KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY,KUMASI

USING HOMOTOPY ANALYSIS METHOD FOR SOLVING FREDHOLM AND VOLTERRA INTEGRAL EQUATIONS

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil Pure mathematics degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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NYANSAPƆ WƆSANE NO BADWENMA

Dedication

I dedicate this work to my lovely wife and children, for them being by my side in all my struggle to bring this work to light.I love you dearly.

I also dedicate this work to my mum,Mma Salamat Ibrahim.You are so dear to my heart,may the Almighty God bless you and give you long life.

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Abstract

In this thesis,we focus on using the Homotopy Analysis Method (HAM) to Fredholm and Volterra integral equation of the second kind.The HAM is based on homotopy,a fundamental concept in topology and differential geometry.After the classification of these integral equations ,we take a look at a review of integral equation and Homotopy Analysis Method (HAM) coupled with theories and definitions of homotopy theory.The description of the method (HAM) to solve Fredholm and Volterra integral equations is analyzed.In this method one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equation.Application of the HAM to some examples of Fredholm and Volterra integral equations is carried out together with the auxiliary parameter } ,which controls the convergence rate of the series solution. After the realization of the exact solution of the various considered equations, MATLAB,a computational software is used to produce graphs of the various exact solutions which shows the convergence of the series solution.

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Chapter 1

INTRODUCTION

Integral equations(IEs) which is encountered in both pure and applied mathematics plays an essential role in modern scientific disciplines such as numerous problem in mass and heat transfer, elasticity, fluid dynamics, oscillation theory, medicine, economics, filtration theory, electrical engineering, and mechanics (Polyanin and Manzhiric, 1998) and (Jerri, 1999). An initial value problems found in equations like ordinary differential as well as partial differential equations is treated in a better way by the methods of integral equations. The subject of integral equations (IEs) has received serous attention by researchers in modelling and in solving physical problems in science. Integral equations (IEs) which presents itself in the form, most essential tools in numerous areas of pure analysis . The exact solutions to integral equations present a useful part for better understanding of many physical problems that affect the environment and man in the field of science (Polyanin and Manzhiric,1998).

1.1 Historical background of integral equations

This concept of IE was introduced by du Bios–Reymond in 1888. Before that however there had been equations of this form. A classical example is the Laplace equation in 1782 which used the integral

$$f(x) = \int_0^1 e^{-xu}\phi(x)du \tag{1.1}$$

The general form is given below

$$\Psi(x) = h(x) + \lambda \int_{f(x)}^{g(x)} T(x,t)\Psi(t)dt \tag{1.2}$$

The following are the different **parameters** in (1.2) for a given integral equation.

Free term:h(x) free or forcing term in the equation. It is known and continuous.

Unknown function: $\Psi(x)$ is to be determined.

Kernel: T(x,t) kernel,is known continuous function defined as appear in the square R: $f \leq x \leq g$, $f \leq t \leq g$ of the (x,t)- plane.

Limits: f(x) and g(x), the limits of integration.

Investigative parameter: λ is a parameter which is introduced to determine variation of the problem . we vary λ to get the solution of the integral equation(Wazwaz, 2011).

An IE is said to be an equation where an unknown function to be determined comes under one and more integral signs of the equation. Differential equations and integral equations have some close connections.

1.2 Integral equations (IEs)

1.2.1 Kinds of Integral Equations (IEs)

Integral equations are of four types. We have two main classes which this work intends to solve (the Fredholm and the Volterra integral equations (FIE and VIE)) and two other types (Singular integral and integro-differential).

(1) The Fredholm's type (FIE)

Here the integral equation presents itself in a way where the limits f and g associated with the integration are fixed (Wazwaz, 1997) or closed bounded set in R^n, for some $n \geq 1$ as shown below

$$g(x)\Psi(x) = f(x) + \lambda \int_f^g H(x,t)\Psi(t)dt \tag{1.3}$$

(i) where g(x)=1, then (1.3) becomes

$$\Psi(x) = f(x) + \lambda \int_{f}^{g} H(x,t)\Psi(t)dt \tag{1.4}$$

this equation is second kind of Fredholm type of integral equation .

(ii) When the function g(x)=0, the (1.3)simply becomes

$$f(x) + \lambda \int_{f}^{g} H(x,t)\Psi(t)dt = 0 \tag{1.5}$$

and this is the first kind of Fredholm type of integral equation.

(iii) Where g(x) is not 0 or 1 then equation (1.3) is said to be the third kind of Fredholm integral equation. Wazwaz(1997, 2011) .

(2).The Volterra's type (VIE)

Here the Volterra type (VIE) is as shown below

$$g(x)\Psi(x) = f(x) + \lambda \int_{a}^{x} T(x,t)\Psi(t)dt \tag{1.6}$$

Thus, Ψ can be represented linearly and nonlinearly under integral sign and this kind of equation have the upper limit being variable.

(i) Where g(x)=1, then equation (1.6) yields

$$\Psi(x) = f(x) + \lambda \int_{a}^{x} T(x,t)\Psi(t)dt \tag{1.7}$$

and (1.7) is the second kind of Volterra equation.

(ii) But when g(x)=0, (1.6) simply yields

$$f(x) + \lambda \int_{a}^{x} T(x,t)\Psi(t)dt = 0 \tag{1.8}$$

And (1.8) is the first kind of Volterra integral equation.

(iii) Where g(x) is not 1 or 0 then (1.6) is the third kind of volterra integralequation as can be found in (1.6) Burton(2005) and Wazwaz(1997,2011)

3. The Singular type

This is the type of integral equation where the limits either one or both limits or the kernel is infinite in the integration

.Then example below

$$\Psi(x) = f(x) + \lambda \int_{-\infty}^{\infty} U\Psi(t)dt \quad (1.9)$$

is the second kind of singular integral equation

(i) It is refered to as **weakly singular integral equation**, if it appears below

$$F(x,t) = \frac{T(x,t)}{|x-t|^{\alpha}} \quad (1.10)$$

or

$$F(x,t) = T(x,t)\ln|x-t|^{\alpha} \quad (1.11)$$

And where T(x,t) is bounded, f ≤ x ≤ g and f ≤ t ≤ g with T(x,t) 6= 0, and a constant $0 < \alpha < 1$.

For example

$$y(x) = \lambda \int_{0}^{x} \frac{1}{(x-t)^{\alpha}} f(t)dt, 0 < \alpha < 1 \quad (1.12)$$

is known to be a generalized **Abel's** IE with a weakly singular kernel.

(ii)The Strongly singular integral equations:

When F(x,t), the kernel is given below

$$F(x,t) = \frac{T(x,t)}{(x-t)^2} \quad (1.13)$$

It is strongly singular and T(x, t)6=0 , is a function which is differentiable of (x,t).

4.The Integro-differential equations:

The unknown function f to be determined comes as both ordinary derivative and under the integral sign. For example, see (1.14)

$$f^1(x) = x - \frac{3}{7}x + \int_{0}^{1} (x-t)f(t), f(0) = 1 \quad (1.14)$$

$$f^{11}(x) = x + \lambda \int_0^x x f(t)dt, f(0) = 0, f^1(0) = 1 \quad (1.15)$$

Equation (1.14) is Fredholm type and 1.15 is Volterra type of the equation (Burton, 2005).

1.2.2 The Linearity of integral equations

Here the linearity of an integral equation is defined with respect to how linear the function Ψ to be determined is in the integral sign

(i) The **IE** is called **linear** if it is seen in (1.16) below

$$\Psi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} H(x,t)\Psi(x)dt \quad (1.16)$$

The unknown function Ψ under integral sign is of exponent one.

As seen in (1.17) below

$$\Psi(x) = \frac{1}{3} + x - \int_0^1 (x-t)\Psi(t)dt \quad (1.17)$$

the function Ψ, the unknown function presents itself in a linear form. That is with power one

(ii). For integral equations to be **Nonlinear**, it is as shown in (1.18) below

$$\Psi(x) = x + \int_0^x (x-t)\Psi^4(t)dt \quad (1.18)$$

thus, Ψ inside the integral sign has exponent other than one, Wazwaz(1997, 2011).

1.2.3 Homogeneity of integral equations

As regard to homogeneity, if we set the free term h(x)=0, the resulting integral equation is homogeneous, for if $h(x)$ 6= 0 is then called a non-homogeneous integral equation.

i. **Homogeneous integral equation** :

If the function h(x), the free term of integral equation is equal to zero, for example

$$\Psi(x) = \lambda \int_{a(x)}^{b(x)} F(x,t)\Psi(t)dt \tag{1.19}$$

this type is called homogeneous ii. **For the Non**

homogeneous integral equation:

Where h(x) of this type of integral equation , we have $h(x)$ 6= 0, then this type is refered to as non-homogeneous.

as shown

$$\Psi(x) = h(x) + \lambda \int_{a(x)}^{b(x)} F(x,t)\Psi(t)dt \tag{1.20}$$

Where $h(x)$ 6= 0.

this type is typical of equations of second kind only.

