KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI-GHANA

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RELATIVE EFFICIENCIES OF GHANAIAN LIFE INSURANCE COMPANIES.

By

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DECLARATION

I hereby declare that this work is the result of my own original research and this thesis has neither in whole nor in part been prescribed by another degree elsewhere. References to other people‟s work have been duly cited.

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DEDICATION

I dedicate this work to my late grandmother Sedonia Naa-ekuu.

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I wish to express my profound gratitude to all persons without whose support this work would not have seen the light of the day.

All the glory to my savior Jesus Christ whose unconditional love and guidance has brought me this far.

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ABSTRACT

Insurance acts as a risk transfer mechanism and investment platform to protect against losses and to provide peace of mind. Insurance penetration which is defined as the contribution of total insurance to GDP is still 1%.The players of the insurance market consist of insurance companies (insurers) and policy holders. The insurance market is divided into Non-Life and Life insurance companies. This study focuses on the efficiency of the life insurance industry in Ghana. Data Envelopment Analysis was employed to evaluate the efficiencies of fourteen life insurance companies in Ghana for the period 2010to 2013. Data Envelopment Analysis, a non-parametric mathematical programming tool, has the capability of evaluating the relative efficiencies of companies or firms that use similar multiple inputs to produce similar multiple outputs. We used capital, commission and management expenses as inputs that are used by life insurers to produce net premiums, investment income and claims as outputs. The results of the study revealed that Ghanaian life insurance companies operated at an average overall efficiency of 82%, average scale efficiency of 93% and an average technical efficiency of 88%. This shows that the efficiency the of life insurers is largely due to their scale efficiency that is scale of operations rather than technical efficiency (managerial skills).The study also tested hypotheses relating to the roles dimension and market share play in the efficiency of Ghanaian life insurers. The study showed that large insurers in terms of capital do not necessarily tend to have higher efficiencies than smaller insurers. Also, the study revealed that life insurers with higher market shares tend to be more efficient than those with lower market shares.

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CHAPTER 1

INTRODUCTION

This chapter captures the background of the study. It briefly explains the problem statement and outlines the objectives of the study. The methodology utilized is briefly considered and the limitations of the study stated.

1.1 Background of the study

The functions of insurance are to act as a risk transfer mechanism and investment platform to protect against losses and to provide peace of mind. Insurance schemes persuade a large number of individuals to pool their risks into a large group to minimize overall risk. Insurance is part of society such that some forms of cover are required by law in the developed nations. In developing countries, the need for such a safety net is much greater, particularly at the poorest levels where vulnerability to risks is much greater and there are fewer opportunities available to recover from a large loss. Therefore, in the developing countries which are characterized by low-income levels, lacking access to social security systems, healthcare and employment opportunities, the need for insurance as a risk transfer mechanism and investment platform is even more imperative. Insurance penetration which is defined as the contribution of total insurance premiums to GDP is still 1%. This can be compared to South Africa (14.8%), Namibia (7.3%), Kenya (2.8%), Nigeria (0.6%) and Malaysia (4.8%) [Source: Swiss Re Sigma Report].

In today‟s world, no insurance company can afford to be an average performer in a highly competitive insurance market. Identifying an optimal performance path leads to benchmarking. Using Data Envelopment Analysis (DEA), not only can we identify top performers in an industry such as insurance, but also discover the alternative ways to make under performing companies become top performers.

The idea behind efficiency measurement is to measure a company‟s performance relative to “best practice” frontiers, which are determined by the dominant, i.e., most efficient companies in the industry. The underlying theory was originally developed by Farrell (1957). Modern frontier efficiency methods, similar to more traditional techniques such as financial ratio analysis, aim at benchmarking firms of an industry against each other. However, these methods are considered superior to other techniques because they integrate different measures of firm performance into a single and thus easily comparable statistic that differentiates between companies based on a sophisticated multidimensional framework (Cummins/Weiss, 2000). This statistic is in most cases standardized between 0 and 1, with the most (least) efficient firm receiving the value of 1 (0). The difference between a company‟s assigned value and the value of 1 determines the company‟s improvement potential in terms of efficiency (Cooper/Seiford/Tone, 2007).

Since the introduction of Data Envelopment Analysis (DEA) by Charnes et al [1978], a lot of efficiency and productivity studies have been carried out by researchers in various fields using different DEA models. DEA has been used to measure the performance of many organizations in recent times. Areas that have received a great deal of attention by researchers are manufacturing, governmental organizations and financial sectors with most studies involving the banking industry. The insurance industry has also gained increased attention.

Most of the efficiency studies that we found in the literature concerning the insurance industry were done in developed countries [Michael Luhnen, 2008]. We found only four efficiency studies, using DEA undertaken in only two West African countries – Nigeria [4,

5] and Ghana [6, 46]. Even though the insurance industry in Ghana dates back to the colonial period and is gaining public attention in recent years, we could find two performance studies of the Ghanaian general insurance using DEA but no performance studies of the Ghanaian life insurance industry specifically that uses this well established and elegant performance measurement tool (DEA). Performance assessment in the private sector is typically based on ratios. The best known ones are financial ratios. The popularity of these ratios is mainly due to their simplicity and ease of calculation, however, each ratio gives only a partial picture of a company's performance outlook. We note however, that the National Insurance Commission normally present in its annual financial reports some performance measures using ratio analysis such as claims ratio, expense ratio, return on equity ratio, combined ratio, retention ratio, gross premium to equity ratio, investment to total assets ratio, return on assets ratio, among others, but these measurements do not allow for proper ranking of the life insurance companies in the country since they give different impressions about the companies. Using state-of-arts methods to study the efficiency of Ghanaian life insurance companies is therefore
essential.

1.2 Statement of the problem

The need for life insurance companies to operate economically and efficiently in order to meet competitive pressures and take advantage of the opportunities to grow cannot be over emphasized. Companies that effectively address these issues will have a competitive advantage over their peers; others do not are likely to struggle and fizzle out.

Data envelopment analysis has the potential to explore and expose the factors that enhance performance and efficiency of life insurance companies. Top performers can therefore serve as benchmarks for the low or relatively inefficient ones.

1.3 Objectives of the study

General Objective

To use data envelopment analysis (DEA) to evaluate the relative efficiencies of some Ghanaian Life insurance companies in the five (5) year period 2010 – 2013.

Specific objectives

(i) To examine whether or not larger (in terms of capital) life insurers are more efficient than smaller ones.

(ii) To find out whether or not life insurers with higher (in terms of premiums collected) market shares are more efficient than those with small market shares.

1.4 Relevance of the study

Efficiency measurement methods can be divided into three main categories: ratio indicators, parametric and non-parametric methods. Ratios rank among the most simple methods but their drawback is that they evaluate just a handful of indicators and cannot influence overall corporate efficiency.

DEA models can generate new alternatives to improve performance compared to other techniques. Linear programming is the backbone of DEA methodology that is based on the optimization platform. Thus, what differentiates the DEA from other methods is that it identifies the optimal ways of performance rather than the averages.

This study will contribute to a stock of literature on insurance in Ghana and might serve as a source of reference for further research in the life insurance industry in Ghana. Life insurance managers who need broader perspectives on efficiency assessment of insurance companies will find this research work helpful since knowledge of efficiency scores (static) and total factor productivity indices (dynamic) relative to a given bench are relevant and could represent the basis for strategic decision making.

1.5 Methodology

In this study we shall consider fourteen life insurance companies for the period 2010 to 2013. Hence we obtain fifty six (56) observations (14 companies in 4 years). Our choice thus conforms to the DEA requirement that, the total number of observations be more than three times the sum of the number of inputs and outputs [Barros, 2008].

The objective of this research is to study the relative efficiencies of Ghanaian life insurance companies and to find out how some variables commonly used in the insurance industry contribute to efficiency. To evaluate the relative efficiency, the DEA model of Hongliang and Michael [2007] will be used and in order to find out how some insurance variables contribute to efficiency, the Mann-Whitney U-test will be used to carry out the hypothesis.

1.5.1 The model of Hongliang and Michael: DEA Model Used

The DEA model used is briefly discussed. In efficiency analysis; the choice of the DEA model to be used usually depends on the type of data under consideration. Analysts usually will have to make sure their models are translation invariant if the data involves negative and/or zero value. They also will have to ensure that the DEA model they use is units invariant if the variables are of different dimensions.

Most studies involving efficiency of insurance industries use claims as output as though claims are desirable to insurers, even though insurers would like to incur fewer claims [Owusu-Ansah etal, 2010]. In this study, we treat claims as an undesirable output by employing the model of Hongliang and Michael [2007]. Hongliang and Michael suggested that when undesirable outputs are produced jointly with desirable outputs, it makes sense to credit a Decision Making Unit (DMU) for producing a desirable output and penalize it for producing undesirable output in calculating the efficiency of the DMU [Owusu-Ansah etal, 2010]. Therefore, we use their model due to the fact that a claim incurred is an undesirable output to insurers.

The model proposed by Hongliang and Michael [2007] is stated as follows:

For a set of N (homogeneous) DMUs using m inputs to produce r outputs among which we have d desirable outputs and u undesirable outputs, the input-oriented DEA model used to evaluate the efficiency of DMU_j is given as follows:

Minimize θ Subject:

$$Y_d \lambda \geq y_{dj}$$

$$Y^u \lambda \leq \theta y_j^u \qquad (1.1)$$

$$X\lambda \leq \theta x_j$$

$$\lambda \geq 0, j = 1, \dots \dots, N$$

Where

θ is the input-oriented efficiency measurement score for DMU j

Y^d is a d-by-N matrix of desirable output

y_j^d is a vector of desirable output of DMU_j

Y^u is a u-by-N matrix of undesirable output

y_j^u is a vector of undesirable output of DMU_j.

X is an m-by-N matrix of inputs and

λ is an N-by-1 vector of coefficients which represents the intensity levels for DMUs in the construction of the reference efficiency frontier.In order to calculate technical efficiencies of the insurance companies during the period of study we also employed the model (1.1) above under variable-returns-to-scale (VRS),which disentangles technical efficiency from overall efficiency.

The VRS model is given as follows:

Minimize θ

Subject to:

Y $_{d}$□□ y_{dj}

Y^{u}□□□y^{u}_{j} (1.2)

X□□□x_{j}

e_{N}□□1

□□ 0, j=1,…, N.

Wheree_{N} is a 1-by-N vector of ones and the other variables are as already defined in equation (1.1) above.

1.5.2 Mann-Whitney U-test

Other analyses that are usually considered in efficiency studies of insurance companies are statistical tests of hypothesis in the following forms.

(i). Scale of operation of an insurance company positively correlates with its efficiency. To test this hypothesis, the life insurance companies are classified by capital (shareholder funds) and then sample is divided into subsets on this basis.

(ii). Market share of a life insurance company positively correlates with its efficiency. To test this hypothesis, the life insurance companies are classified according to the estimated market share by net premium and the sample is divided into two subsets on this basis.

These tests of hypotheses can be carried out using the Mann-Whitney U-test. The MannWhitney U-test is a non-parametric test that is used in place of an unpaired t-test. It is used to test the hypothesis that two samples come from the same population (i.e. have the same median) or, alternatively, whether observations in one sample tend to be larger than observations in the other. Although it is a non-parametric test it does assume that two distributions are similar in shape.

Suppose we have a sample of n observations (x_1, $x_2...x_n$) in one group and a sample of m observations ($y_1,y_2...y_m$) in another group. The Mann-Whitney test is based on a comparison of every observation x_i in the first sample with every observation $y_j$in the other sample. The total number of pair wise comparisons that can be made is nm. If the samples have the same median then each x_i has an equal chance (i.e. probability =1/2) of being greater or smaller than each y_j.

The Mann-Whitney U-test uses the following hypothesis:

Ho: The two samples are from the same population or P $(x_i>y_j) = ½$

H1: The two samples are not from the same population or P $(x_i>y_j) \neq ½$

We count the number of times an x_i from sample 1 is greater than ay_j from sample 2. This number is denoted by U_x. Similarly, the number of times an x_i from sample 1 is smaller than a y_j from sample 2 is denoted by U_y. Under the null hypothesis we would expect U_x and U_y to be approximately equal.

Procedure for carrying out the test:

(i). Arrange all the observations in order of magnitude.

(ii). Under each observation, write down x or y (or some other relevant symbol) to indicate which sample they are from.

(iii). Under each x write down the number of ys which are to the left of it (i.e. smaller than it); this indicates $x_i > y_j$. Under each y write down the number of xs which are to the left of it (i.e. smaller than it); this indicates $y_j > x_i$.

(iv). Add up the total number of times $x_i > y_j$ denoted by Ux, add up the total number of times $y_j > x_i$ denoted by U_y and check that $U_x + U_y = nm$.

(v). Calculate $U = \min(U_x, U_y)$.

(vi). Use statistical table for the Mann-Whitney U-test to find the probability of observing a value of U or lower. If the test is one-sided, this is our p-value. If the test is a two-sided test, double this probability to obtain the p-value.

Making conclusion from the Mann-Whitney test

If the critical p-value read from the statistical table is greater than the calculated p-value, then there is a significant difference between the populations from which the samples were drawn. In other words, the two samples are not from the same population. So we reject the null hypothesis. We fail to reject the null hypothesis otherwise. We note here that for sufficiently large observations (nm > 20), the normal approximation can be used with

$$\mathrm{U} = \frac{nm}{2},$$

$$\delta\mu = \sqrt{\frac{nm(N+1)}{12}}, \text{ where } N = n + m. \quad (1.3)$$

It is possible that two or more observations might be the same.

If this is the case we can still calculate by allocating half the tie to X value and half the tie to the Y value. However, if this is the case then the normal approximation must be used with an adjustment to the standard deviation. This becomes:

$$\delta\mu = \sqrt{\left(\frac{nm}{N(N-1)}\right)\left(\frac{N^3 - N}{12} - \sum_{j=1}^{g}\frac{t_j^3 - t_j}{12}\right)} \quad (1.4)$$

Where g = number of groups of ties t_j

= number of tied ranks in group j.

1.6 Scope of study

This study involves life insurance companies that contribute by gross premiums over 90 % of the life market share during each year of the period of the study, 2010 – 2013 as well as representing over 80% of the life insurance sector, thus being abundantly representative of the Ghanaian life insurance market.

1.7 Limitations of the study

1. Due to unavailability of data on prices of inputs and outputs, efficiency measurements such as cost and allocative efficiencies could not be studied.
2. Missing data as a result of inefficient record keeping slightly affected research work.

1.8 Organization of the study

The study is organized into five chapters. Chapter one provides an overview of the research undertaken in this study. Chapter two entitled "Literature Review" summarizes some research work done in the area of efficiency measurement, data envelopment analysis and linear programming. The research methodologies utilized in this study are described in chapter three. Chapter four presents the data collection and analysis of the study. Chapter five is devoted for the summary of findings, conclusions and recommendation of the study

CHAPTER 2

LITERATURE REVIEW

2.0 INTRODUCTION

This chapter reviews literature in the areas of performance measurement using mathematical programming. Frontier Efficiency analysis in the insurance industry, with emphasis on Data Envelopment Analysis (DEA) is sought to be reviewed. We shall review literature on topics on efficiency beginning from traditional approaches to efficiency analysis through Farrel"s work [1957] to the introduction of Data Envelopment analysis (DEA) by Charnes et al [1978]. We continue to review literature on extensions of the DEA models by Charnes et al [1978] and finally narrow down our review to the application of DEA in the insurance subsector of the financial sector.

2.1 Data Envelopment Analysis (DEA)

Following Farrel"s work [1957] on measurement of productive efficiency, Charnes et al [1978] developed the first DEA model called the CCR (Charnes, Cooper, and Rhodes) model. The CCR model is a mathematical programming model that measures a firm"s efficiency by calculating a ratio of sum of weighted output to sum of weighted inputs and compares it with the efficiencies of all other firms in the set of firms under study. The efficiency therefore is said to be relative. The model finds (for each firm) a set of weights that maximize the firm"s efficiency while allowing other firms to strive for maximum efficiencies.

2.2 Review of DEA Models and some Extensions

After the CCR model, a good number of DEA models have been proposed by researchers in the field. These models all use the same concept as the CCR model but only differ in their specifics of the shape of the frontier and the method of projection of inefficient firms onto the frontier [Amit, 2001]. Next, we discuss some of the commonly used DEA models and some extensions.

