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Stone's theorem and its applications to particle properties.

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

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By

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(BSc. Mathematics)

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Declaration

I hereby declare that, this thesis is the result of my own original research and that no part of it has been submitted to any institution or organization anywhere for the award of a degree. All inclusion for the work of others has been dully acknowledged.

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Dedication

. To my parent, relatives, friends and anyone who motivated and supported me in prayers and financially.

Abstract

Most of the physical laws associated with quantum mechanics are formulated in a mathematical framework where observables are represented as self-adjoint operators in Hilbert space. These self-adjoint operators are unbounded and therefore very hard to work with. Stone’s theorem makes it a little bit easier by establishing a bijection between a strongly continuous one-parameter group and self-adjoint operators.

We began with the needed terminology, and then proved the stones theorem. In addition, we have indicated some applications of Stone’s theorem , particularly those associated with quantum mechanics (dilation and rotation in the Cartesian coordinates).

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Finally, I owe my mum an expression of sincere gratitude for her patience, understanding and encouragement during my research.

Contents

Acronyms

W.T.S	Want to show
$U_a(t)$	Strongly continuous unitary group
H_b	Hilbert space
X_v	Vector space
X_ω	Inner-product space
T_a	operators
$B(H^b)$	Bounded Hilbert Space
$D(T_a)$	Domain of an unbounded operator.
h,i	Inner-product
k · k	Normed linear Space
(kk,X_v)	Normed Vector Space
$C_0(\mathrm{R})$	Continuous function with compact support
$C[a,b]$	Function space
WE	Wave Equation
QM	Quatum mechanics
IDHS	Infinite Dimensional Hilbert Space

RVF	Real-valued function
PDE	Partial differential equation
I	Dilation operator
I_x	Identity operator
SCUG	Strongly Continuous Unitary Group
HS	Hilbert Space

Chapter 1

Introduction

1.1 Overview

This chapter consists of the following: the background of the study, problem statement, objectives of the research and its structure

1.1.1 Background of the study

Most of the physical laws that governed QM can be formulated on the basis of the fundamental laws of mathematics where elements of a mathematical framework are mapped to physical objects. In the study of QM, the Observables (position, energy, momentum, etc.) of a physical system are characterized as self-adjoint operators in HS. However, these self-adjoint operators naturally existing in quantum theory are unbounded, and it is very necessary to provide new and efficient way of dealing with such operators: this is where Stone's theory comes to the fore. Stone's theorem on a strongly continuous one-parameter unitary group is one of the powerful tools that make life a little bit easier by establishing a bijection between self-adjoint operators and one parameter unitary group. In addition, it gives us a way to write down certain families of unitary operators in terms of self-adjoint, possibly unbounded operators. Translated in physical terms, this gives rise to statements such as ”momentum generates translation” or ”angular momentum generates rotation.”

1.1.2 Problem statement

Despite all the existing theories, the lack of boundedness property of the self-adjoint operator make life extremely difficult. Although some attention has been given to the unbounded operators, there exist a limited study to explain the unboundedness of the self-adjoint op-
erators.

1.1.3 Main objective

The main objective was to review a paper by Sven Moller(Mo¨ller (2010)) on Stone’s theorem and it’application. However, in order to do this, we made valuable use of functional calculus of Marcello Porta ((Porta, 2019))

1.1.4 Specific objectives

Based on the problem statement given, these are the objectives

1. To parametrize strongly continuous unitary operators in terms of self-adjoint operators.

2. To formulate the mathematical description of quantum mechanics in a strong theoretical setting.
3. To prove Stone's theory on strongly continuous unitary group using the direct approach.

1.2 Justification

QM observables are described by operators acting on the Wave-functions belonging to the HS of the systems under consideration. However, in contrast to the mathematical literature, where operators are defined by their action (that is, what they do to the functions on) and by their domain (i.e, the set of functions on which they operate) in the physical literature domain are seldom mentioned and operators are defined only by their actions. Operators in IDHS are not defined for all the functions of the space, and this suggests that one should be aware of situations where domains of the operators are so important, even in physics, that we need the operators in quantum mechanics to be self-adjoint and operators are self-adjoint only in well-defined and prescribed domains. Until quite recently, domains or self-adjoint were not mentioned in physical. However, in every recent years, some articles in physics literature begun to point out examples where domains of operators are essential to the full solution of the problems posed. As far as our research is concerned, not enough have been published before those articles that mention domains and self-adjointness. Hence, among other reason, this research thesis seeks to build upon the possibility and importance of self-adjointness within the realm of quantum mechanics.

1.3 Scope of work and outline of project

The project is presented in five chapters.

Chapter one is divided into two parts: the first part presents the background of study. The second part of this chapter present statement of the problem, objectives and justification.

Chapter two: Spaces and Bounded Operators

Chapter three: The methods used in carrying out the entire project. Theory of unbounded operators with emphasis to self-adjoint operators, spectral theory and functional calculus.

Chapter four: Mathematical formulation of Quantum mechanics, prove of stones theory on strongly continuous unitary group and its applications.

Chapter five : conclusions and recommendations.

Chapter 2

Spaces and Bounded Operators

2.1 Overview

To start with, we need to look at the general concept of spaces in purely mathematical settings, including the important concept of Hilbert space. The theory of Hilbert space generalizes the definition of Euclidean space and therefore extends the principle of capturing vector algebra of 2-3 dimensional spaces to either finite or infinite dimensional spaces. Nevertheless, it has an inner-product that allows the measurement of vector length, angle and perpendicularity to be determined. The theory of spaces can, of course, contribute to the study of linear operators and bounded operators. This linkage opens up our study to cover spaces that are specifically classified as domains and co-domains of defined operators. Next to follow closely is the analysis of One Parameter Unitary Group, a group of unitary operators defined on Hilbert spaces with some additional properties, such as continuity and homomorphism. Finally, we will be involved in the study of Fourier Analysis, an important theory that regulates the superposition of wave-function.

2.1.1 Linear Vector Space(LVS)

In practical and theoretical sense, we usually encounter physical situations involving a set $\mathbf{X}_v$, whose nature of elements are vectors either in two or three dimensional space, or a sequence of numbers, or functions. The elements in the set $\mathbf{X}_v$ can be added and multiplied by constants and the resulting is also a member of the set $\mathbf{X}_v$. The constant is considered as generalized field $\mathbf{K}_{sc}$, and whose elements are mainly real and complex numbers. However, such a physical situations proposes the concept of vector space as defined below.

Definition 1. Suppose $\mathbf{X}_v$ 6= $\emptyset$ and K_{sc} be a scalar field. Then a mapping

$\mathbf{+} : X_v \times X_v \longrightarrow X_v$ and

$\times : K_{sc} \times X_v \longrightarrow X_v$

is called addition and scalar multiplication respectively. That is,

$(\forall x_v, y_v \in \mathbf{X}_v$ $x_v + y_v \in \mathbf{X}_v$ and $\alpha \in \mathbf{K}_{sc}$ such that $\alpha x_v \in \mathbf{X}_v)$:

$\mathbf{X}_v$ is an abelian

1. $\forall x_v, y_v, z_v \in \mathbf{X}_v$ and $\alpha, \beta \in \mathbf{K}_{sc}$

   (a) $x_v + y_v = y_v + x_v$ commutativity property

   (b) $(x_v + y_v) + z_v = x_v + (y_v + z_v)$ associativity property

   (c) if $\exists$ $0 \in \mathbf{X}_v : x_v + 0 = x_v$ identity property

   (d) if $\exists$ $(-x_v) \in \mathbf{X}_v : x_v + (-x_v) = 0$ inverse property

2. $(\alpha + \beta)x_v = \alpha x_v + \beta x_v$

3. $\alpha(x_n u + y_v) = \alpha x_v + \alpha y_n u$

4. $\alpha(\beta x_v) = (\alpha\beta)x_v$

5. $x_v \times 1 = x_v$, where $1 \in \mathbf{K}_{sc}$

Thus, we call $\mathbf{X}_v$ linear vector space when the coefficients are real numbers, and complex linear vector space when the coefficients are complex numbers.

Definition 2 (Subspace)**.** Suppose $\mathbf{Y}_v$ is a subspace of linear vector space $\mathbf{X}_v$ if

$\forall y_{v1}, y_{v2} \in \mathbf{Y}_v$ and $\alpha, \beta \in \mathbf{K}_{sc}$ then,

$$\alpha y_{v1} + \beta y_{v2} \in \mathbf{Y}_v \tag{2.1}$$

In addition, $\mathbf{Y}_v$ is a linear Vector space because of the algebraic structure it inherited from $\mathbf{X}_v$

Definition 3 (Linear combination)**.** Given vectors $\{y_i\}_{i=1}^{n} \in \mathbf{X}_\nu$ and the set of scalars $\{\alpha\}_{i=1}^{n} \in \mathbf{K}_{sc}$. An expression of the form $\alpha_1 y_{v1} + \alpha_2 y_{v2} + \alpha_3 y_{v3} +,\dots,+ \alpha_n y_{vn}$ is called linear combination

Definition 4 (Linear independence and Linear dependence)**.** We say $\{y_{\nu i}\}_{i=1}^{n} \in \mathbf{X}_\nu$ is linearly independence if there exist a linear combination of the vectors in $\mathbf{X}_v$ such that

$$\alpha_1 y_{v1} + \alpha_2 y_{v2} + \alpha_3 y_{v3} +,\dots,+ \alpha_n y_{vn} = 0 \tag{2.2}$$

where $\{\alpha\}_{i=1}^{n}$ are scalars and are all zero. Geometrically, Any linear independence set of vectors in a space will always generate another vector under the addition and scalar multiplication, and a such, the resultant vector is coplanar to the plane. However, if there exist at least one non zero scalar say $\alpha_1 \in \{\alpha\}_{i=1}^{n}$ such that the equation(1.2) holds.

Then the set of vectors are linearly dependence. In fact, $\{y_{\nu i}\}_{i=1}^{n}$ is linearly dependent if $\exists y_{\nu 1} \in \{y_{\nu i}\}_{i=1}^{n}$ such that

$$y_{v1} = \alpha_2 y_{v2} + \alpha_3 y_{v3} + \alpha_4 y_{v4} + \cdots + \alpha_n y_{vn} \tag{2.3}$$

or it can be written as linear combination of other vectors in the set (at least one vector is redundant), and a such all the vectors are collinear.

Definition 5 (Span of a Linear Vector Space). Suppose $\mathbf{Y}_v = \{x_{v1}, x_{v2}, x_{v3}, \cdots, x_{vn}\}$ is a subspace to a linear vector space X_v, then the span of $\mathbf{Y}_v$ is the set of linear combination of the vectors in $\mathbf{Y}_v$,

$Span(\mathbf{Y}_v) = \{\alpha_1 x_{v1} + \alpha_2 x_{v2} + \alpha_3 x_{v3}, \cdots + \alpha_n x_{vn}\}$ such that $\forall\{\alpha_i\}_{i=1}^{n} \in \mathbb{K}_{sc}$. The span of $\mathbf{Y}_v$ is denoted by $Span(\mathbf{Y}_v)$. However, if $Span(\mathbf{Y}_v) = \mathrm{X}_v$, then X_v is spanned by $\{x_{v1}, x_{v2}, x_{v3}, \cdots, x_{vn}\}$. Moreover, $\mathbf{Y}_v$ is a spanning set if all the vectors in X_v can be written as a linear combination of vectors in $\mathbf{Y}_v$.

Definition 6 (Basis of a Linear Vector Space). Let $\{e_{\nu i}\}_{i=1}^{n}$ be a set of linearly independent vectors. The set $\{e_{\nu i}\}_{i=1}^{n}$ is called the basis for $\mathbf{X}_v$ if the set

$$Span\{e_{\nu i}\}_{i=1}^{n} = \mathbf{X}_\nu \tag{2.4}$$

In fact, since $\{e_{\nu i}\}_{i=1}^{n}$ is the basis vectors for $\mathbf{X}_\nu$. Then $\forall x_\nu \in \mathbf{X}_\nu$ there is a unique representation called linear combination of the basis vectors, such basis vectors are called **CANONICAL BASIS/STANDARD BASIS** for $\mathbb{R}^n$. Now, if $\{\alpha_i\}_{i=1}^{n}$ are scalars and

$e_{v1} = (1,0,0,...,0)$ $e_{v2} =$

$(0,1,0,...,0)$ $e_{v3} =$

$(0,0,1,...,0)$

...

$e_{vn} = (0,0,0,...,1)$

thus $x_v = \alpha_1 e_{v1} + \alpha_2 e_{v2} + \alpha_3 e_{v3} +,...,+ \alpha_n e_{vn}$. Consider a vector space $\mathbf{Y}_v \subset \mathbf{X}_v$ in which $\forall y_v \in Y_v$ are linearly independent and $y_v = span(\mathbf{X}_v)$. Then $\mathbf{Y}_v$ is called the Hamel basis and obeys equation 2.4

2.1.2 Dimension of Linear Vector Space

The dimension of linear vector space $\mathbf{X}_v$ depends on the cardinality of the basis vectors that spans $\mathbf{X}_v$. However, two types of dimensions are encountered in analysis. Namely, finite and infinite dimensional linear vector space. The infinite dimensional space is of interest and most relevant in analysis.

2.1.3 Finite Dimensional Linear vector space ($\mathbf{X}_\nu$)

X_ν is finite dimensional if $n \in N$, and $\exists\{x_{\nu i}\}_{i=1}^{n} \in \mathbf{X}_\nu$ which are linearly independent vectors such that

$$\mathbf{X}_\nu = Span\{x_{\nu i}\}_{i=1}^{n} \tag{2.5}$$

Furthermore, if we have a fixed amount of vectors that generates all the elements in a space, then the X_ν is finite dimensional.

Example 2.1.3.1. The Euclidean space (R^n): This space consists of the collection of all ordered n-tuples of real numbers. That is, $\forall x_\nu, y_\nu \in R^n$ where $n \in N$, we have $x_\nu = (x_{\nu 1}, x_{\nu 2}, x_{\nu 3}, \cdots, x_{\nu n})$ and $y_\nu = (y_{\nu 1}, y_{\nu 2}, y_{\nu 3}, \cdots, y_{\nu n})$ and form a vector space under two algebraic operations.

$$x_\nu + y_\nu = (x_{\nu 1}, x_{\nu 2}, x_{\nu 3}, \cdots, x_{\nu n}) + (y_{\nu 1}, y_{\nu 2}, y_{\nu 3}, \cdots, x_{\nu n})$$

$$\alpha x_\nu = (\alpha x_{\nu 1}, \alpha x_{\nu 2}, \alpha x_{\nu 3}, \cdots, \alpha x_{\nu n}) \quad \text{where} \quad \alpha \in K_{sc}$$

The Unitary Space C^n: The space is consists of all ordered n-tuples of complex numbers and does form a vector space under the supervision of addition and scalar multiplication.

That is; $\forall z, w \in C^n$, where $z = (z_1, z_2, z_3, \cdots, z_n)$ and $w = (w_1, w_2, w_3, \cdots, w_n)$ then

$$z + w = (z_1, z_2, z_3, \cdots, z_n) + (w_1, w_2, w_3, \cdots, w_n)$$
$$\alpha z = (\alpha z_1, \alpha z_2, \alpha z_3, \cdots, \alpha z_n)$$

The Polynomial Space: It is a vector space consisting of all polynomials of degree 2 or less and it is closed under linear combinations. It is closed because

1. Adding any two such polynomials in the space will yield another polynomial in the space
2. Scaling any such polynomial also produces another polynomial in the space.

In fact, the polynomial space of such degree is finite dimensional since a basis for it consists of the three polynomials, $1, x_\nu, x^2_\nu$. That is to say, every function in this space can be uniquely written as $1 + \alpha x_\nu + \beta x^2{}_\nu$ where $\alpha, \beta \in K_{sc}$

Definition 7 (Convergence)**.** A sequence $< X_{\nu n} >$ of vectors in $\mathbf{X}_\nu$ is convergent if $\forall \epsilon > 0. \exists x_\nu \in \mathbf{X}_\nu$, and $m \in \mathrm{N}$:

$$|X_{\nu n} - x_\nu| < \varepsilon \qquad n \geq m$$

Then x_ν is the limit of $< X_{\nu n} >$. Also if a sequence $< X_{\nu n} >$ is convergent and x_ν is the limit, then we say that the sequence $< X_n >$ converges to x_ν, and in symbols, we write

$$\lim X_n = x_\nu \quad \text{or} \quad X_n \longrightarrow x_\nu \quad \text{as} \quad n \longrightarrow \infty \ {}_{n \longrightarrow \infty}$$

Theorem 1. If the limit of a sequence exist, then it's unique.

Proof. If we set $< X_{\nu n} >$ to converge to two distinct limits say $x_\nu, y_\nu : x_\nu$ 6= y_ν so, $|x_\nu - y_\nu| > 0$ and $\varepsilon = \frac{1}{2}|x_\nu - y_\nu| > 0$

Case 1 let $X_{\nu n}$ converges to a x_ν, then $\forall \varepsilon > 0$ and $m_1 \in \mathrm{N}$ we have

$$|X_{\nu n} - x_\nu| < \varepsilon \qquad \forall n \geq m_1$$

Case 2 let $X_{\nu n}$ converges to a y_ν, then $\forall \varepsilon > 0$ and $m_2 \in$ N we have

$$|X_{\nu n} - y_\nu| < \epsilon \qquad \forall n \geq m_2$$

Let $m = \max(m_1, m_2)$

$\implies |X_{\nu n} - x_\nu| < \epsilon \qquad \forall n \geq m$

$\implies |X_n - y_\nu| < \epsilon \qquad \forall n \geq m$

$\therefore |x_\nu - y_\nu| = |x_\nu - X_{\nu n} + X_n - y_\nu|$

$|x_\nu - y_\nu| \leq |x - X_n| + |X_{\nu n} - y|$

$|x_\nu - y_\nu| < \epsilon + \varepsilon \qquad \forall n \geq m$

$|x_\nu - y_\nu| < 2\varepsilon \quad \forall n \geq m$

$|x_\nu - y_\nu| < |x_\nu - y_\nu|$

Thus, we arrive at a contradiction. The assumption x_ν 6= y_ν is False.

□

Definition 8. A space X$_\nu$ is bounded if it is both bounded below and bounded above. If $\exists (K_1, K_2) \in$ R :

$$K_1 \leq x_\nu \leq K_2 \qquad \forall x_\nu \in \mathrm{X}_\nu$$

where K_1 is the lower bound and K_2 is the upper bound.

Theorem 2. Every convergent sequence is bounded

Proof. Let $\{X_{\nu n}\}_{n \geq 1}$ be a convergent sequence.

W.T.S $\{X_{\nu n}\}_{n \geq 1}$ is bounded

From definition of convergence, $\forall \varepsilon > 0, \exists x_\nu \in \mathbf{X}_\nu$, and $m \in$ N such that

$|X_{vn} - x_v| < \varepsilon \qquad n \geq m$

if we set $\varepsilon = 1 \implies |X_{vn} - x_v| < 1$

$-1 \leq X_{vn} - x_v \leq 1 \implies x_v - 1 \leq X_{vn} \leq x_v + 1$

set $K_1 = \min\{x_{v1}, x_{v2}, x_{v3}, \cdots, x_{vm} - 1, x_v - 1\}$ and

$K_2 = \max\{x_{v1}, x_{v2}, x_{v3}, \cdots, x_{vm} - 1, x_v + 1\}$

$\therefore K_1 \leq X_{vn} \leq K_2$

Hence the sequence $\{X_{vn}\}_{n\geq 1}$ is bounded. □

2.1.4 Normed Linear Space

Here, we will look at norms as a mapping defined on linear vector space, the elements of which are assigned distance or size.This concept is necessary because it improves the understanding of geometry as a distance of wave function from its point of reference to a fixed point. In addition, the concept of completeness will also be introduced by assuming that there is a Cauchy sequence in the space whose elements will converge to a point in space. We will then end by giving some examples of normed linear spaces that are complete and for that matter they are Banach spaces.

Definition 9 (Normed Linear Space (NLVS))**.** Suppose $\mathbf{X}_v$ is a linear vector space A norm defined on $\mathbf{X}_v$ is a real-valued mapping denoted as $\|\cdot\|$ where, $\|\cdot\| : \mathbf{X}_v \longrightarrow [0\ \infty[\ : \forall \xi_v, \varepsilon_v \in \mathbf{X}_v$ and $\exists \alpha, \beta \in \mathbb{K}_{sc}$.Then the following conditions must be satisfied

(NLVS1) $\|\xi_v\| \geq 0$ and $\|\xi_v\| = 0 \iff \xi = 0$ (positive definite).

(NLVS2) $||\alpha\xi_v|| = |\alpha\|\xi_v\|$ (homogenous)

(NLVS3) $\|\xi_v + \varepsilon_v\| \leq \|\xi_v\| + \|\varepsilon_v\|$ (triangle inequality)

Hence, the linear space $\mathbf{X}_v$ with norm define on it is called a normed linear space denoted as $(\mathbf{X}_v, \|\cdot\|)$

Show that ℓ_∞ is a normed linear vector space

Proof. **NB**: To show that a space is normed, then it is necessary to show that the space satisfy the above three conditions 9. Also,the equivalence relation in NLVS1 suggest that we assume one quantity to be true and prove the other and vice versa. Now,

Since l_∞ is a normed linear vector space then

$\forall \xi_v, \varepsilon_v \in l_\infty$, $\exists \alpha \in \mathrm{K}_{sc}$ that is $\xi_v = (\xi_{v1}, \xi_{v2}, \xi_{v3}, \cdots)$ and $\varepsilon_v = (\varepsilon_{v1}, \varepsilon_{v2}, \varepsilon_{v3}, \cdots)$ we define a mapping $\|\cdot\|_\infty : l_\infty \longrightarrow [0,\infty[$ such that $\|\xi_v\|_\infty = \sup_{i\geq 1} |\xi_{vi}|$

from definition,

(NLS1) $\|\xi\|_\infty \geq 0$ (trivial)

Next, suppose $\|\xi_v\|_\infty = 0$ W.T.S $\xi = 0$

since $\|\xi_v\|_\infty = 0 \Rightarrow \sup_{i\geq 1} |\xi_{\nu_i}| = 0$

Also, the supremum of all the absolute value is zero, then it implies that each of elements are made up zero components

$|\xi_{\nu i}| = 0\ \forall i \geq 0\ \xi_{\nu 1} = 0, \xi_{\nu 2} = 0, \xi_{\nu 3} = 0, \cdots$ hence, $\xi_\nu = (\xi_{\nu 1}, \xi_{\nu 2}, \xi_{\nu 3}, \cdots) = 0 \Longrightarrow \xi_\nu = 0$ Next, suppose $\xi_\nu = 0$ W.T.S $\|\xi_\nu\|_\infty = 0$ since $\xi_\nu = 0 \Longrightarrow \xi = (\xi_1, \xi_{\nu 2}, \xi_{\nu 3}, \cdots) = 0, : \forall i \geq 1\ |\xi_{\nu i}| = 0$ taking the supremum norm on both side $\sup_{i\geq 1}|\xi_{\nu i}| = 0 \Longrightarrow ||x_\nu||_\infty = 0$

(NLS2) W.T.S $\|\alpha\xi_\nu\|_\infty = \alpha\|\xi_\nu\|_\infty$

$\|\alpha\xi_\nu\|_\infty = \sup_{i\geq 1}|\alpha||\xi_{\nu i}|$ since α is constant and does not depend on i, so, $\|\alpha\xi_\nu\|_\infty = \sup_{i\geq 1}|\alpha\xi_{\nu i}| = \|\alpha\xi_\nu\|_\infty = |\alpha|\sup_{i\geq 1}|\xi_{\nu i}| \Longrightarrow \|\alpha\xi_\nu\|_\infty = |\alpha|\|\xi_\nu\|_\infty$

(NLS3) W.T.S $\|\xi_{\nu i} + \varepsilon_\nu\|_\infty \leq \|\xi_\nu\|_\infty + \|\varepsilon_\nu\|_\infty$

$\|\xi_\nu + \varepsilon\|_\infty = \sup_{i\geq 1}|\xi_{\nu i} + \varepsilon_{\nu i}| = \|\xi_\nu + \varepsilon\|_\infty \leq \sup_{i\geq 1}\{|\xi_{\nu i}| + |\varepsilon_i|\}$

$\|\xi_\nu + \varepsilon\|_\infty \leq \sup_{i\geq 1}|\xi_\nu| + \sup_{i\geq 1}|\varepsilon_\nu| \Longrightarrow \|\xi_\nu + \varepsilon_\nu\|_\infty \leq \|\xi_\nu\|_\infty + \|\varepsilon_\nu\|_\infty$

□

Definition 10 (Cauchy Sequence)**.** Suppose X_ν is a normed linear vector space. An infinite sequence $\{\xi_\nu\}_{i=1}^\infty \in X_\nu$ is Cauchy if $\forall\epsilon > 0 \quad \exists\delta > 0 : \|\xi_{\nu m} - \xi_{\nu n}\| \leq \epsilon \quad \forall n, m \geq N_L \in \mathbb{N}$

Definition 11 (Convergence)**.** An infinite sequence $\{\xi_{\nu n}\}_n^\infty \in X_\nu$ is convergent if $\exists\xi_\nu \in X_\nu$: $\lim_{n\to\infty}\|\xi_{\nu n} - \xi_\nu\| = 0$ where ξ_ν is the limit of infinite sequence $\lim_{n\to\infty}\xi_{\nu n} = \xi$

Theorem 3. If X_ν is a normed linear vector space Then very convergent sequence is Cauchy.

Proof

suppose $\{\xi_{\nu n}\}_n^\infty \in X_\nu$ then $\exists\xi \in X_\nu$: $\lim_{n\to\infty} X_{\nu n} = \xi_\nu$

Also, From 11 $\forall \epsilon > 0, \quad \exists N_L \in \mathbb{N} : \quad n, m \geq N_L$ Now $\lim_{n \longrightarrow \infty} ||\xi_{\nu n} - \xi_\nu|| \leq \frac{\epsilon}{2}$

by triangle inequality $\|\xi_{\nu m} - \xi_{\nu n}| = \|\xi_{\nu m} - \xi_\nu + \xi_\nu - X_{\nu n}\|$ $\Longrightarrow$ $\|\xi_{\nu m} - \xi_{\nu n}\| \leq$ $\|\xi_{\nu m} - \xi_\nu\| + \|\xi_\nu - \xi_n u_m\| \implies \|\xi_{\nu m} - \xi_{\nu n}\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \therefore \|\xi_{\nu m} - \xi_{\nu n}\| \leq \epsilon$

Definition 12 (Continuous mapping)**.** Suppose X_ν and Y_ν are normed linear vector spaces We define a mapping $\|\cdot\| : X_\nu \longrightarrow Y_\nu$ to be continuous at a point $\xi_\nu \in X_\nu$ if

$$\lim_{n \longrightarrow \infty} \xi_n = \xi_\nu \Longrightarrow \quad \lim \|\xi_n\| = \|\xi_\nu\| \ _{n \longrightarrow \infty}$$

OR

Suppose $\mathbf{X}_\nu$ and $\mathbf{Y}_\nu$ are two linear vector spaces. A map T_a is said to be continuous at a point $x_0 \in \mathbf{X}_v$ if $\forall \epsilon >0$ and $\exists \delta > 0$:

$$\|T_a x_\nu - T_a x_0\| < \varepsilon \quad \text{whenever} \quad |x_\nu - x_0| < \delta \tag{2.6}$$

Theorem 4. The mapping $\xi_\nu \longrightarrow \|\xi_\nu\|$ is continuous in the sense that if $\lim_{n \longrightarrow \infty} \xi_{\nu n} = \xi_\nu$ then $\lim \|\xi_{\nu n}\| = |\xi_\nu\| \ _{n \longrightarrow \infty}$

Proof. Let $\|\cdot\| : X_\nu \longrightarrow [0, \infty[$ W.T.S $\|\cdot\|$ is continuous.

Assume, $\lim \xi_{\nu n} = \xi_\nu$ then from 11 $\forall \epsilon > 0 \quad \exists N_L \in \mathbb{N} : n \geq \mathbb{N}_{n \longrightarrow \infty}$

We have $\|\xi_{\nu n} - \xi_\nu\| < \epsilon$ we can write it as $\left|\|\xi_n - \xi_\nu\|\right| \leq \|\xi_{\nu n} - \xi\| < \varepsilon$ thus

$|\|X_{\nu n} - x_\nu\|| < \varepsilon \Longrightarrow \lim_{n \longrightarrow \infty} \|X_{\nu n}\| \longrightarrow \|x_\nu\| \therefore \|\cdot\|$ is continuous. □

Completeness

A vector space X_ν (either set of vectors or functions) is made into a normed linear space by treating it as metric space which allows the computation of vector length and distance between vectors, and it is related to the normed linear space. i.e $\forall x_\nu, y_\nu \in (X_\nu, d)$ $d(x_\nu, y_\nu) = \|x_\nu - y_\nu\|$. Then we are to prove that the metric space satisfy the desired property of being complete notwithstanding the fact that a Cauchy sequence of vectors always converges to a well defined limit that is within the space. Hence, the general procedures to show completeness are:

1. Construct an x_ν^* which is used as limit of the Cauchy sequence.

2. Prove that x_ν^* is under space of consideration

3. Prove convergence that is $\lim_{n \longrightarrow \infty} x_{\nu n} = x_\nu^*$ (in the sense of the metric under consideration)

Remark Every complete norm linear space is regarded as Banach space

Example 2.1.4.1. Show that the space ℓ_∞ endowed with the supremum norm is complete *Proof.* $\ell_\infty =$

$\{X_v = (x_{v1}, x_{v2}, x_{v3}, \cdots) : x_{vi} \in \mathbb{R}, \quad |x_v| \le M \quad where\ M \in \mathbf{K}_{sc} \quad and \quad \|X_v\|_\infty = \sup|x_{vi}|\}$ Since l_∞

is a vector space, set $\{x_\nu{}^m\}_{m=1}^\infty$ to be a Cauchy sequence in ℓ_∞. Then

1

$x_\nu^1 = (x_{\nu 1}^1, x_{\nu 2}^1, x_{\nu 3}^1, \cdots)_{i\geq}$ $x_{2v} = (x_{2v1}, x_{2v2}, x_{2v3}, \cdots)$

$x_{3v} = (x_{3v1}, x_{3v2}, x_{3v3}, \cdots)$

...

