KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

NUMERICAL SIMULATION OF TWO PHASE OIL-WATER FLOW MODEL USING THE IMPES METHOD

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Declaration

I hereby declare that this submission is our own work towards the award of the Master of Philosophy (Applied Mathematics) and that, to the best of our knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the University, except where due acknowledgment had been made in the text.

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Dedication

I dedicate this work to my two sons: Manuel Donatus Frimpong and Louis Adu-Poku Frimpong for whom I pray that they will grow to achieve higher laurels than I have done.

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I thank the Lord God Almighty for his mercies and how far he has brought me.

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Abstract

The two phase flow model for simulating oil/water flow in petroleum reservoir simulation at the secondary recovery stage is reviewed. The formulation of two phase flow equations using the governing equations; conservation of mass, conservation of momentum, saturation and capillary pressure relations are considered and results in a coupled system of equations. The Implicit Pressure Explicit Saturation (IMPES) solution approach is employed as a numerical technique. Incompressible and slightly compressible reservoir conditions are investigated and compared numerically. The results indicate that slightly compressible conditions are more realistic and the relative errors between the fluids being incompressible and slightly compressible are linear in time.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background Studies

The concept of *reservoir simulation* applies varying areas such as Physics, Mathematics, Reservoir Engineering, Economics and Computer Programming on the same platform in developing a predictive tool for hydrocarbon-reservoir performance management. For many stakeholders of this uniting utility, the primary role of reservoir simulation lies in the optimization of the development and production of hydrocarbon resources(Espedal & Ewing, 1983). At the heart of investors and many engineers, reservoir simulation becomes just not an option but a necessity considering the amount of risk involved in deciding which reservoirs are economically feasible and the time for which this feasibility is valid. These risks emanate from factors such as the complexity of the reservoir (i.e. heterogeneity and anisotropic rock properties, regional variations of fluid properties, relative permeability characteristics, etc.) complexity associated with hydrocarbon mechanism and the application of other methods with limitations that renders the methods inappropriate .

Reservoir simulation is managed using mathematical models that explains different fluid(s) and flow regimes; every specific scenario evolves into a unique mathematical model. For instance, the *single phase* flow gives an equation or the pressure distribution in the reservoir and is used for early-stage and simplified flow studies. Hence, it is used to identify flow direction, connection between producers and injectors, and in preliminary model studies. The *Two phase* flow in petroleum reservoir involves the simultaneous flow of two fluids phase i.e oil and water or oil and gas or water and gas. In the two phase flow the primary variables needed to work with are the oil pressure and water saturation (Chen, Qin, & Ewing, 2000) and (Kwok & Tchelep, 2008). This model is a coupled system of nonlinear time-dependent partial differential equation. The *black oil model* consist of oil, water and gas at all standard conditions. if it is assumed that there is no mass transfer between the phases, each fluid component is contained in its own phase. There is also the *compositional model* within which we have flows in up to three dimensions of solvent, oil and water phases as well as n components in the solvent and oil phases. These models are made to compute the phase equilibrium of the oil and solvent phases (i.e, the equilibrium compositions and relative volumes of the solvent and oil phases) in each gridblock of the simulator. Other models includes thermal model, volatile model, generalized model and many more. All these models are mathematical equations that are normally partial differential equations that are solved using numerical methods (Pruess & Narasimhann, 1985).

In this research our focus will be based on the two phase model in spite its varied level of application even extending beyond petroleum engineering to areas such as pump cavitation (where vapor pressure of the fluid being pumped drops causes phase change in the pump), in marine propellor (when vapor bubble collapses producing pressure spikes which cause damage on the propellor), in climate system such as clouds(Levy, 1999) and in groundwater flow. In petroleum engineering the two phase flow is applied in water coning, oil displacement by water and many more.

There has been an extensive literature on numerical methods for solving the equations that describe fluid flow in porous media (Aziz & Settari, 1979), (Chavent & Jaffre, 1978),(R. E. Ewing, 1983), (Peaceman, 1977). In conjunction with this literature, there has been intensive research into the analysis of these methods (Arbogast & Wheeler, 1995),(J. Douglas, 1980a),(J. Douglas, 1980b),(Ewing & WHeeler, 1980),(Russell, 1980), (Wheeler & Darlow, 1980). Most researchers like (Chou & Li, 1991), (J. Dougjas, 1983), (J. Douglas, Ewing, & Wheeler, 1983), (Jr., Ewing, & Wheeler, 1983), (Jr. & Roberts, 1983), (Jr. & Russell, 1982), (Ewing & Wheeler, 1980), (Ewing & Wheeler, 1984), (Ewing, Yuan, & Li, 1989), (Potempa & Wheeler, 1983) did their study based on incompressible fluid which are fluid though in practical sense do not exist, a slightly compressible miscible displacement problem was treated in (Ewing et al., 1989), (Potempa & Wheeler, 1983), (Jr. & Roberts, 1983), (Chou & Li, 1991) but even these studies were more focused on single phase flow. In spite of this, there has been less research into the analysis of the two-phase flow for both incompressible and slightly compressible fluids in a porous media.

Primarily because of the instability associated with the numerical solution and given that the resulting equations are nonlinear and coupled (Chen, Huan, & Ma, 2006a), using an explicit method is discouraged and it is also well-known that the fully implicit scheme

(Aziz & Settari, 1979), (Collins, Nghiem, Li, & Grabonstotter, May 1992), (Dawson, Kle, Wheeler, & Woodward, 1997), (Monteagudo & Firoozabadi, 2007), (Tan & Kaiogerakis, 1991), (Hickel, Adams, & Domaradzki, 2006) has unconditional stability resulting in a system of nonlinear equation yet maintains the inherent coupling of two-phase flow model. Hence, there is a need to choose a finite difference technique that will make the results show that the method used is more efficient and stable.

There exists a wide range of routines that can be used to solve the resulting equations such as the Simultaneous solution (SS), Sequential and adaptive implicit methods, Finite difference, Control Volume Finite Element (CVFE) method, Implicit in Pressure Explicit in Saturation (IMPES) method, Mixed finite element method and many others. The IMPES method remains a popular method in the petroleum industry and a very powerful method for solving two-phase flow although not simple (Stone & Garder, 1961). It solves the pressure equation implicitly and updates the saturation explicitly and as a time discretization scheme, this method is conditionally stable. This research reviews the Classical IMPES method for as applied to simulating the two phase flow in porous media though there are numerous improved versions of IMPES for two-phase flow. (Chen, Huan, & Ma, 2006b),

(Q. Lu, 2000), (Watts, 1985), (Young & Stephenson, 1983)

1.2 Problem Statement

Every stage and scenario within the reservoir simulation process requires a careful studying and thorough understanding. The study of two phase oil/water flow model in spite its varied level of application even extending beyond petroleum engineering has been biased toward incompressible fluid flow with less attention on slightly compressible and compressible fluids. The problem is that incompressible fluids ideally are nonexistent and are used as simplified forms of fluids for theoretical understanding of the modeling of reservoir fluids (Chou & Li, 1991). The realistic situation of slightly compressible (oil and water) and compressible fluids (gas) are rarely documented. Most literature although introduce two phase flow concentrate more on single phase flow analysis. This has left a mass of difficultly in the attempt to build and investigate sensitive parameters within the two phase flow model.

1.3 Objectives

The general objective of this research is to study the two-phase oil/water reservoir model in porous media in both incompressible and slightly compressible fluid flow. The specific objectives are

1. Review the modeling of the two-phase oil/water model in porous media.

2. Study the finite difference formulation using the IMPES method.

3. Analyze and compare solutions from incompressible and slightly compressible flow conditions.

1.4 Methodology

Modeling fluid flow through a porous medium is a fundamental problem of interest in many areas of science and engineering. This research presents the mathematical models that combine to simulate numerical solutions of the Two-Phase flow. Literature is reviewed on existing models of calculating the two-phase flow in porous media. In this research, we review the two-phase oil/water model and develop a numerical model and solution scheme for the fluids transportation in one dimensions, using the popular Implicit Pressure - Explicit Saturation (IMPES) scheme. The numerical form of the IMPES is programmed using MATLAB 7.10.0 (R2013a). The simulation for the models and error analysis computation is entirely done using MATLAB 7.10.0 (R2013a).

1.5 Justification

Reservoir Simulation is a very sensitive area in Reservoir Engineering.The exploration and development of oil is still ongoing and there is need for knowledge and simulators that explain the dynamics of flow at various levels in reservoir stage. This research apart from being a contribution to literature will present a first hand review of two-phase flow in reservoir simulation and a foundation on which other stages in reservoir simulation to be investigated.

1.6 Structure of the Research

The research will be divided into five chapters. Chapter one deals with the introduction, problem statement, objectives, methodology, justification and the structure of the study. The second chapter will cover the mathematical models related to the subject matter and some fluid and rock properties. Chapter three will examine the details of the methodology used in the research. Chapter four will present the results and discussion of the simulation of the two-phase flow in one dimension using the IMPES method and the final chapter (five) will deal with the conclusions and recommendations of the research.

Chapter 2

Mathematical Models For Two-Phase

2.1 Introduction

In order to model a system mathematically, a thorough understanding of the behaviour of different elements that constitute the system is required. In reservoir simulation the elements of interest that constitute the system of the rock and fluid are oil, water and gas occupying the rock. Apart from including all components and phases in a reservoir model each exclusion of a particular component or assumption made sums up into a different mathematical model in the modeling process. A thorough understanding of the component and dimension in place, pressure dependence of fluid densities, viscosities, gas/ liquid ratio, and the saturated dependence of relative permeability and capillary pressure is very helpful. This chapter reviews basic mathematical models related to modeling two phase flow and highlights the key components in linear geometry.

2.2 Basic reservoir Engineering concepts

This section reviews some very important petroleum reservoir engineering concept that will be used in the modeling process. This includes the momentum equation via Darcys law, types of fluid and steady - unsteadily flow.

Darcys Law: This is an empirical relationship between fluid flow rate through a porous medium and potential gradient. For a flow which can be described as single phase and one dimensional (ID) flow, Darcys law can be expressed in a differential form as

$$\frac{q}{A_x} = \mathbf{u}_x = -\beta_c \frac{k_x}{\mu}\frac{d\phi}{dx} \tag{2.1}$$

Where β_c = unit conservation factor for transmissibility coefficient , k = absolute rock permeability in the direction of flow , μ = fluid viscosity , φ = fluid potential and $\mathbf{u}$ = fluid flow rate per unit cross sectional area perpendicular to the direction of flow (superficial velocity)

2.3 Fluid Types and Steady/Unsteady Flow

Generally, incompressible fluids and porous media are a class of fluids that have solution that are not time dependent primarily because all the derivation with respect to time are zero and are therefore dependent only on space. This is referred to as *steady-state* flow because all the properties are steady or constant with respect to time.

Otherwise fluids may be regarded also as either slightly compressible or compressible. For these kind of fluids, there is transient or time dependent behavior in the porous medium. Therefore apart from having space dependent solutions for slightly compressible or compressible fluids they are also time dependent which is advanced either continuously or directly, such that flow is called transient or unsteady flow

2.4 Fluid and Rock Properties

A significant set of notes to consider in reservoir modeling are the fluid and rock properties and the nature in which they are describe i.e homogeneity or heterogeneity. This section introduce basic reservoir rock properties such as porosity and permeability as well as fluid properties such as fluid compressibility, gas- compressibility factors, fluid density and viscosity. Also discussed are fluid/rock properties such as fluids saturated, capillary pressure and relative permeability.

2.4.1 Porosity

This refers to the measurement of the pore space in a reservoir rock that can contain fluids. Basically the ratio of the pore space in a rock sample to the total volume of the rock sample is called the porosity. It is a measure of the ability of the reservoir rock to store producible fluids in its pores. Porosity is dependent on fluid pressure because of rock compressibility as

$$\varphi = \varphi^{o} [1 + c_{\varphi} (p - p^{o})] \tag{2.2}$$

Where p^0 = reference pressure at which the porosity is φ^0. An alternative form of expression for the porosity is

$$\phi = \frac{V_0}{V_b} \tag{2.3}$$

$$V_b = V_p + V_s \tag{2.4}$$

where V_o = Volume of Oil, V_b = Bulk rock volume and V_s = volume of the solid matrix.

2.4.2 Permeability

This is the capacity of a porous medium to transmit fluids through its interconnected pores. For a porous medium saturated with 100% of a single fluid, the capacity is referred to as absolute permeability. If two or more phases are however saturated at the porous medium, the reservoir capacity to transmit any phase is called effective permeability to that phase.

The permeability is represented principally by k_x,k_y and k_z in the x,y and z directions respectively. In general if

$$k_x = k_y = k_z \tag{2.5}$$

the porous medium is termed isotropic. The media is considered as anisotropic if there is any form of directional bias in the permeability, i.e

$$k_x \neq k_y \neq k_z \tag{2.6}$$

2.4.3 Fluid Compressibility

An incompressible fluid as discussed earlier has zero compressibility and therefore has constant density regardless of pressure. A slightly compressible fluid has a small but constant compressibility and its density varies linearly with pressure. A compressible fluid on the other hand has a higher compressibility than slightly compressible fluids and its density increase as pressure increases but tends to level off at high pressures. Gas is a good example of a compressible fluid; dead oil, under saturated oil and water are examples of slightly compressible fluids, and gas free (or dead) oil and water are considered as idealized examples of incompressible fluids. Fluid compressibility is defined as the relative volumetric change of a given mass to pressure change at constant pressure.It is mathematically expressed as

$$c_f = \left(-\frac{1}{V}\frac{\partial V}{\partial p}\right) = \frac{1}{\rho_L}\frac{\partial \rho_L}{\partial p}\bigg|_T \tag{2.7}$$

Where $L = o$, w or g and p$=\frac{m}{v}$

2.4.4 Formation Volume Factor (FVF)

A fixed mass of a reservoir fluid occupies a different volume at different reservoir pressures. FVFS are used to convert volumes at reservoir pressure and temperature to their equivalent volumes at standard conditions. These factors account for volume changes caused by fluid compressibility for the water and gas phase and those caused by fluid compressibility and mass transfer of solution gas for the oil phase.The phase FVF is the ratio of the volume that the phase occupies at reservoir pressure and temperature to that standard conditions.

$$B = \frac{V}{V_{sc}} \tag{2.8}$$

For a single phase water, dead oil or gas; *B* may be expressed in terms of densities.

$$B_L = \frac{\rho_{lsc}}{\rho_c} \tag{2.9}$$

Where l = o,w or g. In the case of slightly compressible fluids, the effect of temperature may be incorporated and thus approximated to

$$B_l = B_l^0 \left[1 + c_l \left(p - p_0\right)\right] \tag{2.10}$$

Where L = o,w or$g$ and c_{tl} = Coefficient of thermal expansion of the fluid.