2.2.1 The CCR Model

Assuming there are n DMUS, each using m inputs to produce s outputs, to measure the efficiency of say DMUp, Charnes et al [1978] proposed that we solve the following fractional programming problem, which is the CCR model

$$Maximize = \frac{\sum_{k=1}^{s} v_k y_{kp}}{\sum_{j=1}^{m} u_j x_{jp}}$$

Subject to: (2.1)

$$\frac{\sum_{k=1}^{s} v_k y_{ki}}{\sum_{j=1}^{m} u_j x_{ji}} \leq 1, \forall i$$

Where $v_k \geq 0, \forall k$ and $u_j \geq 0, \forall j$

K = 1, 2, ..., S

j = 1, 2,..,, m

i = 1, 2,..., n

y_{ki} = Amount of output k produced by DMU_i

x_{ji} = Amount of input j used by DMU_i

v_k = Weight assigned to output k

u_j = Weight assigned to input j

The fractional programming problem given by equation (2.1) above is converted into a Linear programming problem by letting the weighted sum of inputs of the test DMU be unity and rearranging. Hence the linear form of the CCR model is given by:

$$Maximize = \sum_{k=1}^{s} v_k y_{kp}$$

Subject to (2.2)

$$\sum_{j=1}^{s} u_j x_{jp} = 1$$

$$\sum_{k=1}^{s} v_k y_{kp} - \sum_{j=1}^{m} u_j x_{jp} \leq 0, \forall i$$

$v_k \ge 0, u_j \ge 0 \; \forall k$ and $\forall i$

In order to find the relative efficiencies of all DMUs the linear programming problem (2.2) above is run n times. Each DMU chooses its input and output weights so as to maximize its efficiency. All DMUs obtaining an efficiency score of unity are said to be relatively efficient and inefficient otherwise. The dual of the LP above is preferable from the computational point of view. The dual of the CCR model is given by:

Minimize θ

Subject to

$$\sum_{i=1}^{n} \lambda_i y_{ki} - y_{kp} \ge 0, \forall k \qquad (2.3)$$

$$\sum_{i=1}^{n} \lambda_i x_{ji} - \theta x_{kp} \le 0, \forall j$$

Where $\lambda i \ge 0 \; \forall i$ are dual variables and $\theta \in [0, 1]$ is the overall efficiency of the test DMU.

2.2.3 The BCC Model

A major drawback of the CCR model is that it assumes constant returns to scale for all DMUs. However, this is not always the case in reality. In order to eliminate this drawback, Banker et al. [1984] extended the CCR model to their BCC model, which assumes variable returns to scale and compares DMUs purely on the basis of technical efficiency. In order to generalize the CCR model to all economies of scale, Banker etal [1984] added an additional constraint to the dual of CCR model. The constraint is that, all weights add up to unity. The BCC model thus is given as follows:

Minimize θ

Subject to:

$$\sum_{i=1}^{n} \lambda_i y_{ki} - y_{kp} \geq 0, \forall k .$$

$$\sum_{i=1}^{n} \lambda_i x_{ji} - \theta x_{kp} \leq 0, \forall j \qquad (2.4)$$

$$\sum_{j} \lambda_j = 1$$

This transforms the model from being constant returns-to-scale to variable returns-to scale. The scores θ from this model are sometimes called technical efficiency scores as they eliminate scale-efficiency from the analysis.

2.3 Extensions of the DEA Models

Several modifications have been made to the original CCR and BCC models in the literature by researchers in the field in order to accommodate some situations that the CCR and BCC models could not address. We shall, in the next brief, discuss some of these extensions.

2.3.1 Benchmarking in DEA

Best practice is the set of management and work practices which results in the highest potential. Best practice can be identified at a number of levels including departmental, organizational, national and international. Benchmarking is the process of comparing the performance of an individual firm against a bench mark or ideal level of performance. Benchmarks can be set on the basis of performance over time or across a sample of similar firms or against some externally set standard. Benchmarking appeals most to firms with similar strategic orientations or facing comparable problems and opportunities (Smith, 2005, Collis et al. 2007).It helps to enhance performance and facilitate learning and understanding of new best practices

DEA identifies for all inefficient DMUs, a set of efficient DMUs that can be used as benchmarks for improving the efficiency of the inefficient DMU. These benchmarks are found by solving the dual problem represented by equation (2.5) below [Talluri, 2000].

Minimize θ

Subject to:

$$\sum_{i=1}^{n} \lambda_i y_{ki} - y_{kp} \geq 0 \quad \forall k \qquad (2.5)$$

$$\lambda_i \geq 0 \quad \forall i$$

Where =Efficiency score and λ_i =Dual variable assigned to DMUi

A linear combination of the set of identified efficient DMUs for each inefficient DMU is called a Composite DMU for the inefficient DMU. Based on the model in equation (2.5) above, a DMU is said to be inefficient if we can identify a composite DMU that utilizes less input to produce at least the same amount of output as the test DMU or produces more output whilst using almost the same amount of input as the test DMU. We note here that DEA only identifies targets for improvement but does not provide any strategy to make inefficient DMUs efficient. It is the responsibility of managers to study the operations of the benchmarks in order to identify ways of improving the efficiency of the inefficient

DMUs [Talluri, 2000].The difficulty that arises in the benchmarking process is that the inefficient DMU and its benchmarks may have totally different modes of operations due to the fact that the composite DMU is virtual(does not really exist). In order to evade this problem some researchers such as [John Doyle and Rodney Green, 1994] identified appropriate benchmarks by grouping DMUs that have similar modes of operations into clusters and identify the best DMUs in every cluster that is then used as a benchmark for improvement of their counterpart inefficient DMUs.

2.3.2 Efficiency ranking of DMUs

All the DEA models discussed above do not make room for ranking of DMUs especially the efficient ones. Also, since these models do not restrict the input and output factor weights, it is possible for a DMU to assign very high weight to some input(s)/output(s)(which might even be unimportant) whilst assigning very low weights to (or even ignoring) some very important input(s)/output(s). The resulting effect is that, some DMUs that are better overall performers might be identified to be inefficient while some DMUs which are niche performers might be identified as efficient DMUs. These niche performers take advantage of some few favorable input(s)/output(s) factors to achieve high efficiency scores whilst completely ignoring several input/output factors. Some models have been proposed in the literature to effectively rank efficient DMUs in DEA studies. An example is the Cross-Efficiency Method by Green and Doyle [1994]

2.4 Traditional Efficiency Measurement Concepts

We start our review from traditional concepts of efficiency since DEA became an efficiency measurement tool. Efficiency measurement is very old in academic literature. In fact, some authors trace efficiency studies back to the days of Aristotle (384-422 BC), when he discussed the efficiencies of different military organizations. Leonardo da Vinci also studied performance issues concerning labor effort in shoveling [1994]. The nineteenth century scientist F. Taylor, however, is always crowned as the father of scientific management, even though some authors argue that there were several earlier publications containing many of Taylor‟s theories. In scientific management, laws from the natural sciences are employed in attempting to improve the efficiencies of decisionmaking units. From this perspective of scientific management, other disciplines such as economics, accounting and politics have also developed concepts of performance in management from their own perspectives. Traditionally, efficiency was measured using the average productivity of labor [Farrel M. J, 1957]. In this case, all other factors of production were ignored. Hence, this concept of efficiency was considered inappropriate when multiple inputs were used to produce multiple outputs. Thus the traditional average productivity of labor concept of efficiency assessment had a serious drawback of not being able to handle multiple inputs and outputs situations.

In order to address the major drawback of the average productivity of labor concept, attempts were made toward the development of concepts that included all the inputs and outputs in the efficiency analysis. One of such is the indices of efficiency concept. Using this concept, input and output quantities were converted into dimensionless quantities and their weighted sums were found. The efficiency was then measured as a ratio of weighted sum of dimensionless outputs to that of inputs. Efficiency comparison was therefore considered as a cost comparison and there was a difficulty in choosing appropriate set of weights[Amit, 2001]. This was a major drawback of the indices of efficiency approach.

2.5 The Concept of Frontier Efficiency

In 1957 Farrell developed an efficiency analysis concept which was meant to eliminate the drawbacks of the traditional concepts of efficiency analysis mentioned above. This concept is the frontier efficiency concept, which employs the efficient production function. In this concept, a firm‟s efficiency is measured by comparing it with a hypothetical perfectly efficient firm represented by the production function.

There are several empirical means for measuring efficiency. They are ratio analysis, regression analysis, parametric and non-parametric methods of which the two latter methods are superior. The parametric or econometric method is made up of Stochastic Frontier Approach (SFA), Thick Frontier Approach (TFA), and Distribution Free Approach (DFA).In this approach, the form of the production function is known or statistically estimated but in many cases, however, the functional form of the production is not known. Assumptions about the distribution of inefficiencies and random error are made in this approach.

In the non-parametric method constituting Data Envelopment Analysis (DEA) and Free Disposal Hull (FDH), no assumptions are made about the form of the production function and distributional behaviors in the data. Instead, a best practice function is built empirically from observed inputs and outputs.

The econometric and mathematical approaches have their advantages and disadvantages. The econometric approach specifies a production, cost, and revenue or profit functional form on the frontier and makes assumptions about the distribution of inefficiencies and random error. A major drawback of the econometric approach is the imposition of the functional form which might be invalid. For instance, one might impose a trans log functional form on a frontier that does not assume this functional form. This will lead to incorrect results and consequently wrong deductions. In contrast to the econometric approach the mathematical programming approach has the advantage of not imposing any strong functional form on the frontier.

DEA enables us to compare several decision making units with each other and determine their relative efficiency. It produces a single score for each unit which makes the comparison easy. Unlike ratios, DEA can accommodate multiple inputs and multiple outputs and they can in different units of measurement. In contrast to regression methods,

DEA focuses on individual observation and optimizes the performance measure of each DMU.A prior knowledge of weights or prices for inputs and outputs is not required unlike the parametric approach; however, managerial judgment can be accommodated when desired. Another advantage that attracts analysts and management is its ability to identify the potential improvement for each inefficient DMU. For DMUs enveloped by the frontier, DEA compares the inefficient units with the convex combination of DMUs located on the frontier and enables the analyst to indicate the sources and level of inefficiency for each firm below the efficient frontier.

The mathematical programming approach has a major drawback of not making any assumption about inefficiencies and random error. The mathematical programming approach therefore considers all deviations from the frontier as inefficiencies [Berger and Humphrey, 1997]. Neither of the two approaches has been established to be superior [Cummins and Zi, 1998]. The methodology one employs in efficiency analysis therefore depends on the available data. For instance, it will be inappropriate to conduct frontier efficiency analysis involving "noisy" data using the mathematical programming approach. In order to understand the concept of efficient production function, we illustrate within example. Suppose that a set of firms use two inputs to produce one output under constant returns to scale. By constant returns to scale (CRS), we mean a situation whereby an increase/decrease in firm‟s input leads to a proportional increase/decrease in the firm‟s output. An otherwise situation is said to be Variable Returns to Scale (VRS).An isoquant diagram is one in which all firms producing the same output lie in the same plane. The isoquant is therefore a scatter diagram with each firm representing a point. An efficient production function is represented by a curve joining the most efficient firms in the isoquant.

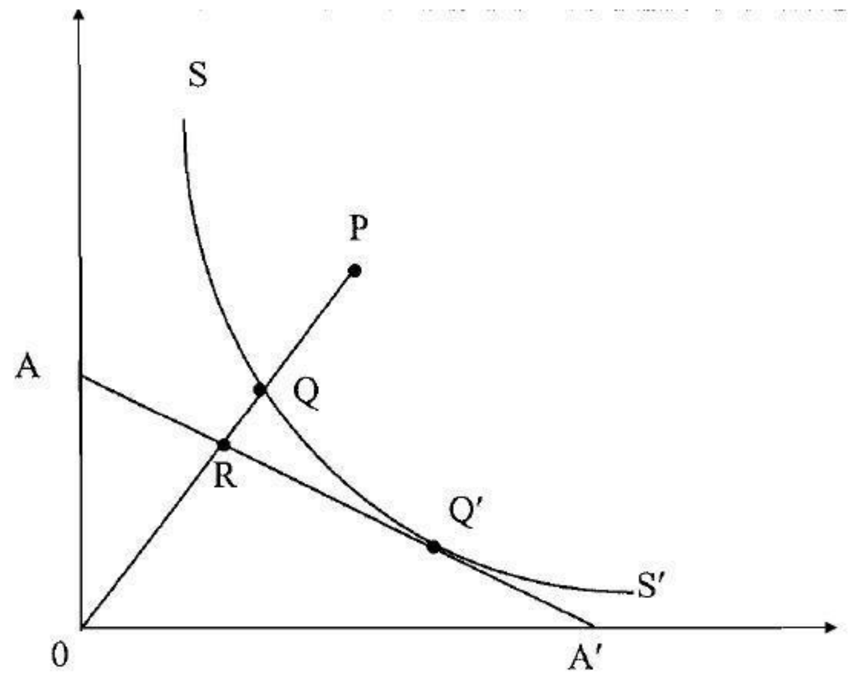

Figure 2.1, the curve SS' represents a production function

All firms located on the curve are said to be efficient and inefficient otherwise. Hence firm Q is efficient whilst firm P is inefficient. This is because both firms P and Q produce the same amount of output but firm P uses a fraction OQ/OP more of each of inputs X and Y. In other words firm P produces the same amount of output as Q but uses OP/OQ less of each of inputs X and Y. The Technical efficiency of firm Q is defined as the ratio OQ/OP. This technical efficiency measure does not include the extent to which each of inputs X and are used by each firm, in the light of their prices. That is, this technical efficiency measure does not include the prices of various factors of production. Another measure of efficiency that makes use of the input prices is the Price (Allocative) efficiency. If the tangent AA" in figure 2.1 has a slope representing the ratio of prices of the two inputs, then instead of Q, Q‟ is an optimal method of production. The cost of production at Q‟ will be OR/OQ of that of Q. This ratio OR/OQ is called the allocative efficiency of firms

P and Q. Overall efficiency is the product of technical efficiency and allocative efficiency. Hence the overall efficiency of firm P is given by the ratio OR/OP as shown in Figure 2.1.

The frontier efficiency concept discussed above is elegant in the sense that it gives one the opportunity to calculate efficiency without having to use any theoretically justified production function and also decomposes efficiency into technical and allocative efficiencies. Technical efficiency measures a firm‟s success in its choice of optimal inputs with minimum cost whilst allocative efficiency measures the firm‟s success in producing maximum output using a given set of inputs.

The production function of the fully efficient firm is not known in practice, and thus must be estimated from observations on a sample of firms in the industry concerned. In this study we use DEA to estimate the efficient frontier.

2.5.1 Input-Oriented Measures

Farrel illustrated his ideas using a simple example involving firms which use two inputs(x_1 and x_2) to produce a single output(y), under the assumption of constant returns to scale. Knowledge of the unit isoquant of the fully efficient firm, represented by SS^1 figure 2.1, permits the measurement of technical efficiency. If a given firm uses quantities of inputs defined by the point P, to produce a unit output, the technical inefficiency of that firm could be represented by the distance QP, which is the amount by which all inputs could be proportionally reduced without a reduction in output. This usually is expressed in percentage terms by the ratio QP/OP, which represents the percentage by which all inputs could be reduced. The Technical Efficiency (TE) of a firm is most commonly measured by the ratio:

$$TE_i = OQ/OP = 1 - QP/OP \qquad (2.6)$$

It takes a value between zero and one, and hence provides an indicator of the degree of technical inefficiency or efficiency of the firm. A value of zero indicates the firm is technically inefficient and a value of one indicates the firm is technically efficient. For example, the point Q is technically efficient because it lies on the efficient isoquant.

If the input price ratio represented by the line AA^1 in figure2.1 is also known, allocative efficiency can also be calculated. TheAllocative Efficiency (AE) of the firm operating at P is defined to be the ratio:

$$AE_i = OR/OQ \qquad (2.7)$$

Since the distance RQ represents the reduction in production costs that would occur if production were to occur at the allocatively (and technically) efficient point Q^1, instead of at the technically efficient, but allocatively inefficient, point Q.

The total economic efficiency (EE) sometimes referred to as overall efficiency is defined to be the ratio:

$EE_i = OR/OP$, where the distance RP can also be interpreted in terms of a cost reduction. The product of technical and allocative efficiency provides the overall economic efficiency.

$$TE_i \times AE_i = (OQ/OP) \times (OR/OQ) = EE_i \qquad (2.8)$$

TE_i, AE_i and $EE_i \in [0,1]$

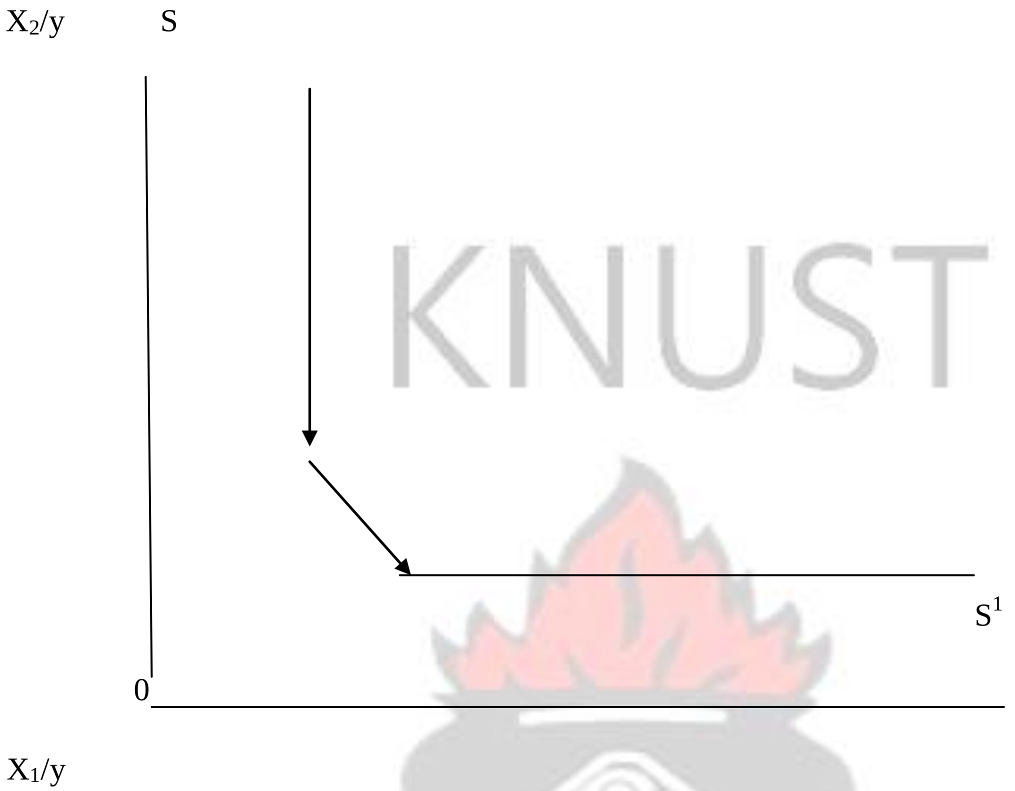

Figure 2.2Piecewise Linear Convex Isoquant

These efficiency measures assume the production function of the fully efficient isoquant is known. In practice this is not the case and the efficient isoquant must be estimated from the sample data. Farrel suggested the use of:

(a) A non-parametric piecewise linear convex isoquant constructed such that no observed point should lie to the left or below it(refer to figure 2.2) or

(b) A parametric function such as the Cobb-Douglas form, fitted to the data, again such that no observed point should lie to the left or below it.