1.3 Kinds of kernels

1. Cauchy kernel. F(x,y) is cauchy when appears as in (1.21) below

$$F(x,y) = \frac{T(x,y)}{x-y} \tag{1.21}$$

thus T(x,y) is a differentiable function of (x,y) with T(x; y)6=0, if the integral equation is seen as a singular equation which has cauchy kernel

(2). We say F(x,y) is **separable or degenerate kernel** where the kernel is presented as the sum of finite number of terms and each of which is the product of a function of x and y only(Wazwaz,1997).

Thus,

$$F(x,y) = \sum_{i=0}^{k} a_i(x) b_i(y) \tag{1.22}$$

The function a_i(x) and b_i(y) are linearly independent.

3. Abel's kernel. If the kernel F(x,y) is of form

$$F(x,y) = \frac{T(x,y)}{|x-y|^{\alpha}} \tag{1.23}$$

Where $0 < \alpha < 1$ and the function T(x,y) can continuously be differentiable. And this integral equations which have this kind of kernel are refered to as **Abel integral equation**.

4. In the case where the kernel is in the(1.24) below is called **Hilbert Kernel**. Where the real variables x and y are **Hilbert Kernel** and are closely connected to the **Cauchy kernel** as they are in the unit circle

$$F(x;y) = cot[y - \frac{x}{2}] \tag{1.24}$$

$$\frac{dt}{t-\tau} = \frac{1}{2}(\cot\frac{y-x}{2} + i)dy, \tag{1.25}$$

where $t=e^{iy}$, $\tau=e^{ix}$

5. When we have the kernel in the form as shown in (1.26) such type of kernel is called **Skew-symmetric kernel.**

$$F(x,y) = -F(y,x) \tag{1.26}$$

6. **Symmetric (or Hermitian) kernel.** The kernel F(x,y) is called symmetric if

$$F(x,y) = F^{*}(y,x) \tag{1.27}$$

the asterisk in (1.27) represents the complex conjugate. This coincides with the definition F(x,y) = F(y,x) for a real kernel.

7. Kernel F(x,y) is **Hilbert-Schmidt kernel.** ,for every x, y , f ≤ x ≤ g and f ≤ y ≤ g, is such that

$$\int_f^g \int_f^g |F(x,y)|^2 dxdy < \infty \tag{1.28}$$

And for each value of x in f≤x≤g, is

$$\int_f^g |F(x,y)|^2 dy < \infty \tag{1.29}$$

Also for each value of y in f≤y≤g, is

$$\int_f^g |F(|x,y)|^2 dx < \infty \tag{1.30}$$

And there is a finite value and the kernel in this instance is called regular kernel and the corresponding integral equation is refered to as regular integral equation.

1.4 Review of spaces

Definition 1.4.1. A space is known to be a vector or linear space if it contains these properties:

1. There should be C a field of scalars.

2. Vectors of a set X of objects.

3. There are vector addition,an operation which gives for each pair of vectors x, y in X a vector x+y in X,refered to as the sum of x and y, such that,

(i) Commutativity of addition ,x+y = y+x.

(ii) Associativity of addition,x+(y+z) = (x+y)+z.

(iii) A unique vector 0 in K, known as zero vector, where a+0=a for all a in X.

(iv) For each vector x in X there is a unique vector -x in X, such that x+(-x) = 0

4. An operation known as scalar multiplication, which takes for each scalar a in C and vector y in Y to a vector ay in Y, this is the product of a and y, which presents:

(i) 1y=y, for every y in Y,

(ii) $(a_1a_2)y = a_1(a_2)y$. (iii) a(x+y) = ax+ay. iv) $(a_1 + a_2)y$

$= a_1y + a_2y$

Definition 1.4.2. Vector norm on X. Let X a vector norm be a function on k.k from X into C, (where k.k represents the norm , X contains objects called a set of vectors and C a scalar field) where object x∈X is represented by kxk which contains these

(i) kxk≥ 0 for all x∈X

(ii) kxk=0, if and only if x=0

(iii) kαxk=|α|kxk,for all α∈C and x∈X

(iv) kx+yk≤ kxk +kyk (triangular inequality)

For instance,as R represents set of all real,there is a vector norms from R^k into R,where

kxk$_\infty$ = max kx$_i$k : 1≤i≤k and the

Euclidean norm

$$\|x\|_2 = (\sum_{i=1}^{k} |x_i|^2)^{\frac{1}{2}} \tag{1.31}$$

for the vector X =$(x_1,...,x_k)$

Definition 1.4.3. Let X a space be a vector space.For if there is a norm space which is defined on the space such that, the space is represented by (X,k.k)

Definition 1.4.4. Cauchy Sequence We say a sequence is Cauchy if the elements in that sequence arbitrary become close to each other as the sequence progresses. A sequence(x_n) can also be said to be a Cauchy sequence, for each

>0 if there is a positive integer N,for all real numbers such

that

$$m, n \geq N \longrightarrow |x_m - x_n| < \epsilon \tag{1.32}$$

In any metric space X,we can define Cauchy sequences as the absolute value $|x_m x_n|$ is replaced by the distance $d(x_m,x_n)$, as $d:X \times X \longrightarrow R$.

Definition 1.4.5. Complete Space or complete Banach space is a complete normed vector space.A Banach spaces is the finite-dimensional vector spaces R^n which contain the maximum norm kxk = max$|x_i|$: $1 \leq i \leq n$ and the Euclidean norm

$$\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}} \tag{1.33}$$

for the vectors $X = (x_1,..., x_n)$

Definition 1.4.6. If X is Banach space for $x_0 \in X$ and $r > 0$, the set

B (x_0, r) $x \in X$: kx - x_0k$\leq r$

is refered to (closed)ball of X with the centre x_0 and the radius r. A set $t \subset X$ is called

Bound if can be found in a ball of X

Open if there is any $x_0 \subset t$ and for r>0 such that $B(x_0, r) \subset t$

Closed if $(x_n) \subset t$, $x_n \longrightarrow x$ implies $x \in t$

It is **relatively compact** for every sequence $(x_n) \subset t$ there is a convergent subsequence and the limit in X not necessarily belong to it. Compact if t is closed and relatively compact.

A set $t \subset X$ is **closure** $\bar{t}$ for the smallest closed set containing S.A set $t \subset X$ is called dense in X,if $\bar{t} = X$.

Theorem1.1.1:The sequence of vectors x_k converges to x in R_n with respect to $\|\cdot\|_\infty$ if $\lim_{k \longrightarrow \infty} x_n^k = x_n$ for each n=1,2,...,k

Definition 1.4.7. Inner product and Inner product space. Consider an inner product on X,which is a function. X is a vector space over K which is either R and C such that h.,.i : $X \times X \longrightarrow K$ Which gives for every h x, yi$\in X^2$ a value in K represented by h x, yi where the following properties fulfilled.

1. **positivity**:h x, xi ≥ 0, for h x, xi=0 if and only if x=0.

2. **Conjugate Symmetry** h x, yi=h*y,x*i if K = R then hx, yi=h y, xi
3. For **Linearity**, Let the vector y∈ X is fixed and for the first variable for all a, b ∈ K, h ax_1+ bx_2, yi = ah x_1, yi + bh x_2, yi and again the pair (X,h.,.i) called inner product space over K. where K=C is complex inner product space,while if K = R is a real inner product space. For instance the L^2 inner product on L^2([a,b]) as shown below

$$(f,g)L_2 = \int_a^b f(x)\overline{g(x)}dx \tag{1.34}$$

for

f,g∈L^2([a,b])

Definition1.4.8. The Linearly independent functions. By linear independence of set of functions a_i's it is meant that, if $c_1a_1+c_2a_2+...+c_na_n=0$ Where c_i's are arbitrary constants, then $c_1=c_2=...=c_n=0$.

Definition 1.4.9. L^2-functions and L^2-spaces. L^2 function is a complex-valued function f(x) of a real variable x on an interval (a, b), and such that in the Lebesgue sense

$$\int_a^b |f(x)|^2 dx < \infty \tag{1.35}$$

In Lebesgue sense, the set of all such functions is referred to as the function space L_2 [a, b],

$$L^2([a,b]) = \{f : [a,b] \longrightarrow C; \int_a^b |f(x)|^2 dx < \infty\} \tag{1.36}$$

And complex numbers C. Two L^2 -function f and g which are equal for ' almost all ' values of x,that is , except for values of x being Lebesgue measure zero, are equivalent. Thus, f and g are equivalent if

$$\int_a^b [f(x) - g(x)]^2 dx = 0 \tag{1.37}$$

And the function h(x) (a ’ null function ’)which is zero almost everywhere will not be different from the zero function

$$h(x) = 0 \leftrightarrow \int_a^b h^2(x)dx = 0 \tag{1.38}$$

For this convection, L^2-function presents a complete inner product space with regard to the inner product(1.34).

