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STRUCTURE TO POLYNOMIAL FUNCTORS IN ORTHOGONAL CALCULUS

BY

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DECLARATION

I declare that this dissertation, which I submit to Kwame Nkrumah University of Science and Technology in consideration of the award of the M. Phil degree, is my own personal e ort, except where due acknowledgement has been made in the text.

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DEDICATION

I humbly dedicate this independent research work to my family especially my loving parents, Alice and Daniel Lawrence osei who has been a source of encouragment throughout my research work.

ABSTRACT

In the study, some algebraic structures and maps (category theory, morphisms and functors) that are inherently tied to the calculus of functors (orthogonal calculus) were explored. I emphasized on linear polynomial functors and generalized it to the n- polynomial functors as in the algebraic and topological settings. Finally some structures of the polynomial and homogeneous functors were analyzed in the orthogonal calculus of functors.

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CONTENTS

LIST OF ABBREVIATION

Top .... Topological space Top_*

.... Pointed Topological Space

Aut ....Automorphism

Obj .... Object Cr_n

....n-th cross e ect Mod_R

.... R-Module grAb

.... Great Abelian Group sp

.... Spectra weq

....weak equivalence dim

.... Dimension holim

.... Homotopy Limit CX

.... Cone Emb

....Embedding mor

....Morphism

LIST OF FIGURES

CHAPTER 1

Introduction

This chapter talks about the history of functors and the main contributers to the development of functors. Also included is the objective of the study as well as the general structure of the entire thesis.

1.1 A Brief Historical Background to the Calculus of Functors

In mathematics especially in the algebraic and di erential topology, the functor calculus I.e. the Goodwillie calculus of homotopy functor, the Weiss calculus of functor and nally the embedding calculus is a technique to studying functors. These functors are well studied by approximating them with sequence of simpler functors. These sequence of approximation is almost same as the Taylor series of smooth function.

There are many objects in algebraic and di erential topology which can be seen as functors. They maybe functors although but it's always di cult to analyze directly, so we think of replacing them with simpler functors which are su ciently good approximation for the functor in question.

The calculus of functors was developed by a mathematician known as Thomas Goodwillie. Goodwillie came out a series of three papers on the calculus of functors(Goodwillie, 1990)(Goodwillie, 1991)(Goodwillie, 2003) in the 1990s and 2000s .He had his inspiration from the work done by Eilenberg and Mac Lane on functors in the 1940s.

This calculus of functors is known as the Goodwillie calculus of homotopy functor which has been the source of motivation for the other calculus of functors.

Micheal Weiss calculus of functor emerged after the papers of Goodwillie were published which is known as the orthogonal calculus of functors, due to (Weiss, 1995) , and this theory is closely related to or he had his inspiration from the Goodwillie calculus of homotopy functor. The orthogonal calculus of homotopy functor is a beautiful tool for calculating the homotopical properties of functors

from the category of vector spaces to pointed spaces or any space enriched over Top_*. With the Weiss calculus we consider covariant funtor from the category of vector spaces(nite dimensional) with an inner product to the category of spaces(Top_*) instead of functors from spaces to spaces as de ned by Goodwillie(Weiss, 1995)

There are Intriguing examples of such functors and they include classical objects in algebraic and geometric topology:

$BAut(V)$

$BTop(V)$

$\Omega^V \left(S^V \Lambda X\right)$

Category of such functors from vector spaces to spaces and natural transformations between them will be called ξ_0. Orthogonal calculus is based on the notion of n polynomial functors (vector spaces at very high dimension), which are well-behaved functors in ξ_0 and which preserves weak equivalences as well. With These n-polynomial functors one can often infer the value at some vector spaces from the values at vector spaces of higher dimension(Barnes and Oman, 2013) . In general sense, orthogonal calculus approximates a functor (locally around) via polynomial functors (approximate into sequence of simpler functors that are homotopy equivalent to the functor in question) and attempts to reconstruct the global functor from the associated in nitesimal information. The orthogonal calculus splits a functor F into a Taylor tower of brations, where we can think of the n-th brations to consist of an arrow(map) from the n polynomial approximation of F to the $(n -$

$1)$−polynomial approximation of F. The homotopy ber or layer (the di erence between n polynomial and $(n-1)$−polynomial approximation) of this map is then an n homogeneous functor and is classi ed by an $O\ (n)$-spectrum up to homotopy which is usually denoted as D_nF.

1.2 Problem Statement

The calculus of functors (orthogonal calculus) can help in solving a lot of interpolating problems from algebraic topology and di erential topology and even in computer science. It is also applied in diverse areas such as category theory, operads, moduli spaces of graphs, manifolds and even in knot theory as well. The orthogonal calculus of functors has many applications, starting with: it giving structure to polynomial functors. It's also used for the construction of the Taylor tower which is approximations by polynomials to the functor in question.

In spite of all this applications, many mathematicians have very little knowledge and insight about the orthogonal calculus of functors and its practical usefulness to our daily day life. Many Mathematicians and scientists nd it very di cult to study, work and apply it to real life situation and other areas of study and hence only concentrating on the Goodwillie calculus of functors from space to spaces. Others see it to be abstract in nature and mere map or theorem and thus, is of no interest and insigni cant to other elds of study.

In fact, it's surprising to see some mathematicians and scientist using and applying the orthogonal calculus of functors without knowing it and how useful it is, even though, some could use it rightly.

1.3 Research Aim and Signi cants

The orthogonal calculus of functors is very important in the area of mathematics, physics and even engineering since its applications to certain activities of life are not yet realized.

This project tries to seek the simplest way for people to understand polynomials functors in orthogonal calculus and also enlighten people on some structure to polynomial and homogeneous functors as well. Since this work is expected to nish within limited time, it cannot touch all the areas under orthogonal calculus of functors. Therefore, it will be recommended at the end of this work for researchers and students for further studies and also develop and obtain formulas for taylor approximation to spaces of smooth embedding's.

1.4 Objective of The Study

This work mainly analyzes the structure to polynomial functor in orthogonal calculus.

1.5 Scope and LImitation

1.5.1 Scope of the Study

This thesis will make use of published researched papers that are centered around the calculus of functors by academicians in a well established journals and bulletins. Videos on youtube by well known academicians such as Gregory Arone,Micheal Ching, David Barnes and many more from well established mathematical conferences will also be accessed.

The research will also involve a study of works on brations,homotopy ber,bundles(ber bundles, vector bundles and sphere bundles), the Yonneda lemma and related topics from well acclaimed authors

1.5.2 Limitation of the Study

This research has been encountered with the challenge of in-depth information on the orthogonal calculus of functors.

1.6 Structure of The Thesis

This research work is in ve(5) chapters

Chapter one provides a general introduction to the research.

Chapter two reviews the various literatures of the calculus of functors.

Chapter three review some maps or tools that are inherently tied to the orthogonal calculus of functors.

Chapter four presents the de nition of polynomial functors with some examples from the algebraic and topological settings. It also gives some insight about the classi cation and approximation (crosse ect/suspensions) of the polynomial functors. It further reviews the orthogonal calculus of functors and also enlightens us on the structures that polynomial functors and homogeneous functors take.ie it gives the basic forms that polynomial functors and homogeneous functors can take.

Chapter ve gives the conclusion and recommendation to the thesis.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Generally we give a brief review of some important literatures which are relevant to the study of the functor calculus. This will include a brief history of functors, categories and polynomial functors.

2.2 A Brief History on Functors

In algebraic and geometric topology, Goodwillie, orthogonal and embedding calculus is a way for doing calculus on functors. This calculus on functors is done by we approximating our functors by a sequence of functors that are simpler to our functor in question.

General functors can be approximated with polynomial functors in two ways: the interpolation polynomials and the Taylor polynomial. But we will move with the Taylor polynomial.

This sequence of approximation is almost same as the Taylor series of smooth function. There are many objects in algebraic and di erential topology which can be seen as functors. They maybe functors although but it's always di cult di cult to check for its connectivity and also to analyze directly, so we think of replacing them with simpler functors which are su ciently good approximation for the functor in question through the Taylor tower.