2.6.2 Output-Oriented Measures

The above input-oriented technical efficiency measure addresses the question "By how much can input quantities be proportionally reduced without changing the output quantities produced?" Alternatively, one could ask the question "By how much can output quantities be proportionally expanded without altering the input quantities used?" This is the output-oriented measure as opposed to the input-oriented measure earlier discussed. The difference between the input-oriented and output-oriented measures can be illustrated using a simple example involving one input and one output. This is depicted in figure 2.3(a) where we have decreasing returns to scale technology represented by f(x) and an inefficient firm operating at the point P. The Farrel input-oriented measure of TE would equal to the ratio AB/AP, while the output-based measure of TE would be CP/CD. Under the existence of constant returns to scale, the output and input oriented measures will be equivalent but unequal when increasing or decreasing returns to scale are present. The constant returns to scale is depicted in figure 2.3(b) where we observe that (AB/AP) = (CP/CD) for any inefficient point we care to choose.

One can consider output oriented measures further by considering the case where production involves two outputs (y_1 and y_2) and a single input(x_1). Again, if we assume constant returns to scale, we can represent the technology by a unit production possibility curve in two dimensions. This example is illustrated in figure2.4 where the line ZZ^1 is the unit production possibility curve and the point A corresponds to an inefficient firm. The inefficient point A lies below the curve in this case because ZZ^1 represents the upper bound of production possibilities set.

(a) DRTS (b) CRTS

0 C x 0 C x

Figure 2.3 Input- and output-oriented Technical Efficiency Measures and Returns to Scale

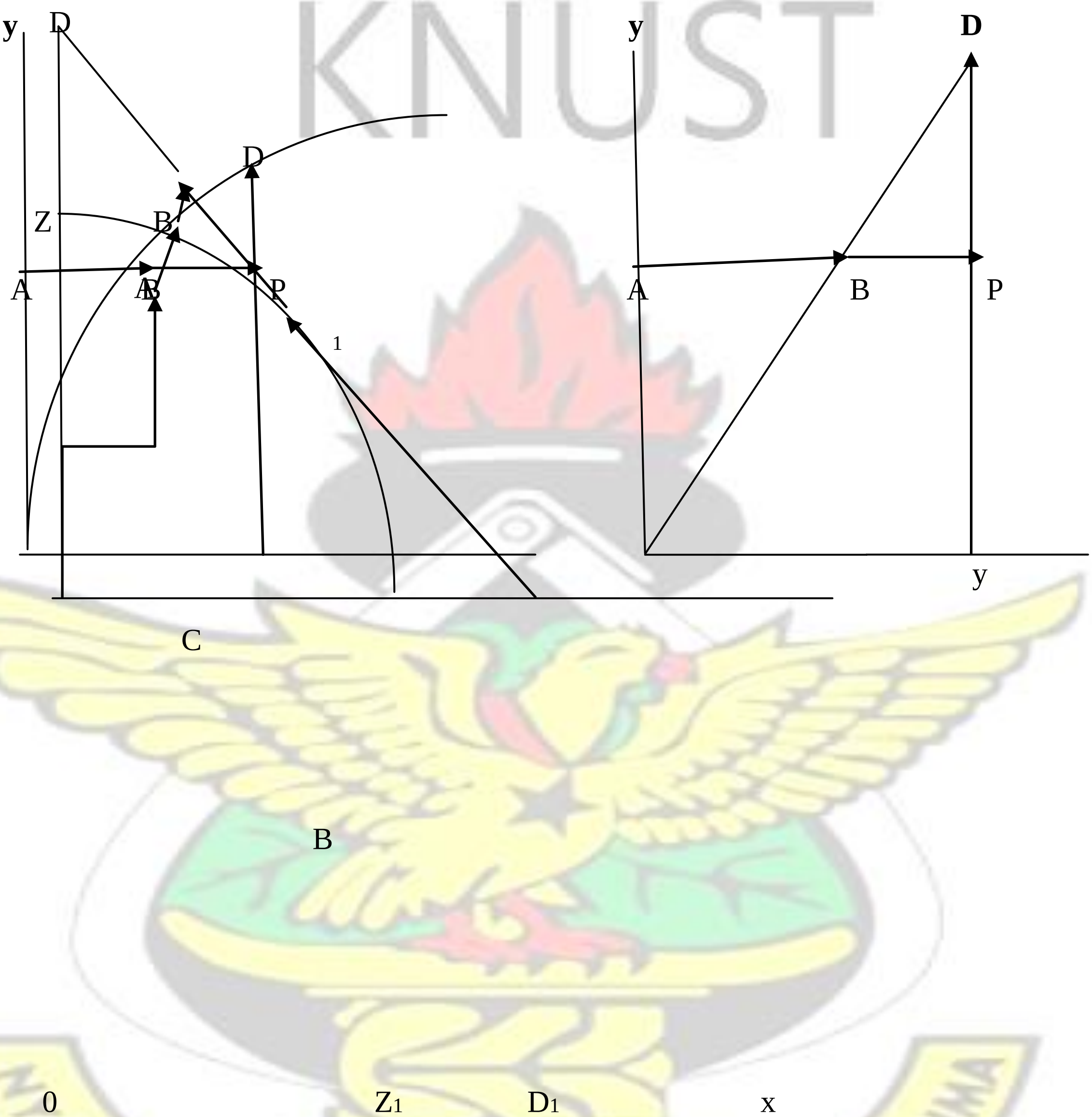

Figure 2.4 Technical and Allocative Efficiency from an output orientation

The Farrel output-oriented efficiency measures would be defined as follows. In Figure 2.4, the distance AB represents technical inefficiency, That is, the amount by which outputs would be decreased without requiring extra inputs. Hence a measure of output-oriented technical efficiency is the ratio:

$TE_o = OA/OB$ (2.10)

If we price information then we can draw the isorevenue line DD^1, and define the allocative efficiency to be:

$AE_o = OB/OC$ (2.11)

Furthermore, overall efficiency can be defined as the product of these two measures.

$EE_o = OA/OC = (OA/OB) \times (OB/OC) = TE_o \times AE_o$. (2.12) All

the above measures are bounded by zero and one.

2.5.3Ways of treating blank or missing data in DEA

Blank or missing data are a chronic problem in DEA [Talluri, 2000]. Researchers are often confronted with the problem of unavailable input(s) or output(s) quantities for some or all of the decision making units in their analysis. Conventionally, what researchers did was that they either ignore those inputs/outputs whose data are not available for some DMUs from the input/output data sets or they ignore those DMUs with missing or blank input/output data from the set of DMUs. Ignoring inputs/output can have a misleading effect on the efficiency studies since very important inputs/outputs might be left out in the study. Ignoring DMUs that fail to provide all input/output data reduces the total number of DMUs and this can have serious impact on the number of efficient DMUs since DEA usually requires a large number of DMUs to effectively discriminate between relatively efficient and inefficient DMUs. Very little has been done in the literature regarding how to handle this problem of missing data. [Kao and Lui, 2000] used fuzzy sets to model the ranges for missing data whilst [Kuosmanen, 2002] proposed a quantitative procedure for handling blank data. We consider in this section the method of Kuosmanen [2002].

Treatment of missing data

Suppose that we have n DMUs utilizing r inputs to produce s outputs. Using the standard notations, let the rxn and sxn matrices, X and Y be the input and output matrices respectively. The standard input-oriented radial DEA model is given by

$$\text{Maximize } \theta(u, v, k) = \sum_{j=1}^{s} Y_{kj} u_j$$

$$\text{Subject to: } \sum_{j=1}^{r} X_{kj} v_j = 1 \qquad (2.6)$$

$$\sum_{j=1}^{s} Y_{ij} u_j - \sum X_{ij} v_j \le 0 \quad \forall i \in \{0,1\}$$

$$u_j, v_j \ge 0$$

If for one reason or another, DMU k does not provide information about output j (that is Y_{kj} blank or missing), traditional DEA models either eliminate DMU k from the data set or ignore a potentially very important output. Kuosmanen [2002] proposed two alternatives for dealing with the situation:

(i). Omit output j in the efficiency measure of DMU k (but keep it in the efficiency measures for other DMUs in the sample) or

(ii). Give output j a value of zero. That is set $Y_{kj} = 0$

Kuosmanen proved that both methods will yield same efficiency scores, arguing that the second method is computationally convenient since it does not require modification of the computation code for every DMU with missing or blank output.

2.5.4 Units Invariance in DEA Models

Data Envelopment Analysis (DEA) models seek to find the efficiencies of DMUs by computing single measures of efficiency. These measures usually involve weighted sums of inputs and outputs which might be of varying units of measurements. A very important property of efficiency measures in DEA is that the score is independent of the dimensions of the input/output variables. The property that efficiency scores be independent of units of measurement of input/output variables is called units invariance.

It is however not all DEA models that give efficiency scores which are independent of the units of measurement of the inputs/output variables. The CCR and BCC DEA models are commonly thought to be units invariant, but they are not. The radial component of the efficiency measure obtained from these models is units invariant, but the slack component is not. The additive model of Charnes et al. and the weighted additive model with constant weights proposed in [Lovell and Pastor, 1995] are also not units" invariant. Some DEA models known to be units invariant are:

(i). The invariant multiplicative model of Charnes etal [1978]. This model, however, requires that input and output data be strictly positive

(ii). The extended additive model of Charnes etal [1978].This model however requires that input and output data be non-negative.

(iii). The normalized weighted additive DEA model proposed by Lovell and Pastor [1995]
(iv). The normalized weighted BCC DEA model of Lovell and Pastor [1995].

2.5.5 Translation Invariance

Traditional DEA models require that input and output data be strictly positive. They are unable to yield meaningfully interpretable efficiency measures when the data involve zero or negative values. When an input and/or output variable data involves negative and/or zero values the data usually will have to be transformed into strictly positive values by adding a sufficiently large value to the data of the variable. This transformation might not yield the same efficiency scores as the untransformed data would.

One of the important properties of DEA models is that the efficiency measures they provide be independent of an affine transformation of the input and output variables. This property is called translation invariance. Not all DEA models are translation invariant. The CCR model of Charnes et al, the variant and invariant multiplicative models, and the extended additive model are known not to be translation invariant. Some DEA models which are known to be translation invariants are:

(i). The additive model

(ii). The weighted additive model with constant weights

(iii). The BCC model is translation invariant in a limited sense, being invariant with respect to translation of inputs or outputs, but not both. The input oriented BCC model is translation invariant with respect to outputs whilst the output-oriented BCC model is translation invariant with respect to translation of inputs.

(iv). The normalized weighted additive model proposed by Lovell and Pastor [1995]. It is worth noting here that no model in its multiplier form is translation invariant. It is only the envelopment forms of the models described above that are translation invariant.

2.6 Applications of DEA in efficiency analysis in the insurance industry.

Frontier efficiency analysis has, in recent years, been applied in several sectors of several economies. One such sector of great application is the financial sector, with the insurance industry gaining increasing attention in recent years. Luhnen[2008] found 82 and Luhnen et al. [2008] found 87 studies applying frontier efficiency analysis in the insurance industry alone. We shall, in this brief, review literature on the topic of frontier efficiency analysis in the insurance industry. We conduct our review considering studies involving methodological issues, Geography of applications, Choice of input and output variables.

2.6.1 Methodological issues in DEA

The insurance industry has, in recent years, seen a number of methodological improvements in frontier efficiency analysis methodology. These improvements seek to address the major drawbacks of the various approaches. In order to address the major drawback of the mathematical programming approach (lack of statistical test), Banker [1992] showed that DEA efficiency estimators could, under some conditions, serve as maximum likelihood estimators, thus providing a statistical base for DEA, even though the sampling distribution of the DEA efficiency estimators is still unknown [Berger and

Humphrey, 1997].Another innovation is the bootstrapping procedure introduced by Wilson et al. [1998]. This procedure seeks to address another drawback of the mathematical programming approach (upward bias in DEA efficiency estimators)[Berger and Humphrey, 1997]. The bootstrapping procedure accounts for various efficiencies

(Technical, cost and revenue) and scale economies (CRS and VRS).

2.6.2 Geography of applications of DEA

Following the implementation of a single European insurance license in 1994, intercountry efficiency studies have started to gain grounds [Diacon etal, 2002]. Diacon et al [2002] in their paper „Size and Efficiency in European Long-Term Insurance

Companies‟ sampled450 insurance companies from 15 EU countries for the period 19961999 and found that insurers in the UK, Spain, Sweden and Denmark had higher technical efficiencies than their counterparts in other parts of Europe and that UK insurers seem to have low levels of allocative and scale efficiencies. We found that almost all the efficiency studies in the insurance industry have been conducted in the developed nations. We found only two papers [Barros and Obijiako, 2007; Carlos etal, 2008] considering the Nigerian insurance market and two papers on the Ghanaian insurance market [Owusu-Ansah etal,

2012] in West Africa as well as one paper on non-life insurance in Kenya [Jackson I. Mdoe etal,2013].Owusu-Ansah etal evaluated the technical efficiencies of Ghanaian general insurance from 2002 to 2007 whilst Ansah-Adu and Andoh evaluated cost efficiency of insurance companies in Ghana. We could not find any studies in the Ghanaian life insurance industry. Hence as far as we know, this research is the first to be conducted on the Ghanaian life insurance industry.

2.6.3 Choice of input and output variables in DEA

The appropriate choice of inputs and outputs plays a major role in the study of efficiency using DEA. We discuss here the choice of inputs and outputs used in our research. There are basically three types of inputs in the insurance industry. These are labor, business services and material, and capital. Labor is divided into home-office labor and agent labor. This input is subdivided because most insurers employ the services of agencies in their marketing and sale of policies. For example most of the insurance companies in Ghana have agencies at most of the regional capitals and some district capitals.

Business services and materials usually involve such items as traveling expenses, fuel expenses and telecommunication expenses. Capital is subdivided into physical capital, debt capital and equity capital. Physical capital is usually in the form of physical assets such as working premises and computers. Capital is used as an input because insurers must keep equity capital to prove their promise to pay claims even if more-than-expected losses are incurred. Due to the problem of unavailability of data (as most of these inputs are not explicitly included in the annual reports of the insurance companies), we found it necessary to simplify the scheme of input choice by combining labor, and business services and materials in the form of management expenses(plus commissions).This simplification is done in most efficiency studies of the insurance industry such as in [Owusu-Ansah etal, 2010] and[Luhnen, 2008]. Furthermore, Ennsfellner et al. [2004] argue that the operating expenses should be treated as a single input in order to reduce the number of parameters that will need to be estimated.

As for capital, we use net assets represented by some insurance companies as total equity. Thus, the input variables in our study are three (3); operating/management expenses, capital and commission. As for output measures, there are basically three approaches in choosing them in the insurance industry-Intermediation approach, User-cost approach and the Value-added approach. The intermediation approach considers insurers as financial intermediaries that collect funds from policy holders, invest them and pay claims, taxes and costs. The user-cost approach determines outputs by considering their net contribution to revenue. The value-added approach selects outputs by considering their contribution to value; those that contribute significantly to value based on cost allocations are considered as outputs. Luhnen et al[2008] found that out of the 87 studies they reviewed 74 of them used the value-added approach to choose their outputs but stating that there is a controversy among researchers as to whether claims or premiums are the most appropriate for value-added. They also found that 40 of the studies reviewed used claims as output whilst 31 use premiums as output. They found that two studies used both claims and premiums and one used neither. They concluded that there is no recognizable trend as to which proxies are most appropriate. In our research, we use both premiums and claims as output measures as [Owusu-Ansah etal, 2010] did. We also used investment income and premium as output measures. Investment income is included as an output variable because insurance companies can be considered as financial institutions seeking to maximize income from investments. Thus, our study uses three output variables; Net premiums earned, Net Incurred Claims and Investment Income. We used panel data of fifteen (15) insurance companies for the period 2009-20013. Thus, we obtained a total of seventy five (75) observations. Therefore, our choice of input and output variables ensures that we conform to the DEA convention that the total number of DMUs be more than three times the number of inputs and outputs.

