Kwame Nkrumah University Of Science And Technology

DIVERGENCE REGULARIZATION METHOD FOR SOLVING ILL-POSED HELMHOLTZ EQUATION

by

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Declaration

I hereby declare that this submission is my own work towards the Doctor of Philosophy (PhD) and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgement has been made in the text.

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Abstract

In this work, we introduce Divergence Regularization Method (DRM) for regularizing the Cauchy problem of the Helmholtz equation where the boundary deflection is not equal to zero in Hilbert space H. The DRM incorporates a positive integer scaler which homogenizes inhomogeneous boundary deflection in Cauchy problem of the Helmholtz equation to ensure the existence and uniqueness of solution for the equation. The DRM employs its regualarization term $(1 + \alpha^{2m})e^{m}$ to restore the stability of the regularized Helmholtz equation, and guarantees the uniqueness of solution of Helmholtz equation when it is imposed by Neumann boundary conditions in the upper half-plane. The DRM gives better stability approximation when compared with other methods of regularization for solving Cauchy problem of the Helmholtz equation where the boundary deflection is zero.

In the process, we introduce Adaptive Wavelet Spectral Finite Difference (AWSFD) method to obtain the approximated solutions of the regularized Helmholtz equation with regularized Cauchy boundary conditions, regularized Neumann boundary conditions in the upper half-plane, and finally with regularized both Dirichlet and Cauchy boundary conditions where the boundary deflection is equal to zero. The AWSFD method captures the boundary points to obtain approximated solution of Helmholtz equation. This method reduces the Helmholtz equation in two dimensions to one dimension which is then solve spectrally using a suitable wavelet basis. The solutions by AWSFD method confirms the analytic solutions of regularized Helmholtz equation by DRM. The norm of relative error between the analytic solution by DRM and the approximated solution by AWSFD method is minimal. Moreover, we introduce interpolation scheme in the AWSFD method to obtain the approximated solutions of the regularized Helmholtz equation with above boundary conditions.

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1.0 Introduction

In this chapter, we discuss the ill-posed second-order linear partial differential equations. We restrict ourselves to homogeneous Helmholtz equation, an elliptic partial differential equation. Ill-posed Helmholtz equation comes about as a result of imprecise readings of the boundary data, given by physical instruments (Bertero et al., 1988). This noise causes distortion in signal transmission. Secondly, the number and type of auxiliary boundary conditions imposed on the equation can lead to ill-posedness. If the number of auxiliary boundary conditions are too many, then the solution to the Helmholtz equation may not exist, but if there are too few, the Helmholtz equation may have more than one solution.

In most practices, experiments with nearly similar boundary conditions do not yield the same results. The solution to Helmholtz equation with boundary conditions may be continuously differentiable, but may suffer numerical instability when solved with finite precision. Numerical simulations of ill-posed Helmholtz equation together with boundary conditions often lead to large errors, wrecks and catastrophes (Petrov and Sizikov, 2005). Undoubtedly, the number of imposed boundary conditions should not only bring about the existence and uniqueness of solution to the Helmholtz equation, but also restore the stability of the solution to the Helmholtz equation.

Solution space of the equation is another important factor which cannot be overemphasized. Inappropriateness of solution space can lead to ill-posed elliptic Helmholtz equation. Spaces like Hilbert space admit classical solution to the Helmholtz equation.

1.1 Background

In this section, we discuss the ill-posed Helmholtz equation with Cauchy boundary conditions where the boundary deflection is not equal to zero, with Neumann boundary conditions in the upper half-plane and finally with imposed both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous. In addition, we identify the cases where if the Helmholtz equation is imposed with one of the above boundary conditions then the Helmholtz equation has no solution, has more than one solution and also, the case where a unique solution exists, but the solution does not depend continuously on the small changes in the boundary conditions. Moreover, we provide proofs of some of the rigorous results related to the issues that have been discussed above. Definitions and theorems which are relevant, necessary and sufficient to establish the claims will be provided in this work.

An ill-posed Helmholtz equation has no practical application or is physically meaningless (Hadamard as cited in Petrov and Sizikov, 2005). However, in vibrating membrane system and laser beam models, ill-posed Helmholtz equation has applications (Petrov and Sizikov, 2005). Thus, such an equation needs regularization for certain imposed boundary conditions. Regularizing ill-posed Helmholtz equation together with the boundary conditions requires additional information in the construction of stable solutions (Lavrent0ev et al., 1997). We prescribe appropriate constraints in the Helmholtz equation, as well as boundary conditions to make the equation wellposed. Regularized Helmholtz equation provides better understanding in the study of stationary processes. Undoubtedly, if the Helmholtz equation is well-posed, then numerical stability is feasible when solved with finite precision, or with errors in the data.

Our method is the Divergence Regularization Method (DRM) for the construction of stable solutions to ill-posed Helmholtz equations with Cauchy boundary conditions where the boundary deflection is not equal to zero, with Neumann boundary conditions in the upper half-plane and then Helmholtz equation with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous in Hilbert space. We achieve this result by applying divergence theorem in two dimensions, Green's first identity and then introduction of homogenization of boundary deflection in the Cauchy boundary conditions.

Also, we show that our solutions of the regularized Helmholtz equation with regularized boundary conditions meet all the three conditions of well-posedness given by Hadamard. Afterwards, we apply the DRM to solve the equations. In the case of Neumann problem for the Helmholtz equation, we apply a shift operator on the $x-$ spatial variable instead of homogenization of the boundary deflection.

In order to confirm our solutions of the regularized Helmholtz equation with above regularized boundary conditions, we introduce a numerical technique, called Adaptive Wavelet Spectral Finite Difference AWSFD method. This (wavelet) method approximates the solutions of the regularized Helmholtz equation in Hilbert space. In addition, we introduce interpolation scheme in the AWSFD method and compare the solutions of regularized Helmholtz equation with the same regularized boundary conditions by DRM and by AWSFD method. We compare the solution of regularized Helmholtz equation by DRM and solutions by existing methods of regularization for cases where if both methods are able to regularize the Helmholtz equation.

We state some definitions and theorems that useful in understanding ill-posed Helmholtz equation with imposed boundary conditions as follows.

Theorem 1.1 (Hadamard Theorem) *If the Laplace operator in the Helmholtz equation*

$A : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *is* $C^2(\Omega)$ *and* $\|Aw(x,y)^{-1}\| \leq \gamma < +\infty$*,* $\forall\, w(x,y) \in \Omega$ *and* $\gamma \in \mathbb{R}$

where

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

then the A is a homeomorphism of Ω onto R^2.

Proof : See for example (Ortega and Rheinboldt, 1970).

We state the Hadamard theorem in the context of operator equation as follows.

Definition 1.1 (ill-posedness) *Let the operator equation*

$$Aw(x,y) = f(x) \tag{1.1}$$

where $w(x,y)$ is the sought solution, f is a known function, Y and F are Hilbert spaces and A is a Laplace operator occurring in the Helmholtz equation. Thus, $A : Y \rightarrow F$ is an operator from a Hilbert space Y into a Hilbert space F. The problem of solving equation (1.1) *presents a well-posed according to Hadamard if 1. for any $f \in F$, there exists an element $w(x,y) \in Y$ such that*

$$Aw(x,y) = f(x).$$ *That is,*

the the range of the operator

$$R(A) = F$$

is closed. Thus, the solution of equation (1.1) exists.

2. if the

$$N(A) = 0.$$

This implies that the null space of A is trivial. Thus, the Helmholtz equation with boundary conditions has a unique solution.

3. the solution $w(x,y)$ depends continuously on the data function $f(x)$. That is, the inverse Laplace operator A^{-1} in the Helmholtz is continuous. Thus, the solution is stable with respect to small perturbations in the data function. Otherwise equation (1.1) *is ill-posed (Petrov and Sizikov, 2005).*

The third condition of well-posedness can be ensured by the bounded inverse theorem.

We see that the Helmholtz equation has a solution if the smoothness requirement is satisfied together with other conditions, which we will give them later.

Definition 1.2 (Smoothness Requirement) *The conventional homogeneous form of equation* (1.1) *is as follows:*

$$Lw = -\sum_{i=1}^{2} \frac{a_i(x)\partial^2 w(x,y)}{\partial x_i^2} + cw(x,y) = 0 \quad in \quad \Omega \tag{1.2}$$

where $x \in \Omega$, $c(x) \in R^2$, $a(x) \in R^{2\times 2}$ *are the coefficients. The coefficients* a_i *and* c *satisfy the following conditions:*

$$a_i \in C^1(\Omega)^-, \quad i = 1,2$$

$$c \in C(\Omega)^-,$$

$$0 \in C(\Omega)^-$$

and

$$\sum_{i=1}^{2} a_i(x)\epsilon_i^2 \geq \tilde{C} \sum_{i=1}^{2} \epsilon_i^2, \quad \forall\, \epsilon = (\epsilon_1,\ \epsilon_2) \quad x \in \bar{\Omega}$$

(Su¨li, 2012).

The Cauchy problem or Neumann problem of the Helmholtz equation will have a solution if the additional condition; data compatibility condition is satisfied.

Theorem 1.2 (Data Compatibility Condition: Duchateau and Zachmann, (1989**))** *Let* Ω *denote a bounded region in* R^2 *having smooth boundary* $\partial\Omega$*. The Neumann problem*

$$\Delta w(x,y) + k^2 w(x,y) = 0 \text{ in } \Omega,$$

$$\frac{\partial w(x,a)}{\partial y} = g(x) \text{ on } \partial\Omega_A$$

$$\frac{\partial w(b,y)}{\partial x} = f(y) \text{ on } \partial\Omega_B$$

has no solution unless the data functions 0, $g(x)$ *and* $f(y)$ *satisfy the compatibility condition*

$$0 = \int_{\partial\Omega_A} g(x)dx = \int_{\partial\Omega_B} f(y)dy.$$

When the compatibility condition is satisfied, then the solution exists for the Helmholtz equation. Thus, the range of the Laplace operator $R(A)$ in the Helmholtz equation is closed. In the case of Dirichlet problem of the Helmholtz equation, no such condition is useful. We state the definitions of the domain, the range and the null space of A as follows:

Definition 1.3 *Let X and Y be normed linear spaces. The operator* $A : X \to Y$ *is said to be a linear operator from a Hilbert space X to another Hilbert space Y if,*

$$A(\mu w_1(x,y) + \eta w_2(x,y)) = \mu A(w_1(x,y)) + \eta A(w_2(x,y)),$$

where $w_1(x,y), w_2(x,y) \in X$ *and* $\mu, \eta \in \mathbb{R}$. *The domain of the operator* $D(A)$ *is defined as*

$$D(A) = \{w \in X \,\big|\, A(w(x,y)) \;\; is \; defined\},$$

the range of the operator $R(A)$ *is*

$$R(A) = \{v \in Y \,\big|\, v = A(w(x,y)), \;\; for \; some \; w(x,y) \in D(A)\}$$

and the null space is

$$N(A) = \{w(x,y) \in X \,\big|\, A(w(x,y)) = 0\}$$

(Atkinson and Han, 2009).

The Riesz representation theorem gives the uniqueness of solution to Helmholtz equation with imposed boundary conditions, on the grounds that the Laplace operator in the Helmholtz equation is bounded on the domain in the Hilbert space. We use other conditions of uniqueness to establish the claim where it is necessary and appropriate.