In the nineteen forty's(1942-1945) there existed two mathematicians known as Saunders Mac Lane and Samuel Eilenberg who came up with natural transformation, functors and categories, in topology, especially as part of what they were studying in algebraic topology. Their study was very useful because it moved us from the intuitive and geometric notion on homology to something more axiomatic. These mathematicians came out with a paper which clearly showed they were working on understanding what the natural transformation was really about. But they explained further that before one can understand the natural transformation required de ning functors since natural transformations are structure preserving maps of functors, and also required understanding categories since functors also preserves categories.

One mathematician Stanislaw Ulam claimed that related ideas of categories, functors and the natural transformation were current in the late 1930s.

Category theory is a continuous work by Mac Lane and Eilenberg from Emmy Noether (Mac Lane's teacher). Emmy Noether studied mathematical structures and came out with the conclusion that if one wants to understand an object of mathematical structures then they really have to know the process that holds the structure or preserves it.Hence with this statement by Emmy Noether, Mac Lane and Eilenberg worked on understanding algebraic structures and how to preserves those structures as well.

2.3 Polynomial Functors

Polynomial functor is Categori cation of the notion of polynomial function. Depending on which properties of polynomial functions exist between the categories, there are di erent notions result which might deserve the name
polynomial functors.

Polynomial is such a basic concept that is quite easy to understand in mathematics, hence not surprisingly that much work have been done on it with people coming up with its categori ed versions called the functors which has been applied in di erent areas, and in di erent names as well.

The careful study of polynomial functors has appeared to be very important in physics, also in mathematics with special areas like topology (Bisson and Joyal, 1995)(Pirashvili, 2000) and in algebra (Macdonald, 1998) and also nd it route in mathematical logic(Girard, 1988),(Moerdijk and Palmgren, 2000) and computer science(theoretical)(Jay and Cockett, 1994), (Abbott et al., 2003), (Setzer and Hancock, 2005).

Since the practical knowledge to polynomial functor is so real and easy to see that is why it was discovered by mathematicians many times and has been used to show di erent notions and also applied in di erent areas of study. The rst intuitive idea was concerned with the category of abelian groups. And was perhaps attributed to Eilenberg Mac Lane(Eilenberg and MacLane, 1945) even though there were some claim that it existed before Eilenberg and mac lane talked about them. Their goal was to understand group homology.

They worked on Abelian-valued functors from the category of abelian groups which preserves the categories with both initial and terminal object i.e. the zero objects.

They came up with the deviation of such covariant functor, which in some sense can be thought of as a di erential quotient; formally it was de ned as a some sort of kernel.

A functor is 1-additive or excisive when its second cross e ect is zero or deviation is zero i.e. $Cr_2(-,-)$.

Polynomial functors is also very useful in the representation theory of symmetric groups (Macdonald, 1998). An endofunctors of the category of inner product space of nite dimension is called a polynomial if for every vector spaces A,B, the map $Hom(A,B) \rightarrow Hom(FA,FB)$ is a polynomial mapping.

The usage of polynomial functor usually involves the actions of the orthogonal groups. However, there were continuous developments of applying symmetric polynomial functors on vector spaces over nite elds.

Individual intuitions about the viewpoint of polynomial functors came up little by little in the 1970s with mathematicians centering on automata ,algebraic theories and data types.

Also category of sets that are abstract in nature was developed by George Cantor to help study in nite quantities. The category of sets that are abstract in nature is the fundamental or basic setting for the polynomial idea. Contents are de ned, respectively, with the right and left adjunction of the pullback functor for the diagram below.

We therefore work with locally Cartesian closed category E which implies, a polynomial functor is speci ed by the data

$W \xleftarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in C.

For each arrow $f : X \to Y$ we dispose of three functors, the pullback functor $f^{*set}/Y \to$ $^{set}/X$, a left adjoint $f_!$, and a right adjoint f_*'. By de nition, a polynomial functor is one canonically isomorphic to any
composite of these kind of functors between slices of E. The
functor below is then composite

$$set/_W \xleftarrow{s^*} set/_X \xrightarrow{P^*} set/_Y \xrightarrow{s_!} set/_Z$$

.

and the functor can be described as those that are canonically isomorphic to one of the functors above.

CHAPTER 3

ALGEBRAIC STRUCTURES AND MAPS

Before looking at polynomials in orthogonal calculus of functors, i will highlight on the category theory and functors that are inherently tied to calculus of functors and which will commonly be encountered.

3.1 Category Theory

Two mathematicians Mac Lane and Eilenberg came up with categories in the 1940's and have been around for about half a century now. Samuel Eilenberg was from Poland and was an algebraic topologist and MacLane was an algebraist.
They later understood it was the same calculations they were doing from di erent areas of mathematics (topology and algebra) and that led them to invent or develop category theory.

De nition 3.1.1 A category ξ consist of

A collection $A,B \in \xi$ where A and B are objects in ξ

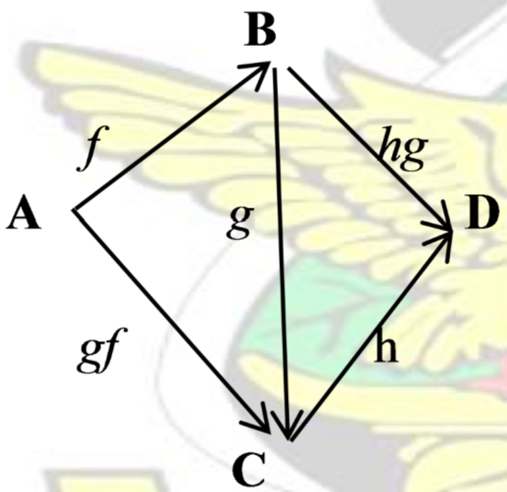
For all $A,B \in obj\ (\xi)$, a collection $\xi\ (A,B)$ of maps(arrows) from the object A to the object B

To every map there exist two objects, its source and target. If f is a map with a source A and target B, then we indicate this by $f : A \rightarrow B$.

For every object A there exist an associated identity map which is written as $1_A : A \rightarrow A$.

Further if $f : A \rightarrow B$ is a map from an object A to the object B and $g : B \rightarrow C$, is also a map from object B to object C then there will exist a composition $gf : A \rightarrow C$, which will satisfy the following relations.

If $f : A \rightarrow B$, $g : B \rightarrow C$, and $h : C \rightarrow D$ then $h(gf) = (hg)f : A \rightarrow D$:



Associativity condition

If $f : A \rightarrow B$, then $f1_A = f = 1_Bf$:

Identity Morphism

Relations such as the associativity and the identity morphism are denoted by saying the gures below commutes,

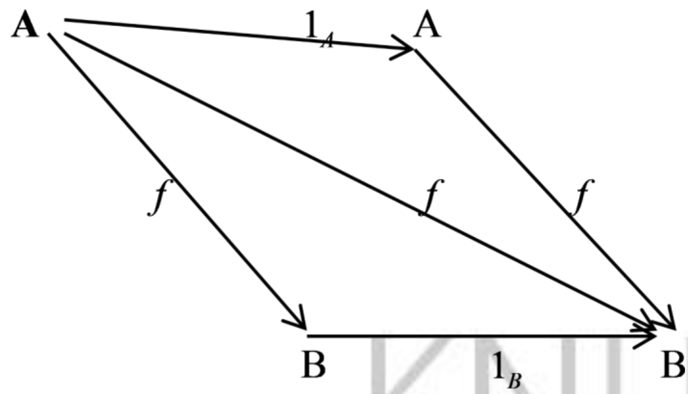


Figure 3.1: Associativity relation of category theory
Figure 3.2: identity morphism of category theory

3.1.1 Sub-Categories

De nition 3.1.2 A category ε is a subcategory of category ξ provided: Each obj (ε) is in obj (ξ)

$f \in \varepsilon(A,B)$, implies $f \in \xi$ (A,B) i.e. if f is a map in the sub-category ε implies f is also a map in the main category ξ.

f : $A \rightarrow B$ and g : $B \rightarrow C$ in ε , implies gf is a composition of f following g in ξ. i.e. composition of maps in sub-categories also holds in the main category.

If 1_A is the morphism(identity) for A in ε , then 1_A is also a morphism(identity) for A in ξ

3.1.2 Special Morphism

Relations among morphisms are often shown with diagrams that commutes, with points or block letters to represent objects and arrows to represent maps or arrows.