Since $\{x_\nu^m\}_{m=1}^\infty$ is Cauchy then $\quad \forall \epsilon > 0 \quad \exists N \in \mathbb{N} : s,t \geq N$

$x_\nu^s, x_\nu^t) = \|x_\nu^s - x_\nu^t\|_\infty \quad = \quad \sup_{i\geq}|x_{\nu i}^s - x_{\nu i}^t| < \epsilon$ d(Now

1

$\sup_{i\geq 1}\{|x_{\nu 1}^s - x_{\nu 1}^t|, |x_{\nu 2}^s - x_{\nu 2}^t|, |x_{\nu 3}^s - x_{\nu 3}^t|, \cdots\} < \epsilon$ it implies that

$|x_{\nu 1}^s - x_{\nu 1}^t| \leq \epsilon \quad |x_{\nu 2}^s - x_{\nu 2}^t| \leq \epsilon \quad |x_{\nu 3}^s - x_{\nu 3}^t| \leq \epsilon \quad \ldots$

since each column is a sequence of real numbers $\mathbb{R}$, and $\mathbb{R}$ is complete, then each column converges

to a point in ℓ_∞ That is, $\lim_{i\longrightarrow\infty} x_\nu^m{}_i = x_{\nu i}^*$

we have, $x_{v1} = (x_{1v1}, x_{1v2}, x_{1v3}, \cdots, x_{1vn})$
$x_\nu^2 = (x_{\nu 1}^2, x_{\nu 1}^2, x_{\nu 3}^2, \cdots)$

$x_{3v} = (x_{3v1}, x_{32}, x_{33}, \cdots)$

...

Hence $x_\nu^* = (x^*\nu_1, x_{\nu 2}^*, x_{\nu 3}^*, \cdots)$

Step(2)

W.T.S $\quad \{x_\nu^*\} \in \ell_\infty$

From definition of ℓ_∞ $\exists M^{\geq 0} \in \mathbb{R} : |x^m_{\nu\ i}| \leq M \quad \forall m \quad \implies \quad |x^m_{\nu\ i}| \leq M \quad \forall i$ Also

$\lim_{i \longrightarrow \infty} x^m_{\nu\ i} = x^*_{\nu i}$ then, $|x^*_{\nu i}| = |x^*_{\nu i} - x^m_{\nu\ i} + x^m_{\nu\ i}|$ by (9) of NLVS

$|x^*_{\nu i}| \leq |x^*_{\nu i} - x^m_{\nu\ i}| + |x^m_{\nu\ i}| \quad \implies |x^*_{\nu i}| < \epsilon + M$

Hence $\{x_{\nu i}{}^*\}_{i=1}$ is a bounded sequence of real numbers.

Step(3)

W.T.S $\lim_{i \longrightarrow \infty} x^m{}_i = x^*{}_i$

from 11 $\forall \epsilon > 0 \quad \exists N \in \mathbb{N} : m \geq N \quad |x^m_{\nu\ i} - x^*_{\nu i}| < \epsilon$ Now, $d(x^s_\nu, x^t_\nu) = \sup_{i \geq 1} |x^m_{\nu\ i} - x^*_{\nu i}| < \epsilon$

$\implies d(x^s_\nu, x^t_\nu) < \epsilon$ Hence, ℓ_∞ is complete. □

2.1.5 Infinite Dimensional Space

Recall, the dimension of vector space is determined by the cardinality of the basis vectors that covered the space. However, not all vector space can be spanned by a finite number of basis vectors. Such a vector space is called infinite dimensional vector space.

Example 2.1.5.1. Consider the expression for the exponential function (e^x_ν) :

$$e^{x_\nu} = 1 + x_\nu + \frac{x^2_\nu}{2!} + \frac{x^3_\nu}{3!} + \cdots \tag{2.7}$$

In fact the exponential function is of infinite degree as the series progresses. Hence, it consists of infinite basis $(1, x_\nu, x^2_\nu, x^3_\nu, \cdots)$ that spanned the series

NB Most of the vector space that are of infinite dimensional are function space.

2.1.6 Function Space

Here, we will systematically look at functions spaces that are most relevant in the formulation of quantum mechanics. However, we shall lay the foundation that will be necessary to categorize spaces and in the process review some basic facts regarding these spaces. Let $f_a : D(f_a) \longrightarrow K_{sc}$ then the support of f_a is define as a set

$$suppf_a = \overline{\left\{x_\nu \in \mathcal{D}(f_a) : f_a(x_\nu) \neq 0\right\}}$$

f_a has a compact support if the $suppf_a$ is bounded i.e if $\exists$ a, b $\in$ R such that $suppf_a \subseteq [a,b]$ outside this interval the support vanishes.

Definition 13 (**The space** C_c(R)**).** Consist of all continuous function with compact support.

Definition 14 (**The space** C_0(R)**).** Let $f_a|D(f_a) \longrightarrow K_{sc}$: f_a is continuous and $\lim\limits_{x_\nu \longrightarrow \infty} f_a(x_\nu) = 0$

Example 2.1.6.1. Suppose $I = (c,d)$ and $f_a : \mathbb{R} \longrightarrow \{0,1\}$:

$$f_a(x_\nu) = \begin{cases} 1 & x_\nu \in I \\ 0 & x_\nu \notin I \end{cases}$$

Show that f_a has a compact support.

solution

$$suppf_a = \overline{\left\{x_\nu \in \mathbb{R} : f_a(x_\nu) \neq 0\right\}} = \overline{(c,d)} = [c,d]$$

f_a has a compact support

$\therefore f_a$ is not continuous but piecewise continuous. so, it is not $C_0(\mathrm{R})$ and $C_c(\mathrm{R})$

2.1.7 The Space L^p where $1 \leq p < \infty$

Definition 15. Suppose $S \subseteq \mathrm{R}$, be measurable space and allow $p \in \mathrm{R}$ where $1 \leq p < \infty$

then

$$L^p(S) = \left\{f_a | f_a : S \longrightarrow \mathbb{C}, \quad f_a \text{ measurable}, \quad \int_S |f_a(x_\nu)|^p \mathrm{d}x_\nu < +\infty\right\}$$

However, we can also define $L^\infty(\mathrm{R})$ as

$$L^\infty(S) = \left\{f_a | f_a : S \longrightarrow \mathbb{C}\right.$$,where f_a is measurable , and $\exists M^{\geq 0} : |f_a(x_d)| \leq M$ a.e

We only defined a norm on $L^\infty(S)$ by taking the essential supremum of f_a. That is

$$\|f\|_{L^\infty(S)} = \inf\left\{M : |f_a(x_\nu)| \leq M \right.$$ a.e in S

2.1.8 Inner product space

Inner-product space enhances the notion of additional structure of geometry called innerproduct. It enables us to define distance, angle, and perpendicularity between two vectors in space, thus matching each pair of vectors in a vector space to a unique quantity called scalars. Hence, we will then define inner-product space and give some basic result associated with the space.

Definition 16 (Dot product)**.** Suppose $x_\nu, y_\nu \in \mathbb{R}^n$ be a linear vector space. We define a mapping $\cdot : \mathbb{R}^n \longrightarrow \mathbb{R}$ as the dot product of x_ν and y_ν denoted by $x_\nu \cdot y_\nu$ i.e $x_\nu \cdot y_\nu = x_{\nu 1}y_{\nu 1} + x_{\nu 2}y_{\nu 2} + x_{\nu 3}y_{\nu 3} + \cdots + x_{\nu n}y_{\nu n}$ where $x_\nu = (x_{\nu 1}, x_{\nu 2}, x_{\nu 3}, \cdots, x_{\nu n})$ and $y_\nu = (y_{\nu 1}, y_{\nu 2}, y_{\nu 3}, \cdots, y_{\nu n})$ or $x_\nu \cdot y_\nu = \sum_{1=1}^{n} x_{\nu i} y_{\nu i}$

Definition 17. Suppose a linear vector space X_ν

A map $\langle \cdot, \cdot \rangle : X_\nu * X_\nu \longrightarrow \mathbb{C}$ is called an inner-product such that $\forall \alpha, \beta \in X_\nu$ *and* $\exists \alpha, \beta \in K_{sc}$ then the following conditions hold:

(IPS1) $\langle x_\nu, x_\nu \rangle \geq 0$ and $\langle x_\nu, x_\nu \rangle = 0 \Longleftrightarrow x_\nu = 0$ (positive definite).

(IPS2) $\langle x_\nu, y_\nu \rangle = \overline{\langle y_\nu, x_\nu \rangle}$ (conjugate symmetry)

The bar denote conjugate

(IPS3) $\langle \alpha x_\nu + \beta y_\nu, z_\nu \rangle = \alpha \langle x_\nu, y_\nu \rangle + \beta \langle y_\nu, z_\nu \rangle$ (linearity in the first slot)

Remark All the spaces enumerated in 2.1.3.1 are also inner product space except the ℓ_∞ which we shall see later in the following through

Example 2.1.8.1. Show that the function space $C[0,1]$ endowed with $\langle f_a, g_a \rangle = \int_0^1 f_a(t)\overline{g_a(t)}$ is an inner product space.

Proof. NB: We are to show that the function space satisfy the above conditions (17)

let $f_a, g_a, h_a \in C[0,1] : \langle f_a, g_a\rangle = \int_0^1 f_a(t)\overline{g_a(t)}$

(IPS1) $\langle f_a, f_a\rangle = \int_0^1 f_a(t)\overline{f_a(t)}\mathrm{d}t \implies \int_0^1 |f_a(t)|^2\mathrm{d}t \geq 0 \therefore \langle f_a, f_a\rangle \geq 0$

Next, suppose $\langle f_a, f_a\rangle = 0$ W.T.S $f_a = 0$

Since $\langle f_a, f_a\rangle = 0 \implies \int_0^1 |f_a(t)|^2\mathrm{d}t = 0 \quad \forall t \in C[0,1]$

$f_a(t) = 0 \quad \forall t \in C[0,1] \therefore f_a = 0$

Next, suppose $f_a = 0$ W.T.S $\langle f_a, f_a\rangle = 0$

Now, $f_a = 0 \implies f_a(t) = 0 \quad t \in C[0,1]$

$$\implies \int_0^1 |f_a(t)|^2\mathrm{d}t = 0 \quad \therefore \langle f_a, g_a\rangle = 0$$

(IPS2) $$\langle f_a, g_a\rangle = \int_0^1 f_a(t)\overline{g_a(t)}\mathrm{d}t \tag{1}$$

$$\langle g_a, f_a\rangle = \int_0^1 g_a(t)\overline{f_a(t)}\mathrm{d}t \implies \overline{\langle g_a, f_a\rangle} = \overline{\int_0^1 g_a(t)\overline{f_a(t)}\mathrm{d}t}$$

$$\overline{\langle g_a, f_a\rangle} = \int_0^1 \overline{g_a(t)}f_a(t)\mathrm{d}t \implies \int_0^i f_a(t)\overline{g_a(t)}\mathrm{d}t \tag{2}$$

comparing (1) and (2)

hence, (1) = (2)

(IPS3) $$\langle \alpha f_a + \beta g_a, h_a\rangle = \int_0^1 (\alpha f_a + \beta g_a)(t)\overline{h_a(t)}\mathrm{d}t \implies \int_0^1 (\alpha f_a(t) + \beta g_a(t)\overline{h_a(t)}\mathrm{d}t$$

$$\langle \alpha f_a + \beta g_a, h_a\rangle = \int_0^1 \left(\alpha f_a(t)\overline{h_a(t)} + \beta g_a(t)\overline{h_a(t)}\right)\mathrm{d}t$$

$$\langle \alpha f_a + \beta g_a, h_a\rangle = \int_0^1 \left(\alpha f_a(t)\overline{h_a(t)}\mathrm{d}t + \beta g_a(t)\overline{h_a(t)}\mathrm{d}t\right)$$

$$\langle \alpha f_a + \beta g_a, h_a\rangle = \alpha\int_0^1 f_a(t)\overline{h_a(t)}\mathrm{d}t + \beta\int_0^1 g_a(t)\overline{h_a(t)}\mathrm{d}t$$

$\langle \alpha f_a + \beta g_a, h_a\rangle = \alpha\langle f_a, h_a\rangle + \beta\langle g_a, h_a\rangle$

Hence, $\left(C[0,1], \langle\cdot,\cdot\rangle\right)$ is an inner product space. $\square$

Definition 18 (properties of inner-product space)**.** Given $x_\nu y_\nu, z_\nu \in X_\nu$ and $\exists \alpha, \beta \in \mathbb{C}$.

If an inner-product in defined on X_ν, then the following properties hold:

1. $\langle x_\nu, y_\nu + z_\nu\rangle = \langle x_\nu, y_\nu\rangle + \langle x_\nu, z_\nu\rangle$

2. $\langle x_\nu, \alpha y_\nu\rangle = \bar{\alpha}\langle x_\nu, y_\nu\rangle$

3. $\langle x_\nu, \alpha y_\nu + \beta z_\nu\rangle = \bar{\alpha}\langle x_\nu, y_\nu\rangle + \bar{\beta}\langle x_\nu, z_\nu\rangle$

4. $\langle 0, x_\nu\rangle = 0$

5. $\langle x_\nu, 0\rangle = 0$

Proof. 1. 1 W.T.S $\langle x_\nu, y_\nu + z_\nu\rangle = \langle x_\nu, y_\nu\rangle + \langle x_\nu, z_\nu\rangle$

$\langle x_\nu, y_\nu + z_\nu\rangle = \overline{\langle y_\nu + z_\nu, x\rangle}$ from 17 $\langle x_\nu, y_\nu + z_\nu\rangle = \overline{\langle y_\nu, x_\nu\rangle + \langle z_\nu, x_\nu\rangle} = \overline{\langle y_\nu, x_\nu\rangle} + \overline{\langle z_\nu, x_\nu\rangle}$

$\langle x_\nu, y_\nu + z_\nu\rangle = \langle x_\nu, y_\nu\rangle + \langle x_\nu, z_\nu\rangle$

2. W.T.S $\langle x_\nu, \alpha y_\nu\rangle = \bar{\alpha}\langle x_\nu, y_\nu\rangle$

$\langle x_\nu, \alpha y_\nu\rangle = \overline{\langle \alpha y_\nu, x_\nu\rangle} = \bar{\alpha}\overline{\langle y_\nu, x_\nu\rangle} = \bar{\alpha}\langle x_\nu, y_\nu\rangle$

3. W.T.S $\langle x_\nu, \alpha y_\nu + \beta z_\nu\rangle = \bar{\alpha}\langle x_\nu, y_\nu\rangle + \bar{\beta}\langle x_\nu, z_\nu\rangle$

$\langle x_\nu, \alpha y_\nu + \beta z_\nu\rangle = \overline{\langle \alpha y_\nu + \beta z_\nu, x_\nu\rangle} = \overline{\langle \alpha y_\nu, x_\nu\rangle} + \overline{\langle \beta z_\nu, x_\nu\rangle} = \bar{\alpha}\overline{\langle y_\nu, x_\nu\rangle} + \bar{\beta}\overline{\langle z, x\rangle} =$

$\bar{\alpha}\langle x_\nu, y_\nu\rangle + \bar{\beta}\langle x_\nu, z_\nu\rangle$

4. W.T.S $\langle 0, x_\nu\rangle = 0$ $\langle 0, x_\nu\rangle \Longrightarrow \langle 0x_\nu, x_\nu\rangle = 0\langle x_\nu, x_\nu\rangle = 0$

5. W.T.S $\langle x_\nu, 0\rangle = 0$ $\langle x_\nu, 0\rangle \Longrightarrow \langle x_\nu, 0x_\nu\rangle = 0\langle x_\nu, x_\nu\rangle = 0$

□

Theorem 5. *(KREYSZIG (1978))* Every normed linear space X_τ is an inner product with

$\|x_\tau\| = \sqrt{\langle x_\tau, x_\tau\rangle}$

Proof. $\forall x_\tau, y_\tau \in X_\tau$, and $\exists \lambda \in \mathbf{K}_{sc}$.

We define a map $\|\cdot\| : X_\tau \longrightarrow [0 \quad \infty[$ by

$\|x_\tau\| = \sqrt{\langle x_\tau, x_\tau\rangle}$

1. From definition $\|x_\tau\|^2 = \langle x_\tau, x_\tau\rangle \Longrightarrow \|x_\tau|^2 = \langle x_\tau, x_\tau\rangle \geq 0$

   $(see 17)$

   Next, suppose $x_\tau = 0$ W.T.S $\|x_\tau\| = 0$

   Since $x_\tau = 0$ $\Longrightarrow \langle x_\tau, x_\tau\rangle = 0 = \sqrt{\langle x_\tau, x_\tau\rangle}$

   $\therefore \|x_\tau\| = 0$

   Assume $\|x_\tau\| = 0$ W.T.S $x_\tau = 0$

   $\|x_\tau\| = 0 \quad \Longrightarrow \quad \sqrt{\langle x_\tau, x_\tau\rangle} = 0$

   $\langle x_\tau, u\rangle = 0 \quad \Longrightarrow \quad x_\tau = 0$

$$\|\lambda x_\tau\| = \sqrt{\langle \lambda x_\tau,\lambda x_\tau\rangle}$$

$$\|\lambda x_\tau\| = \sqrt{\lambda\bar{\lambda}\langle x_\tau,x_\tau\rangle}$$

2. $\|\lambda x_\tau\| = \sqrt{|\lambda\langle x_\tau,x_\tau\rangle}$

$$\|\lambda x_\tau\| = \lambda\sqrt{\langle x_\tau,x_\tau\rangle}$$

$$\|\lambda x_\tau\| = \sqrt{|\lambda|^2\langle x_\tau,x_\tau\rangle}$$

$$\|\lambda\| = \lambda\sqrt{\langle x_\tau,x_\tau\rangle}$$

$$\|\lambda x_\tau\| = \lambda\|x_\tau\|$$

$$\|x_\tau + y_\tau\| = \sqrt{\langle x_\tau + y_\tau,x_\tau + y_\tau\rangle}$$

$$\|x_\tau + y_\tau\|^2 = \langle x_\tau + y_\tau,x_\tau + y_\tau\rangle$$

$$\|x_\tau + y_\tau\|^2 = \langle x_\tau,x_\tau\rangle + \langle y_\tau,x_\tau\rangle + \langle x_\tau,y_\tau\rangle + \langle y_\tau,y_\tau\rangle$$

$$\|x_\tau + y_\tau\|^2 = \|x_\tau\|^2 + 2\Re|\langle x_\tau,y_\tau\rangle| + \|y_\tau\|^2 \leq \|x_\tau\|^2 + 2\|x_\tau\|\|y_\tau\| + \|y_\tau\|^2$$

$$\|x_\tau + y_\tau\|^2 \leq \|x_\tau\|^2 + 2\|x_\tau\|\|y_\tau\| + \|y_\tau\|^2$$

$$\|x_\tau + y_\tau\|^2 \leq (\|x_\tau\| + \|y_\tau\|)^2$$

3. $\|x_\tau + y_\tau\| \leq (\|x_\tau\| + \|y_\tau\|)$

□

Given $x_\tau,y_\tau,z_\tau \in$ X and $\exists \alpha,\beta \in$ C. If an inner-product in defined on X, then the following properties hold:

1. hx_τ,y_τ + z_τi = hx_τ,y_τi + hx_τ,z_τi

2. hx_τ,αy_τi = ¯αhx_τ,y_τi

3. $\langle x_\tau, \alpha y_\tau + \beta z_\tau \rangle = \bar{\alpha}\langle x_\tau, y_\tau \rangle + \bar{\beta}\langle x_\tau, z_\tau \rangle$

4. $\langle 0, x_\tau \rangle = 0$
5. $\langle x_\tau, 0 \rangle = 0$

Proof. 1. W.T.S $\langle x_\tau, y_\tau + z_\tau \rangle = \langle x_\tau, y_\tau \rangle + \langle x_\tau, z_\tau \rangle$

$\langle x_\tau, y_\tau + z_\tau \rangle = \overline{\langle y_\tau + z_\tau, x_\tau \rangle}$ from 17 $\langle x_\tau, y_\tau + z_\tau \rangle = \overline{\langle y_\tau, x_\tau \rangle + \langle z_\tau, x_\tau \rangle} = \overline{\langle y_\tau, x_\tau \rangle} + \overline{\langle z_\tau, x_\tau \rangle}$

$\langle x_\tau, y_\tau + z_\tau \rangle = \langle x_\tau, y_\tau \rangle + \langle x_\tau, z_\tau \rangle$

2. W.T.S $\langle x_\tau, \alpha y_\tau \rangle = \bar{\alpha}\langle x_\tau, y_\tau \rangle$

$\langle x_\tau, \alpha y_\tau \rangle = \overline{\langle \alpha y_\tau, x_\tau \rangle} = \bar{\alpha}\overline{\langle y_\tau, x_\tau \rangle} = \bar{\alpha}\langle x_\tau, y_\tau \rangle$

3. W.T.S $\langle x_\tau, \alpha y_\tau + \beta z_\tau \rangle = \bar{\alpha}\langle x_\tau, y_\tau \rangle + \bar{\beta}\langle x_\tau, z_\tau \rangle$

$\langle x_\tau, \alpha y_\tau + \beta z_\tau \rangle = \overline{\langle \alpha y_\tau + \beta z_\tau, x_\tau \rangle} = \overline{\langle \alpha y_\tau, x_\tau \rangle} + \overline{\langle \beta z_\tau, x_\tau \rangle} = \bar{\alpha}\overline{\langle y_\tau, x_\tau \rangle} + \bar{\beta}\overline{\langle z_\tau, x_\tau \rangle} =$

$\bar{\alpha}\langle x, y_\tau \rangle + \bar{\beta}\langle x_\tau, z_\tau \rangle$

4. W.T.S $\langle 0, x_\tau \rangle = 0$

$\langle 0, x_\tau \rangle \implies \langle 0x, x_\tau \rangle = 0\langle 0, x_\tau \rangle = 0$

5. W.T.S $\langle x_\tau, 0 \rangle = 0$

$\langle x_\tau, 0 \rangle \implies \langle x_\tau, 0x_\tau \rangle = 0\langle x_\tau, x_\tau \rangle = 0$

□

Theorem 6. Every normed linear space X_τ is an inner product space (X_ω) with $\|u\| = \sqrt{\langle x_\tau, x_\tau \rangle}$ *Proof.* $\forall u_\tau, v_\tau \in X_\tau$, and $\exists \lambda \in \mathbf{K}_{sc}$.

We define a map $k{\cdot}k : X_\tau \dashrightarrow [0 \quad \infty[$ by

$$ku_\tau k = \sqrt{hu_\tau,u_\tau i}$$

1. From definition $ku_\tau k^2 = hu_\tau,u_\tau i \Longrightarrow ku_\tau|^2 = hu_\tau,u_\tau i \geq 0$

   $(see 17)$

   Next, suppose $u_\tau = 0$ W.T.S $ku_\tau k = 0$

   Since $u_\tau = 0 \Longrightarrow hu_\tau,u_\tau i = 0 = \sqrt{hu_\tau,u_\tau i} = 0$

   $\therefore ku_\tau k = 0$

   Assume $ku_\tau k = 0$ W.T.S $u_\tau = 0$

   $\|u_\tau\| = 0 \implies \sqrt{\langle u_\tau,u_\tau\rangle} = 0$

   $\langle u_\tau,u_\tau\rangle = 0 \implies u_\tau = 0$

   $\|\lambda u_\tau\| = \sqrt{\langle \lambda u_\tau,\lambda u_\tau\rangle}$

   $\|\lambda u_\tau\| = \sqrt{\lambda\bar{\lambda}\langle u_\tau,u_\tau\rangle}$

   $\|\lambda u_\tau\| = \sqrt{|\lambda\langle u_\tau,u_\tau\rangle}$

   $\|\lambda u_\tau\| = \lambda\sqrt{\langle u_\tau,u_\tau\rangle}$

   $\|\lambda u_\tau\| = \sqrt{|\lambda|^2\langle u_\tau,u_\tau\rangle}$

   $\|\lambda\| = \lambda\sqrt{\langle u_\tau,u_\tau\rangle}$

2. $\|\lambda u_\tau\| = \lambda\|u_\tau\|$

3. $ku_\tau + V_\tau k = \sqrt{hu_\tau + u_\tau,u_\tau + v_\tau i}$

ku_τ + v_τk^2 = hu_τ + v_τ,u_τ + v_τi ku_τ + v_τk^2 = hu_τ,u_τi + hv_τ,u_τi + hu_τ,v_τi + hv_τ,v_τi

ku_τ + v_τk^2 = ku_τk^2 + 2Re|hu_τ,v_τi| + kv_τk^2 ≤ ku_τk^2 + 2k$u_\tau$kkv_τk + kv_τk^2 ku_τ + v_τk^2 ≤ ku_τk^2 + 2k$u_\tau$kkv_τk + kv_τk^2 ku_τ + v_τk^2 ≤ (ku_τk + kv_τk)^2 ku_τ + v_τk ≤ (ku_τk + kv_τk)

Theorem 7.

Let X_v be a linear space

$\forall u_\tau, v_\tau \in X$ and $\lambda \in \mathbf{K}_{sc}$ then the following hold:

1. $\langle u_\tau, v_\tau\rangle = \frac{1}{4}\{\|u_\tau+v_\tau\|^2-\|u_\tau-v_\tau\|^2+i\|u_\tau-v_\tau\|^2+i\|u_\tau+iv_\tau\|^2\}$ Polarization identity

2. ku_τ + v_τk^2 + ku_τ − v_τk^2 = 2(ku_τk^2 + kv_τk^2) parallelogram law

3. kλu_τk = |λ|ku_τk Homogenous

Proof. NB. we will implore result from theorem 6.