2.4.5 Fluid Viscosity

Fluid viscosity is a measure of the ease with which the fluid flows as a result of an applied pressure gradient. For dense fluid, there is high resistance to flow whiles for dilute fluids, there is low resistance. Fluid viscosity is a function of both pressure and time and can be analyzed with respect to the variation of water and gas in consideration of their densities. Water is slightly compressible; therefore as pressure increases water viscosity increases slightly or remains almost constant. Gas on the other hand is a compressible fluid and its viscosity is low at low pressures. The viscosity of gas increases as pressure increases but tends to level off at high pressures because gas under high pressure begins to behave as if it is a liquid.

2.4.6 Fluid Saturation

The fluid saturation of a particular fluid is the fraction of the pore space that is occupied by that fluid . For single- phase flow, the fluid saturation is unity. For two phase flow of oil and water system, the interdependent relationship between the fluid saturation is

$$S_o + S_w = 1 \tag{2.11}$$

Where S_0 = Fraction of the void space occupied by oil phase and S_w = The remaining fraction occupied by the water phase . A gas and oil system has

$$S_o + S_g = 1 \tag{2.12}$$

The water saturation at which the water becomes immobile is called the irreducible water saturated, S_{iw}. During water displacement, the oil saturation at which oil becomes immobile is referred to as the residual oil saturation (ROS) to water S_{orw}. The value

$$S_{wmax} = 1 - S_{orw} \tag{2.13}$$

is the maximum water saturation that can be achieved during water displacement.

2.4.7 Capillary Pressure

This pressure exist whenever pores (capillaries) are saturated with two or more phases. For example, in a two -phase system, the capillary pressure is defined by the pressure of the nonwetting phase minus the pressure of the wetting phase. For a water wet system

$$p_{cow} = p_o - p_w = f(S_w) \tag{2.14}$$

and for a two- phase oil and gas system

$$p_{cgo} = p_g - p_o = f(S_g) \tag{2.15}$$

Capillary Pressure is a function of saturation and saturation history (that is, drainage or imbibitions) for a given reservoir rock and fluids at a constant temperature and composition. Qualitatively,the capillary pressure curves indicate the degree of rock wettability, the nature of pore- size distribution (uniform or nonuniform, large or small pore) and connate- water saturated.

2.4.8 Relative Permeability

The Darcys law discussed earlier was derived for single phase flow. When two or more fluids flow simultaneously through the porous medium, Darcys law is modified in order to calculate the flow rate at each phase. These modifications include the use of effective phase permeability(instead of absolute permeability), phase potential (which includes the effect of phase density and capillary pressure), and phase viscosity. The flow rates for example of two phase oil and water flow system can be expressed as

$$q_{ox} = -\beta_L \frac{k_{ox} A_x}{\mu_o} \frac{d\phi}{dx}\bigg|_o \tag{2.16}$$

and

$$q_{wx} = -\beta_L \frac{k_{wx} A_x}{\mu_w} \frac{d\phi|_w}{d_x}\bigg|_w \tag{2.17}$$

Where the effective permeabilities k_{ox} and k_{wx} are saturated dependent. The effective permeability to oil, for example, can be expressed as

$$k_{ox} = k_x \left(\frac{k_{ox}}{k_x}\right) = k_x k_{row} \tag{2.18}$$

Where k_x = absolute permeability of the porous medium in the x - direction and K_{row} = relative permeability to oil. Similarly for the water phase,

$$k_{wx} = k_x k_{rw} \tag{2.19}$$

By substituting these into earlier definitions (2.16) and (2.17) we obtain

$$q_{ox} = -\beta_c k_x A_x \frac{k_{row}}{\mu_o} \frac{d\phi_o}{d_x} \tag{2.20}$$

$$q_{wx} = -\beta_c k_x A_x \frac{k_{rw}}{\mu_w} \frac{d\phi_w}{d_x} \tag{2.21}$$

2.5 Two Phase Relative Permeability Models

This section discuses two phase relative permeability models that are often used in reservoir simulation to approximate actual relative permeabilities in algebraic form. There are instances where the relative permeabilities are obtained from laboratory measurements on suitable cases. In the case of missing data, these approximation are used.

2.5.1 Coreys Two- phase model

This is applicable for the drainage process in consolidated rocks. The normalized wettingphase is defined as

$$S_{wn} = \frac{S_w - S_{iw}}{1 - S_{iw}} \tag{2.22}$$

Where S_w = water saturated and S_{iw} = irreducible saturation of the wetting phase. The relative permeability of the wetting phase is approximated by

$$k_{rw} = S_{wn}^4 \tag{2.23}$$

While the relative permeability of the non -wetting phase approximated by

$$k_{rnw} = (1 - S_{wn})^2 \left(1 - S_{wn}^2\right) \tag{2.24}$$

2.5.2 Naar and Hendersons Two -Phase model

The Naar and Hundersons model is derived statistically for oil and water systems for the imbibition process. Both oil and water relative permeabilities are approximated as functions of S_{wn}

$$S_{wn} = \frac{S_w - S_{iw}}{1 - S_{iw}} \tag{2.25}$$

The water- phase relative permeability is

$$k_{rw} = S_{wn}^4 \tag{2.26}$$

While the oil- phase relative permeability is approximated by

$$k_{row} = (1 - 2S_{wn})^{\frac{3}{2}} \left[2 - (1 - 2S_{wn})^{\frac{1}{2}}\right] \tag{2.27}$$

2.6 Multiphase Flow Models in Porous Media

The objective of this section is to discuss the rudiments of multiphase flow and specifically consider the scenario of two-phase flow in porous media where oil and water are the fluids within the system. In contrast to *single phase*, multiphase flow is concerned with equation in multiple unknowns for each gridblock. **Multiphase flow** in petroleum reservoirs involves the simultaneous flow of at most three fluid phases i.e. oil, water and gas. The fluid phases are modeled transporting fluid components. For example in black-oil systems, the three fluid components are oil, water and gas at all standard conditions. If it is assumed that there is no mass transfer between the phases, each fluid component is contained in its own phase.

The formulation procedure in multiphase modeling is very similar to single phase flow and involves the *continuity equation* (differential mass balance), an *darcy' s law*(transport equation) and *additional relationships*. The combination of these relations results in a

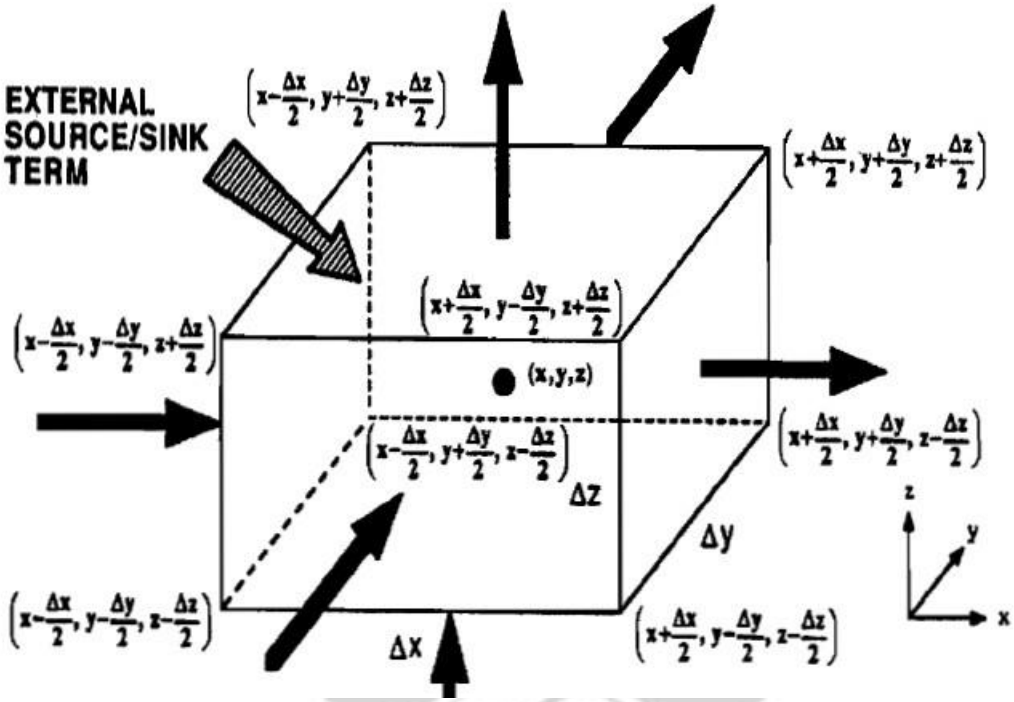

Figure 2.1: Control Volume in Rectangular coordinates

partial-differential system of equations that simultaneously describe multiphase flow in a porous medium.

2.6.1 Conservation of Mass

The rectangular coordinate system is considered here. The material balance equation for a component c written over a finite control volume as shown in Figure (2.1) of the porous reservoir over time interval Δt is

$$(m_i)_c - (m_o)_c + s_c = (m_a)_c \tag{2.28}$$

where $(m_i)_c$ = sum of mass inflows of component c across the control-volume surfaces $x - \Delta x/2$, $y - \Delta y/2$ and $z - \Delta z/2$ over time interval Δt; $(m_o)_c$ = sum of mass outflows of component c across the control-volume surfaces $x + \Delta x/2$, $y + \Delta y/2$ and $z + \Delta z/2$ over time interval Δt; s_c = sum of mass generation and mass depletion through wells of component c over time interval Δt; $(m_a)_c$ = mass accumulation due of component c caused by compressibility and fluid saturation changes in the control volume over time interval Δt.

For the component $c = o,w$ or g, these terms can be expressed mathematically as

$$(m_i)_c = \left[(\dot{m}_{cx}A_x)_{x-\Delta x/2} + (\dot{m}_{cy}A_y)_{y-\Delta y/2} + (\dot{m}_{cz}A_z)_{z-\Delta z/2} \right] \Delta t \qquad (2.29)$$

$$(m_o)_c = \left[(\dot{m}_{cx}A_x)_{x+\Delta x/2} + (\dot{m}_{cy}A_y)_{y+\Delta y/2} + (\dot{m}_{cz}A_z)_{z+\Delta z/2} \right] \Delta t \qquad (2.30)$$

$$s_c = (q_{mtc} + q_{mc})\Delta t \qquad (2.31)$$

$$(m_a)_c = V_b \left[(m_{vc})_{t+\Delta t} - (m_{vc})_t \right] \qquad (2.32)$$

where $\dot{m}_c$ = mass flux; A = the area perpendicular to the flux; m_{vc} = mass per unit volume of porous medium; q_{mtc} = rate of mass transfer between phases, which is positive for generation and negative otherwise; q_{mc} = rate of mass depletion through wells, which is positive for injection and negative for production; and V_b = bulk volume of control volume.

By substituting Equations (2.29), (2.30), (2.31) and (2.32) into (2.28), rearranging and dividing through the resulting equation by Δt and multiplying and dividing the first, second and third paired terms on the left hand side by a non-zero Δx, Δy and Δz respectively yields

$$-\left[\frac{(\dot{m}_{cx}A_x)_{x+\Delta x/2}-(\dot{m}_{cx}A_x)_{x-\Delta x/2}}{\Delta x}\right]\Delta x-\left[\frac{(\dot{m}_{cy}A_y)_{y+\Delta y/2}-(\dot{m}_{cy}A_y)_{y-\Delta y/2}}{\Delta y}\right]\Delta y$$
$$-\left[\frac{(\dot{m}_{cz}A_z)_{z+\Delta z/2}-(\dot{m}_{cx}A_z)_{z-\Delta z/2}}{\Delta z}\right]\Delta z+q_{mt_c}+q_{m_c}=V_b\left[\frac{(m_{v_c})_{t+\Delta t}-(m_{v_c})_t}{\Delta t}\right] \quad (2.33)$$

By taking limits of the terms enclosed as Δx, Δy, Δz and Δt approach zero and substituting the definition of the partial derivative, Equation (2.33) becomes

$$-\frac{\partial}{\partial x}(\dot{m}_{cx}A_x)\Delta x-\frac{\partial}{\partial y}(\dot{m}_{cy}A_y)\Delta y-\frac{\partial}{\partial z}(\dot{m}_{cz}A_z)\Delta z=V_b\frac{\partial}{\partial t}(m_{vc})-q_{m_c}-q_{mt_c} \quad (2.34)$$

2.7 Flows Equations: Oil/Water Formulation

The general model for the conservation of mass can be further expressed by considering specific relationship between the variables. In this section, we consider these relationship and specifically derive conservation of mass equations for each phase in an oil/water system. We further consider Darcy's equation and how it is introduced into the conservation of mass along with other relations such as saturation relation and capillary pressure relations which are needed to complete the formulation.