The identity morphisms is a natural map that every object have.

Aside the natural map(identity) that exist in every type theory, there exist other di erent maps that are useful and interesting to study as well.

i.e. Morphism can have any of the following properties

An Isomorphism

If $f : A \to B$ is a map from the source A to the target B with $f^{-1}f = 1_A$ and $ff^{-1} = 1_B$ as inverses.

If there exist an inverse for f, then it should be unique, to justify the uniqueness of the notation f^{-1}.

To see that f is unique up to isomorphism lets consider g and h to be inverses of f, then

$g = 1_A g = hfg = h1_B = h$.

We will denote an isomorphism with the symbol '$\cong$' and write $f : A \xrightarrow{\cong} B$ to show an isomorphism of the morphism f, and will denote an isomorphism of A and B as $A \cong B$ and say A is isomorphic to B" if A and B are isomorphic to each other.

Note: Morphism that is a section and at the same time a retraction is also called an isomorphism.

Endomorphism

A map $g : B \to C$ is an endomorphism if $B = C$. $end(B)$ denotes the group of endomorphism's of B.

Automorphism

An endomorphism that guarantee a return inverse or also an isomorphism is known as an automorphism. i.e. is an endomorphism that has left and right inverses. The automorphism class of the object C is the group of all automorphism of C. And is usually represented with $Aut(C)$.

Section And Retract

The de nition of an isomorphism can be separated into two parts i.e. isomorphism have both left and right inverses, which is the same as we saying an isomorphism has both a section and a retraction.

Section

For any map $f : A \to B$, a section of f is map $s : B \to A$ Such that $f \circ s = 1_B$.

A section is called a right inverse.

Retract

For any map $f : A \to B$, a retraction of f is a map $r : B \to A$,such that $r \circ f = 1_A$.

A retract is also called a left inverse.

In any category a section is a monic and a retraction is an epic, but the converses are false

Epimorphism (Epic)

In any category theory, the map $e : A \to B$ is an epimorphism or epic morphism in a way that, $\forall, f, g : B \to C, f \circ e = g \circ e$ implies $f = g$.

The equation $f \circ e = g \circ e$ implies f and g are two morphism with Source B and target C.

$$\begin{tikzcd} A \arrow[r, "e"'] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

Figure 3.3: Diagrammatic explanation of an epimorphism

Monomorphism (Monic)

In any category theory, the map $m : B \to C$ is said to be monomorphism or monic morphism for the fact that $\forall, f, g : A \to B, m \circ f = m \circ g$, implies $f = g$.

The equation $m \circ f = m \circ g$ implies f and g are two maps with the source A and the same target point B.

Figure 3.4: Diagramatic explanation of monomorphism

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "m"] & C \end{tikzcd}$$

3.1.3 Dual Category

De nition 3.1.3 C been a category, would imply C^{op} is its dual category and its de ned as follows

They have the same objects as to that of C

The maps are the reversed version of arrows of C, i.e. for every arrow $f : A \rightarrow B$,there exist a maorphism $f^{\#} : B \rightarrow A$ in C^{op}.

The composition of arrows $g^{\#} \circ f^{\#}$ in C^{op} is nothing but $(f \circ g)^{\#}$

3.2 Functors

Functors or covariant functor is morphism or an arrow that preserves the structures between categories.
Functors are now applied almost everywhere in modern mathematics to relate various categories.

De nition 3.2.1 A covariant functor $F : C \rightarrow D$ is a map that preserves the structures that exist between categories C and D and also associates each object A in category C to an object F(A) in category D; and each morphism $f : A \rightarrow B$ in category C to a morphism

$F(f) : F(A) \rightarrow F(B)$ in D, such that

$F(1_A) = 1_{F(A)}$ for every object A in category C; and $F(g \circ_C f) = F(g) \circ_D F(f)$ for every map $f : A \rightarrow B$ and $g : B \rightarrow C$

for which compositions $\circ_C$ and $\circ_D$ are de ned in categories C and category D.

Diagramatic explanation of a functor is giving below:

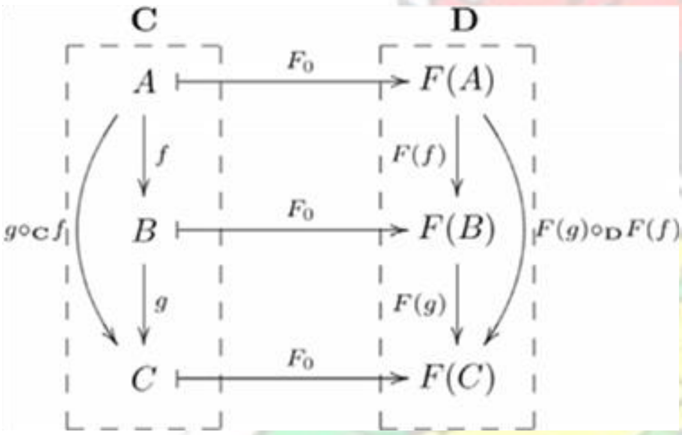


Figure 3.5: diagramatic explanation of a functor

Where we represent the categories with dashed rectangles, and the functors are represented with F_0 and the functor arrows between morphisms are omitted. Composition of Functor and functor isomorphism are de ned analogously to morphisms (above).

I.e. the functor composition of $F : C \rightarrow D$ and $G : D \rightarrow E$ is the functor $G \circ F : C \rightarrow E$ sending all the objects A in category C to objects $G \circ F(A) \in E$

; and morphisms $f : A \rightarrow B \in C$ to morphisms,

$G \circ F(f) : G \circ F(A) \rightarrow G \circ F(B)$

such that identity morphism and composition holds.

I.e. $G \circ F(1_A) = 1_{G\circ F(A)}$ and

$$G \circ F(g \circ_C f) = (G \circ F(g)) \circ_E (G \circ F(f))$$

3.2.1 Contravariant Functor

A functor(contravariant) from category C to category D is a functor from C^{op} to D.Also we can say, F is a contravariant functor if F sends

objects $A \in ob(C)$ to object $FA \in ob(D)$

F sends morphisms $f \in C(A,B)$ to morphisms $Ff \in D(FB,FA)$

The identities are preserved

$F(f \circ g) = Fg \circ Ff$

3.2.2 Forgetful/Underlying Functors

A functor is de ned as an underlying functor or forgetful functor if it drop some or all the input structure or properties.

Examples of forgetful functors are

The functor $U : Top_* \rightarrow Top$ which embeds the category of pointed topological space into the category of topological space by forgetting that the topological space is pointed.

The functor $U : Group \rightarrow Set$ which forgets that a group have more structure than just the underlying set it captured or remembered.

Similarly there exist a functor $U : Ab \rightarrow Grp$ de ned by U(A) = A for A been an abelian group. This functor forgets the property that abelian groups are abelian.

The forgetful functors in this example forget the property on the objects.

CHAPTER 4

POLYNOMIAL FUNCTORS

Polynomial functor is just Categori cation of 'polynomial functions'.

Depending on the properties of polynomial functions one takes as guideline for the Categori cation, di erent notions result which might deserve to be called polynomial functors.

A continuous function f: R → R is linear if $f(a + b) = f(a) + f(b)$ for all a,b ∈ R.

To be precise, we might think of a function been a ne linear if $f(a + b) - f(a) - f(b) + f(0) = 0$

equivalently

$f(a + b) - f(0) = (f(a) - f(0)) + (f(b) - f(0))$.

One of the nice properties of functions of real numbers is the property that

$f(a + b) - f(a) - f(b) + f(0) = 0$

implies that f is actually an a ne linear polynomial in the sense that if we take $f(x) = mx + c$ for some real numbers m and c

Conversely, a function $f(x) = mx + c$ is known as polynomial of degree 1

4.1 Polynomial Linear Functor

Algebraic Setting

Theorem 4.1.1 F is additive i

rstly it takes the no object in C to the no object in D.

i.e. $F(0_C) = 0_D$

Secondly if it preserves nite product or co product i.e.
$F(A+B) \xrightarrow{\cong} F(A)+F(B)$

Example 4.1.1 Given C and D as categories which are both abelian. Thus we can think of C and D as abelian categories of modules over some commutative rings. (Mod_R).