Furthermore, the space L^2,with an appreciate norm and inner product, is an example of a Hilbert Space. We define the L^2 norm of an L^2-function as

$$\|f\|_2 = \{\int_a^b |f(x)|^2 dx\}^{\frac{1}{2}} \tag{1.39}$$

Definition 1.4.10. Delves and Mohammed(1885)**Regularity conditions**.

there is two-dimensional kernel function K(x,y).there exist L^2-function , if these conditions are fulfilled

(i) And values x, y in the square f≤ x≤ g, f ≤ y≤ g,

$$\int_f^g \int_f^g |K(x,y)|^2 dxdy < \infty, \tag{1.40}$$

(ii) And values x in f≤x≤g,

$$\int_f^g |K(x,y)|^2 dy < \infty, \tag{1.41}$$

(iii) And for values of y in f≤y≤g,

$$\int_f^g |K(x,y)|^2 dx < \infty \tag{1.42}$$

And the condition shown is known as the regularity conditions on the kernel K(x,y) .

Definition 1.4.11. Measurable functions

The functions are structure-preserving functions between measurable spaces , and for this reason, a natural context are form for integration theory specially a function which is between measurable spaces is refered to as measurable if the **preimage** of each measurable set is measurable.

Definition 1.4.12. L^p-Space

L^p-functions is refered to as a generalized L^2-spaces if p≥1. The measurable function r must be p-integrable,instead of square integrable, for r to be in L^p. For measure space X, the L^p norm of a function r is as

$$\|r\|L^p = (\int_x |r(x)|^p dx)^{\frac{1}{p}} \tag{1.43}$$

The L^p-functions are the functions where integral above converges.Consider p6=2, the space L^p-functions is known as **Banach space** which is not a **Hilbert space**.

In the case where p=∞, we have $L^\infty(D)$ defined as {*r* : *measurableinDand*krk∞ < ∞},

Where

$$\text{k}f\text{k}_\infty = \inf \sup |f(x)| : x \in S, s \subset D, \tag{1.44}$$

have Lebesgue measure and of the set S=0 .

Definition 1.4.13. A space C(R) is a vector space and r:R−→ K, continuous functions , where K stand for R or C. And C[0,1] consists of all continuous functions r:[0, 1]−→F,

$$\text{k}r\text{k}_{c[0,1]} = \text{k}r\text{k}_\infty = max|r(x)| \tag{1.45}$$

,for

0≤x≤1

Theorem 1.1.2. (Arzela-Ascoli)

Consider set $F \subset c[0, 1]$ is called relatively compact in $c[0, 1]$ if and only if these two conditions are satisfied:

(i) Functions $r \in F$ are uniformly bounded, when there is a constant c such that $|r(x)| \leq c$ for all $x \in [0, 1]$, $r \in F$.

(ii) Functions $r \in F$ called equicontinuous,where there is $\varepsilon > 0$ we have a $\delta > 0$ such that, $x_1, x_2 \in [0, 1]$, $|x_1 - x_2|$

$\leq \delta$

Implies

$|r(x_1) - r(x_2)| \leq \varepsilon$ for all $r \in F$ see Delves and Mohammed(1885).

1.5 Review of Operators

Definition 1.5.1.(Zwiebach, 2013) Let an operator $A: X \longrightarrow Y$ assigns to every function $r \in X$ a function $Ar \in Y$. This is a mapping which is between the two function spaces X and Y. When there is a range on the real line or in the complex plane, here the mapping is usually a functional instead of an operator.

A linear map refers to a kind of function that takes one vector space A to another vector space B. If the linear map takes the vector space A to itself. Such a linear map is a linear operator.

Definition 1.5.2.(Zwiebach,2013). A linear operator T on a vector space A is a function that takes A to A with the properties:

1. $T(x+y) = Tx+Ty$, for all $x, y \in A$.
2. $T(cx) = cTx$, for all $c \in F$ and $x \in A$

L(A) is called the set of all linear operators that act on A. Let consider some examples of linear operators:

(i) Let A denote the space of real polynomials p(x) of a real variable x with real coefficients.Let consider these two linear operators:

- Let T denotes differentiation : Tp = p^0. This operator is linear because $(p_1 + p_2)' = p_1' + p_2' and (ap)' = ap'$.
- Let S denote multiplication by x:Sp=xp. S is also a linear operator.

(ii) In the space F^{∞} of infinite sequences define the left-shift operator L by $L(a_1,a_2,a_3,...) = (a_2,a_3,...)$ and this is consistent with linearity as we loose the first entry. And for the rightshift operator R that acts as follows:
$R(a_1,a_2,...) = (0,a_1,a_2,...)$.
The first entry in the result could not be any other number except zero because the zero element(a sequence of all zeroes)should be mapped to itself and thus, by linearity(Zwiebach, 2013).

Definition 1.5.3. The Linear Operator

Consider X and Y as two vector spaces, if A: X−→Y where A which is defined on the values in X and in Y is linear operator if; A(r + s) = Ar + As, A(αr) = αAr for all r, s ∈X and α∈R or C. Let now consider X and Y as normed spaces. And where an operator A: X−→Y is continuous if kr_n − rk$_x$−→ 0 Implies kAr_n − Ark$_y$−→ 0 A linear operator A: X−→ Y is continuous if it is bounded, and have a constant c where kArk$_y$ ≤ Ckrk .

1.6 Objectives of the study

This work is geared towards using homotopy analysis method (HAM) to solve integral equations. The research intends:

(1) To examine the effectiveness of the homotopy analysis method(HAM).

(2) To demonstrate iteratively the series solution of integral equation using homotopy analysis method(HAM)

(3) To assess the effectiveness of convergence control parameter } to the solution.

1.7 Limitition and Scope

1.7.1 Limitition of Study

This research suffers some contraints of time and a crucial limitation is that of inadequate financial resource as well as material resources such as computer hardware and textbooks to help facilitate the execution of the study.

1.7.2 Scope of Study

This study would be confined to the existing works of researchers and academicians in the field under study, seasoned textbooks as well as some useful internet resource. Knowledge on LATEX which would be used to produce this work as well as MATLAB, a computer software would be of imperative advantage.

1.8 Notations Used

IEs...............Integral equations

FIEs.............Fredholm Integral Equations

VIEs.............Volterra Integral Equations

HAM............Homotopy Analysis Method

H(x, t)...........Kernel function

λ...................Investigative Parameter

}...................Convergence control parameter

L...................Linear Operator N...................Non linear operator

p...................Homotopy parameter

H(x)..............Auxiliary function independent of p

1.9 Organisation of the Thesis

The dessertation of this thesis is composed of the following chapters:

- Chapter 1: Introduction
- Chapter 2: Literature Review
- Chapter 3: The Homotopy Analysis Method(HAM)
- Chapter 4: Application of HAM
- Chapter 5: Summary and Conclusion

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this facet, a brief review of relevant literature is introduced. It encompasses a summary of abstracts on integral equations(IEs) and homotopy analysis method(HAM) and brief gloss over the concept of homotopy function.

2.2 Integral Equation And Homotopy Analysis Method(HAM)

There many biological or physical systems that leads to linear or nonlinear Fredholm and Volterra integral equations(FIEs and VIEs) either first or second kind(Abu jarad, 2009). Solving integral equations(IEs) analytically are usually difficult, so there is a need for a method that will effectively deals with mathematical problems which gives answers to many physical problems confronting humanity.

There have been many publications of various aspects of integral equations(IEs) see(Jerri, 1999), (Cochran, 1972), (Ren, Zhang and Qiao, 1999), (Zabreyko, 1975), (Polyanin and Manzhirov, 1998), (Wazwaz, 2011), (Dariusz Bugajewski, 2003), (Rahman, 2007) . The solution of an integral equation(IEs)can be traced in Abel(Kline, 1972), (Abel, 1823) and (Abel, 1826) see (Abel, 1973). Many numerical and analytical methods for solving IEs, as in (Maleknejad and Mahmoudi, 2004), (Burton, 2005), (Maleknejad and Sohrabi, 2007), (Adawi, Awawdeh and Jaradat, 2009)and recently, (Bazm and Babolian, 2012), (Nemati, Lima and Ordokhani, 2013), (Shirin and Islam, 2010), (Aziz, Siraj-ul-Islam and Khan, 2014), (Aziz and Siraj-ul-Islam, 2013). A review of homotopy analysis method, advances in homotopy analysis theorem(Liao, Abbasbandy, Shivanian, Motsa , Sibanda, Van Gorder, and Vishal, 2013). Introducing a solution for integral equation by HAM (Abu jarad,2009), (Adomain, 1988).