1. W.T.S $\langle u_\tau, v_\tau\rangle = \frac{1}{4}\{\|u_\tau+v_\tau\|^2-\|u_\tau-v_\tau\|^2+i\|u_\tau-v_\tau\|^2+i\|u_\tau+iv_\tau\|^2\}$ ku_τ + v_τk^2 = hu_τ + v_τ,u_τ + v_τi

ku_τ + v_τk^2 = hu_τ,u_τi + hu_τ,v_τi + hv_τ,u_τi + hv_τ,v_τi (*a*)

Also

k$u_\tau - v_\tau$k^2 = h$u_\tau - v_\tau, u_\tau - v_\tau$i

k$u_\tau - v_\tau$k^2 = hu_τ, u_τi − hu_τ, v_τi − hv_τ, u_τi + hv_τ, v_τi substracting (a) from (b) and call it (c) k$u_\tau + v_\tau$k − k$u_\tau - v_\tau$k = hu_τ, v_τi + hu, vi + hv, ui + hv_τ, u_τi (b)

(c)

k$u_\tau + v_\tau$k − k$u_\tau - v_\tau$k = 2hu_τ, v_τi + 2hv_τ, u_τi Now, k$u_\tau + iv_\tau$k^2 = h$u_\tau + iv_\tau, u_\tau + iv_\tau$i k$u_\tau + iv_\tau$k^2 = h$u_\tau, u_\tau$i + h$u_\tau, iv_\tau$i + h$iv_\tau, u_\tau$i + h$iv_\tau, iv_\tau$i k$u$ + iv_τk^2 = hu, u_τi + $\bar{i}$hu_τ, v_τi + ihv_τ, u_τi + $i\bar{i}$hv_τ, v_τi

(d)

ik$u_\tau + iv_\tau$k^2 = ihu_τ, u_τi + $i\bar{i}$hu_τ, v_τi + iihv_τ, u_τi + $ii\bar{i}$hv_τ, v_τi Hence, k$u_\tau - iv_\tau$k^2 = h$u_\tau - iv_\tau, u_\tau - iv_\tau$i k$u_\tau - iv_\tau$k^2 = h$u_\tau, u_\tau$i − h$u_\tau, iv_\tau$i − h$iv_\tau, u_\tau$i + h$iv_\tau, iv_\tau$i k$u_\tau$ − iv_τk^2 = hu_τ, u_τi −$\bar{i}$hu_τ, v_τi − ihv_τ, u_τi + $i\bar{i}$hv_τ, v_τi

(e)

iku_τ − v_τk^2 = ihu_τ, u_τi − $i\bar{i}$hu_τ, v_τi − iihv_τ, u_τi + $ii\bar{i}$hv_τ, v_τi substracting (d) from (e) and call it (f)

iku_τ+iv_τk^2−iku_τ−iv_τk^2 = −hv_τ, u_τi−hv_τ, u_τi+2h$u_\tau . v_\tau$i = −2hv_τ, u_τi+2hu_τ, v_τi (f)

Adding (c) + (f) k$u_\tau + v_\tau$k − k$u_\tau - v_\tau$k = hu_τ, v_τi + hu_τ, v_τi + hv_τ, u_τi + hv_τ, u_τi ku_τ +v_τk−ku_τ

−v_τk+iku_τ +iv_τk^2 −iku_τ −iv_τk^2 = 2hu_τ, v_τi+2hv_τ, u_τi−2hv_τ, u_τi+

$$2\langle u_\tau, v_\tau\rangle = 4\langle u_\tau, v_\tau\rangle$$

$$\therefore \quad \langle u, v\rangle = \frac{1}{4}\{\|u_\tau + v_\tau\|^2 - \|u_\tau - v_\tau\|^2 + i\|u_\tau - v_\tau\|^2 + i\|u_\tau + iv_\tau\|^2\}$$

2. W.T.S $\mathrm{k}u_\tau + v_\tau\mathrm{k}^2 + \mathrm{k}u_\tau - v_\tau\mathrm{k}^2 = 2\mathrm{k}u_\tau\mathrm{k}^2 + 2\mathrm{k}v_\tau\mathrm{k}^2$

$\mathrm{k}u_\tau + v_\tau\mathrm{k}^2 = \mathrm{h}u_\tau + v_\tau, u_\tau + v_\tau\mathrm{i}$

$\mathrm{k}u_\tau + v_\tau\mathrm{k}^2 = \mathrm{h}u_\tau, u_\tau\mathrm{i} + \mathrm{h}u_\tau, v_\tau\mathrm{i} + \mathrm{h}v_\tau, u_\tau\mathrm{i} + \mathrm{h}v_\tau, v_\tau\mathrm{i}\ \mathrm{k}u_\tau - v_\tau\mathrm{k}^2$ (1)

$= \mathrm{h}u_\tau - v_\tau, u_\tau - v_\tau\mathrm{i}$

$\mathrm{k}u_\tau - v_\tau\mathrm{k}^2 = \mathrm{h}u_\tau, u_\tau\mathrm{i} - \mathrm{h}u_\tau, v_\tau\mathrm{i} - \mathrm{h}v_\tau, u_\tau\mathrm{i} + \mathrm{h}v_\tau, v_\tau\mathrm{i}$ (2)

adding (1) and (2) $\mathrm{k}u_\tau + v_\tau\mathrm{k}^2 + \mathrm{k}u_\tau - v_\tau\mathrm{k}^2 =$

$\mathrm{h}u_\tau, u_\tau\mathrm{i} + \mathrm{h}u_\tau, v_\tau\mathrm{i} + \mathrm{h}v_\tau, u_\tau\mathrm{i} + \mathrm{h}v_\tau, v_\tau\mathrm{i} + \mathrm{h}u_\tau, u_\tau\mathrm{i} - \mathrm{h}u_\tau, v_\tau\mathrm{i} - \mathrm{h}v_\tau, u_\tau\mathrm{i} + \mathrm{h}v_\tau, v_\tau\mathrm{i}\ \mathrm{k}u_\tau + v_\tau\mathrm{k}^2 + \mathrm{k}u_\tau - v_\tau\mathrm{k}^2 =$

$2\mathrm{h}u_\tau, u_\tau\mathrm{i} + 2\mathrm{h}v_\tau, v_\tau\mathrm{i}\ \mathrm{k}u_\tau + v_\tau\mathrm{k}_2 + \mathrm{k}u_\tau - v_\tau\mathrm{k}_2 = \mathrm{k}u_\tau\mathrm{k}_2 + \mathrm{k}v_\tau\mathrm{k}_2$

3. W.T.S $\mathrm{k}\lambda u_\tau\mathrm{k} = |\lambda|\mathrm{k}u_\tau\mathrm{k}$ Now, $\mathrm{k}\lambda u_\tau\mathrm{k}^2 = \mathrm{h}\lambda u_\tau, \lambda u_\tau\mathrm{i}\ \mathrm{k}\lambda u_\tau\mathrm{k}^2 = \lambda\bar{\lambda}\,\mathrm{h}u_\tau, u_\tau\mathrm{i}\ \mathrm{k}\lambda u_\tau\mathrm{k}^2 =$

$|\lambda|^2\mathrm{h}u_\tau, u_\tau\mathrm{i}\ \mathrm{k}\lambda u_\tau\mathrm{k}^2 = |\lambda|^2\mathrm{k}u_\tau\mathrm{k}^2$ multiplying the exponent by $\frac{1}{2}$

$\therefore \mathrm{k}\lambda u_\tau\mathrm{k} = |\lambda|\mathrm{k}u_\tau\mathrm{k}$

□

Orthonormal set

Orthonormal sets are very important for dealing with infinite dimensional space since they have helpful characteristics. However, when dealing with infinite-dimensional linear vector spaces which do not posses an algebraic structure but an inner product, the understanding of the Hamel basis is inadequate. In fact, the Hamel base only captures the algebraic structure without taking into account the angle between the vectors. For inner product spaces, orthonormal sets act similarly as Hamel bases and also capture the angle between the vectors.

Definition 19. Let X_ω be an inner product space. Two vectors in X_ω are said to be orthogonal i.e $u_\omega \perp v_\omega$ if $\langle u_\omega, v_\omega\rangle = 0$)

In general, A set $\{e_i\}_{i=1}^{\infty} \in \mathcal{X}_\omega$ is an orthogonal set if $\langle e_m, e_n\rangle = 0 \quad \forall m \neq n$. Then,

$\langle e_i, e_j\rangle = \delta_{ij}$, where δ_{mn} is called kronecker delta and is defined as:

$$\delta_{ij} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$$

hence, we now look at some special result of the above theorem (Broida and Williamson (1989))

Theorem 8. $\|u_\omega + v_\omega\|^2 = \|u_\omega\|^2 + \|v_\omega\|^2$ is achieved if $u_\omega \perp v_\omega$.

Prove that the Pythagoras identity

Proof. $\forall u_\omega, v_\omega \in X_\omega$ and making use of 7 $\|u_\omega + v_\omega\|^2 =$

$\langle u_\omega + v_\omega, u_\omega + v_\omega\rangle$ $\|u_\omega + v_\omega\| = \langle u_\omega, u_\omega\rangle + \langle u_\omega, v_\omega\rangle +$

$\langle v_\omega, u_\omega\rangle + \langle v_\omega, v_\omega\rangle$

From hypothesis, $u_\omega \perp v_\omega$ or $v_\omega \perp u_\omega$ $\Longrightarrow \langle u_\omega, v_\omega\rangle = \langle v_\omega, u_\omega\rangle = 0$

$\therefore \|u_\omega + v_\omega\|^2 = \|u_\omega\|^2 + \|v_\omega\|^2$ □

Example 2.1.8.2.

Show that the two vectors $\cos x_\omega, \sin x_\omega \in C[0,2\pi]$ are orthogonal with the inner-product $\langle f_a \cdot g_a\rangle = \int_o^{2\pi} f_a g_a$

Proof. since sinx_ω,cosx_ω ∈ $C[0,2\pi]$ then'

$$\langle \sin x_\omega, \cos x_\omega \rangle = \int_0^{2\pi} \cos x_\omega (\sin x) \mathrm{d}x_\omega$$

but from trigonometry

$$\sin 2x_\omega = 2 \sin x_\omega \cos x_\nu \quad \Longrightarrow \quad \frac{1}{2} \sin 2x_\omega = \sin x_\omega \cos x_\omega$$

$$\langle \sin x_\omega, \cos x_\omega \rangle = \int_0^{2\pi} \frac{1}{2} \sin 2x_\nu \mathrm{d}x_\nu \quad \Longrightarrow \quad \frac{1}{2} \int_0^{2\pi} \sin 2x_\omega \mathrm{d}x_\omega$$

$$\langle \sin x_\nu, \cos x_\nu \rangle = (-\frac{1}{4} \cos 2x_\omega) \Big|_0^{2\pi}$$

$$\langle \sin x_\omega, \cos x_\omega \rangle = -\frac{1}{4} + \frac{1}{4} = 0$$

$$\langle \sin x_\omega, \cos x_\omega \rangle = 0$$

□

Definition 20. A subset **S** of an inner-product X$_\omega$ is a said to be orthonormal if hu_ω,v_ωi =

0 ∀u_ω,v_ω ∈ X$_\omega u_\omega$ 6= 0 and ku_ωk = 1

Example 2.1.8.3.

Let $L^2[0,2\pi]$, the set $S = \left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\}_{n=1}^{\infty}$ is an orthonormal set

Proof. Let $f_a, g_a \in L^2[0, 2\pi]: \quad f = \frac{1}{\sqrt{2\pi}} e^{int}$ and $g_a = \frac{1}{\sqrt{2\pi}} e^{-imt}$

Now

$$\langle f_a, g_a \rangle = \int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}} e^{int} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-imt} \right) \mathrm{d}t$$

$$\langle f_a, g_a \rangle = \frac{1}{2\pi} \int_0^{2\pi} (e^{int} e^{-imt}) \mathrm{d}t$$

$$\langle f_a, g_a \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)t} \mathrm{d}t \qquad (*)$$

if $m = n$

$$\langle f_a, g_a\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^0 \mathrm{d}t$$

$$\langle f_a, g_a\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t$$

$$\langle f_a, g_a\rangle = \frac{t}{2\pi}\Big|_0^{2\pi}$$

$$\langle f_a, g_a\rangle = 1$$

Also if

$m \neq 0$

from (*)

$$\langle f_a, g_a\rangle = \frac{1}{2\pi}\left(\frac{e^{i(n-m)t}}{i(n-m)}\right)\Big|_0^{2\pi}$$

But from DeMoivre's theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{1}{2\pi}\left[\frac{\cos(n-m)t + i\sin(n-m)t}{i(n-m)}\right]\Big|_0^{2\pi} = (1-1) = 0$$

$$\langle f_a, g_a\rangle = 0$$

Hence

$$\langle f_a, g_a\rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

□

Definition 21. Orthogonal Complement $S^{\perp}$, (Direct Sum)

1. Suppose $S_\omega \subset X_\omega$ be an inner-product space. Then $\forall x_\omega \in X_\omega$ are orthogonal to S_ω i.e

   $\mathcal{S}_\omega^\perp = \{v_\omega \in \mathcal{X}_\omega : \langle v_\omega, u_\omega \rangle = 0\}, \quad \forall u_\omega \in \mathcal{S}_\omega$

   Meaning, if S_ω is any line R^3 that passes through the origin, then $S_\omega^\perp$ is the plane that passes through the origin and its perpendicular to S_ω.

2. Given two subspaces U_ω , $V_\omega \in H_\omega$, the sum $U_\omega + V_\omega$ is defined by

   $U_\omega + \mathscr{V}_\omega = \{w_\omega \in \mathcal{H}_\omega^b : w_\omega = u_\omega + v_\omega$ for some $u_\omega \in U_\omega, v_\omega \in V_\omega\}$. This space is called the direct sum of U_ω and V_ω denoted by $U_\omega \bigoplus V_\omega \; \forall w_\omega$. $\therefore w_\omega = U_\omega + V_\omega$ is uniquely expressed as $w_\omega = u_\omega + v_\omega$

Characteristics of Complete Orthonormal system

Given any orthonormal system $\{e\}_{i=1}^\infty$ the following results can be established.

1. $\{e_{\omega i}\}_{i=1}^\infty$ is an orthonormal basis

$u_\omega = \sum_{i=1}^{\infty} \langle u_\omega, e_i \rangle \quad \forall u_\omega \in \mathcal{H}_\omega^b$

$\langle u_\omega, v_\omega \rangle = \sum_{i=1}^{\infty} \langle u_\omega, e_{\omega i} \rangle \langle e_{\omega i}, v_\omega \rangle \quad \forall u_\omega, v_\omega \in \mathcal{H}_\omega^b$

2.

3.

4. $\sum_{i=1}^{\infty} |\langle u_\omega, e_i|^2 = \|u_\omega\|^2 \quad \forall \in \mathcal{H}_\omega$

5. Span$\{e_\omega\}_{i=1}^\infty = \mathcal{H}_\omega$

6. if $u_\omega \in \mathcal{H}_\omega^b$ and hu_ω, e_{vi}i = 0 $\forall i \in N$ then $u_\omega = 0$

Theorem 9. Suppose $S_\omega \subset X_\omega$ is orthonormal set. Then S_ω is linearly independent.

Proof. let $\{u_i\}_{i=1}^{n} \in \mathcal{S}_\omega \quad \exists\{\alpha_i\}_{i=1}^{n} \in \mathcal{K}_{sc}$ since S_ω is linearly independent then

$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n = 0 \implies \sum_{i=1}^{n} \alpha_i u_i = 0$ from properties of inner product space

$$\langle 0, u_j \rangle = 0 \quad \Longrightarrow \left\langle \sum_{i=1}^{n} \alpha_i u_i, u_j \right\rangle = 0$$

$$\sum_{i=1}^{n} \alpha_i \left\langle u_i, u_j \right\rangle = \delta_{ij}$$

but $\langle u_i, u_j \rangle = \delta_{ij}$

$\therefore$ **S** is linearly independent. □

Theorem 10 (Cauchy-Schwartz inequality)**.** *(Pinchuck (2011))* Suppose X_ω be an inner product space.

An inner-product $\langle \cdot, \cdot \rangle : X_\omega \times X_\omega \longrightarrow \mathbb{C}$ is defined

$|\langle x_\omega, y_\omega \rangle| \le \|x_\omega\| \|y_\omega\|$ such that $\forall x_\omega, y_\omega \in X_\omega$ and the equality holds $\iff x_\omega$ and y_ω are linearly dependent

Proof. Let $x_\omega, y_\omega \in X_\omega : x_\omega \neq 0$ other than the above theory holds trivially. However, let assume $\exists \alpha \in \mathbb{R}$ we compute $\|x_\omega - \alpha \langle x_\omega, y_\omega \rangle y_\omega\|^2 = \langle x_\omega - \alpha \langle x_\omega, y_\omega \rangle y_\omega, x_\omega - \alpha \langle x_\omega, y_\omega \rangle y_\omega \rangle$

Expanding the right hand quantity using IPS3 from definition 17.

$$0 \leq \|x_\omega - \alpha\langle x_\omega, y_\omega\rangle y_\omega\|^2 = \langle x_\omega, x_\omega\rangle - \alpha\langle x_\omega, y_\omega\rangle\langle x_\omega, y_\omega\rangle - \alpha\langle x_\omega, y_\omega\rangle\langle y_\omega, x_\omega\rangle + \alpha^2\langle x_\omega -, y_\omega\rangle\langle x_\omega, y_\omega\rangle\langle y_\omega, y_\omega\rangle$$

$$0 \leq \|x_\omega - \alpha\langle x_\omega, y_\omega\rangle y_\omega\|^2 = \|x_\omega\|^2 - \alpha\langle x_\omega, y_\omega\rangle\overline{\langle y_\omega, x_\omega\rangle} - \alpha\langle x_\omega, y_\omega\overline{\rangle\langle y_\omega, x_\omega\rangle} + \alpha^2\langle x_\omega, y_\omega\rangle\overline{\langle y_\omega, x_\omega\rangle}\langle y_\omega, y_\omega\rangle$$

$$0 \leq \|x_\omega - \alpha\langle x_\omega, y_\omega\rangle y_\omega\|^2 = \|x_\omega\|^2 - 2\alpha\langle x_\omega, y_\omega\rangle\overline{\langle y_\omega, x_\omega\rangle} + \alpha^2\langle x_\omega, y_\omega\rangle\overline{\langle y_\omega, x_\omega\rangle}\langle y_\omega, y_\omega\rangle$$

$$0 \leq \|x_\omega - \alpha\langle x_\omega, y_\omega\rangle y_\omega\|^2 = \|x_\omega\|^2 - 2\alpha|\langle x_\omega, y_\omega\rangle|^2 + \mu^2|\langle x_\omega, y_\omega\rangle|^2\|y_\omega\|^2$$

since $\alpha \in$ R then we set $\alpha = \frac{1}{ky_\omega k2}$

$$\|x_\omega - \alpha\langle x_\omega, y_\omega\rangle y_\omega\|^2 = \|x_\omega\|^2 - \frac{2}{\|y_\omega\|^2}|\langle x_\omega, y_\omega\rangle|^2 + \frac{1}{\|y_\omega\|^4}|\langle x_\omega, y_\omega\rangle|^2\|y_\omega\|^2 \geq 0$$

multiplying through by kyk^2 $kx_\omega k^2 ky_\omega k^2$ –

$2|hx_\omega,y_\omega i|^2 + |hx_\omega,y_\omega i|^2 \geq 0$

$|hx_\omega,y_\omega i|^2 \leq kx_\omega k^2 ky_\omega k^2$ theorem 6 $\Longrightarrow$

$|hx_\omega,y_\omega i| \leq kx_\omega kky_\omega k$

Proving the if and only if statement in the theorem

<u>Case 1</u>

Assuming x_ω and y_ω are linearly independent W.T.S $|hx_\omega,y_\omega i| = kx_\omega kky_\omega k$ From

hypothesis, x_ω and y_ω are linearly independent then $x_\omega = \lambda y_\omega$ where $\lambda \in$ K$_{sc}$

Now, $|h\lambda y_\omega,y_\omega i| = k\lambda y_\omega kky_\omega k$

$|h\lambda y_\omega,y_\omega i| = |\lambda|ky_\omega k^2$ $\qquad\qquad (a)$

$k\lambda y_\omega kky_\omega k = |\lambda|ky_\omega kky_\omega k$

$|h\lambda y_\omega.y_\omega i| = \lambda|hy_\omega,y_\omega i|$

$|h\lambda y_\omega, y_\omega i| = |\lambda| k y_\omega k^1$ $\qquad (b)$

comparing (a) and (b)

Hence, (a) and (b) are equal

Case 2

Suppose $|hx_\omega, y_\omega i| = kx_\omega k k y_\omega k$ W.T.S x_ω and y_ω are linearly dependent Now $kx_\omega - \alpha h x_\omega, y_\omega i y_\omega k^2 = 0$ and applying NLS1 from 9. $\Longrightarrow x_\omega - \alpha h x_\omega, y_\omega i y_\omega = 0$ set $\lambda = \alpha h x_\omega, y_\omega i$ we have

$$x_\omega - \alpha h x_\omega, y_\omega i y_\omega = 0 \qquad \Longrightarrow x_\omega - \lambda y_\omega = 0 \qquad x_\omega = \lambda y_\omega$$

$\therefore x_\omega$ and y_ω are linearly dependent

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[1] 2.1.9 Hilbert Space

The generalization of finite-dimensional space to infinite dimension was the work of an excellent German mathematician David Hilbert.This concept generalizes the understanding of Euclidean space and therefore expands the principle of capturing vector algebra of 2-3-dimensional Euclidean spaces to spaces with either finite or infinite dimension. However, Hilbert space is an abstract space that has an inner-product framework that enables the length, angle, and perpendicularity of vectors to be measured. In the mathematical formulation of quantum mechanics, the possible states (more precisely, pure states) of the quantum mechanic system are represented by unit vectors (called state vectors) that reside in the complex separable Hilbert space known as state space, well-characterized up to the complex number of norm 1 (phase factor). Moreover, in quantum mechanics, Hilbert space (a complete inner-product space) plays a key role in the interpretation of wave functions: the absolute value of each wave function is interpreted as a probability distribution function.

Example 2.1.9.1. The space L^2 of square integrable function has an important result when it comes to mathematical formulation of quantum mechanics. Thus, it turns out to be a Hilbert space. That is $\forall f_a, g_a \in L^2$, we define the inner-product as $\langle f_a, g_a\rangle = \int |f_a(x_\omega)||g_a(x_\omega)|dx_\omega$ it can be seen that the dot product $\langle f, g_a\rangle$ is well defined $\forall f_a, g_a \in L^2$

$$|\langle f_a, g_a\rangle| \leq \int |f_a(x_\omega)||g_a(x_\omega)|dx_\omega = \frac{1}{2}\int |f_a(x_\omega)|^2 dx_\omega + \frac{1}{2}\int |g_a(x_\omega)|^2 dx_\omega = \frac{1}{2}\|f_a\|_{L^2} + \frac{1}{2}\|g_a\|_{L^2} < +\infty$$

It is easy to see that it satisfies conditions 17

Definition 22 (Basic properties of Hilbert space). 1. It's a linear vector space

2. It has an inner-product operations that satisfies 17

3. Are separable, so they contains a countable dense subset.

Lemma 1 (paul Garrett (2016)). *let* W_a *be a closed convex subspace of a Hilbert space* H^b.

Then, $\forall x^b \in H^b \qquad \exists! y^b \in W : kx^b - y^bk$ *is minimize.*

Proof. Since H^b is Hilbert, then we are at liberty to use the parallelogram law.Now, W is closed then $\exists\{x_i^b\}_{i=1}^{\infty} : \quad \lim_{i\longrightarrow\infty} x_i^b = x^{b^*}$

Also,

W is convex, then $\forall x^b, y^b \in W$ and $\alpha \in [0,1]$ we have, $\alpha x^b + (1-\alpha)y^b \in W$ Let $\nabla =$

$\inf\{kx^bk : x^b \in W\}$ but W is closed, then $\lim_{i\rightarrow\infty} kx^b{}_ik = \nabla$

By the parallelogram law

$$\|x_i^b - x_j^b\|^2 = 2(\|x_i^b\|^2 + \|y_j^b\|^2) - \|x_i^b + y_j^b\|^2$$

$$\|x_i^b - x_j^b\|^2 = 2(\|x_i^b\|^2 + \|y_j^b\|^2) - \frac{\|2(x_i^b + y_j^b)\|^2}{2}$$

$$\|x_i^b - \xi_j\|^2 = 2(\|x_i^b\|^2 + \|y_j^b\|^2) - 4\frac{\|(x_i^b + y_j^b)\|^2}{2} i, j \underset{\longrightarrow 0}{\longrightarrow} \infty$$

$$\|x_i^b - x_j^b\|^2 = 2\nabla^2 + 2\nabla^2 - 4\nabla^2 = 0$$

since $\frac{(x_i^b + y_j^b)}{2} \in \mathcal{W}$ is convex then $\frac{\|x_i^b + y_j^b\|}{2} \geq \nabla$

Hence, $\{x_i^b\}_{i=1}^{\infty}$ is a Cauchy sequence. Also W_a is closed then the Cauchy sequence converges to a

point $x^{b*} \in W_a$ $kx^{b*}k = k\lim_{i\rightarrow\infty} x^b{}_ik = \lim_{i\rightarrow\infty} kx^b{}_ik = \nabla$

Let $y^b \in W \qquad y^b \, 6{=}\, x^{b*}$ and set $k\kappa k = \nabla$

$$0 \leq \|x^{b^*} - y^b\|^2 = 4\nabla^2 - 4\frac{\|x^{b^*} + y^b\|^2}{2} \text{ from parallelogram } = \Rightarrow \frac{\|x^{b^*} + y^b\|^2}{2} < \nabla \qquad (1)$$

by convexity $\frac{x^{b^*} + y^b}{2} \in \mathcal{W}$ is false $\therefore x^{b*} \, 6{=}\, y^b$ □

Definition 23 (Isometric spaces)**.** Two spaces $\mathcal{H}_1^b$ and $\mathcal{H}_2^b$ are isometric if $\exists L_a$, linear operator : $\forall x_\nu, y_\nu \in \mathcal{H}_1^b \quad \langle \mathcal{L}_a x_\nu, \mathcal{L}_a y_\nu \rangle_{\mathcal{H}_2^b} = \langle x_\nu, y_\nu \rangle_{\mathcal{H}_1^b}$

2.2 Linear Operator

The concept of linear operators, combined with normed linear space plays a vital role in almost all aspects of mathematics as well as its application to modern physics. However, we will consider more specific spaces such as Banach space and a Hilbert space with particular interest to operators (mappings) that preserve the algebraic structure of vector space.

Definition 24 (Linear operator)**.** Suppose X_v and Y_v are any two linear vector spaces over a scalar field K_{sc}. An operator $T_{op} : X_v \longrightarrow Y_v$ is linear if $\forall x_v, y_v \in X_v \; \exists \alpha, \beta \in K_{sc}$ such that

$$T_a(\alpha x_v + \beta y_v) = \alpha T_a x_v + \beta T_a y_v \tag{2.8}$$

OR

$$T_a(\alpha x_v) = \alpha T_a x_v \tag{2.9}$$

$$T_a(x_v + y_v) = T_a x_v + T_a y_v \tag{2.10}$$

If the co-domain Y_v is replaced by a scalar field K_{sc}, then the linear operator T_a under this case, is called linear functional on X_v,

Example 2.2.0.1. Consider the function space $X = \mathbf{C}[\mathbf{a},\mathbf{b}]$ on a closed and bounded interval [a,b] and let $T_a : X \longrightarrow \mathrm{R}$ be defined by

$(T_af_a)(t) \quad = \quad \int_o^t f_a(s)\mathrm{d}s \quad \forall f_a, \in X$ and $t \in [a,b]$

Then T_a is a linear functional on X

Verification

Since **C**[**a**,**b**] is a function space then it contains that is $\forall f_a, g_a \in X$ and $t \in [a,b]$

$\exists \alpha, \beta \in \mathbf{K}_{sc}$ such that $T_a : X \longrightarrow \mathrm{R}$

$(T_af_a)(t) = \int_0^t f_a(s)\mathrm{d}s$

$(T_a(\alpha f_a + \beta g_a))(t) = \int_o^t (\alpha f_a + \beta g_a)(s)\mathrm{d}s$

$(T_a(\alpha f_a + \beta g_a))(t) = \int_o^t (\alpha f_a + \beta g_a)(s)\mathrm{d}s$

$(T_a(\alpha f_a + \beta g_a))(t) = \int_o^t (\alpha f_a(s) + \beta g_a(s))\mathrm{d}s$

$(T_a(\alpha f_a + \beta g_a))(t) = \int_o^t \alpha f_a(s)\mathrm{d}s + \int_0^t \beta g_a(s)\mathrm{d}s$

$(T_a(\alpha f_a + \beta g_a))(t) = \alpha \int_o^t f_a(s)\mathrm{d}s + \beta \int_0^t g_a(s)\mathrm{d}s$

$(T_a(\alpha f_a + \beta g_a))(t) = \alpha(T_af_a)(t) + \beta(T_ag_a)(t)$

$(T_a(\alpha f_a + \beta g_a))(t) = (\alpha(T_af_a) + \beta(T_ag_a)(t)$

$(T_a(\alpha f_a + \beta g_a))(t) - (\alpha(T_af_a) + \beta(T_ag_a)(t) = 0$

$((T_a(\alpha f_a + \beta g_a)) - (\alpha(T_af_a) + \beta(T_ag_a))(t) = 0$

But t 6= 0 since $t \in [a,b]$

$((T_a(\alpha f_a + \beta g_a)) - (\alpha(T_af_a) + \beta(T_ag_a)) = 0$

$T_a(\alpha f_a + \beta g_a) = \alpha T_af_a + \beta T_ag_a \therefore T_a$ is

a linear operator.

Proposition 2.2.0.2. Let X_v and Y_v be any two linear spaces over a scalar field K_{sc} and let an operator $T_{op} : \mathbf{X} \longrightarrow \mathbf{Y}$ be linear operator. Then

1. $T_a(\vec{0}) = \vec{0}$
2. $R_a(T_a) = \{y_v \in Y_v : T_a x_v = y_v \quad \exists x_v \in X_v\} \subseteq Y_v$
3. T_a is injective $\Longleftrightarrow$ $T_a x_v = 0$ $\Longrightarrow x_v = 0$
4. T_a is injective, then T_a^{-1} exists in $R(T_a)$ and $T_a^{-1} : \mathcal{R}(T_a) \longrightarrow \mathbb{X}_\nu$

Proof. 1. From hypothesis, X_v is a linear space, then, $\forall x_v \in X_v$ and $\alpha \in K_{sc}$ by the linearity of T_a, we have,

$T_a(\alpha x_v) = \alpha T_a x_v$ set $\alpha = 0$ $T_a(0x_v) = 0$ $T_a x_v = 0 \Longrightarrow$

$T_a(\vec{0}) = \vec{0}$

2. suppose $x_{v1}, x_{v2} \in R(T)_a$ and $\exists \alpha, \beta \in K_{sc}$. W.T.S $\alpha y_{v1} + \beta y_{v2} \in R(T)_a$ Since $x_{v1}, x_{v2} \in R(T_a) \Longrightarrow$

$\exists x_{v1}, x_{v2} \in X_v : T_a x_{v1} = y_{v1}$ $T_a x_{v2} = y_{v2} \Longrightarrow \alpha x_{v1} + \beta x_{v2} \in X_v$ also, T_a is linear, then $T_a(\alpha x_{v1} +$

$\beta x_{v2} = \alpha T_a x_{v1} + \beta T_a x_{v2} =$

$\alpha y_{v1} + \beta y_{v2}$

$\therefore, \alpha x_{v1} + \beta x_{v2} \in X_v \Longrightarrow \alpha y_{v1} + \beta y_{v2} \in R_a(T_a)$ $\quad R_a(T_a)$ is a subspace of Y_v

3. Case 1 suppose T_{op} is injective and $T_a x_v = 0$ W.T.S $x_v = 0$

Now, $T_{op} x_v = 0$ from (1), $T_a x_\nu = T_a \vec{}(0)$ since T_a is injective $\Longrightarrow x_v = \vec{0}$

Case 2

Next, suppose $T_a x_{vi} = 0$ and $x_v = 0$ W.T.S T_a is injective in the sense that $\forall x_{v1}, x_{v2} \in X_v : T_a x_v = T_a y_v \Longrightarrow x_v = y_v$ $x_v - y_v = 0$ $T_a x_v - T_v y_v = 0 \Longrightarrow T(x - y) = 0$ $x_v = y_v \therefore T_v$ is injective

4. suppose $T_v : X_v \longrightarrow Y_v$ be injective, and $T_a^{-1} : \mathcal{R}(T_a) \longrightarrow \mathbf{X}$ exists W.T.S T_a^{-1} is

a linear operator

$\forall y_{\nu 1}, y_{\nu 2} \in R(T_a)$ and $\alpha, \beta \in K_{sc}$ then $R(T_a)$ is a subspace of Y_ν. Now, $\alpha y_{\nu 1} + \beta y_{\nu 2} \in$

$R(T_a) \Longrightarrow \exists x_{\nu 1}, x_{\nu 2} \in X_\nu : T_a x_{\nu 1} = y_{\nu 1}$ and $T_a x_{\nu 2} = y_{\nu 2}$ since T_{op} is injective, then $x_{\nu 1} = T_a^{-1} x_{\nu 1}$ and $x_{\nu 2} = T_a^{-1} x_{\nu 2}$, by the linearity T_a. We have, $T_a(\alpha x_{\nu 1} + \beta x_2) =$

$\alpha T x_{\nu 1} + \beta T_a x_{\nu 2} = \alpha x_{\nu 1} + \beta y_{\nu 2}$

$T_a^{-1}(\alpha y_{\nu 1} + \beta y_{\nu 2}) = \alpha T_a^{-1} x_{\nu 1} + \beta T_a^{-1} y_{\nu 2}$ Hence T_a^{-1} is a linear operator.