CHAPTER 3

METHODOLOGY

3.0 Introduction

In this section we consider the concepts of linear programming discussing the methods with emphasis on interior point methods. We briefly explain further the models and methodologies utilized in this study.

3.1 Linear programming

Linear programming is a relatively new discipline in the mathematical spectrum. It was developed as mathematical models and introduced for economic and military planning in the years immediately following the end of World War II. The realization of its usefulness came simultaneously with the development of a solution method, the simplex method. The introduction of the first computer calculators was crucial to the blossoming and increase of this newly born area of study. Historical accounts of the birth and development of linear programming can be drawn from many sources, such as [Berghen, 2004] and [Schrijver, 1986].

A broad definition of linear programming has been given by Dantzig [1947]:

"Linear programming can be viewed as part of the great revolutionary development which has given mankind the ability to state general goals and to lay out a path of detailed decisions to take in order to best achieve its goals when faced with practical situations of great complexity." Further, Dantzig [1947] mentions the essential components of linear programming: "Our tools for doing this are ways to formulate real-world problems in detailed mathematical terms (models), techniques for solving the models (algorithms), and engines for executing the steps of algorithms(computers and software)."

An optimization problem can be described in terms of an objective function, decision variables, and a set of constraints. The investigation of optimization problems stems from the natural desire to solve a problem in the best possible way. It is interesting to note that while the need for an objective function is obvious now, it was not clear when the first problems were modeled: the set of feasible solution used to be investigated with some adhoc criteria, instead of being guided by the optimization of some quantity [Dantzig, 1947].

A linear programming problem is an optimization problem in which the objective function and constraints are linear. Linear programming problems arise directly from real-life applications (economics, transportation, finance, logistics, and other areas), or as approximations to more complex formulations, as most real-life relationships are nonlinear. Another important source of linear programs is the continuous relaxation of integer programming problems [Schrijver, 1986].Among the class of convex optimization problems; linear programming has a peculiar feature which is described by the following Theorem.

3.1.1Fundamental theorem of linear programming.

For a linear programming problem with a feasible domain P containing at least one extreme point, the optimal objective value is either unbounded or is achievable at one extreme point of P.

The set of linear constraints defines a polyhedron that constitutes the feasible region.

According to Theorem 3 1.1, in looking for a solution we can restrict our attention to the vertices of this polyhedron. The polyhedron corresponding to a linear system of m constraints in n variables ($m < n$) has a number of vertices equal to

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \qquad (3.0)$$

This number is an overestimate, as not all of these choices correspond to feasible points. The fact that the number of vertices is finite guarantees termination of any algorithm that explores all vertices. However this number is exponential, as can be clearly seen by further manipulating (3.0):

$$\left(\frac{n}{m}\right)^m \geq 2^m, \; 2m \text{ for } n \geq 2m:$$

This observation gives rise to the need of defining an algorithm that uses an intelligent way to discover an optimal vertex among the multitude of non-optimal ones. An important feature of any algorithm is its efficiency, which is how much effort is needed for the algorithm to provide an answer for some given input. The concept of computational complexity was introduced in the 70s, as the greater availability of computing machines required a deeper insight on the computational performance of different algorithms. The computational complexity of an algorithm can be used as a measure of the growth in the computational effort as a function of the size of the problem. Therefore, it provides a worst-case measure. The formal notion of efficiency is that a problem has an algorithm running in time proportional to a polynomial function of its input size. That is, we consider an algorithm efficient if it runs in time Θ (nk) on any input of size n, for some constant k > 0.

Complexity proofs rely on two assumptions that are necessary simplifications:

(i). Computations are performed in exact arithmetic;

(ii). The numerical data of a problem for instance is rational.

Computational complexity is measured by the number of elementary operations required to perform the algorithmic steps until termination. It often depends on the size of the binary representation of the input, usually denoted by L.Many algorithms have been proposed for solving a variety of optimization problems. However, despite their diversity, they are based on the same general framework which is summarized in the steps of Algorithm 1.1.

Algorithm 1.1 Generic optimization algorithm

Given: An initial iterate ω;

Repeat: Determine a search direction□□.

Compute the distance □ of how far to move along the search direction.

Move to the next point□□□□□.

Until Some termination criteria are met.

Each element of this generic framework (starting point, search direction, stepsize, and termination criteria) has to be accurately specified in order to define a particular algorithm. In what follows, we will introduce the main ideas behind three different solution methods for linear programming: the simplex method, the ellipsoid method, and the class of interior point methods with much emphasis on the latter.

3.1.2 The Simplex method

The simplex method was introduced in 1947 by Dantzig [1963]. The introduction of the simplex method happened simultaneously with the realization of linear programming as an efficient modeling tool for practical decision making. The simplex method exploits the insight provided by the fundamental theorem of linear programming (1.1), which states that the optimal solution, if it exists, is at one of the vertices of the feasible polytope. Thus it reaches a solution by visiting a sequence of vertices of the polyhedron, moving from each subsequent vertex to an adjacent one characterized by a better objective function value (in the non-degenerate case). Since the number of vertices is finite, termination is guaranteed.

Moreover, given the monotonic method of choosing the next vertex, the set of possible vertices decreases after each iteration, in the non-degenerate case. Degeneracy occurs when a vertex in R^m is defined by p > m constraints, and a step of length zero may be produced. In such a case, the simplex method does not actually move away from the current vertex, and thus no improvement in the objective function value can be achieved. In terms of practical efficiency, the simplex algorithm has long been considered the undisputed method for solving linear programming problems. However, the simplex method has exponential complexity. It is possible that all the vertices of the feasible polyhedron have to be visited before an optimal solution is reached.

Klee and Minty [Klee and Minty, 1972] were the first to provide an example of pathological behavior of the simplex method. In their example, a linear program with n variables and2n inequalities, the simplex method visits each of the 2n vertices. However, no cases of exponential number of iterations have been encountered in real-life problems, and usually only a fraction of the vertices are actually traversed before the optimal one is found. Moreover, in most cases the simplex algorithm shows polynomial behavior, being linear in m and sub linear in n [Shu-Cherng and Puthenpura, 1993].

A survey on the efficiency of the simplex method is done by Shamir [1987], where a probabilistic analysis (as opposed to worst-case analysis) is also presented. The gap between the observed and theoretical worst-case performances of the simplex method is still unexplained. Given this theoretical drawback, a great deal of effort has been put into finding an algorithm for linear programming which mischaracterized by a polynomialtime bound.

3.1.3 The Ellipsoid method

In 1979 a breakthrough occurred, as Khachiyan showed how to adapt the ellipsoid method for convex programming to the linear programming case, and determined the computational complexity of linear programming. In Khachiyan's ellipsoid method, the feasible polyhedron is inscribed in a sequence of ellipsoids of decreasing size. The first ellipsoid has to be large enough to include a feasible solution to the constraints; the volumes of the successive ellipsoids shrink geometrically. Therefore it generates improving iterates in the sense that the region in which the solution lies is reduced at each iteration in a monotonic fashion. The algorithm either finds a solution, as the centers of the ellipsoids converge to the optimal point, or states that no solution exists.

The exciting property of the ellipsoid method is that it finds a solution in□ (n^2L) iterations, and thus has polynomial complexity. However, since the ellipsoid algorithm generally attains this worst-case bound [Shu-Cherng and Puthenpura, 1993], its practical performance is not competitive with other solution methods. Besides, it displays other drawbacks related to large round-off errors and a need for dense matrix computation.

Nevertheless, the ellipsoid method is often used in the context of combinatorial optimization as an analytic tool to prove complexity results for algorithms [Nemhausa and Wolsey, 1988].

3.1.4 Interior point methods

Interior point methods were being developed in the 60s and the beginning of the 70s as methods to solve nonlinear programming problems with inequality constraints. However, they fell from favor and received less and less attention because of their inefficiency and the presence of strong competitors such as sequential quadratic programming [Wright, 1992].Since their reintroduction, this time to solve linear programs, following Karmarkar's groundbreaking paper [Karmarkar,1984], interior point methods have attracted the interest of a growing number of researchers. This algorithm was also proved to have polynomial complexity: indeed, it converges in □ (nL) iterations. As opposed to Khachiyan's ellipsoid method, in practice, Karmarkar's algorithm actually performs much better than its worst-case bound states. The main idea behind interior point methods is fundamentally different from theory that inspired the simplex algorithm. Here, the optimal vertex is approached by moving through the interior of the feasible region. This is done by creating a family of parameterized approximate solutions that asymptotically converge to the exact solution. Therefore, by embedding the linear problem in a nonlinear context, an interior point method escapes the “curse of dimensionality" characteristic of dealing with the combinatorial features of the linear programming problem.

In the simplex method, the current solution is modified by introducing a nonzero coefficient for one of the columns in the constraint matrix. The interior point method allows the current solution to be modified by introducing several columns at once.

Karmarkar announced that his method was extremely successful in practice, claiming to beat the simplex method by a large margin 50 times, as reported in[Wright, 1992]. A variant of Karmarkar's original algorithm was then proposed and implemented by Adler etal [1989]. Since then, the theoretical understanding has considerably improved, many algorithmic variants have been proposed and several of them have shown to be computationally viable alternatives to the simplex method.

Over the last two decades, an impressive wealth of theoretical research has been published, and computational developments have brought life to the field of linear programming. Among the positive consequences of the renewed interest in linear programming are the improvements to the implementations of simplex-based solvers [Bixby, 1994; Todd and Ye, 1990].There are classes of problems that are best solved with the simplex method, and others for which an interior point method is preferred. Size, structure and scarcity play a major role in the choice of algorithm for computations. As a rule of thumb, with the increase of problem dimension, interior point methods become more efficient and effective [Zhang, 1995]. However, this does not hold in the hyper-parse case, where the simplex method is virtually unbeatable [Todd and Ye, 1990; Colombo, 2007], and for network problems, where the specialized network simplex method can exploit the structure in an extremely efficient manner [Mdoe etal, 2013]. They can be applied to a wide range of situations with no need of major changes. In particular, they have been successfully applied to complementarity problems, quadratic programming, convex nonlinear programming, second-order cone programming and semi-definite programming.

3.2 Derivation of Primal-Dual Interior-point Methods

Consider the following primal-dual pair of linear programming problems in standard form:

$$\min_{x} C^T x \qquad \max_{(y,s)} b^T y$$

$$\text{(P)} \quad \text{s.t} \quad Ax=b \qquad \text{(D)} \quad \text{s.t} \quad A^T y + s = c \quad (3.1)$$

$$x \geq 0; \qquad s \geq 0,$$

Where $A \in \mathbb{R}^{m \times n}$, $x,s,c \in \mathbb{R}^m$ and $y,b \in \mathbb{R}^n$, $m \leq n$ .We assume, without loss of generality, that A has full row rank, as linearly dependent rows can be removed without changing the solution set. This implies that a feasible $s \geq 0$ determines in a unique way the value of y. In fact, the y variables can be eliminated thus producing the symmetric combined primal-dual form studied by Todd and Ye [2007].

We define some terminologies here:

Definition: The set of primal feasible points is the set of feasible solutions to the primal problem (3.1a) defined by $P = \{x : Ax = b, x \geq 0\}$.

Definition: The set of dual feasible solutions is the set of feasible solutions to the dual problem (3.1b) defined by $D = \{(y,s) : A^T y + s = c, s \geq 0\}$

Definition: The set **P** ^{0} of primal interior points is the set of primal feasible points excluding those on the boundary of the feasible region, defined as $P^0 = \{x \in P : x > 0\}$

Definition: The set D^0 of dual interior points is the set of dual feasible points excluding those on the boundary of the feasible region. This set is defined by $D^0 = \{(y,s) \in D : s > 0\}$

Definition: The set of feasible primal-dual points is the Cartesian product of the set of primal feasible points and the set of dual feasible points. This set is defined by $F = P \times D$

Definition: The set of primal-dual interior points is the set of feasible primal-dual points that are not on the boundary of the feasible region. This set is a subset of the set of feasible primal-dual points and is defined by $Ŧ^0 = \{(x, y, s) \in F : (y,s) > 0\}$.

We state here some lemmas concerning the primal-dual feasible interior points.

Lemma 3.1.2 (Weak Duality): For every $(x, y, s) \in Ŧ$, we have $c^T x \geq b^T y$. That is the primal and dual feasible interior points bound each other.

Proof: Since x ∈ P and (x, y) ∈ D, then we have:

$$c^T x - b^T y = x^T c - x^T A^T y = x^T (c - A^T y) = x^T s \geq 0$$

$c^T x - b^T y \geq 0$ and thus $c^T x \geq b^T y$

The difference $c^T x - b^T y$ is called the duality gap. When the duality gap is zero then the primal-dual solution is optimal.

Lemma 3.1.3 (Strong Duality): A point x $\in$ P is an optimal solution if and only if there exists (y, s) $\in$ D such that $c^T x = b^T y$

The problem (P) has a solution if P $\neq \emptyset$ and D $\neq \emptyset$ then both (P) and (D) have an optimal solution $\omega^* = (x^*, y^*, s^*)$. If either P or D is empty then either the solution of one or the other is empty too or it is unbounded. The difference $c^T x - b^T y = x^T s$ is called the complementarity gap which measures the distance of the current primal-dual point from the optimal solution. Thus a zero complementarity gap implies optimality. Hence, the duality gap and the complementarity gap achieve equal values at optimal feasible solutions.

To proceed with the development of the primal-dual interior-point methods we make the assumption that P $\neq \emptyset$ and D $\neq \emptyset$. This assumption is known as the interior point assumption. The primal-dual solutions to (3.1) can be shown to satisfy the following set of equations known as the Karush-Kuhn-Tucker conditions.

$$
\begin{aligned}
A^T y + s &= c \\
Ax &= b \\
x_i s_i &= 0 \quad i = (1, n) \\
(x, s) &\geq 0
\end{aligned}
\tag{3.2}
$$

Primal-dual methods solve the problem (3.1a) by finding the solution (x*, y*, s*) of the system (3.2) whilst modifying the search direction and length so as to ensure that the solution is strictly in the interior of the feasible region(That is by ensuring that the inequalities $x_i s_i \geq 0$ in (3.2) are strictly satisfied; that is $x_i s_i > 0$)

The system (3.2) can be restated in a slightly different form by defining a mapping F from R^{2n+m} to R^{2n+m} as

$$F(x, y, s) = \begin{bmatrix} A^T y + s - c \\ Ax - b \\ XSe \end{bmatrix} = 0 \quad (3.3)$$

$$(x, s) \geq 0 \quad (3.4)$$

Where X =diag($x_1, x_2, ..., x_n$), S= diag($s_1, s_2, ..., s_n$) and e=$(1,1,...,1)^T$.Primal-dual interiorpoint methods solve the primal and dual problems (3.1)by solving the system (3.3) using either line search or trust-region methods whilst ensuring that the iterates are strictly in the interior of the feasible region. Thus, interior point methods use line search methods or trust-region methods to find appropriate values for α_K and Δx_K and modifying the current feasible interior-point x_K using $x^{K+1} = x^K + \alpha_K \Delta x_K$.

Some line search methods used in solving (3.3) are; Newton‟s method, Quasi-Newton methods and Coordinate Descent methods. Some trust-region methods that can be used to solve (3.3) are; the Dogleg method, two-dimensional subspace minimization method and the Steihaug‟s Approach.

Primal-dual interior-point methods use a modified version of the Newton‟s method for solving non-linear systems, to solve the system (3.2) by finding a search direction(Δx, Δy, Δs) which is the solution to the linear system:

$$J(x,y,s)\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta s\end{bmatrix} = -F(x, y, s)$$

where J is the Jacobian of F.

Thus we have:

$$\begin{bmatrix}0 & A^T & 1\\ A & 0 & 0\\ S & 0 & X\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta s\end{bmatrix} = \begin{bmatrix}0\\ 0\\ -XSe\end{bmatrix} \quad (3.5)$$

The new iterate then is given by $(x, y, s)^{k+1} = (x, y, s)^k + \alpha_k(\Delta x,\Delta y,\Delta s)$ for some line search parameter $\alpha_k \in (0, 1]$. α_k is kept in the half-open interval (0,1] in order to ensure that the new iterate is kept within the interior of the feasible region. The question one might ask now is how we can determine an appropriate value for α_k for each iteration and how can we solve the system (3.5)?

Before we proceed to the answer the question of how to an appropriate value of α_k solve the system (3.5), we consider some few concepts which are important prerequisites.

3.2.1 The Central path

Recall that our emphasis is on path-following interior-point methods. Restricting our attention to the locus of points described by the logarithmic barrier function, we can write, for every linear program in standard form, a corresponding problem known as the barrier problem given by:

$$(P_u)\ \min c^T x - u\sum_{i=1}^{n} In\, x_i \qquad \text{s.t} \qquad x_i > 0$$

The problem (P_u) defines a family of problems characterized by the scalar $u > 0$, called the barrier parameter. The logarithmic term in (Pu) ensures that all iterates stay in the interior of the feasible region by penalizing all iterates that tend to move close to the boundary of the feasible region. We note that the limit of the problem (Pu)as $u \to 0$ is the problem (P).

