\beta_1 I_H + \beta_2 I_M)}{(\beta_1 I_H^* + \beta_2 I_M^*)} - \frac{E_H I_H^*}{E_H^* I_H} - \frac{S_M^*}{S_M} - \frac{I_M}{I_M^*} - \frac{E_M I_M^*}{E_M^* I_M} - \frac{S_M E_M^* I_H}{S_M^* I_H^* E_M}\right] \leq 0.$$

Then $\mathcal{E}_0^*$ *is GAS.*

Proof The Lyapunov function is of the form,

$$\begin{aligned} \mathbb{V} = &\int_{S_H^*}^{S_H} \left(1 - \frac{S_H^*}{x}\right)dx + \int_{E_H^*}^{E_H} \left(1 - \frac{E_H^*}{x}\right)dx + \int_{I_H^*}^{I_H} \left(1 - \frac{I_H^*}{x}\right)dx \\ &+ \int_{S_M^*}^{S_M} \left(1 - \frac{S_M^*}{x}\right)dx \int_{E_M^*}^{E_M} \left(1 - \frac{E_M^*}{x}\right)dx + \int_{I_M^*}^{I_M} \left(1 - \frac{I_M^*}{x}\right)dx. \end{aligned} \tag{29}$$

So multiplying with some specific term and taking the derivative of the above equation we have,

$$\begin{aligned} \mathbb{V}' = &\mu S_M^* I_H^* \left[\left(1 - \frac{S_H^*}{S_H}\right)S_H' + \left(1 - \frac{E_H^*}{E_H}\right)E_H' + \frac{(\beta_1 I_H^* + \beta_2 I_M^*)S_H^*}{\sigma_1 E_h^*}\left(1 - \frac{I_H^*}{I_H}\right)I_H'(t)\right] \\ &+ S_H^*(\beta_1 I_H^* + \beta_2 I_M^*)\left[\left(1 - \frac{S_M^*}{S_M}\right)S_M' + \left(1 - \frac{E_m^*}{E_m}\right)E_M'(t)\right. \\ &\left. + \frac{\mu S_M^* I_H^*}{\sigma_2 E_M^*}\left(1 - \frac{I_M^*}{I_M}\right)I_M'(t)\right]. \end{aligned} \tag{30}$$

Initially we compute the following results, by using the solution (28),

$$\begin{aligned} \left(1 - \frac{S_H^*}{S_H}\right)S_H' &= \left(1 - \frac{S_H^*}{S_H}\right)\left[\Lambda_H - l_1 S_H - S_H(\beta_1 I_H + \beta_2 I_M)\right], \\ &= \left(1 - \frac{S_H^*}{S_H}\right) + \left[(\beta_1 I_H^* + \beta_2 I_M^*)S_H^* - S_H(\beta_1 I_H + \beta_2 I_M) + l_1 S_H^* - l_1 S_H)\right], \\ &= S_H^*(\beta_1 I_H^* + \beta_2 I_M^*)\left(1 - \frac{S_H^*}{S_H} + \left(1 - \frac{S_H}{S_H^*}\right)\frac{(\beta_1 I_H + \beta_2 I_M)}{(\beta_1 I_H^* + \beta_2 I_M^*)}\right) \\ &\quad + \left(2 - \frac{S_H}{S_H^*} - \frac{S_H^*}{S_H}\right)l_1 S_H^*. \end{aligned} \tag{31}$$

$$\begin{aligned} \left(1 - \frac{E_H^*}{E_H}\right)E_h' &= \left(1 - \frac{E_H^*}{E_H}\right)\left[S_H(\beta_1 I_H + \beta_2 I_M) - l_2 E_H\right], \\ &= S_H(\beta_1 I_H + \beta_2 I_M) - S_H(\beta_1 I_H + \beta_2 I_M)\frac{E_H^*}{E_H} - l_2 E_H + l_2 E_H^*, \\ &= S_H^*(\beta_1 I_H^* + \beta_2 I_M^*)\left(1 - \frac{E_H}{E_H^*} + \left(1 - \frac{E_H^*}{E_H}\right)\frac{S_H}{S_H^*}\frac{(\beta_1 I_H + \beta_2 I_M)}{(\beta_1 I_H^* + \beta_2 I_M^*)}\right). \end{aligned} \tag{32}$$

$$\frac{(\beta_1 I_H^* + \beta_2 I_M^*)S_H^*}{\sigma_1 E_h^*}\Big(1 - \frac{I_H^*}{I_H}\Big)I_H' = \frac{(\beta_1 I_H^* + \beta_2 I_M^*)S_H^*}{\sigma_1 E_h^*}\Big(1 - \frac{I_H^*}{I_H}\Big)\big[\sigma_1 E_H - l_3 I_H\big],$$
$$= (\beta_1 I_H^* + \beta_2 I_M^*)S_H^*\Big(1 - \frac{I_H}{I_H^*} - \frac{E_H I_H^*}{E_H^* I_H} + \frac{E_H}{E_H^*}\Big). \tag{33}$$

$$\Big(1 - \frac{S_M^*}{S_M}\Big)S_M' = \Big(1 - \frac{S_M^*}{S_M}\Big)\Big[\Lambda_M - \mu S_M I_H - l_4 S_M\Big],$$
$$= \Big(1 - \frac{S_M^*}{S_M}\Big)\Big[\mu S_M^* I_H^* - \mu S_M I_H + l_4 S_M^* - l_4 S_M\Big],$$
$$= \Big(2 - \frac{S_M}{S_M^*} - \frac{S_M^*}{S_M}\Big)l_4 S_M^* + \mu S_M^* I_H^*\Big(1 - \frac{S_M^*}{S_M} + \frac{I_H}{I_H^*} + \frac{S_M}{S_M^*}\frac{I_H}{I_H^*}\Big). \tag{34}$$

$$\Big(1 - \frac{E_M^*}{E_M}\Big)E_M' = \Big(1 - \frac{E_M^*}{E_M}\Big)\Big[\mu S_M I_H - l_5 E_M\Big],$$
$$= \mu S_M I_H - \mu S_M I_H \frac{E_M^*}{E_M} + \mu S_M^* I_H^* - \frac{\mu S_M^* I_H^*}{E_M^*}E_M,$$
$$= \mu S_M^* I_H^*\Big(1 - \frac{E_M}{E_M^*} - \frac{S_M I_H E_M^*}{S_M^* I_H^* E_M} + \frac{S_M I_H}{S_M^* I_H^*}\Big). \tag{35}$$

$$\frac{\mu S_M^* I_H^*}{\sigma_2 E_M^*}\Big(1 - \frac{I_M^*}{I_M}\Big)I_M' = \frac{\mu S_M^* I_H^*}{\sigma_2 E_M^*}\Big(1 - \frac{I_M^*}{I_M}\Big)\Big[\sigma_2 E_M - l_4 I_M\Big],$$
$$= \mu S_M^* I_H^*\Big(1 - \frac{I_M}{I_M^*} - \frac{E_M I_M^*}{E_M^* I_M} + \frac{E_M}{E_M^*}\Big). \tag{36}$$

Therefore (30) follows the results from (31)–(36),

$$\mathbb{V}' = \mu S_M^* I_H^* S_H^*(\beta_1 I_H^* + \beta_2 I_M^*)$$
$$\times\Big(6 - \frac{S_H}{S_H^*} + \Big(1 - \frac{S_H E_H^*}{S_H^* E_H}\Big)\frac{(\beta_1 I_H + \beta_2 I_M)}{(\beta_1 I_H^* + \beta_2 I_M^*)} - \frac{E_H I_H^*}{E_H^* I_H} - \frac{S_M^*}{S_M} - \frac{S_M I_H E_M^*}{S_M^* I_H^* E_M} - \frac{I_M}{I_M^*} - \frac{E_M I_M^*}{E_M^* I_M}\Big)$$
$$+\mu l_1 S_M^* I_H^* S_H^*\Big(2 - \frac{S_H}{S_H^*} - \frac{S_H^*}{S_H}\Big) + l_4 S_M^* S_H^*(\beta_1 I_H^* + \beta_2 I_M^*)\Big(2 - \frac{S_M}{S_M^*} - \frac{S_M^*}{S_M}\Big). \tag{37}$$

In (37), it is obvious that,

$$\Big(2 - \frac{S_H}{S_H^*} - \frac{S_H^*}{S_H}\Big) \le 0, \quad \Big(2 - \frac{S_M}{S_M^*} - \frac{S_M^*}{S_M}\Big) \le 0,$$

and

$$\Big[6 - \frac{S_H}{S_H^*} + \Big(1 - \frac{S_H E_H^*}{S_H^* E_H}\Big)\frac{(\beta_1 I_H + \beta_2 I_M)}{(\beta_1 I_H^* + \beta_2 I_M^*)} - \frac{E_H I_H^*}{E_H^* I_H} - \frac{S_M^*}{S_M} - \frac{S_M I_H E_M^*}{S_M^* I_H^* E_M} - \frac{I_M}{I_M^*} - \frac{E_M I_M^*}{E_M^* I_M}\Big] \le 0.$$

Thus $\mathbb{V}' \le 0$, the $\mathcal{E}^*$ is GAS by LaSalle's invariance principle when $\mathcal{R}_0 > 1$. □

5.7 Sensitivity analysis

The sensitivity analysis is performed to investigate the most sensitive parameter and shows its impact on the transmission and persistence of the disease. The obtained information enables the public health authorities to take necessary action to eliminate the disease and control its spread. The direct differentiation approach is applied that examined the sensitivity of $\mathcal{R}_0$ about model parameters.

$$\mathbb{S} = \frac{\mathcal{P}_i}{|\mathcal{R}_0|} \times \frac{\partial \mathcal{R}_0}{\partial \mathcal{P}_i}.$$