□

Examples of Linear operator

1. Identity operator : An operator $I_x : \mathbf{X}_\nu \longrightarrow \mathbf{X}_\nu$ is an identity operator if $\forall x \in \mathbf{X}$,

$$I_x x_\nu = x_\nu \tag{2.11}$$

2. An operator $\rho : C[a,b] \longrightarrow \mathbf{Y}$ is called differential operator if $\forall x_\nu \in C[a,b]$ and $t \in C[a,b]$ then

$$\rho(x_\nu(t)) = x_\nu(t) \tag{2.12}$$

3. Integral operator: suppose $T_a : C[a,b] \longrightarrow Y_\nu$ is called integral operator if $y_\nu \in C[a,b]$ and $t \in [a,b]$ then

$$T_a((x_\nu(t))) = \int_0^t x_\nu(t)\mathrm{d}t \tag{2.13}$$

4. Matrix operator: let $T_a : M_n(\mathrm{C}) \longrightarrow M_n(\mathrm{C})$ be a matrix operator if $\mathrm{A} \in M_n(\mathrm{C})$ and

$x_v = (x_{v1}, x_{v2}, x_{v3}, \cdots, x_{vn})^T, \qquad y_v = (y_{v1}, y_{v2}, y_{v3}, \cdots, y_{vn})^T :$

$$y_v = Ax_v \tag{2.14}$$

(2.14) can be written out as

$$\begin{pmatrix} y_{v1} \\ y_{v2} \\ \vdots \\ y_{vn} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_{v1} \\ x_{v2} \\ \vdots \\ x_{vn} \end{pmatrix}$$

$A \in M_n(\mathbb{C})$ is a linear operator since matrix multiplication is linear.

2.2.1 Bounded operator

Definition 25. suppose X_v and Y_v are linear vector spaces. An operator $T_a : X_v \longrightarrow Y_v$ is bounded

if $\exists M^{\geq 0} \in \mathbb{R} : \forall x_v \in X_v$ and $\exists \alpha, \beta \in K_{sc}$, :

$$\|T_a x_\nu\| \leq M\|x_\nu\| \tag{2.15}$$

$\Longrightarrow \|T_a\| = \inf\{M > 0 : \|T_a x_\nu\| \leq \|x_\nu\|\}$

From (2.15) $\frac{\|T_a x_\nu\|}{\|x_\nu\|} \leq M$, we noticed that the smallest value of M which will make the expression in (2.15) holds, is the supremum and it is expressed as $\|T_a\| = \sup_{x_\nu \neq 0} \frac{\|T_a x_\nu\|}{\|x_\nu\|}$ if we set $\|x_\nu\| = 1$ then we have $\|T_a\| = \sup_{x_\nu \neq 0} \|T_a x_\nu\|$

NB. we denote **B**(**X**,**Y**) as the collection of all bounded linear operators from **X** to **Y**, and forms a normed linear space under supervision of the supremum norm. That is, $\forall T_a \in$ **B**(**X**,**Y**)

Extension of Bounded linear operator

Let $\mathtt{U}_\nu$ be a closed subspace of a normed linear space $\mathtt{X}_\nu$ and suppose T_a be a bounded linear operator on $\mathtt{U}_\nu$ onto a Banach space $\mathtt{Y}_\nu$. Then the operator T_a(defined on $\mathtt{U}_\nu$) is said to be extended if there exists unique bounded operator $T_a^* : \mathtt{X}_\nu \longrightarrow \mathtt{Y}_\nu$ such that

$T_a^*(x_\nu) = T_a(x_\nu) \quad \forall x \in \mathtt{U}_\nu$ thus the operator T_a^* satisfies $\|T_a^*\| = \|T_a\|$

2.2.2 Bounded linear operators on Hilbert space

Definition 26 (Adjoint Operator (T_a^*)). Let $\mathcal{H}_1^b$ and $\mathcal{H}_2^b$ be Hilbert spaces. A map $T_a : \mathcal{H}_1^b \longrightarrow \mathcal{H}_2^b$ is such that it's adjoint operator (T_a^*) $T_a^* : \mathcal{H}_2^b \longrightarrow \mathcal{H}_1^b : \quad \forall x_\nu, y_\nu \in \mathcal{H}_1^b$, $\langle T_a x_\nu, y_\nu\rangle = \langle x_\nu, T_a^* y_\nu\rangle$

Sesquilinear form Let X_v and Y_v be linear spaces over a set of scalar fields K_{sc}, then the sequilinear form $T_a : X_v \times Y_a \longrightarrow K_a \; \forall x_{v1}, x_{v2} X_a$ and $y_{v1}, y_{v2} \in Y_v \; \exists \alpha, \beta K_{sc}$:

1. $T_a(x_{v1} + y_{v2}, y_v) = M_a(x_1, y) + L_a(x_2, y)$
2. $L_a(x, y_1 + y_2) = L_a(x_v, y_{v1}) + L(x_v, y_{v2})$
3. $L_a(\alpha x_v, y_v) = \alpha L_a(x_v, y_v)$
4. $L_a(x_v, \beta y_v) = \beta \bar{L}_a(x_v, y_v)$

Then L_a is called bilinear operator. However, the boundedness of L_a is given by

$$|L_a(x_\nu, y_\nu)| \leq M\|x_\nu\|\|y_\nu\| \quad \implies \|T_a\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|L_a(x_\nu, y_\nu)|}{\|x\|\|y_\nu\|} = \sup_{\substack{\|x_\nu\|=1\\ \|\varepsilon\|=1}} |L_a(x_\nu, y_\nu)| \qquad (2.16)$$

We now establish the fact that given any bounded linear operator its adjoint always exist.

Theorem 11. *(KREYSZIG (1978))* Let $\mathcal{H}_1^b$ and $\mathcal{H}_2^b$ be Hilbert spaces. An adjoint operator H^{b*} of L_a exists in $\mathcal{H}_2^b$ and it is bounded with norm $\|T_a^*\| = \|T_a\|$

Proof. from eqn(2.15),

$$\|L_a\| \leq \|T_a^*\| \qquad (1)$$

Also, $\|L_a\| = \sup_{\substack{x_\nu \neq 0 \\ y_\nu \neq 0}} \frac{\|T_a^* y_\nu, x_\nu\|}{\|x_\nu\|\|y_\nu\|} \geq \sup_{\substack{y_\nu \neq 0 \\ T_a^* y_\nu \neq 0}} \frac{|\langle {}^* y, T_a^* y_\nu \rangle|}{\|T_a^* y_\nu\|\|y_\nu\|} = \|T_a^*\| \quad \|L_a\| \geq \|T_a\|$ (2)

Comparing (1) and (2)

$\|M_a\| \leq \|\mathcal{L}_a^*\| \leq \|y_\nu\| \qquad \implies \|\mathcal{L}_a^*\| = \|L_a\|$ similarly

kT_ak = kL_ak then, we have $\|T_a^*\| = \|L_a\| = \|T_a^*\| \implies \|T_a^*\| = \|T_a\|$ □

Proposition 2.2.2.1. Suppose $\mathcal{H}_1^b$ and H_2^b are Hilbert spaces. An operator $\mathcal{K}_a : \mathcal{H}_1^b \longrightarrow \mathcal{H}_2^b$ and $\mathcal{L}_a : \mathcal{H}_1^b \longrightarrow \mathcal{H}_2^b$ are two distinct bounded operators: $\forall x_\nu, y_\nu \in \mathcal{H}_1^b$ and ∃α,β ∈ K$_{sc.}$.

Then the following results are established.

1. $\langle \mathcal{L}_a^* y_\nu, x_\nu \rangle = \langle y_\nu, \mathcal{L}x \rangle$

2. $(\mathcal{K}_a + \mathcal{L}_a)^* = \mathcal{K}_a^* + \mathcal{L}_a^*$

3. $(\alpha\mathcal{L}_a)^* = \bar{\alpha}\mathcal{L}_a^*$

4. $(\mathcal{L}_a^*)^* = \mathcal{L}_a$ 5.

$(\mathcal{K}_a\mathcal{L}_a)^* = (\mathcal{L}_a^*\mathcal{K}_a^*)$

Proof. 1. Using condition (IPS2) OF inner product space (see 17)

$\overline{\langle \mathcal{L}_a^* y_\nu, x_\nu \rangle} = \langle y_\nu, \mathcal{L}_a x_\nu \rangle = \overline{\langle \mathcal{L}_a x_\nu, y_\nu \rangle} = \langle y_\nu, \mathcal{L}_a x_\nu \rangle \implies \langle \mathcal{L}_a^* y_\nu, x_\nu \rangle = \langle y_\nu, \mathcal{L}_a x_\nu \rangle$

2. $\langle (\mathcal{K}_a + \mathcal{L}_a)^* x_\nu, y_\nu \rangle = \langle x_\nu, (\mathcal{K}_a + \mathcal{L}_a y_\nu) \rangle = \langle x_\nu, \mathcal{K}_a y_\nu + \mathcal{L}_a y_\nu \rangle = \langle x_\nu, \mathcal{K}_a y_\nu \rangle + \langle x_\nu, \mathcal{L}_a y_\nu \rangle =$

$\langle \mathcal{K}_a^* x_\nu, y_\nu \rangle + \langle \mathcal{L}_a^* x_\nu, y_\nu \rangle$

$\langle (\mathcal{K}_a^* + \mathcal{L}_a^*) x_\nu, y_\nu \rangle \quad \therefore (\mathcal{K}_a + \mathcal{L}_a)^* = \mathcal{K}_a^* + \mathcal{L}_a^*$

3.W.T.S $(\alpha\mathcal{L}_a)^* = \bar{\alpha}\mathcal{L}_a^*$

$$\langle(\alpha\mathcal{L}_a)^* y_\nu, x_\nu\rangle = \langle y_\nu, (\alpha\mathcal{L}_\nu x_\nu)\rangle = \overline{\alpha\langle y_\nu, \mathcal{L}_a x_\nu\rangle} = \bar{\alpha}\langle\mathcal{L}_a^* y_\nu, x_\nu\rangle = \langle\bar{\alpha}\mathcal{L}_a^* y_\nu, x_\nu\rangle \quad \Longrightarrow (\alpha\mathcal{L}_a)^* = \bar{\alpha}\mathcal{L}_a^*$$

4. $\text{h}(\text{L}_{*a})_* x_\nu, y_\nu \text{i} = \text{h}x_\nu, \text{L}_{*a} y_\nu \text{i} = \text{hL}_a x_\nu, y_\nu \text{i} \quad =\Rightarrow \text{L}_{**a} = \text{L}_a$

5. $\text{h}(\text{K}_a\text{L}_a)_* x_\nu, y_\nu \text{i} = \text{h}x_\nu, \text{L}_{*a} y_\nu \text{i} = \text{hK}_{a*} x_\nu, \text{L}_{*a} y_\nu \text{i} = \text{h}(\text{K}_{a*}\text{L}_{*a}) x_\nu, y_\nu \text{i} \quad =\Rightarrow (\text{K}_a\text{L}_{*a}) = \text{K}_{a*}\text{L}_{*a}$

□

NB. All the operators mentioned in (see 2.2) are bounded operators except the differential operator. However, to show that the differential operator is not bounded, we set $\rho_n \in C[0,1] : t \in [0,1]$ and define $\rho_n(t) = t^n$ then $\text{k}\rho_n\text{k} = \sup_{t \in C[a,b]} |t^n| = 1$ Also,

$$\text{L}_a \rho_n(t) = \rho_{n^0}(t) = nt^{n-1} \qquad \forall n \in \text{N} =\Rightarrow \text{kL}_a \rho_n(t)\text{k} = \sup_{t \in C[a,b]} |nt^{n-1}| = n \text{ where } n \in \text{N}$$

Hence, L_a called the differential unbounded operator, because there is no fixed value that will make the expression in (1.24) to hold.This will lead to the notion of unbounded linear operators in the next section

Definition 27 (Self-adjoint, unitary and normal operators)**.** Given H^b to be Hilbert space and $\text{T}_a : \text{H}^b \longrightarrow \text{H}^b$ to be a bounded adjoint operator. Then

1. The operator T_a is self adjoint or Hermitian if $T_a = T_a^*$

2. The operator T_a is unitary if T_a is bijective and $T_a^* = T_a^{-1}$

3. The operator T_a is normal if $TT_a^* = T_a^* T_a = \mathcal{I}_a$

NB. We are to note that, given a bounded operator T_a if is self adjoint then $\langle T_a, x_\nu, y_\nu \rangle =$ $\langle x_\nu, T_a^* y_\nu \rangle \quad \forall x_\nu, y_\nu \in \mathcal{H}^b$ and if is unitary $\langle T_a x_\nu, T_a \rangle = \langle x_\nu, y_\nu \rangle$ From definition, the operator existing between any two isometric spaces is the unitary operator.

Theorem 12. *(KREYSZIG (1978))* Let H^b be a Hilbert space. A bounded operator on Hilbert space is self adjoint if the conditions hold:

1. If T_a is self-adjoint then $\langle T_a x_\nu, x_\nu \rangle$ is real $\quad \forall x_\nu \in H^b$

2. If H^b is complex Hilbert space, then $\langle T_a x_\nu, x_\nu \rangle$ is real $\quad \forall x_\nu \in H$ the operator T_a is self adjoint

Proof. 1. From the hypothesis, T_a is self adjoint. Now, $\langle L_a x_\nu, x_\nu \rangle = \overline{\langle x_\nu, L_a x_\nu \rangle} =$ $\langle L_a x_\nu, x_\nu \rangle$

An operator which is equal to its complex conjugate is self adjoint.

2. let $\langle T_a x_\nu, x_\nu \rangle$ be real, $\forall x_\nu$, then $\langle T_a x_\nu, x_\nu \rangle = \overline{\langle T_a x_\nu, x_\nu \rangle} = \overline{\langle x_\nu, \mathcal{L}_a^* x_\nu \rangle} = \langle T_a^* x_\nu, x_\nu \rangle$
$\langle T_a x_\nu, x_\nu \rangle - \langle T_a^* x_\nu, x_\nu \rangle = 0 \quad \langle \mathcal{L}_a x_\nu - T_a^* x_\nu, x_\nu \rangle = 0 \quad \langle (T_a - T_a^*) x_\nu, x_\nu \rangle = 0$

$(T_a - T_a^*) = 0 \quad \implies T_a = T_a^*$

□

2.2.3 Matrix operator

Matrix operators are of practical importance in the application of quantum theory, where observables are treated as matrices with a special interest in symmetry and rotation of eigenvectors in an orthonormal system. However, we will examine some classes of matrix operators with emphasis on geometric symmetry.

Definition 28 (General Linear Group **GL(N)**). The set of all square invertible matrix

i.e $\forall A \in \mathbf{GL(N)} : \det A \neq 0$

Definition 29 (Special Unitary Group **SU(N)**). The set of all square matrices

i.e $\{\forall A \in \mathbf{SU(N)} : \det A = +1\}$

Definition 30 (matrix lie group). Let G be a closed subspace of **GL(N)** with the property that given there exist a converging sequence $\{\mathcal{A}_n\}_{n=1}^{\infty} \in G$ for which $\lim_{n \to \infty} A_n = A$ for some $A \in \mathbf{GL(N)}$ then, $A \in \mathbf{GL(N)}$ or A is not invertible

Definition 31 (Lie algebra). Let X_v be vector space over a scalar field K_{sc}. A bilinear operator $[\cdot,\cdot] : X_v \times X_v \longrightarrow X_v$ is called Lie algebra together with the following conditions:

1. $[x_v, x_v] = 0 \quad \forall x_v \in X_v$
2. $\left[x_v, [y_v, z_v]\right] + \left[y_v, [z_v, x_v]\right] + \left[z_v, [x_v, y_v]\right] = 0$ (*Jacobian Identity*)

$[x_v, y_v]$ are called commutator of x_v and y_v

Definition 32 (Exponential Operator)**.** Given A ∈ **GL(N)** define e^{A} ∈ **GL(N)** as

$$e^{\mathcal{A}} = \sum_{n=0}^{\infty} \frac{\mathcal{A}^n}{n!} = 1 + \mathcal{A} + \frac{1}{2!}\mathcal{A}^2 + \frac{1}{3!}\mathcal{A}^3 + \cdots \tag{2.17}$$

Definition 33 (Properties exponential operator)**.** 1. $e^0 = I$

2. e^{A} is invertible

3. if A,B commute, then $e^{A+B} = e^{A}(e^{B})$

4. if S ∈ **GL(N)** then $e^{SAS^{-1}} = Se^{A}S^{-1}$

5. $\frac{d}{dt}e^{t\mathcal{A}} = \mathcal{A}e^{t\mathcal{A}}$

2.2.4 One Parameter Unitary Group

Definition 34. *(Baker (2000))* Set $U_a(t) \in B(H^b)$ to be a family of unitary operators then $U_a(t)$ is strongly continuous one-parameter if

(i) $U_a(t) : \mathbb{R} \longrightarrow B(H^b)$ where $t \mapsto U_a(t)$ is strongly continuous.
(ii) $U_a(t+s) = U_a(t)U_a(s)$ $\forall t,s \in \mathbb{R}$

Definition 35. *(Infinite Generator of Unitary Group)* A well-defined densely linear operator T_a with domain $D(T_a) \subset H^b$ is said to be a generator of $U_a(t)$ if the following conditions are met:

$$x_\nu \in \mathrm{H} \quad \left\{ ^b : t \mapsto \mathcal{U}_a(t)x_\nu \text{ is differentiable} \right\} \quad 1.\ \mathcal{D}(T_a) =$$

2. $\forall x_\nu \in \mathcal{D}(\mathcal{T}_a) \implies i\frac{d}{dt}\mathcal{U}_a(t)x_\nu = \mathcal{U}_a(t)T_a x_\nu$

Lemma 2 (Properties of Infinite Generator)**.** If T_a is an infinite generator of $\mathcal{U}_a(t)$ then it has the following properties:

(a) $\mathcal{D}(T_a)$ is invariant. i.e $\forall t \in \mathbb{R}$ $\mathcal{U}_a(t)\mathcal{D}(T_a) = \mathcal{D}(T_a)$

b It commutes with $\mathcal{U}_a(t)$ i.e

$$[T_a, \mathcal{U}_a(t)]x_\nu = T_a\mathcal{U}_a(t)x_\nu - \mathcal{U}_a(t)T_a x_\nu = 0 \qquad x_\nu \in \mathcal{D}(T_a) \tag{2.18}$$

(c) T_a is symmetric, i.e

$$\langle T_a x_\nu, y_\nu\rangle = \langle x_\nu, T_a y_\nu\rangle \qquad \forall x_\nu, y_\nu \in \mathcal{D}(T_a) \tag{2.19}$$

(d) $\mathcal{U}_a(t)$ is uniquely determined by T_a

Proof. (a) *W.T.S* $\mathcal{D}(T_a)$ is invariant

Now, if we set

$$s \mapsto \mathcal{U}_a(s)\mathcal{U}_a(t)x_\nu = \mathcal{U}_a(s+t)x_\nu \tag{2.20}$$

is differentiable if and only if

$$s \mapsto \mathcal{U}_a(s)x_\nu = \mathcal{U}_a(-t)\mathcal{U}_a(s+t)x_\nu \quad \text{is differentiable.} \tag{2.21}$$

consider (2.20) At $s = 0$, we have $(-i)\mathcal{U}_a(t)T_a x_\nu$

Also, consider (2.21) at $s = 0$ we have $(-i)\mathrm{U}_a(-t)\mathrm{U}_a(t)T_a x_\nu$. Hence $x_\nu \in \mathrm{D}(T_a) \Longleftrightarrow x_\nu \in \mathrm{U}_a(t)\mathrm{D}(T_a)$

(b) If we set $x_\nu \in \mathrm{D}(T_a)$ and allow $\mathcal{U}_a(t)T_a x_\nu = \mathcal{U}_a(t) i \frac{d}{ds}\mathcal{U}_a(s)x_\nu = i\frac{d}{ds}\mathcal{U}_a(t)\mathcal{U}_a(s)x_\nu = i\frac{d}{ds}\mathcal{U}_a(s)\mathcal{U}_a(t)x_\nu = T_a\mathcal{U}_a(t)x_\nu$ evaluated at $s = 0$

$\therefore \mathrm{U}_a(s + t) = \mathrm{U}_a(s)\mathrm{U}_a(t) \Longrightarrow \mathrm{U}_a(t)x_\nu \in \mathrm{D}(T_a)$

(c) To show symmetry, By 27 hx_ν, y_νi = h$T_a x_\nu, T_a y_\nu$i it preserve in inner-product. Now,

$$\frac{d}{dt}\langle x_\nu, y_\nu\rangle = 0 \implies \frac{d}{dt}\langle \mathcal{U}_a(t)x_\nu, \mathcal{U}_a(t)y_\nu\rangle = \langle -iT_a\mathcal{U}_a(t), \mathcal{U}_a(t)\rangle + \langle \mathcal{U}_a(t)x_\nu, -iT_a\mathcal{U}_a(t)y_\nu\rangle$$

$-i$kU$_a(t)$k^{2}h$T_a x_\nu, y_\nu$i + ikU$_a(t)$k^{2}h$x_\nu, T_a y_\nu$i but kU$_a(t)$k^2 = 1 $\Longrightarrow$ $-i$h$T_a x_\nu, y_\nu$i + ih$x_\nu, T_a y_\nu$i $\therefore$ h$T_a x_\nu, y_\nu$i = h$x_\nu, T_a y_\nu$i

(d) set T_a to be a generator of $\hat{\mathrm{U}_a}(t)$. Then by symmetry of T_a we have

$$\frac{d}{dt}\|\mathcal{U}_a(t)x_\nu - \hat{\mathcal{U}_a}(t)x_\nu\|^2 = \frac{d}{dt}\Big[\langle \mathcal{U}_a(t)x_\nu - \mathcal{U}_a(\hat{t})x_\nu, \mathcal{U}_a(t)x_\nu - \mathcal{U}_a(\hat{t})x_\nu\rangle\Big]$$

$$\frac{d}{dt}\|\mathcal{U}_a(t)x_\nu - \hat{\mathcal{U}_a}(t)x_\nu\|^2 = \frac{d}{dt}\Big[\langle \mathcal{U}_a(t)x_\nu, \mathcal{U}_a(t)x_\nu\rangle - \langle \mathcal{U}_a(t)x_\nu, \hat{\mathcal{U}_a}(t)\rangle - \langle \mathcal{U}_a(\hat{t})x_\nu, \mathcal{U}_a(t)x_\nu\rangle +$$
$$\langle \mathcal{U}_a(\hat{t})x_\nu\rangle\Big] = \frac{d}{dt}\Big[\|\mathcal{U}_a(t)x_\nu\|^2 - 2\Re\langle \mathcal{U}_a(t)x_\nu, \hat{\mathcal{U}_a}(t)\rangle + \|\hat{\mathcal{U}_a}(t)x_\nu\|^2\Big] = \frac{d}{dt}\Big[\|x_\nu\|^2(\|\mathcal{U}_a(t)\|^2 +$$

k$\hat{\mathrm{U}_a}(t)$k^2) − 2<hU$_a(t)x_\nu$, $\hat{\mathrm{U}_a}(t)$i$^{\mathrm{i}}$ but kU$_a(t)$k^2 = k$\hat{\mathrm{U}_a}(t)$k^2 = 1

$$\frac{d}{dt}\|\mathcal{U}_a(t)x_\nu - \hat{\mathcal{U}_a}(t)x_\nu\|^2 = \frac{d}{dt}\Big[2\|x_\nu\|^2 - 2\Re\langle \mathcal{U}_a(t)x_\nu, \mathcal{U}_a(\hat{t})x_\nu\rangle\Big] = -2\Re\Big[\frac{d}{dt}\langle \mathcal{U}_a(t)x_\nu, \hat{\mathcal{U}_a}(t)x_\nu\rangle\Big]$$

$$= \frac{d}{dt}\langle \mathcal{U}_a(t)x_\nu, \mathcal{U}_a(t)x_\nu\rangle = \langle -iT_a\mathcal{U}_a(t)x_\nu, \hat{\mathcal{U}_a}(t)x_\nu\rangle + \langle \mathcal{U}_a(t)x_\nu, -iT_a\hat{\mathcal{U}_a}(t)x_\nu\rangle$$

$$\Big[\langle -iT_a\mathcal{U}_a(t)x_\nu, \hat{\mathcal{U}_a}(t)x_\nu\rangle + \langle \mathcal{U}_a(t)x_\nu, -iT_a\hat{\mathcal{U}_a}(t)x_\nu\rangle\Big] = -2\Re[0] = 0$$

$$\frac{d}{dt}\|\mathcal{U}_a(t)x_\nu - \hat{\mathcal{U}_a}(t)x_\nu\|^2 = 0$$

Hence, $\forall x_\nu \in \mathrm{D}(T_a)$ and $\mathrm{U}_a(0) = \hat{\mathrm{U}_a}(0) \Longrightarrow \mathrm{U}_a(t) = \hat{\mathrm{U}_a}(t) \forall t \in \mathrm{R} \therefore \mathrm{D}(T_a) = \mathrm{H}^b$, and $U_a(t) = \hat{U}_a(t)$

□

Proposition 2.2.4.1. *The necessary and sufficient condition for* $U_a(t) \in B(H^b)$ *to be unitary is* $\mathcal{U}_a^*(t) = \mathcal{U}_a^{-1}(t)$

Proof. necessary condition suppose $U_a(t)$ is unitary then by

definition we have $\forall x_\nu, y_\nu \in H^b$,

$\langle \mathcal{U}_a^*(t)\mathcal{U}_a(t)x_\nu - x_\nu, y_\nu\rangle = \langle \mathcal{U}_a(t)x_\nu, \mathcal{U}_a(t)y_\nu\rangle - \langle x_\nu, y_\nu\rangle = 0$ but $\mathcal{U}_a^*(t)\mathcal{U}_a(t) = I$ by the surjectivity of $U_a(t)$ we define $\forall y_\nu \in \mathcal{H}^b \quad \exists x_\nu \in \mathcal{H}^b : \mathcal{U}_a(t)x_\nu = y_\nu \implies \mathcal{U}_a^*(t)\mathcal{U}_a(t)y_\nu = \mathcal{U}_a(t)\mathcal{U}_a^*(t)\mathcal{U}_a(t)x_\nu = \mathcal{U}_a(t)x_\nu = y_\nu \quad \therefore \mathcal{U}_a(t)\mathcal{U}_a^*(t) = 1$

Hence $\mathcal{U}_a^*(t) = \mathcal{U}_a^{-1}(t)$

sufficient condition

suppose $\mathcal{U}_a^*(t) = \mathcal{U}_a^{-1}(t)$ by the surjectivity of $U_a(t)$ we have

$\langle \mathcal{U}_a(t)x_\nu, \mathcal{U}_a(t)y_\nu\rangle = \langle \mathcal{U}_a^*(t)\mathcal{U}_a(t)x_\nu, y_\nu\rangle = \langle \mathcal{U}_a^{-1}(t)\mathcal{U}_a(t)x_\nu, y_\nu\rangle = \langle x_\nu, y_\nu\rangle \implies \mathcal{U}_a(t)$ is unitary. □

We now consider the classes of matrix operators with emphasis on complex matrices.

Definition 36. Let $\mathbf{M_n}(\mathrm{C})$ be a square matrix and let $A \in \mathbf{M_n}(\mathrm{C})$ with entries a_{ij} where i = *rows* and j = *columns* Now, $\bar{A} = \overline{a_{ij}} = a_{ji}$ conjugate of A

$A^{\intercal}_{ij} = \overline{a_{ji}}$ called the transpose conjugate of A and is a square matrix , denoted by A^* :

$A^* = \overline{(A^{\intercal})} = \overline{(A)}^{\intercal}$ if A is real then $A^* = A^{\intercal}$

1. A matrix $A \in \mathbf{M_n}(\mathrm{C})$ is Hermitian if $A^* = A$

2. If A ∈ $\mathbf{M_n}$(R) then A is symmetric, that is $A^{|} = A$

3. A ∈ $\mathbf{M_n}$(C) is normal, if $AA^* = A^*A$

4. A matrix U_a ∈ $\mathbf{M_n}$(C) is unitary if $\mathcal{U}_a\mathcal{U}_a^* = \mathcal{U}^*\mathcal{U} = \mathcal{I}$

5. A matrix Q ∈ $\mathbf{M_n}$(R) is orthogonal if $QQ^{|} = Q^{|}Q = I$

6. A = a_{ij} ∈ $\mathbf{M_n}$(C) then $tra(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$, sum of its diagonal element and it is a linear map.That is

   (a) $tra(\alpha A) = \alpha tra(A)$

   (b) $tra(A + B) = tra(A) + tra(B)$

2.2.5 Spectral Thoery

Given any square matrix A ∈ $\mathbf{M_n}$(C) and λ ∈ C called an eigenvalue of A if $\exists x_v = 06 \in C$ called the eigenvector of x_v such that $Ax_v = \lambda x$, however, if we set λ ∈ C to be the eigenvalue of the square complex A with the corresponding set of eigenvalues. Hence, if

$\lambda_i \in C$ $\forall i$ is a set of an eigenvalues of a complex square matrix A, then $\forall x_\nu \in \mathbb{C}^n, \quad \forall i$ is a also a set of eigenvectors corresponding to each $\lambda^0{}_i s$ are called the eigenvectors of A with respect to λ. However, in the presence of zero vector, the eigenvectors forms a subspace called eigenspace. Also, if we set λ ∈ C to be eigenvalue of A. Then the following holds if

1. $Ax_v = \lambda x_v : x_v 6= 0, \quad x_v \in C^n$

2. $(\lambda I - A)x_v = 0$

3. $(\lambda I - A)$ defines a linear operator which has a non zero kernel.