Theorem 1.3 (Riesz representation theorem) *Let A be a bounded linear operator defined on a subspace Ω of a Hilbert space H. Then there is a unique solution $w_u(x,y) \in \Omega \subset H$ such that*

$$A(w(x,y)) = \langle w(x,y), w_u(x,y)\rangle, \qquad \forall\, w(x,y) \in \Omega.$$

In addition,

$$|A|_{\Omega^* \subset H^*} = \| w_u(x,y) \|$$

(Coleman, 2012).

Proof : Set $S = KerA(w(x,y))$. If $H = S$, then

$$w(x,y) = 0,$$

we set

$$w_u(x,y) = 0.$$

If we set $S \neq H$, there exists $w(x,y) \in S$ such that

$$A(w(x,y)) \neq 0.$$

However, we observe that

$$w(x,y) = w_1(x,y) + w_2(x,y),$$

where $w_1(x,y) \in S$ and $w_2(x,y) \in S_\perp$ and so $A(w_2(x,y)) \neq 0$. For $w(x,y) \in S$, we obtain

$$
\begin{aligned}
A\Big(w(x,y) - \frac{A(w(x,y))}{A(w_2(x,y))}\Big).w_2(x,y) &= A(w(x,y)) - \frac{A(w(x,y))}{w_2(x,y)}.A(w_2(x,y)) \\
A\Big(w(x,y) - \frac{A(w(x,y))}{A(w_2(x,y))}\Big).w_2(x,y) &= 0,
\end{aligned}
$$

so $Aw(x,y) - \frac{A(w(x,y))}{A(w_2(x,y))}.w_2(x,y) \in S$. This implies that

$$\langle w(x,y) - \frac{A(w(x,y))}{A(w_2(x,y))}, w_2(x,y)\rangle = 0.$$

We then set

$$w_u(x,y) = \frac{A(w_2(x,y))}{\|w_2(x,y)\|^2}.w_2(x,y),$$

then we obtain

$$\langle w(x,y), w_u(x,y)\rangle = A(w(x,y)).$$

By contradiction, suppose there are two solutions $w_{u1}(x,y)$ and $w_{u2}(x,y)$ satisfy the first condition of the Riesz representation theorem, then

$$\Big\langle w(x,y), w_{u1}(x,y) - w_{u2}(x,y)\Big\rangle = 0.$$

This implies that

$$w_{u1}(x,y) = w_{u2}(x,y).$$

Thus, $w_u(x,y)$ is unique.

Finally, we show that

$$\|A\|_{\Omega^*} = \|w_u(x,y)\|.$$

To see this, we set

$$A = 0.$$

This follows from the previous result. We set $w_u(x,y)$. If $\|w(x,y)\| \leq 1$, then

$$\begin{aligned} \|A(w(x,y))\| &= \left|\Big\langle w(x,y), w_u(x,y)\Big\rangle\right| \\ &\leq \|w(x,y)\|\|w_u(x,y)\| \\ &\leq \|w_u(x,y)\|. \end{aligned}$$

Thus,

$$\|A\|_{\Omega^*} \leq \|w_u(x,y)\|.$$

In addition, we see that

$$\|\frac{w_u(x,y)}{\|w_u(x,y)\|}\| = 1,$$

and

$$\left|A\Big(\frac{w_u(x,y)}{\|A\|}\Big)\right| = \left|\Big\langle \frac{w_u(x,y)}{\|w_u(x,y)\|}, w_u(x,y)\Big\rangle\right|$$
$$= \|w_u(x,y)\|.$$

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Hence,

$$|A|_{\Omega^*} = \|A\|.$$

The homeomorphism of the A from Ω into $\mathbb{R}^2$, which implies well-posedness of the Helmholtz equation with imposed boundary conditions.

Definition 1.4 *Let X and Y be two normed spaces. An operator* $A : X \to Y$ *is said to be homeomorphism if A is continuous from X to Y and the inverse operator* $A^{-1} : Y \to X$ *is continuous (Oden, 1979).*

1.2 Helmholtz Equation with Different Boundary Conditions

In this section, we discuss three different kinds of ill-posedness of Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous, with both Cauchy and Dirichlet boundary conditions where the boundary deflection is homogeneous and finally, when the equation is imposed with Neumann boundary conditions in the upper half-plane. In summary, the mixed boundary conditions, as well as Neumann boundary conditions are imposed on Helmholtz equation.

Definition 1.5 (Boundary of the Domain) *Let* Ω *be a domain in a Hilbert space H. A point* $(x_1,y_1) \in H$ *is called a boundary point of* Ω *if every neighbourhood of* (x_1,y_1) *intersects both* Ω *and its complement*

$$\Omega^c = H \setminus \Omega.$$

The set of all boundary points of Ω is called the boundary of Ω and is denoted by ∂Ω. (Akcoglu et al., 2009).

1.2.1 Helmholtz Equation with Cauchy Boundary Conditions where the Boundary Deflection is Inhomogeneous

In this subsection, we show that when Cauchy boundary conditions are imposed on homogeneous Helmholtz equation where boundary deflection is not equal to zero, then the equation has no solution. The Cauchy problem of the Helmholtz equation where the boundary deflection is not equal to zero is as follows.

$$\begin{aligned} \frac{\partial^2 w(x,y)}{\partial x^2} &+ \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) = 0, \quad 0 \le x \le \frac{\pi}{2}, \quad 0 \le y \le 2\pi \\ \frac{\partial w(x,0)}{\partial y} &= \frac{1}{n}\sin(nx), \quad 0 \le x \le \frac{\pi}{2} \qquad (1.3) \\ w(x,0) &= 0, \quad 0 \le x \le \frac{\pi}{2} \end{aligned}$$

By the method of separation of variables, we obtain

$$w(x,y) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{(n^2-k^2)}} \sin(nx)\sinh(\sqrt{(n^2-k^2)}y).$$

For the above function $w(x,y)$ to be called a solution to equation (1.3) together with Cauchy boundary conditions it must satisfy the smoothness requirement condition as well as data compatibility condition. In equation (1.3), we can see that the integral of the boundary deflection ______ $\frac{\partial w}{\partial y}(x,0)$ over $[0,\frac{\pi}{2}]$ is

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx = \frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})]$$

Thus,

$$\frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})] = \begin{cases} \frac{1}{n^2}, & \forall\, n = 1,3,5,\ldots \\ \frac{2}{n^2}, & \forall\, n = 2,6,10,\ldots \\ 0, & \forall\, n = 4,8,12,\ldots \end{cases}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx \neq 0, \quad \forall\, n = odd \text{ } or \text{ } n = 2,6,\ldots$$

Equation (1.3) does not satisfy the data compatibility condition. This implies that the function

$$w(x,y) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2-k^2}} \sin(nx)\sinh(\sqrt{n^2-k^2}\,y), \qquad \forall\, n \in \mathbb{I}^+$$

is not a solution of equation (1.3). Hence, equation (1.3) is ill-posed in the sense of Hadamard.

1.2.2 Helmholtz Equation with Neumann Boundary Conditions in the Upper Half-plane

In this subsection, we show that the Neumann problem on the upper half-plane for the Helmholtz equation has solution but not unique. We provide the rigorous proof for our claim. We impose Neumann boundary conditions on homogeneous Helmholtz equation on the upper half-plane as follows:

$$\begin{aligned}
&\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) = 0, && -1 \le x \le 1,\ 0 \le y \le 1\\
&\frac{\partial w(-1,y)}{\partial x} = 0 = \frac{\partial w(1,y)}{\partial x}, && 0 \le y \le 1\\
&\frac{\partial w(x,0)}{\partial y} = 0, && -1 \le x \le 1\\
&\frac{\partial w(x,1)}{\partial y} = \cos(2\pi x), && -1 \le x \le 1
\end{aligned} \tag{1.4}$$

Using the method separation of variables, we obtain the following result from equation (1.4) as follows:

$$w(x,y) = X(x)Y(y) \tag{1.5}$$

$$X''(x)Y(y) + X(x)Y''(y) + k^2X(x)Y(y) = 0$$

$$\frac{Y''(y) + k^2Y(y)}{Y(y)} = \frac{-X''(x)}{X(x)} = \lambda, \tag{1.6}$$

$Y(y)$

where $\lambda = constant > 0$. From equation (1.6), we obtain

$$X^0(x) = -\sqrt{\lambda}A_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}A_2 \cos(\sqrt{\lambda}x)$$

$$X^{00}(x) + \lambda X(x) = 0$$

$$X(x) = A_1 \cos(\sqrt{\lambda}x) + A_2 \sin(\sqrt{\lambda}x) \quad (1.7)$$

When $X_0(-1) = 0$, we obtain

$$X^0(-1) = -\sqrt{\lambda}A_1 \sin(\sqrt{\lambda}.-1) + \sqrt{\lambda}A_2 \cos(\sqrt{\lambda}.-1) = 0 \Rightarrow A_1 \sin(\sqrt{\lambda}) + A_2 \cos(\sqrt{\lambda}) = 0 \quad (1.8)$$

When $X_0(1) = 0$, we obtain

$$X^0(1) = -\sqrt{\lambda}A_1 \sin(\sqrt{\lambda}.1) + \sqrt{\lambda}A_2 \cos(\sqrt{\lambda}.1) = 0 \Rightarrow -A_1 \sin(\sqrt{\lambda}) + A_2 \cos(\sqrt{\lambda}) = 0 \quad (1.9)$$

Summing equation (1.8) and equation (1.9) gives

$$2A_2 \cos(\sqrt{\lambda}) = 0$$

For trivial solution, we let

$$\cos(\sqrt{\lambda}) = 0, \quad A_2 = 0 6$$

$$\lambda = \left(\frac{n\pi}{2}\right)^2, \quad n = 1,3,5,... \quad (1.10)$$

are eigenvalues. Substituting equation (1.10) into equation (1.7) yields

$$X(x) = A_1 \cos\left(\frac{n\pi x}{2}\right) + A_2 \sin\left(\frac{n\pi x}{2}\right)$$

$$X_n(x) = \cos(\frac{2}{2}\_\_\_) + \sin(\frac{2\, n\pi x \;\; n\pi x}{2}\_\_\_) \qquad n = 1,3,5,... \qquad (1.11)$$

are the eigenfunctions which correspond to eigenvalues $\lambda = (\frac{n\pi}{2})^2$.