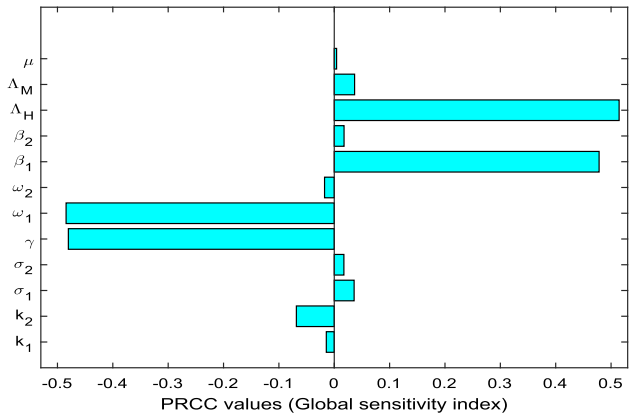


Fig. 3 Global sensitivity for the parameters in $\mathcal{R}_0$ through PRCC

A reasonable variation was observed by introducing a slight change in the most sensitive parameter. The parameters involved in the sensitivity of $\mathcal{R}_0$ are $\beta_1, \beta_2, \Lambda_H, \Lambda_M, \mu, \sigma_1, \sigma_2, \omega_1, \omega_2, \gamma, k_1$, and k_2. The sensitivity of these parameters suggests possible prevention measures such as decreasing the biting rate of mosquitoes and covering a body. The sensitivity of $\mathcal{R}_0$ concerning these parameter yields,

$$
\begin{aligned}
\mathbb{S}_{\beta_1} &= \frac{\beta_1}{R_0}\frac{\partial \mathcal{R}_0}{\partial \beta_1} = \frac{1}{2}\frac{\mathcal{R}_H}{\mathcal{R}_M} > 0,\\
\mathbb{S}_{\beta_2} &= \frac{\beta_2}{R_0}\frac{\partial \mathcal{R}_0}{\partial \beta_m} = \frac{1}{2}\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) > 0,\\
\mathbb{S}_{\Lambda_H} &= \frac{\Lambda_H}{R_0}\frac{\partial \mathcal{R}_0}{\partial \Lambda_H} = \frac{1}{2}\Big(1 + \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) > 0,\\
\mathbb{S}_{\Lambda_M} &= \frac{\Lambda_M}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \Lambda_M} = \frac{1}{2}\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) > 0,\\
\mathbb{S}_{\mu} &= \frac{\mu}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \mu} = \frac{1}{2}\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) > 0,\\
\mathbb{S}_{\sigma_1} &= \frac{\sigma_1}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \sigma_1} = \frac{1}{2}\Big(1 - \frac{\sigma_1}{k_1+\sigma_1}\Big)\Big(1 + \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) > 0,\\
\mathbb{S}_{\sigma_2} &= \frac{\sigma_2}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \sigma_2} = \frac{1-\sigma_2}{2}\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) > 0,\\
\mathbb{S}_{\omega_1} &= \frac{\omega_1}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \omega_1} = -\frac{\omega_1}{4(k_1+\omega_1)}\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) < 0,\\
\mathbb{S}_{\omega_2} &= \frac{\omega_2}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \omega_2} = -\frac{\omega_2}{2}\Big(1 + \frac{k_2+\omega_2}{k_2+\sigma_2+\omega_2}\Big)\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) < 0,\\
\mathbb{S}_{\gamma} &= \frac{\gamma}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \gamma} = -\frac{\gamma}{2(k_1+\gamma)}\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) < 0,\\
\mathbb{S}_{k_1} &= \frac{k_1}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial k_1} = -\frac{k_1}{2}\Big(\frac{1}{k_1+\omega_1} + \frac{2k_1+\omega_1+\sigma_1}{(k_1+\sigma_1)(k_1+\gamma)}\Big)\Big(1 + \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) < 0,\\
\mathbb{S}_{k_2} &= \frac{k_2}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial k_2} = -\frac{k_2}{2}\Big(1 + \frac{k_2+\omega_2}{k_2+\sigma_2+\omega_2}\Big)\Big(1 - \frac{\mathcal{R}_H}{2\mathcal{R}_M}\Big) < 0,
\end{aligned}
$$

where $\mathbb{S}_{\beta_1}, \mathbb{S}_{\beta_2}, \mathbb{S}_{\Lambda_H}, \mathbb{S}_{\Lambda_M}, \mathbb{S}_{\mu}, \mathbb{S}_{\sigma_1}, \mathbb{S}_{\sigma_2}, \mathbb{S}_{\omega_1}, \mathbb{S}_{\omega_2}, \mathbb{S}_{k_1}, \mathbb{S}_{k_2}$ and $\mathbb{S}_{\gamma}$ denote the normalizer of sensitivity index with respect to $\beta_1, \beta_2, \Lambda_H, \Lambda_M, \mu, \sigma_1, \sigma_2, \omega_1, \omega_2, k_1, k_2$ and γ, respectively. As a result increase in positive parameters may cause increase in $\mathcal{R}_0$ and increase in negative one causes decrease in $\mathcal{R}_0$. The global sensitivity graph is shown in Fig. 3.

6 Optimal control problem

This section presents an optimal control problem for the model (1). The aim of the optimal is to minimize Zika virus infection by providing three control strategies. The suggested controls for our model include:

- $u_1(t)$ control minimizing contact between mosquitoes and humans by wearing light colored clothes, using bed nets and window screens.

Table 1 Model parameters values with description.

Parameter	Description	Value/week	Refrence
Λ_H	Birth rate of Human	0.014957	Assumed
Λ_M	Recruitment rate of mosquitoes	119.96	Assumed
β_1	Contact rate b/w S_H and I_H	0.01599	Fitted
β_2	Transmission rate b/w S_H and I_M	0.00014874	Fitted
ω_1	Awareness rate in the host population	0.35809	Fitted
ω_2	Constant rate of effective mosquito control	0.00071429	Fitted
σ_1	Progress rate from E_H to I_H	0.35809	Fitted
σ_2	Rate at which E_M will become I_M	0.020706	Fitted
k_1	Mortality rate of Human	0.00019204	Fitted
k_2	Natural death rate of mosquitoes	0.020531	Fitted
γ	Humans recovery rate	0.07098	Fitted
μ	Probability of transmission per biting of S_M with I_H	0.08134	Fitted