Since the objective function of problem (Pu) is convex it either has one minimizer or it is unbounded. If the minimizer exists then it satisfies the following Karush-Khun-Tucker conditions.

$$Ax = b \qquad uX^{-1}e + A^T y = c \qquad (3.6) \qquad x > 0$$

Substituting s = $uX^{-1}e$ in equation (2.1) we obtain.

$$Ax = b$$

$$A^T y + s = c$$

$$XSe = ue \qquad (3.7)$$

$(x, s) \geq 0)$

If there exists some $u > 0$ that solves (3.7) then every $u > 0$ solves (3.7). As $u \rightarrow 0$ the solution of system (3.7)is uniquely defined by a continuous smooth curve given by $\gamma(u)$ = ($x(u)$, $y(u)$,$s(u)$). This curve is known as the Central path.

The duality gap for a particular $u > 0$ that solves (3.7) can be written as.

$$g(u) = c^T x(u) - b^T y(u) = x^T(u) s(u) \quad (3.8)$$

But since $XSe = ue$corresponds to $x_i s_i = u$, we have

$$X^T(u) s(u) = nu \quad (3.9)$$

The parameter u measures the average of the pairwise product $x_i s_i$. As $u \rightarrow 0$ we observe in equation (3.9) that $g(u) \rightarrow 0$.

If x* is the feasible solution of primal problem (P) and (y*,s*) is the feasible solution to the dual problem (D) then $c^T x(u) \geq c^T x^* = b^T y^* \geq b^T y(u)$ and we can conclude that as $u \rightarrow 0$, $c^T x(u) \rightarrow c^T x^*$ and $b^T y(u) \rightarrow b^T y^*$. Thus the objective function values of the perturbed problem (3.7) converge to the values achieved by an optimal solution(x*, y*, s*) of the original problem. Thus, the following theorem holds.

Theorem 2.7.4: Let the primal and dual problems (P) and (D) respectively be feasible and the matrix A be of full row rank. Then as $u \rightarrow 0$ we have:

$x(u) \rightarrow x^*$and $(y(u), s(u)) \rightarrow (y^*, s^*)$. This theorem suggests that under primal and dual feasible coupled with full row rank of A; the central path converges to an optimal solution of the problem (3.1). The central path can thus be used to reach the optimal solution of the problem (3.1). Path-following interior-point algorithms use the central path in finding solutions to the problem (3.1).

3.2.2 Neighborhoods of the Central Path

Recall that path-following interior-point methods seek the optimal solutions to the original problem (3.1) by searching along the central path. However it is practically impossible to traverse strictly on the central path. Hence, the iterates are allowed to stay within some neighborhood of the central path. It is these neighborhoods that form the topic of this subsection. Several neighborhoods can be defined around the central path for any particular $u > 0$.

However, there exist two most important neighborhoods that are of practical and theoretical interest. These are the 2-norm (or the tight) neighborhood $N_2(\theta)$ and the onesided norm (or wide) neighborhood $N_{-\infty}(\gamma)$ defined as follows. x

$$N_2(\theta) = \{(x, y, s) \in F^0 : \|XSe - ue\|_2 \le \theta u\} \quad (3.10)$$

$$N_{-\infty}(\gamma) = \{(x, y, s) \in F^0 : x_i s_i \ge \gamma u\} \quad (3.11)$$ for some $\theta, \gamma \in (0, 1]$.

Typical values of θ and γ are $\theta = 0{\cdot}5$, $\gamma = 10^{-3}$

$N_2(\theta)$ defines points that lie very close to the central trajectory than those points definedby $N_{-\infty}(\gamma)$.

Path-following algorithms seek iterates that lie in the neighborhood N by ensuring thatthe pair-wise products $x_i s_i$ are reduced to zero at the same rate.

3.2.3 Practical Implementation of Interior-point Methods

The general framework of path-following interior-point algorithms is as follows.

Modify the mapping F in equation (3.3) so that the primal-dual iterates $\omega_k = (x_k, y_k, s_k)$ lie strictly in some neighborhood of the central path. That is, modify F to obtain

$$F(x, y, s) = \begin{pmatrix} A^T y + s - c \\ Ax - b \\ XSe - \mu e \end{pmatrix} = 0 \qquad (3.12)$$

The Newton"s method is then used to linearize the system (3.12). The linearization yields

$$\begin{pmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = \begin{pmatrix} -(A^T y + s - c) \\ -(Ax - b) \\ -(XSe - \mu e) \end{pmatrix} = \begin{pmatrix} -r_c \\ -r_b \\ -r_u \end{pmatrix} \qquad (3.13)$$

3.2.4 Termination Criteria for Interior-Point methods

Recall that path-following interior-point algorithms seek optimal solutions to linear programs by traversing a neighborhood of the central path, which ensures that the iterates always lie in the interior of the feasible region. Thus attainment of exact optimal solutions is never possible. A criterion for termination of the algorithm is thus necessary for proper implementation. Some common termination criteria suggested in the literature are as follows:

$$\frac{\|Ax - b\|}{1 + \|x\|_{\infty}} \le 10^{-p}, \quad \frac{\|A^T y + s - c\|}{1 + \|s\|_{\infty}} \le 10^{-p}, \quad \frac{|c^T x - b^T y|}{1 + |b^T y|} \le 10^{-q} \qquad (3.14)$$

$$\frac{\|Ax - b\|}{1+\|b\|} \le 10^{-p}, \quad \frac{\|A^T y + s - c\|}{1+\|s\|} \le 10^{-p}, \quad \frac{u}{1+\|b^T y\|} \le 10^{-q} \tag{3.15}$$

The values of p and q in the above criteria are problem dependent. However, some common values are $p = q = 8$ for criteria in (3.14) and $p = q = 10$ for criteria in (3.15).

3.2.5 **Solving the Newton's System**

In the implementation of the primal-dual interior-point algorithms, the most tedious computation is that of solving the step equations in system (3.13). The coefficient matrix in this system is usually very huge and sparse. Hence, it is worth restructuring the coefficient matrix into a more compact symmetric one which will allow for easier solution of the system than the original matrix.

In order to reformulate the system (3.13) we note from the last row that

$S\Delta x + X\Delta s = \xi_b$ and hence

$$\Delta s = X^{-1}(\xi_b - S\Delta x)$$

We can therefore eliminate Δs and thus obtain

$$\begin{bmatrix} -X^{-1}S & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \xi_c - X^{-1}\xi_U \\ \xi_b \end{bmatrix} \tag{3.16}$$

Also from the first row in (3.16) we observe that

$$\Delta x = -S^{-1}X[\xi_c - X^{-1}\xi_u - A^T\Delta y].$$

Substituting this into the second row yields the set of normal equations given by

$$AD^2A^T\Delta y = AD^2(\xi_C - x^{-1}\xi_u - A^T\Delta y) \tag{3.17}$$

Where, $D^2 = S^{-1}X$

Thus, solving the Newton's step equations is equivalent to solving the normal equations given by:

$$AD^2A^T\Delta y = AD^2\left(\xi_c - X^{-1}\xi_u\right) + \xi_b \qquad (3.18)$$

$$\Delta x = -S^{-1}X\left(\xi_c - X^{-1}\xi_u - A^T\Delta y\right) \qquad (3.19)$$

$$\Delta s = X^{-1}\left(\xi_b - S\Delta x\right) \qquad (3.20)$$

Most primal-dual interior-point methods use direct sparse Cholesky algorithms to factor the coefficient matrix AD^2A^T in (3.18) and then perform triangular back solutions to find the step Δy. The other steps x and s are then found using (3.19) and (3.20) respectively.

The computation of the Cholesky factors dominates the cost of each iteration. As thesis usually a major computational task, the efforts in the theory and practice of interior point methods concentrate on reducing the number of times the Newton system matrix (3.13) has to be factorized. In particular, it is worth adding more (cheap) back solves if this reduces the number of (expensive) factorizations. Mehrotra predictor corrector algorithm and multiple centrality correctors are two algorithms that have proven to be very efficient in reducing the number of iterations in practical algorithms.

3.2.6 Mehrotra's Predictor-Corrector Algorithm

The material in this section is based on [MdSaad and Idris, 2011; Berghen, 2004].The Mehrotra predictor-corrector algorithm forms the basis of most existing interior-point codes for general-purpose Linear Programming Problems. This is because; the Mehrotra's algorithm is usually very fast and reliable. The algorithm has three main steps; the predictor, the corrector step and the Centering step. The predictor step is used to make advancement towards optimality. The corrector step is used to remedy the error in the predictor step and the centering step is used to move the iterate near the central path.

3.2.6The Predictor Direction

The predictor direction $\Delta w^{aff} = (\Delta x^{aff}, \Delta y^{aff}, \Delta s^{aff})$ is calculated by solving the Newton system (3.13) with the right-hand-side given by

$$r = \begin{pmatrix} b - Ax \\ c - A^T y - s \\ -XSe \end{pmatrix} \quad (3.21)$$

This direction is often called the affine scaling direction.

In order to measure the effectiveness of this direction we findα_{aff}^{pri} and α_{aff}^{dual}(with anupper bound of 1) to be the longest possible step lengths that can be taken along this direction before exiting the interior of the feasible region. That is before violating thenon-negativity constraint (x, s) > 0.

$$\alpha_{aff}^{pri} \stackrel{def}{=} \min\left(1, \min_{i \mid \Delta x_i^{aff} < 0} \frac{x_i}{-\Delta x_i^{aff}}\right) \quad (3.22)$$

$$\alpha_{aff}^{dual} \stackrel{def}{=} \min\left(1, \min_{i \mid \Delta s_i^{aff} < 0} \frac{s_i}{-\Delta s_i^{aff}}\right) \quad (3.23)$$

We define u_{aff} to be the value of u that would be obtained by a full step to the boundary. That is

$$u_{aff} = \left(x + \alpha_{aff}^{pri}\Delta x_{aff}\right)^T \left(s + \alpha_{aff}^{pri}\Delta s_{aff}\right) / n \qquad (3.24)$$

and set the damping parameter σ to be $\sigma = \left(\frac{u^{aff}}{u}\right)^3$

The Corrector Step

The corrector step is obtained by replacing the right hand side of (3.13) by

$$\left(0, 0, -\Delta X^{aff}\Delta S^{aff} e\right)$$

The Centering Step

The Centering step is obtained by replacing the right hand side of (3.16) by $(0, 0, \sigma ue)$.

The complete Mehrotra‟s step can be obtained by combining all the three steps above (by adding all the right hand sides in the above three steps) into one step and solving the following system:

$$\begin{pmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = \begin{pmatrix} b - Ax \\ c - A^T y - s \\ -XSe - \Delta X^{aff}\Delta S^{aff} e + \sigma ue \end{pmatrix} \qquad (3.25)$$

The maximum steps that can be taken along these directions before violating the nonnegativity condition are obtained using the following formulae.

$$\alpha_{max}^{pri} = \min\left(1, \min_{i:\Delta x_i < 0}\left(-\frac{x_i^k}{\Delta x_i^{deff}}\right)\right) \qquad (3.26)$$

$$\alpha_{max}^{dual} = \min\left\{1, \min_{i:\Delta s_i < 0} \left(-\frac{s_i^k}{\Delta s_i}\right)\right\} \qquad (3.27)$$

The primal and dual step lengths are then chosen as follows:

$$\alpha_k^{pri} = \min\left\{1, \eta\,\alpha_{max}^{pri}\right\} \text{ and } \alpha_k^{dual} = \min\left\{1, \eta\,\alpha_{max}^{dual}\right\} \qquad (3.28)$$

Where $\eta \in (0{\cdot}9,\ 1{\cdot}0)$ is chosen so that $\eta \to 1$ near the solution, to accelerate asymptotic convergence.

3.2.7 Termination Criterion

Recall that path-following interior-point algorithms seek optimal solutions to linear programs by traversing a neighborhood of the central path, which ensures that the iterates always lie in the interior of the feasible region. Thus attainment of exact optimal solutions is never possible. A criterion for termination of the algorithm is thus necessary for proper implementation.

Some common termination criteria suggested in the literature are stated above (3.14 and3.15).

.The Mehrotra‟s algorithm is presented in algorithm 1.

Initialization: Given (x^0, y^0, s^0) with $(x^0, s^0) > 0$;

Repeat **Set** $(x, y, s) = (x^k, y^k, s^k)$ and solve (3.21) for $(\Delta x^{aff}, \Delta y^{aff}, \Delta s^{aff})$;

Calculate α_{aff}^{pri}, α_{aff}^{dual} and u_{aff} using (3.22),(3.23) and (3.24);

Set (3.25) for $(\Delta x, \Delta y, \Delta s)$;

Calculate α_k^{pri} and α_k^{dual} using (3.28);

Set $x_{k+1} = x_k + \alpha_{kpri}\Delta x, (y_{k+1}, s_{k+1}) = (y_k, s_k) + \alpha_{kdual}(\Delta y, \Delta s)$**;**

Until Convergence Criterion (3.28) is achieved for some p and q ;

Algorithm 1: Mehrotra‟s Algorithm

3.2.8 An Interior-Point Algorithm Implemented on MATLAB

We discuss, in this brief, the Linear Interior-Point Solver (LIPSOL) proposed by Zang Yi in 1995, which was implemented on MATLAB by University of Maryland BaltimoreCounty and the Math works Inc.LIPSOL is a variant of the Mehrotra‟s algorithm proposed in [Mehrotra S, 1992].

Consider the linear optimization problem:

Min $c^T x$

subject to:

$Aeq \cdot x = beq$ (3.29) A *

$x \leq b$

$l \leq x \leq u$

In order to apply the algorithm, the following are performed on the problem (3.29).

(i). Transform the bounds $l \leq x \leq u$ into the form $x \geq 0$

(ii). Add slacks to all inequality constraints

(iii). Remove all variables with equal upper and lower bounds are removed.

(iv). Remove rows of all zeros in the constraint matrix.

(v). Remove columns of all zeros in the constraint matrix.

(vi). When a significant number of singleton rows exist in the constraint matrix, the associated variables are solved for and the rows removed.

After the above operations, the problem becomes of the form

Min $c^T x$

Subject to :(3.30)

$A\square b \square B$

$0 \leq x\square\ u$

Adding primal slack variables changes problem (3.30) into one of the form

Min $c^T x$

Subject to:

$A\square x \square b$(3.31) $x\square s\square u$

$x \geq 0, s \geq 0$

(Problem (3.31) is a primal problem with x being the primal variable and s being the primal slack variable.

The associated dual of problem (3.31) is given by

Max $b^T x$

subject to:

$A^T \square y\square\square\square z \square c$(3.32)

$z \square 0, \square\square 0$

Where y and □ are dual variables and z is a dual slack variable.

The Karush-Khun-Tucker equations (Optimality conditions) of the problem (3.32) are given by:

$$F(x,y,z,s,w) = \begin{pmatrix} Ax - b \\ x + s - u \\ A^T y - w + z - c \\ x_i z_i \\ s_i w_i \end{pmatrix} = 0 \qquad (3.33)$$

$x \geq 0$, $z \geq 0$, $s \geq 0$, and $w \geq 0$

Where the linear equations in (3.33) above are called the feasibility conditions and the quadratic equations $x_i z_i = 0$ and $s_i w_i = 0$ are called complementarity conditions.

The quantity $x^T z + s^T w$ is called the duality gap.

The algorithm solves the problem (3.30) by solving the non-linear problem (3.33) as follows:

For the iterate $v = (x, y, z, s, w)^T$, calculate the prediction direction

$$\Delta v_p = -\left(F^T(v)\right)^{-1} F(v) \qquad (3.34)$$

and then the corrector direction

$$\Delta v_c = -\left(F^T(v)\right)^{-1} F(v + \Delta v_p) - u\hat{e} \qquad (3.35)$$

where $u \geq 0$ is called the centering parameter and $\hat{e}$ is a zero-one vector with zeros corresponding to the linear equations and ones corresponding to the quadratic equations in (3.35).The next improved interior-point iterate is then given by □□□□□□□□$_p$ □□□$_i$ □ with□ chosen such that □ remains in the interior of the feasible region.

The convergence criterion for the algorithm is given by:

$$\frac{\|r_b\|}{\max\{1, \|b\|\}} + \frac{\|r_c\|}{\max\{1, \|c\|\}} + \frac{|c^T x - b^T y + u^T w|}{\max\{1, |c^T x|, |b^T y - u^T w|\}} \le tol \quad (3.36)$$

Where r_b □ Ax □b, Primal residual r_c □

A^T y□□□z□c, Dual residual r_u □

x□s□u, upper-bound feasibility

This algorithm is elegantly implemented on MATLAB as the lipsol function.

The following are the equations from the model used. These have solved using the algorithm above and the results are analyzed in chapter 4.