Also, subtracting equation (1.9) from equation (1.8) yields

$$\sqrt{2}A_1 \sin(\lambda) = 0$$

$$\Rightarrow \sin(\sqrt{\lambda}) = 0, \qquad A_1 \neq 0$$

$$\lambda = (n\pi)^2, \qquad n = 0,1,2,3,... \qquad (1.12)$$

are eigenvalues. Substituting equation (1.12) into equation (1.7) yields

$$X(x) = A_1 \cos(n\pi x) + A_2 \sin(n\pi x)$$

$$X_n(x) = \cos(n\pi x) + \sin(n\pi x), \qquad n = 0,1,2,3,... \qquad (1.13)$$

are the eigenfunctions which correspond to eigenvalues $\lambda = (n\pi)^2$. Again,

in equation (1.6), we obtain

$$Y(y) = B_1 \cosh(\sqrt{[\lambda^2 - k^2]}\, y) + B_2 \sinh(\sqrt{[\lambda^2 - k^2]}\, y)$$

When $Y'(0) = 0$, we obtain

$$Y(y) = B_n \cosh(\sqrt{(\lambda^2 - k^2)}y) \qquad (1.14)$$

Substituting equations (1.11) and (1.14) into equation (1.5) yields

$$w_1(x,y) = \sum_{n=1,3,5}^{\infty} c_n \cosh(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y)(\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})),$$

$$\frac{\partial w(x,y)}{\partial y} = \sum_{n=1,3,5}^{\infty} \sqrt{((\frac{n\pi}{2})^2 - k^2)}.c_n \sinh(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y)(\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})) \Rightarrow$$

$$\frac{\partial w(x,1)}{\partial y} = \sum_{n=1,3,5}^{\infty} \sqrt{((\frac{n\pi}{2})^2 - k^2)}.c_n \sinh(\sqrt{((\frac{n\pi}{2})^2 - k^2)}.1)(\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2}))$$

$$= \cos(2\pi x)$$

$$\sqrt{((\frac{n\pi}{2})^2 - k^2)}.c_n \sinh(\sqrt{((\frac{n\pi}{2})^2 - k^2)})(\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2}))$$

$$\sum_{n=1,3,5} \ldots = \cos(2\pi x).$$

Since the eigenfunctions

$$X_n(x) = (\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2}))$$

are orthogonal under the inner product, we obtain

$$\sqrt{((\frac{n\pi}{2})^2 - k^2)}.c_n \sinh(\sqrt{((\frac{n\pi}{2})^2 - k^2)}) \times \int_{-1}^{1} [\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})]\,[\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})]\, dx$$

$$= \int_{-1}^{1} \cos(2\pi x)[\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})]\, dx$$

$$\Rightarrow \sqrt{((\frac{n\pi}{2})^2 - k^2)}.c_n \sinh(\sqrt{((\frac{n\pi}{2})^2 - k^2)}) \times \{\int_{-1}^{1} [\cos^2(\frac{n\pi x}{2}) + \sin^2(\frac{n\pi x}{2})]\, dx + 2\int_{-1}^{1} \cos(\frac{n\pi x}{2})\sin(\frac{n\pi x}{2})dx\}$$

$$= \int_{-1}^{1} \cos(2\pi x)\cos(\frac{n\pi x}{2})dx + \int_{-1}^{1} \cos(2\pi x)\sin(\frac{n\pi x}{2})dx. \tag{1.15}$$

We observe that

$$\int_{-1}^{1} \cos(\frac{n\pi x}{2}) \sin(\frac{n\pi x}{2})dx = 0 = \int_{-1}^{1} \cos(2\pi x) \sin(\frac{n\pi x}{2})dx, \quad \forall n = 1, 2, \ldots$$

and

$$\sqrt{((\frac{n\pi}{2})^2 - k^2)}.c_n \sinh(\sqrt{((\frac{n\pi}{2})^2 - k^2)})\int_{-1}^{1} 1\,dx = \int_{-1}^{1} \cos(2\pi x)\cos(\frac{n\pi x}{2})dx.$$

$$c_1 = \begin{cases} \dfrac{2n\pi}{(n^2-16)\pi^2.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{((\frac{n\pi}{2})^2-k^2)})} & \forall n = 1,5,\ldots \\ \dfrac{-2n\pi}{(n^2-16)\pi^2.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{((\frac{n\pi}{2})^2-k^2)})} & \forall n = 3,7,\ldots \end{cases} \quad (1.16)$$

Substituting for c_1, we obtain

$$w_1(x,y) = \begin{cases} \sum_{n=1,5,}^{\infty} \frac{2n\pi.\cosh(\sqrt{((\frac{n\pi}{2})^2-k^2)}y)(\cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2}))}{(n^2-16)\pi^2.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{((\frac{n\pi}{2})^2-k^2)})}, & n = 1,5,\ldots \\ \sum_{n=3,7,}^{\infty} \frac{-2n\pi.\cosh(\sqrt{((\frac{n\pi}{2})^2-k^2)}y)(\cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2}))}{(n^2-16)\pi^2.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{((\frac{n\pi}{2})^2-k^2)})}, & n = 3,7,\ldots \end{cases} \quad (1.17)$$

When

$$n = 0 \quad \Rightarrow \lambda = 0.$$

We obtain

$$Y_o(y) = c_o\cos(ky) + c_{o1}\sin(ky)$$

$$Y_o^0(y) = -kc_o\sin(ky) + kc_{o1}\cos(ky)$$

When $Y_0(0) = 0$, we obtain

$$\Rightarrow c_{01} = 0$$

Substituting equations (1.12), (1.13) and (1.15) into equation (1.5) yields

$$\begin{aligned} w_2(x,y) &= c_0\cos(ky) + \sum_{n=1}^{\infty} c_n\cosh(\sqrt{((n\pi)^2-k^2)}y))(\cos(n\pi x)+\sin(n\pi x)) \\ \frac{\partial w(x,1)}{\partial y} &= -kc_0\sin(k.1) \\ &+ \sum_{n=1}^{\infty}\sqrt{((n\pi)^2-k^2)}.c_n\sinh(\sqrt{((n\pi)^2-k^2)}.1))(\cos(n\pi x)+\sin(n\pi x)) \\ &\Rightarrow -kc_o\sin(k) + \sum_{n=1}^{\infty}\sqrt{((n\pi)^2-k^2)}.c_n\sinh(\sqrt{((n\pi)^2-k^2)}))(\cos(n\pi x)+\sin(n\pi x)) \\ &= \cos(2\pi x) \qquad (1.18) \end{aligned}$$

Since the eigenfunctions

$$X_n(x) = (\cos(n\pi x) + \sin(n\pi x))$$

are orthogonal under the inner product, equation (1.18) becomes

$$\sqrt{((n\pi)^2-k^2)}.c_n\sinh(\sqrt{[(n\pi)^2-k^2]})\times\int_{-1}^{1}[\cos(n\pi x)+\sin(n\pi x)][\cos(n\pi x)+\sin(n\pi x)]\,dx$$

$$=\int_{-1}^{1}\cos(2\pi x)[\cos(n\pi x)+\sin(n\pi x)]\,dx$$

$$\Rightarrow\sqrt{((n\pi)^2-k^2)}.c_n\sinh(\sqrt{((n\pi)^2-k^2)})\left\{\int_{-1}^{1}1dx+2\int_{-1}^{1}\cos(n\pi x)\sin(n\pi x)dx\right\}$$

$$=\int_{-1}^{1}\cos(2\pi x)\cos(n\pi x)dx+\int_{-1}^{1}\cos(2\pi x)\sin(n\pi x)dx$$

But we observe that

$$\int_{-1}^{1}\cos(n\pi x)\sin(n\pi x)dx=0=\int_{-1}^{1}\cos(2\pi x)\sin(n\pi x)dx,\quad \forall n=1,2,\ldots$$

which implies that

$$\Rightarrow c_2=\frac{1}{2\sqrt{((n\pi)^2-k^2)}.\sinh(\sqrt{((n\pi)^2-k^2)})} \tag{1.19}$$

Substituting c_2 into $w_2(x,y)$, we obtain

$$w_2(x,y)=\sum_{n=1}^{\infty}\frac{\cosh(\sqrt{((n\pi)^2-k^2)}y))(\cos(n\pi x)+\sin(n\pi x))}{2\sqrt{((n\pi)^2-k^2)}.\sinh(\sqrt{((n\pi)^2-k^2)})} \tag{1.20}$$

By equations (1.16) and (1.20), we obtain

$$w(x,y)=\begin{cases}\sum_{n=1,5,9,\ldots}^{\infty}\frac{2n\pi.\cosh(\sqrt{((\frac{n\pi}{2})^2-k^2)}y)(\cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2}))}{(n^2-16)\pi^2.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{((\frac{n\pi}{2})^2-k^2)})}, \\ \sum_{n=3,7,11,\ldots}^{\infty}\frac{-2n\pi.\cosh(\sqrt{((\frac{n\pi}{2})^2-k^2)}y)(\cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2}))}{(n^2-16)\pi^2.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{((\frac{n\pi}{2})^2-k^2)})} \\ \sum_{n=1}^{\infty}\frac{\cosh(\sqrt{((n\pi)^2-k^2)}y))(\cos(n\pi x)+\sin(n\pi x))}{2\sqrt{((n\pi)^2-k^2)}.\sinh(\sqrt{((n\pi)^2-k^2)})}\end{cases} \tag{1.21}$$

We can see that all the coefficients of partial derivatives appearing in equation (1.4) are continuously differentiable and the expression on the right hand side of equation (1.4) is zero which continuous. Thus, equation (1.4) meets the smoothness requirement condition. In addition, we observe that all the boundary conditions are zero except $\frac{\partial w(x,1)}{\partial y} = \cos(2\pi x)$. We can see that

$$\int_{-1}^{1} \cos(2\pi x)dx = 0$$

This implies that the function that appears in equation (1.21) is a solution of equation (1.4).

1.2.2.1 Uniqueness of Solution to Helmholtz Equation with Neumann Boundary Conditions in the Upper Half-plane

In this subsection, we show that equation (1.4) has more than one solution in $L^2([-1,1]\times[0,1])$. Let

Proof: By contradiction, Let $u(x,y)$ and $v(x,y)$ be two different solutions of equation (1.4) such that

$$w(x,y) = u(x,y) - v(x,y).$$

Multiplying both sides of equation (1.4) by $w(x,y)$ and integrating over $([-1,1]\times[0,1])$, we obtain

$$\int_0^1 \int_{-1}^1 w(x,y)\Delta w(x,y)dxdy + k^2 \int_0^1 \int_{-1}^1 |w(x,y)|^2 dxdy = 0$$

Applying the Green's first identity to the first term on the left hand side of the above equation, we obtain

$$\int_0^1 \int_{-1}^1 w(x,y)\Delta w(x,y)dxdy = 0 - \int_0^1 \int_{-1}^1 |\nabla w(x,y)|^2 dxdy$$

$$\Rightarrow \int_0^1\int_{-1}^1 w(x,y)\Delta w(x,y)\,dxdy + k^2\int_0^1\int_{-1}^1 |w(x,y)|^2\,dxdy = k^2\int_0^1\int_{-1}^1 |w(x,y)|^2\,dxdy - \int_0^1\int_{-1}^1 |\nabla w(x,y)|^2\,dxdy = 0$$

$$\Rightarrow \quad k^2\int_0^1\int_{-1}^1 |w(x,y)|^2\,dxdy - \int_0^1\int_{-1}^1 |\nabla w(x,y)|^2\,dxdy = 0$$

The above equation holds if $w(x,y)$ is zero in the domain $([-1,1] \times [0,1])$. But, we can see that

$$w(x,y) \neq 0 \qquad \forall\, x,y \in ([-1,1] \times [0,1]).$$

Thus, the solution $w(x,y)$ in equation (1.21) alternates sign from negative to positive in the domain $([-1,1] \times [0,1])$. This implies that the equation(1.4) has more than one solution. Hence, equation (1.4) is ill-posed in the sense of Hadamard.