2.7.1 Conservation of Mass: Oil/Water Model

For oil, water and gas(free gas), it is possible to express the mass flux as a product of density and Darcy's velocity for a particular phase; mass per unit volume of the reservoir as the product of porosity, phase saturation and phase density; and mass flow rate as a product of the volumetric(phase) flow rate and phase density. The described relations are given mathematically as follows:

$\dot{m}_c = \alpha_c \rho_c \mathbf{u}$ which implies that $\dot{m}_{cx} = \alpha_c \rho_c u_{cx}$; $\dot{m}_{cy} = \alpha_c \rho_c u_{cy}$ and $\dot{m}_{cz} = \alpha_c \rho_c u_{cz}$ and $m_{vc} = \varphi \rho_c S_c$; $q_{mc} = \alpha \rho_c q_c$; and $q_{mtc} = 0$. Based on the following expressions, the mass conservation for the oil component for example may be expressed after dividing through the equation by $\alpha_c \rho_{osc}$ using the definition of $B_o = \rho_{osc}/\rho_o$, we arrive at

$$-\frac{\partial}{\partial x}(\frac{A_x}{B_o}u_{ox})\Delta x - \frac{\partial}{\partial y}(\frac{A_y}{B_o}u_{oy})\Delta y - \frac{\partial}{\partial z}(\frac{A_z}{B_o}u_{oz})\Delta z = \frac{V_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right) - q_{osc} \quad (2.35)$$

Similarly, the mass conservation for the water component is given by

$$-\frac{\partial}{\partial x}(\frac{A_x}{B_w}u_{wx})\Delta x - \frac{\partial}{\partial y}(\frac{A_y}{B_w}u_{wy})\Delta y - \frac{\partial}{\partial z}(\frac{A_z}{B_w}u_{wz})\Delta z = \frac{V_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right) - q_{wsc} \quad (2.36)$$

2.7.2 Darcy's Law

To obtain the flow equations for multiphase generally, Darcy's law is substituted into the conservation of mass for each phase. For instance, the oil phase given the introduction of Darcy's law becomes

$$\frac{\partial}{\partial x}\left[\beta_c k_x A_x \frac{k_{ro}}{\mu_o B_o}\left(\frac{\partial p_o}{\partial x} - \gamma_o \frac{\partial Z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\beta_c k_y A_y \frac{k_{ro}}{\mu_o B_o}\left(\frac{\partial p_o}{\partial y} - \gamma_o \frac{\partial Z}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\beta_c k_z A_z \frac{k_{ro}}{\mu_o B_o}\left(\frac{\partial p_o}{\partial z} - \gamma_o \frac{\partial Z}{\partial z}\right)\right] = \frac{V_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right) - q_{osc} \quad (2.37)$$

Similarly, for the water component we obtain

$$\frac{\partial}{\partial x}\left[\beta_c k_x A_x \frac{k_{rw}}{\mu_w B_w}\left(\frac{\partial p_w}{\partial x} - \gamma_w \frac{\partial Z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\beta_c k_y A_y \frac{k_{rw}}{\mu_w B_w}\left(\frac{\partial p_w}{\partial y} - \gamma_w \frac{\partial Z}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\beta_c k_z A_z \frac{k_{rw}}{\mu_w B_w}\left(\frac{\partial p_w}{\partial z} - \gamma_w \frac{\partial Z}{\partial z}\right)\right] = \frac{V_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right) - q_{wsc} \quad (2.38)$$

2.7.3 Additional Relationships

In addition to the conservation of mass and Darcy's law just discussed, additional relations are necessary to complete the flow description together with initial and boundary conditions. The additional relations include the phase saturation constraint and capillary pressures as functions of phase saturations.

The phase saturation equation for an oil-water system which is a constraint on the sum of the phase saturation is

$$S_o + S_w = 1 \quad (2.39)$$

and the oil-water capillary pressure relation is

$$P_{cow} = p_o - p_w \quad (2.40)$$

Equation (2.37), (2.38), (2.39) and (2.40) constitute the **oil-water flow**. These equations contain four unknowns i.e. p_o,p_w,S_o and S_w. The relations in Equations (2.39) and (2.40) are usually used to eliminate two of the unknowns in the flow equations. In this research, we are will concentrate the simulation on p_o and S_w and therefore will eliminate the other two variables by expressing them in the model as

$$S_o = 1 - S_w \tag{2.41}$$

and

$$p_w = p_o - P_{cow} \tag{2.42}$$

When (2.41) and (2.42) are substituted in to (2.37) and (2.38), we obtain

$$\frac{\partial}{\partial x}\left[\beta_c k_x A_x \frac{k_{ro}}{\mu_o B_o}\left(\frac{\partial p_o}{\partial x} - \gamma_o \frac{\partial Z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\beta_c k_y A_y \frac{k_{ro}}{\mu_o B_o}\left(\frac{\partial p_o}{\partial y} - \gamma_o \frac{\partial Z}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\beta_c k_z A_z \frac{k_{ro}}{\mu_o B_o}\left(\frac{\partial p_o}{\partial z} - \gamma_o \frac{\partial Z}{\partial z}\right)\right] = \frac{V_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi(1 - S_w)}{B_w}\right) - q_{osc} \tag{2.43}$$

and

$$\frac{\partial}{\partial x}\left[\beta_c k_x A_x \frac{k_{rw}}{\mu_w B_w}\left(\frac{\partial p_o}{\partial x} - \frac{\partial P_{cow}}{\partial x} - \gamma_w \frac{\partial Z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\beta_c k_y A_y \frac{k_{rw}}{\mu_w B_w}\left(\frac{\partial p_o}{\partial y} - \frac{\partial P_{cow}}{\partial y} - \gamma_w \frac{\partial Z}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\beta_c k_z A_z \frac{k_{rw}}{\mu_w B_w}\left(\frac{\partial p_o}{\partial z} - \frac{\partial P_{cow}}{\partial z} - \gamma_w \frac{\partial Z}{\partial z}\right)\right] = \frac{V_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right) - q_{wsc} \tag{2.44}$$

Given initial and boundary conditions, Equation (2.43) and (2.44) can be solved for the unknows p_o and S_w. The other two variables S_o and p_w can be obtained by substituting the principal unknowns into (2.41) and (2.42).

Chapter 3

Methodology

3.1 Introduction to Numerical Methods

Numerical methods involves the study of methods of computing numerical data particularly involving the design and analysis of techniques to give approximate but accurate solutions to hard problems. No matter how accurate they are, they do not in most cases, provide an exact solution. In some instances working out the exact solution by a different approach may not be possible or may be too consuming and it is in these cases numerical methods are often used.

The two main methods used in numerical methods are the direct method and the iterative method. The direct method computes the solution to a problem in a finite number of steps. These methods would give the precise answer if they were performed in infinite precision arithmetic. In contrast to direct method, iterative methods are not expected to terminate in a number of steps. Starting from an initial guess, iterative methods from successive approximation that converge to the exact solution only to the limit. A convergence test is specified in order to decide when a sufficiently accurate solution has been found.

Iterative methods are more common than direct methods in numerical methods. The classification of numerical methods can be categorized under Finite Differences Method (FDM), Finite Element Method (FEM), Finite Volume Method (FVM) and Spectral Methods.)

In general terms, finite difference method represent the problem through a series of points or nodes. Expressions for the unknowns are derived via replacing the derivative terms in the model equation with truncated or approximate Taylor series expressions. The earliest numerical schemes were based on finite difference schemes and are conceptually intuitively one of the easiest methods amongst the given classifications to implement. Most texts describe finite difference as the most traditional and oldest of numerical methods. However, fundamentally, such techniques require a high degree of regularity with the mesh and so this limits, their application to complex problems. Finite element method employs the use of dividing the domain into elements such as triangles or quadrilaterals and to place with each element nodes at which the numerical solution is determined. The solution at any position is then represented by a series expansion of the nodal values within the local vicinity of that position. Spectral methods can be considered as a subset of the finite element method in which the basis functions are defined globally as opposed to the more common approach whereby the basis functions are local and so one zero outside the neighboring of the associated node. The finite volume method is based on forming a discretization from an integral form of the model equation, and entails subdividing the domain in a number of finite volumes. Within each volume, the integral relationships are applied locally and so the exact conservation within each volume is achieved. The resulting expressions for the unknowns often appear similar to finite difference approximations and the depending on the particular method chosen, it may be considered as a special case of either the finite difference or the finite element techniques.

The general requirements in addressing flow problems and setting up a numerical scheme involves;

1. Selection of a discretization method for the equations in the model. This implies the selection between finite difference, finite volume, spectral or finite element methods as well as selection of the order of accuracy of the spatial and eventually the time discretization.

2. Analysis of the selected numerical algorithm. This step concerns the qualities of the scheme in terms of stability and convergence properties as well as the investigation of generated errors.

3. Selection of a resolution method for the system of ordinary/partial differential equations in time, for the algebraic system of equations and for the iterative treatment of eventual non-linearities.

The equations for two-phase flow through porous media are partial differential equations (PDEs) that are second order in space and first order in time. Primarily because of the nonlinear nature of these set of equations, analytical solutions cannot be obtained; numerical methods are therefore employed in obtaining solutions to these equations. The most popular method currently in use in the petroleum industry is the *finite difference method* which is implemented by superimposing a finite-difference grid over the reservoir to be modeled. The chose grid system is then used to approximate the spatial derivatives in the continuous equations. These finite-difference approximations are obtained by truncating Taylor's series expansion of the unknown variables. A similar procedure is used in the time domain.

3.2 Discretization of Two-phase Flow Equation

Numerical discretization of a partial differential equations is the process of transferring the continuous functions into discrete forms. This study considers the use of a set of finite difference equations to represent, or approximate the continuous differential equations by using algebraic approximations. The algebraic equations used to discretized the flow equations are rendered for the second-order derivatives in the spatial domain and first order derivatives with respect to time. The resulting set of equations will be explicit or implicit depending on the choice of approximation of the finite difference scheme. A system that involves both implicit and explicit schemes is also possible as will be demonstrated with reasons in this study.

The concept of discretization in multiphase flow equations involves three mathematical operations of or general steps. First is the finite difference approximation of the second order derivative terms in the x,y and z directions if a three dimensional system is considered . This normally takes care of the left hand-side of the general two-phase flow system of equations discussed in the previous chapter which is the main feature of this study. The system may be expressed in a more compact form using a second-order differential operator $(\partial/\partial x)[\xi_x^*(\partial\eta/\partial x)]\Delta x$ where any of the expressions in the form

$$\xi^* = \beta_c k_x A_x \frac{k_{rl}}{\mu_l B_l} \tag{3.1}$$

and η may assume $p_o$or P_{cow}. Based on this reconciliation, the compact form of the flow equation may be written in two dimensions as

$$\Delta(\xi\Delta\eta)_{i,j} = \Delta x(\xi_x\Delta_x\eta)_{i,j} + \Delta_y(\xi_y\Delta_y\eta)_{i,j} \tag{3.2}$$

with$\xi_x = \frac{\xi_x^*}{\Delta x}$ and$\xi_y = \frac{\xi_y^*}{\Delta y}$. The second step is to consider finite difference approximations of the first order derivatives with respect to time, which appear on the right hand side of the two-phase flow equations. The general expression for this is $(\partial/\partial t)(\varphi S_l/B_l)$ for l = w or o. Which may be further expressed generally as $(\partial/\partial t)(f)$ where $f = (\varphi S_l)/B_l$.

The third and final step involves the conservation expression of $\Delta_t(f)$ also known as the accumulation term which is normally expressed in terms of the unknowns p_o and S_w in the case of the two-phase flow equations.

Given that the reservoir is represented by a discrete set of grid point in linear geometry in two dimension (i,j) $i = 1,2,3,.....,n_x$; $j = 1,2,...,n_y$; then the finite difference approximations to the flow equation of the two-phase flow model may be written for each grid block and further expressed using transmissibilities defined by

$$T_{ld} = \beta_c \frac{k_d A_d}{\Delta d} \frac{k_{rl}}{\mu_l \beta_l} \tag{3.3}$$

where $l = o, w$ and $d = x, y$.

3.3 Spatial Discretization

Spatial discretization or discretization in space in the context of two-phase flow refers to the process of applying the finite difference approximation to the second order derivative terms in the *x* dimension. These forms of discretization are considered in this section in the *x* direction , we have ($(\partial/\partial x)[\xi_x^*(\partial\eta/\partial x)]\Delta x$ at grid point (i); consider a grid plate with dimension Δx_i centered at (i) with the distance between grid point (i) and ($i - 1$) being $\Delta x_{i-\frac{1}{2}}$, and the distance between the grid point (i) and (i+1) being the $\Delta x_{i+\frac{1}{2}}$, the following relationships hold;

$$\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

$$\Delta x_{i-\frac{1}{2}} = x_{i,j} - x_{i-1}$$

$$\Delta x_{i+\frac{1}{2},j} = x_{i+1,j} - x_i$$

Let τ be defined as $\tau = \xi_x^* \partial\eta/\partial x$, then

$$\frac{\partial}{\partial x}\left(\xi_x^* \frac{\partial \eta}{\partial x}\right)_i \Delta x_i = \left(\frac{\partial \tau}{\partial x}\right)_i \Delta x_i \tag{3.4}$$

By approximating the first derivatives, $\left(\frac{\partial \tau}{\partial x}\right)_i$ using the central difference approximation with

τ evaluated at the grid plate boundaries ($i+\frac{1}{2}$) and ($i-\frac{1}{2}$) in the x direction we have

$$\left(\frac{\partial \tau}{\partial x}\right)_i = \frac{\tau_{i+\frac{1}{2},j} - \tau_{i-\frac{1}{2},j}}{\Delta x_i} \tag{3.5}$$

Where

$$\tau_{i\pm\frac{1}{2},j} = \xi^*_{xi\pm\frac{1}{2}} \left(\frac{\partial \eta}{\partial x}\right)_{i\pm\frac{1}{2}} \tag{3.6}$$

