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Optimal Control of Mutual Fund Portfolio Choice in the Presence of Dynamic Flow with Partial Information

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

Firstly, I dedicate this work to the Lord God Almighty who helped me to overcome this feat. Indeed, his kindness and mercies abound.

Secondly to my brothers Habel, Lawrence and Philemon.

Abstract

We study the utility maximization problem of an economic agent who maximizes the expected utility of the terminal value of his wealth resulting from an investment in a mutual fund and a risk-less asset over a finite time interval in a financial market with partial information. We use continuous-time ordinary differential equation and stochastic differential equation to model security prices. The novelty of this work is that we restrict the assumption that market participants can observe the drift vector process used in the specification of stochastic differential equation for the asset prices. Kalman-Bucy filtering method is used to estimate the drift vector process. Stochastic control theory and dynamic programming methods were used to address the maximization problem. The main results are the explicit representation of the optimal trading strategies and value functional for the economic agent and the fund manager for three different utility functions: namely, the log, power and negative exponential. The optimal trading strategies of the economic agent are independent of wealth level except for the negative exponential utility function whose optimal trading strategy is a decreasing function of wealth. They inversely related to the fee rate for all the utility functions. Optimal trading strategies for the fund manager are also autonomous of wealth level, but directly correlated to the fee rate charged. The accumulated fee process of the mutual fund is autonomous of the fee rate charged and directly related to wealth level (except for the negative exponential utility function where it is independent of wealth). We found that the product of the economic agent's optimal trading strategy and the fund manager's optimal trading strategy results in the optimal trading strategy that the economic agent would have selected if he had been allowed to trade freely in the risky securities.

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Contents

List of Abbreviation

SDE .... Stochastic Differential Equation

BM ....Brownian motion

GMB ....Geometric Brownian motion

DP .... Dynamic Programming

DPP ....Dynamic Programming Principle

HJB ....Hamilton-Jacobi-Bellman

MPT .... Modern Portfolio Theory

CAPM ....Capital Asset Pricing Model

CRRA .... Constant Relative Risk-Aversion

IRRA .... Increasing Relative Risk-Aversion

ODE ....Ordinary Differential Equation

CSP ....Controlled State Process

RHS ....right hand side

LHS ....left hand side

w.r.t ....with respect to

DE ....differential equation

SP .... stochastic process

CV ....Control Variable

PDE .... partial differential equation

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Chapter 1

Introduction

1.1 Background

A *portfolio* can be defined as a collection of investment instruments; namely stocks, shares, mutual funds, cash, bank account, money market instruments and so on depending on economic agent’s wealth (or income), budget, convenient time frame and the risk tolerance level. Economic agents can either create a market portfolio or a zero-investment portfolio. The technique of picking out the appropriate investment policy for economic agents in terms of the lowest risks and highest profits within a stipulated time frame given all the market conditions is known as portfolio management (MSG, Accessed Dec. 2015). A portfolio can be managed personally or by a hired person. People who manage investments for other investors are sometimes called fund managers, money managers or portfolio managers. A *mutual fund* refers to an investment medium that is made up of a lump of funds received from many investors for the purposes of buying return yielding assets. Mutual funds are operated by money or fund managers, who invest the fund’s capital and attempts to produce capital gains and income for the contributors of the fund. In the mutual fund industry, economic agents are charged fees for the services rendered to them by the fund managers. Some of these fees fall into the following categories: fraction-of-fund fees, convex performance fees, fulcrum fees, asymmetric performance fees and high-water-mark fees as documented in Hugonnier and Kaniel (2010). Traditionally, economic agents who invest in mutual funds make one time investment and are allowed to increase their investment share in the fund as they have additional fund. Thus, exogenous infusion to the mutual fund is permitted, but withdrawal from the fund is not permitted before the investment matures. The latter only granted with a fine or forfeiting any return. The fund manager is then allowed to manage the fund’s capital over the term of the investment until maturity to meet the future consumption and investment needs of their clients.

A classical problem in financial economics is the utility maximization problem of an economic agent who has the “delima” of what proportion his wealth should be allocated to a risky-free asset and risky asset(s) purposely for maximizing the expected satisfaction of terminal wealth at some terminal period given financial market uncertainties. According to Markowitz (1952), “The exercise of determining a portfolio may be split into two steps. The first step begins with surveying and practical knowledge and ends with judgment about the future earning potential of the available securities. The second step starts with the relevant judgment about future earning potential and ends with the selection of a portfolio.” Based on the second process, Markowitz establish a geometric relationship between judgement and choice of portfolio based on “expected returns-variance of returns" criteria in his paper “Portfolio Selection” that won him the Noble Prize in Economics. This approach to portfolio selection is known in modern finance literature as Mean-Variance Portfolio Theory. Optimal portfolio choice problems are studied using static and dynamic models. The masterpiece papers of Merton (1969, 1971) were the pacesetters that studied this problem in the framework of continuous-time financial models. In the simplest form, he showed that closed-form solution exists for an investor who decides on the proportion of his wealth to allocate between consumption and investment when his preferences over consumption bundles and terminal wealth can be modelled by a constant relative risk-averse utility function. Merton’s milestone attracted many researchers into the application of both deterministic and stochastic continuous-time models to the study of varied optimal investment choice problems. The work by Cox et al (1985), Karatzas et al (1987, 1991), He and Pearson (1991), Duffie and Zame (1898), Cox and Huang (1989) are some significant ones. These papers studied the utility maximization of an economic agent who seeks to maximize the expected satisfaction of final value of his portfolio on a finite time horizon given that he allocates his fortune among one risk-free asset and *n* risky securities. Thus, the traditional models of portfolio maximization pioneered by Markowitz and expounded in a multi-period continuous-time domain by Merton (1969, 1071) and the others mentioned above assume economic agents invest in risky assets personally. According to Kremer (2012), "This arrangement depicts the governing state of affair on the capital markets long-ago. The year 1950 for example, record about ninety percent of corporate equity was held by private households. Going forward, more and more households have entrusted the management of their portfolios to professional portfolio managers. In 2000, less than forty percent of private households managed their equity portfolios on their own. In contradiction, the market for entrusted investment policies has experienced long-term, continuous growth over the last five decades. In 2000, equity holdings of mutual funds, the most prominent form of delegated investment decisions, accounted for approximately twenty percent in the US. Together with private pension funds and state and local pension funds, both forms for delegated investment decisions, they reached over forty percent of total equities outstanding. Consequently, direct equity investments constitute only a minority of all equity investments today and transference of equity investments has become the practice. The time-consuming and high opportunity cost of trading equities is responsible for economic agents shift from direct investment in equities to mutual funds." The application of continuous-time models to the study of mutual fund portfolio choice can also be found Cetin (2006), Breton et al (2010) and Hugonnier and Kanniel (2010). A common feature of the above papers is the massive usage of *full information* availability to economic agents in the construction of the financial market models. Lakner (1995, 1998), Feldman (2007), Monoyios (2007, 2010), Bj¨ork, Davis and Land´en (2010), Liu and Muhle-Karbe (2013) are some papers that study continuous-time portfolio choice under *partial information*.

This current work discusses the utility maximization problem of an economic agent maximizing expected satisfaction of final wealth on the finite horizon $[0,T]$ by investing his wealth, $W(t)$ at anytime $t \in [0,T]$ in a risky-free asset and a mutual fund consisting of n risky assets plus one risk-free asset. The innovation here is that we restrict the assumption that all market participants can directly observe the return vector process $\mu(t)$ as well as the Brownian motion (BM) $B(t)$ employed in the mathematical formulation of the evolution of risky asset prices. In finance literature, this situation is termed *partial information*. This feature distinguishes this work from that of Hugonnier and Kanniel (2010). This is tenable since the drift vector, $\mu(t)$ and the BM, $B(t)$ are fictitious mathematical objects, and certainly not directly observable to agents in the economy

1.2 Problem Statement

Economic agents over the years seek to optimally allocate their scarce resources among different financial assets so as to fulfill certain investment objectives. For instance, an economic agent allocating his or her wealth among $(n + 1)$ or between a risk-less asset and a mutual fund because of the celebrated adage that says “never put your all your eggs into one basket” would want to do so to attain certain desired future wealth. The question, however, still remains: “What proportion of wealth should be devoted to each of these assets at an time t amid all market or economic conditions so as to attain the target objectives?” “Which of these proportion of wealth allocation among securities maximize the performance indicator?” “For the best proportion of wealth allocation among securities, what value does the performance indicator assume?” Should I create my own portfolio or delegate my investment decision to a second party?” “What are the consequences of a particular choice?” Are the market parameters directly available for model creation?” In an attempt to answer these questions, Mutual Fund Portfolio Choice problem has been widely studied. In most of those studies, researchers used the simplifying assumption that market participants have full information on all the market parameters and variables. However, in reality, market participants do not have full information on all the market parameters and variables that are used for model creation. What are the implications of full information and partial information assumptions on the investment decision rules?. To this end, this thesis seeks to study the utility maximization problem of an economic agent in an economy with partial information.

1.3 Study Objective

1.3.1 General Study Objective

The central task of this work is the maximization of the expected satisfaction (or utility) of the terminal wealth of an economic agent who dynamically allocates his or her wealth between a risk-less asset and a mutual fund over a finite time span.

1.3.2 Specific Objectives

- Derive an optimal trading policy (strategy) for the economic agent;
- Derive an optimal trading policy (strategy) for the mutual fund manager;
- Find the optimal value functional for the economic agent;
- Find the total wealth accumulation of the mutual fund manager.

1.4 Methodology

Our attempt to address the expected utility maximization problem of the economic agents who invest in a risk-less asset and a mutual fund shall be done using quantitative research method. This will use continuous-time financial models such as ordinary and stochastic differential equations. We shall model the price dynamics of the risk-less asset by an ordinary differential equation (ODE). The price dynamics of the risky assets would be modelled by a stochastic differential equation (SDE) known as geometric Brownian motion (GBM) model with directly unobservable drift parameter. The drift parameter would be estimated within the Kalman-Bucy framework. Economic agent's welfare or satisfaction will be model using utility functions. Tools of dynamic programming (DP) and stochastic control theory would be used to solve the problem of the economic agents.

1.5 Justification or Significance of study

Many would argue that the technology breakthrough plus financial globalization has made economic information readily available and accessible to all economic agents. However, the economic complexity and varied needs of economic agents to some "extent" have "defeated" this argument. In particular, according to Bank of Ghana Statistical Release (Nov. 2004), "Mutual Funds assets currently constitute 61.6 per cent of all Collective Investment schemes' assets. The total value of mutual funds stood at 285.8 billion old Ghana cedis, more than three times its December 2003 position. These positive developments in the financial sector, however, have come with some challenges. A fundamental one is the portfolio decisions of banks and other financial intermediaries and their associated risks, in response to the emerging unstable macroeconomic situation." To this end, the subject of this work warrant discussion. The study of the subject is to provide knowledge to the economic agents on how to allocate income between an asset and a mutual fund. It presents the best investment plan to individuals, corporations, associations and fund managers. It also add to the existing literature on the application of DP methods to the study of mutual fund portfolio choices.

1.6 Limitation of the study

Our inability to solve the formulated optimal control problem using numerical methods is the main limitation to this study.

1.7 Organization of Thesis

This thesis is divided into five chapters. Chapter one gives a brief introduction to the subject of this study. It also talks about problem statement, objective, methodology, justification and limitation of the thesis. Chapter two focuses on the the review of the relevant literature on portfolio choice in the framework of both full and partial information. Exposition on the mathematics needed to address the optimization problem is discussed in chapter three. Chapter four begins with the economic set -up of the model and spans through mathematical specification of the model and solving of the utility maximization problem. The last chapter includes the thesis' findings, limitations, conclusions and areas of further research.

Chapter 2

Literature Review

2.0.1 Introduction

This chapter reviews various literature on optimal portfolio choice problems. It will open with pricing or valuing assets using Brownian motion. Ideas on full information and partial information in an economy are then presented. It closes with the discussion of the findings of relevant literature on optimal portfolio choice problem in the framework of both full and partial information.

2.0.2 Pricing Asset Using Brownian motion

In 1827, Robert Brown, a Scottish botanist, discerned under a microscope that pollen grains in water appear to be in a constant state of distress. Initially, postulate that there might be something 'alive', but demonstrate otherwise by observing the same kind of motion in particles of dust. He was unable to adequately decipher the observations. Subsequently, it was unearth that such uneven patterns in motion come from the eminently enormous number of knocks of the suspended pollen grains with molecules of the fluid. Einstein (1905) and Wiener (1923) were credited to have given the mathematical specification of the Brownian motion (BM) even though Bachelier (1900) started the study of diffusion processes five years before the former and decades before the latter. The application of Brownian motion to valuing or pricing assets dated back to the year 1900 with the pioneering PhD thesis of Bachelier (1900). Bachelier used continuous-time models to study the price movement of stocks and remarked that the price movements can be likened to the motion of small particles suspended in liquids. Bachelier independently determined the underlying equation of the movement of the stock price and found pricing formulas for “put and call” options on stocks. Bachelier’s formulation of the movement in the fluctuation of the price process of a stock is defined as

$$dX(t) = \sigma dB(t), \tag{2.1}$$

where σ is a measure of the volatility of the stock price. Thus, the price of an asset, $X(t)$ at time, t has an infinitesimal increment $dX(t)$ that proportionate the increment of the Brownian motion, $B(t)$. A major weakness of Bachelier’s model is that the price of the asset, $X(t)$ can take negative values. This does not support reality since actual stock price can not be negative. Despite this flaw, his findings began a new paradigm in the finance circles and gave birth to the field of mathematical finance. To address this flaw, his successors like Samuelson (1965, 1969) assumed that the relative price of a stock is proportional to the increment $dB(t)$:

$$dX(t) = \sigma X(t)dB(t). \tag{2.2}$$

If $\sigma = 1$ then equation (2.2) is called standard BM. Adding the expected percentage return of the stock to the right hand side (RHS) of equation (2.2)
gives

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t). \tag{2.3}$$

Equation (2.3) is called the BM with drift term or the Geometric Brownian motion (GBM). The use of the GBM model for pricing assets gained popularity in application in financial economies and financial mathematics after Black and Scholes (1973) and Merton (1973) used it to derived consistent formulas for the fair price of European options. Merton (1973) noted that equation (2.3) is sometimes viewed as It´o SDE because It´o provided the tool called It´o calculus for obtaining explicit representation for the stock price. By It´o calculus, equation

(2.3) has an analytic formula

$$X(t) = X(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}$$

for some an arbitrary initial value X(0).

2.0.3 Full information and Partial information

The terms full and partial information are widely used in financial economics and mathematical finance as well as in other fields. These two terms are sometimes linked to certainty and uncertainty. Uncertainty arises in partially observable and/or stochastic environment, as well as due to ignorance and/or indolence. In financial economics and mathematical finance uncertainty in an economy is mostly represented on a filtration which refers to the flow (or revelation) of information over time. For instance, in Liptser and Shiryayev (1977), the idea of partial information is presented in the following context: Let $(\theta,\xi) = (\theta_t,\xi_t), t \geq 0$ be a partially observable random variables defined on the triple (Ω,F,P) called a probability space. We require that only the second element $\xi = (\xi_t)$, $t \geq 0$ can be observed. Then, we are required based on $\xi_0^t = \{\xi_s, 0 \leq s \leq t\}$ to approximate the unobservable state process θ. Lakner (1998) and Bj¨ork, Davis and Land´en (2010) both expressed the idea of partial information by stating that the only information available to the economic agent is the one generated by security prices and in particular, the expected return processes cannot be observed. The idea that the return processes (and sometimes the volatility process as well) are (or not) directly observable by the market participants characterizes utility maximization problems under full (or partial information). According to Bj¨ork et al (2010), the standard approach to solving utility maximization problems under partial information case can be formulated as stated below:

1. Suppose that the mean return process have a particular process (say, “hidden Markov model”, “Bayesian process” or “Gaussian process”).

2. Formulate the securities price evolution on the observable filtration to obtain an entirely observable system.

3. Formulate the filtering equations for the expected return process and adjoin the estimate processes.
4. Address the reconstructed problem by employing standard dynamic programming techniques or alternatively use martingale techniques.