4. $(\lambda I - A)$ is not invertible when $\det(\lambda I - A = 0$

$\det(\lambda I - A)$ is a polynomial of degree n of the form

$$\lambda^n = tra(A)\lambda^{n-1} + \cdots + \det A \tag{2.22}$$

Thus , $\lambda^0 s$ of A are the zeros of (2.22). Equation (2.22) is called the characteristic polynomial of A

Definition 37 (Spectrum)**.** The set of n-roots of the characteristic equation in (2.22) are called the spectrum

Let $\rho(A) = \max_{1 \le i \le n} |\lambda_i|$ is called the spectral radius

Definition 38 (Properties of square complex matrix)**.** 1. $A \in \mathbf{M_n}(C)$ is a complex square matrix, if the eigenvalues of A are real and forms orthogonal matrix as

$A = Q^| dig(\lambda_1, \lambda_2, \lambda_3, \cdots \lambda_n) Q$

2. $A \in \mathbf{M_n}(C)$ is Hermitian if the eigenvalues are real and form an orthogonal matrix as $A = U_a^| dig(\lambda_1, \lambda_2, \lambda_3, \cdots \lambda_n) U_a$

Definition 39 (Diagonalizable matrix operator).

A Square matrix A is diagonalizable if it is similar to a diagonal matrix. That is, $\exists P : P^{-1}AP = D$

Definition 40 (Diagonalization). It is the process of finding a corresponding diagonal matrix or linear operator

Proposition 2.2.5.1. If $A \in \mathbf{M}_n(C)$ is Hermitian and $U_a(t) = e^{itA}$.Then the following are established

1. $U_a(t)$ is unitary
2. $U_a(t)U_a(s) = U_a(t + s)$
3. For any arbitrary collection of $U_a(t)_{t\in R}$ is abelian under the supervision of multiplication.
4. $\lim_{t\longrightarrow 0} \frac{\mathcal{U}_a(t) - I}{t} = i\mathcal{A}$

Proof. 1. To show $U_a(t)$ is unitary then W.T.S $(U_a(t))^*U_a(t) = I$

$$e^{\mathcal{A}} = \sum_{n=0}^{\infty} \frac{\mathcal{A}^n}{n!} = 1 + \mathcal{A} + \frac{1}{2!}\mathcal{A}^2 + \frac{1}{3!}\mathcal{A}^3 + \cdots$$

Depicting the above equation, we have

$$\mathcal{U}_a(t) = \sum_{n=0}^{\infty} \frac{(it\mathcal{A})^n}{n!}$$

∞ $* n$ ∞ $* n$

Now, $(\mathcal{U}_a(t))^* = \sum \frac{((\text{itA})\ )}{n!} = \sum \frac{(-\text{itA}\ )}{n!}$ $\mathcal{A}$ ________ ________ is Hermitian $\implies$ A* = A $n=0$ $n=0$

$$\sum_{n=0}^{\infty} \frac{(-it\mathcal{A})^n}{n!} = \mathcal{U}_a(-t) \tag{2.23}$$

From a singular value decomposition, since A is Hermitian, we can find a unitary operators V and V* : A = VDV* where V are the entries of eigenvectors of A and D is diagonal matrix. So,

$$\mathcal{U}_a(t) = \sum_{n=0}^{\infty} \frac{(it\mathcal{A})^n}{n!} = \sum_{n=0}^{\infty} \frac{(it\mathcal{V}D\mathcal{V}^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\mathcal{V}^*(it)\mathcal{V})^n}{n!} = \mathcal{V}^* \sum_{n=0}^{\infty} \frac{(itD)^n}{n!}\mathcal{V}$$

since matrix multiplication is point-wise then

$U_a(t) = V^* diag(e^{itd1}, e^{itd2}, e^{itd3}, \cdots, e^{itdn})V$ but $U_a(t)(U_a(t))^* = U_a(t)U_a(-t)$

$U_a(t)(U_a(t))_* = [V_* diag(e_{itd1}, e_{itd2}, e_{itd3}, \cdots, e_{itdn})V][V_* diag(e_{-itd1}, e_{-itd2}, e_{-itd3}, \cdots, e_{-itdn})V]$

$U_a(t)(U_a(t))^* = V^*V = I$

2 $\mathcal{U}_a(t) = \sum_{n=0} \frac{(it\mathcal{A})}{n!} = e^{it\mathcal{A}}$ ∞ n

$\mathcal{U}_a(s) = \sum_{n=0}^{\infty} \frac{(is\mathcal{A})^n}{n!} = e^{is\mathcal{A}} \implies \mathcal{U}_a(t)\mathcal{U}_a(s) = (e^{it\mathcal{A}})(e^{is\mathcal{A}}) = e^{it\mathcal{A}+is\mathcal{A}} = e^{i(t+s)\mathcal{A}} =$ similarly

$$\sum_{n=0}^{\infty} \frac{[i(t+s)\mathcal{A}]^n}{n!} = \mathcal{U}_a(t+s)$$

(3) *W.T.S* the set $\{U_a(t) : t \in \mathrm{R}\}$ is abelian.

(i) commutativity property

$\forall s,t \in \mathrm{R}$ we have

$$\mathcal{U}_a(t) = \sum_{n=0}^{\infty} \frac{(it\mathcal{A})^n}{n!} = e^{it\mathcal{A}} \implies \mathcal{U}_a(t)\mathcal{U}_a(s) = e^{i(s+t)} = \mathcal{U}_a(s)\mathcal{U}_a(t)$$

(ii) Identity property

$\exists (U_a(t))^*$ such that $\mathcal{U}_a(t)\mathcal{U}_a^*(t) = I$ but $(U_a(t))^* = U_a(-t) \Longrightarrow = e^{itA}e^{-itA} = e^0 = I$

(iii) Inverse property if $\exists$ $U_a(0)$ such that $U_a(0) = I$

So, $U_a(t)U_a(0) = (e^{itA})(e^{i0A}) = U_a(t)$

(4) $W.T.S \quad \lim_{n \longrightarrow 0} \frac{\mathcal{U}_a(t) - I}{t} = i\mathcal{A}$

$\mathcal{U}$

$$\text{where } I = \mathcal{U}_a(0) = \frac{e^{it\mathcal{A}} - 1}{t} = \lim_{t \longrightarrow 0} \left(\frac{e^{it\mathcal{A}} - 1}{t} \quad \frac{{}_a(t) - I}{t} \right.$$

!

Using the L'Hopitals rule lim (iAe^{itA}) By the linearity of limit function we have $t \longrightarrow 0$

$$i\mathcal{A} \lim_{t \longrightarrow 0} e^{it\mathcal{A}} = i\mathcal{A}$$

$$\Longrightarrow \lim_{t \longrightarrow 0} \frac{\mathcal{U}_a(t) - 1}{t} = i\mathcal{A}$$

□

2.3 Fourier Analysis

To comprehend the phenomenon governing superposition of wave-function, it will be of utmost benefit to begin with the review of Fourier analysis

Fourier series

Consider a periodic function $f_a(x_v)$ of infinite dimension space with periodicity of 2π such that $f_a(x_v + 2\pi) = f_a(x_v)$. However, in the mathematical treatment of Quantum mechanics, the wave-function is considered as complex wave-function. So it will be of particular interest to treat the Fourier expansion in exponential function.

Now

$$f_a(x_\nu) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx_\nu} \tag{2.24}$$

where α_n is the coefficient of expansion which will be determined shortly.

By the orthogonality of the wave-function, we have

$$\int_{-\pi}^{\pi} e^{i(n-m)x'_\nu} \mathrm{d}x' = 2\pi\delta_{nm} \tag{2.25}$$

which is expressed in-terms of Kronecker delta

Since we have introduced the orthogonality relation, we can then proceed to find the coefficient of expansion by multiplying $f_a(x_\nu)$ by the conjugate of its function and then integrate the resultant function along the interval $-\pi$ to π

$$\alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_a(x'_\nu) e^{-mx'_\nu} \mathrm{d}x'_\nu \tag{2.26}$$

put eqn(1.25) into eqn(1.23) we obtain the Fourier expansion for $f(x)$ as

$$f(x) = \sum_{i=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') e^{in(x-x')} \mathrm{d}x' \tag{2.27}$$

To introduce the orthonormal condition for the function $f(x)$. let's consider $f_n(x) = \frac{1}{2\pi} e^{inx}$ we can write eqn(2.26) as:

$$f(x) = \sum_{i=-\infty}^{\infty} \gamma_n f_n(x)$$

$$\int_{-\pi}^{\pi} f_n^*(x')f_m(x')\mathrm{d}x' = \delta_{nm}$$

$$\text{set} \quad \gamma_n = \int_{-\pi}^{\pi} f_n^*(x')f_m(x')\mathrm{d}x' = \delta_{nm}$$

Hence
$$f_a(x_\nu) = \sum_{i=-\infty}^{\infty} \int_{-\pi}^{\pi} f_a(x'_\nu) f_n(x) f_n^*(x'_\nu)\mathrm{d}x'_\nu$$

We now consider the periodic interval of $2l$, where l is the length of the wave :$f_a(x_\nu+2l) = f_a(x_\nu)$.

However, we can introduce the orthonormal function as;

$$\frac{1}{2\pi} e^{\frac{in\pi x_\nu}{l}}$$

$$\Longrightarrow f_a(x_\nu) = \sum_{i=-\infty}^{\infty} \lambda_n \frac{1}{\sqrt{2l}} e^{\frac{in\pi n}{l}}$$

$$\frac{1}{2\pi}\int_{-l}^{l} e^{i(n-m)}\frac{\pi x'_\nu}{l}\mathrm{d}x'_\nu = \delta_{nm}$$

$$\lambda_n = \frac{1}{2\pi}\int_{-l}^{l} f_a(x')e^{-\frac{in\pi x'_\nu}{l}}\mathrm{d}x'_\nu \quad \Longrightarrow f_a(x_\nu) = \frac{1}{2l}\sum_{n=-\infty}^{\infty}\int_{-l}^{l} f_a(x'_\nu)e^{\frac{in\pi}{l}(x_\nu - x'_\nu)}\mathrm{d}x'_\nu$$

Now, to introduce the wave number k_n

$$k_n = \frac{n\pi}{l} = \frac{2\pi}{\lambda_n} \quad \text{with} \quad \lambda_n = \frac{2l}{n}$$

Hence the orthonormal function is $f_n(x_\nu) = \frac{1}{\sqrt{2l}} e^{ik_n x_\nu}$ This relationship enables the transition from the Fourier series to the Fourier integral to a finite-dimensional space wave packet.

Fourier Integral

If we consider a periodic function $f_a(x_\nu)$ with periodic interval $2l$ which repeat its functional value from $-\infty$ to ∞. Setting $l \longrightarrow \infty$ and keeping the wave packet fixed. Then taking $l \longrightarrow \infty$ under the condition that $f_a(x_\nu) = 0$ sufficiently rapidly as $x_\nu \longrightarrow \pm\infty$. In fact, we can make a transition from an infinite wave to a wave packet of finite dimensional. As $l \longrightarrow \infty$, then the spectrum of possible values of k_n goes from discrete spectrum to continuous one.

Now the Fourier expansion for $f(x)$ becomes

$$f_a(x_\nu) = \frac{1}{2\pi l}\int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}x'_\nu f_a(x'_\nu) e^{ik(x_\nu - x'_\nu)}$$

we then obtain the Fourier amplitude as:

$$f_a(x_\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k g(k) e^{ikx_\nu}$$

The Fourier amplitude is called the Fourier transform of $f_a(x_\nu)$ and is given by

$$g_a(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_a(x'_\nu) e^{-ikx'_\nu} \mathrm{d}x'_\nu$$

Hence the orthonormal integral diverges where $k = k^{0}$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix'_\nu(k-k')} \mathrm{d}x'_\nu = \delta(k - k')$$

The Kronecker delta now becomes the Dirac delta function.

2.4 Dirac Delta Function

Consider the Fourier series to be a limit sum over a fixed number of terms, then :

$$f_a(x_\nu) = \lim_{N \longrightarrow \infty} \int_{-\infty}^{\infty} f(x'_\nu) \sum_{n=-N}^{N} f_n(x_\nu) f_n^*(x'_\nu) \mathrm{d}x'_\nu \tag{2.28}$$

The corresponding Fourier integral

$$f_a(x_\nu) = \lim_{k_0 \longrightarrow \infty} \int_{-\infty}^{\infty} f_a(x'_\nu) \frac{1}{2\pi} \int_{-k_0}^{k_0} e^{ik(x_\nu - x'_\nu)} \mathrm{d}k \tag{2.29}$$

set

$$k(x_\nu, x'_\nu) = \sum_{n=-N}^{N} f_n(x_\nu) f_n^*(x'_\nu) \quad \text{OR} \quad k(x_\nu, x'_\nu) = \frac{1}{2\pi} \int_{-k_0}^{k} e^{ik(x_\nu - x'_\nu)} \mathrm{d}k$$

If N or k_o becomes very large then the function is strongly peaked at $x_\nu = x'_\nu$ However, if $x_\nu \neq x'_\nu$ then the oscillation is of a very small amplitude.Notwithstanding the fact that the limit point of this wave are the equations (2.28) and (2.29). Now change the infinite sum or the k-integral with $x'_\nu -$ integral through the definition of Dirac delta function.

$$\sum_{n=-\infty}^{\infty} f_n(x_\nu) f_n^*(x_\nu) = \delta(x_\nu - x_{\nu 0}) \tag{2.30}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x - x_0)} \mathrm{d}x_0 = \delta(x_\nu - x_0) \tag{2.31}$$

NB The Dirac delta function is not a function but a distribution.

Definition 41 (Properties of Dirac delta function). 1. $\delta(x_\nu - x_{\nu 0}) = 0$ for $x \neq x_\nu^{'}$ if $x_\nu = x_\nu^{'}$, the Dirac delta function becomes infinite. That is

$\int_{r}^{r_0} \delta(x_\nu - x_{\nu 0}) dx_{\nu 0}$ provided $x_\nu = x_{\nu 0}$ is in the space (r, r_0)

Also

$$\int_{-\infty}^{\infty} f_a(x_{\nu 0})\delta(x_\nu - x_{\nu 0}) = f_a(x_\nu)$$

The limiting process from eqn(1.28)

$$\delta(x_\nu - x_{\nu 0}) = \lim_{n \longrightarrow \infty} \frac{1}{2\pi} \int_{k_0}^{k} e^{ik(x_\nu - x_{\nu 0})} \mathrm{d}k = \lim_{k_0 \longrightarrow \infty} \frac{\sin k_0 (x_\nu - x_{\nu 0})}{\pi (x_\nu - x_{\nu 0})}$$

Graph

2. Dirac delta function is an even function

$$\delta(-x_\nu) = \delta(x_\nu) \implies x\frac{d}{dx_\nu} = -\delta(x_\nu)$$

Also

$$\int_{r_0}^{r} x_\nu \delta^{'}(x_\nu) = \Big[x\delta(x_\nu) \Big]_{r_0}^{r} - \int_{r_0}^{r} \delta(x_\nu)\mathrm{d}x_\nu = -\int_{r_0}^{r} \delta(x_\nu)\mathrm{d}x_\nu$$

3. if $a \in \mathrm{R}$, then

$$\delta(a x_\nu) = \frac{1}{|a|}\delta(x_\nu)$$

Chapter 3

UNBOUNDED OPERATORS

3.1 Overview

This chapter will provide the necessary background to be able to work with unbounded and self-adjoint operators, which includes the important concept of graph an operator. Next will develop a functional calculus to give meaning to the expression $f(T_a)$, where f is a function on the real line and T_a is an operator. For this we will need the spectral theorem for unbounded, self-adjoint operators. Finally, we state some criteria necessary for an operator to be self-adjoint and essentially self-adjoint.

3.1.1 Introduction

As it turns out, most of the operators occurring in the mathematical formulation of quantum mechanics under certain conditions are linear, self-adjoint and unbounded. In particular, many quantum operators expressed in terms of ordinary and partial differentiation are generally unbounded. More often than not, these operators admit another type of property which in a way, makes up for the fact that they are unbounded. Consequently, these operators are closed. We will now look at the sufficient condition for an operator to be bound by defining everywhere-operator on a Hilbert space (H^b) **NB** we will state the theorem without proof.

Theorem 13 (Helinger-Toeplitz)**.** Suppose A be everywhere defined linear operator on a Hilbert space(H^b) : hx_v,Ay_vi = hAx_v,y_vi

We define an operator everywhere in Hilbert when it is symmetrical and bounded. However, in the case of unbound operators, we can not define the symmetric operator for the whole space, but rather for the subspace of the space. This is referred to as the domain of an unbound operator D(T_a).

Definition 42 (Graph)**.** suppose $\mathcal{H}_1^b$ and $\mathcal{H}_2^b$ are Hilbert spaces and $T_a : \mathcal{H}_1^b \longrightarrow \mathcal{H}_2^b$ an operator. Then graph of T_a(Gr(T_a)

$$Gr(T_a) = \left\{ (\psi^b, T\psi^b) \quad \forall \psi^b \in \mathcal{H}_1^b \right\} \tag{3.1}$$

we observe that $Gr(T) \subset \mathcal{H}_1^b \times \mathcal{H}_2^b$ and $(\psi^b,\varphi^b) \in Gr(T_a) \Longleftrightarrow \varphi^b = T_a\psi^b$

Definition 43 (Closed Graph)**.** Given $\mathcal{H}_1^b$ and $\mathcal{H}_2^b$ as Hilbert spaces and $T_a : \mathcal{H}_1^b \longrightarrow \mathcal{H}_2^b$ as linear operator. Then T_a is closed if $\overline{Gr(T_a)} \subset \mathcal{H}_1^b \times \mathcal{H}_2^b$

In general, not all operators are closed but it is possible to include their extensions.

Proposition 3.1.1.1. Suppose $T_a : \mathcal{H}_1^b \longrightarrow \mathcal{H}_2^b$, where $\mathcal{H}_1^b, \mathcal{H}_2^b$ are Hilbert spaces Then T_a is closed $\Longleftrightarrow \ \exists \ \{\psi_n^b\}_{n=1}^{\infty} \subset \mathcal{D}(T_a)$ with $\psi_n \dashrightarrow \psi$ and $T_a\psi_n^b \longrightarrow \phi^b$ as $n \longrightarrow \infty$ we have

(i) $\psi \in D(T_a)$ (ii) $T_a\psi^b = \varphi^b$

Proof. Suppose T_a is closed subset of $\mathcal{H}_1^b \times \mathcal{H}_2^b$. Let $\{\psi_n^b\}_{n=1}^{\infty} \subset \mathcal{D}(T_a)$ be such that $\lim_{n\longrightarrow\infty} \psi_n^b = \psi^b$ and $T_a\psi_n^b \longrightarrow \phi^b$ W.T.S (i) $\psi^b \in D(T_a)$ (ii) $T_a\psi^b = \varphi^b$

Now, $\lim_{n\to\infty} \psi_n{}^b = \psi^b$ and $\lim_{n\to\infty} T_a\psi_n{}^b = \varphi^b \Longrightarrow (\psi_n{}^b, T_a\psi_n{}^b) \longrightarrow (\psi^b, \varphi^b)$ as $n \longrightarrow \infty$

moreover, $(\psi_n^b, T\psi_n^b) \in Gr(T_a) \quad \forall n \in \mathbb{N}$ and since T_a is closed, we have (ψ^b, φ^b)
$\Longrightarrow \psi^b \in \mathcal{D}(T_a), \quad \phi^b = T_a\psi_n^b$

Next, suppose $\psi_n^b \in \mathcal{H}_1^b$, $\psi_n^b \longrightarrow \psi^b$ and $T_a\psi_n^b \longrightarrow \phi^b$ as $n \longrightarrow \infty$

(i) $\psi^b \in D(T_a)$ and (ii) $T_a\psi^b = \varphi^b$ W.T.S $Gr(T_a)$ is closed subset $\mathcal{H}_1^b \times \mathcal{H}_2^b$

Suppose $\{\psi_n^b, T_a\psi_n^b\}_{n=1}$ be any arbitrary sequence in $Gr(T_a) : (\psi_n^b, T_a\psi_n^b) \longrightarrow (\psi^b, \phi^b)$ as $n \longrightarrow \infty$. To conclude that $Gr(T_a)$ is closed it suffices to show that $(\psi^b, \varphi^b) \in Gr(T_a)$

But $(\psi_n^b, T_a\psi_n^b) \longrightarrow (\psi^b, \phi^b)$ as $n \longrightarrow \infty \Longrightarrow \lim_{n\longrightarrow\infty} \psi_n^b = \psi^b$ and $\lim_{n\longrightarrow\infty} T_a\psi_n^b = \phi^b$

From hypothesis, $\psi^b \in D(T_a)$ and $T\psi_n^b = \phi^b \Longrightarrow (\psi^b, T_a\psi_n^b) \overset{n\longrightarrow\infty}{\longrightarrow} (\psi^b, \phi^b) \in Gr(T_a)$ and so $Gr(T_a)$ is closed.

□

We now give an example of an operator that is linear and unbounded. This is to appreciate the importance of the closed Graph Theorem.

Example 3.1.1.2. Consider the sup*norm* defined on the function space $C[0,1]$ is given by $Dom = \left\{ f_a : f_a \in C[0,1], f_a' < \infty \right\}$ and $T_a : D(T_a) \longrightarrow C[0,1]$ is also defined by $T_af_a = f_a'$ is a differential operator. Then

(i) T_a is linear

(ii) T_a is closed

(i) To show linearity of T, we assume that $\exists f_a, g_a \in C[0,1]$ and $\alpha,\beta \in K_{sc}$ such that

$T_a(\alpha f_a + \beta g_a) = (\alpha f_a + \beta g_a)' = \alpha f_a' + \beta g_a' = \alpha T_a f_a + \beta T_a g_a \implies T_a$ is linear.

(ii) W.T.S T_a is closed

Suppose $\{f_n\}_{n\geq 1} \subset C[0,1] : \lim_{n\longrightarrow\infty} f_n = f_a$ and $T_a f_n = f_a' \longrightarrow \phi^b$ as $-\to \infty$ But

$\lim_{n-\to\infty} T_a f_n = \varphi^b \implies$ convergence in the norm

So, $\|T_a f_n - \varphi^b\| = \sup_{t\in[0,1]} |T_a(f_n)(t) - \varphi^b(t)| = \sup_{t\in[0,1]} |f_n'(t) - \varphi^b(t)| -\to 0$ as $n -\to \infty$

The convergence is uniform ($t \in [0,1]$) and $\varphi^b(t) = \lim_{n-\to\infty} f_n'(t)$,

By the uniform convergence, we have

$ds = \int_0^t \lim_{n\longrightarrow\infty} f_n'(s)ds = \lim_{n\longrightarrow\infty} \int_0^t f_n'(s)ds = f_a(t) - f(0)$ (fundamental theorem of calculus)

$f(t) = f(0) + \int_o^t \phi(s)ds$ Applying the Leibniz rule

$f_a'(t) = \phi^b(t), \quad \forall t \in [0,1]$ but $f_a \in D$ and $T_a f_a = f_a' = \phi^b$ $\forall t \in [0,1] \therefore (f_a, T_a f_a) \in$

$Gr(T_a) \implies T_a$ is closed. (iii) W.T.S T_a is bounded

We set $f_n(t) = t^n$ then $\|f_n\| := \sup_{t\in[0,1]} |t^n| = 1 \implies f_n'(t) = nt^{n-1}$ so that

$\|T_a f_n\| = \sup_{t\in[0,1]} |nt^{n-1}| = n \quad \forall n \in \mathbb{N} \therefore T_a$ is not bounded.

We will now look at the scenario where a map is continuous by carefully examine its bounded and linearity property. In other words, a map is said to be continuous if we can show its boundedness and linearity property.

Theorem 14. Suppose X_v, Y_v are Banach spaces, and $T_a : X_v \longrightarrow Y_v$ is linear. Let $Gr(T_a)$ be closed. Then T_a is continuous

Proof. Consider the space $X \times Y$ with norm

$$\|(\psi_b, \varphi_b)\|_{X_v \times Y_v} = \|\psi_b, \psi_b\|_x + \|\varphi_b\|_p \tag{3.2}$$

Since X_v and Y_v are Banach space, it follows that $X \times Y$ in Banach endowed with the norm (3.2) $Gr(T_a)$, closed in $X_v \times Y_v \Longrightarrow Gr(T_a)$ is Banach.

Consider the projection map $\prod_1 : Gr(T_a) \longrightarrow X$ is defined by $\prod_1 (\psi^b, T_a\psi^b) = \psi^b$ then $\prod_1$ is a bijective map, continuous as well. Then $\prod_1^{-1} : X_\nu \longrightarrow Gr(T_a)$ is continuous. Hence, $\exists M^{>0} \in \mathbb{R} : \| \prod_1^{-1}(\psi)\|_{Gr(T_a)} \leq M\|\psi\|, \quad \forall \psi^b \in X \quad WLOG$ assume $M > 1$. Then $\| \prod_1^{-1}(\psi)\|_{Gr(T_a)} < M\|\psi^b|_\psi^b \implies |(\psi^b, T_a\psi^b)\|_{Gr(T_a)} < M\|\psi^b\|_\psi^b$ by 3.2 $\|(\psi_b, T_a\psi_b) = \|\psi_b\|_{b\psi} + \|T_a\psi_b\|_{b\varphi}$ so $\|\psi^b\|^b_\psi + \|T\psi^b\|^b_\varphi \leq M\|\psi\|^b_\psi$ so, $\|T_a\psi^b\|^b_\varphi \leq (M-1)\|\psi^b|^b_\psi$, $\forall \psi^b X \Longrightarrow T_a$ is continuous (i.e linear and bounded)

□

Definition 44 (Unbounded operators). 1. T_a is an unbound operator if $T_a : D(T_a) \subset H^b \longrightarrow H^b : \overline{D(T_a)} = H^b$. Then T_a is densely defined

2. $(S_a, D(S_a))$ is an extension of $(T_a, D(T_a))$ if $D(S_a) \supset D(T_a) : T_a \subset S_a$
3. $(T_a, D(T_a))$ is symmetric if $\forall \psi^b, \varphi^b \in D(T_a)$ then $\langle \psi^b, T_a\varphi^b \rangle = \langle T_a\psi^b, \varphi^b \rangle$.

Definition 45 (The adjoint of an unbounded operator). If T_a is densely defined on H^b.

Then the domain $\mathcal{D}(T_a^*)$ of the adjoint T_a^* is given by

$$D(T_a^*) = \left\{ \psi^b \in \mathcal{H}^b : \exists \gamma \in \mathcal{H}^b | \langle \psi^b, T_a \phi^b \rangle = \langle \gamma, \phi^b \rangle \quad \forall \phi^b \in \mathcal{D}(T_a) \right\}$$

$D(T_a)$ is dense and γ is uniquely defined $\forall \psi^b \in \mathcal{D}(T_a^*)$. Now

$$T_a^* : \mathcal{D}(T_a^*) \longrightarrow \mathcal{H}^b, \qquad \psi^b \longmapsto T_a^* \psi^b = \gamma$$

Definition 46 (Self-adjoint operator). Suppose $T_a : D(T_a) \subset H^b \longrightarrow H^b$.

$D(T_a^*) = \mathcal{D}(T_a)$ and $T_a^* = T_a$ Then $(T_a, D(T_a))$ is called self-adjoint operator.