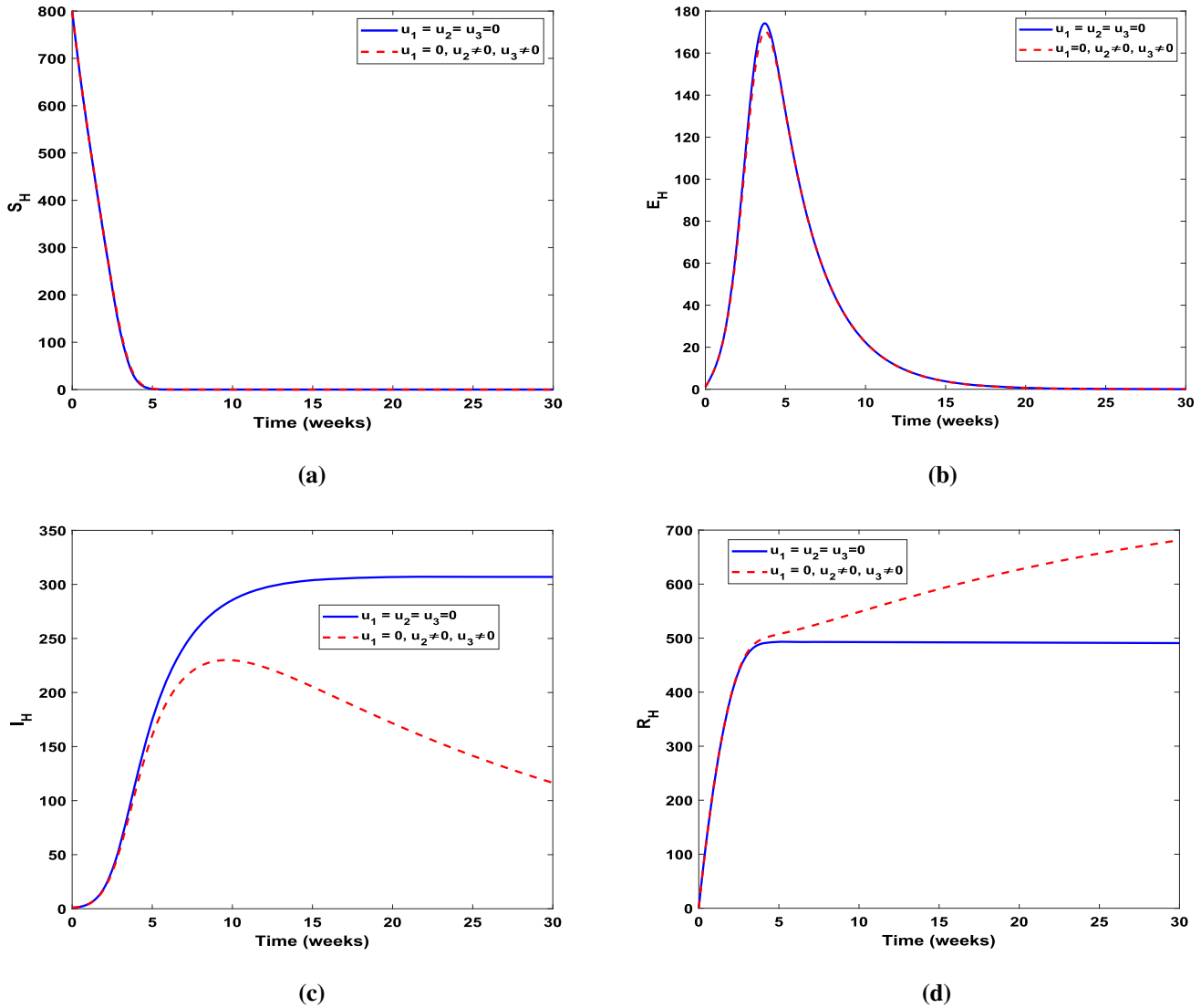


Fig. 4 Prevention through treatment and insecticides spray

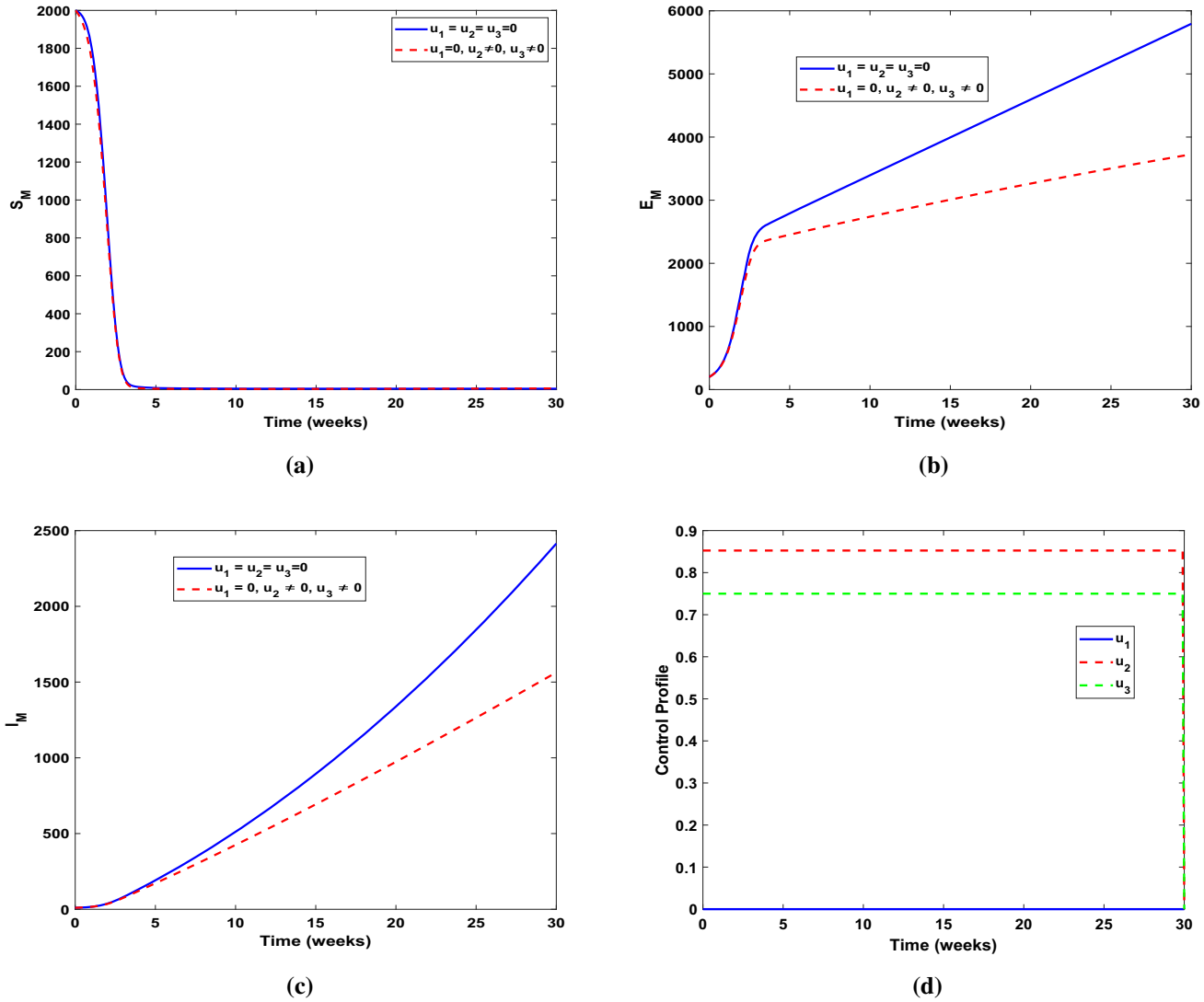


Fig. 5 The implementation of strategy $u_2 = u_3 \neq 0$ on mosquitoes compartment and control profile

- $u_2(t)$ increasing auto immunity through enough rest and taking WHO proposed medicine.
- $u_3(t)$ measures the increase of death rate of mosquitoes by spraying of insecticide.

Applying these three controls, we can rewrite the Zika virus model (1) in the form,

$$\begin{cases} \frac{dS_H}{dt} = \Lambda_H - (1-u_1)S_H\left(\beta_1 I_h + \beta_2 I_M\right) - (k_1+\omega_1)S_H, \\ \frac{dE_H}{dt} = (1-u_1)S_H\left(\beta_1 I_H + \beta_2 I_M\right) - (k_1+\sigma_1)E_H, \\ \frac{dI_H}{dt} = \sigma_1 E_H - (k_1+u_2\gamma)I_H, \\ \frac{dR_H}{dt} = u_2\gamma I_H - k_1 R_H + \omega_1 S_H, \\ \frac{dS_M}{dt} = \Lambda_M - (1-u_1)\mu S_M I_H - u_3(k_2+\omega_2)S_M, \\ \frac{dE_M}{dt} = (1-u_1)\mu S_M I_H - u_3(k_2+\omega_2)E_M - \sigma_2 E_M, \\ \frac{dI_M}{dt} = \sigma_2 E_M - u_3(k_2+\omega_2)I_M, \end{cases} \tag{38}$$

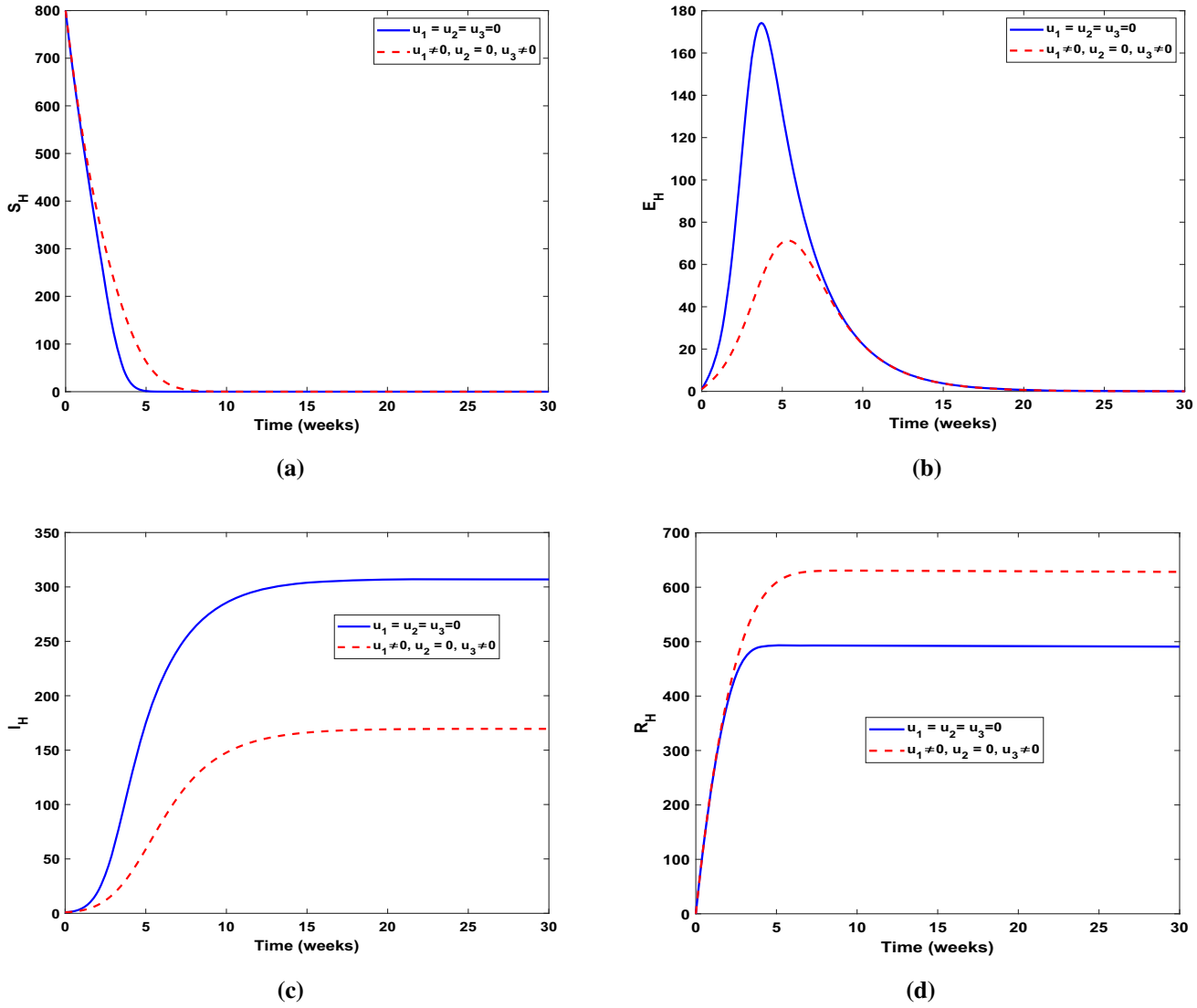


Fig. 6 Prevention through bed nets and insecticide spray

having initial conditions. The three control parameters are $u(t) = u_i \in U$ for control problem (38) associated with the state variables that is bounded above and measure with,

$$U(t) = (u_1, u_2, u_3) : u_i \; is \; Lebesgue \; measurable \; in \; [0, 1],$$
$$0 \leq u_i(t) \leq 1, \; t \in [0, T], \; i = 1, 2, 3, \tag{39}$$

where the objective functional is,

$$\mathbb{J}(u_1, u_2, u_3) = \int_0^T \left[AE_H + BI_H + CE_M + DI_M + \frac{1}{2}(Eu_1^2 + Fu_2^2 + Gu_3^2) \right]. \tag{40}$$

While A, B, C, and D are scalarizing or balancing constants associated with suggested variables of the objective function, the associated cost factor is represented by E, F, and G, which shows, respectively, the use of full clothes and bed nets to protect yourself from mosquitoes, treatment of infected individuals with best available treatment including rest and WHO recommended medicine, spraying insecticide on mosquitoes and their larva to stop their further reproduction. We find an optimal control $u_i^*(t)$, for $i = 1, 2, 3$ such that,

$$\mathbb{J}(u_1^*, u_2^*, u_2^*) = \min_U \mathbb{J}(u_1, u_2, u_3). \tag{41}$$

The existence of control system (38) can be proved by writing it as,

$$Z' = G^* Z + F(Z), \tag{42}$$

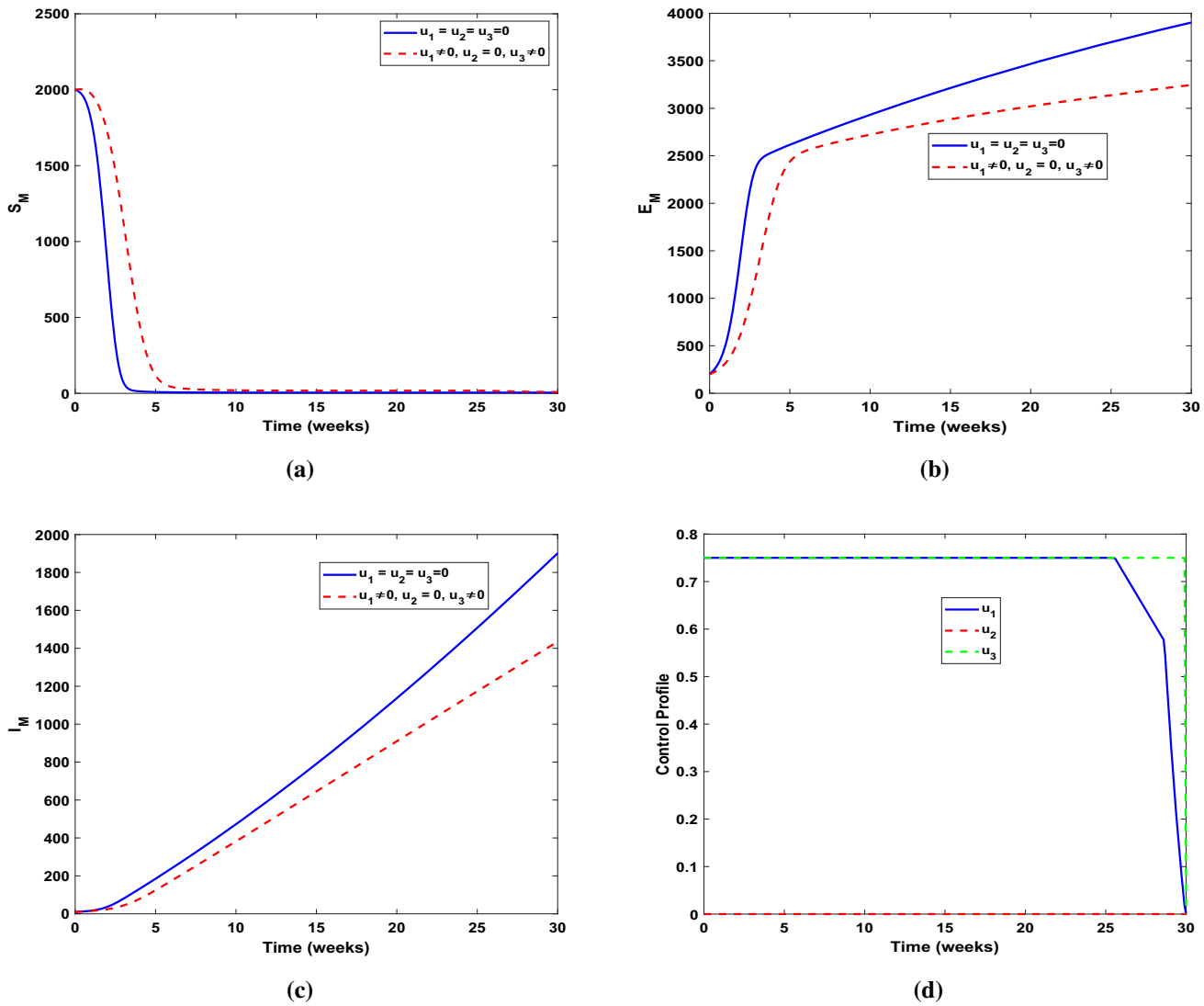


Fig. 7 Implementation of strategy $u_1 = u_3 \neq 0$ on mosquitoes and control profile