2.0.4 Mutual Funds

According to Wikipedia (Accessed Dec. 2015), “a mutual fund is a competently managed investment fund that lumps money from many investors to purchase securities". The pooled economic resources are used to acquire assets such as stocks, bonds, money market instruments, real estate and so on. Mutual funds are standardized investment vehicles for delegated portfolio management (Pozen, 2012). These investment vehicles are principally aimed at primary economic agents. Primary economic agents who customarily invest in mutual funds do so owing to the fact they want to engage in the equity market but refrain or does not posses the skills to form their own portfolio. Perhaps due to the time-consuming and high opportunity cost involved in trading equities directly. Primary investors can trade their ownership in a mutual fund at its quoted current prices-the price of the underlying investments at the end of trading period, rendering mutual funds liquid investment (Kremer, 2013). Some mutual funds allow investors to make contributions and these contributions are managed for a specified period of time. During this stated period, investors restrained from withdrawing their contribution. Others also allow both infusion and withdrawal of resources during the life of the investment. This possibility has made mutual funds very liquid. An example of mutual fund that does not allow withdrawal is an insurance policy that is bought by many for the purposes of mitigating loses against unforeseen event such as death. In this instance, investors are allowed to make regular contribution called a premium but cannot withdraw from the policy before the maturity date which is contingent on death. A similar one is the pension fund where beneficiaries are entitled to their benefits only upon retirement, casualty or death (where benefits are paid to the next of kin).

According to the Bank of Ghana Statistical Release (Nov. 2004), “There has also been a surge in Collective Investment schemes. From only three in 2003, the number had risen to eight by the end of September 2004, providing individuals and companies with alternative investment vehicles for their financial savings. These positive developments in the financial sector, however, have come with some challenges. A fundamental one is the portfolio decisions of banks and other financial intermediaries and their associated risks, in response to the emerging stable macroeconomic situation.” The statistics in support of the fact that the demands for mutual funds over the decades are on the surge is given below: The value of assets held by eight collective investment schemes at end September 2004 stood at 456.4 billion or 0.57% of GDP. This may be compared with the Insurance sector with assets of GHS1.2 trillion or 1.84% of GDP (December 2003), and banking sector assets of GHS27.6 trillion or 35.62% of GDP. While the size of collective investment schemes sub-sector may look insignificant, the rate of growth of the sector, especially in 2004 should be of interest. From just over GHS70 billion in 2002 (3 funds), the sector’s assets rose to GHS164.9 billion in 2003 (134.6%) before increasing further (152.2%) to the present level.

In particular, the Bank of Ghana Statistical Release shows that Mutual Funds assets currently constitute 61.6% of all Collective Investment schemes’ assets. The total value of mutual funds stood at GHS285.8 billion, more than three times its December 2003 position. The expansion was driven mainly by Epack, as two of the mutual funds became operational in August 2004. Unit trusts are also dominated by the HFC Unit Trust, which has close to 76.0% of the sub-sector’s assets of GHS159.7 billion. The HFC REIT has grown rather slowly over the last year recording a yield of 10%, compared to 20% in 2002, because of a slow down in the property market and a legal suit the Trust had to battle with. The other two funds have barely started operations.

Figure 2.1 shows the growth in mutual funds assets in Ghana

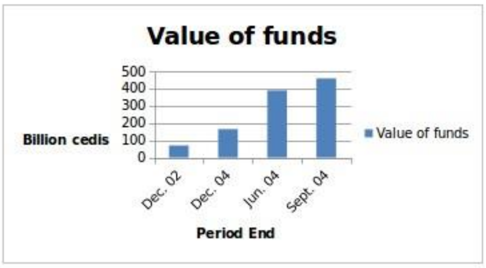

Figure 2.1: Growth of Mutual Fund. (Source: Bank of Ghana Statistical Release, Nov. 2004)

eStock Analysis (Apr. 2013) documented 22 Mutual Funds companies in Ghana. A more general detailed statistics supporting the insurgency in the mutual fund industry can be found in Kremer (2013).

In the mutual fund industry, fund managers adapt special investment strategies. Their investment strategy has the optimization criteria that they must maximize the their compensation. The compensation takes the form of explicit and implicit incentives, performance fee, fee charge on the asset under management, fulcrum fees and so on. The optimization criteria takes into account the quantity of asset under the management of the fund manager as determined by the the contribution and withdrawal of funds by the primary economic agents as well as the effect of price changes in the securities traded. Some factors affecting fund flow to the mutual fund are mutual fund performance, general market condition and management fee. Fund performance is typically determined by using fund's Sharpe ratio and other performance measures. For instance, a fund may be tipped as performing well relative others when its Sharpe ratio computed with respect to the overall mutual industry rank highest. In the case of the management fee, any rational primary investor will shy away from a fund manager who charges higher fees that do not commensurate return.

2.0.5 Review of portfolio choice literature

In ancient civilization, economic agents formed their portfolios solely by evaluating the return and risks of individual securities in isolation. This led to the construction of portfolio of the same return and risk characteristics. Markowitz (1952) argued that economic agents must hold portfolios based on their overall risk-return characteristics by showing how to compute the mean return and variance of a given portfolio in his paper "Portfolio Selection". Markowitz illustrated the mean and variance of the portfolio return geometrically and described a concept called efficient frontier. The efficient frontier is a graphical illustration of a collection of feasible portfolios yielding different return for their level of risk. The efficient frontier shows the trade off between risk and return. Markowitz ideas laid the groundwork for the Modern Portfolio Theory (MPT). Despite the theoretical and practical importance of MPT it was criticized not to be an ideal investment strategy merely because of its numerous assumptions (see Merton and Samuelson (1992). These criticisms paved the way for the independent and collaborative researches. Among them are the Super-efficient Portfolio and the Capital Market Line (CML) of James Tobin (1958) and also Jack Treyor, William Sharpe, John Lintner and Jan Mossin independent formalization of the capital asset pricing model (CAPM). According to Merton and Samuelson (1992), the one period mean-variance model of Markowitz, Jobin's super-efficient portfolio and its equilibrium version, the Sharpe-Lintner-Mossin CAPM, remained the primary capital-market models for optimal portfolio selection up to the late 1960s. To them, the main reason for their limited application was a widespread belief that the mean-variance principle deviates from the generally accepted von Neumann-Morgenstein axioms of choice unless either assets prices follow Gaussian distributions or the investors have quadratic preferences. History of mathematical finance traced the application of continuous-time financial models to Bachelier (1900). His research was unknown in the circles of economics and finance until Samuelson (1969) rediscovered it and gave a more precise specification of the stock price dynamics to what is known in finance parlance is GMB. A major breakthrough in optimal portfolio choice problem using continuous-time models came when Merton published his two revere papers "Life-time Portfolio Selection under Uncertainty: The Continuous-time Case" and "Optimum Consumption Rules in a Continuous-time Model". In Merton's opinion, the continuous-time model is a watershed between the static and the dynamic models of finance. Merton showed in the investment-consumption problem that for a constant relative risk-aversion (CRRA) class of von Neumann-Morgenstein utility functions close-form results exist for the optimal trading strategy and optimal consumption strategy. He found that the proportion of wealth level held in the risky asset rises as wealth rises. Merton also showed that for an increasing relative risk-aversion (IRRA) utility functions as the wealth of the risk-averse economic agent rises his holdings in the risky asset dwindles. Oksendal (2003) found that for a CRRA utility functions, economic agents who are risk-averse invest the same proportion of their wealth level is held in the risky asset as wealth rises when he looked at two asset portfolio choice problem. Both Merton and Oksendal modelled the price of the risky asset by GBM model with constant market coefficients and that of the risk-less asset modelled by ordinary differential equation (ODE). Investors were also allowed to take only long position in the assets. These two researchers used techniques of dynamic programming to solve the problem of the economic agent.

Lakner (1995) tackled the dual maximization problem of maximizing expected satisfaction from consumption and final wealth. In this paper, he require the consumption strategy and the trading strategy to be adapted to the natural filtration triggered by the price process of the stock and assume that the Brownian motion and the drift term are not observable by the investors. By requiring the drift vector to be an F_0-measurable d-dimensional random variable with a known distribution he derived explicit solutions for the terminal wealth, optimal consumption strategy and trading strategy for a logarithmic utility function. Again, Lakner (1998) derived explicit solution for the optimal trading policy for an economic agent who maximizes expected utility of terminal wealth by investing in $(n + 1)$ securities. This was achieved by using martingale methods for portfolio optimization and modelling the directly unobservable drift process as Gaussian process. The drift was modelled as a SDE with Brownian motion independent of those appearing in the stock prices and solution to this SDE was found within the Kalman-Filter framework. He found that for the log utility function the optimal trading policy can be expressed in the feedback-form directly from the optimal trading strategy obtained under the full information assumption by substituting the estimated drift process for the observable constant drift. A similar research by Dothan and Fieldman (1986) modelled the mean return as function of a finite state markov chain, resulting in Wonham filter estimation. Also by using dynamic programming methods, B¨auerle and Rieder (2004, 2005) studied the situation of an underlying markov chain. They compare the optimal the trading strategy to that of the case of full information model. Hugonnier and Kaniel (2010) studied

"Mutual Fund Portfolio Choice in the Presence of Dynamic Flows". They considered economic agents who invest in a risk-less asset and a mutual fund which also invest in *n*-risky securities plus a risk-less asset. In building their model, economic agents are assumed: to have logarithmic utility function, to be restricted from holding equities directly-largely due to the assumption that they face high opportunity cost trading in equities directly, to be myopic and have long position in their portfolio. However, the economic agents are allowed to change position in the fund dynamically, because they can trade the mutual fund based on its net asset value. All agents are assumed to have full information and observe each others action. The fund manager receives a fraction-of-fund fees as a remuneration for his services. They scrutinize the effect of dynamic flows on the mutual fund's trading strategy by viewing the economic agent as someone who perceives that his wealth allocation verdicts may not influence the mutual fund's trading strategy and the fund manager is assumed to be strategic. Hugonnier and Kaniel solved the optimization problem for the two agents using the theory of backward SDE and tools of utility maximization theory in incomplete markets. Taking these strands, the authors were able to demonstrate that in the stability state, the equity constituent of the fund changes as the the fee rate charged also changes, depicting a positive relation between the fund's fee rate charged and its variability. The economic agent's demand of the fund moves in opposite direction to the fee rate. They also found that for a non-stochastic investment opportunity sets, an economic agent's demand for the fund is proportional to the mutual fund's net-of-fees Sharpe ratio and conjectured that the fund trading strategy should be the one that maximizes the fund's net-of-fees Sharpe ratio. They showed that further that this conjecture does not hold for a stochastic investment opportunity set. Thus, for a constant directly observable market coefficients, as fee rate rises, a larger proportion of investor's wealth is apportioned directly in bond. This they explained occurred through two different routes. Firstly, the higher the fee rate, the smaller the wealth devoted to the fund. Secondly, because the fee is charged on the assets under the jurisdiction (or control) of the manager, it is exercise on the equity as well as the bond part of the mutual fund. Thus demanding the bond via the fund becomes costly for the economic agent compared to demanding it personally. The authors stated that the increased allocation to the bond reduces the investor's indirect holding of the bond abating his response to higher fees. According to the authors, the fund manager's power to choose the fund's trading strategy and the ability of the investor to vary his demand of the fund atone each other. Meaning that the two are equally well off regardless of who dictates the fee rate. A similar study by Breton et al (2010) titled "Mutual Fund Competition in the Presence of Dynamic Flow" shows similar results as those found by Hugonnier and Kaniel (2010). They considered more than two mutual funds, each charging fraction-of-fund fees and all the funds can freely trade identical bundle of securities. Taking the economic agent's investment decisions as given, the managers engage in a Nash game. Vis-a-vis these postulations the authors are able to illustrate the existence of a unique Pareto optimal equilibrium whereby each manager chooses the same fund trading strategy as found by Hugonnier and Kaniel (2010). This means that all mutual funds present equal risk-reward trade off. It therefore suffices that the economic agent is unconcerned about having a collection of mutual funds. Breton et al (2010) just like Hugonnier and Kaniel (2010) assumed that the market coefficient are directly observable by agents in an economy.

Chapter 3

Methodology

This chapter reviews definitions and concepts in probability theory and stochastic processes. We define a financial market using the concepts and results thereafter. We overview elements of the theory of stochastic differential equations, based on Brownian motion, for use in subsequent chapters. Existence and uniqueness conditions for solutions to SDE are presented. We discuss stochastic integration using Ito calculus. The final section to this chapter discusses Stochastic Controlˆ Theory. Specifically, the concept Dynamic Programming Principle (DPP) technique for solving optimal control problems as well the demonstration of how the DPP leads to the derivation of a non-linear partial differential equation known as the Hamilton-Jacobi-Bellman (HJB) equation from optimal control problems.

3.1 Mathematical Preliminaries

3.1.1 Probability Theory

Definition 1 (sigma-algebra (simply denoted σ-algebra)) *Let Ω be a given set, then a σ-algebra denoted* F *defined on Ω is a collection of subsets of Ω satisfying the properties:*

- *the empty set* $\emptyset \in F$, *i.e,* F *is non-empty;*
- *closed under complement:* $F \in F$ *implies that* $F^{0} \in F$ *where* $F' = \Omega \setminus F$*and the superscript '0' denotes complement;*
- *closed under countable unions:* $F_1, F_2, \ldots \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} F_i \in \mathcal{F}$.

Definition 2 (Smallest σ-algebra) *Given* $C \subset 2^{\Omega}$, *then the σ-algebra triggered by* C *denoted by* $\sigma(C)$ *is the "smallest" σ-algebra containing* C. *That is,* $\sigma(C) = T_{F \supset C} F$.

Remark: $\sigma(C)$ always exist

because:

1. 2^{Ω} is a σ-algebra;
2. any intersection of σ-algebra is also a σ-algebra.

Definition 3 (Measurable Space, Measure and Probability measure)
The combination (Ω,F) *is referred to as a measurable space. A measure* U *on* Ω *with* σ*-algebra* F *is a function* $U : F \to [0,\infty)$ *satisfying*

- $U(\emptyset) = 0$;
- $\mathcal{U}(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mathcal{U}(F_i)$ *for any pairwise disjoint sets* $F_1,F_2,F_3,\cdots \in F$.

Given the function $P : F \to [0,1]$ *with the property*

- $P(\emptyset) = 0$ *and* $P(\Omega) = 1$;
- *if* $F_1,F_2,... \in F$ *and* $\{F_i\}_{i\geq 1}$ *has no intersection (i.e.,* $F_i \cap F_j = \emptyset$, *for* $i =6$ j*) then* $P(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} P(F_i)$;

then we call P a probability measure. Thus a measure plus the property $P(\Omega) = 1$ *is a probability measure.*

Definition 4 (Probability Space) *The ordered triple (or the three-tuple)* (Ω,F,P) *is known as a probability space. It is said to be complete if* F *include* 2^{Ω} *of* Ω.

We shall suppose that our ordered triple are complete. Note that the subsets of Ω which belongs to F are known as F-measurable sets. In probability terms these sets are termed as events and we shall use the notation $P(F)$ to mean "the probability that the event F happens". We say that "F happens with probability 1" or "almost surely" if $P(F) = 1$.

Definition 5 (Measurable and Random variable) *Let* (Ω,F) *and* (L,M) *be two measurable spaces. Then,*

- *A function* $\psi : \Omega \to L$ *is called* (F,M)*-measurable if* $\psi^{-1}(A) \in F$ *for all* $A \in M$. *If* $\psi : \Omega \to L$ *is a* (F,M)*-measurable map we write more succinctly* $\psi : (\Omega,F) \to (L,M)$.
- *A measurable map* $X : (\Omega,F) \to (L,M)$ *is called a random variable.*

  A random variable X is called an n-dimensional random variable if $L = \mathbb{R}^n, \ \mathcal{M} = \mathcal{C}(\mathbb{R}^n)$.

It is vital to explain that the random variable defined above is a real-valued random variable that assigns a real value to each basic occurrence say ω. This value can be viewed as the outcome of an investigation. In particular, the random variables we will be considering will represent the values of financial instruments and wealth relying on the state of the world.

Definition 6 (Distribution) *Suppose P denotes its usual probability measure meaning on* $(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$. *Then the function*

$$\begin{aligned} F_X(x_1,x_2,\ldots,x_n) &:= F_{X_1}(x_1) \times F_{X_2}(x_2) \times \ldots \times F_{X_n}(x_n), \\ &= P(X_1 < x_1) \times P(X_2 < x_2) \times \ldots \times P(X_n < x_n) \end{aligned}$$

is called an n-dimensional distribution function of X. If $n = 1$, *then*

$$F_X(x) := P(X < x)$$

is called the distribution function of X.