1.2.2.2 Stability of Solution to Helmholtz Equation with Neumann Boundary Conditions in the Upper Half-plane

In this subsection, we show that solutions of the same Helmholtz equation with small changes in boundary conditions that are close to each other remain close for some values of x. In equation (1.4), we choose $x_1 = \epsilon$, where $0 < \epsilon \ll \frac{1}{2}$, in the initial deflection as given below.

$$
\begin{aligned}
\frac{\partial^2 w(x,y)}{\partial x^2} &+ \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) = 0, \quad -1 \leq x \leq 1, \;\; 0 \leq y \leq 1 \\
\frac{\partial w(-1,y)}{\partial x} &= \frac{\partial w(1,y)}{\partial x} = 0, \qquad\qquad 0 \leq y \leq 1 \\
\frac{\partial w(x,0)}{\partial y} &= 0 \qquad\qquad -1 \leq x \leq 1 \\
\frac{\partial w(\epsilon,1)}{\partial y} &= \cos(2\pi\epsilon) \qquad\qquad -1 \leq x \leq 1
\end{aligned}
$$

Equations (1.16) and (1.20) become

$$
\begin{aligned}
&\sqrt{((\frac{n\pi}{2})^2 - k^2)}.c_n \sinh(\sqrt{((\frac{n\pi}{2})^2 - k^2)})\{\int_{-1}^{1} 1dx + 2\int_{-1}^{1} \cos(\frac{n\pi x}{2})\sin(\frac{n\pi x}{2})dx\} \\
= &\int_{-1}^{1} \cos(2\pi\epsilon)\cos(\frac{n\pi x}{2})dx + \int_{-1}^{1} \cos(2\pi\epsilon)\sin(\frac{n\pi x}{2})dx \\
and \quad &\sqrt{((n\pi)^2 - k^2)}.c_n \sinh(\sqrt{((n\pi)^2 - k^2)})\{\int_{-1}^{1} 1dx + 2\int_{-1}^{1} \cos(n\pi x)\sin(n\pi x)dx\} \\
= &\int_{-1}^{1} \cos(2\pi\epsilon)\cos(n\pi x)dx + \int_{-1}^{1} \cos(2\pi\epsilon)\sin(n\pi x)dx,
\end{aligned}
$$

respectively. The corresponding solution is

$$
w_1(x,y) = \begin{cases}
\sum_{n=1,5,9,\dots}^{\infty} \frac{2\cos(2\pi\epsilon).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2 - k^2]}\ y)[\ \cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{((\frac{n\pi}{2})^2 - k^2)}\sinh(\sqrt{[\ (\frac{n\pi}{2})^2 - k^2]}\ )}, \\
\sum_{n=3,7,11,\dots}^{\infty} \frac{-2\cos(2\pi\epsilon).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2 - k^2]}\ y)[\ \cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{[\ (\frac{n\pi}{2})^2 - k^2]}\ \sinh(\sqrt{[\ (\frac{n\pi}{2})^2 - k^2]}\ )}, \\
0, & n = 2, 4, \dots \\
0, & n = 1, 2, \dots
\end{cases}
$$

Again, we perturb from $x_1 = \epsilon$ to $x_2 = \delta$, where 0$< \delta \ll \frac{1}{2},\ \delta > \epsilon$. We obtain the following results:

∂x^2 ∂y^2 − ≤ ≤ ≤ ≤ $\partial_2 w(x,y)$ +

$\partial w(-1,y)$ $\partial w(1,y)$ $\partial_2 w(x,y)$ + $k_2 w(x,y) = 0$,

1 x 1, 0 y 1

= = 0, $0 \leq y \leq 1$

∂x ∂x

∂y

$\partial w(\delta,1)$ − ≤ ≤

$\partial w(x,0)$ ——— = 0 1 x 1 ∂y − ≤ ≤

= $\cos(2\pi\delta)$ 1 x 1

and the corresponding solution is given below

∞ 2cos(2πδ).cosh((( ) −k)y)[cos(

)+sin( )],

P$P_{n=1,5,9,\ldots}$ nπ.√((√ nπ √ 2)nπ√22−nπ2 2k²2nπ)2sinh2 22(√2[(√nπ2nπx)nπ222−2knπx22])2 nπx2 nπx2

∞n=3,7,11,... −2cos(2πδnπ.). cosh((()[(−k))sinh−k (] y)][[(cos() −k)+])sin(

)],

$w_2(x,y)$ =

0, n = 2,4,...

0, n = 1,2,...

The change in the boundary deflection $\frac{\partial w}{\partial y}(x,0)$ is as follows

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\partial w_1(\epsilon,1)}{\partial y}-\frac{\partial w_2(\delta,1)}{\partial y}\right| &= \lim_{n\to\infty}|\cos(2\pi\epsilon)-\cos(2\pi\delta)| \\ &\leq \lim_{n\to\infty}(\|\cos(2\pi\epsilon)\|+\|\cos(2\pi\delta)\|)\end{aligned}$$

But

$$\begin{aligned}\cos(2\pi\epsilon) &\leq 1 \\ \cos(2\pi\delta) &\leq 1\end{aligned}$$

$$
\begin{aligned}
\lim_{n\to\infty}|\frac{\partial w_1(\epsilon,1)}{\partial y}-\frac{\partial w_2(\delta,1)}{\partial y}| &\leq \lim_{n\to\infty}(1+1)\\
&= 2\\
\therefore \lim_{n\to\infty}|\frac{\partial w_1(\epsilon,1)}{\partial y}-\frac{\partial w_2(\delta,1)}{\partial y}| &\leq 2
\end{aligned}
$$

This is implies that there is a small change in boundary deflection. The corresponding change in the solution $w(x,y)$ is

$$
\begin{aligned}
&\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)|\\
=\ &\lim_{n\to\infty}|\{\sum_{n=1,5,9,\ldots}^{\infty}\frac{2\cos(2\pi\epsilon).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)[\ \cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )}\\
-\ &\sum_{n=3,7,11,\ldots}^{\infty}\frac{2\cos(2\pi\epsilon).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)[\ \cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )}\}\\
-\ &\{\sum_{n=1,5,9,\ldots}^{\infty}\frac{2\cos(2\pi\delta).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)[\ \cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )}\\
-\ &\sum_{n=3,7,11,\ldots}^{\infty}\frac{2\cos(2\pi\delta).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)[\ \cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )}\}|\\
\leq\ &\lim_{n\to\infty}\|\sum_{n=1,3,5,\ldots}^{\infty}\frac{2\cos(2\pi\epsilon).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)[\ \cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )}\|\\
+\ &\|\sum_{n=1,3,5,\ldots}^{\infty}\frac{2\cos(2\pi\delta).\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)[\ \cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2})]}{n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )}\|\\
\leq\ &\lim_{n\to\infty}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\|2\cos(2\pi\epsilon)\|.\|\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)\|[\ \|\cos(\frac{n\pi x}{2})\|+\|\sin(\frac{n\pi x}{2})\|]}{\|n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\|\|\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )\|}\\
+\ &\sum_{n=1,3,5,\ldots}^{\infty}\frac{\|2\cos(2\pi\delta)\|.\|\cosh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ y)\|[\ \|\cos(\frac{n\pi x}{2})\|+\|\sin(\frac{n\pi x}{2})\|]}{\|n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}\|\|\sinh(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )\|}\\
\leq\ &\lim_{n\to\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{8e^{(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )y}}{n\pi.\sqrt{((\frac{n\pi}{2})^2-k^2)}e^{(\sqrt{[\ (\frac{n\pi}{2})^2-k^2]}\ )}}
\end{aligned}
$$

∞ 8

$_n$lim$_{\to\infty}$ ||w_1(.,1) −− w_2(.,1)| ≤| →$_n$lim$_{\to\infty}$ X,3→ ∞,... $n\pi$.p(($\frac{n\pi}{2}$)2 − k_2)

n=1

lim w_1(.,1) w_2(.,1) 0 *as n* . ^{n}→∞

This implies that a small change in the boundary deflection $\frac{\partial w}{\partial y}(x,1)$ from $x_1 = \epsilon$ to $x_2 = \delta$ results in a small change in solution. Thus, the solution (1.21) to equation (1.4) is stable. The equation (1.4) violates the second condition of well-posedness.

Hence, equation (1.4) is ill-posed in the sense of Hadamard.

1.2.3 Helmholtz Equation with both Dirichlet and Cauchy Boundary Conditions where the Boundary Deflection is Zero

In this subsection, we show that when Helmholtz equation is imposed with both Dirichlet and Cauchy boundary condition where the boundary deflection is homogeneous then the equation has one solution, but this solution does not depend continuously on the changes in the boundary deflection of the Cauchy boundary conditions. The Dirichlet and Cauchy problem of the Helmholtz equation where the boundary deflection is homogeneous is given below:

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) = 0, \quad 0 \le x \le 2\pi,\ 0 \le y \le 1 \tag{1.22}$$

$$w(0,y) = w(2\pi,y) = 0, \quad 0 \le y \le 1$$

$$w(x,0) = \frac{1}{n}\sin(nx), \quad 0 \le x \le 2\pi$$

$$\frac{\partial w(x,0)}{\partial y} = 0, \quad 0 \le x \le 2\pi,$$

By the method of separation of variables, we obtain

$$w(x,y) = \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n} \tag{1.23}$$

We can see that equation (1.22) satisfies smoothness requirement. Also, we observe that the boundary deflection condition is

$$\frac{\partial w(x,0)}{\partial y} = 0$$

This implies that the function that appears in equation (1.23) is a solution of equation (1.22). In order to show that the function in (1.23) is the only solution to equation (1.22), for example, see [42].

We then show that solutions of the same Helmholtz equation with small changes in boundary conditions that are close to each other remain close for some values of x. In equation (1.22), we choose $x_1 = \epsilon$ in boundary condition $w(\epsilon, 0) = \frac{1}{n}\sin(n\epsilon)$, where $0 < \epsilon \ll \frac{\pi}{12}$ and the corresponding solution is as follows:

$$w(x,y) = \sum_{n=1,3...}^{\infty} \frac{4\sin(n\epsilon)\cosh\left(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n^2\pi}$$

Again, we perturb from $x_1 = \epsilon$ to $x_2 = \delta$ in the boundary condition $w(\delta,0) = \frac{1}{n}\sin(n\delta)$, where $0 < \delta \ll \frac{\pi}{36}$ and the corresponding solution is given as follows:

$$w(x,y) = \sum_{n=1,3...}^{\infty} \frac{4\sin(n\delta)\cosh\left(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n^2\pi}$$

The change in the boundary deflection is as follows:

$$\begin{aligned}\lim_{n\to\infty} |w(\epsilon,0) - w(\delta,0)| &= \lim_{n\to\infty} |\frac{1}{n}\sin(n\epsilon) - \frac{1}{n}\sin(n\delta)| \\ &\leq \lim_{n\to\infty} \frac{1}{n}(\|\sin(n\epsilon)\| + \|\sin(n\delta)\|) \\ &\leq \frac{2}{n}\end{aligned}$$

$$\frac{2}{n} \to 0 \;\; as \;\; n \to \infty$$

This is implies that there is a small change in the boundary deflection.