A powerful technique such as Reducing Initial Problem and Boundary Value Problem to VIE and FIE and Solution of Initial Value Problem by Habeeb Khudhur Kadhim see (Kadhim,2015). This report employs an analytic technique known as (HAM)for both linear and nonlinear types equations(Abu jarad,2009), (Khader, Kumar and Abbasbandy, 2013). The proposed homotopy analysis technique for solving nonlinear problem see (Liao, 1992). As introduced by Liao on the definitions, theorems and notes on the homotopy analysis method see (Liao, 2009), and the usage the method for the treatment of integral and integrodifferential problems see(Hossein Zadeh and Jafari and Karimi, 2010). In the application of HAM (Vahdati and Zulkifly Abass and Ghasemi, 2010) analytically solve problems involving integral equations . The method ensures convergence of series solution easily which presents the (HAM) as an effective method for complicated problems(Liao, 2009). And the (HAM)(Abu jarad,2009) solves the equations by its own techniques, and it does not necessarily have to resort to any advanced mathematical tools, that is, its methodology is understandable and can easily be implemented, and should be accepted easily in the field of engineering(Abu jarad, 2009).

This method (HAM) does not need any modification in moving from linear to nonlinear case. The (HAM) which is derived from homotopy, a basic concept in topology and differential geometry. In dealing with the (HAM)(Abu jarad,2009) a map is contructed continuously starting with an initial value, iterating the result from each step to the exact answer of considered equations. For the series solution to converge, an auxiliary parameter is introduced. There is a great laxity in selecting auxiliary linear operators and initial approximation as result a nonlinear problem can be changed into infinite number of simplier linear sub problems see(Abu jarad, 2009), (Vahdati et al, 2010), (Hossein Zadeh et al, 2010) (Liao, et al., 2013) and (Liao and Tan, 2007).

2.3 Homotopy function

In homotopy,when two continuous functions from a particular topological space to another is refered to as homotopic,means one the function can be deformed continuously into another function, and this kind of deformation is known as homotopy between the two functions.Here some definitions are introduced that precisely define homotopy.

Definition 2.3.1.(Adams and Franzosa, 2008). Consider Y as a topological space and B as a subset of Y. A continuous function $g : Y \to B$ is refered to as a retraction from Y onto B, where g(b)= b for each $b \in B$. In other words, where there is a retraction of $g : Y \to B$, then B is called a retract of Y.

Examples 2.3.1.(Adams and Franzosa, 2008).Consider m, n : R−→R be two continuous, real functions. Then m ' n. To show , define a function H: R×[0, 1]−→R by H(x, t) = (1-t).m(x) + t.n(x). Cleary, H is continuous,which is a composite of continuous functions. Thus, H(x,0) =(1-0).m(x) + 0.n(x) = m(x), and H(x,1) = (1-1).m(x) + 1.n(x) = n(x). Thus , H is a homotopy between m and n. And this explains why any continuous map h : R−→R being **nullhomotopic** If f is homotopic to a constant map, that is, if f'consty , for some y∈Y, then we say that f is **nullhomotopic**.

In generalizing the example, let consider the following definitions.

Definition 2.3.2.. Consider f, g :X−→Y a continuous functions. Let I=[0, 1] contains a subspace topology it inherits from R and that X×I which has a product topology. The f and g are homotopic if there exists a continuous function F: X×I−→Y such that F(x, 0) = f(x) and F(x, 1) = g(x), such a function F is called a homotopy between f and g. The expression f'g shows f and g are homotopic, as shown in figure 2.1 below

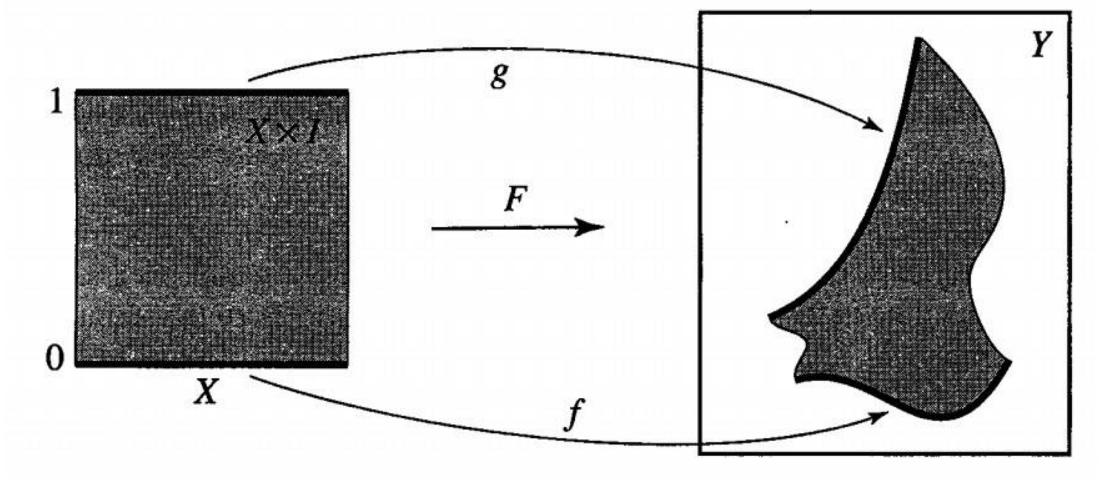

Figure 2.1: A homotopy deforms map f to map g

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Here g and f are two functions that deforms the space X into space Y and g is homotopic to f (Adams and Franzosa, 2008). **Theorem** 2.3.1 Homotopy is said to be an equivalence relation on map(X, Y).

Proof: Here look at the relation being **reflexive**, **symmetric**, and **transitive**. To show that ' is **reflexive**, let f : X−→ Y be a continuous function.Define a homotopy F : X×I−→Y by F(x, t) = f(x). Then F(x, 0) = f(x) and F(x, 1) =

f(x), implies f ' f.

To show ' is symmetric follows that if F(x, t) is a homotopy between f and g, then G : X×I−→Y , defined by G(x, t) = F(x, 1-t), is a homotopy between g and

f. So f ' g implies g'f.

To show that ' is transitive, Let f, g and h be continuous maps from X to Y. Suppose that f is homotopic to g via the homotopy F and g is homotopic to h via the homotopy G. Then , as illustrated in Figure 2.2, define a function H :

X×I−→ Y by

$$H(x,y) = \begin{cases} F(x,2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

For t = $\frac{1}{2}$, the expressions F(x, 2t) and G(x, 2t-1) both equal g(x) and hence they agree on the set where they are both used in the definition. It follows

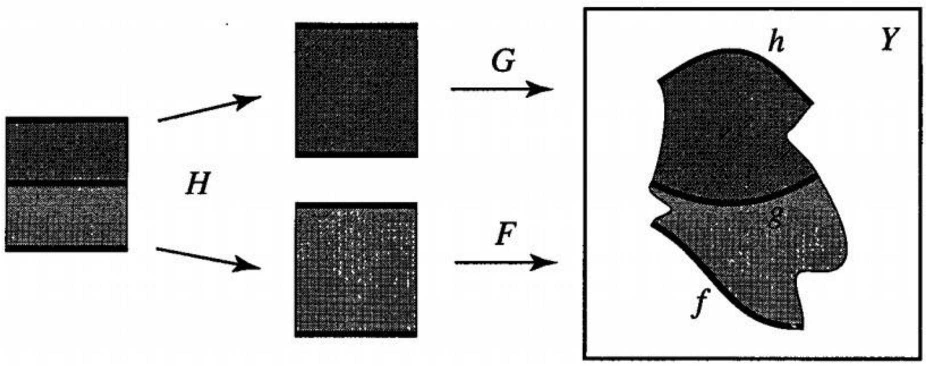

Figure 2.2: Forming H from F and G

that H is continuous by the Pasting Lemma. Since H(x, 0) = f(x) and H(x, 1)= h(x), then the prove that f = g and g = h imply f = h, shown in Figure 2.2 above and proved in Adams and Franzosa(2008)

Proof,By example 2.3.1, and by convexity. Let's take X, Y spaces to be two topological spaces, then(X, Y)a map is the set of all continuous maps from X to Y.

Definition 2.3.3.(Adams and Franzosa, 2008). For a subset B⊂R^n to be convex, when given any two points m, n ∈B refer to as straight line segment from m to n is contained in B, where (1-t)m + tn ∈ B, for every t∈[0, 1].

Preposition 2.3.1.(Adams and Franzosa, 2008) Consider A to be a convex subset of R^n which contains the sub-space topology, and X be any topological space.

Thus any two continuous maps f , g: X−→A are homotopic.

Preposition 2.3.2. If the space Y is contractible, then any map to this space f: X −→Y is **nullhomotopic.**

Proof : By composing the map f with the homotopy taking Y to a point and obtain a homotopy of f and the constant map to that point.