Definition 7 (Density function) *Suppose* $F_X(x)$ *denotes the distribution function of X. If* $F_X(x)$ *is differentiable, then we define*

$$f(x) := \frac{\partial}{\partial x} F_X(x)$$

as the density function of X.

Definition 8 (Independence of random variable) *Let* $X : (\Omega,\mathcal{F}) \longmapsto (L,\mathcal{M})$ *and* $Y : (\Omega,\mathcal{F}) \longmapsto (L,\mathcal{M})$ *be two random variables. We say that X and Y are independent if for all* $A, B \in \mathcal{M}$ *the events* $X^{-1}(A)$ *and* $Y^{-1}(B)$ *are independent.*

Definition 9 (Mathematical Expectation) *Let define X as a real-valued function on* $(\Omega,\mathcal{F},P)$*.*

- *Suppose that X is P-integrable, we define the expectation of X by*

$$\mathbb{E}^P[X] = \int_\Omega X dP$$

.

- *Given* $F_i \in \mathcal{F}$ *with P defined, then we can define the conditional expectation of X under* F_i *by*

$$\mathbb{E}^P[X|F_i] := \frac{1}{P(F_i)} \int_{F_i} X dP.$$

3.1.2 Continuous-time Deterministic Process and Stochastic Process (SP)

A process is an event that evolves overtime intending to achieve a goal. The time period may be given as finite (i.e., $[0,T]$, $T < \infty$), indefinite (i.e., $[0,\tau]$ for some stopping time τ) and infinite (i.e., $[0,\infty)$).

A continuous-time deterministic process is a process in which no randomness (a measure of the uncertainty of an outcome) is entail in the growth of the future paths of the process. For some given starting criterion or initial state, deterministic model will thus always yield the same trajectory. A deterministic process has the property that most of the times, the entire input and output relation of the process is conclusively determined. In passing, a deterministic process is one that evolves in one direction (a notable example is the solutions of ordinary differential equations (ODEs)). A continuous-time SP is a group of random variables, denoting the path evolution of some system of random values over time. It is the probabilistic analogue to a deterministic process. In a SP, for the same initial value there are several directions in which the process can evolve overtime.

Definition 10 (Stochastic Processes) *The collection* $X = \{X_t : t \in [0,\infty)\}$ *of random variables*

$$X_t : (\Omega,\mathcal{F}) \longmapsto (M,\mathcal{M})$$

is called a continuous-time SP. The collection X may be interpreted as a

$$X_t : [0,\infty) \times \Omega \longmapsto M \text{ such that } X(t,\omega) := X_t(\omega), \forall\ (t,\omega) \in [0,\infty) \times \Omega.$$

For fixed $t \in [0,\infty)$ *we view* X_t *as the outcome of an investigation at each t. The stochastic process assigns a path to every* $\omega \in \Omega$*. For a fixed* $\omega \in \Omega$*, the path* $X(\cdot,\omega) := \{t,X(t,\omega)|t \in [0,\infty)\}$ *is a series of realizations of a the random investigation* X_t *associated with state* ω*.*

Definition 11 (Path) *Let X denote a SP. Given a fixed* $\omega \in \Omega$ *the mapping* $t \longmapsto X(t,\omega)$ *is called a path of X.*

Definition 12 (Equality of Stochastic Processes) *Let* X_1 *and* X_2 *denote two stochastic processes.*

- *We say* X_1 *and* X_2 *are equal if*

$$P(X_1(t) = X_2(t) : \forall\ 0 \leq t < \infty) = 1.$$

- *We say* X_2 *is a modification of* X_1 *if*

$$P(X_1(t) = X_2(t)) = 1 : \forall\ 0 \leq t < \infty.$$

Definition 13 (Filtration (Information revelation)) *Let* $(\Omega,\mathcal{F})$ *denote a measurable space. An increasing collection of σ-algebras* $\{\mathcal{F}_t\}_{t\geq 0}$ *where* $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ *for* $0 \leq s \leq t$ *is* *called a filtration on* (Ω,F). $\{\mathrm{F}_t\}_{t\geq 0}$ *represents the availability of information over time and* F_t *is the information available at time t.*

Definition 14 (Generated filtration) *Let X denote a SP on* (Ω,F). *Then*

$$\mathcal{F}_t^X := \sigma(X_s, s \in [0,t])$$

is called the smallest σ-algebra with respect to (w.r.t) which X_s *is measurable* $\forall\ s \in [0,t]$.

Definition 15 (Natural Filtration) *Given a SP* $X = \{X_t : t \in [0,T]\}$, *the natural filtration induced by this process is the filtration where* F_t *is generated by*
all values of X_s *up to time* $s \leq t$, *i.e.,* $\mathrm{F} = \sigma(\{X^{-1}(A) : s \leq t, A \in \mathrm{M}\})$

Definition 16 (Adapted Process) *Let X denote a SP on* (Ω,F) *and* $\{\mathrm{F}\}_{t\geq 0}$
a filtration on (Ω,F). *Then the process X is called* $\{\mathrm{F}_t\}$*-adapted, if* X_t is F_t*measurable for all* $t \geq 0$.

Definition 17 (Conditional expectation of a process) *Given* $\{\mathrm{F}_t\}_{t\geq 0}$, *a filtration with the characteristic* $\mathrm{F}s \subseteq \mathrm{F}_t \subseteq \mathrm{F}, 0 \leq s \leq t$ *and Y an* F *measurable random variable. Then*

$$X_t := \mathrm{E}[Y \mid \mathrm{F}_t]$$

is an $\{\mathrm{F}_t\}$*-adapted process.*

Definition 18 (Martingales) *We define* $\{X(t)\}_{t\geq 0}$ *as a SP on* (Ω,F,P) *and let* $\mathrm{M} = \{\mathrm{M}(t)\}_{t\geq 0}$ *be a filtration of σ-algebras* $\mathrm{M}(t) \subset \mathrm{F}$ *on* (Ω,F)*:* $\mathrm{M}(s) \subset$
$\mathrm{M}(t)$ *for* $0 \leq s \leq t$. *Then* $\{X(t)\}_{t\geq 0}$ *is referred to as a martingale w.r.t*
$\{\mathrm{M}(t)\}_{t\geq 0}$ *and measure P if*

1. $X(t)$ *is* $\mathrm{M}(t)$*-measurable* $\forall\ t$.
2. $\mathrm{E}|X(t)| < \infty\ \forall\ t$.
3. $\mathrm{E}[X(t)|\mathrm{M}(s)] = X(s)\ \forall\ s \leq t$.

Example 3.1.1 (Martingale property) *If the price*, P_A *of an asset at time* t_1 *is* p_1, *then the expected price at time* t_2 *is* $\mathrm{E}[P_A(t_2)|P_A(t_1) = p_1] = p_1$.

3.2 Parameter Estimation Under Partial information

This section would concern itself with the discussion on how the drift vector process $\mu(t)$ would be estimated.

3.2.1 Introduction: Kalman Filtering

Filtering refers to the process of estimating an unobservable state variable from an observable one. According to Wikipedia, Kalman Filtering, also known as linear quadratic estimation, is a technique that employs a sequence of measurements observed over time, containing statistical noise and other inaccuracies and produces estimates of the unobservable variables. These approximations (or estimations) are preferred or more precise than those based on a single measurement value. Following Liptser and Shiryayev's (1977) text on the subject of filtering things become pretty clearer. Let $(\theta,\xi) = (\theta_t,\xi_t), t \geq 0$ be a partially observable random variable on (Ω,F,P). We assume that only the second element $\xi = (\xi_t), t \geq 0$, is observable. At any time, t, the onus, relying on $\xi_0^t = \{\xi_s, 0 \leq s \leq t\}$, is to approximate the unobservable state process θ. The task of approximating θ from ξ_0^t is known as the *filtering problem*. If $\mathrm{E}[|\theta_t|] < \infty$, the optimal mean square filter (or estimate) of θ_t from ξ_0^t is the *a posteriori* mean $m_t = \mathrm{E}[\theta_t|\mathrm{F}_t^\xi]$. $\mathrm{F}_t^\xi = \sigma(\omega : \xi_s, s \leq t)$ is the σ-algebra triggered by ξ_0^t. As a consequence the answer to the problem of optimal filtering collapsed to deriving the mathematical conditional expectation $m_t = \mathbb{E}[\theta_t|\mathcal{F}_t^\xi]$. From the computational perspective it is appropriate that the formula defining the filter $m_t, t \geq 0$ should be of recurrent nature. This mean that $m_{t+\Delta}, \Delta > 0$, must be determined up from m_t and observations $\xi_t^{t+\Delta} = \{\xi_s, s \in [t, t+\Delta]\}$. In continuous-time models, the filter m_t is represented by the following SDE

$$dm_t = a(t,m_t)dt + b(t,m_t)d\xi \tag{3.1}$$

According to Shiryayev and Liptser (1977), without specific postulates regarding the form of the process (θ,ξ) it is hard to think that optimal filter m_t would obey recurrence relation of type specified in equation (3.1). Therefore to concretely address the filtering problem of (θ,ξ) we have to describe the structure. Assume that $(\theta,\xi) = (\theta_t,\xi_t), t \leq 0$ is Gaussian and governed by SDEs

$$\begin{aligned} d\theta_t &= a(t)\theta_t dt + b(t)dW^1(t) \\ d\xi_t &= A(t)\theta_t dt + \beta(t)dW^2(t) \end{aligned} \tag{3.2}$$

where $W^1(t)$ and $W^2(t)$ are standard BM with the property $\text{cov}[W^1(t),W^2(t)] = 0$ (i.e. mutually independent) as well as independent of (θ,ξ) and $\beta(t) \geq D > 0$. Taking the element $\theta = (\theta_t), t \geq 0$, as unobservable. The filtering task becomes the optimal estimation of θ_t form ξ_0^t in the mean square sense for any $t \geq 0$. Since we are assuming that (θ,ξ) is Gaussian; the optimal filter (or estimate) $m_t = E[\theta_t|F_t^\xi]$ depends linearly on $\xi_0^t = \{\xi_s, s \leq t\}$.

3.2.2 The Kalman-Bucy Method

One-Dimensional Filtering Equations

Given $\{\Omega,F,(F_t),P\}$ where $(F_t), t \leq T$ is an increasing family of σ-algebras, we will deal with two Gaussian random processes $(\theta_t,\xi_t), 0 \leq t \leq T$ satisfying the 1-dimensional SDEs

$$d\theta_t = a(t)\theta_t dt + b(t)dW^{(1)}(t) \tag{3.3}$$

$$d\xi_t = A(t)\theta_t dt + \beta(t)dW^{(2)}(t) \tag{3.4}$$

where $W^{(1)}(t) = (W^{(1)}(t),F_t)$ and $W^{(2)}(t) = (W^{(2)}(t),F_t)$ are two independent Brownian motion and (θ_0,ξ_0) are F-measurable. We assume that the measurable functions $a(t)$, $b(t)$, $A(t)$, and $\beta(t)$ satisfy the following

$$\int_0^T |a(t)|dt < \infty, \qquad \int_0^T b^2(t)dt < \infty, \tag{3.5}$$

$$\int_0^T |A(t)|dt < \infty, \qquad \int_0^T \beta^2(t)dt < \infty. \tag{3.6}$$

Theorem 3.2.1 *(Liptser and Shiryayev (1977), Theorem 4.10). Let the components of the vector function* $a_0(t) = [a_{01}(t), a_{02}(t), \cdots, a_{0n}(t)]$ *and the matrices* $a_1(t) = \| a_{ij}(t) \|$, $b_1(t) = \| b_{ij}(t) \|$, $i,j = 1,2,\cdots,n$, *be measurable functions satisfying the following conditions*

$$\int_0^T |a_{0j}(t)|dt < \infty, \quad \int_0^T |a_{ij}(t)|dt < \infty, \quad \int_0^T b^2(t)dt < \infty.$$

Then the vector SDE

$$dx_t = [a_0(t) + a_1(t)x_t]dt + b(t)d\tilde{W}(t), \qquad x_0 = \eta_0 \tag{3.7}$$

with the BM process $\tilde{W}(t) = (\tilde{W}_1(t), \tilde{W}_2(t), \cdots, \tilde{W}_n(t))$ *has a unique strong solution given by the formula*

$$x_t = \varphi(t)[\eta_0 + \int_0^t \varphi^{-1}(s)a_0(s)ds + \int_0^t \varphi^{-1}(s)b(s)d\tilde{W}(s)] \tag{3.8}$$

where $\phi(t)$ *is defined by the fundamental matrix* $(n \times n)$

$$\varphi(t) = E_{n\times n} + \int_0^t a_1(s)\varphi(s)ds.$$

By Theorem 3.2.1, equation (3.3) has a distinct, continuous solution of the form

$$\theta_t = \exp\left[\int_0^t a(u)du\right]\left[\theta_0 + \int_0^t \exp\left\{-\int_0^s a(u)du\right\} b(s)dW^{(1)}(s)\right]. \tag{3.9}$$

The filtering problem examined by Kalman and Bucy can be stated as follows. Given that $\theta_t, t \in [0,T]$ cannot be measured for values and the only realize observations are those of $\xi_t, t \in [0,T]$, including incomplete information on the values of θ_t. One is suppose to approximate in an optimal way the values θ on the basis of observed process: $\xi_0^t = \{\xi_s, s \in [0,t]\}$. If we consider the optimality of approximation in the mean square sense, then the optimal filter for θ_t given $\xi_0^t = \{\xi_s, s \in [0,t]\}$ agrees with the mathematical conditional expectation

$$m_t = \mathrm{E}[\theta_t|\mathrm{F}_t^{\xi}] \tag{3.10}$$

with the associated error of estimation (called the variance) denoted by

$$\rho_t = \mathrm{E}[(\theta_t - m_t)^2]. \tag{3.11}$$

The technique devised by Kalman and Bucy to derive m_t and ρ_t results in dynamical systems (Liptser and Siryayev, 1977). The process $(\theta_t,\xi_t),t \in [0,T]$, within the Kalman-Busy framework is Gaussian. Hence, the optimal approximation $m_t = \mathrm{E}[\theta_t|\mathrm{F}_t^{\xi}]$ is linear by the theorem below:

Theorem 3.2.2 *(Liptser and Shiryayev (1977), Theorem 10.1). Let* $(\theta_t,\xi_t),t \in [0,T]$ *be two Gaussian processes satisfying the equations in* (3.3) *and* (3.4). *Let* (3.5) *and* (3.6) *also be satisfied and require further that*

$$\int_0^T A^2(t)dt < \infty, \quad \beta^2(t) \geq D > 0,\ 0 \leq t \geq T. \tag{3.12}$$

Then $m_t = \mathbb{E}[\theta_t|\mathcal{F}_t^{\xi}]$ *and the error of the estimate* $\rho_t = \mathrm{E}[(\theta_t - m_t)^2]$ *fulfill the following equations respectively*

$$dm_t = a(t)m_t dt + \frac{\rho_t A(t)}{\beta^2(t)}(d\xi_t - A(t)m_t dt), \tag{3.13}$$

$$\dot{\rho}_t = 2a(t)\rho_t - \frac{A^2(t)\rho_t^2}{\beta^2(t)} + b^2(t) \tag{3.14}$$

with $m_0 = \mathrm{E}[\theta_0|\xi_0]$ *and* $\rho_0 = \mathrm{E}[(\theta_0 - m_0)^2]$.

Equations in (3.13) and (3.14) are obtained using the theorem on normal correlation (See Liptser and Shiryayev (1977)). The equations in (3.13) and (3.14) have distinct continuous solution evident in the following lemmas

Lemma 3.2.1 *(Liptser and Shiryayev (1977), Lemma 10.2). Let the assumption of Theorem 3.2.2 be fulfilled and* $m_0 = 0$ *(P-a.s.). Then for each* $t \in [0,T]$, *the function* $G(t,s)$, $s \in [0,t]$, *satisfies a Wiener-Hopf integral equation for almost all* $u \in [0,t]$,

$$K(t,u)A(u) = \int_0^t G(t,s)A(s)K(s,u)A(u)ds + G(t,u)\beta^2(u), \tag{3.15}$$

where $K(t,u) = \mathrm{E}[\theta_t\theta_u]$.