The corresponding change in the solution $w(x,y)$ is

$$\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| = \lim_{n\to\infty}|\sum_{n=1,3...}^{\infty}\frac{4\sin(n\epsilon)\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)\sin(\frac{nx}{2})}{n^2\pi}$$

$$- \sum_{n=1,3,...}^{\infty}\frac{4\sin(n\delta)\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)\sin(\frac{nx}{2})}{n^2\pi}|$$

$$\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| \leq 4\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{\|\sin(n\epsilon)-\sin(n\delta)\|\|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)\|\|\sin(\frac{nx}{2})\|}{\|n^2\pi\|}$$

$$\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| \leq 8\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{e^{(\sqrt{((\frac{n}{2})^2-k^2)}y)}}{n^2\pi}$$

$$\lim_{n\to\infty}|w_1(.,1)-w_2(.,1)| \leq 8\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{e^{\sqrt{((\frac{n}{2})^2-k^2)}}}{n^2\pi}$$

We observe that the numerator $8e^{\sqrt{((\frac{n}{2})^2-k^2)}}$ grows faster than n^2, which in turn, produces a large growth in

$$\frac{8e^{\sqrt{((\frac{n}{2})^2-k^2)}}}{n^2\pi}$$

Thus,

$$\lim_{n\to\infty}|w_1(.,1)-w_2(.,1)| \to \infty \text{ as } n\to\infty.$$

This implies that a small change in the boundary deflection $w(x,0)$ from $x_1 = 0$ to $x_2 = \delta$ results in a large change in solution. Thus, the solution (1.23) to equation (1.22) is unstable. The equation (1.22) violates the third condition of well-posedness. Hence, equation (1.22) is ill-posed in the sense of Hadamard.

1.3 Statement of Problem

We showed that Helmholtz equation with Cauchy boundary conditions where boundary deflection is inhomogeneous has no solution. Thus, all the three requirements of existence, uniqueness and continuous dependence of small changes in the Cauchy boundary conditions do not hold. Hence, the Helmholtz equation is ill-posed in the sense of Hadamard. In addition, the Neumann problem in the upper half-plane has no more than one solution. Thus, second condition of well-posedness is violated. Revealing the literature, we observed that the existing methods of regularization can be used to restore the well-posedness of the Helmholtz equation with Cauchy boundary conditions, as well as Neumann problem in the upper half-plane. These existing methods of regularization are insufficient and inefficient for solving ill-posed Helmholtz equation with imposed Cauchy boundary condition where boundary deflection is inhomogeneous as well as Neumann boundary conditions in the upper half-plane.

1.4 Objectives of Study

In this thesis, we introduce Divergence Regularization Method (DRM) for regularizing Cauchy problem of Helmholtz equation, where the boundary deflection is inhomogeneous. The Helmhotz equation occurs in laser beam models, vibrating membrane problem. To ensure the existence of a solution, the DRM incorporates a positive integer scaler which homogenizes the inhomogeneous boundary deflection in the Cauchy problem of the Helmholtz equation. By this method , the uniqueness of the of solution for the regularized problem is guaranteed. Furthermore, the DRM employs its regularization term $(1 + \alpha^{2m})e^{m}$ to restore the stability of the solution of the regularized Helmholtz equation. Nevertheless, the DRM guarantees the uniqueness of solution of Helmholtz equation when it is imposed by Neumann boundary conditions in the
upper half-plane.

Also, in order to solve Helmholtz equation in irregular domain, we introduce an Adaptive Wavelet Spectral Finite Difference (AWSFD) method to obtain the approximated solutions of the regularized Helmholtz equation with Cauchy boundary conditions, then with regularized Neumann boundary conditions on the upper plane and finally with regularized Dirichlet and regularized Cauchy conditions where the boundary deflection is equal to zero.

1.5 Organization of Thesis

In chapter 1, we discuss the ill-posed homogeneous Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous, with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous and then finally with Neumann boundary conditions in upper half-plane. The type of ill-posedness associated with Helmholtz equation together with each of these boundary conditions is presented by following the conditions of ill-posedness given by Hadamard. This chapter also deals with introduction, objectives, relevance and organization the the work.

In chapter 2, we discuss available literature on regularization of ill-posed Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous, and with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous and then with Neumann boundary conditions in the upper half-plane. Thus, we present the problems for the Helmholtz equation which cannot be solved by the existing methods of regularization as well as the problem for the Helmholtz equation which can be solved by these methods. In addition, any existing method of regularization which does not give minimal error in regularizing the problem for Helmholtz equation will be given.

The DRM is introduced for solving ill-posed Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous, then with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous, and finally with Neumann boundary condtions in the upper half-plane is discussed in chapter 3. Then we apply the DRM to solve equations (1.3) and (1.22). In this chapter, we make use of shift operator on the $x-$ spatial variable instead of homogenization of boundary deflection in the DRM to solve Neumann problem of the Helmholtz equation in the upper half-plane. For each regularized Helmholtz equation with regularized boundary conditions, we show that the solution exists, is unique and depends continuously on the small changes in the imposed boundary conditions in Hilbert space.

Chapter four of this thesis contains is the Adaptive Wavelet Spectral Finite Difference (AWSFD) method for approximation of solution of regularized Helmholtz equation together with above boundary conditions. In addition, an interpolation scheme is introduced for the AWSFD method. Comparison of the solutions of the regularized Helmholtz equation by DRM and by AWSFD method are provided in this chapter. Comparison of results are done for regularized Helmholtz equation with the same regularized boundary conditions. Also, we compare the results by DRM with the results by the existing methods of regularization. Thus, the strength of the DRM is discussed.

Chapter 5 contains a summary of main the findings, conclusions and suggestions for future research.

Chapter 2

Existing Methods of Regularization of Helmholtz Equation and Their Demerits

In this chapter, we discuss the methods for regularizing Helmholtz equation with both Dirichlet and Cauchy boundary conditions where the boundary deflection is equal to zero, Cauchy boundary conditions where the boundary deflection is not equal to zero and Neumann boundary conditions in the upper half-plane. The ill-posedness of Helmholtz equation comes as a result of boundary conditions imposed on it. The existing methods of regularization such as Tikhonov regularization method (Tikhonov, 1963), spectral regularization method (Xiong and Fu, 2007), quasi-reversibility method (Lattes and Lions, 1967), quasi-boundary value method (Clark and Oppenheimer, 1994) and an iterative regularization method (Cheng et al., 2014), are well-known for restoration of well-posedness of Helmholtz equation. These existing regularization methods regularize only Helmholtz equation or the boundary conditions or the solution.

Existing methods of regularization cannot restore the well-posedness of Helmholtz equation when Cauchy boundary conditions where boundary deflection not at zero are imposed on it. We give rigorous proof of how each of these methods works and also show where each of the methods fails to regularize Neumann problem for the Helmholtz equation in the upper half-plane.

2.1 Tikhonov Regularization Method

In this section, we consider the Tikhonov regularization method (TRM) and show how the TRM can be applied in regularizing the class of boundary conditions, which when imposed on the Helmholtz equation and also, the class of boundary conditions, which when imposed on the Helmholtz equation that cannot be solved by TRM (Tikhonov,
1963).

The following theorem and definitions are useful in applying the TRM to regularize Helmholtz equation with boundary conditions. The boundedness of Laplace operator in the Helmholtz equation A ensures the range of A is closed and the null space of A is trivial.

Definition 2.1 (Boundedness) *Let H be a Hilbert space. A (linear) Laplace operator* $A : \Omega \subset H \to H$ *is said to be bounded provided there is a constant* $M > 0$ *for which*

$$\|A(w(x,y))\|_H \leq M\|w(x,y)\|_\Omega, \qquad \forall\, w(x,y) \in \Omega.$$

The infimum of all such M is called the operator norm of A and is denoted by $\|A\|$. *The collection of bounded linear Laplace operators from* Ω *to Y is denoted by* $L(\Omega,H)$ *(Royden and Fitzpatrick, 2010).*

We then state the bounded inverse theorem (BIT) that ensures that a bounded below Laplace operator in the Helmholtz equation with boundary conditions has a continuous inverse operator. The proof of the BIT requires the definition of bounded below Laplace operator. We state them below.

Definition 2.2 (Bounded below operator) *The Laplace operator* $A : \Omega \subset H \to$ *H is bounded below if and only if there exists a constant* $C > 0$ *such that*

$$\|Aw(x,y)\|_H \geq C\|w(x,y)\|_\Omega, \qquad \forall\, w(x,y) \in \Omega$$

(Oden, 1979).

Definition 2.3 (Compact operator) *Let X and Y be two normed spaces. An operator A : $X \to Y$ is said to be compact if it maps bounded sets in X into relatively compact sets in Y (Chidume, 1989).*

Theorem 2.1 (Bounded Inverse Theorem) *Let A be a bounded linear below Laplacetype operator in the Helmholtz equation from a subspace Ω in a Hilbert space H into a Hilbert space H. Then A has a continuous inverse operator A^{-1} from its range $R(A)$ into Ω. Conversely, if there is a continuous inverse operator*

$$A^{-1} : R(A) \to \Omega,$$

then there is a positive constant C such that

$$\|Aw(x,y)\|_H \geq C\|w(x,y)\|_\Omega, \qquad \forall\ w(x,y) \in \Omega$$

(Oden, 1979).

Proof : Suppose that A is bounded below. If an inverse Laplace-type operator exists, then A is an injective operator of Ω onto its range. By definition (2.2), we see that it is certainly onto $R(A)$. The Laplace-type operator is injective if its range,

$$R(A) = 0.$$

This follows from definition (2.2). To see this, let

$$Aw_1(x,y) = Aw_2(x,y) = v,$$

then

$$\begin{aligned} \|Aw_1(x,y) - Aw_2(x,y)\|_H &= 0 \\ &\geq C\|w_1(x,y) - w_2(x,y)\|_\Omega. \end{aligned}$$

This implies that

$$w_1(x,y) = w_2(x,y).$$

Thus, A^{-1} exists on $R(A)$. To show that A^{-1} is continuous from the $R(A)$ to Ω, we see that

$$\begin{aligned} \|A^{-1}v(x,y)\|_\Omega &= \|w(x,y)\|_\Omega \\ &\leq \frac{1}{C}\|Aw(x,y)\|_H \\ &= \frac{1}{C}\|v(x,y)\|_H. \end{aligned}$$

Hence A^{-1} is bounded.

Assume that A^{-1} exists and is continuous on $R(A)$. Then there is a constant $C > 0$ such that

$$\|A^{-1}v(x,y)\|_\Omega \leq \frac{1}{C}\|v(x,y)\|_H.$$

Setting

$$v(x,y) = Aw(x,y)$$

shows that A is bounded below. The BIT encompasses both definitions (2.1) and (2.2). Thus, the BIT is useful is studying the three well-posedness conditions given by Hadamard. In this section, the BIT is used in studying the stability of unique solution of the Helmholtz equation. For the sake of brevity, we will use theorem 1.2 in the studying the existence of the solution to the Helmholtz equation.

The Tikhonov regularization method is also called variational regularization method [65]. Using the TRM, we assume that there is a linear bounded operator

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

from a Hilbert space X into another Hilbert space Y . We regularize the Laplace operator, where $x,y \in D(A)$, that appears in the Helmholtz equation. Thus, we put constraints on both the (exact) solution and on the variations of the data function in the Helmholtz equation so as to control the blow-up of the data function from its emanated errors (Tikhonov and Arsenin, 1977).