By considering the first derivative, $(\partial\eta/\partial x)_{i\pm\frac{1}{2}}$ using the central difference approximation

$$\left(\frac{\partial \eta}{\partial x}\right)_{i+\frac{1}{2},j} = \frac{\eta_{i+1} - \eta_{i,j}}{\Delta x_{i+\frac{1}{2}}} \tag{3.7}$$

and

$$\left(\frac{\partial \eta}{\partial x}\right)_{i-\frac{1}{2}} = \frac{\eta_i - \eta_{i-1}}{\Delta x_{i-\frac{1}{2},j}} \tag{3.8}$$

By substituting Equation (3.7) and Equation (3.8) into Equation (3.6) gives

$$\tau_{i+\frac{1}{2}} = \xi^*_{xi+\frac{1}{2}} \left(\frac{\eta_{i+1} - \eta_i}{\Delta x_{i+\frac{1}{2},j}}\right)$$

and

$$\tau_{i-\frac{1}{2}} = \xi^*_{xi-\frac{1}{2}} \left(\frac{\eta_i - \eta_{i-1}}{\Delta x_{i-\frac{1}{2}}} \right.!$$

When substituted in (3.5) results in

$$\left(\frac{\partial \tau}{\partial x}\right)_i \Delta x_i = \left(\frac{\xi^*_x}{\Delta x}\right)_{i+\frac{1}{2}} (\eta_{i+1} - \eta_i) - \left(\frac{\xi^*_x}{\Delta x}\right)_{i-\frac{1}{2}} (\eta_i - \eta_{i-1}) \tag{3.9}$$

By putting (3.9) and (3.4) together and simplifying using $\xi_x = \xi^*_x/\Delta x$ we have

$$\begin{aligned} \frac{\partial}{\partial x}\left(\xi^*_x \frac{\partial \eta}{\partial x}\right)_i \Delta x_i &= \xi_{x_{i+\frac{1}{2},j}} (\eta_{i+1} - \eta_i) - \xi_{x_{i-\frac{1}{2}}} (\eta_i - \eta_{i-1}) \\ &= \xi_{x_{i-\frac{1}{2}}} (\eta_{i-1} - \eta_{i,j}) + \xi_{x_{i+\frac{1}{2},j}} (\eta_{i+1,j} - \eta_{i,j}) \\ &= \Delta x(\xi_x \Delta_x \eta)_i \end{aligned} \tag{3.10}$$

The discretization in the *y*-axis follows an analogous fashion to the procedure in the *x*-axis thus resulting in

$$\begin{aligned} \frac{\partial}{\partial y}\left(\xi^*_y \frac{\partial \eta}{\partial y}\right)_j &= \Delta y \left(\xi_y \Delta_y \eta\right)_j \\ &= \xi_{y_{j-\frac{1}{2}}} (\eta_{j-1} - \eta_j) + \xi_{y_{j+\frac{1}{2}}} (\eta_{j+1} - \eta_j) \end{aligned} \tag{3.11}$$

Then let $\Delta(\xi\Delta\eta)_{i,j}$ denote the operator which defines the seconds order finite difference operation in 2D Cartesian space when the subscripts *i*, and *j* indicate the *x* and *y* directions respectively. Therefore

$$\Delta(\xi\Delta\eta)_{i,j} = \frac{\partial}{\partial x}\left(\xi_x^*\frac{\partial \eta}{\partial x}\right)_{i,j}\Delta x_{i,j} + \frac{\partial}{\partial y}\left(\xi_y^*\frac{\partial \eta}{\partial y}\right)_{i,j}\Delta y_{i,j} \tag{3.12}$$

where

$$\Delta(\xi\Delta\eta)_{i,j} = \Delta x(\xi_x\Delta_x\eta)_{i,j} + \Delta y(\xi_y\Delta_y\eta)_{i,j} \tag{3.13}$$

3.3.1 Time Discretization

The second step was the consideration of the discretization in time which is the finite difference approximation of the first order derivative term in time. Consider the differential operator in time $(\partial/\partial t)(f)$ at gridpoint (i,j). By fixing the spatial domain at a given point, the partial derivative with respect to time becomes an ordinary derivative evaluated at the given point.

$$\left[\frac{\partial}{\partial t}(f)\right] = \frac{d}{dt}(f_{i,j}) = \frac{1}{\Delta t}\Delta_t(f_{i,j}) \tag{3.14}$$

where the differential operator Δ_t is well defined as

$$\frac{1}{\Delta t}\Delta_t(f_{i,j}) = \frac{1}{\Delta t}\left(f_{i,j}^{n+1} - f_{i,j}^{n}\right) \tag{3.15}$$

Therefore the right hand side of the two phase flow equation can be expressed as

$$\frac{V_b}{\alpha_c\Delta t}\Delta_t\left(\frac{\phi S_l}{B_l}\right) = \frac{V_b}{\alpha_c\Delta t}\left[\left(\frac{\phi S_l}{B_l}\right)^{n+1} - \left(\frac{\phi S_l}{B_l}\right)^{n}\right] \tag{3.16}$$

The equation in (3.16) shows a forward difference (explicit) operator in time. In compact form (3.16) when substituted into the two phase flow equation model gives

$$\Delta[T_l(\Delta p_l - \gamma_l \Delta Z)]^n = \frac{V_b}{\alpha_c \Delta t}\left[\left(\frac{\phi S_l}{B_l}\right)^{n+1} - \left(\frac{\phi S_l}{B_l}\right)^{n}\right] \tag{3.17}$$

within which every term on both sides of the equation is dated at time level n. The equation can be expressed as the back difference (implicit) in time where every term on both sides is captured at time level $n + 1$ as shown in (3.18) below

$$\Delta[T_l(\Delta p_l - \gamma_l \Delta Z)]^{n+1} = \frac{V_b}{\alpha_c \Delta t}\left[\left(\frac{\phi S_l}{B_l}\right)^{n+1} - \left(\frac{\phi S_l}{B_l}\right)^{n}\right] \tag{3.18}$$

3.3.2 Accumulation Terms and Expansions

The final section within the build up of the model is the featuring of the accumulation terms and the process by which the time difference operator considered as a function is expressed in terms of the difference of the unknowns in the preservation of mass conservation. The form of identity expansion employed within this setup is given by

$$\Delta_t f = f^{n+1} - f^n \tag{3.19}$$

As discussed earlier the accumulation terms encountered with the two phase flow is of the form $\Delta_t[(\varphi S_l/B_l)]$ for $l = o,w$. On a general note, these time difference functions in two phase flow may be expressed as

$$f = UVXY \tag{3.20}$$

where $f = (\varphi S_l/B_l)$ and $U = \varphi$, $V = 1/B_l$, $X = 1$ and $Y = S_l$. By substituting into (3.18) yields the general form for time difference of a function

$$\Delta_t(UVXY) = (UVXY)^{n+1} - (UVXY)^n \tag{3.21}$$

for which we derive the conservation expansion formula

$$\begin{aligned} \Delta_t(UVXY) &= (VXY)^n\Delta_t U + U^{n+1}(XY)^n\Delta_t V \\ &+ (UV)_{n+1}Y_n\Delta_t X + (UVX)_{n+1}\Delta_t Y \end{aligned} \tag{3.22}$$

The key note here firstly in the expansion of the accumulation terms is to express the time difference of U,V,X and Y ($\Delta_t U, \Delta_t V, \Delta_t X$ and $\Delta_t Y$) in terms of the time difference of the unknowns which are $\Delta_t p_o$ and $\Delta_t S_w$. To achieve this and preserve the conservative property of the expansion $\Delta_t U, \Delta_t V, \Delta_t X$ and $\Delta_t Y$ must be expressed as a product of the chord slope between successive time levels (n and $n+1$) and the time difference of the unknowns. That is

$$\Delta_t U = \Delta_t \phi \equiv \phi' \Delta_t p_o \quad (3.23)$$

$$\Delta_t V = \Delta_t \left(\frac{1}{B_l}\right) \equiv \left(\frac{1}{B_l}\right)' \Delta_t p_o \quad (3.24)\ (3.25)$$

$$\Delta_t Y = \Delta_t S_l \quad l = o, w$$

$$\Delta_t Y = \Delta_t S_l = -\Delta_t S_w \quad l = o \quad (3.26)$$

The chord slopes are noted as φ^0, $(1/B_l)^0$ so that $\Delta_t\varphi$ and $\Delta_t(1/B_l)$ are exact. That is

$$\phi' = \frac{\phi^{n+1} - \phi^n}{p_o^{n+1} - p_o^n} \quad (3.27)$$

$$\left(\frac{1}{B_l}\right)' = \frac{(1/B_l)^{n+1} - (1/B_l)^n}{p_o^{n+1} - p_o^n} \quad (3.28)$$

where $l = o,w$.

The conservation expansion scheme for $[\Delta_t(\varphi S_l)/B_l]$ for $l = o,w$ is given by

$$\Delta_t \left(\frac{\phi S_l}{B_l}\right) = \left[\frac{\phi}{B_l^n} + \phi^{n+1}\frac{1}{B_l}\right] S_l^n \Delta_t p_o + \left(\frac{\phi}{B_l}\right)^{n+1} \Delta_t S_l \quad (3.29)$$

And that for $\Delta_t[(\varphi S_w)/B_w]$ specifically is given as follows: first consider the general conservative expansion formula given in (3.22) where $f = UV\,XY = (\varphi S_w)/B_w$. This becomes equal to $UV\,Y$ by setting $X = 1$ and the simplification results in

$$\Delta_t(UVY) = (VY)^n\Delta_t U + U^{n+1}Y^n\Delta_t V + (UV)^{n+1}Y^n\Delta_t(1) + (UV)^{n+1}\Delta_t Y \tag{3.30}$$

Observing that $\Delta_t(1) = 0$, we have (3.30) reducing to

$$\Delta_t(UVY) = (VY)^n\Delta_t U + U^{n+1}Y^n\Delta_t V + (UV)^{n+1}\Delta_t Y \tag{3.31}$$

By substituting the defined relations U,V and Y into (3.31), we have

$$\Delta_t\left[\phi\left(\frac{1}{B_l}\right)S_w\right] = \left(\frac{S_w}{B_w}\right)^n\Delta_t\phi + \phi^{n+1}S_w^n\Delta_t\left(\frac{1}{B_w}\right) + \left(\frac{\phi}{B_w}\right)^{n+1}\Delta_t S_w \tag{3.32}$$

Expressing the time differences that occur (3.32) of the functions φ and $(1/B_w)$ in terms of the unknowns p_o and S_w, we have

$$\Delta_t\phi = \phi'\Delta_t p_o \tag{3.33}$$

$$\Delta_t\left(\frac{1}{B_w}\right) = \left(\frac{1}{B_w}\right)'\Delta_t p_o \tag{3.34}$$

where φ^0 and $(1/B_w)^0$ are the chord slopes. By substituting (3.33) and (3.34) into (3.32) we arrive at

$$\Delta_t\left(\frac{\phi S_w}{B_w}\right) = \left(\frac{S_w}{B_w}\right)^n\phi'\Delta_t p_o + \phi^{n+1}S_w^n\left(\frac{1}{B_w}\right)'\Delta_t p_o + \left(\frac{\phi}{B_w}\right)^{n+1}\Delta_t S_w \tag{3.35}$$

After further simplification and factoring, (3.35) can be expressed as

$$\Delta_t\left(\frac{\phi S_w}{B_w}\right) = \left[\left(\frac{\phi'}{B_w^n}\right) + \phi^{n+1}\left(\frac{1}{B_w}\right)'\right]S_w^n\Delta_t p_o + \left(\frac{\phi}{B_w}\right)^{n+1}\Delta_t S_w \tag{3.36}$$

An important note here is that based on the relation in (3.32), we can derive a general relation using the same approach to arrive at

$$\Delta_t\left(\frac{\phi S_l}{B_l}\right)=\left[\left(\frac{\phi'}{B_l^n}\right)+\phi^{n+1}\left(\frac{1}{B_l}\right)'\right]S_l^n\Delta_t p_o+\left(\frac{\phi}{B_l}\right)^{n+1}\Delta_t S_l \tag{3.37}$$

where $l = o,w$. In the case where $l = o$ we have

$$\Delta_t\left(\frac{\phi S_o}{B_o}\right)=\left[\left(\frac{\phi'}{B_o^n}\right)+\phi^{n+1}\left(\frac{1}{B_o}\right)'\right]S_o^n\Delta_t p_o+\left(\frac{\phi}{B_o}\right)^{n+1}\Delta_t S_o \tag{3.38}$$

Therefore for a $p_o - S_w$ formulation, we express S_o in terms of S_w using $S_o = 1 - S_w$ and $\Delta_t S_o = -\Delta_t S_w$ which when substituted into (3.38) gives

$$\Delta_t\left(\frac{\phi(1-S_w f)}{B_o}\right)=\left[\left(\frac{\phi'}{B_o^n}\right)+\phi^{n+1}\left(\frac{1}{B_o}\right)'\right](1-S_w^n)\Delta_t p_o-\left(\frac{\phi}{B_o}\right)^{n+1}\Delta_t S_w \tag{3.39}$$

This maneuvering allows us to have a system with just two unknowns, p_o and S_w. Based on the development issued in the face of the conservative terms, we can represent the finite difference equations for the two phase flow in compact form as using the steps

$$\Delta[T_o(\Delta p_o-\gamma_o\Delta Z)]^{n+1} = C_{op}\Delta_t p_o+C_{ow}\Delta_t S_w-q_{osc}^{n+1} \tag{3.40}$$

$$\Delta[T_w(\Delta p_o-\Delta_t P_{cow}-\gamma_w\Delta_t Z)]^{n+1} = C_{wp}\Delta_t p_o+C_{ww}\Delta_t S_w-q_{wsc}^{n+1} \tag{3.41}$$

discussed in the foregoing sections as firstly discretizing the second order partial derivative terms with respect to the spatial variables, secondly expanding the first order partial derivative terms with respect to time and finally expanding the time difference terms with a conservative expansion scheme where the coefficients C_{lu} (where $l = o, w$ and $u = p_o$ or S_w) are obtained as follows

$$C_{op} = \frac{V_b}{\alpha_c \Delta t}\left[\left(\frac{\phi'}{B_o^n}\right) + \phi^{n+1}\left(\frac{1}{B_o}\right)'\right](1 - S_w^n) \quad (3.42)$$

$$C_{ow} = -\frac{V_b}{\alpha_c \Delta t}\left(\frac{\phi}{B_o}\right)^{n+1} \quad (3.43)$$

$$C_{wp} = \frac{V_b}{\alpha_c \Delta t}\left[\left(\frac{\phi'}{B_w^n}\right) + \phi^{n+1}\left(\frac{1}{B_w}\right)'\right] S_w^n \quad (3.44)$$

$$C_{ww} = \frac{V_b}{\alpha_c \Delta t}\left(\frac{\phi}{B_w}\right)^{n+1} \quad (3.45)$$

3.4 CVFD Representation

It is obvious that writing the finite difference equation in the previous section for two or three dimensions becomes a very challenging task. The Control Volume Finite Difference (CVFD) method introduced by (Abou-Kassem, 1981) and (Lutchmansingh, 1987) as means of expressing equations in a compact form.