Lemma 3.2.2 *(Liptser and Shiryayev (1977), Lemma 10.3). Let* $t \in [0,T]$ *be fixed. The solution* $G(t,s), s \in [0,t]$, *of the equation* (3.15) *is unique and is given by the formula*

$$G(t,s) = \varphi_s^t G(s,s), \text{ where } G(s,s) = \frac{\rho_t A(s)}{\beta^2(s)} \tag{3.16}$$

and φ_s^t *solves the differential equation (DE)*

$$\frac{d\varphi_s^t}{dt} = \left[a(t) - \frac{\rho_t A^2(t)}{\beta^2(t)}\right]\varphi_s^t, \quad \varphi_s^s = 1. \tag{3.17}$$

By Lemmas 3.2.1 & 3.2.2, equations (3.13) & (3.14) permit a solution for the filter, m_t in terms of φ_s^t and ρ_s defined by

$$m_t = \varphi_s^t \int_0^t (\varphi_s^t)^{-1} \frac{\rho_s A(s)}{\beta^2(s)} d\xi_s. \tag{3.18}$$

Multi-Dimensional Filtering Equations

Let us now take the two Gaussian random processes to be *n*-dimensional vector processes such that $(\theta_t, \xi_t) = [(\theta_1(t), \theta_2(t), \cdots, \theta_n(t))^\top, (\xi_1(t), \xi_2(t), \cdots, \xi_n(t))^\top]$.

Theorem 3.2.3 *(Liptser and Shiryayev (1977), Theorem 10.2). Let the ndimensional Gaussian processes* (θ_t,ξ_t), $t \in [0,T]$, *evolve according to the following SDEs*

$$d\theta_t = a(t)\theta_t dt + b(t)dW^{(1)}(t), \tag{3.19}$$

$$d\xi_t = A(t)\theta_t dt + \beta(t)dW^{(2)}(t), \tag{3.20}$$

where $a(t), b(t), A(t)$ *and* $\beta(t)$ *are* $n \times n$ *matrices satisfying linear growth conditions. Then by Theorem 3.2.2,* m_t *and* ρ_t *solve the system of equations*

$dm_t = a(t)m_t dt + \rho_t A^{>}(t)(\beta(t)\beta^{>}(t))^{-1}(d\xi_t - A(t)m_t dt)$ (3.21) $\dot{\rho}_t = a(t)\rho_t + \rho_t a^{>}(t) -$

$\rho_t A^{>}(t)(\beta(t)\beta^{>}(t))^{-1}A(t)\rho_t + b(t)b^{>}(t)$ (3.22)

with the initial conditions $m_0 = \mathrm{E}[\theta_0|\xi_0]$, $\rho_0 = \mathrm{E}[(\theta_0 - \xi_0)(\theta_0 - \xi_0)^{>}]$.

The system of equations (3.21) & (3.22) have distinct solution. By Lemmas 3.2.1 & 3.2.2, just as in the one dimensional case, equations (3.21) & (3.22) permit a solution for the optimal filter m_t in terms of φ_s^t and ρ_s defined by

$$m_t = \varphi_s^t \int_0^t (\varphi_s^t)^{-1}\rho_t A^\top(t)(\beta(t)\beta^\top(t))^{-1}d\xi_s \tag{3.23}$$

where φ_s^t is a matrix solution of the DE

$$\frac{d\varphi_s^t}{dt} = [a(t) - \rho_t A^\top(t)(\beta(t)\beta^\top(t))^{-1}A(t)]\varphi_s^t, \quad \varphi_s^s = E_{n\times n}. \tag{3.24}$$

In Lemma 3.2.1, it was assumed that $m_0 = 0$. If m_0 6= 0, then (3.23) can be written as

$$m_t = \varphi_s^t\left\{m_0 + \int_0^t (\varphi_s^t)^{-1}\rho_t A^\top(t)(\beta(t)\beta^\top(t))^{-1}d\xi_s\right\} \tag{3.25}$$

3.3 Stochastic Calculus

3.3.1 Stochastic Differential Equation (SDE)

Definition 19 (Brownian motion (BM)) *Let* $B(t) : [0,\infty)\times\mathrm{R}^d \to \mathrm{R}^d$ *denote a SP satisfying the following properties:*

1. $B(0) = 0$ *(P-almost surely);*
2. *The map* t 7−→ $B(t,)$ *is continuous (P-almost surely). Thus* $B(t)$ *has continuous sample paths;*
3. *For given* $t_0 < t_1 < \cdots < t_k$ *the increments* $B(t_1) - B(t_0)$, $B(t_2) - B(t_1)$,;..., $B(t_k) - B(t_{k-1})$ *are mutually independent;*

4. *For all* $0 \leq s \leq t$ *we have* $(B(t) - B(s)) \sim N(0, (t-s)I_d)$ *- meaning the increments are normally distributed with zero-mean and covariance* $(t-s)I_d$. I_d *represents* $d \times d$ *identity matrix.*

Then $B(t)$ *is called (d-dimensional) BM or Wiener process.*

Here the idea behind the Brownian motion is the same as the one behind the Gaussian distribution. When a phenomenon adds a large number of independent components, the result should be driven by a Gaussian distribution. Thus, combining points 1 and 4 of Definition 19, $B(t)$ is zero-mean Gaussian variable with covariance matrix $(t-s)I_d$.

Definition 20 (Stochastic differential equation (SDE for short)) *Let* $X(t)$ *and* $B(t)$ *denote stochastic processes defined on* $(\Omega,\mathcal{F},P)$ *then the equation*

$$dX(t) = a(t,X(t))X(t)dt + b(t,X(t))X(t)dB(t), \qquad X(0) = x, \tag{3.26}$$

where $x \in \mathbb{R}^n$, $a : [0,\infty)\times\mathbb{R}^n \mapsto \mathbb{R}^n$, $b : [0,\infty)\times\mathbb{R}^n \mapsto \mathbb{R}^n\times\mathbb{R}^d$*, the space of real-valued* $n\times d$ *matrix and* $B(t)$ *is the standard (d-dimensional) BM is called a SDE.*

It is worth noting that (3.26) is also referred as a GBM in literature. Equation (3.26) means that the change $dX(t) = X(t+dt)-X(t)$ is due to the drift factor $a(t,X(t))$ multiplied by the change dt of time plus the diffusion (volatility) factor $b(t,X(t))$ multiplied by the change $dB(t) = B(t+dt) - B(t)$. The first and the second terms on the RHS of equation (3.26) are the deterministic and random (stochastic) components of equation (3.26). Suppose that $X(t)$ denote asset price and t denote trading periods, then the trajectory of $X(t)$ is shown in the figure below

Let $a_i,x_i,X_i(t),B_j(t)$ denote the elements of the column vectors $a,x,X(t),B(t)$ respectively and $b_{ij}, i = 1,2,\cdots,n-1,n$, $j = 1,2,\cdots,n-1,n$ denoting the entries of the matrix b. Equation (3.26) can be written as

$$dX_i(t) = a_i(t,X(t))dt + \sum_{j=1}^{d} b_{ij}(t,X(t))dB_j(t), \; X_i(0) = x_1, \; i = 1,2,\cdots,n.$$

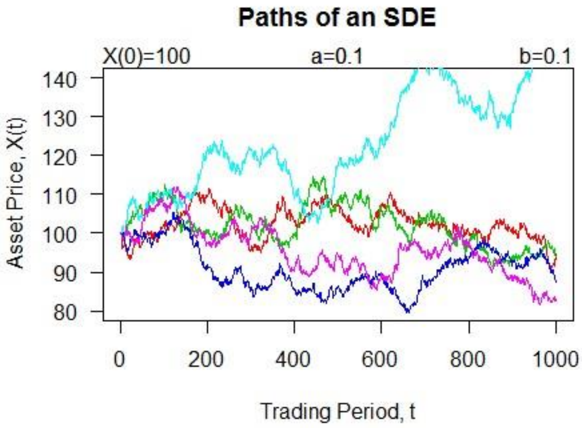

Figure 3.1: Five Simulated Paths of X(t). The

stochastic process $X(t)$ solves equation (3.26) provided

$$X(t) = x + \int_0^t a(m, X_m)dm + \int_0^t b(m,\ X_m)dB(m), \quad 0 \leq t < \infty, \qquad (3.27)$$

exists.

3.3.2 Ito Calculus

In the sequel, we discuss Ito's theory of stochastic integration. This is a broadˆ subject. However, our goal is rather modest: we will discuss this theory only generally enough for later applications.

Definition 21 (Ito Process)ˆ *Let* $B(t)$ *denote an (d-dimensional) BM on* (Ω, F, P) *and let* $a(t,X(t)) \in R^n$ *and* $b(t,X(t)) \in R^n \times R^d$ *be* F_t*-adapted processes. Then the stochastic process* $\{X(t), t \geq 0\}$ *is an n-dimensional Itoˆ process of the form:*

$$X(t) = x + \int_0^t a(m, X(m))dm + \int_0^t b(m,\ X(m))dB(m), \quad x = X(0). \quad (3.28)$$

Equation (3.28) can be represented in a differential notation as

$$dX(t) = a(t,X(t))dt + b(t,X(t))dB(t), \qquad X(0) = x. \qquad (3.29)$$

Clearly, the Ito process in differential and integral forms are analogous toˆ equations (3.26) and (3.27) respectively. This implies that the existence of a solution to equation

(3.28) means a solution to equation (3.27). Pham (2009) explains the concept of a strong solution and a weak solution to equation (3.29).

Definition 22 (Existence and Uniqueness of Solution) *Suppose that the coefficients* $a : [0,T] \times \Omega \rightarrow \mathrm{R}^n$ *and* $b : [0,T] \times \Omega \rightarrow \mathrm{R}^n \times \mathrm{R}^d$ *are continuous. Then* $\exists$ *a constant* $C \in (0,\infty) \mid \forall\, t \in [0,\infty)$ *and any* $x, y \in \mathrm{R}^n$*, the following Lipschitz conditions are satisfied:*

$$|a(t,x) - a(t,y)| + ||b(t,x) - b(t,y)|| \leq C|x - y|, \tag{3.30}$$

$$|a(t,x)|^2 + ||b(t,x)||^2 \leq C^2(1 + |x|^2). \tag{3.31}$$

where $||b(t,x)||^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij}$. These two conditions are called *Ito conditions*ˆ . These conditions ensure that the solution $X(t)$ does not grow without bound in finite time. Theorem 3.3.1 below illustrates that equations (3.30) and (3.31) give the sufficient conditions on the drift term a and volatility term b under which there exists a unique solution to the SDE in equation (3.26).

Theorem 3.3.1 *Let the three-tuple* (Ω,F,P) *be equipped with a filtration* $\{\mathrm{F}_t\}$ *and* $\{\mathrm{F}_t\}-$*BM* $B(t)$*. Assume that a and b satisfy equations (3.30) and (3.31), then there exist a unique solution to equation (3.26). Meaning, there exist a SP* $\{X_t\}$ *on* (Ω,F,P)*, with continuous sample paths almost surely, which fulfill the stochastic integral equation*

$$X(t) = x + \int_t^T a(m, X_m)dm + \int_t^T b(m, X_m)dB(m).$$

The unique solution $X(t)$ *of equation (3.26) is called an Ito process by Definition*ˆ *21.*

Example 3.3.1 *If we consider the SDE of the form*

$$dX_t = X_t dB_t,\ X_0 = 1,$$

where $a(t,X_t) = 0$ *and* $b(t,X_t) = X_t$ *then the Ito conditions are met, hence the*ˆ *existence of a strong solution for the SDE.*

Example 3.3.2 *Let us consider the price* p_t *of a stock defined according to the*

SDE of the form

$$dp_t = \mu p_t dt + \sigma p_t dB t,$$

where $a(t,X_t) = \mu p_t$ *and* $b(t,X_t) = \sigma p_t$. *Since* μ *and* σ *are constants the Itô conditions are satisfied. Hence, a strong solution exists for the stock price.*

Definition 23 (Itô Integral) *Let* $\{X(t), t \in [0,T]\}$ *denote a SP that is measurable w.r.t* F_t *of the BM* $\{B(t), 0 \leq t \leq T\}$. *Then*

$$I(t) = \int_0^t X(m)dB(m), \quad 0 \leq t \leq T, \tag{3.32}$$

called a stochastic integral w.r.t $B(t)$ *is known as an Itô Integral.*

Visiting calculus theory, the Riemann integral $\int_0^T f(t)dt$ is defined for a continuous function f over the bounded interval $[0,T]$. By partitioning the interval into n small subintervals of the form $[t_0 = 0, t_1], [t_1,t_2],...,[t_{n-1}, t_n = T]$ and summing up all the area of the associated rectangles we obtain the

Riemann sum

$$\int_0^T f(t)dt \approx \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}).$$

The natural question to ask is whether we can still define the integral in equation (3.32) with the Riemann approach.

Oksendal (2003) explains that since the BM $\{B(t), 0 \leq t \leq T\}$ defined on (Ω, F, P) is nowhere differentiable and cannot possibly be a bounded variation (especially of the first order) the calculus rules cannot be used to evaluate the integral in equation ((3.32)) from Riemann-Stieltjes sense.

Now there is a glaring need to make sense out of the integral in equation (3.32).

To make sense of equation (3.32) let us consider the following as in Oksendal (2003)

$$I_1(t) = \int_0^t B(m)dB(m), \qquad 0 \leq t \leq T. \tag{3.33}$$

By partitioning $[0, T]$ into n equal subintervals $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$ we can write equation (3.33) as

$$I_1(t) = \int_0^t B(m)dB(m) = \sum_{i=1}^{n} B(t_{i-1})(B(t_i) - B(t_{i-1})),$$

where t_{i-1} is the left end point of each interval of integration. Now

$$\begin{aligned}\sum_{i=1}^{n} B(t_{i-1})(B(t_i) - B(t_{i-1})) &= \sum_{i=1}^{n} \left(\frac{1}{2}(B^2(t_i) - B^2(t_{i-1})) - \frac{1}{2}(B(t_i) - B(t_{i-1}))^2\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}(B^2(t_i) - B^2(t_{i-1})) - \frac{1}{2}\sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))^2.\end{aligned}$$

,

The term $\sum_{i=1}^{n}(B^2(t_i) - B^2(t_{i-1}))$ is a telescoping sum. Therefore by Definition 19

$$\sum_{i=1}^{n}(B^2(t_i) - B^2(t_{i-1})) = B^2(T) - B^2(0) = B^2(T)$$

.

As the partition gets finer and finer, it turns out that the squared variation converges to T. That is

$$V(B)[0,T] \overset{\text{def.}}{=} \lim_{n\to\infty} \sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))^2 = T.$$

Recall that by Definition 19, $(B(t_i) - B(t_{i-1})) \sim N(0, t_i - t_{i-1})$. It therefore suffices that

$$\text{var}(B(t_i) - B(t_{i-1})) = \text{E}[(B(t_i) - B(t_{i-1}))^2] = t_i - t_{i-1},$$

so that

$$\mathbb{E}\left[\sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))^2\right] = \sum_{i=1}^{n}\mathbb{E}\left[(B(t_i) - B(t_{i-1}))^2\right] = \sum_{i=1}^{n}(t_i - t_{i-1}) = T,\ T = n.$$

Substituting these results into $I(t)$ we have

$$I_1(t) = \int_0^t B(m)dB(m) = \frac{1}{2}(B^2(T) - T) \quad (3.34)$$

Equation (3.34) extends to integrals of the form $\int_0^t X(m)dB(m)$ and $\int_0^t b(m,\ X(m))dB(m)$.

Hence, the Itˆo integral in equation (3.32) can be written as

$$I(t) = \int_0^t X(m)dB(m) = \sum_{i=1}^{n} X(t_{i-1})(B(t_i) - B(t_{i-1})).$$

Definition 24 (Ito-integrable)ˆ *Let* $\{X(t), t \geq 0\}$ *denote a SP defined on* (Ω, F, P)*. We say that* $\{X(t), t \geq 0\}$ *is called Ito integrable on*ˆ $[0,T]$ *if it satisfies:*

1. $X(t)$ *is adapted i.e.,* $X(s)$ *is* F_s*-measurable for* $0 \leq s \leq t$*.*
2. $\int_0^t \mathbb{E}[X^2(m)]dm < \infty$.

The following holds by the Itˆo integral defined in equation (3.32) 1. The

integral $I(t)$ is a martingale w.r.t the filtration $\{\mathrm{F}_t, t \geq 0\}$.