A covariant functor $F: C \to D$ is additive if it respects the enrichment of C and D in abelian groups.

If we look at $Hom(A,B) \xrightarrow{F} Hom(FA,FB)$ is an Abelian group homomorphism for every A and B.

i.e. The covariant functor F is an enriched over the category of Abelian group.

Topological Algebraic Settings

Theorem 4.1.2 lets think of C to be pointed category with co-products, and D be an Abelian group.

$F: C \to D$ Is additive if it preserves

$F(0_C) = 0_D$ and

$$F(A \textstyle\coprod B) \xrightarrow{\cong} F(A) + F(B)$$

Example 4.1.2 The reduced homology

$$\widetilde{H}_* : Top_* \to grAb$$

.

I.e. $\widetilde{H}_*(*) = 0$ and $\widetilde{H}_*(A \vee B) \cong \widetilde{H}_*(A) + \widetilde{H}_*(B)$.
satis es this property

Remark. The additivity of the reduced homology group is captured by
Figure 4.1: Pushout squares of pointed category with coproduct and abelian groups

Which preserves this kind of pushout?

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{Po}"] & A \arrow[d] \\ B \arrow[r] & A \vee B \end{tikzcd} \rightsquigarrow \begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{po}"] & \tilde{H}_* A \arrow[d] \\ \tilde{H}_* B \arrow[r] & \tilde{H}(A \vee B) \end{tikzcd}$$

Hence is additive when the reduced homology group preserves this kind of pushout.

Homology also has interesting property when applied to di erent types of pushout squares produces the Mayer-victoris sequence.

$$\text{Ie } \begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{hpo}"] & A \arrow[d] \\ B \arrow[r] & A \vee B \end{tikzcd} \rightsquigarrow$$

MAYER VICTORIS SEQUENCE

$$\to \tilde{H}_*(X) \to \tilde{H}_*(A) \oplus \tilde{H}_*(B) \to \tilde{H}_*(A \vee B) \to$$

Figure 4.2: homotopy pushout to Mayer Victoris Sequence

This is a stronger property than pure additivity condition.

Hence $\tilde{H}_*$ is excisive since it has the Mayer-victoris sequence for homotopy pushout squares.

Topological Settings (Goodwillie Case)

Example 4.1.3 Consider the homotopy functor $F : Top_* \to sp$ (f weq ⇒ Ff weq).

F is additive (reduced degree ≤ 1) if

$$F(*) = *$$

$$F(A) \vee F(B) \xrightarrow{\sim} F(A \vee B)$$

Excisive condition.

The covariant functor F is 1-excisive if it preserves homotopy pushout squares. Equivalently F takes homotopy pushout squares to homotopy pullback squares. (in this case π_*^s has the Mayer-Victoris sequence)

Example 4.1.4 A homotopy functor $F : Top_* \to Top_*$ is excisive if the covariant functor F takes homotopy pushout to homotopy pullback squares. (If we take homotopy group of the functor F this will have the Mayer- victoris sequence a rming the excisive condition of the functor F).

Manifold Calculus

Example 4.1.5 A contravariant functor $F : \vartheta(\mathrm{R}^n) \to Top$. Where we can think of F to be a functor on the category of open subsets of R^n.

Hence F is is excisive or degree≤ 1 if we consider the homotopy pushout of this category.

$$\begin{tikzcd} U \wedge V \arrow[r] \arrow[d] \arrow[dr, phantom, "hpo"] & U \arrow[d] \\ V \arrow[r] & U \vee V \end{tikzcd} \longrightarrow \begin{tikzcd} F(U \wedge V) \arrow[dr, phantom, "hpb"] & F(U) \arrow[l] \\ F(V) \arrow[u] & F(U \vee V) \arrow[l] \arrow[u] \end{tikzcd}$$

Figure 4.3: excisive diagram of manifold calculus

Hence the contravariant functor F is 1-excisive since it preserves the homotopy pushout squares. Equivalently F takes homotopy pushout squares to homotopy pullback squares.

4.2 Constructing Approximation

4.2.1 Approximation Via Cross-E ect

Example 4.2.1 Considering the settings $F : C \to D$ with C being pointed with co-product and D as Abelian group.

$F : C \to D$ Reduced implies $F(0_C) \to 0_D$. The second cross-e ect measures the failure of F to be additive.

Hence we can de ne the linear cross-e ect of the covariant functor $F : C \to D$ as

$Cr_2F(A,B) := ker(F(A \vee B) \to FA + FB)$

$\therefore F(A \vee B) \cong FA + FB + Cr_2F(A,B)$

Therefore F is additive i $Cr_2F(A,B) = 0, \forall A,B \in C.$

Example 4.2.2 Considering $F : Top_* \to Sp$.

We can de ne the linear cross-e ect of the functor $F : Top_* \to Sp$ as

$$Cr_2F(A,B) = hofiber(F(A \vee B) \to FA \vee FB)$$

$$\therefore F(A \vee B) \cong FA \vee FB \vee Cr_2F(A,B)$$

Hence F is additive i $Cr_2F(A,B) = 0$

4.2.2 Approximation Via Suspension (To get Excisive Functors)

Example 4.2.3 Considering $F : Top_* \to Top_*$ reduced homotopy functor. Want to naively force F to be 1-excisive or excisive.

Note. The di erence between additive functors and excisive functors is that one can take push out squares that don't just have the initial object in this top hand corner.

For any base space X, there is a nice homotopy pushout that takes the form below,

$$\mathrm{F\,X} \longrightarrow T_1F = \operatorname{holim}\left\{ \begin{array}{ccc} & & \mathrm{F(CX)} \\ & & \downarrow \\ F(CX) & \longrightarrow & F(\Sigma X) \end{array} \right\}$$

Figure 4.4: Homotopy pushout of excisive functors

Where CX is the cone and $\mathrm{P}X$ is the suspension(reduced).

And from de nition of excisive functors, a functor is excisive if it takes homotopy pushout squares to homotopy pull back squares.

Hence if F is excisive then the output of the gure 9 will be a pullback and FX will be equivalent to the pullback of the remaining parts of the square.

$$\begin{tikzcd} \mathrm{X} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "hpo"] & \mathrm{CX} \arrow[d] \arrow[r, squiggly] & \mathrm{FX} \arrow[r] \arrow[d] & F(CX) \arrow[d] \\ \mathrm{CX} \arrow[r] & \Sigma X & F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

Hence FX should be a pullback of the remaining square

I.e.

If F is excisive then $F \xrightarrow{\sim} T_1F$. But T_1F need not be excisive.

However T_1F is closer to being excisive than the original functor FX.

If we iterate this construction then we will eventually be arriving at something excisive.

Thus the essence of Goodwillie construction.

Hence $P_1F = \mathrm{hocolim}(F \to T_1F \to T_1T_1F \to \dots)$.

$F \to P_1F$is an excisive approximation.

4.2.3 Higher degree polynomial

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is quadratic if $f(+y+z) = f(x+y) + f(y+z) + f$

$(z+x) - f(x) - f(y) - f(z) + f(0)$

4.2.4 Higher Cross-E ects

We have talked about the second cross-e ect being the basic object of additivity. Hence to talk about the higher versions of additivity we would need higher cross e

ect to measure the failure of the functor being n-additive.

From the setting $F : C \to D$ where C was pointed with co-product and D being

an abelian group.

Hence we can de ne the nth cross-e ect of the functor $F : C \to D$ as

$Cr_nF(A_{1,\dots\dots A_n}) = Cr_2(Cr_{n-1}F(A_1,\dots A_{n-2,}-)(A_{n-1},A_n))$.

Then

$$F(A_1 \vee \dots A_n) ' \quad \underset{S=\{1,\dots n\}}{Q} \quad Cr_{|S|}F(\{A_i\})_{i\in S}$$

F is degree $\leq n$ (n-additive) if $Cr_{n+1}F ' 0$.

4.3 Orthogonal Calculus

Introduction

There exist another brand of functors Calculus, which emerged after the papers of Goodwillie were published, which is known as the orthogonal calculus of functors, due to Weiss, and this theory is closely related to or he had his inspiration from the Goodwillie calculus of homotopy functor.