2010

DMU$_1$□*DONEWELL LIFE*□ *Minimize* □ *subject to* :

□□$_1$ □
□ □
□^{2} □
□
□□$_3$ □
□ □
□□$_4$ □
□□$_5$ □

947.6 5906.7 192.7 1597.8 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8 947.61 84.8 1883.5 50.6 443.1 215.7 277.5 760.4 86.3 226.1 138.1 32.4 434.6 117.2 13.8 84.82

420.6 1011.1 28.7 492 258.1 163.4 1629 86.4 588.7 1.3 23.7 1096.2 326.1 75.4 420.6

33596.8.113102607..9921944..12862249..86 197389.1.61264.3 2770.2 223.1 589.5 219.6 237.5 3860112.4 38470..96 22217.6.3 33576.8.1

170.3 328 37.5 94.7 11.9 51.7

507.2 1365.4 86.9 640.8 313.4 858.9 1603.7 270.5 538.8 305.1 389.5 876.9 308.9 191.9 507.2

□□$_{13}$□

□□□$_{14}$□□

□$_j$ □ 0; j □1,...,14 *and* □ *is the iput* □ *oriented efficiency measurement score for DMU* $_j$

DMU$_1$ *ENTERPRISE*

LIFE *Minimize* *subject to* :

λ_1 λ_2 λ_3 λ_4 λ_5 λ_6

947.6 5906.7 192.7 1597.8 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8 λ_7 5906.7 84.8 1883.5 50.6

443.1 215.7 277.5 760.4 86.3 226.1 138.1 32.4 434.6 117.2 13.8 λ_8 1883.5

λ_9 λ_{10} λ_{11} λ_{12} λ_{13} λ_{14}

λ_1 λ_2 λ_3 λ_4 λ_5 λ_6

420.6 1011.1 28.7 492 258.1 163.4 1629 86.4 588.7 1.3 23.7 1096.2 326.1 75.4 λ_7 1011.1

λ_8

λ_9 λ_{10} λ_{11} λ_{12} λ_{13} λ_{14}

λ_1

947.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

596.1 2607.9 219.2 862.8 1973.6 1264.3 2770.2 223.1 589.5 219.6 237.5 3860.4 384.6 222.3

13102607..99

33.8 1310.9 44.1 249.6 89.1 170.3 328 37.5 94.7 11.9 51.7 112 70.9 17.6

507.2 1365.4 86.9 640.8 313.4 858.9 1603.7 270.5 538.8 305.1 389.5 876.9 308.9 191.9

1365.4

$\lambda_j \geq 0;\ j = 1,\ldots,14$ *and* θ *is the iput* $-$*oriented efficiency measurement score for DMU* $_j$

DMU$_1$ *GHANA UNION* *Minimize* θ

subject to :

947.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

84.8 5906.7 192.7 1597.8 1345.6 277.5 226.1 138.1 32.4 434.6 117.2 192.7

1883.5 50.6 443.1 215.7 760.4 86.3 13.8 50.6

420.6 1011.1 1629 23.7 1096.2 326.1

588.7 75.4 28.7

1264.3 589.5 219.6 237.5

170.3 94.7 11.9 51.7

858.9 538.8 305.1 389.5 219.2

28.7 492 258.1 163.4 86.4 1.3

596.1 2607.9 219.2 862.8 1973.6 33.8 2770.2 223.1

1310.9 44.1 249.6 89.1 328 37.5 44.1

1603.7 270.5

507.2 1365.4 86.9 640.8 313.4 222.3 86.9

17.6

3860.4 384.6 191.9

λ_1

λ_2

λ_3

λ_4

λ_5

λ_6

947.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

112 70.9

876.9 308.9

λ_{10}

λ_{11}

λ_{12}

λ_{13}

λ_{14}

$\lambda_j \geq 0; j = 1,...,14$ *and* θ *is the iput* $-$*oriented efficiency measurement score for DMU* $_j$

DMU$_1$ *METROPOLITAN LIFE* *Minimize* θ *subject to* :

5906.7 192.7 1597.8 1345.6 $_7$ 1597.8

84.8 1883.5 50.6 443.1 215.7 277.5 760.4 86.3 226.1 138.1 32.4 434.6 117.2 13.8 λ_8 443.1

λ_9

λ_{10}

λ_{11}

947.6					2694.4	7173.8	401.3	2519.9	81.2	389.5	3159.5	582.9	207.8		
420.6	1011.1	28.7	492	258.1	163.4	1629	86.4	588.7	1.3	23.7	1096.2	326.1	75.4	492	
596.1	2607.9	219.2	862.8	1973.6	1264.3	2770.2	223.1	589.5	219.6	237.5	3860.4	384.6	222.3	76	862249..86
33.8	1310.9	44.1	249.6	89.1	170.3	328	37.5	94.7	11.9	51.7	112	70.9	17.6		

λ_1 λ_2 λ_3 λ_4 λ_5 λ_6

947.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

507.2 1365.4 86.9 640.8 313.4 858.9 1603.7 270.5 538.8 305.1 389.5 876.9 308.9 191.9 λ_{89}

640.8

λ_{10} λ_{11} λ_{12} λ_{13} λ_{14}

λ_j 0; j 1,...,14 *and* θ *is the iput oriented efficiency measurement score for DMU* $_j$

DMU$_1$ *PROVIDENT LIFE* *Minimize* θ *subject to* :

84.8 5906.7 192.7 1597.8 1345.6 277.5 226.1 138.1 32.4 434.6 117.2 7 91345.6

1883.5 50.6 443.1 215.7 760.4 86.3 13.8 λ_8 215.7

λ_9 λ_{10} λ_{11} λ_{12} λ_{13} λ_{14}

λ_1 λ_2 λ_3 λ_4 λ_5

28.7 492 258.1 163.4 86.4 588.7 1.3

947.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

420.6 1011.1 1629 23.7 1096.2 326.1

1264.3 589.5 219.6 237.5 75.4 258.1

170.3 94.7 11.9 51.7

858.9 538.8 305.1 389.5

596.1 2607.9 219.2 862.8 1973.6 33.8 2770.2 223.1 1973.6

1310.9 44.1 249.6 89.1 328 37.5 89.1

1603.7 270.5

507.2 1365.4 86.9 640.8 313.4 313.4

3860.4 384.6 222.3

112 70.9 17.6

876.9 308.9 191.9

$\lambda_j \geq 0;\ j = 1,\dots,14$ *and* θ *is the iput* $-$*oriented efficiency measurement score for DMU*

947.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

j

*DMU*1 STAR LIFE *Minimize*

subject to :

5906.7 192.7 1597.8 1345.6 7 2694.4

84.8 1883.5 50.6 443.1 215.7 277.5 760.4 86.3 226.1 138.1 32.4 434.6 117.2 13.8 8 277.5

9 10 11 12 13 14

1 2 3 4 5 6

420.6 1011.1 28.7 492 258.1 163.4 1629 86.4 588.7 1.3 23.7 1096.2 326.1 75.4 7 163.4

8

947.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

33596.8.1 13102607..9921944..12 862249..86197389.1.61264.32770.2 223.1 589.5 219.6 237.5 3860112.4

38470..9622217.6.3 76 1264170..33

170.3 328 37.5 94.7 11.9 51.7

507.21365.4 86.9 640.8 313.4 858.9 1603.7 270.5 538.8 305.1 389.5 876.9 308.9191.9 98

858.9

$\lambda_j \geq 0;\ j = 1,\ldots,14$ *and* θ *is the iput-oriented efficiency measurement score for DMU*$_j$

947.6

2694.4

7173.8

401.3

2519.9

81.2

389.5

3159.5

582.9 207.8

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

DMU$_1$ (*SIC LIFE*)

Minimize θ *subject to* :

84.8 5906.7 192.7 1597.8 215.7 277.5 226.1 138.1 32.4 434.6 117.2 7173.8

1883.5 50.6 443.1 760.4 86.3 13.8 760.4

420.6 1011.1 163.4 1629 23.7 1096.2 326.1

588.7 75.4 1629.0

1264.3 589.5 219.6 237.5

170.3 94.7 11.9 51.7

858.9 538.8 305.1 389.5

28.7 492 258.1 86.4 1.3 2770.2

596.1 2607.9 219.2 862.8 1973.6 33.8 2770.2 223.1

1310.9 44.1 249.6 89.1 328 37.5 328.0

1603.7 270.5

507.2 1365.4 86.9 640.8 313.4 222.3

17.6

3860.4 384.6 191.9 1603.7

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

112 70.9

876.9 308.9

$\lambda_j \geq 0;\ j = 1,\ldots,14$ *and* θ *is the iput* $-$*oriented efficiency measurement score for DMU* $_j$

DMU$_1$ = *UNIQUE LIFE*: *Minimize* θ

subject to :

947.6 5906.7 192.7 1597.8 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8 401.3 84.8 1883.5 50.6 443.1 215.7 277.5 760.4 86.3 226.1 138.1 32.4 434.6 117.2 13.8 86.3

420.6 1011.1 28.7 492 258.1 163.4 1629 86.4 588.7 1.3 23.7 1096.2 326.1 75.4 7 86.4

33596.8.1 13102607..9921944..12 862249..86197389.1.61264.32770.2 223.1 589.5 219.6 237.5 3860.4

384.6222.3 76 22337.5.1

170.3 328 37.5 94.7 11.9 51.7 112 70.9 17.6

507.21365.4 86.9 640.8 313.4 858.9 1603.7 270.5 538.8 305.1 389.5 876.9 308.9191.9 98

270.5

947.6

1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

$\lambda_j \geq 0;\ j = 1,\ldots,14$ *and* θ *is the iput* $-$*oriented efficiency measurement score for DMU* $_j$

DMU$_1$ $=$ *VANGUARD LIFE* $\Rightarrow$ *Minimize* θ *subject to* :

84.8 5906.7 192.7 1597.8 215.7 277.5 226.1 138.1 32.4 434.6 117.2 2519.9

1883.5 50.6 443.1 760.4 86.3 13.8 226.1

420.6 1011.1 163.4

28.7 492 258.1 86.4 588.7 1.3 75.4 588.7

1629 23.7 1096.2 326.1 □□8 □ □

□□9 □□

□

□□10□

□ □

□□11□

□□12□

□ □

□□13□

□□14□□

□

□□596.1 2607.9 219.2 862.8 1973.6 □ 33.8 2770.2 223.1

1310.9 44.1 249.6 89.1 1264.3 328 37.5 589.5 219.6 237.5

□ 170.3 1603.7 270.5 94.7 11.9 51.7

□□ 507.2 1365.4 86.9 640.8 313.4 858.9 538.8 305.1 389.5

3860.4 384.6 □□1 □

112 70.9 □ □

876.9 308.9 □

□ 2 □

□□3 □

□ □

□□4 □

□□5 □

□ □

□□6 □

□□□ □ □589.5□

□□ 7 □□□□ 94.7 □

□□□8 □ □ □

□□□□9 □□

222.3 □□538.8□□

17.6 □

191.9 □□10□

□ □

□□11

□

□□12

□

□ □

□□13

□

□□□14□ □

□j □ 0; j □1,...,14 *and □ is the iput □oriented efficiency measurement score for DMU j*

947.6

1345.6

2694.4

7173.8

401.3

2519.9

81.2

389.5

3159.5

582.9 207.8

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

DMU$_1$ *EXPRESS LIFE* *Minimize* *subject to* :

84.8 5906.7 192.7 1597.8 50.6 277.5 226.1 138.1 32.4 434.6 117.2 13.8 81.2

1883.5 443.1 215.7 760.4 86.3 138.1

420.6 1011.1

75.4

28.7 492 258.1 163.4 86.4 588.7 1.3 1.3

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

1629 23.7 1096.2 326.1

596.1 2607.9 219.2 862.8 1973.6 33.8
1310.9 44.1 249.6 89.1

1264.3 2770.2 223.1
328 37.5 589.5 219.6 237.5
170.3 1603.7 270.5 94.7 11.9 51.7
858.9 538.8 305.1 389.5

507.2 1365.4 86.9 640.8 313.4

3860.4 384.6
112 70.9
876.9 308.9

219.6
11.9

222.3 305.1
17.6
191.9

947.6

1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \ldots, \lambda_{12}, \lambda_{13}, \lambda_{14}$

$\lambda_j \geq 0;\ j = 1,...,14$ *and θ is the iput oriented efficiency measurement score for DMU* $_j$

DMU$_1$ *= PHOENIX LIFE: Minimize θ subject to :*

5906.7 192.7 1597.8 50.6 443.1 760.4 86.3 7 389.5

1 2 3 4 5 6

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

84.8 1883.5 215.7 277.5 226.1 138.1 32.4 434.6 117.2 13.8 8 32.4

9 10 11 12 13 14

1 2 3 4 5 6

420.6 1011.1 163.4 1629 23.7 1096.2 326.1

588.7 75.4 7 23.7

8 9 10 11 12 13 14

1264.3 589.5 219.6 237.5
170.3 94.7 11.9 51.7
858.9 538.8 305.1 389.5

1 2 3 4 5 6 237.5

28.7 492 258.1 86.4 1.3

596.1 2607.9 219.2 862.8 1973.6 33.8 2770.2 223.1

1310.9 44.1 249.6 89.1 328 37.5

1603.7 270.5

7 51.7 8

507.2 1365.4 86.9 640.8 313.4

9

3860.4 384.6 222.3 389.5
112 70.9 17.6
876.9 308.9 191.9

10 11 12

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

$\lambda_j \geq 0; j = 1,...,14$ *and* θ *is the iput* $-$*oriented efficiency measurement score for* DMU_j

DMU$_1$ *GLICO LIFE* *Minimize* θ

subject to :

84.8 5906.7 192.7 1597.8 215.7 277.5 226.1 138.1 32.4 434.6 117.2 3159.5

1883.5 50.6 443.1 760.4 86.3 13.8 434.6

420.6 1011.1 163.4

28.7 492 258.1 86.4 588.7 1.3

1 2 3 4 5 6

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

1629 23.7 1096.2 326.1 75.4 7 1096.2

8 9 10 11 12 13 14

596.1 2607.9 219.2 862.8 1973.6 33.8
1310.9 44.1 249.6 89.1

1264.3 2770.2 223.1 589.5 219.6 237.5
170.3 328 37.5 94.7 11.9 51.7
858.9 1603.7 270.5 538.8 305.1 389.5

507.2 1365.4 86.9 640.8 313.4

3860.4 384.6
112 70.9
876.9 308.9

1 2 3 4 5 6

6860.4
7 112.0
8
9 876.9

222.3
17.6
191.9

10 11 12 13 14

947.6 1345.6 2694.4 7173.8

401.3 2519.9 81.2

389.5 3159.5 582.920

7.8

$\lambda_j \geq 0;\ j = 1,...,14$ *and* θ *is the iput* $-$*oriented efficiency measurement score for* DMU_j

DMU_1 $-$ *GHANA LIFE* $-$ *Minimize* θ

subject to :

5906.7 192.7 1597.8

50.6 443.1 760.4 86.3 7

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

84.8 1883.5 215.7 226.1 138.1 32.4 434.6 117.2 13.8

582.9

117.2

277.5

420.6 1011.1 163.4 23.7 1096.2 326.1

588.7 75.4 326.1

1629

589.5 219.6 237.5

94.7 11.9 51.7

538.8 305.1 389.5

684.6

28.7 492 258.1 86.4 1.3

596.1 2607.9 219.2 862.8 1973.6 33.8 2770.2 223.1

1310.9 44.1 249.6 89.1 328 37.5

1603.7 270.5

70.9

507.2 1365.4 86.9 640.8 313.4

1264.3 3860.4 384.6 222.3 308.9

170.3 112 70.9 17.6

858.9 876.9 308.9 191.9

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

λ_1 λ_2 λ_3 λ_4 λ_5 λ_6 λ_{13} λ_{14}

$\lambda_j \geq 0; j = 1,...,14$ *and θ is the iput oriented*

efficiency measurement score for DMU $_j$ DMU$_1$ (UT LIFE)

Minimize θ subject to :

5906.7 192.7 1597.8 7 207.8

50.6 443.1 760.4 86.3

1
2
3
4
5
6

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

84.8 1883.5 215.7 277.5 226.1 138.1 32.4 434.6 117.2 13.8 8 13.8

9
10
11
12
13
14

1
2
420.6 1011.1 163.4 1629 23.7 1096.2 326.1
3
4
5
6
7

492 588.7 75.4 75.4

8
9

1264.3 589.5 219.6 237.5 10
170.3 94.7 11.9 51.7 222.3
858.9 538.8 305.1 389.5 11

12
13
14

1
2
384.6 3
70.9
3860.4 308.9 4
112 5
876.9

28.7 258.1 86.4 1.3
862.8 1973.6 2770.2 223.1 222.3 6

947.6 1345.6 2694.4 7173.8 401.3 2519.9 81.2 389.5 3159.5 582.9 207.8

596.1 2607.9 219.2 249.6 89.1 328 37.5 17.6 17.6

33.8 1310.9 44.1 640.8 313.4 1603.7 270.5

507.2 1365.4 86.9 191.9 191.9

is the iput oriented efficiency measurement score for DMU $_j$

$\lambda_j \geq 0;\ j = 1,\ldots,14$ *and*

This is replicated for each Decision making unit for each year. The equation above represents that of all DMU$_s$ for 2010. For the entire period there are a total of 56 equations in matrix representation. Scaled data has been used in model construction and analysis.

CHAPTER 4

DATA ANALYSIS AND RESULTS

We present the results of the study in this chapter vis-à-vis technical, scale, pure, and overall efficiencies. Results of the tests of hypotheses concerning the dimension and market shares of Ghanaian Life Insurance Companies on their efficiencies using the Mann Whitney U test are also presented.