2. $\mathbb{E}[I(t)] = \mathbb{E}\left[\int_0^t X(m)dB(m)\right] = 0$.
3. $\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^t X^2(m)dm\right] = \mathbb{E}\left[\left(\int_0^t X(m)dB(m)\right)^2\right]$ for all t. This is called Ito Isometry.ˆ
4. $\mathrm{var}(I(t)) = \int_0^t \mathbb{E}(X^2(m))dm$.
5. $V[I,I](t) = \int_0^t X^2(m)dm$. This is called the second-order variation (or quadratic variation) of $I(t)$.

Lemma 3.3.1 (Ito lemma)ˆ *Let X denote an Ito process and any function*ˆ $G : [0,\infty) \times \mathrm{R}^n \longmapsto \mathrm{R}$ *be of class* $C^{1,2}$ *(i.e., twice continuously differentiable). Then for all t* $\in [0,\infty)$*, almost surely*

$$\begin{aligned} dG(t,X(t)) &= \left(\frac{\partial G(t,X(t))}{\partial t} + \sum_{i=1}^{n} \frac{\partial G(t,X(t))}{\partial x_i} a(t,\ X(t)) \right. \\ &\quad \left. + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 G(t,X(t))}{\partial x_i\,\partial x_k} \hat{a}_{ik}(t,X(t))\right) dt \qquad (3.35) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{d} \frac{\partial G(t,X(t))}{\partial x_i}\, b_{ij}(t,X(t))dB(t), G(0,X(0)) = G(0,x), \end{aligned}$$

where $\hat{a}_{ik} = b(t,X(t))b^{>}(t,X(t))$, *where the superscript > denote transposition. For G of class* $C^{1,2}$, *let* LG *be define as*

$$\begin{aligned}\mathcal{L}G(t,X(t)) &= \sum_{i=1}^{n}\frac{\partial G(t,X(t))}{\partial x_i}a(t,X(t)) \\ &+\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 G(t,X(t))}{\partial x_i\,\partial x_k}\hat{a}_{ik}(t,X(t))\end{aligned}, \qquad (3.36)$$

where $\hat{a}_{ik}(t,X(t)) = b_{ij}(t,X(t))b^{T}_{ij}(t,X(t))$ *is an* $n \times d$ *matrix. Using equation* (3.36) *we can rewrite equation* (3.35) *as*

$$dG(t,X(t)) = \mathcal{L}G(t,X(t))dt + \sum_{i=1}^{n}\sum_{j=1}^{d}\frac{\partial G}{\partial x_i}(t,X(t))b_{ij}(t,X(t))dB(t)$$

Example 3.3.3 (Demonstration of Ito’s lemma to one dimensional SDE)ˆ *Let* $y(t)$ *denote the price of a stock that follows an Ito process of the form*ˆ

$$dy(t) = \mu y(t) + \sigma y(t)dW(t),$$

where $W(t)$ *is a GMB and let the price of a forward contract be defined as follows*

$$f(t,y(t)) = y(t)e^{r(T-t)}, \qquad \text{for } t < T.$$

*Then by the Ito lemma in equation (3.35) the forward price evolves according to*ˆ *the following equation*

$$\begin{aligned} df(t,y(t)) &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\mu y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\sigma^2 y^2\right)dt + \frac{\partial f}{\partial y}\sigma y dW(t), \\ &= \left(-ry(t)e^{r(T-t)} + e^{r(T-t)}\mu y\right)dt + e^{r(T-t)}\sigma y dW(t) \\ &= (\mu - r)\,f dt + \sigma f dW(t). \end{aligned},$$

Observe that just like the stock price, forward price follows a GBM with drift coefficient $(\mu - r)$ and volatility σ.

3.4 Optimal Control Theory

Wikipedia defines stochastic control or stochastic optimal control as “a branch of Control Theory that investigates the existence of randomness either in observations or the noise that drives the evolution a system.” Stochastic control seeks to characterize the time path of the controlled variables that perform the desired control task with the lowest cost or maximum output, somehow defined, despite the inclusion of the noise.

3.4.1 Basic Elements of Stochastic Control

We explained a process as an event that evolves overtime with the objective of achieving a specified goal. Most of these processes in stochastic control problems are formulated with the following features:

- Time horizon: The commonly used time horizon in stochastic control problems are infinite time horizon (i.e, $t \in [0,\infty)$); indefinite time horizon (i.e, $t \in [0,\tau]$ for some stopping time τ) and finite time period (i.e, $t \in [0,T]$ for a fixed terminal time $T < \infty$). The scope of this work is finite time horizon optimal control problem in portfolio selection.

- Controlled Variable or Controlled State Process (CSP): The CSP is a SP that express the state of a physical system of choice. The CSP is usually a solution of a controlled SDE of the form in equation (3.37). The evolution of the CSP is determined by a control variable (CV). The CSP assumes values in a set called the state set. Typical examples of CSP’s are total wealth of an economy, growth rate of a bacteria, total travel time among others.

- Control Variable or Control Process (CV): It is also SP, determined by the controller to steer or drive the path of the system. The control process assumes values in the control space. In some control problems, the control variable may be bounded or unbounded. The choice of the control variable depends on the type of problem confronted with. For instance, in portfolio control problems without short selling the control could be specified as $0 \leq u(t) \leq 1$ and for those with short selling (short position) the control could allowed to take values between $-a_1 \leq u(t) \leq a_2$, $(a_1, a_2 > 0)$. That is, most often, only controls that fulfill certain admissibility criteria are accepted by the system driver. We have *adapted controls, feedback controls and Markov controls* (Oksendal, 2003).

- Admissible controls: A control process satisfying certain constraints is called an admissible control. Let A denote all the set of admissible controls which depend on time and the state of the initial value of the controlled process.
- Cost (or reward) function: The process always comes with a cost (or reward) which often rely on both the controlled process and the control process. The reward function is usually denoted by $J(x,u(t))$, characterizing the expected total reward beginning from the initial controlled process x if the control $u(t)$ is chosen.
- Value functional: The value functional explain the value of the maximum possible reward (or minimum possible cost) from the controlled process. The aim of a stochastic control problem would then be to characterize the value functional and derive the CV $u^*(t)$ whose reward (or cost) attains a maximum (or minimum) value $V(t,x) = J(x,u^*(t))$ for some initial value x of the CSP.

3.4.2 Controlled SDE

We continue our discussion of stochastic optimal control problem by looking at a process evolving in time called the CSP with the dynamics given by an SDE. Here, the coefficient terms in the deterministic and random components of the SDE may not solely be time t, state process X_t independent (i.e., constant) or dependent but also depend a control process (or control processes). In most applications, the CSP is influenced by a controller who seeks to optimize a performance function that depends on the CSP. The controller influences the dynamics of the controlled state process through the SDE by choosing the control process(es). A typical controlled SDE is: $dX(t) = a(t,X(t),u(t))X(t)dt + b(t,X(t),u(t))X(t)dB(t)$, $X(0) = x$ (3.37) where $u(t)$ denotes a *control variable*. Given $u(t)$, the controlled variable X evolves like a diffusion, but the controller can steer the behaviour of X through the choice of $u(t)$. Let U denote a control space, then we require that $u(t)$ assumes values in U. Therefore, we need to strengthen the Itˆo conditions to guarantee the existence of a unique solution of equation (3.37). As a result, we need the Itˆo conditions to hold uniformly over all $\alpha \in$ U. This means we must have

$$|a(t,x,\alpha) - a(t,y,\alpha)| + |b(t,x,\alpha) - b(t,y,\alpha)| \leq K|x - y|, \quad (3.38)$$

$$|a(t,x,\alpha)|^2 + \|b(t,x,\alpha)\|^2 \leq K^2(1 + |x|^2). \quad (3.39)$$

Theorem 3.4.1 below illustrates that equations (3.38) and (3.39) provide sufficient conditions on the drift term a and the volatility term b under which ∃ a unique solution to the SDE in equation (3.37).

Theorem 3.4.1 *Let* $(\Omega,\mathcal{F},P)$ *be equipped* $\{\mathcal{F}_t\}$ *and a (d−dimensional)* $\{\mathcal{F}_t\}-$*BM* $B(t)$. *Let u be an* $\{\mathcal{F}_t\}-$*adapted process assuming values in* U. *Assume that a and b satisfy equations* (3.38) *and* (3.39). *Then ∃ a unique (strong) solution X to equation (3.37). Given the solution X to equation (3.37), we define a collection of operators* $A^\alpha : \alpha \in$ U *such that for suitable class of twice differential function* $G : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ *we have*

$$A^\alpha G(t,x) = \frac{\partial G(t,\ x)}{\partial t} + \mathcal{L}G(t,\ x), \quad (3.40)$$

where $\mathcal{L}G(t, x)$ *is given in equation* (3.36).

3.4.3 Dynamic Programming Principle (DPP)

Dynamic programming (DP) is a mathematical maximization and minimization and computer programming method. In both setting, it is concern with simplifying a complex problem by splitting it into sub-problems in a recursive manner. Though some decisions problems cannot split apart this way, decisions that stretch several points in time do often split apart recursively. A typical optimization problem has some goal, say, maximizing profit, maximizing output, minimizing travel time, minimizing cost, et cetera. The mathematical function that describes this goal is called the value (objective) function.

We now formalize the DP approach. The key hallmark of the DP technique is the building of optimization problems into several steps (or stages):

$$i \text{ from } 1 \text{ to } n,$$

which are solved one after the other at each stages. Besides each one-stage problem solved as an ordinary optimization problem, its solution helps to define the characteristics of the next one-stage problem in the sequence. The stages denote different time periods in the problem planning horizon as discussed under elements of stochastic control in Section 3.4.1. Related with each stage i of the optimization problem are the states of the process ω. The states reflect the information needed to fully scrutinize the consequences that the current decision has upon future actions. The last universal feature of the DP approach is the development of a recursive optimization procedure, which builds to a solution of the overall n-stage problem by first solving a one-stage problem and sequentially including one stage at a time and solving one-stage problems until the overall optimum is reached. The methodology can be based on a backward or forward induction process. In the backward induction process, the first stage to be analyzed is the final stage of the problem and problems are solved moving back one stage at a time until all stages are added. For example, if we define a series of value functional

$$V_1(\cdot),V_2(\cdot),\cdots,V_n(\cdot).$$

Now $V_n(\cdot)$ is the value obtained in the last stage n of the decision making process.

The value functional $V_i(\cdot)$ at the stages $i = n-1,n-2,\cdots,2,1$ can be determined by working backwards, using a recursive relationship. Let $i = 2,3,\cdots,n-1,n$. Then $V_{i-1}(\cdot)$ at any stage is computed from $V_i(\cdot)$ by maximizing a simple function of the gain from a decision at stage $i-1$ and the function $V_i(\cdot)$ at the new state of the system if any decision is taken. Since $V_i(\cdot)$ has already been computed for the needed states, the above operation yields $V_{i-1}(\cdot)$ for those states. Finally, $V_1(\cdot)$ at the initial stage of the decision process is the value of the optimal solution. Similarly, the recursive procedure can be done using a forward induction process, where the first stage to be solved is the initial stage of the problem and problems are solved moving forward one stage at a time, until all stages are included. For optimal control problems involving uncertainties, DP works only through the backward induction technique. The justification of the recursive optimization technique is the principle of optimality.

Definition 25 (Principle of Optimality) *An optimal policy has the property that, whatever the current state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the current decision.*

In the context of controlled stochastic processes described earlier, we give a formulation of DP for stochastic optimal control problems. Observe that until now, we have explained DPP based on discrete stages. Replacing the discrete state space with a continuous state space the procedures described above still hold. The optimal value function for DP under uncertainty is then defined in the following recursive form:

$$V(t,x) = \sup_{u} \mathbb{E}\left[\int_t^{\tau} g(s,X(s),u(s))ds + V(\tau,X(\tau))\right], s \in [t,T], x \in \mathbb{R}^n, \quad (3.41)$$

for any stopping time $\tau \in [t,T]$, running reward function $g(s,X(s),u(s))$ and terminal reward function $V(\tau,X(\tau))$.

Definition 26 (The reward function) *Let* $J : \mathrm{R}^n \times [0,T] \times \mathrm{U} \longmapsto \mathrm{R}$ *denote the reward function for a general maximization problem. Then we define J by*

$$J(t,x,u(t)) = J^{u(t)}(t,x) = \mathbb{E}\left[\int_t^{T} g(s,X(s),u(s))ds + h(X(T))\right], \quad (3.42)$$

where $g(\cdot)$ *is the running reward function and* $h(\cdot)$ *is the terminal reward function.*

Definition 27 (Value function) *Let* $V : [0,T] \times \mathrm{R} \longmapsto \mathrm{R}$ *denote a value function. Then* $V(\cdot)$ *is given by*

$$V(t,x) = \sup_{u \in \mathrm{A}(t,x)} J(t,x,u(t)). \quad (3.43)$$

We say that the CV $u^* \in A(t,x)$ is an optimal control if $V(t,x) = J(t,x,u^*)$ for the initial conditions $x \in R$ and $t \in [0,T]$.

3.4.4 Hamilton-Jacobi-Bellman (HJB) Equation

The Hamilton–Jacobi–Bellman (HJB) equation is a partial differential equation (PDE) which is central to optimal control theory. The solution of the HJB equation is the ‘value function’ which gives the minimum cost (or maximum reward) for a given dynamical system with an associated cost (or reward) function. When solved locally, the HJB is a necessary condition, but when solved over the whole of state space, the HJB equation is a necessary and sufficient condition for an optimum. The solution is open loop, but it also permits the solution of the closed loop problem. The HJB method can be generalized to both deterministic and stochastic systems. HJB equation results from the theory of DP which was originated by Richard Bellman (1950) .

Derivation of HJB Equation

Choose $\tau = t + \Delta t$ and an arbitrary constant control $u \in U$. Then by equation (3.41) of the DPP we obtain

$$V(t,x) \geq \sup_{u \in \mathcal{A}(t)} \mathbb{E}\left[\int_t^{t+\Delta t} g(s,X(s),u(s))ds + V(t+\Delta t, X(t+\Delta t))\right]. \quad (3.44)$$

Assuming that V is of class $C^2([0,T]\times R)$, then by using Itˆo’s lemma on equation (3.37) between t and $t + \Delta t$ we have

$$\begin{aligned} V(t+\Delta t,\ X(t+\Delta t)) &= V(t,x) + \int_t^{t+\Delta t} \left(\frac{\partial V}{\partial t}(s,X(s)) + \mathcal{L}^u V(s,X(s))\right) ds \\ &\quad + \frac{\partial V}{\partial x}(s,X(s))b(s,X(s),u(s))dB(t), \qquad (3.45) \end{aligned}$$

where L^u defined by equation (3.36). By adding $\int_t^{t+\Delta t} g(s,X(s),u)ds$ to both sides of equation (3.45) and taking expectation yields

$$\begin{aligned} &\mathbb{E}\left[\int_t^{t+\Delta t} g(s,X(s),u(s))ds + V(t+\Delta t, X(t+\Delta t))\right] = \\ &V(t,x) + \mathbb{E}\left\{\int_t^{t+\Delta t} \left[\left(\frac{\partial V}{\partial t} + \mathcal{L}^u V\right)(s,X(s)) + g(s,X(s),u(s))\right] ds\right\} \quad (3.46) \end{aligned}$$

From equations (3.44) and (3.46) we obtain

$$0 \geq \mathbb{E}\left\{\int_t^{t+\Delta t}\left[\left(\frac{\partial V}{\partial t}+\mathcal{L}^u(V)\right)(s,X(s))+g(s,X(s),u(s))\right]ds\right\}.$$

Dividing the above inequality by Δt and sending Δt to 0, then by the mean-value theorem we obtain

$$0 \geq \frac{\partial V}{\partial t}(t,x)) + \mathcal{L}^u\, V(t,x) + g(t,x,u).$$

Since this is true for any $u \in$ U, we obtain the following inequality:

$$-\frac{\partial V}{\partial t}(t,x)) - \sup_{u\in\mathcal{A}(t)}\{\mathcal{L}^u\, V(t,x)+g(t,x,u))\} \leq 0. \quad (3.47)$$

Now, suppose that u^* is an optimal control, then by equation (3.41) we have

$$V(t,x) = \mathbb{E}\left[\int_t^{t+\Delta t} g(s,X^*(s),u^*(s))ds + V(t+\Delta t, X^*(t+\Delta t))\right],$$

where X^* solves equation (3.37) for initial state value x at time t. By using arguments similar to those above in equations (3.45) and (3.46), we have

$$\frac{\partial V}{\partial t}(t,x) + \mathcal{L}^{u^*}V(t,x) + g(t,x,u^*(t)) = 0. \quad (3.48)$$

By combining equations (3.47) and (3.48) we derive the HJB equation

$$\frac{\partial V}{\partial t}(t,x)) + \sup_{u\in\mathcal{A}(t)}\{\mathcal{L}^u\, V(t,x)+g(t,x,u))\} = 0, \quad \forall\ t\in[0,T], \quad x\in\mathbb{R}. \quad (3.49)$$

Thus we have derived the dynamic programming equation (known as HJB equation in stochastic continuous-time models) associated with the above finite time control problem. The HJB equation is sometimes written as

$$0 = \sup_{u\in \mathrm{U}}\{A^u V(t,x) + g(t,x,u)\},\ t \in [0,T],\ x \in \mathrm{R},$$

where $A^u V(t,x)$ is given in equation (3.40) of Theorem 3.4.1.

$$\mathcal{A}^u V(t,x) \;=\; \frac{\partial V}{\partial t}(t,x) + a(t,x,u)\,\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\,b^2(t,x,u)\,\frac{\partial^2 V}{\partial x^2}(t,x).$$

At the terminal time T we are rewarded $h(X(T))$; subsequently no extra control is chosen and no extra reward is accrued. Thus, the terminal condition associated with the PDE in equation (3.49) is

$$V(T,x) = h(x),\ x \in \mathrm{R}, \tag{3.50}$$

which suffices from the very definition (3.43) of the value function $V(\cdot)$ considered at the time period T. Equations (3.49) and (3.50) provide us with the tool for finding optimal controls and value functions by the techniques of optimization.