The orthogonal calculus of functor is a beautiful tool for calculating the homotopical properties of functors from the category of inner product space spaces to pointed spaces or any space enriched over Top_*.

Interesting examples of such functors abound and include classical objects in algebraic and geometric topology:

1. $\Omega^V\left(S^V \Lambda X\right)$.
2. $BAut(V)$
3. $Emb(M \times N, N \times V)$
4. $BTop(V)$.

Category of such functors from vector spaces to spaces and natural transformations between them will be call ξ_0

These functors satisfy an extrapolation condition, which allows one to identify the value at some vector space from the values at vector spaces of greater dimension.(Barnes and Oman, 2013)

Orthogonal calculus is based on the notion of n polynomial functors (vector spaces at very high dimension), which are well-behaved functors in ξ_0 and which preserves weak equivalences as well.

With These n-polynomial functors one can often infer the value at some vector spaces from the values at vector spaces of higher dimension.

In geometric sense, orthogonal calculus approximates a functor (locally around R^{∞}) via polynomial functors (approximate into sequence of simpler functors that are homotopy equivalent to the functor in question) and attempts to reconstruct the global functor from the associated in nitesimal information.

The orthogonal calculus splits a functor F in ξ_0 into a Taylor tower of

brations, where our n-th brations will consist of maps from the n .polynomial approximation of F to the $(n-1)$−polynomial approximation of F.

The homotopy ber or layer (the di erence between n polynomial and $(n-1)$−polynomial approximation) of this map is then an n homogeneous functor and is classi ed by an $O(n)$−spectrum up to homotopy which is usually denoted as D_nF.(Barnes and Oman, 2013)

4.3.1 Continuous Functors

Let consider = to be the category of vector space with an inner product and that is nite dimensional with linear maps to preserve the internal structure of the vector space.

To see our category is nitely small let's assume our vector spaces belongs to some larger space R^{∞}, since orthogonal calculus is based on the notion of n polynomial functors (vector spaces at very high dimension), which are well-behaved functors in ξ_0 and which preserves weak equivalences as well.(Barnes and Oman, 2013) With These n-polynomial functors one can often infer the value at some vector spaces from the values at some vector spaces of higher dimension.

Orthogonal calculus is concerned with covariant functors that are continuous i.e. E from = to spaces. A functor been Continuous implies

I.e. ev:$mor\,(V,W) \times E\,(V) \rightarrow E\,(W)$

is continuous, for every $V,W \in$ =. (Weiss, 1995)

Some examples are

$E(V) = BO(V), E(V) = BTop(V), E(V) = BG(S(V))$, Suggesting that orthogonal groups are associated with classical spaces, like BO, BTop, BG equipped with a sophiscated ltration indexed by nite dimension linear subspaces V of R^∞.

4.3.2 The Tower And The Classi cation

For a covariant functor $F \in =_0Top$, Weiss calculus constructs the n-polynomial approximations T_nF and the n-homogeneous approximations D_nF. These can be clearly shown in a tower of brations. For all $n \geq 0$ there exist a sequence of bration

$D_nF \to T_nF \to T_{n-1}F$,

Which can be arranged as below?

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & T_nF \arrow[d] & D_nF \arrow[l] \\
 & T_{n-1}F & D_{n-1}F \arrow[l] \\
 & \vdots \arrow[d] & \\
 & T_2F \arrow[d] & D_2F \arrow[l] \\
F \arrow[uuuur] \arrow[uuur] \arrow[ur] \arrow[r] \arrow[dr] & T_1F \arrow[d] & D_1F \arrow[l] \\
 & T_0F &
\end{tikzcd}
$$

Figure 4.5: Tower of Fibration

For this tower of bration to be useful we must understand the functor F, the polynomial approximation of the functor F and also the homogeneous functors as well. (Barnes and Oman, 2013)

4.3.3 Derivatives Of A Functor

We will denote R^{∞} with μ (as in nite-dimensional vector space with a positivede nite inner product) with the standard inner product, and regard all nite dimensional vector spaces R^n as subspaces of μ, inheriting its inner product.

Througout our work we will denote our nite dimensional vector spaces with object U,V,W and denote the one point compacti cation of V with V^C.

We write $R^n \otimes V$ to mean $n.V$.

Let'think of R^n to be a suitable subspace of R$^\infty$, so that

R$^0 \subset$ R$^1 \subset$ R$^2 \subset$ ^{3}.....R$^\infty$ then $0 \cdot V \subset 1 \cdot V \subset 2 \cdot V \subset 3 \cdot V \subset ....n \cdot V$

this will also denote the one point compacti cation.

Let's consider $mor(V,W)$ to be a linear isometries from V to W which preserves the inner product. Also lets consider the category = of vector spaces preserving inner product with objects been U,V,W such that $mor(V,W)$ is the set of maps from V to W.

Also let $=_n$ be of the same object as = with the category of objects U,V, W.... and with $mor_n\,(U,V)$ as the set of maps from U to V.

$=_n$ is considered as a topological category which is pointed with class of objects that are discrete.

The morphisms set are topological spaces that are pointed. Just as the non equivariant case, we can form inclusions $=_0 \subset =_1 \subset =_2 \subset =_3 \subset$ ..... and a notion of derivatives.

More that $=_0$ di ers slightly from =, such that $mor_0\,(V,W)$ is $mor(V,W)$ with an added base point.

We now concentrate on functors that are continuous; i.e. if E is a covariant functor $=_0 \to (Top_*)$ pointed spaces, then it has a derivative $E^{(1)} : =_1 \to Top_*$ which itself has a derivative $E^{(2)} : =_2 \to Top_*$ which will also have the derivative $E^{(3)} : =_3 \to Top_*$ and so on as in the non equivariant case.

The derivative is de ned in terms of the adjoint to the restriction functor. Thus restriction from ξ_n to ξ_m for m,n with $m \leq n$ gives us a natural transformation res_m^n. we can think of m,n as positive integers.

more generally we can obtain a restriction map $res_m^n : \xi_n \to \xi_m$ for $m \leq n$ by successive composition.

There exist a map which helps to transform one functor to the other or preserves the structure of functor which is known as the natural transformation. Hence the natural transformation above will be continuous if it is invertible and a homomorphism. E.g. $res^n_m : nat_n(E,F) \to nat_m(res^n_m E, res^n_m F)$is continuous.

Proposition 4.3.1 A functorres^n_m has

$$ind^n_m : \varepsilon_m \to \varepsilon_n$$

as its right adjoint, and its de ned as
$(ind^n_m X)(V) = Nat_{\varepsilon_m}(Mor_n(V,-), X)$

With the right hand side denoting the topological space of the morphism between two objects of ε_m.

Proposition 4.3.2 For all V and $W \in =_0$ and all $n \geq 0$ there is a natural homotopy co ber sequence

$Mor_n(\mathrm{R} \oplus V,W) \wedge S^n \to Mor_n(V,W) \to Mor_{n+1}(V,W)$.

Proof. Identifying S^n as the closure of the subspace

$(i,x) \in \gamma_n(V,\mathrm{R} \oplus V)$,

where i is the standard inclusion, the composition map

$Mor_n(\mathrm{R} \oplus V,W) \wedge Mor_n(V,\mathrm{R} \oplus V) \to Mor_n(V,W)$

Restricts to a morphism

$Mor_n(\mathrm{R} \oplus V,W) \wedge S^n \to Mor_n(V,W)$.

The homotopy co ber of the restriction is then the quotient of

$[0,\infty] \times \gamma_n(\mathrm{R} \oplus V,W) \times \mathrm{R}^n$

The desired homeomorphism, away from the base point, is indeed by the association below.

Consider a quadruple

$(t \in [0,\infty], f \in Mor\,(\mathrm{R} \oplus V, W), y \in \mathrm{R}^n \otimes (W - f\,(\mathrm{R} \oplus V)), z \in \mathrm{R}^n)$

We send this to the element

$(f|v, x) \in Mor_{n+1}\,(V,W)$ Where $x = y$

$+ (f|_{\mathrm{R}*})(z) + t\omega\,(f|_{\mathrm{R}*}\,(1))$, And

$\omega : W \to \mathrm{R}^{n+1} \otimes W$

Identi es

$W \sim= (\mathrm{R}_n \otimes W)_\perp \subset \mathrm{R}_{n+1} \otimes W$

From this co ber sequence we can make a ber sequence by applying the functor Nat_{ε_n}

$(-,F)$ for $F \in \varepsilon_n$.