where

$$Z = \begin{pmatrix} S_H \\ E_H \\ I_H \\ R_H \\ S_M \\ E_M \\ I_M \end{pmatrix}, \quad G^* = \begin{pmatrix} -l_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -l_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_1 & -k_1 - u_2\gamma & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & u_2\gamma & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u_3 l_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -u_3 l_4 - \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2 & -u_3 l_4 \end{pmatrix},$$

and

$$F(Z) = \begin{pmatrix} \Lambda_H - (1-u_1)S_H(\beta_1 I_H + \beta_2 I_M) \\ (1-u_1)S_H(\beta_1 I_H + \beta_2 I_M) \\ 0 \\ 0 \\ -(1-u_1)\mu S_M I_H \\ (1-u_1)\mu S_M I_H \\ 0 \end{pmatrix}.$$

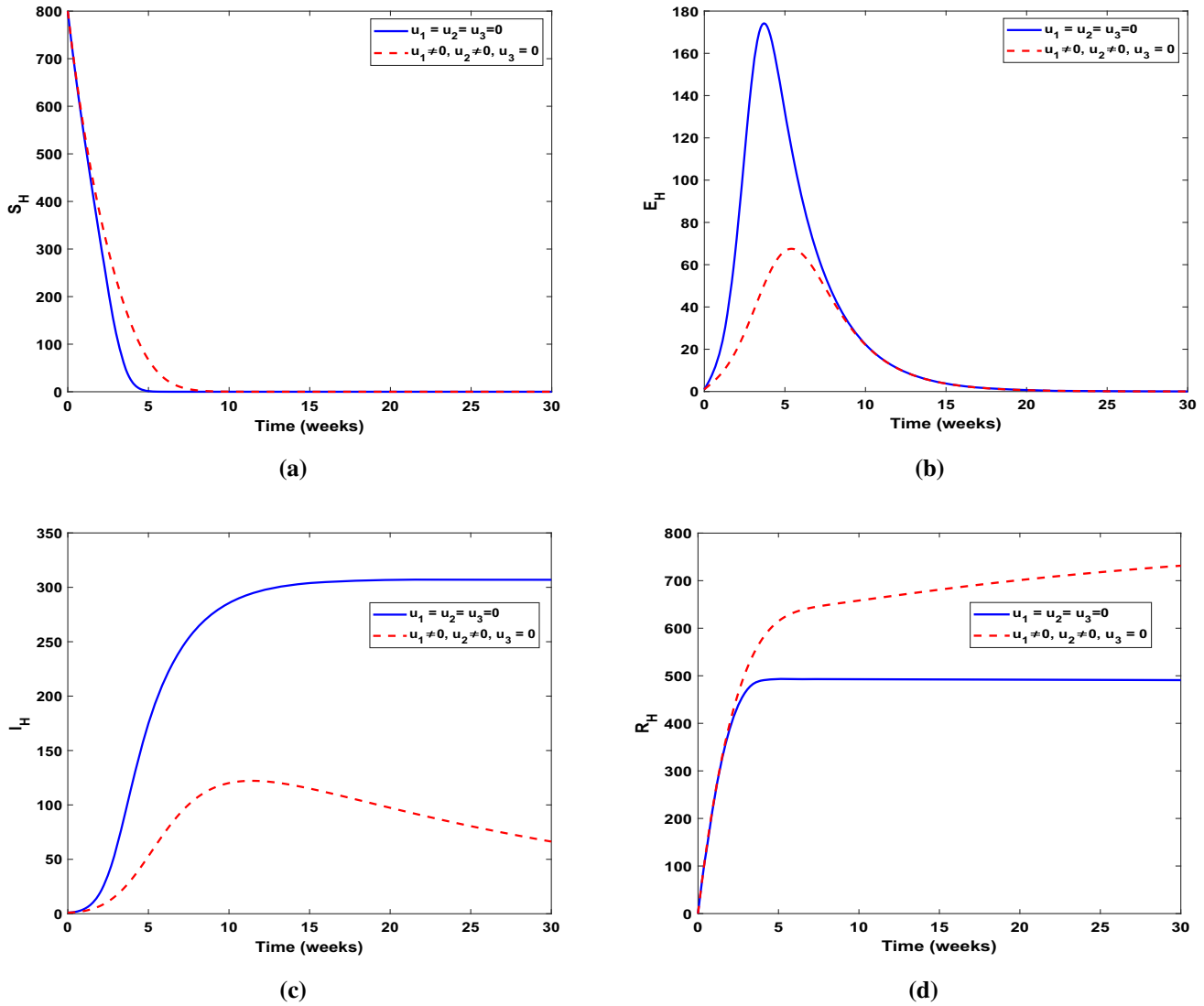


Fig. 8 Prevention through bed nets and treatment

We set

$$H(Z) = G^* Z + F(Z) \tag{43}$$

In above equation the term $F(Z)$ satisfies,

$$\begin{aligned} |F(Z_1) - F(Z_2)| \leq\ & Q_1|S_{1H}(t) - S_{2H}(t)| + Q_2|E_{1H}(t) - E_{2H}(t)| + Q_3|I_{1H}(t) - I_{2H}(t)| \\ & + Q_4|R_{1H}(t) - R_{2H}(t)| + Q_5|S_{1M}(t) - S_{2M}(t)| + Q_6|E_{1M}(t) - E_{2M}(t)| \\ & + Q_7|I_{1M}(t) - I_{2M}(t)|, \\ \leq\ & Q\Big\{|S_{1H}(t) - S_{2H}(t)| + |E_{1H}(t) - E_{2H}(t)| + |I_{1H}(t) - I_{2H}(t)| \\ & + |R_{1H}(t) - R_{2H}(t)| + |S_{1M}(t) - S_{2M}(t)| + |E_{1M}(t) - E_{2M}(t)| \\ & + |I_{1M}(t) - I_{2M}(t)|\Big\}, \end{aligned}$$

where $Q = max(Q_i)$ for $i = 1, 2, ...7$ is a positive constant and is free of the state variables. Further we have

$$|F(Z_1) - F(Z_2)| \leq Q|Z_1 - Z_2|,$$

where $Q = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + \|K\| < \infty$, the above results show that Lipschitz function is uniformly continuous. Next Pontryagin's maximum principle [25] is applied to obtain conditions necessary for optimal control problem (38).

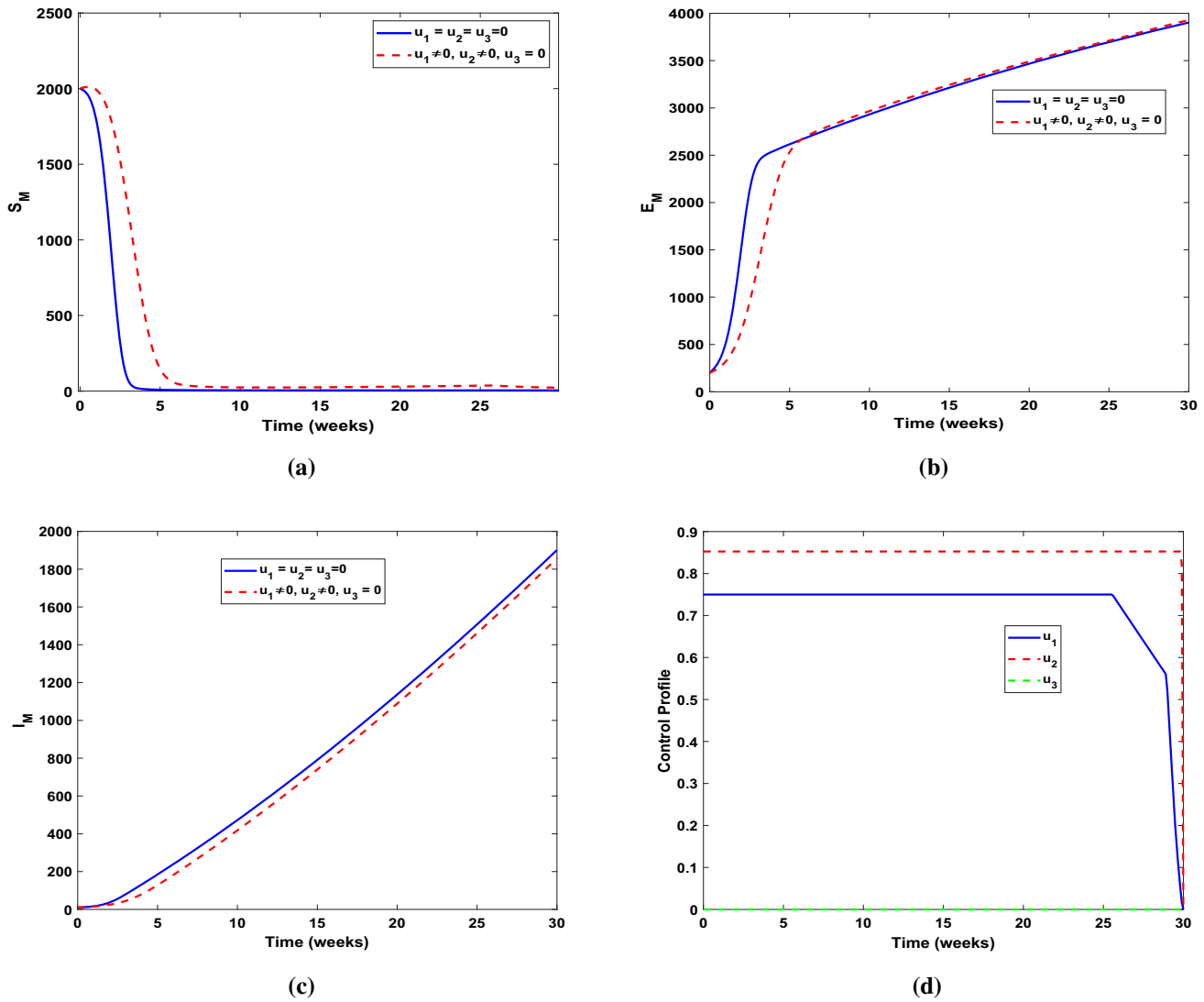


Fig. 9 The impact of coupled strategy $u_1 = u_2 \neq 0$

6.1 Existence of solution

The existence and boundedness of control system (38) are shown by using the results [26], if the following conditions satisfy:

i. The state and control variables are nonempty.
ii. The set of control variables given by (39) is closed and convex.
iii. The equations on the control problem right side are continuous, bounded, and represented as a linear function of u, with coefficients that depend on state variables and time.
iv. There exist positive constants θ_1, θ_2 and $p > 1$ that makes the integrand in (40) convex and satisfy.