Let consider the inhomogeneous operator Helmholtz equation as:

$$Aw(x,y) = f(x),$$

where $w(x,y)$ is the unknown solution in X. Based on the principles of TRM, we formulate the above equation as:

$$\|Cw(x,y)\|_Y^2 = \sum_{r=0}^{\gamma} \iint_{\Omega} c_r(x,y)|w^{(r)}(x,y)|^2 dxdy,$$

where C is the constraint (linear) operator from X to Y , c_r are weights (positive functions) and $w^{(r)}(x,y)$ denotes the *rth* partial order derivative of $w(x,y)$. In order to solve the above equation, we combine two minimal conditions; the Gauss least-squares method (LSM) and the Moore-Penrose pseudo-inverse matrix method (PIMM). Using the Lagrange method of undetermined multipliers, we obtain

$$\|Aw(x,y) - f(x)\|^2_Y + \alpha\|Cw(x,y)\|^2_Y = \min_{x,y}, \tag{2.1}$$

where $\alpha > 0$ is a regularization parameter, is called Euler-Tikhonov equation (Tikhonov, 1963).

We minimize the equation (2.1), which means that the derivative of the discrepancy

$$\|Aw(x,y) - f(x)\|^2_Y + \alpha\|Cw(x,y)\|^2_Y$$

with respect to the spatial variable y and equate it to zero, which yields regularized solution

$$w_\alpha(x,y) = (A^*A + \alpha I)^{-1}A^*f(x),$$

where

$$C^*C = I,$$

I is a unit operator from a Hilbert X to another Hilbert space Y and $w_\alpha(x,y)$ is the regularized solution. As the regularization parameter increases, the discrepancy increases and the normed solution decreases. Thus, the normed solution becomes more and more stable until a moderate α is obtained. The regularization parameter is chosen in a posteriori way (Engl et al., 1990).

There are different ways of selecting α in equation (2.1). The regularization parameter can be selected in a manner that among the whole set of solutions $w_\alpha(x,y)$, the

$$\|Aw(x,y) - f(x)\|_Y \leq \varepsilon,$$

where ε are errors in setting the boundary conditions on the Helmholtz equation. We find the solution $w(x,y)$ that minimizes

$$\|Cw(x,y)\|.$$

Thus, we search for a unique α that solves

$$\|Aw_\alpha(x,y) - f(x)\|_Y = \varepsilon.$$

This is called the Morozov's discrepancy principle (Morozov, 1966). But if, the regularization parameter is chosen in a way that the minimal solution satisfies

$$\|Aw_\alpha(x,y) - f(x)\|_Y \leq \varepsilon.$$

and

$$\|Cw(x,y)\| \leq E,$$

where E is the priori bounded solution. Thus, in this case, we choose

$$\alpha = (\frac{\varepsilon}{E})^2$$

(Miller, 1970). Moreover, we choose

$$\alpha = 0$$

then equation (2.1) becomes

$$kAw(x,y) - f(x)k^2_Y = \min_{x,y}.$$

The TRM coincides with Gauss LSM. In this case, the solution to the Helmholtz equation becomes unstable. Thus, the TRM fails to regularize the Helmholtz equation with boundary conditions (Petrov and Sizikov, 2005). Using the TRM, in [29], they showed that the conditions that guarantee the convergence rate for the regularized solution of the Helmholtz equation. Moreover, their result confirmed the compactness of the nonlinear Laplace operator in the Helmholtz equation in order to ensure the stability of its solution.

A closely related approach of the TRM is one given by [32,35,57,84]. In this method, A is a linear, self-adjoint and compact Laplace-type operator from $L^2[a,b] \to L^2[a,b]$ for $c \leq y \leq d$. The operator equation is formulated from a known unstable unique solution $w(x,y)$ of the Helmholtz equation with given boundary conditions as

$$A(y)w(x,y) = \varphi(x), \qquad c \leq y \leq d, \tag{2.2}$$

where $\varphi(x) \in Y$ is the (noisy) boundary condition at x−axis. Then we find the expression of the inner product of $w(x,y)$ and the eigenfunctions X_n of the solution. The express for the Laplace-type operator is then obtained from above two equations and use it to form Euler-Tikhonov equation. We solve Euler-Tikhonov equation by assuming that

$$A^* = A,$$

see [67], which yields regularized operator equation

$$\alpha I w_\alpha(x,y) + A^* A w_\alpha(x,y) = A^* \varphi(x). \tag{2.3}$$

We then solve the above equation to obtain the regularized solution below:

$$w_\alpha(x,y) = (\alpha I + A^1)^{-1} A\varphi(x) \tag{2.4}$$

Unlike the approach by Tikhonov (1963), this method works for both homogeneous and inhomogeneous Helmholtz equation. In [84], they applied the TRM to regularize the Cauchy problem of the homogeneous linear elliptic partial differential equation with variable coefficients where the boundary deflection is homogeneous.

Recently, [17] modified TRM by considering the variables in three different spaces. In their method, we define

$$S : D(S) \to U$$

denotes the operator from the parameter space to state space, and

$$B : U \to Z,$$

is a linear bounded operator from the state to the corresponding data function. We assume that the operator from the parameter to state is well-posed whiles the from the state to the data function is ill-posed. In this case, we minimize the quadratic functional

$$\|Aw(x,y) - f(x)\|^2_Z + \alpha\|S(y) - \bar{U}\|^2_U = \min(x,y),$$

all the variables and parameter have usual meanings except U being the parameter space.

We show the class of boundary conditions, which when imposed on the Helmholtz equation is regularized by TRM. Using the approach by [57], we write equation (1.22) as

$$w(x,y) = \sum_{n=1}^{\infty} \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\langle\phi(x), X_n\rangle X_n ,$$

where,

[1]

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[1] $X_n = \sqrt{\frac{1}{\pi}}\sin(\frac{nx}{2}), \quad \forall\ 0 \le x \le 2\pi.$

$$\varphi(x) = \sin(\quad)$$
n 2

We obtain both the operator Helmholtz equation and the regularized Helmholtz equation as

$$A(y)w(x,y) = \frac{1}{n}\sin(\frac{nx}{2}), \quad 0 \le y \le 1.$$

and

$$\alpha I w_\alpha(x,y) + A^2 w_\alpha(x,y) = A \times \frac{1}{n}\sin(\frac{nx}{2}), \tag{2.5}$$

respectively. We see from equation (1.22), that the inner product of $w(x,y)$ and X_n is

$$\langle w(x,y), X_n\rangle = \langle \phi(x), X_n\rangle \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)$$

From the operator and above equation, we obtain

$$A(y)w(x,y) = \sum_{n=1}^{\infty} \frac{\langle w(x,y), X_n\rangle X_n}{\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)}$$

and

$$A(y) = \frac{1}{\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)}.$$

Using assumption by [67], and substituting $A(y)$ and $\varphi(x)$ into equation (2.4), we obtain the regularized solution as:

$$w_\alpha(x,y) = \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n(1 + \alpha\cosh^2\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right))} \tag{2.6}$$

We observe that equation (2.5) satisfies both the smoothness requirement in definition (1.2) and data compatibility condition in theorem (1.2), respectively. Therefore, the function (2.6) is a solution to equation (2.5).

We show that the function (2.6) is the only solution to equation (2.5).

Proof: By contradiction, Let $u(x,y)$ and $v(x,y)$ be two different solutions of equation (2.5) such that

$$w(x,y) = u(x,y) - v(x,y),$$

Multiplying both sides of equation (2.5) by the solution (2.6) and integrating over ([0,2π] × [0,1]), we obtain

$$\int_0^1\int_0^{2\pi} \alpha I|w_\alpha(x,y)|^2dxdy + \int_0^1\int_0^{2\pi} w_\alpha(x,y)\Delta^2 w_\alpha(x,y)dxdy - \frac{1}{n}\int_0^1\int_0^{2\pi} w_\alpha(x,y)\Delta\sin(\frac{nx}{2})dxdy = 0$$

Applying the Green's first identity to the second and third terms on the left hand side of the above equation, we obtain

$$\int_0^1\int_0^{2\pi} \alpha I|w_\alpha(x,y)|^2dxdy + \int_0^1\int_0^{2\pi} w_\alpha(x,y)\nabla^3 w_\alpha(x,y)dxdy - \int_0^1\int_0^{2\pi}|\Delta w_\alpha(x,y)|^2dxdy + \frac{2}{n}\int_0^1\int_0^{2\pi} w_\alpha(x,y).\nabla\cos(\frac{nx}{2})dxdy - \frac{2}{n^2}\int_0^1\int_0^{2\pi} \nabla w_\alpha(x,y).\nabla\cos(\frac{nx}{2})dxdy = 0$$

Since the $\nabla\cos(\frac{nx}{2})$ vanishes on both the boundary and in the domain ([0,2π]×[0,1]). The fourth and fifth terms of the above equation are zero, we have

$$\Rightarrow \int_0^1\int_0^{2\pi} \alpha I|w_\alpha(x,y)|^2dxdy + \int_0^1\int_0^{2\pi} w_\alpha(x,y)\nabla^3 w_\alpha(x,y)dxdy - \int_0^1\int_0^{2\pi}|\Delta w_\alpha(x,y)|^2dxdy = 0$$

We observe that the solution (2.6) $w_\alpha(x,y) = 0 \; \forall \; x,y$ *on* $\partial([0,2\pi] \times [0,1])$ *and* x,y

in $([0,2\pi] \times [0,1])$.

This implies that solution (2.6) is unique to equation (2.5).

We show that the regularized solution (2.6) is stable with respect to changes in the boundary condition. We observe that if $x_1 = \epsilon$ in $w(\epsilon, 0) = \frac{1}{n}\sin(n\epsilon)$, where $0 < \epsilon \ll \frac{\pi}{12}$ and the corresponding solution is

$$w_1(x,y) = \sum_{n=1}^{\infty} \frac{-4\sin(n\epsilon) n \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right) \sin(\frac{nx}{2})}{n\left(1 + \alpha\cosh^2\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right)\right)}$$

But, when x is perturbed from $x_1 = \epsilon$ to $x_2 = \delta$, where $0 < \delta \ll \frac{\pi}{24}$, the corresponding solution is obtained as

$$w_2(x,y) = \sum_{n=1}^{\infty} \frac{-4\sin(n\delta) n \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right) \sin(\frac{nx}{2})}{n\left(1 + \alpha\cosh^2\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right)\right)}$$

The change in boundary condition is as follows:

$$\begin{aligned} \lim_{n\to\infty} |w(x_1,0) - w(x_2,0)| &= \lim_{n\to\infty} |\frac{1}{n}\sin(n\epsilon) - \frac{1}{n}\sin(n\delta)| \\ &\leq \frac{2}{n} \\ \therefore |w(x_1,0) - w(x_2,0)| &\rightarrow 0 \;\; as \;\; n \rightarrow \infty \end{aligned}$$

This implies that there is a small change in the boundary condition. The corresponding change in the solution is as:

$$\begin{aligned} &\lim_{n\to\infty} |w_1(x,y) - w_2(x,y)| = \\ &\lim_{n\to\infty} |\sum_{n=1}^{\infty} \frac{-4\sin(n\epsilon) n \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right) \sin(\frac{nx}{2})}{n\left(1 + \alpha\cosh^2\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right)\right)} \\ &- \sum_{n=1}^{\infty} \frac{-4\sin(n\delta) n \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right) \sin(\frac{nx}{2})}{n\left(1 + \alpha\cosh^2\left(\sqrt{((\frac{n}{2})^2 - k^2)} y\right)\right)}| \\ &\lim_{n\to\infty} |w_1(x,y) - w_2(x,y)| \rightarrow 0 \;\; as \;\; n \rightarrow \infty. \end{aligned}$$

Hence, the operator equation (2.3) is well-posed in the sense of Hadamard.