In one dimension (1D), we can consider the expansion of the second order finite difference operator at a given point say $\Delta(\xi \Delta \eta)_i$ defined by

$$\Delta(\xi\Delta\eta)_i = \Delta_x(\xi\Delta\eta)_i$$

$$= \xi_{x_{i-\frac{1}{2}}}(\eta_{i-1} - \eta_i) + \xi_{x_{i+\frac{1}{2}}}(\eta_{i+1} - \eta_i) \quad (3.46)$$

From the expansion given in (3.46) it can be seen that the expansion at gridpoint is an algebraic combination of the differences $(\eta_{i+1} - \eta_i)$ and $(\eta_{i-1} - \eta_i)$ where $i-1$ and $i+1$ are the immediate neighbors of i. We can then define ψ_x as the set of gridblocks associated with (but excluding) gridblock i in the x direction i. e. $\psi_x = \{i-1, i+1\}$; the coefficient of $(\eta_{i+1} - \eta_i)$ as $\xi_{i,i-1}(\xi_{i-1,i})$; and $(\eta_{i-1} - \eta_i)$ as $\xi_{i,i+1}(\xi_{i+1,i})$; Based on this we can write for

$m = i - 1$

$$\xi_{i,m} = \xi_{m,i} = \xi_{x_{i-\frac{1}{2}}} \quad (3.47)$$

and for $m = i + 1$

$$\xi_{i,m} = \xi_{m,i} = \xi_{x_{i+\frac{1}{2}}} \quad (3.48)$$

Then Equation (3.46) may be expressed as

$$\Delta(\xi\Delta\eta)_i = \Delta_x(\xi\Delta\eta)_i$$

$$= \xi_{i,i-1}(\eta_{i-1} - \eta_i) + \xi_{i,i+1}(\eta_{i+1} - \eta_i)$$

$$= \sum_{m \in \psi_x} \xi_{i,m}((\eta_m - \eta_i)) \quad (3.49)$$

This can be applied in multiphase flow specifically for the two phase problem under study. For example, the expansion of $\Delta(\xi\Delta\eta)$ at gridpoint (i,j) in the x−y plane can be obtained by

$$\Delta(\xi\Delta\eta)_{i,j} = \Delta_x(\xi_x\Delta_x\eta)_{i,j} + \Delta_y(\xi_y\Delta_y\eta)_{i,j}$$

$$= \sum_{m\in\psi_x} \xi_{(i,j),m}(\eta_m - \eta_{i,j}) + \sum_{m\in\psi_y} \xi_{(i,j),m}(\eta_m - \eta_{i,j})$$

$$= \sum_{m\in\psi_{i,j}} \xi_{(i,j),m}(\eta_m - \eta_{i,j}) \qquad (3.50)$$

where $\psi_{i,j} = \psi_x \cup \psi_y = \{(i,j-1),(i-1,j),(i+1,j),(i,j+1)\}$ representing the set of gridblocks associated with but excluding gridblock (i,j); η_m is variable η for element m contained in set $\psi_{i,j}$ and $\xi_{(i,j),m}$ represents the interactions between gridblock (i,j) and its neighboring gridblock, i. e. gridblock m.

There is an ordering scheme known as the *Natural Ordering* within which the gridblocks are numbered along the x direction first, the y direction second and the z direction last for a three dimensional system. Within this natural ordering scheme, the general form of the the spatial operator at gridblock n in multidimensional space may be expressed with the CVFD method as

$$\Delta(\xi\Delta\eta)_n = \sum_{m\in\psi_n} \xi_{n,m}(\eta_m - \eta_n) \qquad (3.51)$$

where $\psi_n = \psi_x \cup \psi_y \cup \psi_z$ and $\psi = \{n-1, n+1\}$; $\psi_y = \{n - n_x, n + n_x\}$; and $\psi_z =$

$\{n-n_xn_y, n+n_xn_y\}$. The variable η_m is the value of η defined only at gridblock $m \in \psi_n$. The coefficient of $\xi_{n,m}$ is the ξ interaction between gridblock n and its immediate neighboring gridblock; i.e. gridblock $m \in \psi_n$ and is defined by

$$\xi_{n,m} = \xi_{m,n} = \xi_{x_{i+\pm\frac{1}{2},j,k}} \quad \text{for } m = n \pm 1 \tag{3.52}$$

$$\xi_{n,m} = \xi_{m,n} = \xi_{y_{i,j+\pm\frac{1}{2},k}} \quad \text{for } m = n \pm n_x \tag{3.53}$$

$$\xi_{n,m} = \xi_{m,n} = \xi_{x_{i,j,k+\pm\frac{1}{2}}} \quad \text{for } m = n \pm n_xn_y \tag{3.54}$$

Based on the foregoing expressions and introduction of CVFD with the natural ordering scheme, the two-phase flow model can be expressed as follows; For the the oil equation

$$\sum_{m\in\psi_n} T_{o_{n,m}}^{n+1}\left(\Delta_m p_o^{n+1} - \gamma_{o_{n,m}}^n \Delta Z\right) = C_{o,p_n}\Delta_t p_{o_n} + C_{ow_n}\Delta_t S_{w_n} - q_{osc_n}^{n+1} \tag{3.55}$$

For the water equation

$$\sum_{m\in\psi_n} T_{w_{n,m}}^{n+1}\left(\Delta_m p_o^{n+1} - \Delta P_{cow}^{n+1} - \gamma_{w_{n,m}}^n \Delta Z\right) = C_{wp_n}\Delta_t p_{o_n} + C_{ww_n}\Delta_t S_{w_n} - q_{osc_n}^{n+1} \tag{3.56}$$

where

$$\Delta_m p_o = p_{o_m} - p_{o_n} \tag{3.57}$$

$$\Delta_m P_{cow} = P_{cow_m} - P_{cow_n} \quad \text{and} \tag{3.58}$$

$$\Delta_m Z = Z_m - Z_n \tag{3.59}$$

This instrumentation will be deployed in formulation of the IMPES methodology. The compact form of the equations aids in reducing the challenge with the manipulation of the equations.

3.5 The Classical IMPES Method

The simulation of multiphase flow is involving particularly considering the fact that in contrast to single phase flow, multiple finite difference equations have to be written for each gridblock (one equation for each component for each gridblock).The previous sections have sequentially elaborated the coupled and nonlinear nature of these multiphase equations; particularly the two-phase oil/water model which is the highlight of this study. There exists a variety of approaches for solving these equations such as the Implicit In Pressure Explicit In Saturation (IMPES), Simultaneous Solution (SS), Sequential and Adaptive Implicit Methods, of these methods, the IMPES remains the most popular in the petroleum industry and a very powerful method for solving two phase flow (particularly for incompressible or slightly compressible fluids). (Chen et al., 2006a).

The concept of the IMPES method was originally developed by (J. W. Sheldonand B. Zondek & Cardwell, 1959) and (Stone & Garder, 1961) which basically involves the separation of the computation of pressure from that of the saturation. The coupled two phase system is first split into a pressure equation and a saturation equation; the pressure equation is solved using implicit time approximation whiles the saturation equation is solved using an explicit time approximation (B. Lu & Wheeler, 2009) and (Lacroix, Vassilevski, Wheeler, & Wheeler, 2003). This structure is efficient and requires less computer memory compared to the other method for solving the two phase flow models such as the SS method(Douglas, Peaceman, & Rachford, 1959). The stability of the IMPES method however requires very small time steps for the saturation. In steps, the first procedure comprises of writing the pressure equation for each gridblock $n = 1,2,....,N$ and solving the resulting equation directly or iteratively, for the oil phase pressure distribution. The second step involves the explicit solution for the saturation unknowns by substituting pressures at t^{n+1} into the appropriate flow equations for individual gridblocks. The next section considers a step by step methodology for solving the two phase oil/water flow model using the IMPES method.

3.6 IMPES Method for an Oil/ Water Model

As discussed previously the finite difference equations for the oil/water flow model in CVFD for explicit transmissibilities, flow rates, and capillary pressures for gridblock n is given by

$$\sum_{m\in\psi_n}\left[T^n_{o_{n,m}}\left(\Delta_m p_o^{n+1}-\overline{\gamma}^n_{o_{n,m}}\Delta_m Z\right)\right]=C_{op_n}\Delta_t p_{o_n}+C_{ow}\Delta_t S_{w_n}-q^n_{osc_n} \tag{3.60}$$

and

$$\sum_{m\in\psi_n}\left[T^n_{w_{n,m}}\left(\Delta_m p_o^{n+1}-\Delta_m P^n_{cow}-\overline{\gamma}^n_{w_{n,m}}\Delta_m Z\right)\right]=C_{wp_n}\Delta_t p_{o_n}+C_{ww_n}\Delta_t S_{w_n}-q^n_{wsc_n} \tag{3.61}$$

The first step in the implementation of the IMPES occurs here where these two equations are combined such that each term containing $\Delta_t S_{w_n}$ vanishes; thus this produces the pressure equation for each gridblock. This is achieved by multiplying (3.61) by a constant A such that, adding the result to (3.60) and equating the coefficient of the $\Delta_t S_{w_n}$ term gives us the desired results. When this is done, the value of A is given by

$$A = -\frac{C_{ow_n}}{C_{ww_n}} \tag{3.62}$$

and the combined oil and water equation is given by

$$\sum_{m\in\psi_n} \left\{ \left[T^n_{o_{n,m}} \left(\Delta_m p_o^{n+1} - \overline{\gamma}^n_{o_{n,m}} \Delta_m Z \right) \right] + A \left[T^n_{w_{n,m}} \left(\Delta_m p_o^{n+1} - \Delta_m P_{cow} - \overline{\gamma}^n_{w_{n,m}} \Delta_m Z \right) \right] \right\}$$

$$= \left(C_{op_n} + A C_{wp_n} \right) \Delta_t p_{o_n} - \left(q^n_{osc_n} + A q^n_{wsc_n} \right) \tag{3.63}$$

As for the introduction of constant A, the alternative expression is

$$A = -\frac{C_{ow_n}}{C_{ww_n}} = \frac{B^{n+1}_{w_n}}{B^{n+1}_{o_n}} \tag{3.64}$$

By substituting (3.64) into equation (3.63) and multiplying the resulting equation by $B^{n+1}_{o_n}$ results in

$$\sum_{m\in\psi_n}\left\{B_{o_n}^{n+1}\left[T_{o_{n,m}}^{n}\left(\Delta_m p_o^{n+1}-\overline{\gamma}_{o_{n,m}}^{n}\Delta_m Z\right)\right]\right.$$

$$\left.+B_{w_n}^{n+1}\left[T_{w_{n,m}}^{n}\left(\Delta_m p_o^{n+1}-\Delta_m P_{cow}^{n}-\overline{\gamma}_{w_{n,m}}^{n}\Delta_m Z\right)\right]\right\}$$

$$=\left(B_{o_n}^{n+1}C_{op_n}+B_{w_n}^{n+1}C_{wp_n}\right)\Delta_t p_{o_n}-\left(B_{o_n}^{n+1}q_{osc_n}^{n}+B_{w_n}^{n+1}q_{wsc_n}^{n}\right) \tag{3.65}$$

The equation may be written as

$$\sum_{m\in\psi_n}\left(B_{o_n}^{n+1}T_{o_{n,m}}^{n}+B_{w_n}^{n+1}T_{w_{n,m}}^{n}\right)p_{o_m}^{n+1}$$

$$-\left[\left(B_{o_n}^{n+1}C_{op_n}+B_{w_n}^{n+1}C_{wp_n}\right)+\sum_{m\in\psi_n}\left(B_{o_n}^{n+1}T_{o_{n,m}}^{n}+B_{w_n}^{n+1}T_{w_{n,m}}^{n}\right)\right]p_{o_n}^{n+1}$$

$$=-\left(B_{o_n}^{n+1}C_{op_n}+B_{w_n}^{n+1}C_{wp_n}\right)p_{o_n}^{n}$$
$$-\left(B_{o_n}^{n+1}q_{osc_n}^{n}+B_{w_n}^{n+1}q_{wsc_n}^{n}\right)+\sum_{m\in\psi_n}B_{w_n}^{n+1}T_{w_{n,m}}^{n}\Delta_m P_{cow}^{n}$$

$$+\sum_{m\in\psi_n}\left(B_{o_n}^{n+1}T_{o_{n,m}}^{n}\overline{\gamma}_{o_{n,m}}^{n}+B_{w_n}^{n+1}T_{w_{n,m}}^{n}\overline{\gamma}_{w_{n,m}}^{n}\right)\Delta_m Z \tag{3.66}$$

This (3.66) is the pressure equation for gridblock n, which will be written for each gridblock n = 1,2,3,.....,N. and solving directly or iteratively the system of N coupled equation for the oil-phase pressure distribution at time level n + 1.