Lemma 4.3.0.1 For all $V \in =_n$ and $F \in \varepsilon_n$,

there is a natural homotopy ber sequence
$res_n^{n+1} ind_n^{n+1} F\,(V) \to F\,(V) \to \Omega^n F\,(\mathbb{R} \oplus V)$

4.4 Structure to polynomial functors

Which functor E in $=_0 Top$ deserves to be called polynomial functors of degree $\leq n$?

This question has to be certainly answered at some point in time if we want do calculus. One easy to see requirement of the n-polynomial functor is that it's $(n + 1)$-Th derivative of the functor E should vanish.

However this does not hold for all cases especially the case $n = 0$ shows that this de nition is not enough.

A functor E deserves to be called polynomial of degree 0 i E(f) is a homotopy equivalence for all nonzero morphisms f in $=_0$.

4.5 Polynomial Functor

We want to study a well-behaved collection of functors in ξ_0; those whose derivatives are eventually trivial. By analogy with functions on the real numbers, we call these functors polynomial.

In this section we introduce this class of functors and examine their structures.

De nition 4.5.1 For $E \in =_0 Top$ or for $E \in \xi_0$ de ne

$$\tau_n E(V) = \underset{06=U\subset R^{n+1}}{holim} E(U \oplus V)$$

We can think of the covariant functor E to be n-polynomial if the canonical map $\rho^n{}_E(V) : E(V) \rightarrow \tau_n E(V)$.

Is homotopy equivalence for every genre vector space V of =.

CommentThe non-zero linear subspace $U \subset R^{n+1}$ form a poset P where $T \leq U$ means $T \subset U$.

With the above theorem we sometimes think of such functor E to be n-polynomial.

The value of the functor E which is n-polynomial at the vector space V is determined up to homotopy by the values $E\ (U \oplus V\ )$ and the arrows between them for the nonlinear subspace $U \subset \mathrm{R}^{n+1}$.

This de nition captures the idea of the value of the functor E at some vector space V being recoverable from the value of E at vector spaces of higher dimension.

I.e. we can think of the n-polynomial functor as one where it is possible to extrapolate the information of $E(V\ )$ from the spaces $E\ (U \oplus V\ )$. The homotopy ber of $\rho^{n}{}_{E}\ (V\ )$: $E\ (V\ ) \to \tau_n E\ (V\ )$

measures how far E is from being n-polynomial, its always helpful for us identifying what the bers are. Also let's recall that a sphere bundle

$S\gamma_{n+1}\ (V,W) \to \overset{\rho}{mor}\ (V,W)$, if we x V and vary W, we will get a natural transformation

$S\gamma_{n+1}\ (V,-) \to mor\ (V,-)$, We then have a map ρ^* : $nat(mor\ (V,-),E) \to nat(S\gamma_{n+1}\ (V,-),E)$ Hence by the yonneda lemma we get $\rho^* : E\ (V\ ) \to nat(S\gamma_{n+1}$ $(V,-),E)$ And its polynomial of degree $\leq n$ i $\rho^* : E\ (V$ $) \to nat(S\gamma_{n+1}\ (V,-),E)$ is homotopy equivalence for all V.

De nition 4.5.2 FOR $E \in \xi_0$, we de ne $\tau_n E \in \xi_0$

Such that

$$(\tau_n E)\,(V) = Nat_{\xi_0}\left(S\gamma_{n+1}(V,-)_+, E\right)$$

We also have natural transformation of self- functors on ξ_0 :

$\rho_n : Id \to \tau_n$

This natural transformation comes from the map $S\gamma_{n+1}(V,W)_+ \to\ Mor_0\ (V,W)$ And by yonneda lemma.

By Michael Weiss there is another description of

$$S\gamma_{n+1}(-,-)$$

. It is the homotopy colimit:

$$S\gamma_{n+1}(V,A)_+ \cong \underset{0\neq U\subset\mathbb{R}^{n+1}}{hoco\lim} E(U\oplus V),$$

Where the right hand side is the Bous eld-Kan formula for the homotopy colimit of the functor $U \to Mor_0(U \oplus V)$ as U varies over the topological category of nonzero subspace of R^{n+1} and inclusions. Thus we see that

$$\tau_n E(V) = \underset{06=U\subset R^{n+1}}{ho\lim} E(U \oplus V)$$

We choose to de ne τ_n in terms of

$$S\gamma_{n+1}(-,-)$$

and we then de ne polynomial functors in terms of τ_n. [Barnes and Oman 2013]

Proposition 4.5.1 For any $E \in \xi_0$, and any $n \in N$, the sequence,

$$res_0^{n+1} ind_0^{n+1} E(V) \xrightarrow{\mu} E(V) \xrightarrow{\rho} \tau_n E(V)$$

Is a bration sequence up to homotopy? Hence

$res_0^{n+1} ind_0^{n+1} E(V)$ vanishes if E is a

polynomial of degree $\leq n$. Proof Let's de ne

$$res_0^{n+1} ind_0^{n+1} E(V) = Nat_{\xi_0}(mor_{n+1}(V,-),E)$$

then for the natural co- ber sequence

$S\gamma_{n+1}(V,A)_+ \to Mor_0(V,A)_+ \to Mor_{n+1}(V,A)$

Which is natural in A with respect to $=_0$. This converges to give a co ber sequence of : $=_0 - spaces$.

$S\gamma_{n+1}(V,-)_+ \to Mor_0\,(V,-) \to Mor_{n+1}\,(V,-)$

Considering the induced maps of spaces

$Nat_{\xi_0}\left(Mor_{n+1}\left(V,-\right),E\right)$

$Nat_{\xi_0}\left(S\gamma_{n+1}(V,-)_+,E\right).$ $\to$ $Nat_{\xi_0}\,(Mor_0\,(V,-),E)$ $\to$

Hence the above can be identi ed with

$\left(res_0^n ind_0^{n+1}E\right)(V) \to E\,(V) \to (\tau_n E)\,(V)$

Which is a bration sequence up to homotopy for all vector space V?

(Barnes and Oman, 2013)

Proposition 4.5.2 If E in ξ is polynomial of degree ≤ $n-1$, then it is polynomial of ≤ n degree .

Proof. We will actually show that any S_n−$equivalence$ is an S_{n-1}−$equivalence$. Thus we are to prove that

$S_n = \left\{S\gamma_{n+1}(V,-)_+ \to Mor_0\,(V,-)\,|V \in \Im_0\right\}$

is an S_{n-1} − $equivalence$ for any V. We can reduce this to proving that the map $\alpha : S\gamma_n(V,-)_+ \to S\gamma_{n+1}(V,-)_+$ is an S_{n-1} - equivalence.

The standard inclusion $\mathrm{R}^n \to \mathrm{R}^{n+1}$ induces a map of vector bundles $\gamma_n\,(V,W) \to \gamma_{n+1}\,(V,W)$ and hence a map of their respective unit spheres bundles:

$\alpha : S\gamma_n(V,-)_+ \to S\gamma_{n+1}(V,-)_+$

We can write $S\gamma_{n+1}(V,-)_+$ as the berwise product over $Mor_0\,(V,-)$ (denoted $\otimes$) of $S\gamma_n(V,-)_+$ and $S\gamma_1(V,-)_+$.

Thus we can write $S\gamma_{n+1}(V,-)_+$ as the homotopy pushout of the following diagram

$$S\gamma_n\,(V,-) \xleftarrow{\rho^1} S\gamma_n\,(V,-) \otimes S\gamma_1\,(V,-) \xrightarrow{\rho^2} S\gamma_1\,(V,-).$$

Where ρ_1 and ρ_2 are the projection maps. Now we can identify the codomain ρ_2 with the stiefel manifold Mor (R $\oplus$ V,−), and in fact ρ_2 itself is just the bundle. Writing $\in^n$ for the n-dimensional trivial bundle, it clear that there is a pullback

square:

Figure 4.6: pullback squares diagram

$$\begin{tikzcd}
\left(\in^n \oplus \gamma_n(\mathbb{R} \oplus V, -)\right)_+ \arrow[r] \arrow[d] & \gamma_n(V,-)_+ \arrow[d] \\
Mor_0(\mathbb{R} \oplus V, -) \arrow[r] & Mor_0(V,-)
\end{tikzcd}$$

The projection map ρ_2 can be identi ed with

$S(\in^n \oplus \gamma_n(\mathrm{R} \oplus V,-))_+ \to Mor_0(\mathrm{R} \oplus V,-)$.