$$L(y, u, t) \geq \theta_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^{\frac{p}{2}} - \theta_2.$$

Using the results [27], the above conditions (i-iv) are easily satisfied. The condition (iii) holds due to bilinear property of control system in control variables and condition (iv) satisfies by writing,

$$AE_H + BI_H + CE_M + DI_M + \frac{1}{2}(Eu_1^2 + Fu_2^2 + Gu_3^2) \geq \vartheta(|u_1|^2 + |u_2|^2 + |u_3|^2)^{\frac{r}{2}} - \vartheta,$$

where $A, B, C, D, E, F, G > 0$ and $p > 1$, so we come across the following results.

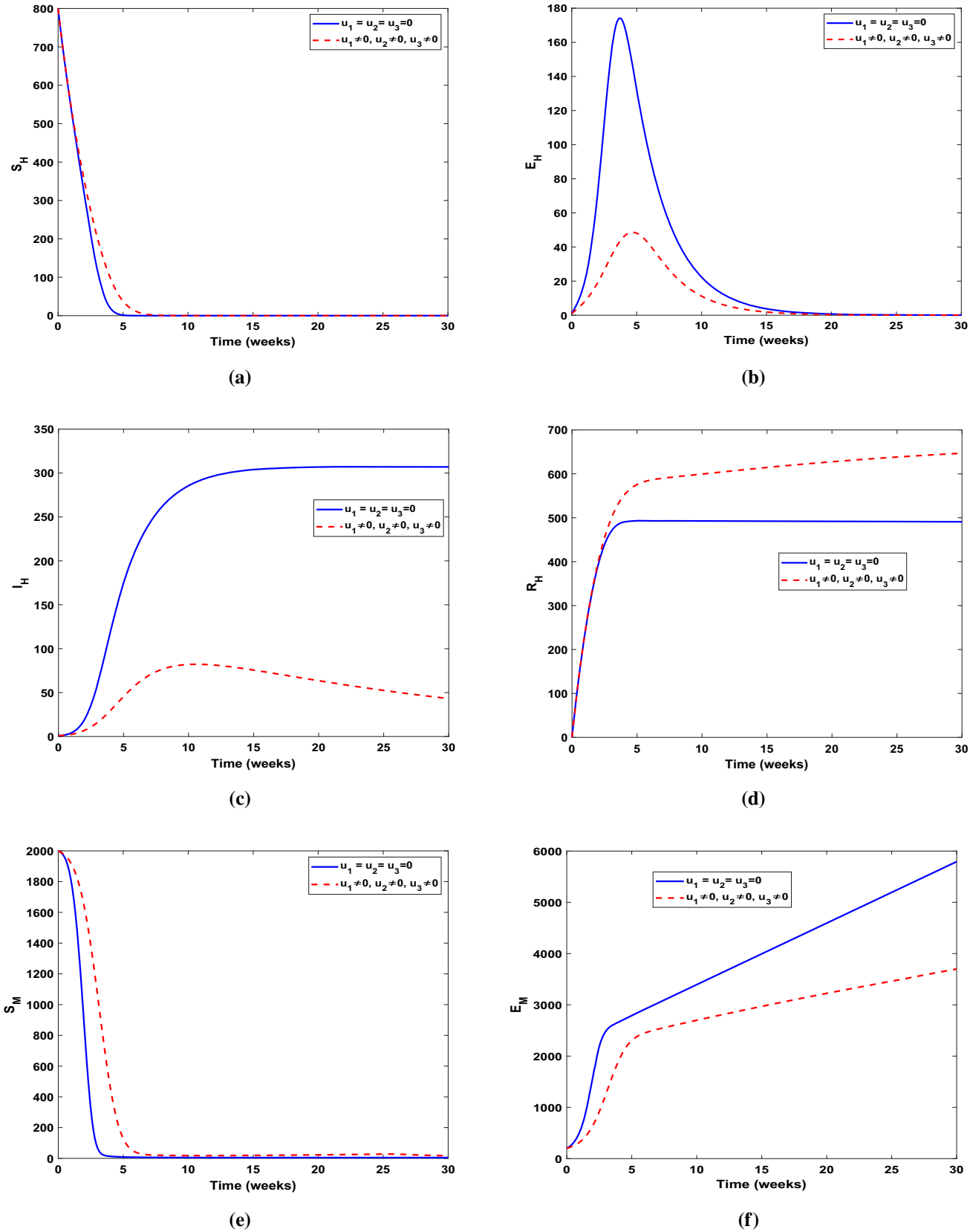
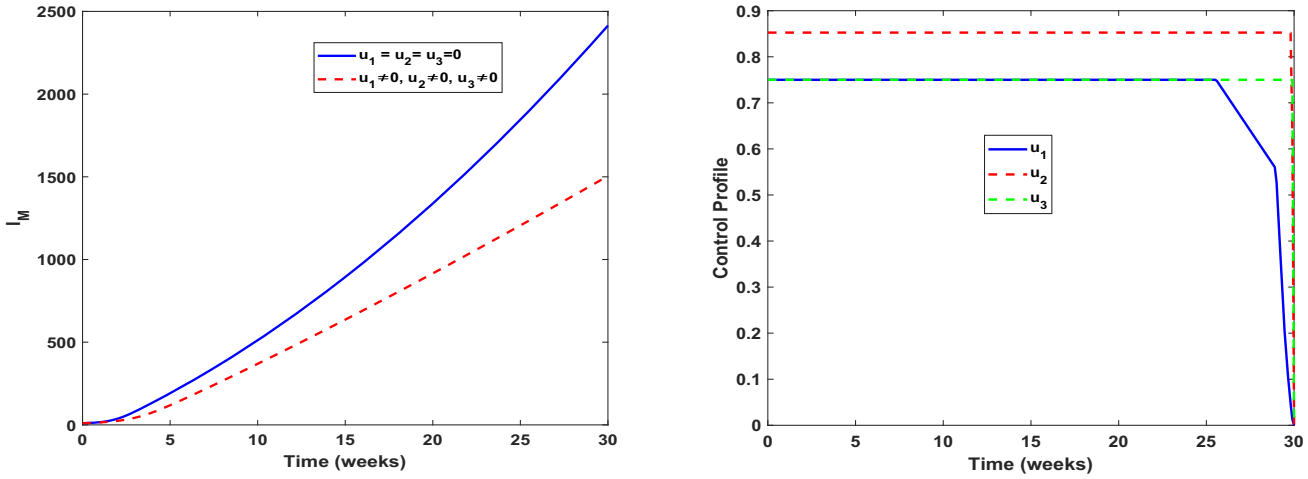
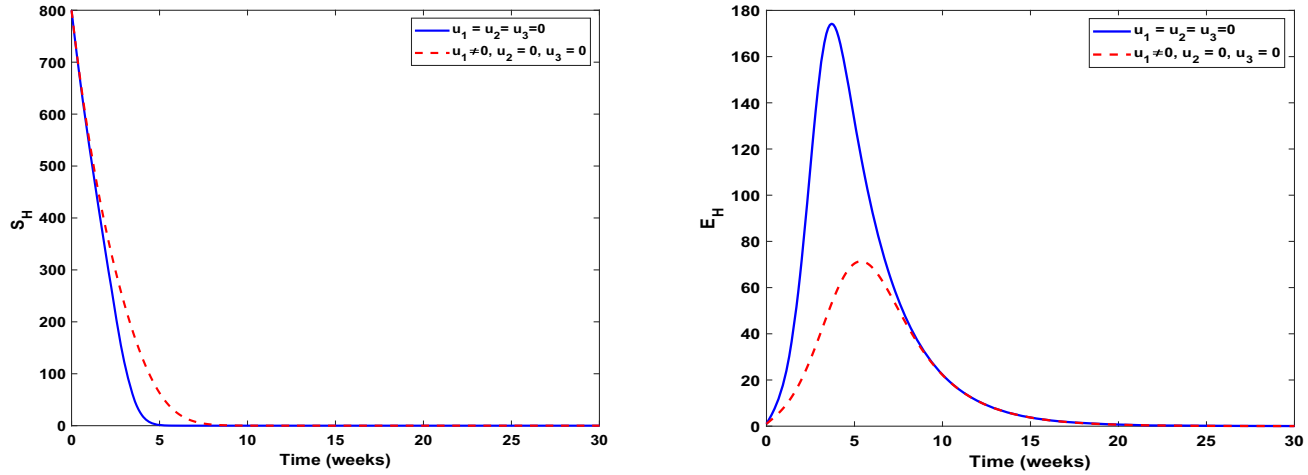


Fig. 10 Prevention through all three controls $u_1 = u_2 = u_3 \neq 0$

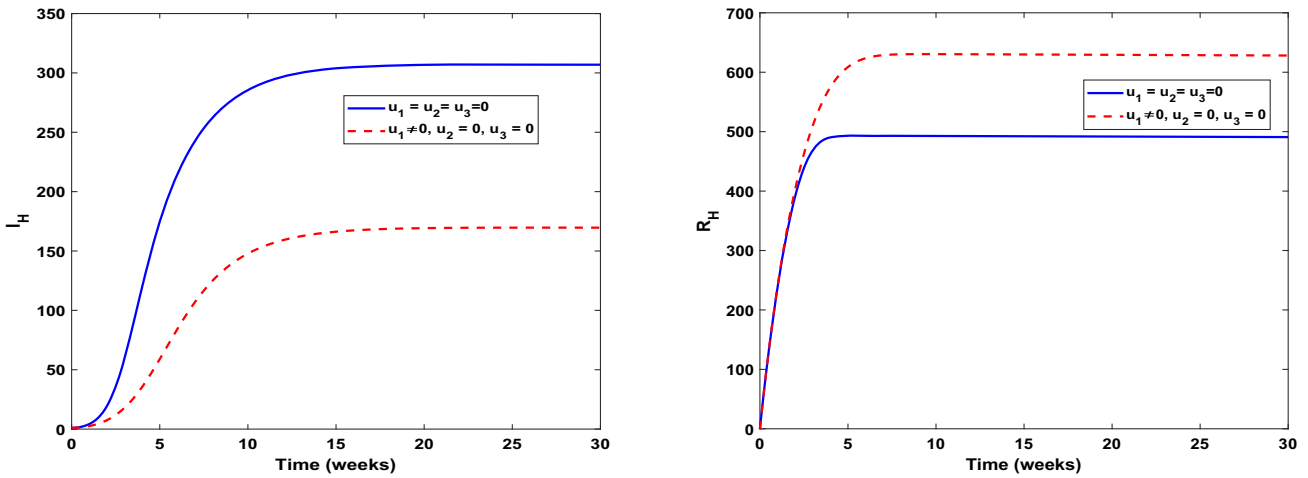


(a) (b)