Chapter 4

Optimal Control of Mutual Fund Portfolio Choice in the Presence of Dynamic Flow with Partial Information

4.1 Economic Set-up of the Model

The study shall assume a continuous-time financial system. The financial market would be open on the time interval $[0,T]$ where $T < \infty$. Thus the time horizon $[0,T]$ is finite. Uncertainty in the market is specified by the three- tuple (Ω,F,P). As the market opens at different dates $t \in [0,T]$, prices of securities, returns processes and wealth processes are sequences of random variables written to have time as their argument. The market consist of one risk-less asset (called a bank account or T-bills) and n risky securities (called stocks). The risk-less asset $S_0(t)$ has a fixed yield rate r and the n risky securities $S(t)$ have their evolution governed by an n-dimensional standard BM $B(t)$. Trading activities are done continuously on the market and there are no frictions. Definitions, notations and assumptions regarding the market parameters and variables are given below:

- $S(t) = (S_1(t),S_2(t),\cdots,S_n(t))^{>}$, $t \in [0,T]$ denotes the n-dimensional vector price process of the risky securities and as before the superscript $>$ denotes transposition;

- $\mu(t) = (\mu_1(t),\mu_2(t),\cdots,\mu_n(t))^{>}$, $t \in [0,T]$ denotes an n-dimensional drift vector process of the risky securities;

- $\sigma(t) = (\sigma_{ij}(t))$, $i,j = 1,\cdots,n$ is an $n \times n$ matrix with non-zero determinant;

- $B(t) = (B_1(t),B_2(t),\cdots,B_n(t))^{>}$, $t \in [0,T]$ is an n-dimensional vector BM;

- We assume that economic agents cannot observe (or measure) the drift vector process $\mu(t)$ directly and the BM $B(t)$. This situation is called partial information.

This distinguish the our framework for the study of optimal control of mutual fund portfolio choice from those of full information;

- To build consistent financial models, it is then necessary to chose a relevant definition of the information relation over time. In particular, at a given date t, economic agents observe date-t prices which are no more random at future dates. We shall assume that the quantity of information held by economic agents increases over time-this is synonymous to the idea that economic agents do not forget anything;
- Let $\mathrm{F}^S = \{\mathrm{F}_t^S : 0 \leq t \leq T\}$ be the augmented filtration triggered by the asset price process;
- We suppose that only F^S-adapted processes are observable;
- The fact that market participants have only partial information will be represented, F^S, which is smaller than the original filtration;
- The positive constant yield rate r, the positive constant initial stock price $S_i(0), i = 1,2,3,\cdots ,n$ where i denotes the ith stock and the volatility matrix σ are available to all market participants.

4.2 Mathematical Specification of the Model

4.2.1 Price Dynamics of the Risk-less Security

The price dynamics of the bank account obeys the following deterministic ordinary differential equation:

$$dS_0(t) = S_0(t)r_t dt;\ S_0(0) = s_0, \tag{4.1}$$

$r_t > 0$, is the risk-free rate of return on the government bond or the bank account.

4.2.2 Price Dynamics of Risky Securities

The price process of the risky securities follow the SDE

$$dS_i(t) \;=\; \mu_i(t)S_i(t)dt + \sum_{j=1}^{n}\sigma_{ij}(t)S_i(t)dB_j(t),\; S_i(0) = s_i,\; i = 1, \dots, n. \quad (4.2)$$

Considering each *i*-th stock we have

$$\begin{aligned}
dS_1(t) &= S_1(t)\left[\mu_1(t)dt + \sum_{j=1}^{n}\sigma_{1j}(t)dB_j(t)\right], \; S_1(0) = s_1,\\
dS_2(t) &= S_2(t)\left[\mu_2(t)dt + \sum_{j=1}^{n}\sigma_{2j}(t)dB_j(t)\right], \; S_2(0) = s_2,\\
&\dots\\
dS_n(t) &= S_n(t)\left[\mu_n(t)dt + \sum_{j=1}^{n}\sigma_{nj}(t)dB_j(t)\right], \; S_n(0) = s_n
\end{aligned}$$

In matrix notation, we have

$$dS(t) = D[S(t)]\mu dt + D[S(t)]\sigma(t)dB(t).$$

$D[S(t)]$ in the above equation is a diagonal matrix with $S(t)$ along its diagonal and $\mu(t)$, $\sigma(t)$, and $B(t)$ are as defined above.

4.2.3 Dynamics of Mutual Fund Return

Let the vector process $\eta(t) = \{[\eta_i(t))_{i=1}^{n}]^{\top}, t \in [0,T]\}$ be the mutual fund trading strategy representing the proportion of wealth held in risky securities. Then fund's portfolio denoted by υ becomes

$$\upsilon = \begin{bmatrix} 1 - \eta^{\top}(t)\mathbf{1} \\ \eta(t) \end{bmatrix}, \; 0 \le \eta^{\top}(t)\mathbf{1} \le 1, \; 0 < \eta_i(t) < 1$$

where **1** is conformable vector of ones. Given that the fund manager trades *n* risky assets plus a bond and charges a fee rate, denoted by γ on assets under management, the return for investment in the mutual fund follows the dynamics (see Appendix A, Section 5.0.11 for derivation):

$$\frac{dF(t)}{F(t)} = [1-\eta^\top(t)\mathbf{1}]\frac{dS_0(t)}{S_0(t)} + \sum_{i=1}^{n}\eta_i(t)\frac{dS_i(t)}{S_i(t)} - \gamma dt \tag{4.3}$$

Using equations (4.1) and (4.2), equation (4.3) becomes

$$\begin{aligned} \frac{dF(t)}{F(t)} &= [1-\eta^\top(t)\mathbf{1}]r_t dt + \sum_{i=1}^{n}\eta_i(t)\left\{\mu_i(t)dt + \sum_{j=1}^{n}\sigma_{ij}(t)dB_j(t)\right\} \\ &\quad -\gamma dt \\ &= [1-\eta^\top(t)\mathbf{1}]r_t dt + \eta^\top(t)\mu(t)dt + \eta^\top(t)\sigma(t)dB(t) - \gamma dt \\ &= [(r_t-\gamma) + \eta^\top(t)(\mu(t)-r_t\mathbf{1})]dt + \eta^\top(t)\sigma(t)dB(t) \qquad (4.4) \end{aligned}$$

The reward-to-variability ratio associated with the investment in stocks and a bond is given by

$$\xi_t = \sigma(t)^{-1}(\mathrm{E}[F(t)] - r_t). \tag{4.5}$$

By the risk averse nature of economic agents $\mathrm{E}[F(t)] - r > 0$.

4.2.4 Dynamics of the Wealth Process of the Economic Agent

For any given level of wealth, $W(t)$ of an economic agent (hereafter also refered to as an investor), he or she is suppose to allocate all this wealth between the risk-less asset and the mutual fund such that

total wealth(t) = value of risk-less asset(t) + value of mutual fund(t).

This implies that the investor's portfolio, denoted by π at anytime $t \in \mathrm{T}$ is

$$\pi = \begin{pmatrix} 1-\varphi(t) \end{pmatrix},$$

□ □ $\varphi(t)$

where $\varphi(t)$ is the proportion of wealth held in the mutual fund and $(1 - \varphi(t))$ is the remaining proportion of wealth held in the risk-less asset. Let $W(0) = w_0 > 0$ denote the initial wealth of the investor and refrain the investor from short selling the actively managed fund so that $0 \leq \varphi(t) \leq 1$, $t \in \mathrm{T}$.

We suppose that there are no exogenous infusion or withdrawal of money and that the purchase of more of a given security must be financed by lower investment in another security. Then the wealth process of the investor evolves according to the following (see Appendix A, Section 5.0.10 for derivation)

$$dW(t) = [1 - \phi(t)]W(t)\frac{dS_0(t)}{S_0(t)} + \phi(t)W(t)\frac{dF(t)}{F(t)}, \;\; W(0) = w_0 . \qquad (4.6)$$

By equations (4.1) and (4.4), equation (4.6) becomes to

$$\begin{aligned} dW(t) &= [1 - \phi(t)]W(t)r_t dt + \phi(t)W(t)\left\{[(r_t - \gamma) + \eta^\top(\mu_t - r_t\mathbf{1})]dt \right. \\ &\quad \left. + \eta^\top \sigma_t dB(t)\right\}, \\ dW(t) &= [r_t + \phi(t)(\eta^\top(t)(\mu(t) - r_t\mathbf{1}) - \gamma)]W(t)dt \\ &\quad +[\varphi(t)\eta^{>}(t)\sigma(t)]W(t)dB(t). \qquad (4.7) \end{aligned}$$

4.2.5 Dynamics of the Wealth Process of the Fund Manager

For a remuneration, the fund manager is paid fees that results from the investor's demand of the fund. Given that the investor chooses the trading strategy, $\varphi(t)$, the manager is rewarded $\gamma\varphi(t)W(t)$ per trading period and over the time span T , we have the associated total fee process given by the following ODE

$$\Phi(t) := \int_0^T \gamma\phi(t)W(t)dt. \qquad (4.8)$$

4.3 Utility maximization

We consider an economic agent whose choices over terminal consumption bundles and wealth are represented by a strictly increasing, strictly concave and continuously differentiable von Neumann-Morgenstern utility function

$$U : (0,\infty) \to \mathbb{R}. \tag{4.9}$$

Given the utility function U, then for any $(t,W) \in [0,T] \times R$ the problem is to find the control strategies $\eta(t)$ (the fund manager's allocation of resources under his management to the n risky securities) and $\varphi(t)$ (the agent's wealth allocation to the mutual fund) which maximize the expected utility of the open and bounded set of terminal wealth $W^{\eta(t)\,\varphi\,w_0}(T) = W(\eta(t), \varphi(t), w_0, T)$:

$$\sup_{\eta\in\mathcal{A}_1,\phi\in\mathcal{A}_2} \left\{\mathbb{E}\left[U(W^{\eta\,\phi\,w_0}(T))\right]\right\}.$$

Where $A_1 = [0,1]$ and $A_2 = [0,1]$ are set of non-negative processes of η and φ respectively. We solve the maximization problem of the agent by resorting to the DP technique discussed in Section 3.4.3 of Chapter 3. Without any ambiguity, we use the notations for the reward and value (or performance) functions of that Chapter. By Definitions 26 and 27, we can write value functional for our optimal control problem as follow:

$$\begin{aligned} V(t,\eta,\phi,w_0) &= J(t,\hat{\eta},\hat{\phi},w_0) = \sup_{\eta\in\mathcal{A}_1,\,\phi\in\mathcal{A}_2} J(t,\eta,\phi,w_0) \\ &= \sup\mathbb{E}\left[\int_0^T g(t,W(t),\eta(t),\phi(t))dt + h(T,W(T))\right] \end{aligned} \tag{4.10}$$

subject to the law of motion of the wealth process

$$dW(t) = [r_t + \varphi(t)(\eta^{>}(t)(\mu(t) - r_t\mathbf{1}) - \gamma)]W(t)dt + [\varphi(t)\eta^{>}(t)\sigma(t)]W(t)dB(t)$$

for $t \in [0,T]$ and $W(0) = w_0$ given. Observe that the supremum is taken over all admissible controls A_1 and A_2. If there exist $\hat{\eta}(t), \hat{\varphi}(t)$ such that equation (4.10) is optimised, then they are called “best” *optimal controls* and $V(t,W(t))$ is the corresponding *optimal value functional*. Applying the principle of dynamic programming to (4.10) and using the fact that the wealth process is an It´o process we can achieve our maximization goal by applying Theorem 3.4.1 to obtain

$$\begin{aligned} A^{\eta\,\phi}V(t,w_0) &= V_t + \left\{r_t + \phi(t)[\eta^\top(t)(\mu(t)-r_t\mathbf{1})-\gamma]\right\}W(t)V_W \\ &\quad + \frac{1}{2}\phi^2(t)W^2(t)\mathbf{Tr}[\sigma^\top(t)\eta(t)\eta^\top(t)\sigma(t)]V_{WW}, \end{aligned}$$

where $V_t = \frac{\partial V(t,W(t))}{\partial t}$, $V_W = \frac{\partial V(t,W(t))}{\partial W}$, $V_{WW} = \frac{\partial^2 V(t,W(t))}{\partial W^2}$ and Tr denotes the trace of a matrix. By equation (3.49) the associated HJB equation together with the terminal condition is

$$\begin{aligned} \sup_{\eta\in\mathcal{A}_1,\,\phi\in\mathcal{A}_2}\left\{A^{\eta\,\phi}V(t,W(t))\right\} &= 0, \quad (t,W)\in[0,T]\times\mathbb{R}^+ \\ V(T,\eta,\phi,w_0) &= U(W(T)). \end{aligned} \qquad , \tag{4.11}$$

The HJB equation in equation (4.11) is a nonlinear second-order PDE which is coupled with an optimization over $\varphi(t)$ and $\eta(t)$. Thus, we seek to find the controls $\hat{\eta}(t)$ and $\hat{\varphi}(t)$ that maximize the function

$$y(\phi(t),\eta(t)) = V_t + \left\{r_t + \phi(t)[\eta^\top(t)(\mu(t)-r_t\mathbf{1})-\gamma]\right\}W(t)V_W \tag{4.12}$$

$$+\frac{1}{2}\phi^2(t)W^2(t)\mathbf{Tr}[\sigma^\top(t)\eta(t)\eta^\top(t)\sigma(t)]V_{WW}. \tag{4.13}$$

Equation (4.12) is maximized if the following first order condition (F.O.C) and second order condition (F.O.C) for optimality are satisfied

$$\text{F.O.C :} \qquad \left\{\frac{\partial y}{\partial \eta} = 0; \quad \frac{\partial y}{\partial \phi} = 0\right. , \tag{4.14}$$

$$\text{S.O.C :} \qquad \{y_{\eta\eta}y_{\varphi\varphi} - y_{\eta\varphi}y_{\varphi\eta} > 0 \text{ and } y_{\eta\varphi} < 0,\ y_{\varphi\eta} < 0.$$