Chapter 3

Methodology

The Homotopy Analysis Method (HAM) to Fredholm and Volterra Integral Equations (FIEs and VIEs)

In this thesis, an application of the method known as (HAM) is employed to solve linear and nonlinear second kind of (FIEs and VIEs). In 1992, Liao proposed this method. The method helps us to get the solution of considered equations by summing the infinite solution series, ususlly converges to the exact solution. The method comes from homotopy, an aspect of topology. This HAM, a construction of continuous mapping with an initial guess and iterating with solution series of a given equation to arrive at the solution.The method allows the choosing of auxiliary linear operator as bases for constructing this continuous mapping and the method employs auxiliary parameter which helps to ensure the solution converges(Liao, 1997). The HAM again allows for the selection of initial approximation and auxiliary linear operators to obtain the required result which is valid for small and large parameters. A very important method and here the study treats nonlinear and linear Fredholm and Volterra equations by iteratively deforming the given equations to get series solutions where their sum give the exact solution. (Abu jarad, 2009)and (Adawi et al, 2009). Let

$$y(x) = g(x) + \lambda \int_a H(x,t)dt \tag{3.1}$$

In the equation (3.1) the upper limit is either variable or fixed depending whether it is Fredholm or Volterra type of integral equation(Wazwaz, 1997). λ, H(x, t) and g(x) are all parameters which are known of (3.1), whereas y is to be determined (Vahdati et al, 2010).

3.1 The description of the method

Let the following equation

$$N[y(x)]=0$$

Where N in the equation above is nonlinear operator, y(x) that is unknown function,and $y_0(x)$ represents an initial value of y(x). In order to construct the homotopy,there should be L which is the auxiliary linear operator that have a property L[y(x)] = 0 for y(x)= 0. As p∈[0, 1] is embedding homotopy parameter, (Liao, 1997) and(Adawi et al, 2009).

$$(1 - p)L[\varphi(x : p) - y_0(x)] - p\}H(x)N[\varphi(x : p)] = 0 \tag{3.2}$$

The method allows for the selection of $y_0(x)$ which is an initial guess, L and } which is introduced to ensure convergence(Liao, 2003) and the function H(x).

Then,

$$(1 - p)L[\varphi(x : p) - y_0(x)] = p\}H(x)N[\varphi(x : p)]. \tag{3.3}$$

As p=0,(Adawi et al, 2009 the equation (3.2) yields φ(x:0)= $y_0(x)$ Where for p=1 equation (3.2) as noted by Liao in (Liao, 2003)and (Liao, 1997) is equal to φ(x:1)= y(x). For (3.2) and (3.3) where'p' appreciate in value from 0 to 1, φ(x:p) changes progressively starting from $y_0(x)$ to exact solution y(x)(Hossein Zadeh et al, 2010). Where the changes that are seen are what is refered to as deformation. The exponential series of p of φ(x;p) is expanded by Taylor's theorem in the form as shown below

$$\phi(x : p) = y_0(x) + \sum_{n=1}^{\infty} y_n(x)p^n \tag{3.4}$$

where

$$y_n(x) = \frac{1}{n!}\frac{\delta^n \phi(x : p)}{\delta p^n}|_{p=0} \tag{3.5}$$

For the initial guess $y_0(x)$, L, } and function H(x) if properly chosen(Vahdati etal, 2010) helps for the convergence of the exponential series (3.4)of φ(x:p), where it converges at p=1.Then,the solution is presented in the form below(Adawi et al, 2009)

$$y_n(x) = \phi(x:1) = y_0(x) + \sum_{n=1}^{\infty} y_n(x) \tag{3.6}$$

Therefore,

$$y_n(x) = y_0(x), y_1(x), y_2(x), \ldots, y_n(x) \tag{3.7}$$

from (3.5), $y_n(x)$ is solved by the deformation equation (3.3). By means of differentiating (3.3) in n number of times for q thereby multiplying by $\frac{1}{n!}$ as p=0, the nth-series deformation equation is then realised (Adawi et al,2009).

$$L[y_n(x) - \chi_n y_{n-1}(x)] = \hbar H(x) R_n(\overrightarrow{y_{n-1}}(x)), y_n(0) = 0 \tag{3.8}$$

where

$$R_n(\overrightarrow{y_{n-1}}(x)) = \frac{1}{(n-1)!} \frac{\delta^{n-1} N[\phi(x:p)]}{\delta p^{n-1}}|_{p=0} \tag{3.9}$$

and

$$\chi_n = \begin{cases} 0 & \text{for } n \leq 1 \\ 1 & \text{for } n \geq 1 \end{cases}$$

As noted in (Abu jarad,2009) the equation(3.8) is governed by L,$R_n(\overrightarrow{y_{n-1}}(x))$ as shown in (3.9) for N (Adawi et al,2009). As a result, $y_n(x)$ can be realised by using **MATLAB** software. And the solution y(x) depends on the choice of L, }, H(x) as well as $y_0(x)$(Liao, et al., 2013). Note here that for this analytical techniques, the convergence sometimes are determined as $\sum_{n=0}^{n} y_n(x)$ moves to a limit as $n \longrightarrow \infty$, and as a result, gives the solution(Adawi et al, 2009) and (Vahdati etal, 2010).

3.2 HAM's solution to Fredholm and Volterra integral equations

Let consider equation

Z

$$h(t)u(t) = g(t) + \lambda \int_{r} H(t,x)u(x)dx \qquad (3.10)$$

where the solution to equation (3.10)can be categorized into both Fredholm and Volterra integral equations(Vahdati et al, 2010).

3.2.1 For the first kind of Fredholm and Volterra integral equations(Vahdati et al, 2010)

For instance, by substituting h(t)=0 into equation (3.10),as

$$g(t) + \lambda \int_{r}^{*} H(t,x)u(x)dx = 0, b \leq t \leq c \qquad (3.11)$$

by equation (3.3), the HAM's zeroth-order deformation for equation (3.11) will be

$$(1-p)u(t,p,\hbar) = \hbar p(g(t) + \int_{r}^{*} H(x,t)u(x,p,\hbar)dx \qquad (3.12)$$

As p=0 and p=1, then u(t,0,})=0,u(t,1,})=u(t).

For Maclauin series of u(t,p,})which match to p,

$$u(t,p,\hbar) = u(t,0,\hbar) + \sum_{n=1}^{+\infty} \frac{u_0^{[n]}(t,\hbar)}{n!} p^n \qquad (3.13)$$

which

$$u_0^{[n]}(t,\hbar) = \frac{\partial^n u(t,p,\hbar)}{\partial p^n}|_{p=0} \qquad (3.14)$$

If p=1,equation (3.13), gives

$$u(t) = \sum_{n=1}^{+\infty} \frac{u_0^{[n]}(t,\hbar)}{n!}$$ (3.15) And the nth-order deformation looks like

$$L[u_0^{[n]}(t,\hbar) - \chi_n u_0^{[n-1]}(t,\hbar)] = \hbar R_n(\overrightarrow{u_{n-1}}) \qquad (3.16)$$

The result of nth-order deformation equation where n≥1 is as follows

$$u_0^{[1]}(t,\hbar) = \hbar g(t) \qquad (3.17)$$

and

$$\frac{u_0^{[n]}(t,\hbar)}{n!} = \frac{u_0^{[n-1]}(x,\hbar)}{(n-1)!} + \hbar \int_r^* H(x,t) \frac{u_0^{[n-1]}(x,\hbar)}{(n-1)!} dx \qquad (3.18)$$

3.2.2 The second kind of Fredholm and Volterra integral equations(Vahdati et al, 2010)

If h(t)=1 is substituted into equation(3.10), then

$$u(t) = g(t) + \lambda \int_r^* H(t,x)u(x)dx, b \leq t \leq c \qquad (3.19)$$

Thus construct the zeroth-order deformation for this kind of integral equations as

$$(1-p)(u(t,p,\hbar) - g(t)) = \hbar p(u(t,p,\hbar) - g(t) - \int_r^* H(x,t)u(x,p,\hbar)dx) \qquad (3.20)$$

where p=0 and p=1,

u(t,0,})=g(t) u(t,1,})=u(t)

As Maclaurin series of u(t,p,}) matching to p,

$$u(t,p,\hbar) = u(t,0,\hbar) + \sum_{n=1}^{+\infty} \frac{u_0^{[n]}(t,\hbar)}{n!} p^n \qquad (3.21)$$

which

$$u_0^{[n]}(t,\hbar) = \frac{\partial^n u(t,p,\hbar)}{\partial p^n}|_{p=0} \qquad (3.22)$$

substituting p=1, into (3.21)gives

$$u(t) = g(t) + \sum_{n=1}^{+\infty} \frac{u_0^{[n]}(t,\hbar)}{n!}$$ (3.23) Thus the nth-order equation

$$L[u_0^{[n]}(t,\hbar) - \chi_n u_0^{[n-1]}(t,\hbar)] = \hbar R_n(\overrightarrow{u_{n-1}}) \tag{3.24}$$

And the result of the nth-series for n≥1 yields

$$u_0^{[1]}(t,\hbar) = -\hbar \int_r^* H(t,x) g(x) dx \tag{3.25}$$

and

$$\frac{u_o^{[n]}(t,\hbar)}{n!} = \frac{u_0^{[n-1]}(t,\hbar)}{(n-1)!} + \hbar \frac{u_0^{[n-1]}(t,\hbar)}{(n-1)!} - \hbar \int_r^* H(x,t) \frac{u_0^{[n-1]}(x,\hbar)}{(n-1)!} dx \tag{3.26}$$

The solution of the problem look similar to that of Homotopy Perturbation Method when one choose } = -1 (Liao, et al 2013) and (Vahdati et al, 2010)

Chapter 4

Application of HAM

In this part, the HAM is applied for solving some considered examples of FIE and VIE.