Hence the vector bundle

$S\gamma_{n+1}(V,-)_+$ is the homotopy

pushout of

$S\gamma_n(V,-)_+ \leftarrow S(\in^n \oplus \gamma_n(\mathrm{R} \oplus V,-))_+ \xrightarrow{\rho_2} Mor_0(\mathrm{R} \oplus V,-)$.

If ρ_2 is an S_{n-1} -equivalence, then so is its homotopy pushout, which is α. The unit sphere of the Whitney sum of vector bundles is equal to the berwise join of the unit sphere bundles.

Hence we can write domain of ρ_2 as the homotopy pushout

$S\gamma_n(\mathbb{R} \oplus V, -)_+ \leftarrow S_+^{n-1} \wedge S\gamma_n(\mathbb{R} \oplus V, -)_+ \xrightarrow{\delta} S_+^{n-1} \wedge Mor_0(\mathbb{R} \oplus V, -)$.

The map δ is an S_{n-1} -equivalence, hence the top map in the commutative diagram below is an S_{n-1} -equivalence:

Figure 4.7: S_{n-1} equivalence

Since the diagonal map is an element of S_{n-1}, it follows that ρ_2 is an S_{n-1}-equivalence, as desired.(Barnes and Oman, 2013)

$$\begin{tikzcd}
S\gamma_n(\mathbb{R}\oplus V,-)_+ \arrow[r] \arrow[dr] & S\left(\in^n \oplus\gamma_n(\mathbb{R}\oplus V,-)\right)_+ \arrow[d, "\rho_2"] \\
& Mor_0(\mathbb{R}\oplus V,-)
\end{tikzcd}$$

Proposition 4.5.3 Let $g : F \to E$ be a map in ξ_0, such that $ind_0^{n+1}E$ is object wise contractible and F in n-polynomial.

Then the covariant functor

$$V \mapsto hofiber\left[F(V) \xrightarrow{g} E(V)\right]$$

Is also polynomial of degree $\leq n$.

Comment.In particular, it proves that the homotopy ber of a map between n-polynomial objects is n-polynomial.

Proposition 4.5.4 We say that a functor $E \in \xi_0$ is connected at in nity if the space $hocolim_k E(\mathbb{R}^k)$ is connected.

Comment. Polynomial functors can be determined by their behaviour at very high dimension.

i.e. by considering the behaviour of the vector space V at a very high dimension and which is always the best possible approximation to the functor in question. If a functor E is polynomial functor of degree $\leq n$, then all morphisms in the diagram

$$E \xrightarrow{\rho} \tau_n E \xrightarrow{\rho} \tau_n\tau_n E \xrightarrow{\rho} \ldots$$

are equivalences.

For arbitrary E in ξ_0, the space
$E(\mathbb{R}^{\infty}) := hoco\lim_{i} E(\mathbb{R}^{i})$

And the spectra

$\Theta E(1), \Theta E(2), \Theta E(3), \ldots$

are determined up to homotopy equivalence by the behaviour of E at in nity.

Proposition 4.5.5 For a morphism $g : E \to F$ in ξ_0 such that

h g i $hofiber\ E(V$
$) \to F(V)$

is contractible for all V. lets think of F to be connected at in nity, and that the covariant functors E and F are polynomial of degree $\leq n$.

Then g is a homotopy equivalence.

Proof. The problem lies in the fact that at each stage of V, the homotopy ber is de ned via a xed choice of base point in F(V), but we need an isomorphism of homotopy groups between E(V) and F (V) for all choices of base points. Let $F_b(V)$ be the Subspace of F(V) consisting of only the basepoint component of F(V).

We prove that

$$F_b \to F$$

is an equivalence after applying the functor $\tau_n = hoco\lim_k \tau_n^k$. Note that since E and F are n polynomial, the maps $E \to \tau_n E$ and $F \to \tau_n F$ are objectwise weak equivalences.

Consider the map
$hoco\lim_k \tau_n F_b \to hoco\lim_k \tau_n^k F$.

For each choice of basepoint, the homotopy ber of $\tau_n^k F_b \to \tau_n^k F$ is empty or contractible.

If C is some component in $F(V) \simeq \tau_n^k F(V)$, then because f is connected at in nity, there is some l such that the image of C in $\tau_n^l F(V)$ is in the basepoint component.

This holds since $\tau_n^l F(V)$ is de ned using only the terms $F(V \oplus U)$ for U of dimension greater than or equal to l.

Hence C is contained in $\tau_n^l F_b(V)$ and there can be no empty bers.

We thus have objectwise weak equivalences $T_nF_b \to T_nF$. Consider the map $T_nE(V) \to T_nF(V)$ and choose some basepoint x in $T_nF(V)$, then we see that $x \in \tau_n^k F(V)$ for some k.

As k increases, eventually x is in the same component as the canonical basepoint of $\tau_n^k F(V)$.

Hence by our assumptions, the homotopy bre for this choice x is contractible.

So $T_nE \to T_nF$ is an objectwise weak equivalence and it follows that $E \to F$ is a objectwise weak equivalence.

Now we show from Weiss that τ_m preserves n polynomial functors. The proof is simply that homotopy limits commute, $(\tau_n\tau_m = \tau_m\tau_n)$ and that homotopy limits preserve weak equivalences.

Lemma 4.5.0.1 If E is an n-polynomial object of ξ_0, then so is $\tau_m E$ for any $m \geq 0$. (Weiss, 1995)

Proof. We Start by showing that the canonical map

$$\tau_m E(V) = \underset{0 \neq U \subset \mathbb{R}^{m+1}}{ho\lim} E(U \oplus V) \to \underset{0 \neq W \subset \mathbb{R}^{n+1}}{ho\lim} \underset{0 \neq U \subset \mathbb{R}^{m+1}}{ho\lim} E(W \oplus U \oplus V)$$

Is a homotopy equivalence, for all generic object V in = target can be written as

$$\underset{06=W \subset \mathrm{R}^{m+1}}{ho\lim} \underset{06=U \subset \mathrm{R}^{n+1}}{ho\lim} E(W \oplus U \oplus V)$$

4.6 Homogeneous Functors

When working with actual smooth functions, the n-th Taylor approximation (around 0) to $f: \mathrm{R} \to \mathrm{R}$ is giving by $T_n(x) = \sum_{i=0}^{n} f^{(n)}(0) \frac{x^n}{n!}$

In particular, the di erence between two consecutive Taylor approximations is giving by

$$T_n(x) - T_{n-1}(x) = f^{(n)}(0) \frac{x^n}{n!}$$

The analogue of taking the di erence , when working with (stable) ∞-categories, is to nd the ber of the map $T_nF \to T_{n-1}F$.

The classi cation of homogeneous functors takes a similar form. It is the space of sequence of a bration whose bers are the derivatives $F^{(k)}(\varphi)$ with orthogonal group actions.

Let's consider some examples of homogeneous functors and also de ne what it means for a functor to be homogeneous and consider some examples and also de ne what makes a functor homogeneous.

De nition 4.6.1 Let $F : =_0 \to Top_*$ be a functor. De ne D_nF to be the ber of the natural transformation $T_nF \to T_{n-1}F$, then D_nF is a homogeneous functor of degree n.

If it is a polynomial functor of degree $\leq n$ and $T_{n-1}F(V)$ is contractible for every

$V \in =_0$

i.e. $T_{n-1}D_nF(V)$ ' $*$ for all $V \in =_0$.

Comment. For contravariant functor F, choose a basepoint in $=_0$. This bases F(V) for all $V \in =_0$.

This is then a homogeneous functor of degree n. That is the polynomial of degree $\leq$ n.