Proposition 3.1.1.3. An unbound operator T_a is symmetric $\iff \quad T_a \subset T_a^*$

Proof. Let T_a to be symmetric $\Rightarrow \mathcal{D}(T_a) \subset \mathcal{D}(T_a^*) \quad \forall \psi^b \in \mathcal{D}(T_a)$ if

we set $\gamma = T_a \psi^b = T_a^* \psi^b$

Next, suppose $T_a \subset T_a^*$ *W.T.S* T_a is symmetric

Now, $\forall \psi^b \in \mathcal{D}(T_a) \subset \mathcal{D}(T_a^*)$ and $\langle \psi^b, T_a \phi^b \rangle = \langle T_a^* \psi^b, \phi^b \rangle = \langle T_a \psi^b, \phi^b \rangle$

□

Theorem 15. Suppose T_a is a densely defined linear on $D(T_a)$. Then T_a is a generator of a unitary Group $U_a(t) = e^{-iT_a t} \iff T_a$ is self-adjoint

Remark A symmetric operator is called closable whenever $T_a \subset T_a^*$ and T_a^* is closed. Self-adjoint operators are of course, the most relevant operators in quantum mechanics due to their capacity to produce time evolution. However, we will lay down the necessary requirements that will allow us to verify whether or not an operator is self-adjoint

Criteria for an operator to be self-adoint

Lemma 3. Let $T_a : D(T_a) \subset H^b \longrightarrow H^b$ where H^b is Hilbert. Then

$$h\psi^b, T_a\psi^b i \in R \ \forall \psi^b \in D(T_a) \Longleftrightarrow T_a$$

is symmetric

Proof. Suppose T_a is symmetric Then W.T.S $h\varphi^b, T_a\psi^b i \in R \ \forall \psi^b \in D(T_a)$

Since T_a is symmetric $\Longrightarrow h\varphi^b, T_a\psi^b i$ and $\overline{h\varphi^b, T_a\psi^b i} = hT_a\varphi^b, \psi^b i$

Next, suppose $h\varphi^b, T_a\psi^b i, \quad \forall \psi^b \in D(T_a)$ *W.T.S* T_a is symmetric

Now, $h\varphi^b, T_a\psi^b i \quad \forall \psi^b \in D(T_a) \quad \Longrightarrow h\varphi^b, T_a\psi^b i = hT_a\varphi^b, \psi^b i \quad \forall \psi^b \in D(T_a)$

From polarization identity

$$\langle\phi^b, T_a\psi^b\rangle = \frac{1}{4}\Big(\langle\phi^b+\psi^b, T_a(\phi^b+\psi^b)\rangle - \langle\phi^b-\psi^b, T_a(\phi^b-\psi^b)\rangle' - i\langle\phi^b+i\psi^b, T_a(\phi^b+i\psi^b)\rangle + i\langle\phi^b-i\psi^b, T_a(\phi^b-i\psi^b)\rangle\Big)$$

Also,

$$\langle\phi^b, T_a\psi^b\rangle = \langle T_a\psi^b, \phi^b\rangle = \frac{1}{4}\Big[\langle\phi^b+\psi^b, T_a(\phi^b+\psi^b)\rangle - \langle\phi^b-\psi^b, T_a(\phi^b-\psi^b)\rangle + i\langle\phi^b + i\psi^b, T_a(\phi^b+i\psi^b)\rangle - i\langle\phi^b-i\psi^b, T_a(\phi^b-i\psi^b)\rangle\Big]$$

Interchanging ψ^b with φ^b we have

$$\langle T\phi^b, \psi^b\rangle = \frac{1}{4}\Big[\langle\phi^b+\psi^b, T_a(\phi^b+\psi^b)\rangle - \langle\phi^b-\psi^b, T_a(\phi^b-\psi^b)\rangle + i\langle\psi^b+i\phi^b, T_a(\psi^b+i\phi^b)\rangle - i\langle\phi^b-i\psi^b, T(\psi^b-i\phi^b)\rangle\Big]$$

$$\langle T_a\phi^b, \psi^b\rangle = \frac{1}{4}\Big[\langle\phi^b+\psi^b, T_a(\phi^b+\psi^b)\rangle - \langle\phi^b-\psi^b, T_a(\phi^b-\psi^b)\rangle + i\langle i\psi^b-\phi^b, T_a(i\psi^b-i\phi^b)\rangle - i\langle i\psi^b+\phi^b, T_a(i\psi^b+\phi^b)\rangle\Big] = \langle\phi^b, T_a\psi^b\rangle \qquad \square$$

Definition 47 (Essentially Self-adjoint operator)**.** A symmetric operator is essentially selfadjoint whenever its closure is self-adjoint

We shall state a lemma without proof that will allow us to state the requirements necessary for an operator to be essentially self-adjoint.

Lemma 4. Suppose (T_a,D(T_a)) be densely defined, then

1. $\forall z \in$ C $\Longrightarrow$ ker($T_a^* \pm z) = Ran(T_a \pm z)^\perp$

$$\ker(T_a^* \pm z) = \{0\} \iff Ran(T_a \pm z) = \overline{T_a} \qquad (3.3)$$

2. If T_a is closed and symmetric, then $Ran(T_a \pm i)$ is closed.

Theorem 16 ((Mo¨ller (2010))). Suppose a well-defined symmetric operator T_a together with the D(T_a). Then the following conditions are equivalent.

1. T_a is self-adjoint
2. T_a is closed and ker$\{T_a^* \pm i\} = \{0\}$
3. Ran$\{T_a^* \pm i\} = \{T_a\}$

Proof. (1) =⇒ (2) Suppose T_a is self-adjoint and T_a is closed we set $\psi_\pm^b \in$ ker($T_a^* \pm i$). Then $T_a\psi_\pm = Fi\psi_\pm^b$. But the eigenvalues of a symmetric operators are always real =⇒ $\psi_\pm^b = 0$

(2) =⇒ (3) From (3.3) T_a is closed and symmetric and by (4) =⇒ $RanT_a$ is closed.

(3) =⇒ (1) since T_a is symmetric, =⇒ $T_a \subset T_a^*$ and suppose $\psi^b \in Dom(T^*)$ then by assumption, $Ran(T_a \pm i) = T_a \, \exists \psi^b \in$ D(T_a) :

$$(T_a^* - i)\psi^b = (T_a - i)\phi \tag{3.4}$$

By $T \subset T_a^* \implies (T_a^* - i)\phi = (T_a^* - i)\phi$ i.e $\psi - \varphi \in$ ker($T_a^* - i) \implies \psi^b - \phi^b = 0$
$\psi^b = \phi^b \in \mathcal{D}(T_a) \implies \mathcal{D}(T_a^*) \subset \mathcal{D}(T_a)$. Also, $T_a = T_a^*$ on $\mathcal{D}(T_a)$ □

3.1.2 Spectral Theorem for unbounded operator

To be able to formulate and prove stone theorem, it is necessary to understand the expression of the form $U_a(t) = e^{-itA}$ for an unbounded operator A. We can achieve this by developing a functional calculus with the help of spectral theorem for unbounded selfadjoint operators.

Definition 48 (Resolvent set$\rho(T_a)$)**.** *(Porta (2019))*
Let $T_a : D(T_a) \subset H^b \longrightarrow H^b$. resolvent of T_a is:

$\rho(T_a) :=$ n$z \in C|(T_a - z) : D(T_a) \longrightarrow H^b$ is a bijection with continuous inverse.^{o} (3.5) **Definition 49**

(Resolvent)**.** Considering $z \in \rho(T_a)$, we define the resolvent of T_a at z as:

$$R_z(T) := (T - z)^{-1} \in H^b \tag{3.6}$$

Definition 50 (Spectrum)**.** The spectrum of T_a is given by

$$\sigma(T_a) := C \setminus \rho(T_a) \tag{3.7}$$

Remark The consequence of a graph of an operator is that an operator is closed only if the linear operator is continuous, so the continuously property of the resolvent set can be dropped.(see 3.5)

Proposition 3.1.2.1. $\rho(T_a)$ 6= 0 only if T_a is not closed

Proof. if we set $(T_a - z) : D(T_a) \longrightarrow H^b$ to be bijective and $(T_a - z)$ invertible. Then $\Gamma(T_a) = \Gamma(T_a - z)$

$= \Gamma(T_a - z)^{-1}$. Thus, if $\Gamma(T_a)$ is not closed $\Longrightarrow \Gamma(T_a - z)^{-1}$ is not

closed . Then $\exists \psi_n^b \subset \mathcal{H}^b : \psi_n \longrightarrow 0$ as $n \longrightarrow \infty$ but $(T_a - z)^{-1}$ is not continuous (i.e)

$(T_a - z)\psi_n^b \not\longrightarrow 0$ as $n \longrightarrow \infty$ Hence $\rho(T_a) = \emptyset$ □

Definition 51. Suppose $T_a : D(T_a) \subset H^b \longrightarrow H^b$ is closed. Then its spectrum $\sigma(T_a)$ can

be categorized as follows:

$$\sigma(\rho(T_a) :== \left\{ z \in \mathbb{C} \middle| T_a - z \right.$$ 1.is not injective) is called the point spectrum, and it

coincides with the set of eigenvalues of the operator.

2.is $$\sigma_C(T_a) := \left\{ z \in \mathbb{C} \middle| T_a - z \right.$$ injective not surjective with dense range) is called the

continuous spectrum

3.is $$\sigma_\gamma(T_a) := \left\{ z \in \mathbb{C} \middle| T_a : z \right.$$ injective but not surjective with no dense range) is

called the residual spectrum.

Example 3.1.2.2 (Porta (2019)). (a) Given a position operator $\hat{x}_\nu$ with domain

$$\hat{D} := \left\{ \psi^b \in L^2(\mathbb{R}) \middle| \hat{x_\nu}(\psi(x_\nu)) \in L^2(\mathbb{R}) \right\} \tag{3.8}$$

We defined

$$\hat{x} : \psi \longrightarrow x(\psi) \Longrightarrow (\hat{x}_\nu - z)^{-1}$$

$\therefore \sigma(\hat{x}_\nu) = \mathbb{R}$ Then $(x_\nu - \lambda)$

is the product of the function $(x_\nu - z)^{-1} < \infty$, $\forall z \in \mathbb{C}\backslash\mathbb{R}$ has a dense range $\forall \lambda \in \mathbb{R}$. Also, $\forall \psi^b \in L^2$, we define

$\hat{x} : \psi \longrightarrow x\psi \quad \Longrightarrow \quad (\hat{x}_\nu - z)^{-1}$ is product of the function $(x_\nu - z)^{-1}$ which is bounded

$\forall z \in \backslash R \therefore \sigma(\hat{x}_\nu) = R$

The map $(\hat{x}_\nu - \lambda)$ has a dense range $\forall \lambda \in R$. Now $\forall \psi^b \in L^2$, we defined

$$\psi_n^b := \chi\mathbb{R}\left[\lambda - \frac{1}{n}, \lambda + \frac{1}{n}\right]\frac{\psi^b}{x_\nu - \lambda} \tag{3.9}$$

Then $(x_\nu - \lambda)\psi_n \longrightarrow \psi^b \in L^2$ and hence the range of $x - \lambda$ is dense $\therefore \sigma(\hat{x}) = \sigma_c(\hat{x}) = R$

(b) Let $U_a \in B(H^b)$ be unitary. Then $\sigma(T_a) = \sigma(U_a T_a U_a^{-1}) \quad \Longrightarrow T_a - z$ is injective

$\Longleftrightarrow U_a(T_a - z)U^{-1} = U_a T_a U_a^{-1} - z$ is injective $\therefore$ the momentum operator $\hat{p} = -i\frac{d}{dx_\nu}$ on R^2 has real

continuous spectrum, $\sigma(\hat{p}) = \sigma_c(\hat{p}) = R$, since $\hat{p} = F\hat{x}F^{-1} \Longrightarrow F$ is unitary.

Definition 52 (Neumann series)**.** If we set X to be Banach and $T_a \in B(x)$ with $\|T_a\| < 1$. Then $(1 - T)$ is continuously invertible. i.e

$$(1 - T_a)^{-1} = \sum_{n=0} T_a^n \tag{3.10}$$

and

$$\|(1 - t)^{-1}\| \leq (1 - (\|T_a\|)^{-1}) \tag{3.11}$$

Theorem 17. If $T_a : D(T_a) \subset H^b \longrightarrow D(T_a)$. Then

1. $\rho(T_a)$ is not open.i.e the $\sigma(T_a)$ is closed

2. The resolvent map

$$\rho(T_a) \longrightarrow B(\mathcal{H}^b) : z \mapsto R_z(T_a) := (T_a - z)^{-1} \tag{3.12}$$

3. if we set $T_a \subset B(\mathcal{H}^b) : |z| \leq \|T_a\| \quad \forall \sigma(T_a) \Longrightarrow$ the spectrum is compact
4. $\forall z, w \in \rho(T_a)$ the first resolvent identity holds:

$$R_w(T_a)R_z(T_a) = R_z(T_a)R_z(T_a)$$

Proof. (1) if we set $z_0 \in \rho(T_a)$ and allow $|z - z_0| < \|R_{z_0}\|^{-1}$ we have

$$T_a - z = (T_a - z_0) - (z - z_0) = (T_a - z_0)(1 - (z - z_0)R_{z_0})(T_a) \tag{3.13}$$

$|(z - z_0)R_{z_0}(T_a)| \leq 1 \Longrightarrow (1 - (z - z_0)R_{z_0})$ is continuously invertible. $\Longrightarrow (T_a - z)$ is continuously invertible $z \in \rho(T)$ (2) From 52

$$R_{z_0} = (1 - (z - z_0)R_{z-0}^{-1})R_{z_0} = \sum_{n=0} (z - z_0)^n R_{z_0}^{n+1} \tag{3.14}$$

where the coefficients $R_{z_0}^{n+1} \in B(\mathcal{H})$

(3) set $|z| > \|T_a\|$ then $1 - \frac{1}{z}$ is invertible and $T_a - z$ as well $\therefore z \in \rho(T_a)$

(4) $R_z(T) - R_w(T_a) = R_z(T_a)(T_a - w)R_w(T_a) - R_z(T_a)(T_a - z)R_w = (z - w)R_z(T_a)R_w(T_a)$

□

We shall now look at the projection-valued measure which is a necessary condition in deriving the spectrum of an unbounded self-adjoint operator.

A bounded operator $\hat{p}$ defined on a Hilbert space H^b is called projection-valued measured if it satisfies $\Phi^{\hat{}2} = \Phi$. All the eigenvalues of $\hat{}$ Φ are either 0 or 1. The compliment of a$\hat{}$ projector Φ is also a projector NB$\hat{}$ We denote σ -algebra by B

Definition 53. Considering the map $\Phi^b : B \longrightarrow B(H^b)$ is a projection measure, if $\forall\phi \in B, \Phi(\phi^b)$ is a projection. Then

1. $\Phi(\mathbb{R}) = 1$
2. if we set $\varphi = \bigcup_{n=1} \varphi_n^b$ with $\varphi_m \cap \varphi_n^b = \emptyset, m \neq n$ then

$$\lim_{n \longrightarrow \infty} \sum_{n=1}^{\infty} \Phi(\varphi_n^b)\mu = \Phi(\varphi^b)\mu \quad \forall \mu \in \mathcal{H}^b$$

Lemma 5. Suppose Φ is a projection-valued measured. Then the following properties can be established:

1. $\Phi(\phi^c) = 1 - \Phi(\phi)$ 2. $\Phi(\phi_n \cap$

$\phi_m) = \Phi(\phi_n)(\phi_m)$

3. $\varphi_n \subset \varphi_m \implies \|\Phi(\varphi_n^b)\mu\| \leq \|\Phi^b(\varphi_m)\mu\|$

Proof. (1) $\Phi(\emptyset)\mu = \sum_{n=1} \Phi(\emptyset)\mu \Longrightarrow \Phi(\emptyset) = 0$ Also, $R = \phi^c \cup \phi \cup \emptyset \cup \emptyset \cup, \cdots$

$\Phi(R)\mu = (\Phi(\phi^b) + \Phi(\phi^c)\mu \Longleftrightarrow \Phi(\phi^c)\mu = (1 - \Phi(\phi))\mu, \quad \forall \mu \in H^b$

(2) set φ_n^b , φ_m to be Borel sets and if $\varphi_n^b \cap \varphi_m^b = \emptyset$. Then $\Phi(\varphi_n^b)\Phi(\varphi_m^b)$ = 0. on the other hand, we write $\varphi_n^b = (\varphi_n^b - \varphi_m^b) \cup (\varphi_n^b \cap \varphi_m^b)$ Also, $\Phi(\varphi_n^b)\Phi(\varphi_m^b) = \Big(\Phi(\varphi_n^b - \varphi_m^b) + \Phi(\varphi_n^b \cap \varphi_m)\Big)\Big(\Phi(\varphi_n^b - \varphi_m^b) + \Phi($

(3) suppose $\varphi_n^b \subset \varphi_m$, then $\varphi_m^b = \varphi_n^b \cup (\varphi_m^b - \varphi_n^b)$ partitioned φ_m^b

$\|\Phi(\phi^b{}_m)\mu\|^2 = \langle\Phi(\phi^b{}_m)\mu,\mu\rangle = \langle\Phi(\phi_n{}^b)\mu,\mu\rangle + \langle\Phi(\phi^b{}_m - \phi^b{}_n)\mu,\mu\rangle$

$= \|\Phi(\phi^b{}_n)\|^2 + \|\Phi(\phi^b{}_m - \phi^b{}_n)\mu\|^2 \geq \|\Phi(\phi^b{}_n)\mu\|^2$

□

Corollary 1. Suppose densely well-defined operator $(T_a, D(T_a))$ on H^b. Then, $\emptyset \neq \sigma(T_a)$. If $z \in \Phi(T_a)$, we have

$$\|R_T(z)\| \leq \frac{1}{|\Im(z)|} \tag{3.15}$$

Furthermore, $R_T(\bar{z}) = R_T(z)^*$

see(Troung (2015)) for the proof.

Definition 54 (Nevanlinnna-Herglotz functions)**.** Analytic function $F_a(z)$ is said to be

Herglotz if it maps the upper half-plane H^{+}nz ∈ C : =(z) > 0⁰ into itself

Theorem 18. For all Herglotz functions have the integral representation as:

$$F_a(z)\gamma + \beta z + \int_{\mathbb{R}} \Big(\frac{1}{1-\lambda} - \frac{1}{1+\lambda^2}\Big) d\mu$$

$\forall z \in H^{+}$ for some Borel measure μ satisfying $\int_{\mathbb{R}} \Big(1+\lambda^2\Big) d\mu < \infty$ where α and β are constants to be determined. α = <[(F_a(i))] and $\beta = \lim_{y \longrightarrow \infty} \frac{F_a(iy)}{iy}$. Also, suppose $F_a(z)$ is Herglotz satisfying $|F_a(z)| \leq \frac{M}{\Im(z)}, \quad z \in H^{+}$. Then the integral form is

$$F_a(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu \tag{3.16}$$

for some Borel measure μ satisfying μ(R) ≤ M

Theorem 19 (Stieltjes inversion formula). *Suppose $F(z)$ be Borel transform of a Borel measure μ. Then the Borel measure of an interval(λ_1,λ_2) with r.p.t μ is*

$$\mu(\lambda_1, \lambda_2) = \lim_{\delta \longrightarrow 0} \lim_{\varepsilon \longrightarrow} \frac{1}{\pi} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \Im(F(\lambda + i\varepsilon)) d\lambda \tag{3.17}$$

Definition 55 (Spectral Theory for unbounded self-adjoint operator)**.** If a well-defined self-adjoint operator (T_a,D(T_a)) is define on a Hilbert space H.

Then $\exists ! \Phi_{T_a}$:

$$T_a = \int_{\mathbb{R}} \lambda d\Phi_{T_a}$$

Proof. see proof from(Troung (2015)) □

Theorem 20 (Functional Calculus)**.** Suppose on a Hilbert spaceH^b we can define a selfadjoint A such that $\exists!\Psi^b$ from the Borel function on R into a B(H^b). Then the following results are established.

1. Ψ^A is algebraic- homomorphism

   $\Psi^A(f_a g_a) = \Psi^A(f_a)\Psi^A(g)$, where $f_a, g_a \in \sigma(R)$ and $\alpha \in K_{sc}$

   $\Psi^A(\alpha f_a) = \alpha \Psi^A(f_a)$
2. Ψ^A is continuous. $k\Psi^A(f_a)k_{B(A)} \leq N k f_a k_\infty$ where N is constant.
3. suppose the function $f_a(x_v) = x_v$ then $\Psi^A(f_a) = A$
4. suppose $A\psi^b = \lambda^b \psi^b$
   $\Psi^{\mathcal{A}}(f_a)\psi^b = f_a^(\lambda^b)$
5. suppose $f \geq 0$ then Ψ^A

Proof. see (M¨oller (2010)) for the proof □

Having develop the means of executing any bounded Borel function of a self-adjoint operator such that their algebraic structure is possible on the real line. This is what we called functional calculus.

Chapter 4

MATHEMATICAL FORMULATION OF QUANTUM MECHANICS

4.1 Overview

First of all, we will look at the description of Quantum Mechanics in a strongly mathematical setting. We will also prove the existence and uniqueness of Schroedinger's timedependent particle wave equation, which is the pivot in the formulation and proof of Stone's theorem. Lastly, we will state and prove Stone's theorem using a direct approach and then some applications related to the Dilation and Rotation of unitary Operators.

4.1.1 Introduction

The wave function has been the core and the most relevant to the study of microscopic particles in quantum mechanics. The concept was first established in 1926 by a German Scientist Erwin Schroedinger, in which he proposed that the wave function that describes the particles and are spread out in space with the most particles concentrated at where the wave-function is large. However, Born argued that the description propounded by

Schroedinger was actually a probabilistic amplitude and the square of its absolute value represents the probability density function for finding or locating a particle in space (Gao (2011)). This idea became the most conventional and commonly accepted notion of wave function in the contemporary research of QM. Moreover, the wave function defining the particle was actually a complex-wave function as determined through double-slit experiment that the intensity of the incoming wave ($\psi^b(x,t) = Acos(kx - wt)$) of amplitude (A) and wavelength must be a constant- that is $I = |A^2|$. However, squaring the wave function to achieve the square of the amplitude as resulted in varying the cosine square:

$$I = |\psi^b(x,t)|^2 = |A|^2 \cos^2\left(\frac{p}{h}x\right)$$

. To resolve the problem, complex wave function was adopted $\psi_b(x,t) = Ae^{\frac{ipx}{h}}$ where, $|\psi^b(x,t)|^2 = \psi^*\psi = (Ae^{\frac{-ipx}{h}})(Ae^{\frac{ipx}{h}}) = |A|^2 = I$ which is constant at all points and agrees with the experiment, and therefore the incoming wave was actually a complex-wave with a complex amplitude containing a phase variable. (Energy (E), Momentum (P), Position (X)) the general particle wave equation is $\psi_b(x,t) = Ae^{\frac{i}{h}(px-Et)}$ The configuration of any quantum system is completely described by the phase variable, and the program of quantum mechanics is to look for the particle wave function, in which the notion of finding the particle wave function is captured in the Schrödinger equation:

$$-ih\frac{\partial\psi^b}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2\psi^b}{\partial x^2} + V\psi^b$$

and thereby determines the position of particle by completely solving the differential equation with some initial condition which determines the particle wave function ($\psi^b(x,t)$) at all future time, just as, how Newton classical mechanics determines ($x(t)$) at all future time(Griffiths and Schroeter (2018)). However, In the mathematical formulation of quantum mechanics, the complex wave function(ψ^b) are state vectors in Hilbert space with some geometric properties associated with it. Hilbert space plays a vital role in the formulation of QM, where vectors in space represent the quantum state of the particles and for the matter, any geometric property associated with it can be use to describe the quantum state of the particle in any given system.

The mathematical formulation of Quantum mechanics as to do with the mathematics of linear vector space. Any quantum system is associated with a complex wave function. This wave-function (state vectors) completely described the state or condition of any physical system. Moreover, this state vectors covers all the possibilities of a system, and a such forms a complex linear vector space.

4.1.2 Dirac Notation

The mathematical objects of quantum mechanics are mainly state vectors and linear operators(matrices) which are written in Dirac notation as ket and bra-vectors. Dirac denote the ket-vectors $|\psi^b$i as column vectors and the bra-vectors h$\psi^b|$ as row vectors.

□□□□□□□□□□□□$aa^{b1}{}_{b2}$□□□□□□□□□□□□

$|\psi^b\rangle =$, $\langle\psi^b| = (a_1^b, a_2^b, a_3^b, \cdots, a_n^b)^\dagger$

□□□□□□□□□□□□$a_{\dots bn}$□□□□□□□□□□□□

4.1.3 Scalar Product

Usually, in Quantum theory, we often compute wave amplitude of any physical system in a state $|\psi^b\rangle$ as

$$\int_{-\infty}^{\infty} \psi^{b^\dagger}(x)\psi^b(x)\mathrm{d}x = \int_{-\infty}^{\infty} |\psi^b(x)|^2 \mathrm{d}x \tag{4.1}$$

However, generalization of the concept conforms with the mathematics of inner-product

$$\Big(|\psi_1^b\rangle, |\psi_2^b\rangle\Big)$$

The complex inner-product returns a complex number and a such its value depends on the order of the state vectors

$$\langle\psi^b|\varphi^b\rangle = \langle\varphi^b|\psi^b\rangle^\dagger \tag{4.2}$$

Hence, (4.2) is designed in a such a way that its value will always be real Although the inner-product is positive definite

$$|\psi\rangle \geq o$$

It allows us to define length or norm on a complex linear vector space.

$$\|\psi^b\| = \sqrt{\langle\psi^b|\psi^b\rangle} \tag{4.3}$$

Also, (4.1) made it possible to also look at the limit and convergence of an infinite sequence.

Two state vectors are orthogonal if

$$\langle\psi^b|\varphi^b\rangle = 0$$

However, this can be made into an orthonormal system by first showing that the two vectors are orthogonal and each state vector has a norm of 1. That is

$$\langle\psi^b|\varphi^b\rangle = 0$$

$$|\varphi^b\rangle = 1 \quad |\varphi^b\rangle = 1$$

In generalization,

$$\langle\psi_m|\psi_n\rangle = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

where δ_{mn} is called the Kronecker delta

Also, since $|\psi_n\rangle$ represents basis vectors in a state $|\psi\rangle$, then $|\psi_n\rangle$ spanned $|\psi\rangle$ that is , if $\exists \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n \, \forall n \in \mathbb{N}$ we have,

$$|\psi^b\rangle = \sum_n^{\infty} a_n^b |\psi_n^b\rangle \tag{4.4}$$

From (4.4),

$$\langle\psi^b|\psi^b\rangle = \left(\sum_m a_m^b |\psi_m^b\rangle\right)^\dagger \left(\sum_n a_n^b |\psi_n^b\rangle\right)$$

$$\langle\psi^b|\psi^b\rangle = \sum_m \sum_n a^{b\dagger}_m a_n^b \langle\psi_m^b|\psi_n^b\rangle$$

If $m = n$ then

$$\langle\psi^b|\psi^b\rangle = \sum_n |\alpha_n|^2 \quad \text{but} \quad \langle\psi_n^b|\psi_n^b\rangle = 1 \tag{4.5}$$

Meaning there is the possibility of locating the particle somewhere with certainty. However, if $m \neq n$ then

$$\langle\psi^b|\psi^b\rangle = 0$$

In fact (4.5) is complete, provided that $\sum_n |a_n|^2 < \infty$ and the expansion from (4.4) converges to a vector in a vector space. A complete complex linear is called a HS. The HS is a mathematical structure that serves as probability space in QM where all the information of a PS can be based.

QM is based on two fundamental concepts: state vectors and operators. The state vectors describe the state of any quantum system, and the observables are represented as operators. These state vectors satisfy the conditions for abstract vectors and the operators act as a linear transformation on them. Thus, the mathematical formulation of QM is centered around HS, where state vectors reside, and is often reserved for an infinite-dimensional inner product space having the property that is complete or closed.

$$\mathcal{H}^b = \left(|\psi_1^b\rangle, |\psi_2^b\rangle, |\psi_3^b\rangle, \cdots \right)$$

4.1.4 The Probability Interpretation of Wave-function

In any given quantum system, we are only interested in physical quantities that can be measured such as position, momentum, and energy called observables. These observables are random variables, and their values as a result of measurement are completely described by the quantum state of the particle. However, the observables do not commute – meaning the order in which the values are obtained will not influence the outcome of the measurement, which is a clear deviation from classical probability theory. Hence, there's the need to appeal to the non-commutativity probability theory in other to give meaning to the values of the observables (Go (2016)). The Hilbert space serves as the probabilistic space that contains state vectors, and at a particular time, contains all the statistical information that anybody needs about the particle. But the wave function($\psi^b(x,t)$) itself has no physical interpretation . It is not measurable. However, the square of the absolute value of the wave function has a physical interpretation. We interpret $|\psi^b(x,t)|^2$ as a probability density, a probability per unit length of finding the particle at a time t at position x which is actually Born statistical interpretation of the wave function.

mathematically

$$\int_a^b |\psi^b(x,t)|^2 dx \tag{4.6}$$

which is the probability of finding a particle within the interval *a* and *b*. The probability density function immediately established predict with certainty the position of any particle in a given domain of interpretation. The probability is just an area under the graph $|\psi^b|^2$, and this can be computed by dividing the finite interval into segment and adding together the contributed each segment.

Note: The wavefunction represents a bunch of identical prepared system called Ensemble. It is this system that contains all the information one needs about a particle..

4.1.5 Normalization of wave function

Recall that the wave function is a state vector residing in a complex linear space and it allows us to carry out some useful mathematical operations including the inner product. Space is either finite or infinite dimensional based on the physical state of the system. Therefore, it is required in quantum mechanics, that the state vector should be square integrable in other to have a physical meaning to the quantum system. However, this state vector ψ^b is a solution of the Schro¨dinger WE, and so any constant multiplying the state vector is also a solution. We will later treat the Schro¨dinger equation into details in other to give meaning to the wave vector We can, therefore, define normalization as multiplying a constant to a state vector to ensure that the sum of the possibility of finding the particle is one. mathematically,

$$\int_{-\infty}^{\infty} |\psi^b(x)|^2 dx = 1 \tag{4.7}$$

This is the probability of finding a particle if we look everywhere. We notice that the wave function that we have been mostly dealing with, the wave function of a free particle of given energy and momentum $\psi^b = A\sin(kx - \omega t)$, $A\cos(kx - \omega t)$, $Ae^{i(kx-\omega t)}$ does not satisfy the normalization condition Eq (1.2) – the integral of $|\psi^b(x,t)|^2$ is infinite. Hence, it appears that there is an irregularity in the way we handle the wave equation. However, there is a place for such wave functions in the greater scheme of things, though this is an issue that cannot be considered here. It is sufficient to interpret this wave function as saying that because it has the same amplitude everywhere in space, the particle is equally likely to be found anywhere.