4.1 Fredholm integral equation (FIEs) of the second kind

Let first consider the following FIEs

$$\phi(x) = g(x) + \int_r^s H(x,t)\phi(t)dt \tag{4.1}$$

where H(x, t) refers to as kernel of the integral equation

Example 1. Let look at the following Fredholm integral equations

$$\phi(x) = 3x + \frac{1}{\alpha} \int_0^1 xt\phi(t)dt \tag{4.2}$$

To begin with, choose

$$\varphi_0(x) = 3x \tag{4.3}$$

Choose the linear operator L

$$L[\varphi(x,p)] = \varphi(x,p) \tag{4.4}$$

And defining N as

$$N[\phi(x,p)] = \phi(x,p) - 3x - \frac{1}{\alpha}\int_0^1 xt\phi(t)dt \tag{4.5}$$

where nth-series deformation is constructed as

$$L[\phi_n - \chi_n\phi_{n-1}] = \hbar R_n(\overrightarrow{\phi_{n-1}}) \tag{4.6}$$

And

$$R_n(\overrightarrow{\phi_{n-1}}) = \phi_{n-1}(x) - (1-\chi_n)3x - \frac{1}{\alpha}\int_0^1 xt\phi_{n-1}(t)dt \tag{4.7}$$

where the solution of the nth-series (4.6) becomes

$$\phi_n(x) = \chi_n\phi_{n-1}(x) + \hbar L^{-1}[R_n(\overrightarrow{\phi_{n-1}})] \tag{4.8}$$

Finally,

$$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty}\phi_n(x) \tag{4.9}$$

where

$\phi_0(x) = 3x$

$\phi_1(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_1(t)dt = -\hbar\frac{x}{\alpha}$

$\phi_2(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_1(t)dt = \hbar\frac{x}{3\alpha^2}$

$\phi_3(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_2(t)dt = -\hbar\frac{x}{9\alpha^3}$

$\phi_4(x) = -\frac{1}{\alpha}\int_0^1 xt\phi_3(t)dt = \hbar\frac{x}{27\alpha^4}$

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Hence

$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x)+\ldots$

$=3x + -\hbar\frac{1}{\alpha}x + \hbar\frac{1}{3\alpha^2}x + -\hbar\frac{1}{9\alpha^3}x + \hbar\frac{1}{27\alpha^4}x+\ldots$

If }=-1

$\phi(x) = 3x + \frac{1}{\alpha}x - \frac{1}{3\alpha^2}x + \frac{1}{9\alpha^3}x - \frac{1}{27\alpha^4}x+\ldots$

$$= 3x + \sum_{n=1}^{+\infty} \frac{(-1)^n}{3^{n-1}(\alpha)^n}x \tag{4.10}$$

which is the exact solution of equation(4.2)as shown in figure 4.1 below

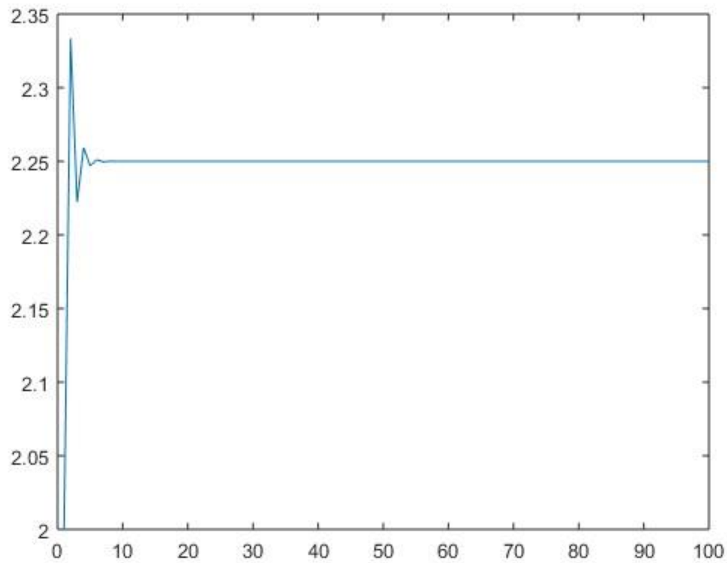

Figure 4.1: Example 1. Exact solution to equation (4.2)

Example 2. Let look at this FIE

$$\varphi(x) = 2x + x\int_0^1 y\varphi(y)dy \tag{4.11}$$

To begin with, choose

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$$\varphi_0(x) = 2x \tag{4.12}$$

the linear operator is presented as

$$L[\varphi(x,p)] = \varphi(x,p) \tag{4.13}$$

the nonlinear operator N is constructed on (4.11) as

$$N[\phi(x,p)] = \phi(x,p) - 2x - x\int_0^1 y\phi(y)dy \tag{4.14}$$

where (4.15) is expressed as follows

$$L[\phi_n - \chi_n\phi_{n-1}] = \hbar R_n(\overrightarrow{\phi_{n-1}}) \tag{4.15}$$

And

$$R_n(\overrightarrow{\phi_{n-1}}) = \phi_{n-1}(x) - (1-\chi_n)2x - x\int_0^1 y\phi(y)dy \tag{4.16}$$

Where the solution of (4.15) is

$$\phi_n(x) = \chi_n\phi_{n-1}(x) + hL^{-1}[R_n(\overrightarrow{\phi_{n-1}})] \tag{4.17}$$

Finally,

$$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty}\phi_n(x) \tag{4.18}$$

Then

$\phi_0(x) = 2x$

$\phi_1(x) = -x\int_0^1 y\phi_0(y)dy = -x[\frac{2}{3}y^3]_0^1 = -\hbar\frac{2}{3}x$

$\phi_2(x) = -x\int_0^1 y\phi_1(y)dy = -x[-\frac{2}{9}y^3]_0^1 = \hbar\frac{2}{9}x$

$\phi_3(x) = -x\int_0^1 y\phi_2(y)dy = -x[\frac{2}{27}y^3]_0^1 = -\hbar\frac{2}{27}x$

$\phi_4(x) = -x\int_0^1 y\phi_3(y)dy = -x[-\frac{2}{81}y^3]_0^1 = \frac{2}{81}x$

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Hence $\varphi(x)=\varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x)+\ldots$

$= 2x + -\hbar\frac{2}{3}x + \hbar\frac{2}{9}x + -\hbar\frac{2}{27}x + \hbar\frac{2}{81}x + \ldots$

If }=-1

$\phi(x) = 2x + \frac{2}{3}x - \frac{2}{9}x + \frac{2}{27}x - \frac{2}{81}x + \ldots$

$$= 2x + \sum_{n=1}^{+\infty}\frac{(-2)}{3^n}(x) \qquad (4.19)$$

Which is the exact solution to equation (4.11) which is represented in figure 4.2 below

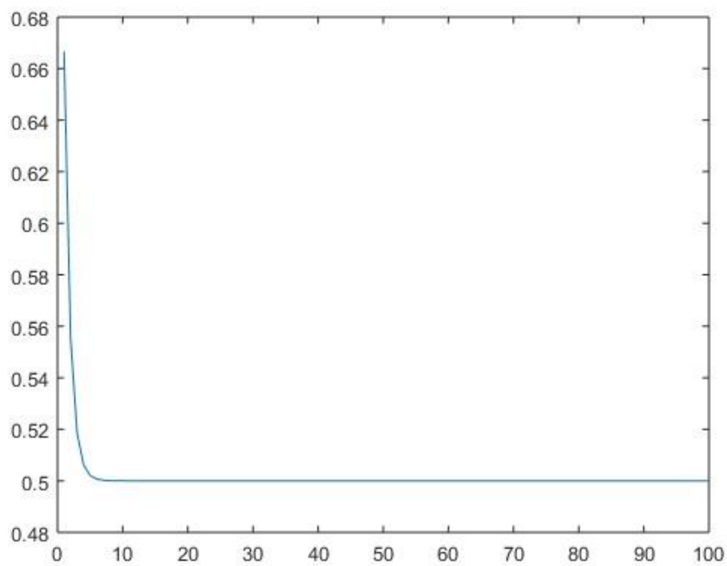

Figure 4.2: Example 2. Exact solution to equation (4.11)

4.2 The second kind of Volterra integral equation

Let's look at Volterra type, which reads

$$\phi(x) = g(x) + \int_a^x H(x,t)\phi(t)dt \tag{4.20}$$

where H(x,t), the kernel in (4.20)