To see that $T_{n-1}D_nF(V)$ ' $*$ for every V, rst observe that T_{n-1} commutes with homotopy bers and next observe that $T_{n-1}T_nF$ ' $T_{n-1}F$.

Theorem 4.6.1 The full subcategory of n-homogenous functors inside $Ho(=_0Top)$ is equivalent to the homotopy category of spectra with the orthogonal group action on n.

For a given spectrum Ψ_E with orthogonal group action on n the functor below is an n-homogeneous functor of $=_0Top$.

$$V \mapsto \Omega^\infty\left[\left(S^{\mathbb{R}^n \otimes V} \wedge \Psi_E\right)/ho\,(n)\right]$$

We can think of, $S^{\mathrm{R}^n \otimes V}$ from the theorem as the one-point compacti cation of $\mathrm{R}^n \otimes V$

.

This has orthogonal group action(O(n)-action) induced from the regular

$$\gamma_n(V, \mathrm{W}) \times \gamma_n(\mathrm{U}, \mathrm{V}) \to \gamma_n(\mathrm{U}, \mathrm{W})$$
$$(g, y) \qquad (f, x) \mapsto \left(g \circ f, y + \left(\mathbb{R}^n \otimes g\right)x\right)$$

representation of the smash product is equipped with the diagonal action of $O(n)$,Ψ_E indicates a spectrum with the orthogonal group action $O(n)$. $O(n)$

denotes homotopy orbits alias the Borel construction.

We now look at how to obtain the spectra Ψ_E. We begin by recalling that = denotes the category of nite dimensional inner product space with maps the linear maps that preserves the internal structures. Let de ne a vector bundle over $=(U,V)$, for $U,V \in =$

$\gamma_n(U,V) = \{(f,x) | f : U \to V, x \in {}^n \otimes (V - f(U))\}$.

The total space of the vector bundle has a natural action of O(n) due to the R^n factor. We assume $=_n(U,V) := T\gamma_n(U,V)$, the associated Thom space. Hence this is the co ber in the sequence:

Recall that

$T(\mathrm{R}^n \to *) = S^n$ and $T(X = X) = X_+$

$$\begin{array}{ccccccc} S\gamma_n(U,V)_+ & \to & D\gamma_n(U,V)_+ & \to & T\gamma_n(U,V) & \to & T\gamma_n(U,V) \\ \wr\| & & \wr\| & & & & \\ \{(f,x) \mid \|x\| = 1\} & & \{(f,x) \mid \|x\| \leq 1\} & & & & \end{array}$$

as de ned already. In particular if we choose $n = 0$, then

$=_0(U,V) = =(U,V)_+$

When looking at the vector bundles there exist a natural composition

Where

$(\mathbb{R}^n \otimes g) : \mathbb{R}^n \otimes (V - f(U)) \to \mathbb{R}^n \otimes W.$

This composition induces associative and unital maps

$$=_n (V,W) \wedge =_n (U,V) \to =_n (U,W)$$

Which are O(n) equivariant and functorial in the inputs.(?)

CHAPTER 5

CONCLUSION AND RECOMMENDATION

This chapter gives a summary of the result of the study, also discussing the conclusions arrived at and nally giving recommendation that would be necessary for further research in the orthogonal calculus of functors

5.1 Conclusion

Orthogonal calculus of homotopy functors is important in diverse areas of study more precisely in algebraic and di erential topology. The applications to the calculus of functor (orthogonal calculus) are realistic in computer science, engineering, physics and many other elds.

The study explains that calculus is not only about derivatives or uxions but is also about approximation by polynomials.

The study focuses on linear polynomial functors. I.e. the study explained polynomial functors in the algebraic and topological settings with the topological setting focusing on the Goodwillie case, the embedding case and the orthogonal case.(thus concentrating on the linear case and generalizing it to the n-polynomial case)

The study reviewed continuous functors, Taylor Tower of Fibrations, derivatives of orthogonal calculus of functors by concentrating on categories of vector spaces to pointed spaces or any space that is enriched over Top_* .

Finally our research work has reviewed some structures of polynomial and homogeneous functors in the orthogonal calculus.

5.2 Recommendation

It is recommended that future research can be geared towards developing and obtaining a formula for Taylor approximation to spaces of smooth embedding's.

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APPENDIX A

We require E to be continuous $mor\ (V,W) \times E\ (V) \to E\ (W)$

We will consider $E : \mathfrak{I} \to Top$

$$\wr\wr$$

{Finite dimensional inner product subspace of $\mathbb{R}^{\infty}$}

EXAMPLE OF FUNCTORS

1. $O(n)$
2. $BO(V)$
3. $conf(n,V)$
4. $Emb(M,N)$
5. $\Omega^{\infty}(V^{c} \wedge \theta)$
6. $\Omega^{\infty}((R^{n} \otimes V)^{c} \wedge \theta)_{hO(n)}$

De nition

$$E(V) \to ho\lim_{0 \neq U \subseteq \mathbb{R}^{n+1}} E(U \oplus V) \text{ is a homotopy equivalence for all } V \in \mathfrak{I}.$$

$$= \tau_n E(V)$$

UNIVERSATILITY OF T_n

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & P_n \\
T_nE \arrow[ur, "\exists! \text{ up to homotopy}"'] &
\end{tikzcd}$$

Let $\varepsilon = cat(=,Top)$. $E \in \varepsilon$ is a polynomial of degree n, if Figure 5.1: Universality of T_n

$$T_nE = hoco\lim\left(E \to \tau_nE \to \tau^2{}_nE \to \ldots\right)$$

The nth taylor polynomial is $T_nE : \varepsilon \to \varepsilon$

$$\eta_n : T \to T_n$$

Remark

a. Every polynomial of degree $n-1 \Rightarrow$ it is polynomial of degree n

b. T_nE is a polynomial of degree n

c. If E is a polynomial of deg n, then $\eta_n : E \to T_nE$ is an equivalence

d. $T_n(\eta_n) : T_nE \to T_n^2E$ is an equivalence

Existence

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & P_n \arrow[d, "\simeq"] \\
T_nE \arrow[ur] \arrow[r] & T_nP_n
\end{tikzcd}$$

Figure 5.2: Existence

Figure 5.3: Uniqueness condition

Uniqueness:

$$\begin{tikzcd}
E \arrow[r, "\eta_n"] \arrow[d, "\eta_n"'] & T_nE \arrow[r] \arrow[d, "\simeq\ \eta_n(\eta_n)"] & P_n \arrow[d, "\simeq"] \\
T_nE \arrow[r, "\simeq"] & T_n^2E \arrow[r, "g"] & T_nP_n
\end{tikzcd}$$

Theorem. If $E \in \varepsilon_0$, then $D_nE \simeq \Omega^\infty\Big(\big(\mathbb{R}^n \underset{\substack{\uparrow\\ \simeq X^n}}{\otimes} V\big)^c \underset{\substack{\uparrow\\ \simeq f^{(n)}}}{\wedge} \theta\Big)hO\big(\underset{\substack{\uparrow\\ \simeq \frac{1}{n!}}}{n}\big)$

Corollary . E is homogeneous of degree n if $E\tau_n E$ and $T_{n-1}E' *$ Figure 5.4: Taylor Tower
De nition $E^{(n+1)} = hofib(E(V) \to \tau_n E(V))$

$E^1(V) = hofib(E(V) \to E(V \oplus \mathrm{R}))$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & T_nF \arrow[d] & D_nF \arrow[l] \\
 & T_{n-1}F & D_{n-1}F \arrow[l] \\
 & \vdots \arrow[d] & \\
 & T_2F \arrow[d] & D_2F \arrow[l] \\
F \arrow[uuuur] \arrow[uuur] \arrow[ur] \arrow[r] \arrow[dr] & T_1F \arrow[d] & D_1F \arrow[l] \\
 & T_0F &
\end{tikzcd}$$

Example. $E(V) = BO(V)$

$$\begin{array}{ccccc} E^1(V) = hofib\big(& BO(V) & \to & BO(V\oplus\mathbb{R}) & \big) = \dfrac{O(n+1)}{O(n)} = \\ & \| & & \| & \\ & \dfrac{EO(\mathbb{R}\oplus V)}{O(n)} & & \dfrac{EO(\mathbb{R}\oplus V)}{O(n+1)} & \end{array}$$