From (4.12) the first and second order partial derivatives are (here, we used vector, matrix and trace differentiation rules in Petersen and Pedersen (2012)):

$$y_{\eta(t)} = \varphi(t)(\mu(t) - r_t 1)W(t)V_W + \varphi^2(t)W^2(t)[\sigma(t)\sigma^{>}(t)]\eta(t)V_{WW},$$

$$y_{\phi(t)} = \left[\eta^\top(\mu(t) - r_t\mathbf{1}) - \gamma\right] W(t)V_W + \phi(t)W^2(t)\mathbf{Tr}[\sigma^\top\eta(t)\eta^\top(t)\sigma(t)]V_{WW},$$

$$y_{\eta\eta} = \varphi^2(t)W^2(t)[\sigma(t)\sigma^{>}(t)]V_{WW}\,; y_{\varphi\varphi} = W^2(t)\mathrm{Tr}[\sigma^{>}\eta(t)\eta^{>}(t)\sigma(t)]V_{WW},$$

$$y_{\eta\varphi} = y_{\varphi\eta} = [\mu(t) - r_t 1]W(t)V_W + 2\varphi(t))W^2(t)[\sigma(t)\sigma^{>}(t)]\eta(t)V_{WW}.$$

Applying the F.O.C's given in (4.14) on the first two partial derivatives we have

$$\begin{aligned} \phi(t) &= -\frac{\{\eta^\top(t)[\mu(t) - r_t\mathbf{1}] - \gamma\}\, V_W}{W(t)V_{WW}\mathbf{Tr}[\sigma^\top(t)\eta(t)\eta^\top(t)\sigma(t)]}, \\ \eta(t) &= -\frac{(\sigma(t)\sigma^\top(t))^{-1}[\mu(t) - r_t\mathbf{1}]V_W}{\phi(t)W(t)V_{WW}}. \end{aligned} \tag{4.15}$$

Simplifying the equations in (4.15) yield the two optimal control candidates

$$\begin{aligned} \hat{\phi}(t) &= -\frac{(a-b)V_W}{\gamma W(t)V_{WW}}, a > b, \\ \hat{\eta}(t) &= -\frac{([\sigma(t)\sigma^\top(t)]^{-1}[\mu(t) - r_t\mathbf{1}]\gamma}{(b-a)}, \quad a > b, \end{aligned} \tag{4.16}$$

where a and b in the expressions for $\hat{\varphi}(t)$ and $\hat{\eta}(t)$ are defined by

$$a = [\mu(t) - r_t 1]^{>}(\sigma(t)\sigma^{>}(t))^{-1}[\mu(t) - r_t 1], \tag{4.17}$$

$$b = \mathrm{Tr}[\sigma^{-1}(t)[\mu(t) - r_t 1][\mu(t) - r_t 1]^{>}(t)\sigma^{->}(t)]. \tag{4.18}$$

Suppose that $\sigma(t)$ is not invertible (as assumed above), then its generalized inverse (or pseudo inverse) can be used. This extends to the product $\sigma(t)\sigma^{>}(t)$. Substituting equations in (4.16) into equation (4.11) yields the following nonlinear boundary value problem

$$\begin{aligned} V_t + rW(t)V_W - \frac{1}{2}\frac{bV_W^2}{V_{WW}} &= 0, \quad \forall\, (t, W(t)) \in [0,T] \times \mathbb{R}^+, \\ V(T,\cdot) &= U(W(T)), \end{aligned} \tag{4.19}$$

with b is given by (4.18). To proceed with solving the HJB equation in equation (4.19) and verify the admissibility of optimal control candidates we have to be pretty clear about the nature of the utility function. This is necessary because it is naturally difficult to obtain solutions with the general utility function described in equation (4.9). We shall consider *von Neumann-Morgenstern utility functions* such as the *logarithmic*, *power* and *negative exponential* utility functions.

4.3.1 Logarithmic Utility Function

The logarithmic utility function (or simply log utility function) given as a function of wealth is

$$V(\cdot) = U(W(t)) = \ln(\epsilon W(t)) = \ln(W(t)),$$

where the risk-aversion coefficient, is unity. is a parameter that measures an economic agents risk preference. We show below that the two controls fulfill the admissibility condition and the F.O.C and S.O.C are also met. From the above utility function, we have $V_W = \frac{1}{W(t)} > 0$ and $V_{WW} = -\frac{1}{W^2(t)} < 0$ so that

$\hat{\eta} \geq 0$ and $\varphi(\hat{t}) \geq 0$. By the same argument on the sign of V_W and V_{WW} we have $y_{\eta\eta}y_{\varphi\varphi} - y_{\eta\varphi}y_{\varphi\eta} > 0$ and $y_{\eta\varphi} < 0$, $y_{\varphi\eta} < 0$ verified. In actual sense, the partial derivatives with respect to $W(t)$ should be found using the value function for all $t \in [0,T]$, but the conditions are still met in that case. For this utility function, the boundary value problem in equation (4.19) reduces to

$$\begin{aligned} V_t + rW(t)V_W - \frac{1}{2}\frac{bV_W^2}{V_{WW}} &= 0, \quad t \in [0,T], \ W > 1, \\ V(t,\eta,\phi,w_0) &= \ln(W(t)), \quad t = T. \end{aligned} \tag{4.20}$$

To solve the HJB equation in equation (4.20) for $V(\cdot)$, we make the Ansatz (i.e., try solution)

$$V(\cdot) = f(t)\ln(W(t)).$$

Substituting the partial derivatives of this try solution into the HJB equation in equation (4.20) yields

$$f'(t)\ln(W(t)) + \left(r + \frac{1}{2}b\right) f(t) = 0 \tag{4.21}$$

with the terminal condition *f*(*T*) = 1. Equation (4.21) is a separable ODE which has the solution

$$f(T) = f(t)\exp\left\{\frac{\Gamma_1(T-t)}{\ln(W(t))}\right\}, \quad \Gamma_1 = -\left(r + \frac{1}{2}b\right), \quad W(t) > 1 .$$

Applying the terminal condition *f*(*T*) = 1 yields

$$f(t) = \exp\left\{\frac{-\Gamma_1(T-t)}{\ln(W(t))}\right\}, \quad \Gamma_1 = -\left(r + \frac{1}{2}b\right), \quad W(t) > 1 .$$

The value function can then be succinctly written as

$$V(t, W(t)) = \begin{cases} \exp\left\{\frac{-\Gamma_1(T-t)}{\ln(W(t))}\right\}\ln(W(t)); & \text{for } 0 \le t < T, \\ \ln(W(t)); & \text{for } t = T. \end{cases} \tag{4.22}$$

Substituting the partial derivatives of the value function in equation (4.22) with respect *W* into the expressions for the optimal control candidates yields the optimal controls we desired. From (4.22)

$$\begin{aligned} V_W &= \frac{1}{W}\exp\left\{\frac{-\Gamma_1(T-t)}{\ln(W(t))}\right\}\left\{1 + \frac{\Gamma_1(T-t)}{\ln(W(t))}\right\}, \quad W(t) > 1, \\ V_{WW} &= -\frac{1}{W^2}\exp\left\{\frac{-\Gamma_1(T-t)}{\ln(W(t))}\right\}\left\{1 + \frac{\Gamma_1(T-t)}{\ln(W(t))} - \frac{\Gamma_1^2(T-t)^2}{(\ln(W(t)))^3}\right\}, \quad W(t) > 1 . \end{aligned}$$

However, for $V(t,W(t)$ to jointly fulfill the increasing condition $V_W(t,W(t) > 0$ and concavity condition $V_{WW}(t,W(t) < 0$, an investor must have a wealth level, $W(t) \ge 35$ so that the optimal controls become

$$\hat{\phi} = \frac{(a-b)c}{\gamma d}; \quad \hat{\eta} = \frac{([\sigma(t)\sigma^\top(t)]^{-1}[\mu(t) - r_t\mathbf{1}]\gamma}{(a-b)}, \tag{4.23}$$

where $$c = \left\{1 + \frac{\Gamma_1(T-t)}{\ln(W(t))}\right\}; \quad d = \left\{1 + \frac{\Gamma_1(T-t)}{\ln(W(t))} - \frac{\Gamma_1^2(T-t)^2}{(\ln(W(t)))^3}\right\}.$$

We emphasize that the use of the value function, $V(t,W(t) = \ln(W(t))$ eliminates *c* and *d* in equation (4.23).

4.3.2 Power Utility Function

The power utility function given as a function of wealth has the form

$$U(W) = \frac{W^{\epsilon}}{\epsilon}, \ \ \epsilon < 1, \ \epsilon \neq 0 .$$

The parameter, called risk-aversion coefficient, is a measure of economic agents risk preference. From the above equation it is easy to see that both the first and second order conditions for optimality in equation (4.14) are satisfied. Here the boundary value problem then reduces to

$$\begin{aligned} V_t + rW(t)V_W - \frac{1}{2}\frac{bV_W^2}{V_{WW}} &= 0, \ \ \forall\, (t, W(t)) \in [0,T] \times \mathbb{R}^+ \\ V(t, W(t)) &= \frac{W^{\epsilon}(t)}{\epsilon}, \ \ t = T. \end{aligned} , \tag{4.24}$$

Making use of the Ansatz $V(t, W(t)) = f(t)\frac{W^{\epsilon}}{\epsilon}$, equation (4.24) reduces to the following separable ODE

$$\begin{aligned} f'(t) &= \Gamma_2 f, \quad \Gamma_2 = -\frac{\delta}{1-\delta}\left(r(1-\delta) + \frac{1}{2}b\right) \\ f(t) &= 1, \quad t = T \end{aligned} ,$$

with the solution

$$f(t) = \exp\{-\Gamma_2(T - t)\}.$$

The value function then becomes

$$V(\cdot) = \exp\{-\Gamma_2(T-t)\}\frac{W^{\epsilon}}{\epsilon}, \ \ t \in [0,T], W > 0 . \tag{4.25}$$

The corresponding optimal controls are

$$\hat{\phi}(t) = \frac{(a-b)}{\gamma(1-\epsilon)}; \quad \hat{\eta}(t) = \frac{[\sigma(t)\sigma^{\top}(t)]^{-1}[\mu(t) - r_t\mathbf{1}]\gamma}{(a-b)} . \tag{4.26}$$

4.3.3 Negative Exponential Utility Function

A negative exponential utility function has the form

$$U(W) = -\exp\{-\epsilon W\}, \quad \epsilon > 0. \tag{4.27}$$

The parameter, called risk-aversion coefficient, is a measure of economic agents risk preference. By similar arguments made under the above utility functions, it can easily be verified that both the first and second order conditions for optimality in equation (4.14) are satisfied. The boundary value problem in equation (4.19) then becomes

$$\begin{aligned} V_t + rW(t)V_W - \frac{1}{2}\frac{bV_W^2}{V_{WW}} &= 0, \;\; \forall\,(t, W(t)) \in [0,T] \times \mathbb{R}^+, \\ V(t, W(t)) &= -\exp\{-\rho W(t)\}, \; t = T. \end{aligned} \tag{4.28}$$

Making use of the Ansatz $V(t, W(t)) = -f(t)\exp\{-\epsilon W(t)\}$, the boundary value problem in equation (4.28) reduces to the following separable ODE

$$\begin{aligned} f'(t) &= f(t)\Gamma_3, \;\; \Gamma_3 = \left(r\epsilon W(t) + \frac{1}{2}b\right) \\ f(t) &= 1, \;\; t = T \end{aligned},$$

with the solution

$$f(t) = \exp\{-\Gamma_3(T-t)\}.$$

Given the expression for $f(t)$, the value function then becomes

$$\begin{aligned} V(\cdot) &= -\exp\{-\Gamma_2(T-t)\}\exp\{-\epsilon W(t)\}, \;\; \forall\, t \in [0,T] \\ &= -\exp\left\{-(1+r\varsigma)\epsilon W(t) - \frac{1}{2}b\varsigma\right\}, \;\; \varsigma = T - t > 0 \end{aligned}. \tag{4.29}$$

From equations (4.15) and (4.29), the optimal controls are

$$\hat{\phi} = \frac{(a-b)}{\gamma\epsilon W(t)(1+r_t\varsigma)}; \quad \eta = \frac{[\sigma(t)\sigma^\top(t)]^{-1}[\mu(t) - r_t\mathbf{1}]\gamma}{(a-b)} \,\hat{}\, \tag{4.30}$$

4.3.4 Explicit Representation for the Optimal Trading Strategies and Optimal Value Functions

Until now, we have solved the maximization problem of the individual economic agent and the fund manager pretending we know the drift (mean return) vector process, $\mu(t)$. To obtain explicit formulas for the best controls and their corresponding performance indicators, we devote this section to the estimation of $\mu(t)$.

Let the n-dimensional drift vector process, $\mu(t)$ be a solution of the SDE

$$d\mu(t) = \alpha(\delta - \mu(t))dt + \beta d\bar{B}(t), \tag{4.31}$$

where $\bar{B}(t)$ is BM w.r.t to (F,P) with the property cov$(B(t),\bar{B}) = 0$ under P, δ is a known vector of real numbers, α and β are known matrices of real numbers. We suppose that the initial drift vector process, μ_0 is Gaussian (or normal) such that $\mu_0 \sim N(m_0,\rho_0)$ and also assume that the matrix β is nonsingular. Lakner (1998), noted that equation (4.31) becomes an Ornstein-Uhlenbeck process with mean-reverting drift with the condition that α is a diagonal matrix with entries $\alpha_i > 0$. In this thesis, we assumed that given the probability space (Ω,F,P) the drift vector process, $\mu(t)$ is not directly observable. The task then is to estimate $\mu(t)$ conditioned on a directly observable process on (Ω,F,P). In the financial market under consideration, stock prices, $S_i(t)$ are directly observable and can possibly be used as to estimate $\mu(t)$ at any time, t, $0 \leq t \leq T$. By definition, $\mu(t)$ refers to the mean return vector process, as a consequence it would be more appropriate to use the return vector process, $R(t) = (R_1(t),R_1(t),\cdots,R_n(t))^\top$ to estimate $\mu(t)$ than the price vector process, $S(t) = (S_1(t),S_2(t),\cdots,S_n(t))^\top$.

According to Lakner (1998), $S(t)$ and $R(t)$ generate the same σ-algebra. Thus, $\mathrm{F}_t^S = \sigma(S(s),s \leq t) = \mathrm{F}_t^R = \sigma(R(s),s \leq t)$. This implies that at any time, t, we can estimate the unobservable state $\mu(t)$ based on $R = \{R(t),0 \leq t \leq T\}$. From equation (4.2), we can write the return process as

$$dR(t) = \frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dB(t) . \tag{4.32}$$

By this formulation, we can estimate μ using the Kalman-Bucy method. In Liptser and Shiryayev (1977), if $E[|\mu(t)|^2] < \infty$, then the approximation of $\mu(t)$ from R is

$$m(t) = E[\mu(t)|F_t^R]$$

with estimation error

$$\rho(t) = E[(\mu(t) - m(t))(\mu(t) - m(t))^{>}].$$

By Theorem 3.2.3, $m(t)$ and $\rho(t)$ are unique solutions to the equations

$$dm(t) = [\alpha\delta + (-\alpha - \rho(t)(\sigma\sigma^{>})^{-1})m(t)]dt + \rho(t)(\sigma\sigma^{>})^{-1}dR(t) \tag{4.33}$$

$$\dot{\rho}(t) = -\rho(t)(\sigma\sigma^{>})^{-1}\rho(t) - \alpha\rho(t) - \rho(t)\alpha^{>} + \beta\beta^{>} \tag{4.34}$$

with initial condition m_0, ρ_0. Equation (4.33) is an SDE and equation (4.34) is a first order ODE called Riccati equation. Note that we suppress the argument of σ to simplify notation.

By Lemmas 3.2.1 & 3.2.2, we can determine the estimate (filter), $m(t)$ in terms of $\phi(t)$ and $\rho(t)$ defined by

$$m(t) = \varphi(t)\left\{m_0 + \int_0^T (\varphi(t))^{-1}\rho(t)(\sigma\sigma^{\top})^{-1}dR(t) + \int_0^T (\varphi(t))^{-1}\alpha\delta dt\right\} \tag{4.35}$$

where $\phi(t)$ is the fundamental solution of the deterministic system

$$\dot{\phi}(t) = [\alpha - \rho(t)(\sigma\sigma^{>})^{-1}]\phi(t).$$

$\phi(t)$ is an $(n \times n)$ matrix valued function with the initial condition that $\phi(0)$ is the $(n \times n)$ identity matrix.

4.3.5 Main Results

Definition 28 *Let* $W(0) = w_0 > 0$, $\varphi(t) \geq 0$ *and* $\eta(t) \geq 0$. *We say that the* *trading strategies (controls)* $\varphi(t) \in A_1$ *and* $\eta \in A_2$ *for all* $t \in [0,T]$ *are admissible if* $W(t) \geq 0$.