Fig. 11 The infected mosquito and control profile for strategy $u_1 = u_2 = u_3 \neq 0$



(a) (b)



(c) (d)

Fig. 12 The only bed nets impact on human population

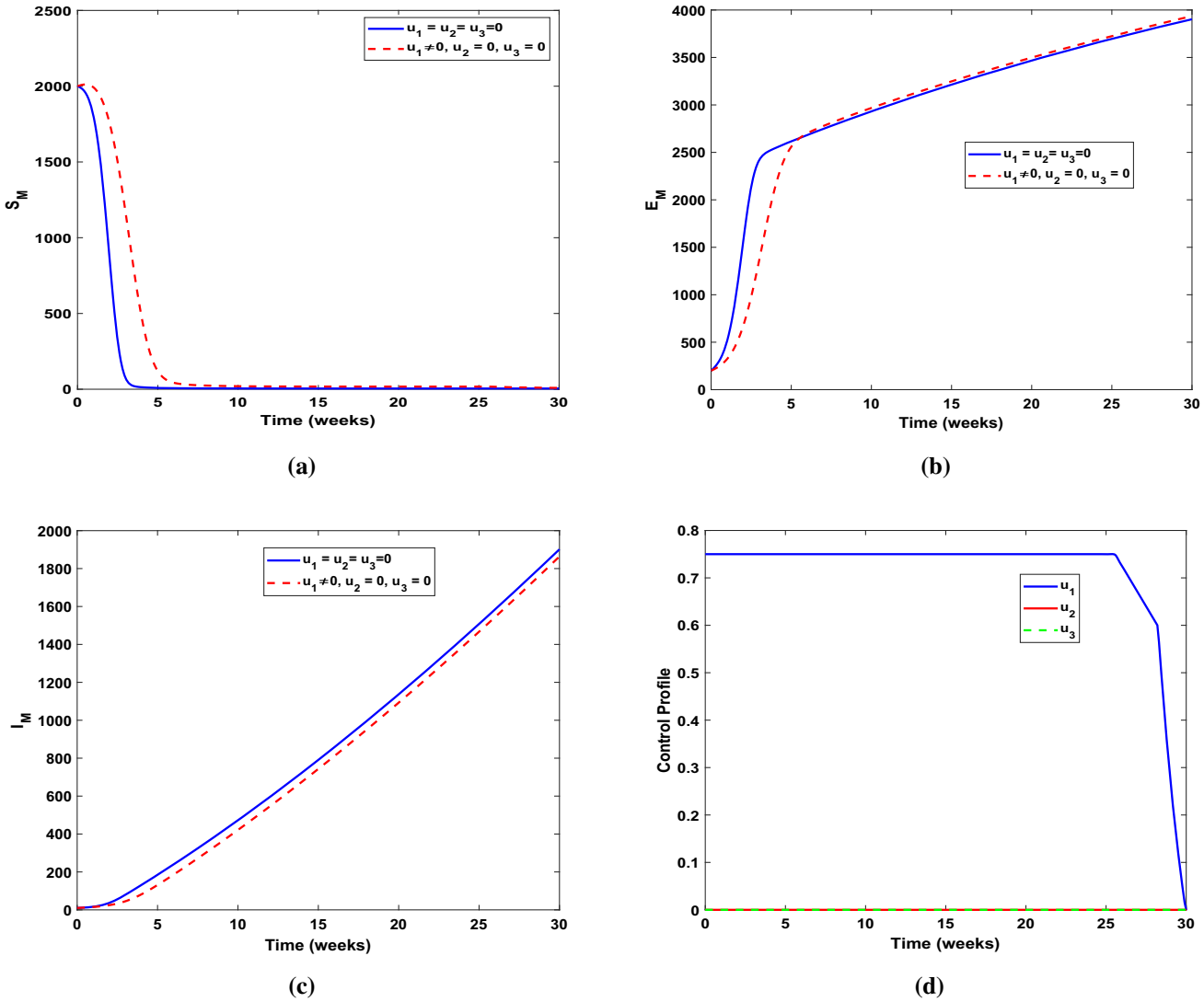


Fig. 13 Single control impact on mosquitoes compartments

Theorem 6.1 *An optimal control* $u^* = (u_1^*, u_2^*, u_3^*)$ *exists for the objective function and control set given in (40) and (39), respectively, such that* $\mathbb{J}(u_1^*, u_2^*, u_3^*) = \min_U \mathbb{J}(u_1, u_2, u_3)$.

Proof To solve the optimal control problem presented in (38), we set the Lagrangian and Hamiltonian. The Lagrangian, indicated by L, is given by

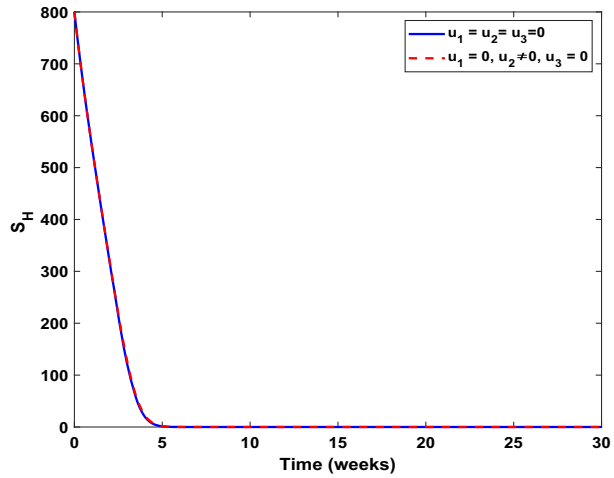
$$L(E_H, I_H, E_M, I_M, u_1, u_2, u_3) = AE_H + BI_H + CE_M + DI_M + \frac{1}{2}(Eu_1^2 + Fu_2^2 + Gu_3^2). \tag{44}$$

The Hamiltonian H is defined as the minimum of the Lagrangian L, whereas

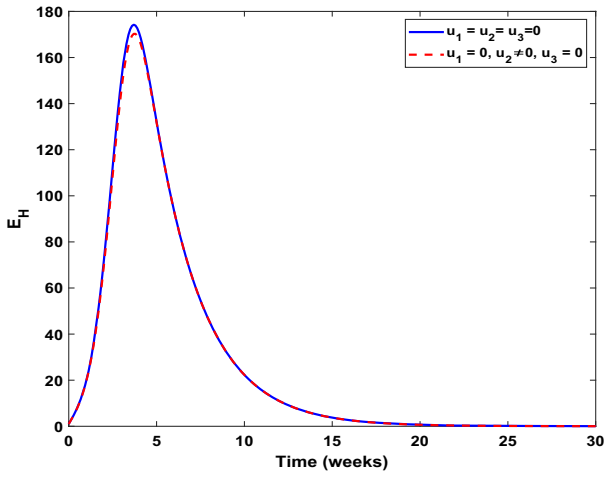
$$Y = (S_H, E_H, I_H, R_H, S_M, E_M, I_M), \quad U = (u_1, u_2, u_3), \quad \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7).$$

So Hamiltonian becomes,

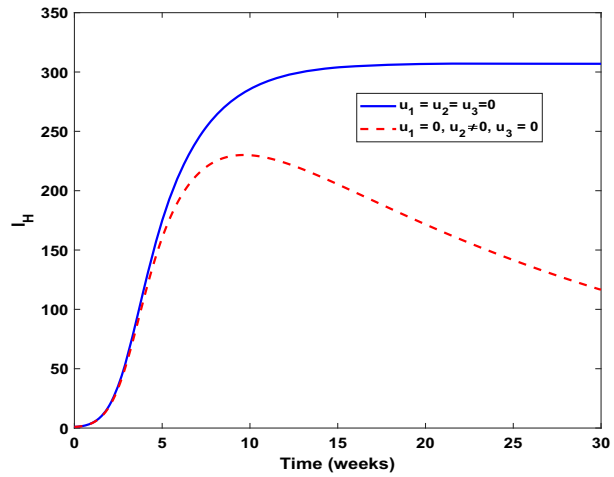
$$\begin{aligned} H(Y, U, \lambda) = {} & L(E_H, I_H, E_M, I_M, u_1, u_2, u_3) \\ & + \lambda_1\left[\Lambda_H - (1-u_1)S_H\left(\beta_1 I_h + \beta_2 I_M\right) - (k_1 + \omega_1)S_H\right] \\ & + \lambda_2\left[(1-u_1)S_H\left(\beta_1 I_H + \beta_2 I_M\right) - (k_1 + \sigma_1)E_H\right] + \lambda_3\left[\sigma_1 E_H - (k_1 + u_2\gamma)I_H\right] \\ & + \lambda_4\left[u_2\gamma I_H - k_1 R_H + \omega_1 S_H\right] + \lambda_5\left[\Lambda_M - (1-u_1)\mu S_M I_H - u_3(k_2 + \omega_2)S_M\right] \end{aligned}$$



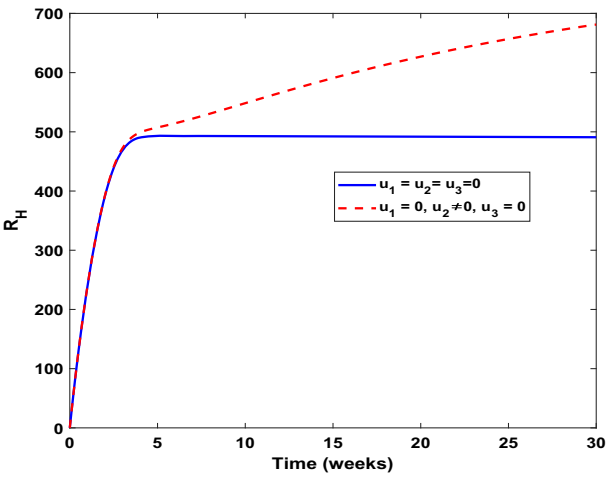
(a)