Example 3. Let take a look at the followig example

$$\phi(x) = x + \int_0^x (3t - x)\phi(t)dt \tag{4.21}$$

To begin,choose

$$\varphi_0(x) = x \tag{4.22}$$

Let L befined as

$$L[\varphi(x,p)] = \varphi(x,p) \tag{4.23}$$

Thus, N is constructed on (4.21) as

$$N[\phi(x,p)] = \phi(x,p) - x - \int_0^x (3t - x)\phi(t)dt \tag{4.24}$$

Where

$$L[\phi_n - \chi_n \phi_{n-1}] = \hbar R_n(\overrightarrow{\phi_{n-1}}) \tag{4.25}$$

And

$$R_n(\overrightarrow{\phi_{n-1}}) = \phi_{n-1}(x) - (1-\chi_n)x - \int_0^x (3x - x)\phi(t)dt \tag{4.26}$$

The solution of (4.23) is

$$\phi_n(x) = \chi_n \phi_{n-1}(x) + \hbar L^{-1}[R_n(\overrightarrow{\phi_{n-1}})] \tag{4.27}$$

Finally,

$$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty} \phi_n(x) \tag{4.28}$$

Then

$\phi_0(x) = x$

$\phi_1(x) = -\int_0^x (3t - x)\phi_0(t)dt = -\hbar \frac{3}{3!}x^3$

$\phi_2(x) = -\int_0^x (3t - x)\phi_1(t)dt = \hbar \frac{9}{5!}x^5$

$\phi_3(x) = -\int_0^x (3t - x)\phi_2(t)dt = -\hbar \frac{27}{7!}x^7$

$\phi_4(x) = -\int_0^x (3t - x)\phi_3(t)dt = \hbar \frac{81}{9!}x^9$

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.

Hence $\varphi(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x) + \ldots$

$$= x + -\hbar \frac{3}{3!}x^3 + \hbar \frac{9}{5!}x^5 - \hbar \frac{27}{7!}x^7 + \hbar \frac{81}{9!}x^9 + \ldots$$

If }=-1

$= x + \frac{3}{3!}x^3 - \frac{9}{5!}x^5 + \frac{27}{7!}x^7 - \frac{81}{9!}x^9 + \ldots$

$$= \sum_{n=0}^{+\infty} \frac{(-3)^n}{(2n+1)!} x^{2n+1} \tag{4.29}$$

Which is the exact solution of equation(4.21)as shown in figure 4.3 below

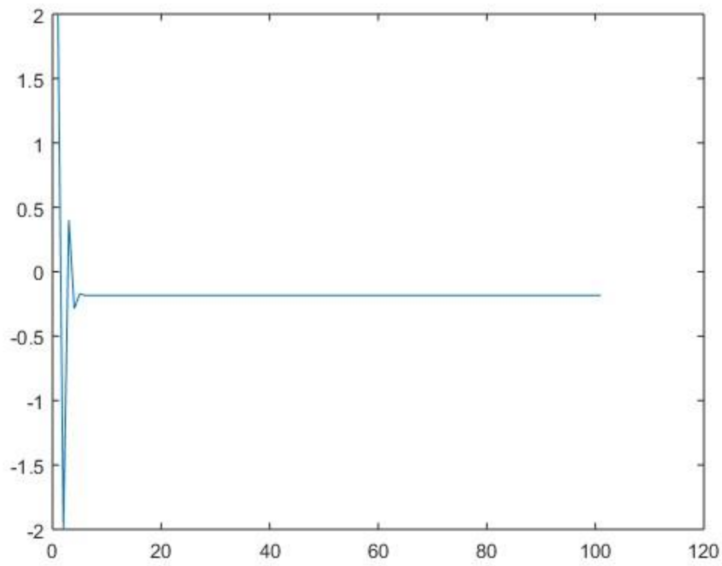

Figure 4.3: Example 3. Exact solution to equation (4.21)

Example 4. Let look at the example below

$$\phi(x) = 2x - x^2 - \int_0^x \phi(t)dt \tag{4.30}$$

To solve equation (4.30),choose

$$\varphi_0(x) = 2x - x^2 \tag{4.31}$$

Where L is constructed as

$$L[\varphi(x,p)] = \varphi(x,p) \tag{4.32}$$

Thus,nonlinear operator is define as

$$N[\phi(x,p)] = \phi(x,p) - 2x + x^2 + \int_0^x \phi(t)dt \tag{4.33}$$

And the nth-series is as follow

$$L[\phi_n - \chi_n\phi_{n-1}] = \hbar R_n(\overrightarrow{\phi_{n-1}}) \tag{4.34}$$

where

$$R_n(\overrightarrow{\phi_{n-1}}) = \phi_{n-1}(x) - (1-\chi_n)2x + x^2 + \int_0^x \phi(t)dt \tag{4.35}$$

Hence the solution of (4.30) presents as

$$\phi_n(x) = \chi_n\phi_{n-1}(x) + \hbar L^{-1}[R_n(\overrightarrow{\phi_{n-1}})] \tag{4.36}$$

Finally

$$\varphi(x) = \varphi_0(x) + \sum_{n=1}^{+\infty} \varphi_n(x) \tag{4.37}$$

And

$\phi_0(x) = 2x - x^2$

$\phi_1(x) = \int_0^x \phi_0(t)dt = \int_0^x (2t - t^2)dt = \hbar(x^2 - \frac{1}{3}x^3)$

$\phi_2(x) = \int_0^x \phi_1(t)dt = \int_0^x (t^2 - \frac{t^3}{3})dt = \hbar(\frac{x^3}{3} - \frac{x^4}{12})$

$\phi_3(x) = \int_0^x \phi_2(t)dt = \int_0^x (\frac{t^3}{3} - \frac{t^4}{12})dt = \hbar(\frac{x^4}{12} - \frac{x^5}{60})$

$\phi_4(x) = \int_0^x \phi_3(t)dt = \int_0^x (\frac{t^4}{12} - \frac{t^5}{60})dt = \hbar(\frac{x^5}{60} - \frac{x^6}{360})$

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$\varphi(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x) + \ldots$

$$=2x - x^2 + \hbar(x^2 - \frac{x^3}{3}) + \hbar(\frac{x^3}{3} - \frac{x^4}{12}) + \hbar(\frac{x^4}{12} - \frac{x^5}{60}) + \hbar(\frac{x^5}{60} - \frac{x^6}{360})\ldots$$

when }=-1,then

$=2x - x^2 - (x^2 - \frac{x^3}{3}) - (\frac{x^3}{3} - \frac{x^4}{12}) - (\frac{x^4}{12} - \frac{x^5}{60}) - (\frac{x^5}{60} - \frac{x^6}{360})\ldots$

$$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x) = 2x - 2x^2 \quad (4.38)$$

Which is the exact solution to equation (4.30)

Example 5. consider this equation

$$\phi(x) = x + \int_0^x \phi^2(t)dt \quad (4.39)$$

To solve equation(4.39),choose

$$\varphi_0(x) = x \quad (4.40)$$

Then the operator L is

$$L[\varphi(x,p)] = \varphi(x,p) \quad (4.41)$$

Thus,the operator N is as

$$N[\phi(x,p)] = \phi(x,p) - x - \int_0^x \phi^2(t)dt \quad (4.42)$$

construct nth-series equation as

$$L[\phi_n - \chi\phi_{n-1}] = \hbar R_n(\overrightarrow{\phi_{n-1}}) \quad (4.43)$$

where

$$R_n(\overrightarrow{\phi_{n-1}}) = \phi_{n-1}(x) - (1-\chi_n)x - \int_0^x \phi^2(t)dt \quad (4.44)$$

The solution of the nth-order deformation(4.39)

$$\phi_n(x) = \chi_n\phi_{n-1}(x) + \hbar L^{-1}[R_n(\overrightarrow{\phi_{n-1}})] \quad (4.45)$$

Finally

$$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty} \phi_n(x) \quad (4.46)$$

And ,

$\phi_0(x) = x$

$\phi_1(x) = -\int_0^x \phi_0^2(t)dt = -[\frac{1}{3!}t^3]_0^x = -\hbar\frac{1}{3!}x^3$

$\phi_2(x) = -\int_0^x \phi_1^2(t)dt = -[\frac{1}{7!}x^7]_0^x = -\hbar\frac{1}{7!}x^7$

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$$\phi_3(x) = -\int_0^x \phi_2^2(t)dt = -[\frac{1}{15!}t^{15}]_0^x = -\hbar\frac{1}{15!}x^{15}$$
$$\phi_4(x) = -\int_0^x \phi_3^2(t)dt = -[\frac{1}{31!}t^{31}]_0^x = -\hbar\frac{1}{31!}x^{31}$$

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Hence
$$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x) + \ldots$$
$$x - \hbar\frac{1}{3!}x^3 - \hbar\frac{1}{7!}x^7 - \hbar\frac{1}{15!}x^{15} - \hbar\frac{1}{31!}x^{31} + \ldots$$
If } = −1
$$x + \frac{1}{3!}x^3 + \frac{1}{7!}x^7 + \frac{1}{15!}x^{15} + \frac{1}{31!}x^{31} + \ldots$$

$$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty}\phi_n(x) \qquad (4.47)$$