4.1.6 Continuous Space

So far we have discussed state vectors as discrete particle. However, we will now considered the state vectors as continuous particle in an infinite dimensional Hilbert space. It is a space where all physical observables take infinite number of values called eigenvalues. Such spaces are continuous real valued functions on a closed and bounded interval $[0,2\pi]$. The function have to be square integrable. That is

$$\int_0^{2\pi} |\psi^b(x)|^2 \mathrm{d}x < \infty \tag{4.8}$$

Addition and multiplication of vectors be done in a natural way.

$$\phi^b(x) = \alpha\psi_1^b(x) + \beta\psi_2^b(x)$$

Note **An Observable is any physical property of a system or particle that can be measured. eg momentum, energy, position, angular momentum etc.**

4.1.7 Quantum Measurements and Observables

1. Suppose through a series of measurements the observables Q of a physical system is found to have values $q_1, q_2, q_3, \cdots, q_n$ we then introduce the basis states or eigen-states $\psi_1, \psi_2, \psi_3, \cdots, \psi_n$ for the respective measured values.
2. The measured valued corresponding to the observables Q are called the eigenvalues of Q.
3. The basis states form an orthonormal basis function set since q_j is associated with ψ_j
4. The total number of eigenvalues or basis states is called the state space.
5. A state ψ_n^b can be prepared for which the value of the observable Q is q_n with certainty

6. If a system is prepared in the basis state ψ_n^b and measurement of Q is made on the system only the value q_n will be produced and never any real value q_m for m 6= n. Hence, q_m 6= q_n we then conclude that $\psi_1^b, \psi_2^b, \psi_3^b \cdots, \psi_n^b$ are orthonormal

7. The basis states $\psi_1^b, \psi_2^b, \psi_3^b \cdots, \psi_n^b$ cover all possibilities of the system and form a complete orthonormal basis set.

For any state ψ^b we have

$$|\psi^b \rangle = \sum_n C_n \psi_n \qquad \text{(Scramble state)}$$

$$\langle \psi^b, \psi^b \rangle = \left(\sum_n C_n \psi_n^b \right)^\dagger \left(\sum_m C_m \psi_m \right)$$

$$\langle \psi, \psi^b \rangle = \sum_n \sum_m C_n^\dagger C_m {\psi^b}_n^\dagger \psi_m = \sum_n |C_n|^2 \quad \text{and} \, C_n = {\psi^b}_n^\dagger \psi^b$$

8. If the system is in state ψ then the probability of obtaining the eigenvalue q_n is $|{\psi^b}_n^\dagger \psi|^2 = |C_n|^2$ provided $\psi^{b\dagger}\psi^b = 1$

9. The Observables Q is represented by a Hermitian operator $\hat{Q}$ whose eigenvalues are the possible values $q_1, q_2, q_3, \cdots, q_n$ of the measurement of Q associated with the basis states $\psi_1^b, \psi_2^b, \psi_3^b, \cdots, \psi_n^b$ which are vectors in Hilbert space. That is

$$\hat{Q}\psi_b = q\psi_b$$

10. If a measuring Q for a system ψ^b the result q_n is obtained then the system immediately after the measurement goes into the basis state ψ_n^b. This called collapse of wave vector.

4.1.8 Quantum Operators

An operator in QM is any function that acts on a state vector in a vector space and transform the state vector into another state vector.

Linear operators

From section(4.1), a linear operator L can be define on $|\psi_n^b\rangle$ as

$$L|\psi_n^b\rangle = \sum_m |\psi_m^b\rangle L_{mn} \tag{4.9}$$

where

$$L_{mn} = \langle\psi_m|L|\psi_n\rangle$$

is the matrix entry of $|\psi_n\rangle$.

In generalization

$$L|\psi\rangle = \sum_n L\alpha_n|\psi_n^b\rangle = \sum_n \alpha_n L|\psi_n^b\rangle$$

$$\sum_n^{b} \alpha_n |\psi_n^{b}\rangle L_{mn} = \sum_n^{b} |\psi\rangle \alpha_n L_{mn}$$

where $\alpha_n L_{mn}$ is a matrix entry of $L|\psi_n\rangle$

Identity operator

An operator is said to be an identity operator if

$$I|\psi^b\rangle = |\psi^b\rangle \tag{4.10}$$

Now,

Given any orthonormal system $|\psi_n\rangle$, the identity operator I acting on $|\psi_n^b\rangle$ is

$$\mathcal{I}|\psi_n^b\rangle = \sum_n |\psi_n\rangle\langle\psi_n|\psi\rangle = |\psi\rangle$$

where $\mathcal{I} = \sum_n |\psi_n\rangle\langle\psi_n|$ (outer product)

Inverse operator

An operator L^{-1} is called an inverse operator if

$$L^{-1}L = LL^{-1}$$

If L^{-1} has o solution then L is singular. However, this always true in finite dimensional space when the $\det L = 0$. Furthermore, in infinite dimensional, the concept of determinant is not always defined.

Definition 56 (Expectation value)**.** If we allow $\hat{Q}$ to be a linear operator define on the state $|\psi^b\rangle$. Then we can find a real number called the expectation value of $|\psi^b\rangle$ if

$$\langle\hat{Q}\rangle = \langle\psi^b|\hat{Q}|\psi^b\rangle = \langle|(\hat{Q}|\psi^b\rangle \quad \text{Componentwise} \quad \langle\hat{Q}\rangle = \sum_{i,j} a^{\dagger}_i a_{i,j} a_j \tag{4.11}$$

In addition, if $\hat{Q}$ is Hermitian then the expectation value is the average value of measurement of quantum system in the state $|psi^b\rangle$

Definition 57 (Normal Operators)**.** we say an operator $\hat{Q}$ is normal if $\hat{Q}^{\dagger}\hat{Q} = \hat{Q}\hat{Q}^{\dagger}$.

However, diagonalisation exists whenever the operator is normal. i.e , for any normal operator, we can find an orthonormal basis $|\psi^b\rangle$ such that

$$\hat{Q} = \sum_i \lambda_i |\psi_i^b\rangle\langle\psi_i^b| \tag{4.12}$$

where λ_i are the eigenvalues of the operator $\hat{Q}$ and $|\psi_i^b\rangle$ are the corresponding eigenvectors.In an experiment where the eigenvalues are degenerate then there is always a unique eigenvectors that correspond to each eigenvalues.

Definition 58. An operator $\hat{H}_m$ is said to be Hermitian if

$$\hat{H}_m = \hat{H}_m^{\dagger} \tag{4.13}$$

Definition 59 (Commutator)**.** Allowing $\hat{Q}$ and $\hat{R}$ to be operators then the matrix multiplication existing between them is non commutative if $\hat{Q}\hat{R} \neq \hat{R}\hat{Q}$

We denote the commutator of $\hat{Q}$ and $\hat{R}$ by $[\hat{Q}, \hat{R}] = \hat{Q}\hat{R} - \hat{Q}\hat{R}$ 0r $[\hat{Q}, \hat{R}] = 0$ It is the direct consequence of the Heisenberg's uncertainty principle which state non-commuting observables cannot be measured at the same time.

Theorem 21 (Heisenberg's uncertainty principle)**.** Suppose $\hat{Q}$ and $\hat{R}$ are any two observables of a quantum system in the state $|\psi^b\rangle$. Then

$$\Delta\hat{Q}\Delta\hat{R} \geq \frac{1}{4}|\langle\psi^b, [\hat{Q}, \hat{R}]\psi^b\rangle|^2$$

Proof. we compute $\langle\psi^b, \hat{Q}\psi^b\rangle - \langle\psi^b, \hat{R}\psi^b\rangle = 0$
Now, $\langle\psi^b, [\hat{Q}, \hat{R}]\psi^b\rangle = \langle\psi^b, \hat{Q}\hat{R}\psi^b\rangle - \langle\psi^b, \hat{R}\hat{Q}\psi^b\rangle = 2\Im\langle\psi^b, \hat{Q}\hat{R}\psi^b\rangle$ By boundedness property $|\langle\psi^b, [\hat{Q}, \hat{R}]\psi^b\rangle \leq 2\Im\langle\psi^b, \hat{Q}\psi^b\rangle \leq 2\langle\psi^b\hat{Q}, \hat{R}\psi^b\rangle$ from(10) $|\langle\psi^b, [\hat{Q}, \hat{R}]\psi^b\rangle| \leq 2|\langle\hat{Q}\psi^b, \hat{R}\psi^b\rangle|$ and by (6) $|\langle\psi^b, [\hat{Q}, \hat{R}\psi^b]\rangle \leq 2\|\hat{Q}\psi^b\|\|\hat{R}\psi^b\|2(\Delta\hat{Q}\psi^b)^{\frac{1}{2}}(\Delta\hat{R}\psi^b)^{\frac{1}{2}}$

$$\frac{|\langle\psi^b, [\hat{Q}\hat{R}]\psi^b\rangle|^2}{4} \leq \Delta\hat{Q}\psi^b\Delta\hat{R}\psi^b$$

□

Definition 60 (Orthogonal Projectors)**.** Every orthogonal projectors is Hermitian if $\hat{P}_r^2 = \hat{P}_r$. In addition, all the eigenvalues are either 1 or 0.

Eigenfunction and Eigenvalue of an Operator

Consider a wave-function $\psi^b(x,t)$ with an operator $\hat{A}$ acting upon. Then $\hat{A}\psi^b(x,t)$ produces a new function say $\varphi^b(x,t)$ .If $\psi^b(x,t)$ is such that $\psi^b(x,t)$ is directly proportional to $\varphi^b(x,t)$. We have

$$\hat{A}\psi^b(x,t) = \alpha\psi^b(x,t)$$

Here, α is the constant of proportionality called the eigenvalue and $\psi^b(x,t)$ is called the eigenfunction of the operator $\hat{A}$

Example 4.1.8.1. *Given a wave-function $\psi^b(x,t) = e^{i(kx-\omega t)}$ of a free particle traveling along the x- trajectory with momentum $\hat{P}_m = \sim k$ and energy* $\hat{E} = \frac{p^2}{2m}$

Momentum Operator($\hat{P}_m$) The

wave-function

$$\psi_b(x,t) = e_{i(kx-\omega t)}$$

multiplying the wave-function by a constant $\frac{\hbar}{i}$ and then differentiate the resultant function with respect to *x*. We obtain

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^b(x,t) = \frac{\hbar}{i}ike^{i(kx-\omega t)}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^b(x,t) = \hbar ke^{i(kx-\omega t)} = \hbar k\psi^b(x,t)$$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^b(x,t) = \hbar k\psi(x,t)$$ but $\hbar k = \hat{P}$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^b(x,t) = P\psi^b(x,t)$$

The momentum is a constant called the eigenvalue and with $\frac{\hbar}{i}\frac{\partial}{\partial x}$ as the momentum operator. Hence $\psi^b(x,t)$ is an eigenstate of $\hat{P}_m$ moving with definite momentum

$$\frac{\hbar}{i}\frac{\partial}{\partial x} = \hat{P}_m \tag{4.14}$$

Energy operator $\hat{E}_m$

For non-relativistic system , the $\hat{E} = \frac{P^2}{2m}$

Now

$$E\psi^b(x,t) = \frac{P^2}{2m}\psi^b(x,t)$$

$$E\psi^b(x,t) = \frac{P}{2m}P\psi^b(x,t)$$ but $P\psi^b(x,t) = \hat{P}\psi(x,t)$

$$E\psi^b(x,t) = \frac{P}{2m}\hat{P}\psi^b(x,t)$$

$$E\psi^b(x,t) = \frac{P}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^b(x,t)\right)$$

$$E\psi^b(x,t) = \frac{\hbar}{i}\frac{1}{2m}P\left(\frac{\partial}{\partial x}\psi^b(x,t)\right)$$

$$E\psi^b(x,t) = \frac{\hbar}{i}\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^b(x,t)\right)$$

$$E\psi^b(x,t) = -\frac{\hbar^2\partial^2}{2m\partial x^2}\psi^b(x,t)$$

Hence $\hat{E} = -\frac{\hbar^2\partial^2}{2m\partial x^2}$ is the energy operator with $\psi(x,t)$ as the eigenstate of definite energy

$$\hat{E}_m = -\frac{\hbar^2\partial^2}{2m\partial x^2} \tag{4.15}$$

Parity or Space-Inversion operator(Π) The parity operator changes x-component to $-x$ component and y to $-y$ and z to $-z$ in the function in which it acts. In fact, it only acts on wave-function which are described by spatial coordinates.

$$\Pi\psi(x,y,z) = \psi^b(-x,-y,-z) \tag{4.16}$$

Position operator($\dot{x}$)

This type of operator corresponds the position of an observable. However, the position for a single particle is simply given by a scalar say x such that the operator $\dot{x}$ acting on a wave-function $\psi^b(\sim x)$ multiplies the wave-function by $\sim x$.That is

$$\dot{x}|\psi^b(\sim x)\mathrm{i} = x|\psi^b(\sim x)\mathrm{i} \tag{4.17}$$

NB All quantum operators are Hermitian

4.1.9 Superposition of plane waves

Consider the plane waves propagating along the positive *x*-direction

$$\psi^b(x,t) = \sin(kx - \omega t)$$

$\psi_1^b(x,t) = \sin\big((k+\Delta)x - (\omega+\Delta\omega)t\big)$ and $\psi_2^b(x,t) = \sin\big((k-\Delta)x - (\omega-\Delta\omega)t\big)$

$$\psi(x,t) = \psi_1^b + \psi_2^b = \sin\big((k+\Delta)x - (\omega+\Delta\omega)t\big) + \psi_2^b(x,t) = \sin\big((k-\Delta)x - (\omega-\Delta\omega)t\big)$$

$$\psi^b(x,t) = 2\cos\big((\Delta k)x - (\Delta\omega)t\big)\sin\big((\Delta k)x - (\Delta\omega)t\big)$$

The second term represent a plane wave whereas the first term is the amplitude of the 2nd term which varies with position and time. It is called the modulating amplitude envelope function

The velocity of the propagating wave sin($kx - \omega t)$ is $v_p = \frac{\omega}{k}$ and is called the phase velocity.However, the velocity of the envelope is $v_g = \frac{d\omega}{dk}$ called the group velocity. Moreover, if we represent a particle by a delta function then the wave-function should be infinite dimensional and continuous in variable *P* and *x*

We now write the wave-function as:

$$\psi^b(x,t) = A(k_1)e^{i(k_1 - \omega_1 t)} + A(k_2)e^{i(k_2 x - \omega_2 t)} + A(k_3)e^{i(k_3 x - \omega_3 t)} + \cdots = \sum_{n=1}^{\infty} A(k_n)e^{i(k_n x - \omega_n t)}$$

countably infinite dimensional

$k - \Delta k \leq k_n \leq k + \Delta k$ width allowed k's is $2\Delta k$

$$\psi^b(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)}\mathrm{d}k \tag{4.18}$$

where

$A(k)$ is a delta distribution function that helps to obtain the envelope

$e^{i(kx-\omega t)}$ is a traveling wave component.

At $t = 0$

$$\psi^b(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\mathrm{d}k \tag{4.19}$$

comparing eqn(4.18) and (4.19) we require that $A(k)$ be a delta function and that

$$\int_{-\infty}^{\infty} A(k)\mathrm{d}k = \delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

which is the integral form of a delta function. Hence, $\delta(k) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{ikx}\mathrm{d}x$ In general,

$\delta(x - x_0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{ik(x-x_0)}\mathrm{d}k$ This leads to the study of Fourier transform.

By the property of the delta function

$\psi^b(x) = \displaystyle\int_{-\infty}^{\infty} \psi^b(x_0)\delta(x - x_0)\mathrm{d}x_0$ using the delta function

$$\psi^b(x) = \int_{-\infty}^{\infty} \psi^b(x_0)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\mathrm{d}k\right]\mathrm{d}x_0$$

$$\psi^b(x) = \frac{1}{2\pi}\iint \psi^b(x_0)e^{ik(x-x_0)}\mathrm{d}x_0\mathrm{d}k$$

$$\psi^b(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}\Big[\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}\psi(x_0)e^{ik(x_0)}\mathrm{d}x_0\Big]e^{ikx}\mathrm{d}k$$

$$\psi^b(x) = \int_{\infty}^{\infty}\delta(x-x_0)\psi(x_0)\mathrm{d}x_0$$

$\psi^b(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}\psi^b(k)e^{ikx}\mathrm{d}k$ is the Laplace transform of $\psi^b(k)$ which is also the envelope in k momentum space.

The inverse transform

$$\psi^b(k) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}\psi^b(x)e^{-ikx}\mathrm{d}x$$

4.1.10 Quantum Dynamics

When setting up an experiment to evaluate the numerical value of any quantum system. The values are expressed as the EV of SAO which is the immediate result of the spectral theorem. Here, we will apply the spectral theorem to study Schroedinger time-dependent equation of the form:

$$i\frac{\partial}{\partial t}\psi^b(t) = \hat{H}_m\psi^b(t) \tag{4.20}$$

where $\hat{H}_m$ is SAO, interpreted as the total energy of the system (Hamiltonian) defined on a domain $D(\hat{H}_m) \in H^b$

Remark. We can see that the Schroedinger equation is typical of PDE, and whose existence and uniqueness of solutions is of particular interest.

Existence and Uniqueness of the solution

Now,From $i\frac{\partial}{\partial t}\psi^b(t) = H\psi^b(t)$. The solution of the equation is of the form:

$$\psi^b(t) = U_a(t)\psi^b(0)$$

where $U_a(t) = e^{-i\hat{H}_m t}$ defined through functional calculus as:

$$e^{-i\hat{H}_m t} = \int \lambda e^{-i\lambda t} d\Phi^b \tag{4.21}$$

Also, Φ^b is the projection-valued measured of the operator $\hat{H}_m$ defines on the $D(\hat{H}_m)$

Theorem 22. If $(\hat{H}_m, D(\hat{H}_m))$ is a densely well-defined linear operator with $U_a(t) = e^{-i\hat{H}_m t}$ Then

1. $U_a(t)$ is a strongly continuous one parameter group.

2. The $\lim\limits_{t\longrightarrow 0} \dfrac{\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(0)\psi^b}{t} < \infty, \iff \psi^b \in \mathcal{D}(\hat{H}^b_m)$ $\lim\limits_{t\longrightarrow\infty} \dfrac{\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(0)\psi^b}{t} = -i\hat{H}^b_m\psi^b$

3. $U_a(t)\mathcal{D}(\hat{H}^b_m) = \mathcal{D}(\mathcal{H}_a)$ and on $\mathcal{D}(\hat{H}^b_m), [\mathcal{U}_a(t), \hat{H}^b_m] = 0 \quad \forall t \in \mathbb{R}$

Proof. (a) $U_a(t)$ is continuous at $t_0 \in R$ such that $\forall\epsilon > 0, \exists\delta >0$ then

$$\|\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi\|^2 = \langle \mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b, \mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b\rangle$$

$$= \langle \mathcal{U}_a(t)\psi^b, \mathcal{U}_a(t)\psi^b\rangle - \langle \mathcal{U}_a(t)\psi^b, \mathcal{U}_a(t_0)\psi^b\rangle - \langle \mathcal{U}_a(t_0)\psi^b, \mathcal{U}_a(t)\psi^b\rangle + \langle \mathcal{U}_a(t_0)\psi^b, \mathcal{U}_a(t_0)\psi\rangle$$

$$= \|\psi^b\|^2\langle \mathcal{U}_a(t), \mathcal{U}_a(t)\rangle - \|\psi^b\|^2\langle \mathcal{U}_a(t), \mathcal{U}_a(t_0)\rangle - \|\psi^b\|^2\langle \mathcal{U}_a(t_0), \mathcal{U}_a(t)\rangle + \|\psi^b\|^2\langle \mathcal{U}_a(t_0), \mathcal{U}_a(t_0)\rangle$$

$$= \|\psi^b\|^2\left\{ \langle \mathcal{U}_a(t), \mathcal{U}_a(t)\rangle - \langle \mathcal{U}_a(t), \mathcal{U}_a(t_0)\rangle - \langle \mathcal{U}_a(t_0), \mathcal{U}_a(t)\rangle + \langle \mathcal{U}_a(t_0), \mathcal{U}_a(t_0)\rangle \right\}$$

Unitary operator preserves inner-product. i.e $\langle U_a(t), U_a(t_0)\rangle = \langle t, t_0\rangle$

$$\|\mathcal{U}_a(t)\psi^b - U(t_0)\psi^b\|^2 = \|\psi^b\|^2\left\{ \langle t, t\rangle - \langle t, t_0\rangle - \langle t_0, t\rangle + \langle t_0, t_0\rangle \right\}$$

$$\lim_{t\longrightarrow t_0} \|\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b\|^2 = \|\psi^b\|^2\left\{ \langle t_0, t_0\rangle - \langle t_0, t_0\rangle - \langle t_0, t_0\rangle + \langle t_0, t_0\rangle \right\}$$

$\lim\limits_{t\longrightarrow t_0} \|U_a(t)\psi^b - U_a(t_0)\psi^b\|^2 = \|\psi\|^2\{0\} = 0$ showing that $U_a(t)$ is strongly continuous

(b) $U_a(t) = e^{-i\hat{H}_m t} = \int e^{-i\lambda t} d\Phi_b$ but $\mathcal{U}_a(t)\mathcal{U}_a(s) = \left(\int \lambda_t e^{-i\lambda t} d\Phi^b\right)\left(\int \lambda_s e^{-i\lambda s} d\Phi^b\right) = \left(\int \lambda_t\lambda_s e^{-i\lambda t}e^{-i\lambda s} d\Phi^b\right)$

$\mathcal{U}_a(t)\mathcal{U}_a(s) = \left(\int \lambda_t\lambda_s e^{-i\lambda(t+s)} d\Phi\right)$ we set $\lambda_s\lambda_t = \lambda$ Then $\left(\int \lambda e^{-i\lambda(t+s)} d\Phi^b\right) = \mathcal{U}_a(t+s)$

(2) suppose $\lim\limits_{t\longrightarrow o} \dfrac{U_a(t)\psi^b - U_a(0)\psi^b}{t} < \infty$ Then W.T.S $\lim\limits_{t\longrightarrow 0} \dfrac{\mathcal{U}_a(t)\psi^b - \psi^b(0)\psi^b}{t} = -i\hat{H}_m\psi^b$

Now, if we set $\psi^b \in D(\hat{H}_m)$ and allowing $\lim^{t\longrightarrow 0}\left[\frac{\left(e^{-i\hat{H}_m t}-1\right)\psi^b + i\hat{H}_m\psi^b}{t}\right] = 0$

$$\Longrightarrow \lim_{t\longrightarrow 0}\left\|\frac{\left(e^{-i\hat{H}_m t}-1\right)\psi^b + i\hat{H}_m\psi^b}{t}\right\|^2 = \lim_{t\longrightarrow 0}\int\left|\frac{\left(e^{-i\lambda t}-1\right)\psi^b + i\lambda\psi^b}{t}\right|^2 d\Phi^b(\lambda) = 0$$

we can bound $|e^{-i\lambda} - 1| \leq |t\lambda|$ by dominated convergence theorem, $\psi^b \in D(\hat{H}_m)$ such that

$$\int \lambda^2 d\Phi^b(\lambda) < \infty$$

Conversely, suppose $\hat{H}_m : D(\hat{H}_m) \longrightarrow H^b$ such that

$$\mathcal{D}(\hat{H}_m) = \left\{\lim_{t\longrightarrow 0}\frac{i[\mathcal{U}_a(t)\psi^b - \psi^b]}{t} < \infty\right\} \tag{4.22}$$

and

$$\hat{H}_m\psi^b = \lim_{t\longrightarrow 0}\frac{i[\mathcal{U}_a(t)\psi^b - \psi^b]}{t} \tag{4.23}$$

Also, $\forall\psi^b \in D(\acute{H}_m) \Longrightarrow \acute{H}_a$ is a generator $U_a(t).\forall\psi^b\varphi^b \in D(\hat{H}_m)$

$$\left\langle \phi^b, \acute{\mathcal{H}}_m\psi^b\right\rangle = \left\langle \phi^b, \lim_{t\longrightarrow 0}\frac{i[U_a(t)\psi^b - \psi^b]}{t}\right\rangle = \lim_{t\longrightarrow 0}\left\langle\frac{-i[U_a(-t)\psi^b - \psi^b]}{t}, \phi^b\right\rangle = \left\langle \hat{H}^b_m\psi^b, \phi^b\right\rangle$$

Hence $\hat{H}^b_m$ is symmetric, since $\hat{H}^{b\dagger}_m = \hat{H}^b_m$ by ((47)) $\Longrightarrow \phi^b \in \mathcal{D}(\hat{H}^b_m)$(3)

suppose $\psi^b \in D(\hat{H}_m)$. we have

$$\mathcal{U}_a(t)\hat{H}_m\psi^b = \mathcal{U}_a(t)i\tfrac{d}{ds}\phi^b|_{s=0} = i\tfrac{d}{ds}\mathcal{U}_a(t)\mathcal{U}_a(s)\psi^b|_{s=0} = i\tfrac{d}{ds}\mathcal{U}_a(t)\mathcal{U}_a(s)\psi^b|_{s=0} = \hat{H}_m\mathcal{U}_a(t)\psi^b$$

hence we used $U_a(t)U_a(s) = U_a(s+t)$ to get third equality, and also $U_a(t)\psi^b \in D(\hat{H}_m)$ Hence,

$U_a(t) = e^{-i\hat{H}_m t}$ is solution to the Schroedinger wave equation with initial valued condition $\psi^b(0) = \psi^b_0$

$$i\frac{\partial}{\partial t}\mathcal{U}_a(t)\psi^b(0) = \lim_{s\longrightarrow 0}\frac{i[\mathcal{U}_a(t+s) - \mathcal{U}(t)]\psi^b(0)}{s} = \lim_{s\longrightarrow 0}\frac{i[\mathcal{U}_a(s) - 1]\mathcal{U}_a(t)\psi^b(0)}{s} = \hat{H}_m\mathcal{U}_a(t)\psi^b_0$$

□

Lemma 6 (Uniqueness)**.** *Consider a particle wave equation*

$$i\frac{\partial}{\partial t}\psi^b(t) = \hat{H}^b_m\psi^b$$

and suppose $\psi^b(t)$ is a solution to the differential with initial value condition $\psi^b(0) = \psi^b_0$ *then* $\psi^b(t) = \mathcal{U}_a(t)\psi^b_0$

Proof. From (4.1.10), since $\varphi^b \in D(\hat{H}_m) \Longrightarrow \varphi^b$ is differentiable. Set $\varphi^b = U_a(-t)\psi^b$

Now, $$i\frac{\partial}{\partial t}\phi^b(t) = \lim_{h\longrightarrow 0}\left[i\Big(\frac{\mathcal{U}_a(-t-h)\psi^b(t+h) - \mathcal{U}_a(t)\psi^b(t)}{h}\Big)\right] = \lim_{h\longrightarrow 0}\left[\tfrac{i}{h}\Big(\mathcal{U}_a(t+h)[\psi^b(t+h) - \psi^b(t)] + \mathcal{U}_a(-t-h)\psi^b(t) - \mathcal{U}_a(-t)\psi^b(t)\Big)\right]$$

$$i\frac{\partial}{\partial t}\phi^b(t) = \lim_{h\longrightarrow 0}\left[\tfrac{i}{h}\Big(\mathcal{U}_a(-t-h)[\psi^b(t+h) - \psi^b(t)] + \mathcal{U}_a(-t)\mathcal{U}_a(-h)\psi^b(t) - \mathcal{U}_a(-t)\psi^b(t)\Big)\right]$$

$$i\frac{\partial}{\partial t}\phi^b(t) = \lim_{h\longrightarrow 0}\left[\tfrac{i}{h}\Big(\mathcal{U}_a(-t-h)[\psi^b(t+h) - \psi^b(t)] + \mathcal{U}_a(-t)\psi^b(t)[\mathcal{U}_a(-h) - 1]\Big)\right]$$

$$i\frac{\partial}{\partial t}\phi^b(t) = \lim_{h\longrightarrow 0}\left[i\Big(\mathcal{U}_a(-t-h)\frac{[\psi^b(t+h) - \psi^b(t)]}{h} + i\mathcal{U}_a(-t)\psi^b(t)\frac{[\mathcal{U}_a(-h) - 1]}{h}\Big)\right]$$

The limit of a function is a linear operator

$$i\frac{\partial}{\partial t}\phi^b(t) = \lim_{h\longrightarrow 0}\frac{i\mathcal{U}_a(-t-h)[\psi^b(t+h)-\psi^b(t)]}{h} + \lim_{h\longrightarrow 0}\frac{i[\mathcal{U}_a(-h)-1]}{h}\mathcal{U}_a(-t)\psi^b(t)$$
$$i\frac{\partial}{\partial t}\phi^b(t) = i\mathcal{U}_a(-t)\psi^{b'}(t) - \hat{H}_m\mathcal{U}_a(-t)\psi^b(t)$$

Hence $\forall t \in$ R we conclude that $\phi^{b'}(t) = 0$ and $\varphi^b(t) = U_a(t)\psi^b(t)$ when we set $t = 0$ we then have $\phi^b(0) = \psi^b(0) = \psi^b_0$ thus $\psi^b(t)' = \mathcal{U}_a(t)\psi^b_0$ □

Theorem 23 (Stone's Theorem)**.** Suppose $\{\mathcal{U}_a(t) : t \in \mathbb{R}\}$ be a strongly continuous one parameter group and a generator A of $U_a(t)$ is defined by

$$A : D(A) \longrightarrow H^b$$

such that

$$\mathcal{D}(\mathcal{A}) = \left\{\psi^b \in \mathcal{H}^b \middle| \lim_{t\longrightarrow 0}\frac{\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(0)\psi^b}{t} < \infty\right\}$$

Then the following results are established

1. $\forall\psi^b \in D(A)$, $U(t)\psi$ is differentiable i.e $\frac{d}{dt}\mathcal{U}_a(t)\psi^b\Big|_{t=0}$