We now consider the Helmholtz equation with both Neumann and Cauchy boundary conditions as follows:

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\begin{aligned}
\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) &= 0, \quad 0 \le x \le 2\pi, \quad 0 \le y \le 1 \\
\frac{\partial w(0,y)}{\partial x} = \frac{\partial w(2\pi,y)}{\partial x} &= 0, \quad 0 \le y \le 1 \qquad (2.7)\\
w(x,0) &= \frac{1}{n}\sin(nx), \quad 0 \le x \le 2\pi \\
\frac{\partial w(x,0)}{\partial y} &= 0, \quad 0 \le x \le 2\pi,
\end{aligned}
$$

Using a similar approach to the both Cauchy and Dirichlet problems of the Helmholtz except that we replace

$$X_n = \sqrt{\frac{2}{\pi}}\sin\left(\frac{nx}{2}\right), \quad \forall\ 0 \le x \le 2\pi$$

with

$$X_n = \sqrt{\frac{2}{\pi}}\cos\left(\frac{nx}{2}\right), \quad \forall\ 0 \le x \le 2\pi.$$

Using the approach by [57], we obtain

$$w_\alpha(x,y) = \sum_{n=odd}^{\infty} \frac{n\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\cos(\frac{nx}{4})}{n\left(1 + \alpha\cosh^2\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\right)}$$

All the three requirements for well-posedness of equation (2.7) are the same as the one shown above. Hence, equation (2.7) is well-posed in the sense of Hadamard.

Thirdly, we consider the Cauchy problem of the Helmholtz equation where the boundary deflection is homogeneous as follows:

$$\begin{aligned} &\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) = 0, \quad 0 \le x \le \frac{\pi}{2}, \; 0 \le y \le 1 \\ &w(x,0) = \frac{1}{n}\sin(nx), \quad 0 \le x \le \frac{\pi}{2} \\ &\frac{\partial w(x,0)}{\partial y} = 0, \quad 0 \le x \le \frac{\pi}{2} \end{aligned} \qquad (2.8)$$

Using the approach by [57], we obtain

$$w(x,y) = \frac{n\cosh\left(\sqrt{(n^2-k^2)}y\right)\sin(nx)}{n(1+\alpha\cosh^2\left(\sqrt{(n^2-k^2)}y\right))}. \qquad (2.9)$$

We observe that regularized equation (2.2) satisfies smoothness requirement in definition (1.2). On the data compatibility condition, we observe that

$$\int_0^{\frac{\pi}{2}} \frac{\partial w(x,0)}{\partial y}\,dx = \int_0^{\frac{\pi}{2}} 0\,dx = 0$$

a constant, which satisfies it. This implies that the solution (2.9) is a solution to equation (2.8).

The proof that equation (2.8) with the boundary conditions has only one solution, follows the proof of uniqueness of equation (2.5).

The stability of equation (2.8) to small changes in boundary deflection is similar to the stability of equation (2.4). Hence, equation (2.5) is well-posed in the sense of Hadamard.

On the contrary, we show a class of boundary conditions which when imposed on Helmholtz equation cannot be solved (regularized) by TRM. Firstly, we apply the TRM to equation (1.3) as follows:

$$A(y)w(x,y) = \varphi(x), \qquad 0 \le y \le 2\pi.$$

$$\Rightarrow \alpha I w_\alpha(x,y) + A^* A w_\alpha(x,y) = A^* \varphi(x). \tag{2.10}$$

and

$$\langle w(x,y), X_n \rangle = \langle \phi(x), X_n \rangle \sinh\left(\sqrt{(n^2 - k^2)}y\right),$$

where,

$$\varphi(x) = \frac{1}{\imath} \sin(nx)$$

$$X_n = \frac{1}{\sqrt{n^2 - k^2}} \sin(nx),$$

and $A(y)$ and I have usual meanings. From the operator and above equation, we obtain

$$A(y)w(x,y) = \sum_{n=1}^{\infty} \frac{\langle w(x,y), X_n \rangle X_n}{\sinh\left(\sqrt{(n^2 - k^2)}y\right)}$$

and

$$A(y) = \frac{1}{\sinh\left(\sqrt{(n^2 - k^2)}y\right)}.$$

Using assumption by [67], and substituting $A(y)$ and $\varphi(x)$ into equation (2.4), we obtain the regularized solution as:

$$w_\alpha(x,y) = \sum_{n=1}^{\infty} \frac{\sinh\left(\sqrt{n^2-k^2}y\right)\sin(nx)}{n\left(1+\alpha\sinh^2\left(\sqrt{(n^2-k^2)}y\right)\right)},$$

α has the usual meaning.

Moreover, we can see from boundary deflection condition of equation (2.10) that

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx = \frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})]$$

Thus,

$$\frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})] = \begin{cases} \frac{1}{n^2}, & \forall\, n = 1,3,\ldots \\ \frac{2}{n^2}, & \forall\, n = 2,6,\ldots \\ 0, & \forall\, n = 4,8,\ldots \end{cases}$$

Thus, $$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx \neq 0, \quad \forall\, n = odd\ or\ n = 2,6,\ldots$$

Equation (2.10) does not satisfy the data compatibility condition, theorem (1.2). Thus, the TRM does not homogenize the boundary deflection in the Cauchy boundary conditions imposed on the Helmholtz equation. This implies that the function

$$w_\alpha(x,y) = \sum_{n=1}^{\infty} \frac{\sinh\left(\sqrt{n^2-k^2}y\right)\sin(nx)}{n\left(1+\alpha\sinh^2\left(\sqrt{(n^2-k^2)}y\right)\right)},$$

is not a solution of equation (2.8). Equation (2.10) has no solution. Hence, equation (2.8) is ill-posed in the sense of Hadamard.

Also, we apply the TRM to regularize equation (1.4). The regularized operator equation (2.3) becomes

$$\Rightarrow \alpha I w_\alpha(x,y) + A^*Aw_\alpha(x,y) = A^*\cos(2\pi x), \qquad (2.11)$$

the varibles and parameter have usual meanings. The inner products of the exact solution and X_n are

$$\langle w_1(x,y), X_n\rangle = \langle \phi(x), X_n\rangle c_1 \cosh\left(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y\right),$$

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where,

$$X_n = \cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2}),$$

and c_1 are provided in equation (1.16), and

$$\langle w_2(x,y), X_n\rangle = \langle \phi(x), X_n\rangle c_2 \cosh\left(\sqrt{((n\pi)^2 - k^2)}y\right),$$

where,

$$X_n = \cos(n\pi x) + \sin(n\pi x),$$

and c_2 are provided in equation (1.19). The operators are then obtained as

$$A(y) = \frac{1}{c_1 \cosh\left(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y\right)}$$

and

$$A(y) = \frac{1}{c_2 \cosh\left(\sqrt{((n\pi)^2 - k^2)}y\right)}.$$

respectively, and substitute each equation into equation (2.4) which yields the regularized solution of the Helmholtz equation as:

$$w_\alpha(x,y) = \begin{cases} \sum_{n=odd}^{\infty} \frac{c_1 \cosh(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y)\left(\cos(\frac{n\pi x}{2}) + \sin(\frac{n\pi x}{2})\right)}{1 + \alpha \cosh^2\left(\sqrt{((\frac{n\pi}{2})^2 - k^2)}y\right)} \\ \sum_{n=1}^{\infty} \frac{c_2 \cosh(\sqrt{((n\pi)^2 - k^2)}y)(\cos(n\pi x) + \sin(n\pi x))}{1 + \alpha \cosh^2(\sqrt{((n\pi)^2 - k^2)}y)} \end{cases} \quad (2.12)$$

We observe that the boundary conditions of equation (2.11) satisfy both the smoothness requirement in definition (1.3) and data compatibility condition in theorem (1.1), respectively. Therefore, the function (2.12) is a solution to equation (2.11).

We now show that the equation (2.11) has more than one solution.

Proof: By contradiction, Let $u(x,y)$ and $v(x,y)$ be two different solutions of equation (2.11) such that

$$w(x,y) = u(x,y) - v(x,y).$$

Multiplying both sides of equation (2.11) by the solution (2.12), using condition by [67] and integrating over $([-1,1] \times [0,1])$, we obtain

$$\int_0^1\int_{-1}^1 \alpha I|w_\alpha(x,y)|^2 dxdy + \int_0^1\int_{-1}^1 w_\alpha(x,y)\Delta^2 w_\alpha(x,y)dxdy - \int_0^1\int_{-1}^1 w_\alpha(x,y)\Delta\cos(2\pi x)dxdy = 0$$

Applying the Green's first identity to the second and third terms on the left hand side of the above equation, we obtain

$$\int_0^1\int_{-1}^1 \alpha I|w_\alpha(x,y)|^2 dxdy + \int_0^1\int_{-1}^1 w_\alpha(x,y)\nabla^3 w_\alpha(x,y)dxdy - \int_0^1\int_{-1}^1 |\Delta w_\alpha(x,y)|^2 dxdy$$

$$-\frac{1}{2\pi}\int_0^1\int_{-1}^1 w_\alpha(x,y).\nabla\sin(2\pi x)dxdy + \frac{1}{2\pi}\int_0^1\int_{-1}^1 \nabla w_\alpha(x,y).\nabla\sin(2\pi x)dxdy = 0$$

$$\Rightarrow \int_0^1\int_{-1}^1 \alpha I|w_\alpha(x,y)|^2 dxdy - \int_0^1\int_{-1}^1 |\Delta w_\alpha(x,y)|^2 dxdy = 0$$

The above equation holds if $w_\alpha(x,y)$ is zero in the domain $([-1,1] \times [0,1])$. But, we observe that

$$w_\alpha(x,y) \neq 0 \qquad \forall\, x,y \in ([-1,1] \times [0,1]).$$

Thus, the regularized solution $w_\alpha(x,y)$ alternates sign from negative to positive in the domain ([−1,1] × [0,1]). In the first expression solution (2.12), the sin($\frac{n\pi x}{2}$)changes sign from positive to negative and also, cos($2n\pi$) in the second expression in the same solution (2.12), which give non-value in the domain ([−1,1] × [0,1]). This implies that the equation (2.11) has more than one solution. The second condition of well-posednes is violated. Hence, the Helmholtz equation with Neumann boundary conditions in an upper half-plane is ill-posed in the sense of Hadamard.

2.2 Spectral Regularization Method

A closely related approach of the TRM is the spectral regularization method (SRM) [81]. In this method, we assume that there is a bounded self-adjoint Laplace-type operator

$$A : X \rightarrow Y,$$

with A^{-1} a self-adjoint inverse operator, which solves equation (2.2). In this case, the norm of A^{-1} and the noise level δ is observed as:

$$|(\delta, A^{-1})| = \sup\{\|A^{-1}\phi^{\delta}(x) - w(x,y)\| \,\Big|\, w(x,y) \in M, \phi^{\delta}(x) \in Y, \|Aw(x,y) - \phi^{\delta}(x)\| \leq \delta\},$$

where $M \in X$ is a bounded set, $\varphi(x)$ and $\varphi^{\delta}(x)$ are noise-free and noisy boundary conditions which satisfies

$$\|\varphi^{\delta}(x) - \varphi(x)\| \leq \delta.$$

To use SRM to regularize Helmholtz equation with boundary conditions, we give a definition of compact inverse operator, which will be useful in studying the stability of the unique solution to the Helmholtz equation with boundary conditions.