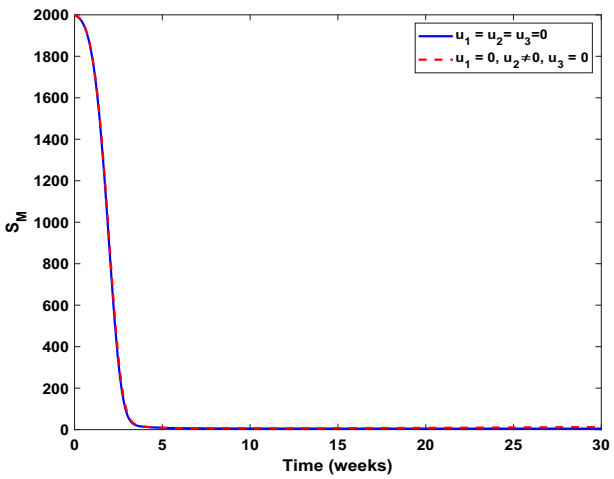
(b)



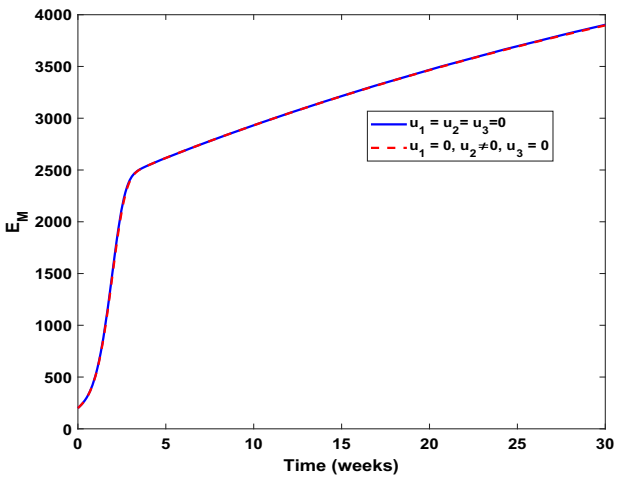
(c)



(d)

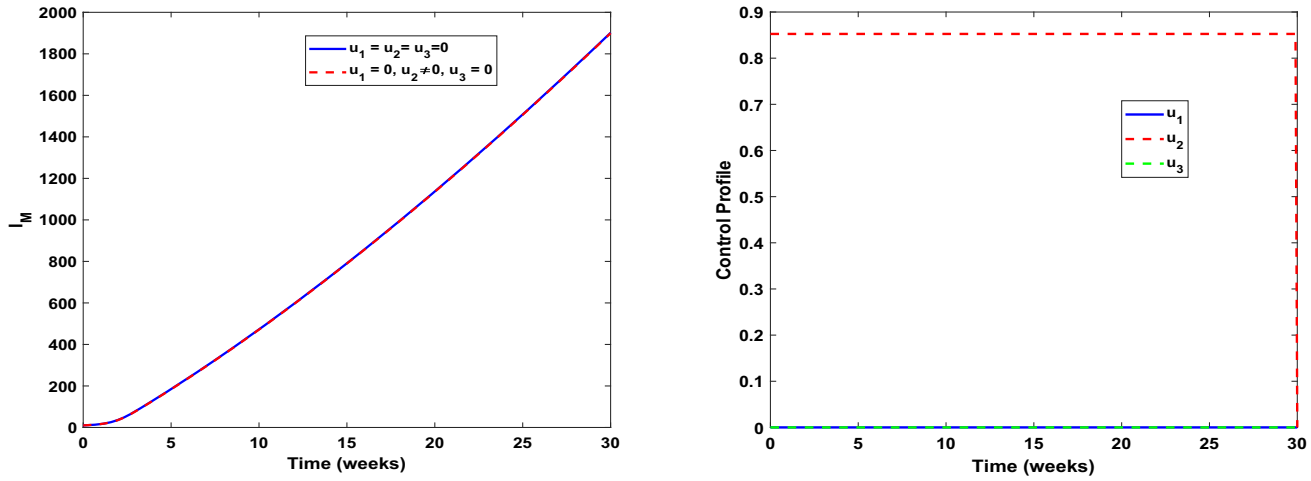


(e)



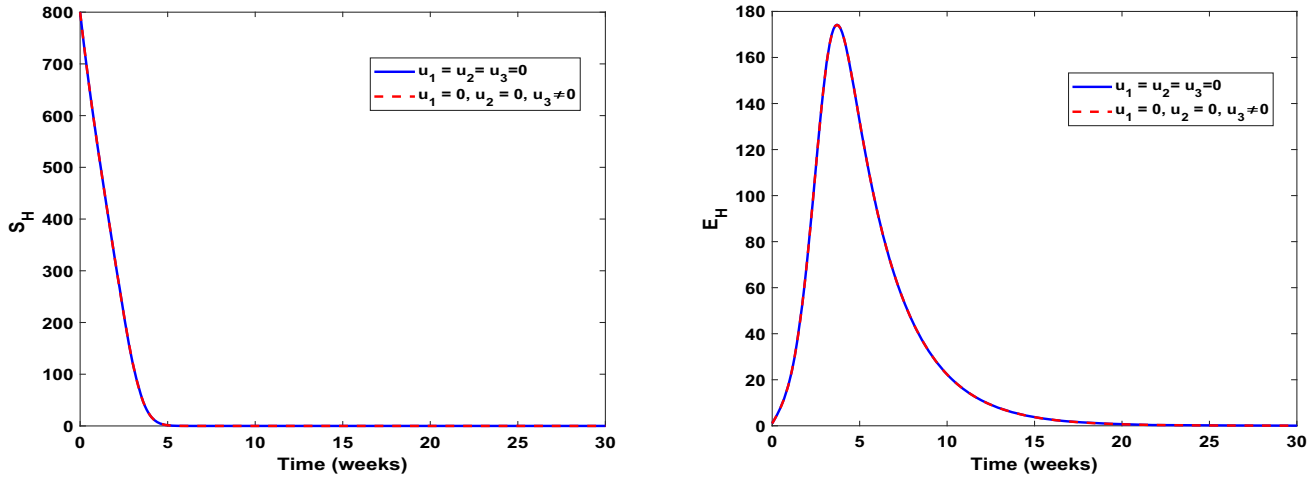
(f)

Fig. 14 Only treatment impact on disease dynamics

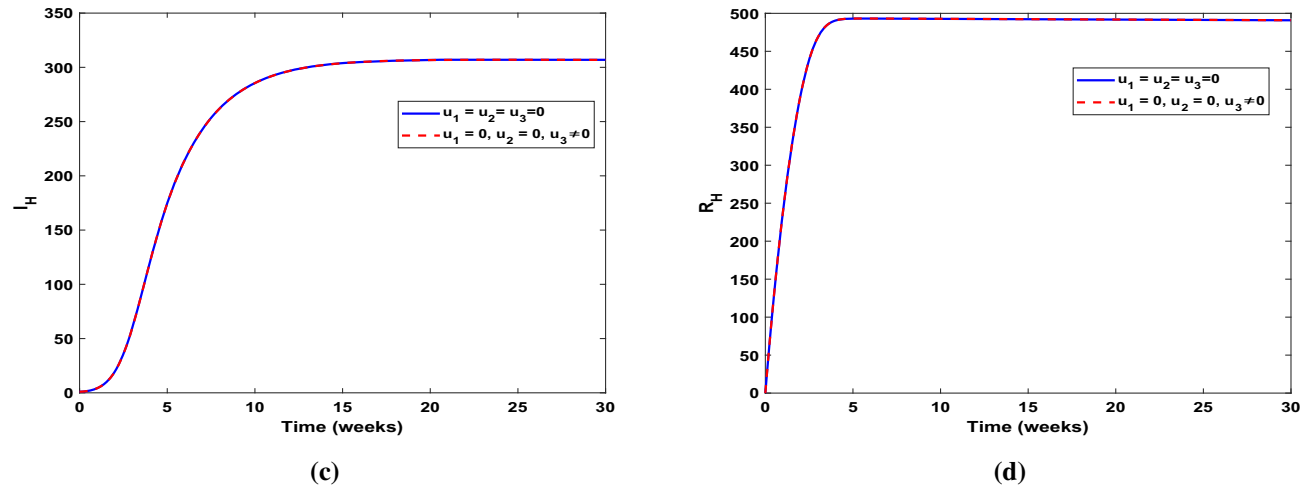


(a) (b)

Fig. 15 Implementation of strategy 6



(a) (b)



(c) (d)

Fig. 16 Impact on human population dynamics through only insecticide spray

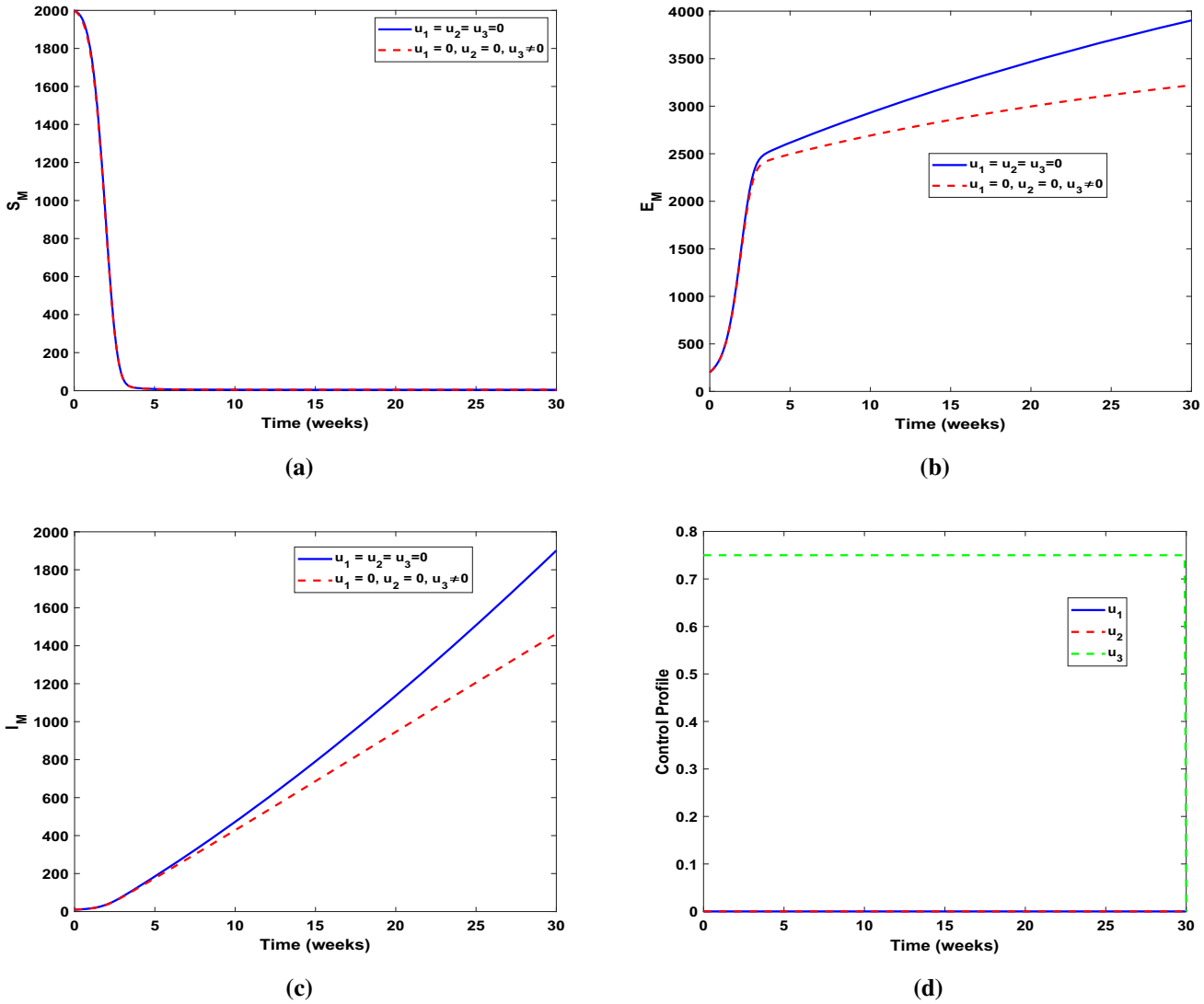


Fig. 17 The only insecticide spray strategy impact on mosquitoes

$$+\lambda_6\big[(1-u_1)\mu S_M I_H - u_3(k_2+\omega_2)E_M - \sigma_2 E_M\big] + \lambda_7\big[\sigma_2 E_M - u_3(k_2+\omega_2)I_M\big].$$