2. A is essential self-adjoint if

$$A\psi^b = i(U_a(0))_0\psi^b$$

3. suppose $T_a(t) = e^{-itA^-}$ then $T_a(t) = U_a(t) = e^{-itA^-}$

Proof. From hypothesis, let $\{U(t) : t \in \mathbb{R}\}$ be strongly continuous, and by definition (12) we set $t_0 \in$ R such that

$$\|\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b\|^2 = \langle \mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b, \mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b\rangle$$

$$= \langle \mathcal{U}_a(t)\psi^b, \mathcal{U}_a(t)\psi^b\rangle - \langle \mathcal{U}_a(t)\psi^b, \mathcal{U}_a(t_0)\psi^b\rangle - \langle \mathcal{U}_a(t_0)\psi^b, \mathcal{U}_a(t)\psi^b\rangle + \langle \mathcal{U}_a(t_0)\psi^b, \mathcal{U}_a(t_0)\psi^b\rangle$$

$= k\psi^b k^2 h\mathrm{U}_a(t),\mathrm{U}_a(t)i - k\psi^b k^2 h\mathrm{U}_a(t),\mathrm{U}_a(t_0)i - k\psi^b k^2 h\mathrm{U}_a(t_0),\mathrm{U}_a(t)i + k\psi^b k^2 h\mathrm{U}_a(t_0),\mathrm{U}_a(t_0)i$

$$= \|\psi^b\|^2 \Big\{ \langle \mathcal{U}_a(t), \mathcal{U}_a(t)\rangle - \langle \mathcal{U}_a(t), \mathcal{U}_a(t_0)\rangle - \langle \mathcal{U}_a(t_0), \mathcal{U}_a(t)\rangle + \langle \mathcal{U}_a(t_0), \mathcal{U}_a(t_0)\rangle \Big\}$$

Unitary operator preserves inner-product. i.e $h\mathrm{U}_a(t),\mathrm{U}_a(t_0)i = ht,t_0 i$

$$\|\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b\|^2 = \|\psi^b\|^2 \Big\{ \langle t, t\rangle - \langle t, t_0\rangle - \langle t_0, t\rangle + \langle t_0, t_0\rangle \Big\}$$

$$\lim_{t \longrightarrow t_0} \|\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b\|^2 = \|\psi^b\|^2 \Big\{ \langle t_0, t_0\rangle - \langle t_0, t_0\rangle - \langle t_0, t_0\rangle + \langle t_0, t_0\rangle \Big\}$$

$\lim_{t \longrightarrow t_0} k\mathrm{U}_a(t)\psi^b - \mathrm{U}_a(t_0)\psi^b k^2 = k\psi k^2\{0\} = 0$ showing that $\mathrm{U}_a(t)$ is strongly continuous

Next, W.T.S D(A) is dense in H^b_0

From (6), we have $\psi^b(t) = \mathrm{U}_a(t)\psi^b$ which implies

$$\psi^b(t) = \int_{\mathbb{R}} \mathcal{U}_a(s)\psi^b ds \tag{4.24}$$

If we set $t > 0$ and allow $g \in C_0(\mathrm{R}), \forall \psi \in \mathrm{H}^b$, then (4.24) becomes

$$\psi_g^b = \psi^b(g(t)) = \int_0^t \mathcal{U}_a(s)\psi^b ds \tag{4.25}$$

Also, by the continuity property of $\mathrm{U}_a(t)$ at a point $t_0 \in \mathrm{R}$ and by definition,

$\forall \epsilon \geq 0, \exists \delta > 0 \quad \|\mathcal{U}_a(t)\psi^b - \mathcal{U}_a(t_0)\psi^b\| \leq \epsilon$ whenever $|t - t_0| < \delta$ From (4.25) it implies that $\lim_{t \longrightarrow 0} \frac{\psi_g^b}{t} = \psi^b$, i.e,

$$\frac{1}{t}\|\psi_g^b - \psi^b\| = \frac{1}{t}\left\| \int_0^t \mathcal{U}_a(s)\psi^b - \psi^b d(s)\right\| \leq \frac{1}{t}\int_0^t \|\mathcal{U}_a(s)\psi^b - \psi^b\| d(s) \leq \epsilon$$

$\Longrightarrow \ t^{-1}\psi_g^b \longrightarrow \psi^b$ as $t \longrightarrow 0$ Hence, we claim $\psi_g^b \in \mathcal{D}(\mathcal{A})$

<u>verification</u>

set $t_0 > 0$ we have

$$\frac{1}{t}(\mathcal{U}_a(t)\psi_g^b - \psi_g^b) = \frac{1}{t}\left[\int_{t_0}^{t_0+t} \mathcal{U}_a(s)\psi^b ds - \int_0^t \mathcal{U}_a(s)\psi^b ds\right],$$ From Leibniz rule

$$= \frac{1}{t}\left[\left(\int_{t_0}^{t+t_0} \frac{\partial}{\partial t} U(s)\psi^b ds + \mathcal{U}_a(t+t_0)\psi^b \frac{\partial}{\partial t}(t+t_0) - \mathcal{U}_a(t)\psi^b \frac{d}{dt}(t_0)\right) - \left(\int_0^t \frac{\partial}{\partial t}\mathcal{U}_a(s)\psi^b ds + \mathcal{U}_a(t)\psi^b \frac{\partial}{\partial t}(t) - \mathcal{U}_a(0)\psi^b \frac{d}{dt}(0)\right)\right]$$

$$\lim_{t \longrightarrow 0} \frac{1}{t}(\mathcal{U}_a(t)\psi_g^b - \psi_g^b) = \frac{\mathcal{U}_a(t+t_0)\psi^b - \mathcal{U}_a(t)\psi^b}{t} \longrightarrow [U(t_0) - 1]\psi^b \text{ as } t \longrightarrow 0$$ thus $\psi_g^b \in \mathcal{D}(\mathcal{A})$

(1) W.T.S differentiability of $\mathrm{U}_a(t)$

Since $U_a(t)$ is continuous and From (4.25), $\psi_g^b(t) = \int_0^t \mathcal{U}_a(s)\psi^b ds$ $\forall t \geq 0$

we have

$$\frac{d}{dt}\psi_g^b(t) = \mathcal{U}_a(t)\psi^b \quad (4.26)$$

The *L.H.S* of (4.26) becomes

$$\frac{d}{dt}\psi_g^b(t) = \lim_{t\longrightarrow 0}\frac{\psi_g^b(t)}{t} = \lim_{t\longrightarrow 0}\frac{\psi_g^b(t)-\psi_g^b(0)}{t-0} = \frac{d}{dt}\psi_g^b(0) \quad (4.27)$$

The *R.H.S* of (4.26) becomes

$$\frac{d}{dt}\psi_g^b(0) = \mathcal{U}_a(0) = 1 \quad \text{at} \quad t = 0 \quad (4.28)$$

$\Longrightarrow \psi_g^b(t_0) \neq 0 \because$ invertible for some $t_0 > 0$

$$\mathcal{U}_a(t) = (\psi_g^b(t_0))^{-1}(\psi_g^b(t_0))\mathcal{U}_a(t) = (\psi_g^b(t_0))^{-1}\int_0^{t_0}\mathcal{U}_a(t+s)\psi^b ds = (\psi_g^b(t_0))^{-1}\int_t^{t+t_0}\mathcal{U}_a(s)\psi^b ds$$

$(\psi_g^b(t_0))^{-1}\Big((\psi_g^b(t+t_0)) - (\psi_g^b(t))\Big)\psi^b \Longrightarrow U_a(t)$ is differentiable,

$$\frac{d}{dt}\mathcal{U}_a(t) = \lim_{t_0\longrightarrow 0}\frac{\mathcal{U}_a(t+t_0)-\mathcal{U}_a(t)}{t_0} = \lim_{t_0\longrightarrow 0}\frac{[\mathcal{U}_a(t_0)-\mathcal{U}_a(0)]U(t)}{t_0}$$

$$\frac{d}{dt}\mathcal{U}_a(t) = \mathcal{U}_a'(0)\mathcal{U}_a(t) \qquad \forall t > 0$$

(2) To show the operator A is essentially self-adjoint. Then it suffices to prove it is

(a) Self-adjoint

(b) $ker(A^* + iI) = \{0\}$ 2(a) suppose $\psi^b, \varphi^b \in D(A)$ and

$f, g \in C_0(\mathbb{R})$ $\mathcal{A}\psi_g^b = i\mathcal{U}'(0)\psi_g^b$

$$\mathcal{A}\psi_g^b = -i \lim_{t \longrightarrow 0} \frac{\mathcal{U}_a(t) - \mathcal{U}_a(0)}{t}\psi_g^b$$

so, $$\langle \mathcal{A}\psi_g^b, \phi_f^b \rangle = \left\langle \lim_{t \longrightarrow 0} -i\frac{[\mathcal{U}_a(t) - \mathcal{U}_a(0)]\psi_g^b}{t}, \quad \phi_f^b \right\rangle = \lim_{t \longrightarrow 0} \left\langle \psi_g^b, \quad \frac{i[(\mathcal{U}_a(t))^* - 1]}{t}\phi_f^b \right\rangle$$

$$= \lim_{t \longrightarrow 0} \left\langle \psi_g^b, \quad \frac{i[(\mathcal{U}_a(-t)) - 1]}{t}\phi_g^b \right\rangle = \left\langle \psi_g^b, \mathcal{A}^\dagger \phi_f^b \right\rangle$$

∴ A is Hermitian.

Suppose A is Hermitian and $\psi^b \in D(A^\dagger)$ such that $iA^\dagger\psi^b = \psi^b$

$$\mathcal{A}\mathcal{U}_a(t)\psi_g^b = \mathcal{U}_a(t)\mathcal{A}\psi_g^b = -\frac{d}{dt}\mathcal{U}_a(t)$$

$$\frac{d}{dt}\langle \psi^b, \mathcal{U}_a(t)\phi^b \rangle = \langle \psi^b, -i\mathcal{A}\mathcal{U}_a(t)\phi^b \rangle = \langle i\mathcal{A}^*\psi^b, \mathcal{U}_a(t)\phi^b \rangle = \langle \psi^b, \mathcal{U}_a(t)\phi^b \rangle$$

$$\frac{d}{dt}\langle \psi^b, \mathcal{U}_a(t)\phi^b \rangle = \langle \psi^b, \mathcal{U}_a(t)\phi^b \rangle \tag{4.29}$$

(4.29) represent differential equation with initial datum $U_a(0) = 1$
We then allow the equation to depict

$$\langle \psi^b, \mathcal{U}_a(t)\phi^b \rangle = \langle \psi^b, \phi^b \rangle e^t \tag{4.30}$$

Recall, $U_a(t)$ is unitary with k$U_a(t)$k = 1 and by the boundedness property, we have

$|\langle\psi^b, \mathcal{U}_a(t)\phi^b\rangle| \leq \|\psi^b\|\|\phi^b\| \quad \forall t \in \mathbb{R} \quad \forall\phi^b \in (D(A))^{\perp}$ which is only possible when hψ^b,φ^bi = 0, thus $\varphi^b = 0$ since D(A) is dense

We then show that $ker(A^* \pm iI) = \{0\}$.. it has been proven in (16). Similarly, $ker(A^* + iI) = 0$. =⇒ A is essentially self-adjoint.

(3) Finally, we are to show that $U_a(t)\psi^b = T_a(t)\psi^b$ that is, coincide at 0 $\forall\psi^b \in D(A)$

=⇒ $U_a(t)\psi^b - T_a(t)\psi^b = 0$ Let $g(t) = U_a(t)\psi^b - T_a(t)\psi^b$

Hence $T_a(t)\psi^b \in D((\bar{A})$ $g^0(t) = U^0(t)\psi^b - T^0(t)\psi^b = iAU_a(t)\psi^b - i\{\bar{A}\}T(t)\varphi^b = i\bar{A}g(t)$

$$\therefore \frac{d}{dt}\|g(t)\|^2 = -i\langle\bar{A}g(t), g(t)\rangle + i\langle g(t), \bar{A}g(t)\rangle = 0 \quad g(0) = 0$$

and thus $g(t) = 0$ $\forall t \in$ R □

4.2 Applications

After the proof , we will look at more interesting results especially physics related issues and whenever possible we will attempt to interpret it in terms of the physical quantities mentioned in the last few chapters.

4.2.1 Dilation

Dilation is one more to the point operation to look at. This of course would be unitary in its nature for each γ 6= 0 the operator I is defined by

$$(I\psi^b)(x^b) = \gamma^{\frac{1}{2}}\psi^b(\gamma x^b) \tag{4.31}$$

(4.31) satisfies the isometry property of unitary operator. i.e L^2(R)

Definition 61. The unitary operator I by definition would be expressed as:

$$(I\phi)(x) = e^{\frac{\gamma}{2}}\phi(e^\gamma x) \tag{4.32}$$

Since $I(\gamma)I(\mu) = I(\gamma + \mu)$ pose SCUG property then we are to show that the dilation operator $\gamma \to I\gamma$ is strongly continuous.

Proposition 4.2.1.1. *(M¨oller (2010))* Every dilation operator is strongly continuous *Proof.*

$$\begin{aligned} ||I(\gamma)\phi - \phi|| &= \left|\left|e^{\frac{\gamma}{2}}\phi(e^\gamma x) - \phi(x)\right|\right| \\ &\leq \left|\left|(e^{\frac{\gamma}{2}} - 1)\phi(e^\gamma x)\right|\right| + \left|\left|\phi(e^\gamma) - \phi(x)\right|\right| \\ &= \frac{e^{\frac{\gamma}{2}} - 1}{e^{\frac{\gamma}{2}}}||\phi|| + \left|\left|\phi(e^\gamma x) - \phi(x)\right|\right| \end{aligned} \tag{4.33}$$

Obviously, the starting term turns to zero as $\gamma \to 1$. Studying the second term. From rotation the term approaches zero as $\gamma \to 0$ $\forall \varphi^b \in C_0$ For those functions we get (since they are uniformly continuous) that $\left|\left|\phi(e^\gamma x) - \phi(x^b)\right|\right|_\infty$ goes to zero and thus $\left|\left|\phi(e^\gamma x^b) - \phi(x)\right|\right|_2$ goes to zero as φ has compact support. So indeed, the $I\gamma$ form a unitary oneparameter strongly continuous group. Again we can apply Stone's theorem and determine the infinitesimal generator A of $I(\gamma)$. We know $iA\varphi$ is given by

$$\lim_{\gamma \to 0} \frac{I(\gamma)\phi - \phi}{\gamma} \tag{4.34}$$

if it exists. Now let us try to determine what limit is. For this, let us first assume that $\varphi \in C^{\infty}$ i.e. φ is continuously differentiable. Then the pointwise limit is for any x given by

$$\begin{aligned}\lim_{\gamma \to 0} \frac{e^{\frac{\gamma}{2}}\phi(e^{\gamma}x) - \phi(x)}{\gamma} \\ &= \lim_{\gamma \to 0 \left(\frac{1}{2}e^{\frac{\gamma}{2}}\phi(e^{\gamma}x) + e^{\frac{\gamma}{2}}(\phi')^{b}(e^{\gamma}x)e^{\gamma}x\right)} \\ &= \frac{1}{2}\phi_x + x \cdot \phi'(c)\end{aligned} \tag{4.35}$$

$_0$where we used l'Hopital's rule and continuity of φ and φ . The next question is under which premises this convergence is also in L^2. For this we have to assume in addition that φ has compact support, so $\phi \in C_{\prime}^{\infty}$. Then:

$$\begin{aligned}\left|\frac{e^{\frac{\gamma}{2}}\phi(e^{\gamma}x) - \phi(x)}{\gamma}\right| &= \left|\frac{e^{\frac{\gamma}{2}}\phi(x + (e^{\gamma} - 1)x) - \phi(x)}{\gamma}\right| \\ &\leq \left|\frac{e^{\frac{\gamma}{2}}(\phi(x) + (e^{\gamma} - 1)x)||\phi'||_{\infty}) - \phi(x)}{\gamma}\right| \\ &= \left|\frac{e^{\frac{\gamma}{2}} - 1}{\gamma}\phi(x) + \frac{(e^{\gamma} - 1)x)||\phi'||_{\infty})}{\gamma}x\right|\end{aligned} \tag{4.36}$$

If we assume that we have a sequence γ_n converging to , then there is a compact support K, such that the left hand side of the equation is supported inside of it for all γ_n. On the other hand, the right hand side is monotonic in γ. So if we set $\Lambda = sup_n(\gamma_n)$, then with

$$\begin{cases} \left|\frac{e^{\frac{\gamma}{2}}-1}{\gamma}\phi(x) + \frac{(e^{\gamma}-1)x)||\phi'||_{\infty})}{\gamma}x\right| & \text{if } x \in K \end{cases}$$

$$g(x) = \begin{cases} \\ 0 & \text{otherwise} \end{cases} \tag{4.37}$$

we have found an L^2-function which dominates the left hand side and thus the left hand side also converges in L^2 p its pointwise limit by the dominated convergence theorem. As we did in other examples, we could also look at the scalar product of any function φ for which the (ref) exits with $\mathcal{C}_0^\infty$-function η. Then

$$\begin{aligned}
\langle \phi, \eta \rangle &= \left\langle \lim_{\gamma \to 0} \frac{I(\gamma)\phi - \phi}{\gamma}, \eta \right\rangle \\
&= \lim_{\gamma \to 0} \left\langle \frac{I(\gamma)\phi - \phi}{\gamma}, \eta \right\rangle \\
&= -\lim_{\gamma \to 0} \left\langle \phi, \frac{I(\gamma)\eta - \eta}{\gamma} \right\rangle \\
&= -\left\langle \phi, \lim_{\gamma \to 0} \frac{I(\gamma)\eta - \eta}{\gamma} \right\rangle \\
&= -\left\langle \phi, \frac{1}{2}\eta + x \cdot \eta' \right\rangle
\end{aligned} \tag{4.38}$$

where the limit in the last step exists by the above calculations and since $\mathcal{C}_0^1 \supset \mathcal{C}_0^\infty$ If ϕ were sufficient differentiable, integration by parts would give us

$$\begin{aligned}
-\left\langle \phi, \frac{1}{2}\eta + x\dot{\eta}' \right\rangle &= -\left\langle \phi, \frac{1}{2}\eta \right\rangle - \left\langle \phi \cdot x, \eta' \right\rangle \\
&= -\left\langle \phi, \frac{1}{2}\eta \right\rangle + \left\langle x \cdot \phi' + \phi, \eta \right\rangle \\
&= \left\langle \frac{1}{2}\eta + x\dot{\eta}', \phi \right\rangle
\end{aligned} \tag{4.39}$$

which means that the $\lim_{\gamma\to 0} \frac{I(\gamma)\phi-\phi}{\gamma}$ is a weak version of

$$\frac{1}{2}\eta + x\eta' \tag{4.40}$$

□

4.2.2 Rotation in Cartesian Coordinate

Now that we have looked at Dilation in $\mathbb{R}^n$, we will focus on linear maps which can be parameterised so that they form a unitary group. Rotations around a fixed axis come immediately into one's mind. Let us for convenience first look at rotation in in $\mathbb{R}^2$ since rotations (around a fix axis) in higher dimensions can be reduced to that case by choosing an appropriate coordinate system.

Definition 62. We define the rotation matrix

$$R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

which rotates a vector in $\mathbb{R}^2$ counterclockwise around the origin by the angle α With this we define the rotation operator on $L^2(\mathbb{R}^2)$

Definition 63. *The rotation operator I_α on* $L^2(\mathbb{R}^2)$ *is defined by*

$$(I_\alpha\varphi)(x) = \varphi(R_\alpha x) \tag{4.41}$$

It rotates the function φclockwise around the origin by an angle of α

Proposition 4.2.2.1. *(M¨oller (2010))* The rotation forms a strongly continuous oneparameter unitary group

Proof. We note that the rotation operator is unitary for all *theta* since it is an isometry, namely

$$\begin{aligned} ||I_\alpha\varphi||^2 &= \int_{R2} |I_\alpha\varphi)(x)|^2 d^2x \\ &= \int_{R_2} |\varphi(R_\alpha x)|^2 d^2(x) \\ &= \int_R |\varphi(\vec{y})|^2 |det(R_{\alpha^{-1}})| d^2y \\ &= \int_R |\varphi(\vec{y})|^2 d^2y \end{aligned} \tag{4.42}$$

and a bijection, namely

$$(I_\alpha I_\alpha \phi) = (I_\alpha \phi)(R_{-\alpha}\vec{x}) = \phi(R_\alpha R_{-\alpha}\vec{x}) = \phi(\vec{x}) = (I_\alpha I_{-\alpha}\phi)(\vec{x}) \tag{4.43}$$

Furthermore

$$(I_\alpha I_\xi \phi) = (I_\alpha \phi)(R_\alpha \vec{x}) = \phi(R_\xi R_\alpha \vec{x}) = \phi(R_{\xi+\alpha}\vec{x}) = (I_{\alpha+\xi})(\vec{x}) \tag{4.44}$$

where we use the well known property that

$$R_\xi R_\alpha = R_{\xi+\alpha} \tag{4.45}$$

which can be easily shown using trigonometric addition theorems. So, if we can show that $\alpha \to I_\alpha$ is strongly continuous, we have shown that the I_α form a strongly continuous one-parameter unitary group.

We need to show that $I_\alpha\varphi \to \varphi$ in $L^2(\mathbb{R}^2)$ as $\alpha \to 0$ for all $\varphi \in L^2(\mathbb{R}^2)$.We can show this again using Lemma 1 and noting that rotations are all bounded with unit norm. Thus, it is enough if we can show that $I_\alpha\varphi \to \varphi$ as $\alpha \to 0$ for all $\varphi \in C_0(\mathbb{R}^2)$ i.e. for all continuous functions with compact support.

Let $\varphi \in C_0(\mathbb{R}^2)$. Then φ is uniformly continuous. Hence there is a $\delta > 0$ such that $||\phi(\vec{x}) - \phi(\vec{y})|| < \epsilon'$ for all $\vec{x}, \vec{y} \in \mathbb{R}^2$ with $||\vec{x} = \vec{y}|| < \delta$. Also note that since φ has compact support, there is an $r > 0$ such that $\operatorname{supp}(\varphi) \subseteq B_r(0)$. Thus, $||R_\alpha \vec{x} - \vec{x}|| \leq r\alpha$. So making α smaller than $\frac{\delta}{r}$ we get $||I_\alpha\phi - \phi||_\infty \leq \epsilon'$ and choosing $\epsilon' = \frac{\epsilon}{(\gamma^2(K))^{1/2}}$ we get $||I_\alpha\phi - \phi||_p \leq (\gamma^2(K))^{1/2}||I_\alpha\phi - \phi||_\infty \leq \epsilon$. Hence the rotation group is strongly continuous, which completes the proof □

We can then once again apply stone's theorem and determine the infinitesimal generator A. Again $D(A)$ is given by all the $\varphi \in L^2(\mathbb{R}^2)$ for which

$$-i \lim_{\alpha\to 0} \frac{I_\alpha\phi - \phi}{\alpha} = A\phi \tag{4.46}$$

exists. It is not directly apparent what the above limit means (it is exists).

4.2.3 Proposition

If the limit $\lim_{\alpha\to 0}\frac{I_\alpha\phi-\phi}{\alpha}$ *exists, it is given by* $\phi = x_1 \cdot D_2\varphi - x_2 \cdot \varphi$*, where* $D_i\varphi$ *is the weak partial derivative of* φ *in the direction of the i-th unit vector. Proof.* To investigate this, let us assume the limit exists and call it ϕ. Now let $\eta \in \mathbb{C}_0^\infty(\mathbb{R}^2)$. Then

$$\langle \varphi, \eta \rangle = \left\langle \lim_{\alpha\to 0} \frac{I_\alpha \phi - \phi}{\gamma}, \eta \right\rangle = \lim_{\gamma\to 0} \left\langle \eta, \frac{I_{-\alpha}\eta - \phi}{\alpha} \right\rangle \tag{4.47}$$

On the other hand since$\eta \in \mathbb{C}_0^\infty(\mathbb{R}^2)$ we get pointwise

$$
\begin{aligned}
\lim_{\alpha\to 0}\frac{I_{-\alpha}\eta(\overrightarrow{x})-\eta(\overrightarrow{x})}{\alpha} &= \lim_{\alpha\to 0}\frac{\eta(R_{-\alpha}\overrightarrow{x})-\eta(\overrightarrow{x})}{\alpha} \\
&= \lim_{\alpha\to 0}\frac{\eta\begin{pmatrix}\cos(\alpha) & \sin(\alpha)\\ -\sin(\alpha) & \cos(\alpha)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}-\eta\begin{pmatrix}x_1\\ x_2\end{pmatrix}}{\alpha} \\
&= \lim_{\alpha\to 0}\frac{\eta\begin{pmatrix}\cos(\alpha)x_1+\sin(\alpha)x_2\\ -\sin(\alpha)x_1+\cos(\alpha)x_2\end{pmatrix}-\eta\begin{pmatrix}x_1\\ x_2\end{pmatrix}}{\alpha} \\
&= \lim_{\alpha\to 0}\frac{\eta\begin{pmatrix}\cos(\alpha)x_1+\sin(\alpha)x_2\\ -\sin(\alpha)x_1+\cos(\alpha)x_2\end{pmatrix}-\eta\begin{pmatrix}x_1\\ \sin(\alpha)x_1+\cos(\alpha)x_2\end{pmatrix}}{\alpha} \\
&\quad + \lim_{\alpha\to 0}\frac{\eta\begin{pmatrix}x_1\\ -\sin(\alpha)x_1+\cos(\alpha)x_2\end{pmatrix}-\eta\begin{pmatrix}x_1\\ x_2\end{pmatrix}}{\alpha} \\
&= \lim_{\alpha\to 0}\frac{\eta\begin{pmatrix}x_1+\alpha x_2\\ \alpha x_1+x_2\end{pmatrix}-\eta\begin{pmatrix}x_1\\ -\alpha x_1+x_2\end{pmatrix}}{\alpha}+\lim_{\alpha\to 0}\frac{\eta\begin{pmatrix}x_1\\ -\alpha x_1+x_2\end{pmatrix}-\eta\begin{pmatrix}x_2\\ x_2\end{pmatrix}}{\alpha} \\
&= \lim_{\alpha\to 0}x_2\frac{\eta\begin{pmatrix}x_1+\alpha x_2\\ x_2\end{pmatrix}-\eta\begin{pmatrix}x_1\\ x_2\end{pmatrix}}{\alpha x_2}+\lim_{\alpha\to 0}x_1\frac{\eta\begin{pmatrix}x_1\\ -\alpha x_1+x_2\end{pmatrix}-\eta\begin{pmatrix}x_2\\ x_2\end{pmatrix}}{\alpha x_1} \\
&= x_2\partial_1\eta(\overrightarrow{x})-x_1\partial_2\eta(\overrightarrow{x}) \qquad (4.48)
\end{aligned}
$$

Since $\eta \in \mathbb{C}_0^\infty(\mathbb{R}^2)$, the pointwise limit is also the L^2-limit and we have

$$\langle\phi,\eta\rangle = -\langle\varphi,x_1\partial_2\eta - x_2\partial_1\eta \tag{4.49}$$

Hence, in the sense of weak derivatives,

$$\lim_{\alpha\to 0}\frac{I_\alpha\phi-\phi}{\alpha} = \varphi = x_1\cdot D_2\phi - x_2\cdot D_1\phi \tag{4.50}$$

if it exists. We have therefore identified the generator A to be $i(T_{x_1}D_1 - T_{x_2}D_1)$ with its domain being all functions for which the limit in (ref) exists and lies again in $L^2(\mathrm{R}^2)$. As final result we can write

$$e^{(x_1D_2-x_2D_1)\alpha} = I_\alpha \tag{4.51}$$

We again find as intriguing physical interpretation of the above result. Formally, the generator $A = i(T_{x_1}D_1 - T_{x_2}D_1)$ of the rotation corresponds to the observable $L_3 = xp_y - yp_x$(making use of the corresponding rules for the momentum and position operator), which is the third component of the angular momentum $\vec{L} = \vec{x} \times \vec{p}$. Since the infinitesimal generator is per see self-adjoint, we can indeed view it as the operator corresponding to L_3.

Hence, one says that ”angular momentum generates rotation”.

Chapter 5

Conclusions

In this research, the theory of unbounded operators on Hilbert spaces was established by targeting important classes of operators like closed, symmetric, or self-adjoint ones. This also considered the relationship and utilization of the properties for the focus areas mentioned above. An in-depth examination of self-adjoint operators and their spectral properties was imperative to meet the intermediate goal of a functional calculus. This in effect, allowed for a clearer understanding to the exponential of an operator. Focus was also placed on strongly continuous groups of unitary operators. Employing functional calculus, parameterization of the unitary operators in terms of self-adjoint operators became possible and making them much more convenient and easier to handle with respect to its manipulations and computation. The resultant was Stone's theorem, the ultimate goal of this research. It came to light as part of the observations that, these unitary groups exerts an essential influence in quantum mechanics if their generator could be related to a physical observable. As a result, many other remarkable features was deduced. The principal conclusion here is that not only is it possible but also easy to put the mathematical description of quantum mechanics on a firm theoretical environment. However, the use of more sophisticated concepts and results such as the spectral theorem for unbounded self-adjoint operators was employed to arrive at this point. It should be however noted that, when dealing with these unitary groups, the level of complexity can increase tremendously.

RECOMMENDATIONS

Additionally, this thesis could be extended to take an insightful look into the examination and proof of the classic Bochner's theorem on positive-definite functions. Also, Stone's theorem could for instance be applied, when considering the time evolution of a particle in a more complicated potential to further extend the research around it.

Furthermore, try to use Fourier transform to recast Stone's where the real line is locally compact abelian group.

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