The data used for the analysis can be found in Appendix A.

4.1 Descriptive Statistics

The descriptive statistics of the outputs and inputs of the life insurance companies are presented here. Within the period of analysis, SIC life and Enterprise life had the highest amount of output: Enterprise life recorded the highest investment income and SIC life maximum in net premiums and claims. Express life recorded the lowest in all the outputs. For the inputs, Enterprise recorded the highest in capital and commission while SIC life recorded the highest in management expenses. Done well life had the lowest capital; Glico life recorded the minimum in management expenses and Express life had the minimum commission.

Table 4.1: Life insurer input/output data statistics for the period of study, 2010-2013

Statistics	Life Insurer Input Data (GH¢)			Life Insurer Output Data (GH¢)		
	Capital	Commission	Management Expenses	Net Premiums	Investment Income	Claims Incurred
Minimum	(8,563,608)	79,300	26,783	541,202	90,063	8,427
Maximum	58,090,930	23,934,676	21,406,152	126,790,109	57,083,110	68,845,574
Mean	10,345,513	2,250,869	5,593,972	22,158,520	4,565,593	7,430,197
Standard Deviation	14732218.08	4374661.357	4956418.194	29697385.89	8853863.397	11848814

4.2 Overall Efficiencies of Ghanaian Life Insurers

Briefly we present the overall efficiencies of the life insurers for the period of study. The overall efficiencies overall efficiencies are obtained when the constant returns to scale assumption is made. That is, to say the performance of an insurer continuous to increase if the insurer continuous to increase inputs. Table 4.2 and Figure 4.2 show the overall efficiencies of the insurers for the period of study.

Table 4.2: Overall Efficiencies of Ghanaian Life Insurers

Company	2010	2011	2012	2013
Donewell	1	1	1	1
Enterprise	1	0.949933431	1	1
Ghana Union	0.79067194	0.628365069	0.608904353	0.731629462
Metropolitan	0.68054626	0.661438165	0.905843013	1
Provident	0.77034	1	1	1
Star Life	1	1	0.779188703	0.746772121
SIC Life	0.88946718	1	0.953307258	1
Unique	0.69712064	0.750410244	0.731448272	0.769712271
Vangaurd	1	0.963144946	0.777982898	0.657893438
Express Life	1	1	1	0.513759159
Phoenix Life	0.84155562	1	0.615473771	0.582752263
Glico Life	1	1	1	1
Ghana Life	0.44218017	0.412715443	0.483802502	0.519251879
UT life	0.49016563	0.497615977	0.54881188	0.616718078

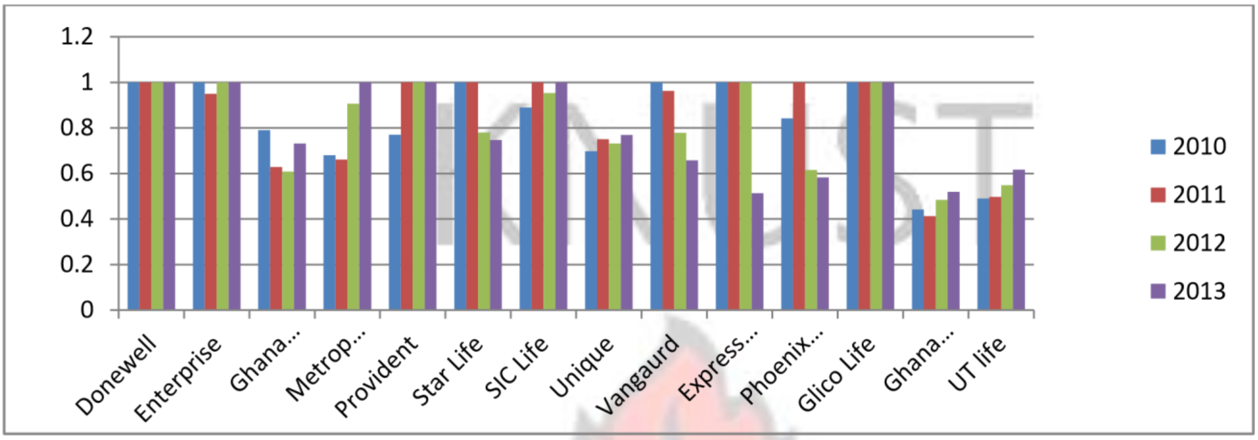

Figure 4.2: A Plot of Overall Efficiencies of Ghanaian Life Insurers for the period

We observe from Figure 4.2 and Table 4.2 that:

1. Ghana Life Insurance Company recorded the minimum overall efficiency for the period of study in 2011. This stood at 41% (0.4127).
2. Glico Life and Donewell Life recorded 100% consistently for the period of the study. Enterprise Life, Provident Life and Express Life also recorded 100% for three different years.
3. Though a few life insurers performed at 100% efficiency, majority of them underperformed during the study.
4. The average overall efficiency of the life insurers for the period of study stood at 82%. This is an indication that Ghanaian life insurers still need to explore avenues for improving and increasing efficiency.

4.3 Technical Efficiencies of Ghanaian Life Insurers

Here, the technical efficiencies of life insurers is briefly considered. The BCC model assumes a convex combination of observed DMUs as the efficient frontier and provides technical efficiency. If a unit is fully efficient in both the CCR and BCC models, it is operating in the most productive scale size (MPSS). If a DMU is BCC efficient but inefficient in the CCR model, then it is locally efficient but not globally. The constant returns to scale model identifies the overall efficiency whereas the variable returns to scale model differentiates between technical efficiency and scale efficiency [Charnes, 1978]. It is computed using the variable returns to scale hypothesis which assumes that firms do not necessarily increase their efficiency by simply increasing inputs. The variable returns to scale decompose overall efficiency into technical efficiency and scale efficiency. The technical efficiencies of the

Ghanaian life insurers are shown in Table 4.3 and graphically in Figure 4.3

Table 4.3: Technical Efficiencies of Ghanaian Life Insurers

Company	2010	2011	2012	2013
Donewell	1	1	1	1
Enterprise	1	0.951	1	1
Ghana Union	1	0.955223	0.826925	0.872068
Metropolitan	0.702164	0.671303	0.908986	1
Provident	0.843587	1	1	1
Star Life	1	1	0.801507	0.823239
SIC Life	0.893797	1	1	1
Unique Life	0.934659	0.914065	0.873697	0.83025
Vanguard	1	0.975725	0.785078	0.674492
Express Life	1	1	1	0.515762
Phoenix Life	1	1	0.688907	0.594028
Glico Life	1	1	1	1
Ghana Life	0.637626	0.501429	0.527924	0.53891
UT life	1	0.796827	0.635899	0.670237

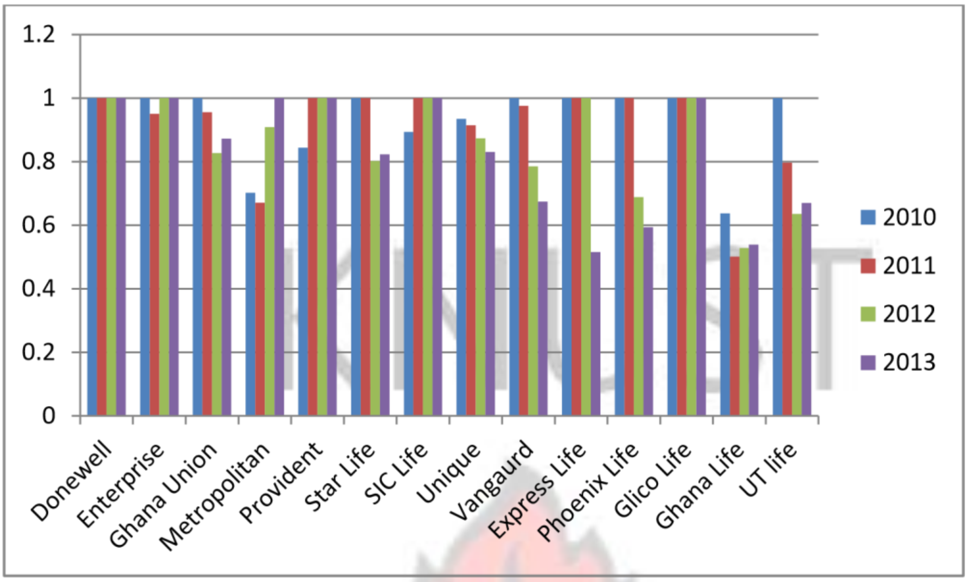

Figure 4.3: A Plot of Technical Efficiencies of Ghanaian Life Insurers for the period

We notice from Table 4.3 and Figure 4.3 that:

1. The minimum technical efficiency of the life insurers for the study period was recorded by Ghana Life in 2011 which stood at 50% (0.5014).
2. The life insurers demonstrated higher technical efficiencies than they did for overall efficiencies.
3. In 2010, Donewell Life, Enterprise Life, Star Life, Vanguard Life, Express Life and Glico Life operated in the most productive scale size whereas Donewell Life, Provident Life, Star Life, SIC Life and Glico Life were fully efficient in 2011. Similarly, in 2012, Donewell Life, Enterprise Life, Provident Life, Express Life and Glico Life were globally efficient while Donewell Life, Enterprise Life, Provident Life, SIC Life and Glico Life were fully efficient.
4. Donewell Life and Glico Life operated in the most productive scale size and thus were globally efficient for the period of study.

4.4 Scale Efficiencies of Ghanaian Life Insurers

Scale efficiency shows the effect of a DMU"s size on efficiency. It indicates inefficiency due to inappropriate size of a DMU and if a DMU moved towards the best size, the overall and technical efficiency can be improved at the same level inputs. It is the extent to which a company can take advantage of returns to scale by altering its size towards optimal scale (which is defined as the region in which there are constant returns to scale in the relationship between outputs and inputs). The scale efficiency of a firm can be computed by taking the ratio of the firm"s overall efficiency to its technical efficiency. The scale efficiencies of the Ghanaian life insurers as evaluated using the DEA model of Holiang and Michael [2007] are shown in table 4.4 and figure 4.4.

Table 4.4: Scale Efficiencies of Ghanaian Life Insurers

Company	2010	2011	2012	2013
Donewell	1	1	1	1
Enterprise	1	0.998878	1	1
Ghana Union	0.790672	0.65782	0.736348	0.838959
Metropolitan	0.969213	0.985304	0.996542	1
Provident	0.913172	1	1	1
Star Life	1	1	0.972154	0.907115
SIC Life	0.995155	1	0.953307	1
Unique	0.745856	0.820959	0.837188	0.927085
Vangaurd	1	0.987107	0.990962	0.975391
Express Life	1	1	1	0.996117
Phoenix Life	0.841556	1	0.893407	0.981018
Glico Life	1	1	1	1
Ghana Life	0.693479	0.823078	0.916425	0.963523
UT life	0.490166	0.624497	0.863049	0.920149

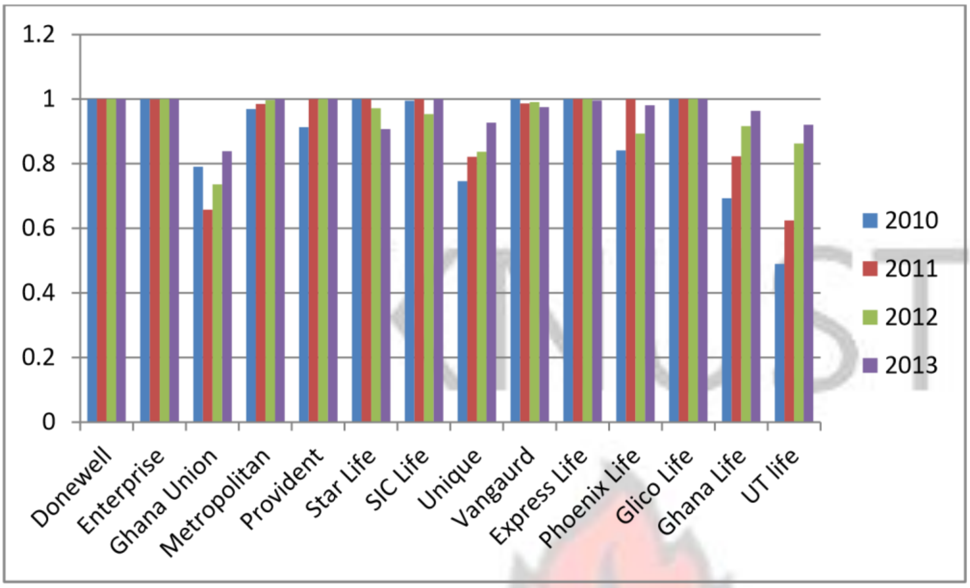

Figure 4.4: A Plot of Scale Efficiencies of Ghanaian Life Insurers for the period We observe from Table 4.4 and Figure 4.4 that:

1. The minimum scale efficiency was recorded by Ghana Union in 2011 which stood at 49 %(0.490166).
2. The life insurance companies exhibited higher scale efficiencies than they did for technical efficiencies. This implies that the various companies could increase their efficiency by among other things altering their size towards optimal scale.
3. The average scale efficiency of the life insurers for the period of study stood at 93%.This shows that, on the average, Ghanaian life insurers operated at increasing returns to scale for the period of the study.

4.5 Average Overall, Technical and Scale Efficiencies of Ghanaian Life Insurers

The average overall, technical and scale efficiencies of the life insurers for the period of the study are shown in Table 4.5 and graphically in Figure 4.5 that, apart from Ghana Life, all the other insurance companies performed above average. There is still need for improvement since they all operated below 100%.

Table 4.5: Overall, Technical and Scale Efficiencies of Ghanaian Life Insurers

Company	Overall efficiencies	Scale efficiencies	Technical efficiencies
Donewell	1	1	1
Enterprise	0.987483358	0.999719517	0.987750098
Ghana Union	0.689892705	0.7559499	0.913553862
Metropolitan	0.820613371	0.987764817	0.820613371
Provident	0.942585001	0.978292913	0.960896824
Star Life	0.881490206	0.969817269	0.906186515
SIC Life	0.960693609	0.98711566	0.973449322
Unique	0.737172857	0.83277208	0.888167523
Vangaurd	0.849755321	0.988365046	0.858823847
Express Life	0.87843979	0.999029343	0.878940418
Phoenix Life	0.759945413	0.928995163	0.820733624
Glico Life	1.000000001	1.000000001	1
Ghana Life	0.464487498	0.849126495	0.551471996
UT life	0.53832789	0.724465077	0.77574085
Average	0.822206216	0.928672377	0.881166304

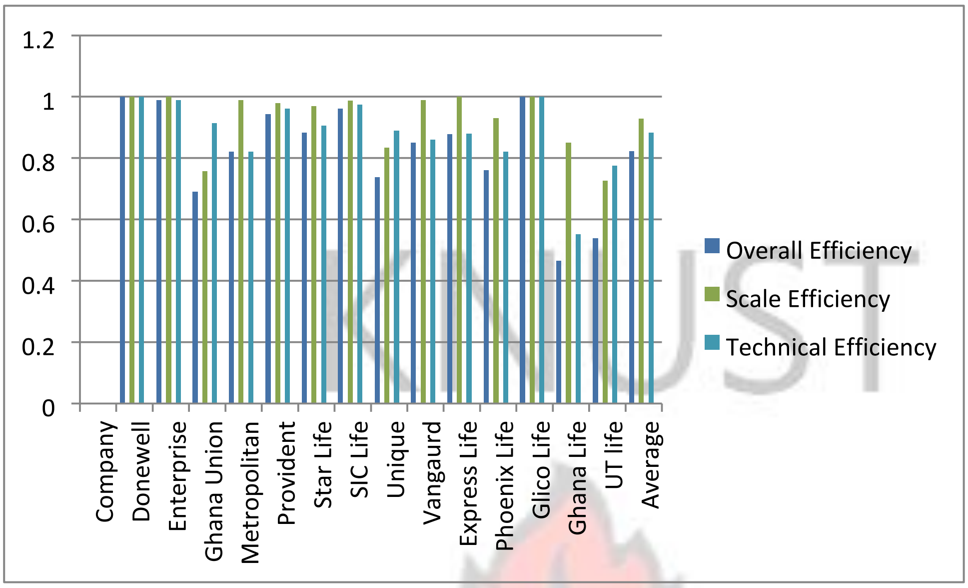

Figure 4.5: A Plot of Average Overall, Technical and Scale Efficiencies of Ghanaian Life Insurers

We observe from Figure 4.5

1. The scale efficiency scores are higher than the technical and overall efficiency scores. This implies that Ghanaian life insurers" inefficiencies are largely due to inefficient management operations.
2. Donewell Life and Glico Life outperformed the other companies. These relative inefficient companies can improve by observing the mode of operations of these efficient ones and implementing those things contributing to the efficiency of the two companies.

4.6 Effects of Dimension and Market Share on Insurer Efficiency

We seek to test some hypotheses related to the efficiency scores of the insurance companies obtained. The Mann Whitney U test is utilized. The Mann Whitney U test tests whether two samples are from the same population and is recommended for the non- parametric analysis of DEA results. It is used because the efficiency scores here do not fit within a standard normal distribution. We use the overall efficiency scores for the test.

4.6.1 Effect of Dimension on Insurer Efficiency

To test this hypothesis, we classify the life insurance companies by capital and then divide the sample into two subsets.