λ_i, for $i = 1, 2, ...7$ denotes adjoint variables. □

6.2 Solution of control problem

To solve the optimal control problem, we use the [28] optimal approach. Let u_i^* be the optimal control system solution (38) for $i = 1, 2, 3$, and there are adjoint variables λ_j, for $j = 1, 2...7$, which satisfies the conditions given as,

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H(t,u_i^*,\lambda_j)}{\partial \lambda}, & i = 1,2,3, \quad j = 1,2...7, \\ 0 = \frac{\partial H(t,u_i^*,\lambda_j)}{\partial u}, & i = 1,2,3, \quad j = 1,2...7, \\ \frac{d\lambda_j}{dt} = \frac{\partial H(t,u_i^*,\lambda_j)}{\partial x}, & i = 1,2,3, \quad j = 1,2...7, \end{cases} \tag{45}$$

To obtain the necessary findings for the optimal control problem (38), we use equations derived in (45) and establish the following results.

Theorem 6.2 *For the solution* $(S_H, E_H, I_H, R_H, S_M, E_M, I_M)$ *and optimal controls given by* u_i^* *for* $i = 1, 2, 3$ *associated with control system (38), we have adjoint variables* λ_j *for* $j = 1, 2, ...7$ *that satisfies,*

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_1(1-u_1)(\beta_1 I_H + \beta_2 I_M) + \lambda_1(k_1+\omega_1) - \lambda_2(1-u_1)(\beta_1 I_H + \beta_2 I_M) - \lambda_4\omega_1, \\ \frac{d\lambda_2}{dt} = -A + \lambda_2(k_1+\sigma_1) - \lambda_3\sigma_1, \\ \frac{d\lambda_3}{dt} = -B + \lambda_1(1-u_1)\beta_1 S_H - \lambda_2(1-u_1)\beta_1 S_H + \lambda_3(k_1+u_2\gamma) - \lambda_4 u_2\gamma \\ \qquad +\lambda_5(1-u_1)\mu S_M + \lambda_6(1-u_1)\mu S_M, \\ \frac{d\lambda_4}{dt} = \lambda_4 k_1, \\ \frac{d\lambda_5}{dt} = \lambda_5(1-u_1)\mu I_H + \lambda_5 u_3(k_2+\omega_2) - \lambda_6(1-u_1)\mu I_H, \\ \frac{d\lambda_6}{dt} = -C + \lambda_6 u_3(k_2+\omega_2) + \lambda_6\sigma_2 - \lambda_7\sigma_2, \\ \frac{d\lambda_7}{dt} = -D + \lambda_1(1-u_1)\beta_2 S_H - \lambda_2(1-u_1)\beta_2 S_H - \lambda_5\Lambda_M(1-u_3) + \lambda_7 u_3(k_2+\omega_2), \end{cases} \tag{46}$$

with transverse conditions

$$\lambda_j(T_f) = 0 \quad for, \quad j = 1, 2, ..7. \tag{47}$$

While writing the optimal control variables in the range $0 \le (u_1^*, u_2^*, u_3^*) \le 1$, we obtain

$$\begin{aligned} u_1^* &= \max\left\{\min\left\{1, \tfrac{(\lambda_2-\lambda_1)S_H(\beta_1 I_H+\beta_2 I_M)+(\lambda_6-\lambda_5)\mu S_M I_H}{E}\right\}, 0\right\}, \\ u_2^* &= \max\left\{\min\left\{1, \tfrac{(\lambda_3-\lambda_4)\gamma I_H}{F}\right\}, 0\right\}, \\ u_3^* &= \max\left\{\min\left\{1, \tfrac{(k_2+\omega_2)(\lambda_5 S_M+\lambda_6 E_M+\lambda_7 I_M)}{G}\right\}, 0\right\}. \end{aligned} \tag{48}$$

Proof Differentiating the Hamiltonian with respect to state variables yields the results (46) and (47). We also derive (48) by differentiating Hamiltonian H with respect to control variables. □

6.3 Numerical simulation of optimal control problem

In this section, we present the numerical simulation of the optimal control problem (38) and without control problem (1) by using the backward Runge–Kutta scheme and different preventive strategies. Table 1 shows the parameter setting for simulation with corresponding contribution for each strategy to minimize the infection with weights $A = 1,\ B = 1\ C = 1,\ D = 1,\ E = 0.5,\ F = 0.5,\ G = 0.5$. The suggested strategies found upon the set of controls are:

6.3.1 Scenario 1: Coupled control

Strategy 1: Treatment and insecticide spray

$(u_1 = 0, u_2 \neq 0, u_3 \neq 0)$

In this strategy, we keep the treatment (u_2) and insecticide spray (u_3) active and calculate our proposed results in Figs. 4 and 5. The human population is shown in Fig. 4, while Fig. 5 unfolds the results for mosquitoes and control profile as well. The variables of the control problem with control and without control are represented by the dashed and solid lines, respectively. Although there is no specific treatment for Zika infection, drinking plenty of fluids to avoid dehydration, getting plenty of rest, and taking acetaminophen may be useful in eradicating the virus from the population.

Strategy 2: Prevention through bed nets and insecticide spray

$(u_1 \neq 0, u_2 = 0, u_3 \neq 0)$

Here the strategy of using bed nets (u_1) and insecticide spray (u_3) is suggested to stop the transmission of the Zika virus. In Fig. 6, the population of infected and exposed humans has decreased significantly, while the population of susceptible and recovered humans has increased. In addition, Fig. 7 depicts a reasonable decrease in the mosquito population.

Strategy 3: Prevention through bed nets and treatment

$(u_1 \neq 0, u_2 \neq 0, u_3 = 0)$

This strategy considers prevention through bed nets (u_1) and treatment (u_2) controls and ignores the third control (u_3). The graphical results after simulation is given in Fig. 8 and 9. So this strategy is much effective on human population.

6.3.2 Scenario 2: Threefold control

Strategy 4: Prevention through bed nets, treatment, and insecticide spray

$(u_1 \neq 0, u_2 \neq 0, u_3 \neq 0)$

It is determined that each strategy is important in its own right and that a strategy may be effective in either human or mosquito compartments. We used this strategy to implement all three controls at the same time, and the simulation results are shown in Figs.

10 and 11. The susceptible and recovered humans increase, while the exposed and infected humans decrease, as shown in Fig. 10(a–f). Similarly, the population of infected mosquitoes reduces dramatically in Fig. 11. For the possible elimination of the Zika virus from the population, using all three controls u_1, u_2, u_3 at the same time is favored over other strategies.

6.3.3 Scenario 3: Single control

Strategy 5: Bed nets only

$(u_1 \neq 0, u_2 = 0, u_3 = 0)$

In this strategy, we assume prevention from Zika virus infected mosquitoes through bed nets (u_1) only and ignore the strategies of treatment and insecticide spray. Figures. 12 and 13 show that this strategy effectively reduces the number of infected individuals by keeping them away from mosquitoes and increases the susceptible in the community.

Strategy 6: Treatment only

$(u_1 = 0, u_2 \neq 0, u_3 = 0)$

The strategy of only treatment (u_2) of infected individuals is applied, while prevention through bed nets and insecticide spray is neglected. Figures 14 and 15 show that the treatment through WHO recommended medicine minimizes the course of infection only in humans and makes the individuals join the susceptible class.

Strategy 7: Insecticide spray only

$(u_1 = 0, u_2 = 0, u_3 \neq 0)$

This strategy makes the use of insecticide spray (u_3) on Zika- infected mosquitoes and does not consider the prevention through bed nets and treatment. Figures 16 and 17 illustrate the number of infected mosquitoes that transmit the infection to humans is decreasing, increasing the number of existing susceptible individuals and the population remaining safe from mosquito bites.

7 Conclusion

In this paper, we analyzed a new Zika virus model that includes bilinear incidence rate and human awareness; we presented its dynamical results and optimal control analysis. Initially, we considered the human sub-model only and determined $\mathcal{R}_H$. The risk-free equilibrium state is locally and globally asymptotically stable if $\mathcal{R}_H < 1$. Furthermore, if $\mathcal{R}_H > 1$, the model is locally asymptotically stable at endemic equilibrium. Moreover, we calculated that the model is globally asymptotically stable at endemic equilibrium using a geometric technique. Similarly, we presented the entire Zika model's fundamental features and stability study. The basic reproduction number is calculated using the next-generation matrices approach; the whole model is locally asymptotically stable at disease-free equilibrium if $\mathcal{R}_0 < 1$, and the model is globally asymptotically stable at DFE using the Castillo-Chavez theory. The model for the existence of endemic equilibrium was discussed. If $\mathcal{R}_0 > 1$, we established that the model is locally asymptotically stable at endemic equilibrium. We concluded that the complete model is globally asymptotically stable at endemic equilibrium after introducing a suitable Lyapunov function. We did a sensitivity analysis of each relevant parameter concerning the basic reproduction number $\mathcal{R}_0$ and specified the most sensitive parameters using a direct differentiation approach. The whole model has no bifurcation at $\mathcal{R}_0 = 1$, according to the backward bifurcation analysis. Finally, we formulated the optimal control problem with three control variables: prevention through bed nets and mosquito repellents, treatment of Zika-infected patients, and insecticide spray. The results of the numerical simulation are obtained by the RK4 numerical scheme, and different combinations of controls are discussed; each combination has its usefulness, but it is suggested that combining all the three controls at a time gives more insight into the disease dynamics and reduces of the number of infection than other control combinations.

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Conflict of Interest The authors have declared no conflict of interest.

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