For this class of controls and the representation of the estimated (filtered) mean return vector process, $m(t)$ in equation (4.35), we state the main results of this thesis based on the utility functions used as follows:

Logarithmic Utility Function

From equation (4.23), we present the explicit representation of the optimal trading strategies of the individual economic agent and the the fund manager as well as their corresponding value functions respectively.

Economic agent's Optimal Trading Strategy and Value Function

$$\hat{\phi}(t) = \frac{(a-b)\left(1+\frac{\Gamma_1(T-t)}{\ln(W(t))}\right)}{\gamma\left(1+\frac{\Gamma_1(T-t)}{\ln W(t)}-\frac{\Gamma_1^2(T-t)^2}{(\ln W(t))^3}\right)},\ W(t) \geq 35, \tag{4.36}$$

$$V(\cdot) = \exp\left\{\frac{-\Gamma_1(T-t)}{\ln W(t)}\right\}\ln W(t),\ W(t) \geq 35 \quad , \tag{4.37}$$

where *a* and *b* are defined in equation (4.17) and (4.18) respectively and

$$\Gamma_1 = \left(r+\frac{1}{2}\mathbf{Tr}[\sigma^{-1}[m-r_t\mathbf{1}][m-r_t\mathbf{1}]^\top\sigma^{-\top}]\right).$$

Fund Manager's Optimal Trading Strategy and Value Function

$$\hat{\eta}(t) = \frac{(\sigma(t)\sigma^\top(t))^{-1}[m-r_t\mathbf{1}]\gamma}{[m-r\mathbf{1}]^\top(\sigma\sigma^\top)^{-1}[m-r\mathbf{1}]-\mathbf{Tr}[\sigma^{-1}[m-r\mathbf{1}][m-r\mathbf{1}]^\top\sigma^{-\top}]}, \tag{4.38}$$

$$\Phi(t) = \int_0^T \gamma\hat{\phi}(t)W(t)dt. \tag{4.39}$$

Power Utility Function

Using equation (4.26), we abreast the optimal trading strategies and the value function of the investor and fund manager.

Economic agent's Optimal Trading Strategy and Value Function

$$\hat{\phi}(t) = \frac{[m-r\mathbf{1}]^\top(\sigma\sigma^\top)^{-1}[m-r\mathbf{1}]-\mathbf{Tr}[\sigma^{-1}[m-r\mathbf{1}][m-r\mathbf{1}]^\top\sigma^{-\top}]}{\gamma(1-\epsilon)} \tag{4.40}$$

$$\tag{4.41}$$

$$V(\cdot) \;=\; e^{-\Gamma_2(T-t)}\frac{W^{\epsilon}}{\epsilon}, \text{ where} \tag{4.42}$$

$$\Gamma_2 = -\frac{\epsilon}{1-\epsilon}\left(r_t(1-\epsilon) + \frac{1}{2}\mathbf{Tr}[\sigma^{-1}[m - r_t\mathbf{1}][m - r_t\mathbf{1}]^{\top}\sigma^{-\top}]\right).$$

Fund Manager’s Optimal Trading Strategy and Value Function

$$\hat{\eta} \;=\; \frac{(\sigma(t)\sigma^{\top}(t))^{-1}[m - r_t\mathbf{1}]\gamma}{[m - r\mathbf{1}]^{\top}(\sigma\sigma^{\top})^{-1}[m - r\mathbf{1}] - \mathbf{Tr}[\sigma^{-1}[m - r\mathbf{1}][m - r\mathbf{1}]^{\top}\sigma^{-\top}]} \tag{4.43}$$

$$\Phi(t) \;=\; \int_0^T \gamma\hat{\phi}(t)W(t)dt. \tag{4.44}$$

Negative Exponential Utility Function

For the negative exponential utility function, the optimal trading strategy and value function of the investor and the fund manager are:

Economic agent’s Optimal Trading Strategy and Value Function

$$\hat{\phi}(t) \;=\; \frac{[m - r\mathbf{1}]^{\top}(\sigma\sigma^{\top})^{-1}[m - r\mathbf{1}] - \mathbf{Tr}[\sigma^{-1}[m - r\mathbf{1}][m - r\mathbf{1}]^{\top}\sigma^{-\top}]}{\gamma\epsilon W(t)(1 + r_t\varsigma)} \tag{4.45}$$

$$V(\cdot) \;=\; -e^{-\Gamma_3(T-t)-\epsilon W(t)}, \tag{4.46}$$

where

$$\Gamma_3 = \left(r\epsilon W(t) + \frac{1}{2}\mathbf{Tr}[\sigma^{-1}(t)[m(t) - r_t\mathbf{1}][m(t) - r_t\mathbf{1}]^{\top}\sigma^{-\top}(t)]\right)$$

Fund Manager’s Optimal Trading Strategy and Value Function

$$\hat{\eta} \;=\; \frac{(\sigma(t)\sigma^{\top}(t))^{-1}[m - r_t\mathbf{1}]\gamma}{[m - r\mathbf{1}]^{\top}(\sigma\sigma^{\top})^{-1}[m - r\mathbf{1}] - \mathbf{Tr}[\sigma^{-1}[m - r\mathbf{1}][m - r\mathbf{1}]^{\top}\sigma^{-\top}]} \tag{4.47}$$

$$\Phi(t) \;=\; \int_0^T \gamma\hat{\phi}(t)W(t)dt. \tag{4.48}$$

4.3.6 Surprising Results

Proposition 4.3.1 *The product of the optimal trading strategy, $\hat{\varphi}(t)$ of the investor and the optimal trading strategy, $\hat{\eta}(t)$ of the fund manager results in optimal trading strategy that are identical to the trading strategy of an individual investor who invest directly in n securities (stocks) plus a risk-less bond.*

$$\hat{\phi}(t)\hat{\eta}(t) = \begin{cases} [\sigma(t)\sigma^{\top}(t)]^{-1}[m(t) - r_t\mathbf{1}] & \textit{log} \\ \frac{[\sigma(t)\sigma^{\top}(t)]^{-1}[m(t)-r_t\mathbf{1}]}{(1-\epsilon)} & \textit{power} \\ \frac{[\sigma(t)\sigma^{\top}(t)]^{-1}[m(t)-r_t\mathbf{1}]}{\epsilon W(t)(1-r_t\varsigma)} & \textit{exponential} \end{cases} \tag{4.49}$$

This means that, the combination of the economic agent's effective equity portfolio is similar to the portfolio he or she would have chosen for by freely trading the stocks directly. For the log utility, this coincides with a result found by Ocone and Karatzas (1991) for full information case and Lakner (1998) established that this result can be extended to the partial information case by mere substitution of $m(t)$ for $\mu(t)$.

4.3.7 Discussion of Main Results

For all the three utility functions, the optimal trading strategy, $\hat{\varphi}(t)$ in equations (4.36), (4.40) and (4.45) respectively, is independent of the level of wealth, $W(t)$, except for the negative exponential utility function where it is a decreasing function of wealth, $W(t)$. This means that the actual amount, $\hat{\varphi}(t)W(t)$ to allocated to the mutual fund changes with changes in $W(t)$ for the log and power utility functions but fixed for the negative exponential utility function. Thus, a rise in the wealth level raises the allocation to the fund and vice versa. It is inversely related to the fee rate, γ as shown in equations (4.36), (4.40) and (4.45) respectively for the three utility functions. This agrees with natural intuition that an investor who is risk averse will sway from a fund that charges exorbitant fee rate. $\hat{\varphi}(t)$ is also inversely related to the volatility of the return on the securities. The fund manager's optimal trading strategy, $\hat{\eta}(t)$ in equations (4.38), (4.43) and (4.47) respectively for the three utility functions, is directly proportional to the fee rate charged per trading period and independent of wealth level. Thus, the actual amount, $\hat{\varphi}(t)W(t)\hat{\eta}(t)$ to be held in each of the securities changes with changes in the level of wealth, except with the negative exponential utility function where $\hat{\varphi}(t)W(t)\hat{\eta}(t)$ is constant.

$\Phi(t)$, the optimal value function of the fund manager given in equations (4.39), (4.44) and (4.48) respectively for the three utility functions, is independent of the fee rate, γ charged at each trading period against "a priori". It is a changing function of the optimal amount of allocation from the investor without the term γ, except for the negative exponential utility function. Thus, as wealth increases, $\Phi(t)$ also increases and the opposite is also true.

Under both the power and exponential utility functions, the investor's optimal trading strategies are inversely related to the risk-aversion coefficient, with their respective associated constraint. This means that, the more risk averse the investor is, the less proportion of his or her wealth that would devoted to the mutual fund. As a consequence, the optimal value functions of the fund manager under these two utility functions decrease with rising risk-aversion coefficient.

The optimal value function, $V(t,W(t))$ given in equations (4.37), (4.41) and (4.46) respectively for the log, power, and negative exponential utility functions, is a product of an exponential growth function $f(t)$ and the utility function. The formal becomes unity at the final time $t = T$ so that $V(t,W(t))$ agrees with the formulation of the HJB equation having terminal condition equal to the latter. $V(t,W(t))$ is a strictly concave function because it is essentially a product of a strictly positive function $f(t)$ and concave preference function.

From the foregoing discussion, an investor who is risk averse should invest the same proportion of his or her wealth in the mutual fund if his preferences are characterized by a log or power utility function and invest a decreasing proportion of his or her wealth in the mutual fund if his or her preferences are characterized by a negative exponential utility function. Thus, for the case of log or power utility function, the dollar or cedis value (or amount) to be held in the mutual fund increases with rising level of wealth, but the dollar value to be held in the mutual fund for an investor whose preferences are characterized by a negative exponential utility function remained fixed for all levels of wealth.

Chapter 5

Conclusion

5.0.8 Introduction

This chapter gives a brief but holistic overview of this thesis. It encompasses the summary of our findings, the meaning and implication of the findings, limitations of the thesis and suggested areas for further investigation.

5.0.9 Summary, Findings and Recommendations

The surge in the demand for Mutual Fund over the years couple with the modern financial market complexities and their uncertainties fuel the need for models and techniques for efficiently managing portfolios. Optimal portfolio management within the framework of static and dynamic model remains an important tool for allocating scarce resources (wealth) among different assets so that an economic agent can maximize satisfaction from terminal wealth. This thesis study the utility maximization problem of an economic agent maximizing the expected satisfaction of the terminal wealth resulting from an investment in a risk-less asset and a Mutual Fund that spans over a finite time period. It investigates the impact of dynamic flow on the portfolio choices on the part of the economic agent and the fund manager in a financial market with partial information. Continuous-time financial models such as ordinary differential equation and stochastic differential equation were used to specify the price dynamics of the risk-less and risky securities respectively. To determine the portfolio rules for the economic agent and the fund manager, the theory of stochastic optimal control, dynamic programming methods and the theory of utility maximization in the continuous-time setting were employed. The assumption of partial information is implicit in the fact that the drift process appearing in the SDE for the asset prices are not directly observable. Using the return vector process as the observation available, we use the continuous-time Kalman-Bucy filter method to obtain the optimal estimate (filter), $m(t)$ (equation (4.35)) which in the mean square error sense is a conditional expectation. Within the stochastic optimal control theory and DPP framework the optimal control problem was formulated. It´o’s lemma together with stochastic calculus mathematical expectation rules, we obtain the HJB equation for the optimal control problem. The HJB equation provided the necessary and sufficient condition for the optimum. Closed-form analytic solutions were obtained for the optimal control policies with their corresponding value functions under the logarithmic utility function (equations (4.36) to (4.39)), power utility function (equations (4.40) to (4.44)) and negative exponential utility function (equations (4.45) to (4.48)).

All other things being equal, we found that the investor's trading strategies are inversely related to the fee rate for all the three utility functions considered. The allocation to the securities by the fund manager increases with a rising fee rate. With an appreciable fee rate, the investor's holding of the risk-less asset through the fund becomes more expensive. For the full information case, Hugonnier and Kaniel (2010) considers an investor with logarithmic utility function and also found that the allocation to the fund is dwindling with dwindling fee rate whereas the allocation to the securities is rising with rising fee rate. Just like Hugonnier and Kaniel (2010), we found that the investor's allocation of wealth to the fund increases with rising level of wealth. The power utility function also shows that an investor's wealth allocation to the mutual fund increases with the level of wealth, but the negative exponential utility function shows that allocation to the mutual is independent of wealth but solely a function of the portfolio weights in the securities and level of risk-aversion coefficient. Even though the fund manager receives a reward equal to the product of fee rate and investor's trading strategy per unit of time, the accumulated reward is independent of the fee rate. This holds for all the utility functions considered. It is a function of the portfolio allocation in securities, volatility of security prices and the risk-aversion coefficient. The optimal control formula for the fund manager has the same structure for all the utility functions. Allocations to assets available to him or her are however different because the allocation to him or her from the investor are different for these utility functions.

While this thesis is vital in deciphering the dynamic portfolio choice made by both investors and fund managers, there are other vital areas that need further investigation.

Firstly, incorporating the cost of re-balancing holdings in the Mutual Fund by the investor would be very interesting. Secondly, the model can be extended to the situation where the volatility market coefficients for model creation are also assumed not be directly observable. Another area that this work can be extended is by requiring the investor to have access to a sub-section of the risky securities as well as restricting the Mutual Fund to have access to only the risky securities. Finally, extending this work to an automated system using tools of pattern recognition and machine learning would be captivating.

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Appendix A

5.0.10 Derivation of the Wealth Dynamics of the Economic Agent

At any time, *t*, the investor must allocate all his wealth between the risky mutual fund and the risk-less asset such that

$$W(t) = S_0(t) + F(t) \Leftrightarrow 1 = \frac{S_0(t)}{W(t)} + \frac{F(t)}{W(t)}, \tag{5.1}$$

where $W(t)$ is the level of wealth, $S_0(t)$ is the price process of the risk-less asset and $F(t)$ is the mutual fund process. Let $\phi(t) = \frac{F(t)}{W(t)}$ denote the proportion of wealth held in the fund at any time, *t*, so that $1 - \phi(t) = \frac{S_0(t)}{W(t)}$ denote the proportion of wealth held in the risk-less asset. Since there are no exogenous infusion and withdrawal from the investment over the investment horizon, it must hold that

$$S_0(t) + F(t) = (1 - \varphi(t))W(t) + \varphi(t)W(t) = W(t). \tag{5.2}$$

From the left hand side (LHS) of equation (5.1), we have

$$dW(t) = dS_0(t) + dF(t) = S_0(t)\frac{dS_0(t)}{S_0(t)} + F(t)\frac{dF(t)}{F(t)}. \tag{5.3}$$

By equation (5.2), equation (5.3) becomes

$$dW(t) = (1 - \phi(t))W(t)\frac{dS_0(t)}{S_0(t)} + \phi(t)W(t)\frac{dF(t)}{F(t)} \tag{5.4}$$

5.0.11 Derivation of the Mutual Fund Return Process

Given that the mutual fund has access to a risk-less asset and *n* stocks as well as charges a fee rate of γ per trading period, if we assumes 1 as a unit of money (for simplicity) and chooses the vector process $\eta(t) = \{(\eta_i(t))_{i=1}^n, \ \forall\, t \in [0, T]\}$ which is a vector with entries being portfolio weights in each of the stocks, then $(1 - \sum_{i=1}^n \eta_i(t))$ is the weight in the risk-less asset. The return process from the investment in the mutual fund then becomes the sum of the all returns from the stocks and the risk-less minus the fee rate:

$$
\begin{aligned}
\frac{dF(t)}{F(t)} &= \left(1-\sum_{i=1}^{n}\eta_i(t)\right)\frac{dS_0(t)}{S_0(t)}+\eta_1(t)\frac{dS_1(t)}{dS_1(t)}+\ldots+\eta_1(t)\frac{S_n(t)}{S_n(t)}-\gamma dt,\\
\frac{dF(t)}{F(t)} &= \left(1-\eta^{\top}(t)\mathbf{1}\right)\frac{dS_0(t)}{S_0(t)}+\sum_{i=1}^{n}\eta_i(t)\frac{dS_i(t)}{S_i(t)}-\gamma dt,
\end{aligned}
\tag{5.5}
$$

where $\eta^{\top}(t)\mathbf{1} = \sum_{i=1}^{n}\eta_i(t),\ \top$ denotes transposition and **1** is conformable *n*dimensional vector of ones.