The second consecutive step involves the water saturation at time level n + 1. This is calculated by substituting p_o^{n+1} which is obtained in the first step into the water equation for each individual gridblock.

$$S_{w_n}^{n+1} = S_w^n + \frac{1}{C_{ww_n}} \left\{ \sum_{m \in \psi_n} \left[T_{w_{n,m}}^n \left(\Delta_m p_o^{n+1} - \Delta_m P_{cow}^n \right. \right. \right.$$

$$\left. \left. \left. - \overline{\gamma}_{w_{n,m}}^n \Delta_m Z \right) \right] - C_{wp_n} \left(p_{o_n}^{n+1} - p_{o_n}^n \right) - q_{wsc_n}^n \right\} \quad (3.67)$$

Where n = 1,2,.....,N. The actual implementation of this popular scheme is dependent on the nature of the fluids and the conditions of flow. Consider for example a homogeneous, ID horizontal oil reservoir which is 1000ft long with a cross-sectional area of 10,000ft^2 discretized into four equal gridblocks. Initially $Sw_i = Si_w = 0.160$ and P_1 = 1000psia everywhere. Water is injected at $x = 0$ at a rate of 75.96B/D at standard conditions, and oil is produced at x = 1000ft at the same rate. The gridblock dimensions and properties Table 3.1: Relative permeability data

S_w	0.16	0.20	0.30	0.40	0.50	0.60	0.70	0.80
k_{rw}	0.000	0.010	0.035	0.060	0.110	0,160	0.240	0.420
k_{ro}	1.000	0.700	0.325	0.140	0.045	0.031	0.015	0,000

are Δx = 250ft, Ax = 10000ft^2, $k\alpha$ = 300md and φ = 0.20. The reservoir fluids are incompressible with $B_w = B_o$ = 1RB/STB and $\mathbf{u}_o = \mathbf{u}_w$ = 1cp. We can use this information together with the relative permeability and saturation data in Table 1. to simulate the pressure and saturation distributions using the IMPES method. Using the time-stability limit resulting explicit transmissibilities, the criteria is to have $\Delta t < \Delta x/0.4453$ or $\Delta t <$ 561$days$. We build four gridblocks with the direction of flow for both oil and water phases from the injection well (Gridblock 1) to the production well (Gridblock 4). This means that fluids flow from *Gridblock*1 to *Gridblock*2 to *Gridblock*3 to *Gridblock*4. Using the parameters given for this illustration, we obtain the pressure and saturation equations from *eq*6 and *eq*7 by substituting $Bo_n = Bw_n = 1$ and $P^n_{cow} = 0$ and $\Delta_m Z = 0$, this results in

$$\sum_{m\in\psi_n}\left(T^n_{o_{n,m}} + T^n_{w_{n,m}}\right)p^{n+1}_{o_m} - \left[C_{op_n} + C_{wp_n} + \sum_{m\in\psi_n}(T^n_{o_{n,m}} + T^n_{w_{n,m}})\right]p^{n+1}_{o_n}$$

$$= -[(C_{op_n} + C_{wp_n})p^n_{o_n} - (q^n_{osc_n} + q^n_{wsc_n})] \quad (3.68)$$

where $n = 1,2,3,4$ and the water saturation equation also given by

$$S^{n+1}_{w_n} = S^n_{w_n} + \frac{1}{C_{ww_n}}\left[\sum_{m\in\psi_n} T^n_{w_{n,m}}\Delta_m p^{n+1}_o - C_{wp_n}\left(p^{n+1}_{o_n} - p^n_{o_n}\right) + q^n_{wsc_n}\right] \quad (3.69)$$

where $n = 1,2,3,4$. Since the fluid is incompressible, it implies $C_{op_n} = 0$, $C_{wp_n} = 0$ and $C_{ww_n} = \frac{V_b\phi}{\alpha_c\Delta t}$. This means that *eqn*9 and *eqn*9 may be express as

$$\sum_{m\in\psi_n}\left(T^n_{o_{n,m}} + T^n_{w_{n,m}}\right)p^{n+1}_{o_m} - \sum_{m\in\psi_n}\left(T^n_{o_{n,m}} + T^n_{w_{n,m}}\right)p^{n+1}_{o_n}$$

$$= -(q^n_{osc_n} + q^n_{wsc_n}) \quad (3.70)$$

and

$$S^{n+1}_{w_n} = S^n_{w_n} + \frac{1}{V_b\phi/\alpha_c\Delta t}\left(\sum_{m\in\psi_n} T^n_{w_{n,m}}\Delta_m p^{n+1}_o + q^n_{wsc_n}\right. ! \quad (3.71)$$

For this particular reservoir it means that

$n = 1, \psi_1 = 2$ $n =$

$2, \psi_2 = 1,3$ $n =$

$3, \psi_3 = 2,4$ $n =$

$4, \psi_4 = 3$

For n=1,

$$\left[(T^n_{w_{1,2}}) + (T^n_{o_{1,2}})_1\right] p^{n+1}_{o_2} - \left[(T^n_{w_{1,2}})_1 + (T^n_{o_{1,2}})\right] p^{n+1}_{o_1} = -q^n_{wsc_1}$$

n=2,

$$\left[(T^n_{w_{2,1}})_1 + (T^n_{o_{2,1}})_1\right] p^{n+1}_{o_1} + \left[(T^n_{w_{2,3}})_2 + (T^n_{o_{2,3}})_2\right] p^{n+1}_{o_3}$$
$$- \left\{\left[(T^n_{w_{2,1}})_1 + (T^n_{o_{2,1}})\right] + \left[(T^n_{w_{2,3}})_2 + (T^n_{o_{2,3}})_2\right]\right\} p^{n+1}_{o_2} = 0$$

n=3,

$$\left[(T^n_{w_{3,2}})_2 + (T^n_{o_{3,2}})_2\right] p^{n+1}_{o_2} + \left[(T^n_{w_{3,4}})_3 + (T^n_{o_{3,4}})_3\right] p^{n+1}_{o_4}$$

$$- \left\{\left[(T^n_{w_{3,2}})_2 + (T^n_{o_{3,2}})_2\right] + \left[(T^n_{w_{3,4}})_3 + (T^n_{o_{3,4}})_3\right]\right\} p^{n+1}_{o_3} = 0$$

n=4,

$$\left[(T^n_{w_{4,3}})_3 + (T^n_{o_{4,3}})_3\right] p^{n+1}_{o_3} - \left[(T^n_{w_{4,3}})_3 + (T^n_{o_{4,3}})_3\right] p^{n+1}_{o_4} = -q^n_{osc_4}$$

The saturation equation in equation (12) for each gridblock n as follow for n = 1. n=1,

$$S_{w_1}^{n+1} = S_{w_1}^{n} + \frac{1}{[(V_b\phi)]/[(\alpha_c\Delta t)]}\left[(T_{w_{1,2}}^{n})_1(p_{o_2}^{n+1} - p_{o_1}^{n+1}) + q_{wsc_1}^{n}\right]$$

n=2,

$$S_{w_2}^{n+1} = S_{w_2}^{n} + \frac{1}{[V_b\phi]/[(\alpha_c\Delta t)]}\left[(T_{w_{2,1}}^{n})_1(p_{o_1}^{n+1} - p_{o_2}^{n+1}) + (T_{w_{2,3}}^{n})_2(p_{o_3}^{n+1} - p_{o_2}^{n+1})\right]$$

n=3,

$$S_{w_3}^{n+1} = S_{w_3}^{n} + \frac{1}{[V_b\phi]/[(\alpha_c\Delta t)]}\left[(T_{w_{3,2}}^{n})_2(p_{o_2}^{n+1} - p_{o_3}^{n+1}) + (T_{w_{3,4}}^{n})_3(p_{o_4}^{n+1} - p_{o_3}^{n+1})\right]$$

n=4.

$$S_{w_4}^{n+1} = S_{w_4}^{n} + \frac{1}{[V_b\phi]/[(\alpha_c\Delta t)]}\left[(T_{w_{4,3}}^{n})_3(p_{o_3}^{n+1} - p_{o_4}^{n+1})\right]$$

This particular problem is ill-posed because there will be no unique solution for the pressure system of equations primarily due to the fact that all the fluid PVT properties are constant for incompressible-flow problems thus the pressure level being immaterial. The variable $p_{o_1}^{n+1}$ is therefore fixed at 1000*psia* and $p_{o_2}^{n+1}$, $p_{o_3}^{n+1}$ and $p_{o_4}^{n+1}$ are determined by uniquely solving the modified system of equations simultaneously.

All the four gridblocks have the dimensions and rock properties of φ = 0.20, Δx = 250*ft*, A_x = 10,000ft^2 and k_x = 0.300*darcy*, $V_b = A_x\Delta x$ = 10,000 ∗ 250 = 2.5 ∗ $10^6 ft^3$.

$$T^n_{w_{n,m}} = G_{n,m} \left(\frac{1}{\mu_w B_w}\right)^n_{n,m} (k_{rw})^n_{n,m}$$

,

$$G_{n,m} = \frac{B_c K_x A_x}{\Delta x}$$

$$T^n_{w_{n,m}} = \frac{(1.127)0.300 * 10,000}{250} \left[\frac{1}{1*1}\right] k^n_{rw_{n,m}} = 13.524 k^n_{rw_{n,m}}$$

similarly,

$$T^n_{o_{n,m}} = G_{n,m} \left(\frac{1}{\mu_o B_o}\right)^n_{n,m} (k_{ro})^n_{n,m}$$

,

$$G_{n,m} = \frac{B_c K_x A_x}{\Delta x}$$

$$T^n_{o_{n,m}} = \frac{(1.127)0.300 * 10,000}{250} \left[\frac{1}{1*1}\right] k^n_{ro_{n,m}} = 13.524 k^n_{ro_{n,m}}$$

With the flow rates being $q^n_{wsc_1} = 75.96B/D$ and $q^n_{osc_4} = 75.96STB/D$, at $t = 100days$,

$\Delta t = 100 - 0 = 100days$ which may be taken as one time step because $\Delta t < 561days$.

$$\frac{V_b \phi}{\alpha_c \Delta t} = \frac{(2.5 * 10^6) * 0.20}{(5.615)(100)} = 890.472$$

$$p^{n+1}_{o_1} = 1,000psia$$

$$p^n_{o_1} = p^n_{o_2} = p^n_{o_3} = p^n_{o_4} = 1000psia$$

and

$$S^n_{w_1} = S^n_{w_2} = S^n_{w_3} = S^n_{w_4} = 0.160$$

Therefore $k^n_{rw_1} = k^n_{rw_2} = k^n_{rw_3} = (k_{rw})_{sw=0.16} = 0.$000 and$k^n_{ro_1} = k^n_{ro_2} = k^n_{ro_3} = (k_{ro})_{sw=0.16} =$ 1.000.

The solution of the resulting system is given in Table 2.

Table 3.2: Table of solutions

Gridblock	$p_o(psia)$	$p_w(psia)$	S_w	S_o
1	1000	1000	0.2453	0.7547
2	994.4	994.4	0.1600	0.8400
3	988.8	988.8	0.1600	0.8400
4	983.2	983.2	0.1600	0.8400

3.7 Boundary Conditions

A boundary condition is a set of additional constraints together with a differential equation on a boundary value problem. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.There are four types of boundary condition, these are Dirichlet boundary condition, Neumann boundary condition,Cauchy boundary condition and Mixed Boundary condition.

3.8 Types of Boundary Condition

3.8.1 Dirichlet Boundary Condition

This is a first- type boundary condition named after Peter Gestau Lejeune Dirichlet (18051859) it specifies the values that a solution needs to take on along the boundary of the domain. Hence it specifies the value of the function itself. Example if one end of an iron rod is held at absolute zero, then the value of the problem would be known at that point in space

3.8.2 Neumann Boundary Condition

This is a second- type boundary condition named after Carl Neumann when imposed on an ordinary or a partial differential equation it specifies the values that the derivative of a solution is to take on the boundary of the domain.That is, it specifies the value of the normal derivative of the function . Example, if there is a heater at one end of an iron rod then energy would be added at a constant rate but the actual temperature would not be known.

3.8.3 Mixed Boundary Condition

A mixed boundary condition for a partial differential equation defines a boundary value problem in which the solution of the given equation is required to satisfy different boundary condition on disjoint parts of the boundary of the domain where the condition is stated. In a mixed boundary value problem, the solution is required to satisfy a Dirichlet or a Neumann boundary condition in a mutually exclusive way on disjoint parts of the boundary.

3.8.4 Caunchy Boundary Condition

It augment on ordinary differential equation or a partial differential equation with conditions that the solution must satisfy on the boundary. Ideally so to ensure that a unique solution exists. A Cauchy boundary condition specifies both the function value and normal derivative on the boundary of the domain. This corresponds to imposing both a Dirichlet and a Neumann boundary condition. It is named after the prolific 19th century French mathematical analyst Augustin Louis Cauchy. if the boundary has the form of a curve or surface that gives a value to the normal derivative and the variable itself then it is a Cauchy boundary condition.

Chapter 4

Results and Discussions

4.1 Introduction

This chapter is a presentation of the results from the simulation of the two phase oil/water flow model using the IMPES method. The algorithm for the solution is simulated first for incompressible flow conditions and then for slightly compressible flow conditions in one dimension by considering the pressure profiles in varying space and time. The algorithm for the IMPES method as discussed in the foregoing chapter is coded in Matlab for the implicit pressure and explicit saturation. The research employs the use of Relative Percentage Error to compare the two cases investigated. The Matlab codes for the simulation are provided in the Appendix of this research.