Which is the exact solution to equation(4.39) as shown in figure 4.4 below

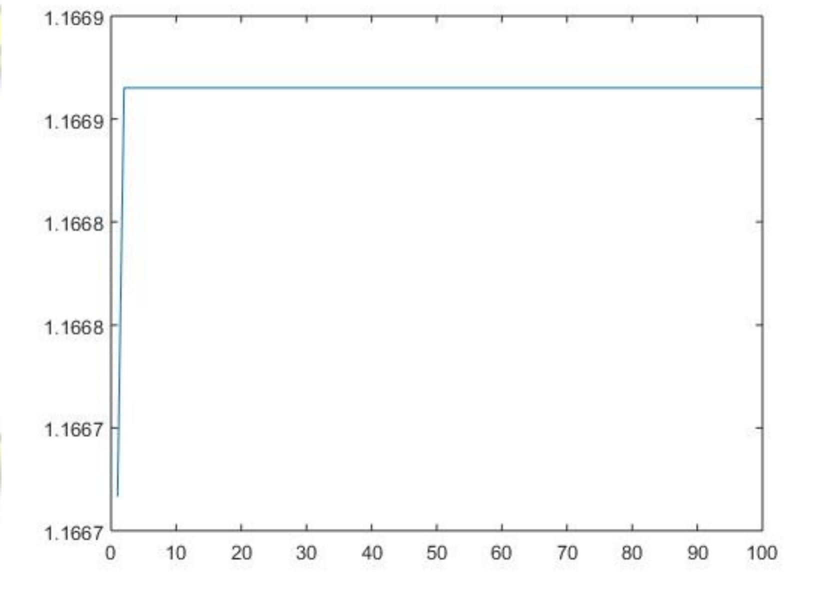

Figure 4.4: Example 5. Exact solution to equation (4.39)

Example 6. Let consider the Volterra integral equations

$$\phi(x) = x + \frac{1}{2}\int_0^x \phi^2(t)dt$$

(4.48)

To solve equation(4.48),choose

$$\varphi_0(x) = x \qquad (4.49)$$

then operator L is defined as

$$L[\varphi(x,p)] = \varphi(x,p) \qquad (4.50)$$

Thus,the N is explain on the equation as

$$N[\phi(x,p)] = \phi(x,p) - x - \frac{1}{2}\int_0^x \phi^2(t)dt \qquad (4.51)$$

Constructing the nth-series equation is represented as

$$L[\phi_n - \chi\phi_{n-1}] = \hbar R_n(\overrightarrow{\phi_{n-1}}) \qquad (4.52)$$

where

$$R_n(\overrightarrow{\phi_{n-1}}) = \phi_{n-1}(x) - (1-\chi_n)x - \frac{1}{2}\int_0^x \phi^2(t)dt \qquad (4.53)$$

The solution of the nth-order deformation(4.48)

$$\phi_n(x) = \chi_n\phi_{n-1}(x) + \hbar L^{-1}[R_n(\overrightarrow{\phi_{n-1}})] \qquad (4.54)$$

Finally

$$\varphi(x) = \varphi_0(x) + \sum_{n=1}^{+\infty}\varphi_n(x) \qquad (4.55)$$

And ,

$\phi_0(x) = x$

$\phi_1(x) = -\frac{1}{2}\int_0^x \phi_0^2(t)dt = -\frac{1}{2}[\frac{(t)^3}{3}]_0^x = -\hbar\frac{1}{6}x^3$

$\phi_2(x) = -\frac{1}{2}\int_0^x \phi_1^2(t)dt = -\frac{1}{2}[\frac{t^{10}}{10}]_0^x = -\hbar\frac{1}{20}x^{10}$

$\phi_3(x) = -\frac{1}{2}\int_0^x \phi_2^2(t)dt = -\frac{1}{2}[(\frac{1}{101})t^{101}]_0^x = -\hbar\frac{1}{202}x^{101}$

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Hence

$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \ldots$

$x - \hbar\frac{1}{6}x^3 - \hbar\frac{1}{20}x^{10} - \hbar\frac{1}{202}x^{101} + \ldots$

If } = −1

$$x + \frac{1}{6}x^3 + \frac{1}{20}x^{10} + \frac{1}{202}x^{101} + \ldots$$

Which is the exact solution to equation(4.48)as shown in figure 4.5 below.

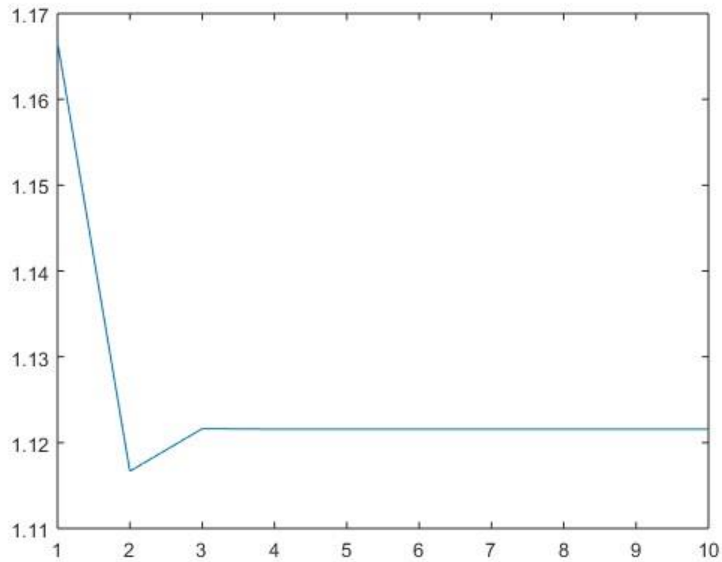

Figure 4.5: Example 6. Exact solution to equation (4.48)

Chapter 5

Conclusion

5.1 Summary and Conclusion

This work focused on an analytical method proposed by Liao, in the name homotopy analysis method (HAM) to deal with linear and nonlinear integral equations (I.E)of FIE and VIE of the second kind. A powerful method in seaking analytical answers for integral equations. We begun by exploring historical background of integral equations, classification of integral equations and glossing over kinds of kernel as well as review of spaces and operators that have been subsequently brought forth in this work. A review of integral equations and HAM as well as the concept of homotopy function are looked at. The research presented a description of HAM and subsequently followed it by analytical solutions of the method. Where a MATLAB codes generated give a graphical representation of the various solutions of considered equations.

In analysising a given equation by HAM, the research considered the Fredholm(FIE) and Volterra(VIE) in the process of the analysis. The analysis of the homotopy solution series of specific examples are iteratively determined. As the perturbation method is dependent of small/large physical parameters, this method does not relie on such. The controlling tool } in the solution series helps greatly in influencing the solution converging but this control parameter is to be chosen properly to influence the convergence. In order to obtain the exact solution, the infinite series solutions are summed. Apparently, as seen from the report, an integral equation is best analysed by HAM a powerful analytical method. The HAM presents an easy approach for ensuring the convergence of the approximation solution series. It also makes it easy and more flexible in choosing the type of equation. It is apparently seen from the result that this analytical method is an efficient approach of getting solutions of linear and nonlinear equations.

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Appendix A

```
function [x,sumc] = solplot1(x,alpha,n) sumc(1) =
3*x; for i=1:n num = (-1)^i; m=i-1; den = (3^m) *
(alpha^i); rsult = num*x/den; sumc(i+1) = sumc(i)
+ rsult; end plot(1:n,sumc(2:end)) end function
[x,sumc] = solplot2(x,n) sumc(1) = 2*x; for i=1:n
num = (-2); den = 3^i; rsult = num*x/den;
sumc(i+1) = sumc(i) + rsult; end
     %plot(1:n+1,sumc)
plot(1:n,sumc(2:end)) end function
[x,sumc] = solplot3(x,n) sumc(1) = x;
    for i=1:n num = (-3)^i; den = factorial(2*i+1); rsult =
num/den; rsult = rsult*x^(2*i+1); sumc(i+1) = sumc(i) + rsult;
end plot(1:n+1,sumc) %plot(1:n,sumc(2:end)) end function
[x,sumc] = solplot4(x,n) sumc(1) = x; m=1; for i=1:n m =
2*m+1; num = 1; den = factorial(m); rsult = (num*x^m)/den;
sumc(i+1) = sumc(i) + rsult; end plot(1:n,sumc(2:end)) end
function [x,sumc] = solplot5(x,n) sumc(1) = x; m=sqrt(2); for
i=1:n
        m = m^2 + 1; num
        = x^m; den = 2*m;
        rsult = (-1)^(i+1) *
        (num/den);
        sumc(i+1) =
        sumc(i) + rsult;
        end
```

```
plot(1:n,sumc(2:e
nd)) end

% % Script to run
solplot1(1,1,100)

solplot2(1,100)

solplot3(0.5,100)

solplot4(1,100)

solplot5(1,100)
```

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