(i) H_0: Large life insurers are not more efficient than small life insurers H_1: Large life insurers are more efficient than small life insurers.

(ii). Significance level: $\alpha = 0.05$

(iii). Rejection Region: This is a two-tailed test with $n_1=7$ and $n_2=7$. Reject the null hypothesis if p-value < 0.05

(iv). The test statistic, p-value= Assymp.Sig.(2-tailed) = 0.142

Table 4.6.1: Mann Whitney U test on differences in Life Insurers based on Dimension

Ranks

	Dimension	N	Mean Rank	Sum of Ranks
Overall Efficiencies	Large	7	9.14	64.00
	Small	7	5.86	41.00
	Total	14		

Test Statistics

	Overall Efficiencies
Mann-Whitney U	13.000
Wilcoxon W	41.000
Z	-1.469
Asymp. Sig. (2-tailed)	.142
Exact Sig. [2*(1-tailed Sig.)]	.165[b]

(v). Decision: Since the p-value= 0.142 >0.05=α, we retain the null hypothesis.

(vi). Conclusion: At the α =0.05 level of significance, there is enough evidence to show that large life insurers in terms of capital are not more efficient than small life insurers.

4.6.2 Effect of Market share on Insurer Efficiency

To test this hypothesis, the insurance companies are classified by net premiums to determine estimated market share. We further divide the sample into two constituting large and small insurers.

(i). H_0: Life insurers with higher market shares are not more efficient than life insurers with lower market shares.

H_1: Life insurers with higher market shares are more efficient than those with lower shares.

(ii). Significance level: α=0.05

(iii).Rejection region: This is a two-tailed test with n_1=7 and n_2=7.Reject the null hypothesis if the p-value < 0.05.

(iv). The test statistic, p-value= Assymp. Sig.(2-tailed) = 0.035

Table 4.6.2: Mann Whitney U test on differences in Life Insurers based on Market share

Ranks

	Market share	N	Mean Rank	Sum of Ranks
Overall efficiencies	Large	7	9.86	69.00
	Small	7	5.14	36.00
	Total	14		

Test Statistics

	Overall efficiencies
Mann-Whitney U	8.000
Wilcoxon W	36.000
Z	-2.108
Asymp. Sig. (2-tailed)	.035
Exact Sig. [2*(1-tailed Sig.)]	.038[b]

(v), Decision: Since the p-value = 0.035< 0.05= α, we reject the null hypothesis.

(vi). Conclusion: At the α = 0.05 level of significance, there is enough evidence to show that life insurers with higher market shares are more efficient than life insurers with lower market shares.

Thus, from the above hypotheses tests performed at 5% significance level, we can state that:

1. Large life insurers in terms of capital do not necessarily tend to have higher efficiency than small life insurers. Conclusions about size efficiency could be significantly affected by methodological choice. Yuengert (1993) found that efficiency and size were statistically unrelated.
2. Life insurers with higher market shares do tend to more efficient than those with lower market shares. This result is consistent with previous study in the insurance industry [4, 6] and the findings of Gardner and Grace (1993).

CHAPTER 5

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary

The study revealed that Ghanaian life insurance companies with higher scale-size are relatively efficient which is evidenced by the scores in table 4.4. The average scale efficiency or the Ghanaian life insurance companies stood at 93% whereas the average technical efficiency stood at 88%. This indicates that although life insurers operated at high technical efficiency, the life insurers operated with higher scale efficiencies. Thus, there is need for Ghanaian life insurers to develop and upgrade their technical and managerial skills to enable them get to the efficient frontier.

Also, we note that large life insurers with high capital do not necessarily tend to have higher efficiency scores than life insurers with lower capital. Moreover, life insurers with higher market shares tend to be more efficient than life insurers with lower market shares, an effect that is explained by economies of scale in this particular activity [Owusu-Ansah etal, 2010; Cummins and Zi,1998].

5.2 Conclusions

In this research, we sought to evaluate relative efficiencies of Ghanaian life insurance companies. We used Data Envelopment Analysis, a mathematical programming tool that allows for the incorporation of multiple inputs and outputs in determining relative efficiencies of life insurance companies. The general conclusion is that a good number of the life insurers operated with relatively high managerial skills and higher scale efficiency. Also size-efficiency of Ghanaian life insurance companies was significantly affected by choice of classification and efficiency and size are statistically unrelated.

5.3 Recommendations

We give the following recommendations

1. Though a good number of life insurance companies are doing well in Ghana, they can improve their performance by employing managerial best practices from top performers and increase efficiency through altering their size towards optimal scale.
2. There is need for more research into performance and productivity analysis of the life insurance industry as well as other areas of the financial sector such as the banking industry. This will enlighten the public about the life insurance industry in Ghana and also help life insurer know the opportunities and threats to their effectiveness and efficiency to enable them position themselves for higher performance
3. Data Envelopment Analysis should be employed as a technique for the measuring the efficiency and productivity of government service delivery such as health, education, etc.

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APPENDIX A

Data Used in the Study

Company	Capital	Commission	Management expenses	Net Premiums	Investment Income	Claims incurred
Donewell	(3,840,526)	225,026	3381142	6317379	565471	2804248
Enterprise	17,385,799	8,739,098	9102447	39377727	12556776	6740895
Ghana Union	1,461,597	293,940	579255	1284369	337209	191299

Metropolitan	5,752,000	1,664,000	4272000	10652000	2954000	3280000
Provident	13,157,606	593,716	2089038	8970448	1437908	1720592
Star Life	8,428,655	1,135,656	5726656	17962779	1850076	1089173
SIC Life	18,468,028	2,186,936	10691090	47825527	5069108	10860265
Unique	1,487,488	250,284	1803281	2675531	575083	576301
Vangaurd	3,930,182	631,195	3591691	16799683	1507497	3924947
Express Life	1,464,307	79,300	2033984	541202	920978	8427
Phoenix Life	1,583,134	344,756	2596494	2596494	216303	158284
Glico Life	25,735,743	746,887	5846012	21063014	2897566	7307842
Ghana Life	2,564,330	473,041	2059760	3886491	781392	2173875
UT life	1,481,848	117,147	1279642	1385578	92033	502893

TableA.1: Input/output Data of life insurers for 2010 (GH¢) TableA.2: Input/output Data of life insurers for 2011 (GH¢)

Company	Capital	Commission	Management expenses	Net Premiums	Investment Income	Claims incurred
Donewell	(6,110,671)	289,447	2669439	7065168	922859	3150348
Enterprise	25,939,529	13,740,608	13275345	60608946	4049455	10463796
Ghana Union	1,410,787	276,383	727960	1292772	281793	207573
Metropolitan	5,533,000	1,996,000	4956000	13825000	3712000	5322000
Provident	24,109,813	668,758	2620632	15991601	1685233	1374585
Star Life	9,117,664	1,762,941	7654803	27249777	2486483	2615414

SIC Life	21,304,013	2,777,233	12489574	71494097	6980012	15238945
Unique	361,351	262,489	1987398	3299766	620808	940230
Vangaurd	3,948,275	563,267	4052328	13212747	2668415	6382227
Express Life	1,571,981	86,226	6451823	1164195	886500	39095
Phoenix Life	1,858,445	558,556	3562146	3562146	314334	155147
Glico Life	29,392,766	1,234,574	7035977	34215551	3097366	12858276
Ghana Life	4,271,278	781,876	3456993	6168288	637538	2508700
UT life	1,550,605	283,805	1984663	2458784	201838	596917

TableA.3: Input/output Data of life insurers for 2012 (GH¢)

Company	Capital	Commission	Management expenses	Net Premiums	Investment Income	Claims incurred
Donewell	(7,841,172)	272,522	8382557	7073375	2046557	5358224
Enterprise	39,718,819	18,643,122	16608983	87481395	17970770	15506699
Ghana Union	1,562,747	434,853	867105	1975773	464054	422836
Metropolitan	6,302,000	2,555,000	6148000	19383000	6359000	5454000
Provident	6,092,823	713,822	2870894	15713378	3750936	4572069
Star Life	10,678,128	2,974,549	10210546	39687950	4398579	11403633
SIC Life	26,684,132	4,324,558	18409188	99660986	14229886	43196739

Unique	492,623	252,664	2061881	3409919	566569	958421
Vangaurd	4,636,875	683,966	5218587	12271055	3136177	6572310
Express Life	7,638,244	115,009	2033984	1781307	90063	92398
Phoenix Life	2,013,293	830,990	2483740	5841694	826181	1638980
Glico Life	47,142,000	1,936,000	26783	35970000	6131000	14149000
Ghana Life	3,457,162	967,534	4068982	8278234	809000	3202148
UT life	5,995,912	725,445	3468816	6337182	664548	815682

TableA.4: Input/output Data of life insurers for 2013 (GH¢)

Company	Capital	Commission	Management expenses	Net Premiums	Investment Income	Claims incurred
Donewell	(8,563,608)	301,512	8479128	8060266	3148623	4867158
Enterprise	58,090,930	23,934,676	20048637	122331166	57083110	32217309
Ghana Union	1,672,041	947,978	1098542	3515811	722089	833844
Metropolitan	9,922,000	3,313,000	6113000	21161000	12071000	10285000
Provident	3,332,410	835,096	4167679	15308194	5381644	15395554
Star Life	15,799,649	4,020,286	12698590	53211010	8238867	24447156
SIC Life	37,651,373	6,267,473	21406152	126790109	30340078	68845574
Unique	209,599	194,585	2754155	3691697	924675	1689678

Vangaurd	5,810,291	824,838	4982339	11069780	3241398	7239814
Express Life	3,563,752	1,361,560	6451823	5375023	401515	986454
Phoenix Life	2,399,784	788,088	3134847	6892858	1700246	3395073
Glico Life	56,727,000	2,253,000	40033	53710000	7471000	23367000
Ghana Life	3,820,588	1,445,350	4664664	10533671	1140978	4107884
UT life	11,020,323	1,368,034	6385250	11414222	2058636	1878118

APPENDIX B

MATLAB Codes Used for the Study

```
Function    [Windows Average Windows table]=...

WindowHongliangMichael CRS (X, Y_d,Y_u, DMUs,Years)

p=6; w=3; n=14; wr=(p-w+1)*n; nw=p-w+1;

%p is the number of windows, n is number of DMUs, wr is the number of

%individual DMUs in a window and nw is

Window=cell (p-w+1,p,n);
```

%Create a cell array of empty entries to contain window results

NumericWindow=zeros (p-w+1,p,n); for i=1:p-w+1

% For every window Compute the efficiency scores and assign it to efficiency efficiency=

HongliangMichaelCRS (X(1+n*(i-1):n*(w+i-1),:),...

Y_d (1+n*(i-1):n*(w+i-1),:),Y_u(1+n*(i-1):n*(w+i-1),:));

Window1=[efficiency(1:n,:) efficiency(n+1:2*n,:) efficiency(2*n+for j=1:n

%Decompose the efficiencies obtained above into their respective Window (i,i:w+i1,j)=num2cell(Window1(j,:));

Numeric Window(i,i:w+i-1,j)=Window1(j,:);

end end

Windows=Window (:,:,1);

Numeric Windows=Numeric Window(:,:,1);

for i=2:n

Windows= [Windows; Window(:,:,i)];

Numeric Windows=[NumericWindows;NumericWindow(:,:,i)];

end temp1=cell(n*(p-w+1)+n,1); for i=1:n

temp1{1+(1+nw)*(i-1)}=DMUs{i+1};

temp1{(1+nw)*i}="Moy/An"; end for

i=1:n for j=1:p

WindowsAverage(i,j)=mean(NumericWindows(...find(NumericWindows(1+nw*(i-1):nw*i,j)>0)+nw*(i-1),j)); end temp2{i}=[Windows(1+nw*(i-

1):nw*i,:);num2cell(WindowsAverage(i,:))]; end temp3=temp2{1};

```
for i=2:n temp3=[temp3;temp2{i}]; end temp3=[temp1
temp3];temp3=[[DMUs(1) num2cell(Years)];temp3];
Windowstable=temp3;
function[WindowsAverageWindowstable]=...
WindowHongliangMichaelVRS(X,Y_d,Y_u,DMUs,Years)
p=6;w=3;n=14;wr=(p-w+1)*n;nw=p-w+1;
%Create a cell array of empty enteries to contain window results
Window=cell(p-w+1,p,n); NumericWindow=zeros(p-w+1,p,n);
for i=1:p-w+1 efficiency=HongliangMichaelVRS(X(1+n*(i-
1):n*(w+i-1),:),...
Y_d(1+n*(i-1):n*(w+i-1),:),Y_u(1+n*(i-1):n*(w+i-1),:));
Window1=[efficiency(1:n,:) efficiency(n+1:2*n,:) efficiency(2*n+for j=1:n
Window(i,i:w+i-1,j)=num2cell(Window1(j,:));
NumericWindow(i,i:w+i-1,j)=Window1(j,:);
end end
Windows=Window (:,:, 1);
NumericWindows=NumericWindow(:,:,1);
for i=2:n
Windows=[Windows;Window(:,:,i) ];
NumericWindows=[NumericWindows;NumericWindow(:,:,i) ]; end
```

```
temp1=cell(n*(p-w+1)+n,1); for

i=1:n

temp1{1+(1+nw)*(i-1)}=DMUs{i+1};

temp1 {(1+nw)*i}="Mean"; end for

i=1:n for j=1:p

WindowsAverage(i,j)=mean(NumericWindows(find...(NumericWindows(1+nw*(i1):nw*i
,j)>0)+nw*(i-1),j)); end temp2{i}=[Windows(1+nw*(i-

1):nw*i,:);num2cell(WindowsAverage(i,:))]; end temp3=temp2{1};

for i=2:n temp3=[temp3;temp2{i}]; end temp3=[temp1

temp3];temp3=[[DMUs(1) num2cell(Years)];temp3];

Windowstable=temp3;

function
[WindowsAverageWindowstable]=...WindowHongliangMichaelScale(X,Y_d,Y_u,DMUs,

Years) p=6; w=3;n=14;wr=(p-w+1)*n;nw=p-w+1;

Window=cell(p-w+1,p,n);%Create a cell array of empty enteries to contain 59

NumericWindow=zeros(p-w+1,p,n); for i=1:p-w+1

efficiency=HongliangMichaelScale(X(1+n*(i-1):n*(w+i-1),:),...

Y_d(1+n*(i-1):n*(w+i-1),:),Y_u(1+n*(i-1):n*(w+i-1),:));

Window1=[efficiency(1:n,:) efficiency(n+1:2*n,:) efficiency(2*n+for j=1:n

Window(i,i:w+i-1,j)=num2cell(Window1(j,:));

NumericWindow(i,i:w+i-1,j)=Window1(j,:);

end end
```

```
Windows=Window(:,:,1);

NumericWindows=NumericWindow(:,:,1);

for i=2:n

Windows=[Windows;Window(:,:,i) ];

NumericWindows=[NumericWindows;NumericWindow(:,:,i) ];

end temp1=cell(n*(p-w+1)+n,1); for i=1:n

temp1{1+(1+nw)*(i-1)}=DMUs{i+1};

temp1{(1+nw)*i}=‟Moy/An‟; end for

i=1:n

for j=1:p

WindowsAverage(i,j)=mean(NumericWindows(find...(NumericWindows(1+nw*(i-
1):nw*i,j)>0)+nw*(i-1),j)); end temp2{i}=[Windows(1+nw*(i-

1):nw*i,:);num2cell(WindowsAverage(i,:))]; end temp3=temp2{1};

for i=2:n temp3=[temp3;temp2{i}]; end temp3=[temp1

temp3];temp3=[[DMUs(1) num2cell(Years)];temp3];

Windowstable=temp3;

%We seek to develop a code that generates latex tables from tables

%imported from Excel

% To use this function first launch notebook so as to get the table % on word for you to copy and paste in your latex file.

function excel2latex1(A,label,caption)

ifisnumeric(A) A=num2cell(A); end
```

```
[r,c]=size(A);

p=1;columnstyle=‟l‟; for i=1:c-1

columnstyle=[columnstyle ‟l‟];

end deleteTableName.tex

diaryTableName.tex

disp([‟\begin{longtable}{‟

columnstyle ‟}‟])

disp(strcat(‟\caption[‟,caption,‟]{‟

,caption,‟}\\‟))

disp(‟\endfirsthead‟)

disp(‟\caption{(continued)}\\‟)

disp(‟\hline‟)

disp(‟\endhead\hline‟)

disp(‟\endfoot\hline‟)

disp(‟\endlastfoot\hline‟)

disp(strcat(‟\label{‟,label,‟}‟)) for

i=1:r temp=‟‟‟‟; for j=1:c-1 if

(isnan(A{i,j})) temp=[temp ‟&‟];

else ifisnumeric(A{i,j})

temp=[temp

num2str(A{i,j}) ‟&‟];
```

```
else temp=[temp A{i,j}

‟&‟]; end end end

if(isnan(A{i,c}))

temp=[temp ‟\\‟]; else

ifisnumeric(A{i,c})

temp=[temp

num2str(A{i,c}) ‟\\‟];

else temp=[temp

A{i,c} ‟\\‟]; end end

disp(temp) end

disp(‟\end{longtable}‟

) end
```