4.2 Numerical Example of Two-Phase Oil/Water Flow

The research considers a two-phase oil/water water injection problem. The reservoir is treated as a two-phase oil/water system with up to ten equally spaced wells. The wells are tagged as follows: W_1, W_2,...,W_{10} the wells are active with W_1 being the injection well. The reservoir considered is 1000ft long with a cross-sectional area of 10,000ft^2. The flow direction therefore for both the oil and water phases is from the injection well to the other wells which are all production wells. Experimental results of a real reservoir **A-1**, documented in Turcay et. al. (2001) is used for the simulation. The results feature experimental saturation and relative permeability values as shown in Table 4.1

Table 4.1: Relative Saturation and Permeability Data

Water Saturation, S_w	0.18	0.21	0.24	0.27	0.30
Water Relative Permeability k_{rw}	0.00000	0.00000	0.00002	0.00014	0.00045
Water-Oil Relative Permeability k_{row}	1.00000	0.92692	0.85441	0.79288	0.71312
Water Saturation, S_w	0.33	0.36	0.39	0.42	0.45
Water Relative Permeability k_{rw}	0.00111	0.00232	0.00430	0.00733	0.01175
Water-Oil Relative Permeability k_{row}	0.64526	0.57980	0.51709	0.45744	0.40110

In the case of unavailable saturation relative permeability tallies, the model by *Naar and Henderson* is used to estimate the relative permeabilities given the water saturation values. The results for each of the cases (i.e. incompressible and slightly compressible cases) considered are plotted as lines and surfs for illustrative examinations.

4.2.1 Incompressible flow

The oil/water fluids are both considered as incompressible fluids in this scenario. The water saturation is initially set to 0.5 and the pressure of water and oil both set to 7000psi everywhere. The oil/water capillary pressure is zero. The gridblock dimensions and properties are Δx = 100ft, cross-sectional area, A_x = 10,000ft^2. The absolute permeability, k_x

= 300md and the porosity φ = 0.20. The reservoir fluids being incompressible implies B_w = B_o = 1RB/STB and the water and oil viscosities μ_o = μ_w = 1cp. The flow rates are given as follows: Water is injected at a rate of 75.96B/D at standard conditions and oil is produced at the same rate at the last well W_{10} which is 1000ft away from the production

well.

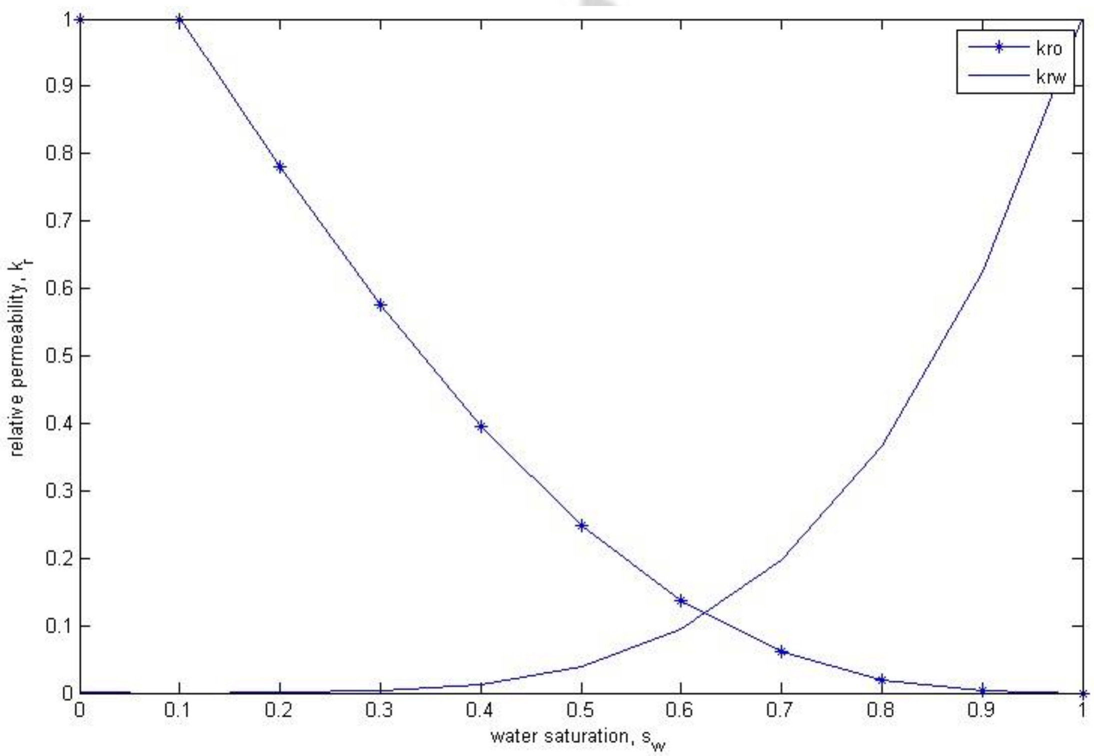

Figure 4.1: Relative Permeability-Saturation Profile

The relative permeabilities k_{row} and k_{rw} are shown in Figure (4.1) both as functions of the water saturation, S_w for the oil/water system considered. The relative permeabilities simulated were between zero (0) and one (1) as in literature. This relationship is required to describe the two-phase oil/water flow and also correlates with laboratory results on suitable cores.

The one-dimensional plot of the pressure of oil is shown in figure (4.2). The results

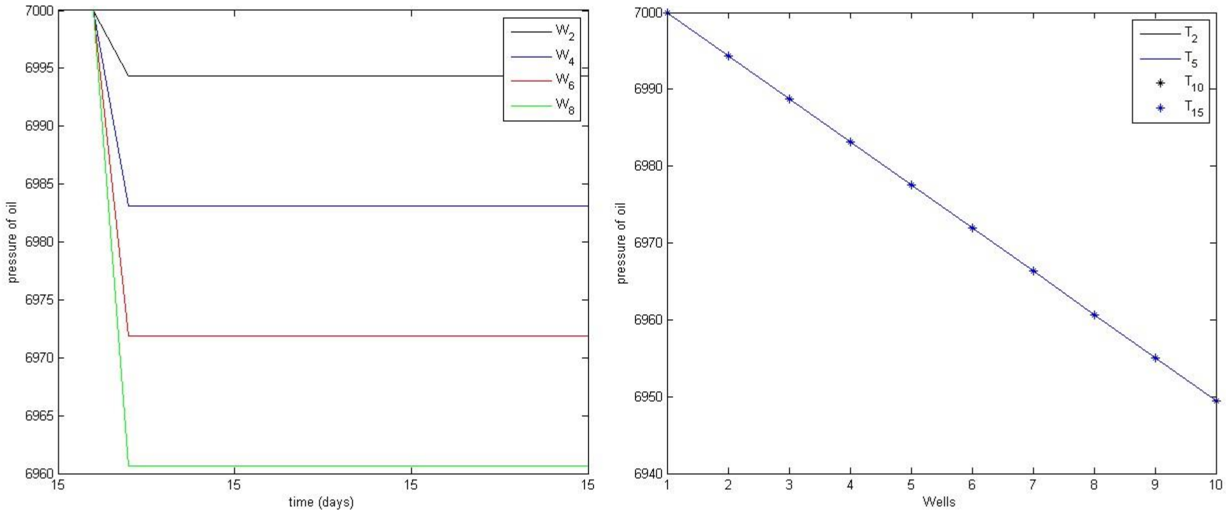

Figure 4.2: Oil Pressure Profile for Two Phase Flow in Time (left) & at Wells (right)

of the simulation of the eight wells are shown for some of the times recorded during the 500 ($T_1 - T_{15}$) days; intervals of 35 days. The flow of the fluids is in the direction from $W -$ 1 to $W -$ 10. The oil pressure-time (left) of Figure (4.2) shows that the pressure is initially set to 7000psi for each of the wells. The second well W_2 however drops to 6995psi from the initial 7000psi at 35 days due to production. All the other wells experience the increasing levels of pressure drops from the initial within the first 70 days. The pressures after the first 70 days remain constant throughout the 500 day period (4.2). The dynamics for the different times investigated are similar with sharp pressure drops and then constant pressure until the end of the 500 days considered.

A three dimensional plot of the profile for the oil pressure for all the wells is shown in Figure (4.3). The first well, W_1 remains at 7000psi; this is primarily due to the fact that the oil pressure is constant by injection at that level. The remainder of the wells however experience sharp drop in level of pressures after which the oil pressure levels

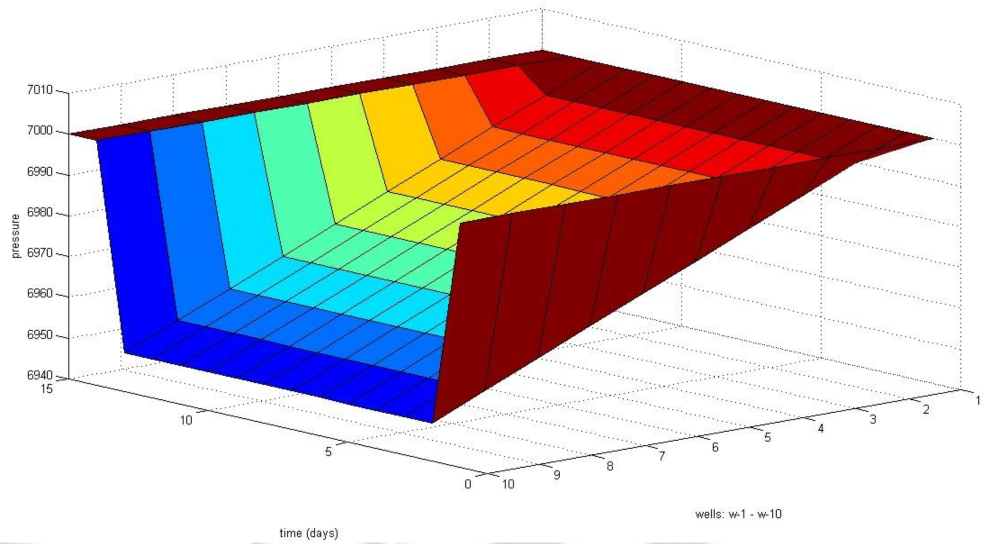

Figure 4.3: Surf plot for Oil Pressure Profile

remain constant.

4.2.2 Slightly compressible flow

The injection problem is considered for oil/water being slightly compressible fluids. The water saturation is initially set to 0.5 as in the previous scenario and the pressure of water and oil are both 7000psi everywhere. The capillary pressure is set to zero thus implying

Table 4.2: Experimental Results of Oil Pressure and Oil Viscosity in **A-1** Reservoir

Pressure(psia)	Oil Viscosity(cp)
5,000.0	0.9200
5,500.0	0.9243
6,000.0	0.9372
6,500.0	0.9494
6,700.0	0.9650
7,500.0	0.9812
8,000.0	1.0019

that the oil pressure and water pressure will be the same throughout the simulation.

The absolute permeability, k_x = 300md and the porosity φ = 0.20. The other fluid conditions are given as follows: Rock compressibility, $C_R = 5.9 * 10^6$, $B_o = 1.0/[1.0 + 5.0 * 10^6(p-14.7)]RB/STB$, $B_w = 1.0/[1.0+1.0*10^6(p-14.7)]$, viscosity of water, μ_w = 0.52cp. The oil viscosity, μ_o is estimated from the experimental results data shown in Table 4.2.

A graphical examination of the relationship between the oil viscosity and the pressure shows that there is a positive linear relationship between the two variables as shown in Figure (4.4). A simple linear regression estimation with oil viscosity, μ_o as the dependent variable and pressure(psia) as the independent variable shows that the linear relationship

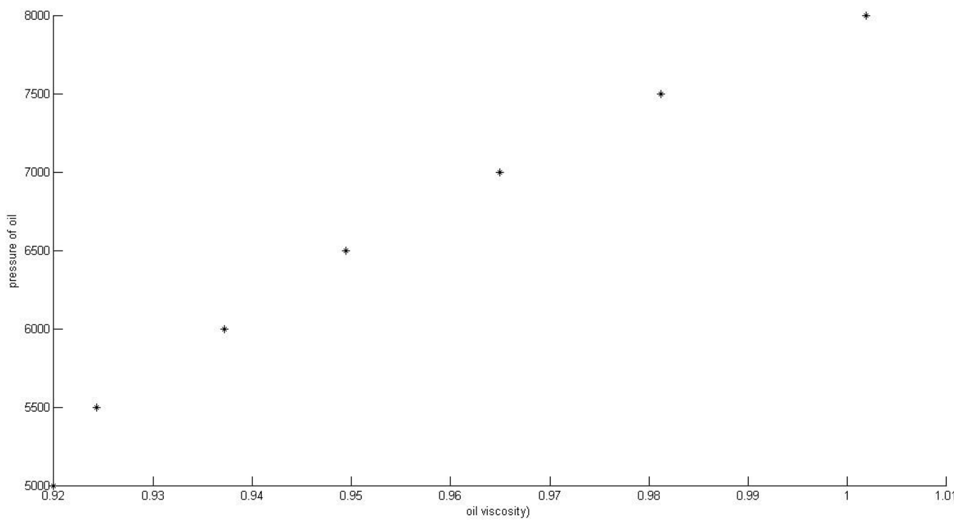

Figure 4.4: Oil Pressure vrs Viscosity Relationship

between the two variables is

$$\mu_o = 2.76643 * 10^{-5} p + 0.774325 \qquad (4.1)$$

This is used to compute the reservoir oil viscosity. The injection and production rated for the rest of the wells are given in Tables 4.1 and Table 4.2. Table 4.2 represents the distribution of oil production at W_{10}.

The simulation from the slightly compressible assumption shows a much realistic profile of the oil pressure. There is a pressure drops between the first 140 days similar to the incompressible flow profile. The pressure however does not remain constant but then to rise because of the injection provided at the various wells. The final wells i.e. from W_8 to W_{10} are maintained at 7000psia three periods before the final 500 days as shown in Figure

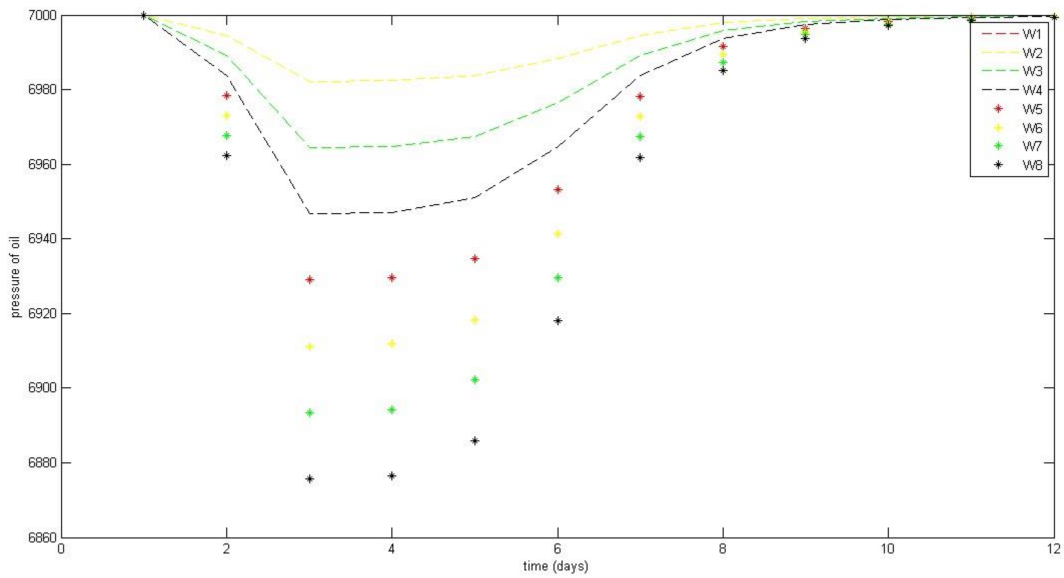

Figure 4.5: Oil Pressure Profile vrs Time For Slightly Compressible Flow

(4.5).

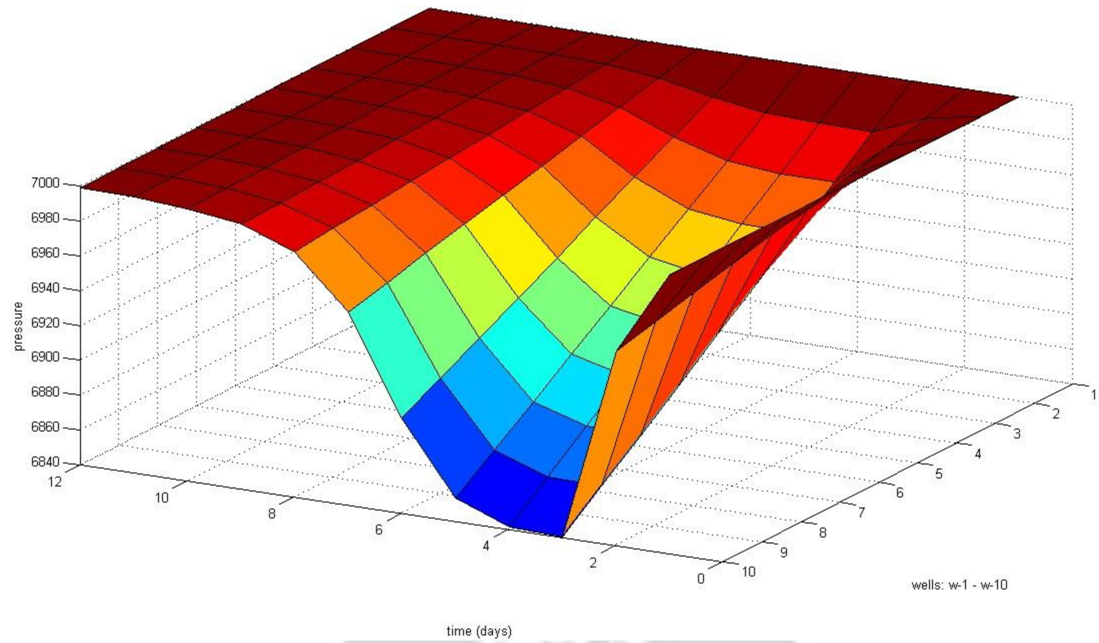

Figure 4.6: Surf plot of Oil Pressure Profile Slightly Compressible Flow

Figure (4.6) shows a surface plot of the oil pressure simulated in the two phase oil/water model by considering slightly compressible fluids. The plot gives a much realistic scenario showing how the pressure is maintained at the injection and production wells in contrast to the incompressible case where the decrease in pressure from the injection well is constant until the duration of investigation.

4.3 Error Analysis

The research considers numerical comparison of the incompressible flow and slightly compressible flow scenarios by considering the pressures investigated only, i.e. oil and water pressure. The results of the pressure simulated in the incompressible flow scenario is compared with the results from the slightly compressible scenario.

Table 4.3: Relative Percentage Errors for Differences in Two Cases (%)

Wells	T_1=35 days	T_2=70 days	T_5=178 days	T_{10}=357 days	T_{15}=500 days
W_1	0.0000	0.0000	0.0000	0.0000	0.0000
W_2	0.0000	0.0031	0.1623	0.0685	0.0797
W_3	0.0000	0.0062	0.3253	0.1371	0.1595
W_4	0.0000	0.0093	0.4891	0.2056	0.2392
W_5	0.0000	0.0125	0.6538	0.2742	0.3190
W_6	0.0000	0.0156	0.8192	0.3428	0.3988
W_7	0.0000	0.0187	0.9855	0.4114	0.4785
W_8	0.0000	0.0282	1.4892	0.6174	0.7178
Average Error	0.0000	0.0117	0.6156	0.2571	0.2991

The Relative Percentage Absolute Errors (RPAE) are computed for the two cases investigated using the slightly compressible case as the benchmark as follows:

$$\text{RPAE} = \frac{\text{Pressure}_{sc} - \text{Pressure}_{ic}}{\text{Pressure}_{sc}} \times 100\% \qquad (4.2)$$

where Pressure_{sc} represents the pressure of the slightly compressible flow and Pressure_{ic} represents the pressure of the incompressible flow. Sampled results of the error computation for T_1=35 days T_2=70 days T_5=178 days T_{10}=357 days and T_{15}=500 days have been summarized in Table 4.3. The results indicate that there are increasing differences in time with peaks occurring some few wells away from the final production well. The average relative percentage errors on the final row of Table 4.3 are confirmatory to the generally increasing in time and subsequently increasing towards the production well.

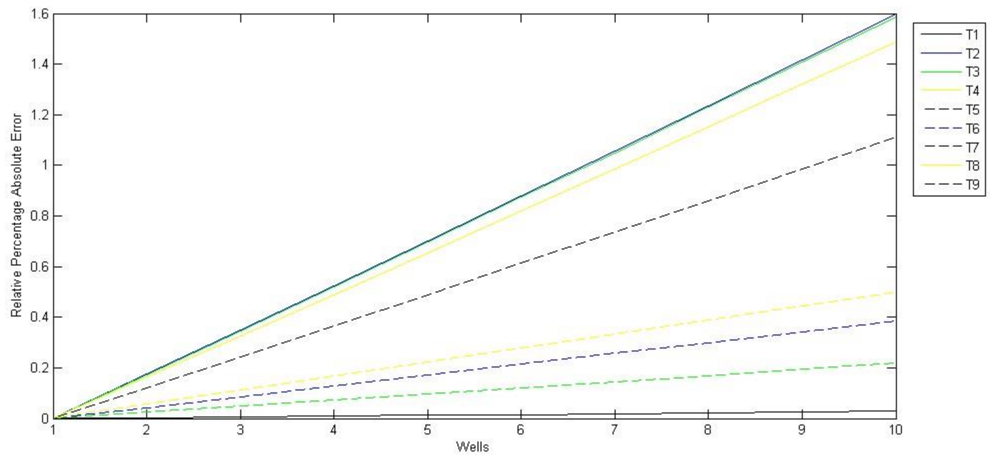

Figure 4.7: Relative Percentage Absolute Errors for Wells at Considered Times

Figure (4.7) shows the error plot for sampled times in order of increment (T_1 – T_9). These turn out to be linear plots of the percentage absolute differences of the different times between the incompressible and slightly compressible routines that are simulated.

Chapter 5

Conclusion And Recommendations

5.1 Introduction

The key objectives of this research were to the two phase oil/water flow model in rectangular geometry using the IMPES finite difference scheme, examine the incompressible flow and slightly compressible flow conditions and draw inferences based on the differences. The research considers a complete review of the two phase modeling in rectangular geometry and the numerical algorithms are programmed in Matlab; the matlab code is presented in the appendix. All the objectives set for the research were successfully met. This chapter presents the conclusions and recommendations of the research.

5.2 Conclusions

The conclusions for the research are summarized based primarily on the objectives formulated and their related outcomes. The conclusions are given as follows:

The two phase oil/water flow model in rectangular geometry was reviewed in this research from the formulation stage using the governing equation i.e. the conservation of mass, conservation of momentum, saturation and capillary pressure relations. The IMPES scheme was successfully employed in discretizing the two phase flow model after the four set of equations were compressed into two equation with algebraic manipulations. The conditions of stability for the application of the IMPES method on the model are clearly examined and reported.

The incompressible flow and slightly compressible flow in one dimension are investigated for a reservoir, **A-1** with up to ten active wells. The results indicate that the incompressible flow assumption is found to be deficient of the actual dynamics of two phase oil/water simulation. The results are shown with relative percentage absolute errors indicating the deviations of the incompressible flow from the slightly compressible flow; the deviations are shown to be linear in nature.

5.3 Recommendation

The following recommendations were made in the research: Other solution schemes like the Improved IMPES Method and Fully Implicit Finite Volume method should be used to solve the two-phase flow equation. Other numerical methods such as the Finite Element Method which has received a lot of attention in recent times within the petroleum reservoir simulation industry should be used to render solutions to the two phase flow methods. The solutions can also be considered in two and three dimensional flow.

Appendix

Matlab Code

```
clear clc clear all clc %% This is a two phase oil/water linear flow model: slightly compressible
%% fluids


%% time and space
xa=0; xb=1000; n=10; % number of nodes in spatial domain dx=(xb-xa)/n; x=dx:dx:xb; ta=0;
tb=1000; m=15; % number of time nodes dt=linspace(0,500,m);% this should less than the
minimum stability criterion
```

```
%% parameters

alphac=5.615; bc=1.127;

kx=.300; Ax=10000; dx=250;

Vb=Ax*dx; % bulk volume


    kr1=@(s,lmd)(s.^((2+3*lmd)/lmd)); ko1=@(s,lmd)((1-s).^2.
*(1-s.^((2+lmd)/lmd)));


%% Porosity distribution

phi=ones(n,m); phi=0.20*phi;


%% injection qw=zeros(n,m);

qo=zeros(n,m);

qw(1,1:end)=75.96;

qo(end,1:end)=-75.96;


    %% permeabilities

krw=zeros(n,m); kro=ones(n,m);


%% initial saturation

sw=zeros(n,m); so=zeros(n,m);

sw(:,1)=0.5; so(:,1)=1-sw(:,1);
```

```
%% pressure distribution

p=zeros(n,m); p(1,:)=7000;

p(:,1)=7000;


%% FVF

Bo=zeros(n,m); Bw=zeros(n,m);


fBw=@(p)(1./(1+5*10^(−6)*(p −14.7))); fBo=@(p)(1./(1+1*10^(−6)*(p

−14.7)));


Bw(:,1)=fBw(p(:,1));

Bo(:,1)=fBo(p(:,1));


%%Other computations

Tw=zeros(n,1); To=zeros(n,1);

Gnm=bc*kx*Ax/dx;


%% Functions comm=@(h1,h2,j)((h1(:,j)-h1(:,j-1))./(h2(:,j)-h2(:,j-1))); %% viscosity

muw=.52; muo=@(p)(2.76643*10^(−5)* p +0.774325);


for j=2:m;

Cww=Vb/(alphac*dt(j)).*(phi(:,j-1)./Bo(:,j-1));

Cop=Vb/(alphac*dt(j)).*(comm().*(1-sw(:,j-1))

%pressure of oil computation
```

```
Tw=Gnm./(muw.*fBw(p(:,j-1))).*krw(:,j-1); To=Gnm./(muo(p(:,j-1)).*fBo(p(:,j-
1))).*kro(:,j-1);
Tw=Bw(:,j-1).*Tw;
To=Bo(:,j-1).*To;
Tow=Tw+To; Tow(1)=0;Tow(end)=0;

l=n; aa=sparse(2:l,1:l-1,Tw(2:end)+To(2:end),l,l);
bb=sparse(1:l,1:l,-(Tw+To+Tow),l,l); cc=sparse(1:l-
1,2:l,Tw(1:end-1)+To(1:end-1),l,l); A=aa+bb+cc;
AA=A(2:end,2:end); rhs=-(Bo(:,j-1).*qo(:,j-1)+Bw(:,j-1).*qw(:,j-
1)); rhs(2,1)=-p(1,j-1)*A(2,1); rhs=rhs(2:end);
p(2:end,j)=inv(AA)*rhs;

%% Explicit calculation of Saturation v sTw2=Tw; sTw2(1)=0;
sTw1=Tw; sTw1(end)=0; ps1=zeros(n,1);ps1(1:end-1)=p(2:end,j)-
p(1:end-1,j); ps2=zeros(n,1);ps2(2:end)=p(1:end-1,j)-p(2:end,j);

        sw(:,j)=sw(:,j-1)+1./Cww.*(sTw1.*ps1+sTw2.*ps2+qw(:,j-1)); so(:,j)=1-
sw(:,j); krw(:,j)=kr1(sw(:,j),2); kro(:,j)=ko1(sw(:,j),2);

        Bw(:,j)=fBw(p(:,j)); Bo(:,j)=fBo(p(:,j));
end surf(p)
xlabel('time (days)');
ylabel('wells: w-1 - w-10');
zlabel('pressure')
```

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