Definition 2.4 (Compactness) *Let a linear self-adjoint Laplace-type operator in the Helmholtz equation*

$$A : X \rightarrow Y,$$

where X and Y are Hilbert spaces. Let M be any bounded set in Y, A^{-1} is bounded if,

$$\|A^{-1}w_1(x,y)\| \leq \|A^{-1}\||w_1(x,y)|, \qquad \forall\, w_1(x,y) \in Y.$$

The set $A^{-1}(M)$ is bounded in Y . Hence, A^{-1} is compact in Y (Attouch et al., 2005).

Unlike the TRM, we restore the stability of unique solution to Helmholtz equation by A^{-1} is compact on $M \in Y$.

In order to regularize equation (2.2) using the SRM, we restore the boundedness of A^{-1}. Thus, the exact (unstable) solution has to satisfy the source condition as follows. Let

$$M_{\varphi,E} = \left\{ w(x,y) \in X \middle| w(x,y) = [\varphi(A^*A)]^{\frac{1}{2}} v, \|v\| \le E \right\}$$

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be a source set, where the operator function

$$\varphi(A^*A) = \int_0^a \varphi(\lambda) dE_\lambda,$$

the spectral decomposition of

$$A^*A = \int_0^a \lambda dE_\lambda,$$

the constant

$$a \ge \|A^*A\|,$$

$\{E_\lambda\}$ denotes a set of the spectral of the A^*A and E is a priori bounded solution which satisfies

$$\|w(.,0)\|_X \le E.$$

Moreover, the following assumptions are made in using SRM in regularizing equation (2.2). The function

$$\phi : (0,a] \to (0,\infty)$$

is continuous and satisfies

$$\lim_{\lambda \to 0} \phi(\lambda) = 0,$$

$\phi(\lambda)$ is monotonically increasing on $(0,a]$, and

$$\phi(\lambda) = \lambda\phi^{-1}(\lambda) : (0,\phi(a)] \to (0,a\phi(a)]$$

is convex. We obtain the regularized form of equation (2.2) as:

$$A^{-1} = \varphi_\alpha(A^*A)A^*, \tag{2.13}$$

where $\varphi_\alpha(\lambda)$ satisfies

$$\lim_{\alpha \to 0} \phi_\alpha(\lambda) = \frac{1}{\lambda},$$

α is the regularization parameter. We solve equation (2.12) to obtain the regularized solution

$w_\alpha(x,y) = A^{-1}\varphi^\delta(x)$ (2.14) (Xiong and Fu, 2007).

The SRM can be classified into four types. They are spectral method 1, which is defined as:

$$\phi_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha \\ \frac{1}{\alpha}, & \lambda < \alpha \end{cases}$$

and the second spectral method is given by

$$\phi_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha \\ \frac{1}{\sqrt{\alpha\lambda}}, & \lambda < \alpha \end{cases}$$

In the SRM, the spectral method 3 is also called Tikhonov regularization and singular value decomposition (TSVD) method, which is given by

$$\phi_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha \\ 0, & \lambda < \alpha \end{cases}$$

and the fourth method is called Tikhonov regularization method

$$\phi_\alpha(\lambda) = \frac{1}{(\alpha + \lambda)}.$$

In [73], they extended the SRM to the Neumann problem of the Laplace equation. In [31], they used pseudo-differential operator instead of Laplace operator in regularizing the solution of the Helmholtz equation. In [19,54,55,59], the authors applied the SRM to regularize the Cauchy problem of the elliptic equations where the boundary deflection is homogeneous.

Next, we apply the spectral method 1 to regularize Helmholtz equation. The spectral methods 2 and 3 are similar to the spectral method 1. The spectral method 4 (TRM) has been discussed above. We regularize equation (1.22) by applying SRM, which gives regularized solution

$$w(x,y)=\begin{cases}\sum_{n=1}^{\infty}\frac{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)\sin(\frac{nx}{2})}{n}, & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)|^2}\geq\alpha\\ \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{n\sin(\frac{nx}{2})}{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)}, & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)|^2}<\alpha\end{cases}$$

45 (2.15) We observe that function (2.15) satisfies smoothness requirement in definition (1.2) as well as data compatibility condition in theorem (1.2). This implies that the function (2.15) is a solution to equation (2.13). The proof of uniqueness of regularized solution (2.15) is similar to that of equation (2.5).

We prove the stability of solution (2.15) to equation (2.13) as follows. In equation (1.22), we choose $x_1=\epsilon$, where $0<\epsilon\ll\frac{\pi}{24}$ in the boundary condition $w(x,0)=\frac{1}{n}\sin(nx)$ and the corresponding solution is

$$w_1(x,y)=\begin{cases}\sum_{n=1,3}^{\infty}\frac{4\sin(n\epsilon)\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)\sin(\frac{nx}{2})}{n^2\pi} & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)|^2}\geq\alpha\\ \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{n^2\pi\sin(n\epsilon)\sin(\frac{nx}{2})}{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)}, & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)|^2}<\alpha\end{cases}$$

We perturb $w(x,y)$ from $x_1=\epsilon$ to $x_2=\delta$, where $0<\delta\ll\frac{\pi}{12}$ and $<\delta$. The corresponding solution is:

$$w_2(x,y)=\begin{cases}\sum_{n=1,3}^{\infty}\frac{4\sin(n\delta)\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)\sin(\frac{nx}{2})}{n^2\pi} & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)|^2}\geq\alpha\\ \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{n^2\pi\sin(n\delta)\sin(\frac{nx}{2})}{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)}, & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)|^2}<\alpha\end{cases}$$

The change in boundary condition is observed as:

$$\lim_{n\to\infty}|w(x_1,0)-w(x_2,0)| = \lim_{n\to\infty}|\frac{1}{n}\sin(n\epsilon)-\frac{1}{n}\sin(n\delta)|$$
$$\therefore |w(x_1,0)-w(x_2,0)| \to 0 \ \ as \ \ n\to\infty$$

is small.

Also, the corresponding change in the solution $w(x,y)$ for $\frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)y}\right)|^2}<\alpha$

is as follows:

$$
\begin{aligned}
\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| &= \lim_{n\to\infty}|\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{n^2\pi\sin(n\epsilon)\sin(\frac{nx}{2})}{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)} \\
&- \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{n^2\pi\sin(n\delta)\sin(\frac{nx}{2})}{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)}| \\
\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| &\leq 2\lim_{n\to\infty}|\frac{1}{\alpha}|\sum_{n=1}^{\infty}\frac{|n^2\pi|\|\sin(n\delta)\|\|\sin(\frac{nx}{2})\|}{\|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)\|} \\
\lim_{n\to\infty}|w_1(.,1)-w_2(.,1)| &\leq 2\lim_{n\to\infty}\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{n^2\pi}{e^{\sqrt{((\frac{n}{2})^2-k^2)}}} \\
\lim_{n\to\infty}|w_1(.,1)-w_2(.,1)| &\to 0 \;\; as \;\; n\to\infty.
\end{aligned}
$$

This implies that a small change in the boundary deflection $w(x,0)$ from $x_1 = \epsilon$ to $x_2 = \delta$ results in a small change in solution. Hence, using the SRM, equation (1.22) is well-posed in the sense of Hadamard.

We apply the SRM to regularize equation (2.7) as:

$$
w(x,y) = \begin{cases} \sum_{n=odd}^{\infty}\frac{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)\cos(\frac{nx}{4})}{n}, & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)|^2}\geq\alpha \\ \frac{1}{\alpha}\sum_{n=odd}^{\infty}\frac{n\cos(\frac{nx}{4})}{\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)}, & \frac{n}{|\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)|^2}<\alpha. \end{cases}
$$

All the three requirements for well-posedness of equation (2.7) are the same as the one shown above. Hence, by SRM, equation (2.7) is well-posed in the sense of Hadamard. Thirdly, we apply SRM to regularize equation (2.8) to obtain equation (2.13). This gives the regularized solution as:

$$
w(x,y) = \begin{cases} \frac{1}{n}\sin(nx)\cosh\left(\sqrt{(n^2-k^2)}y\right), & \frac{1}{|\cosh\left(\sqrt{(n^2-k^2)}y\right)|^2}\geq\alpha \\ \frac{1}{\alpha n}\frac{\sin(nx)}{\cosh\left(\sqrt{(n^2-k^2)}y\right)}, & \frac{1}{|\cosh\left(\sqrt{(n^2-k^2)}y\right)|^2}<\alpha, \end{cases} \tag{2.16}
$$

We observe that equation (2.16) satisfies both the smoothness requirement in definition (1.3). On the data compatibility condition, we observe that

$$\int_0^{\frac{\pi}{2}} \frac{\partial w(x,0)}{\partial y} dx = \int_0^{\frac{\pi}{2}} 0 dx = 0$$

which implies that equation (2.16) is the solution to equation (2.8).

The proof that regularized equation (2.13) has only one solution, follows the proof of uniqueness of equation (2.5). In addition, the stability of equation (2.13) is similar to the stability of solution (2.16). Hence, equation (2.13) is well-posed by SRM.

We show the kinds of boundary conditions, which when imposed on Helmholtz equation cannot be regularized by the SRM. By the applying SRM to equation (1.3), we obtain regularized equation (2.13), where

$$\phi_\alpha(x) = \frac{1}{n}\sin(nx),$$

which yields the regularized solution

$$w(x,y) = \begin{cases} \frac{1}{n}\sin(nx)\sinh\left(\sqrt{(n^2-k^2)}y\right), & \frac{1}{|\sinh\left(\sqrt{(n^2-k^2)y}\right)|^2} \geq \alpha \\ \frac{1}{\alpha n}\sinh\left(\sqrt{(n^2-k^2)}y\right), & \frac{1}{|\sinh\left(\sqrt{(n^2-k^2)y}\right)|^2} < \alpha \end{cases}$$

We observe that

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx = \frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})]$$

which implies that

$$\frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})] = \begin{cases} \frac{1}{n^2}, & \forall\ n = 1,3,5,\ldots \\ \frac{2}{n^2}, & \forall\ n = 2,6,10,\ldots \\ 0, & \forall\ n = 4,8,12,\ldots \end{cases}$$

Thus,

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx \neq 0, \quad \forall\, n = odd\ or\ n = 2, 6, \ldots$$

Since equation (2.13) with above boundary condition does not satisfy data compatibility condition. This implies that equation (2.13) with above boundary condition has no solution. Hence, by SRM, equation (2.13) with above boundary condition is ill-posed in the sense of Hadamard.

Next, we apply the SRM to equation (1.4), we obtain the regularized equation (2.13) with

$$\varphi_\alpha(x) = \cos(nx)$$

and the corresponding regularized solution as:

PP∞ n=1,5,... 2(nπ.n2−cosh(16)π2.√√((((nπ2nπnπ2 2)2)−2 2k−2k)2y2))(sinhcos((nπx√2nπx((2)+nπ2sin)2(−nπxk2nπx22))))' $|\tau_1 1|^2 \geq \alpha$

∞n=3,7,... −(2nnπ.2−cosh(16)π2.√((((nπ2))2−−kk)2y))(sinhcos((√(()+nπ2 sin)2−(k2)))), $|\tau_1 1|^2 \geq \alpha$

$w(x,y) =$ PP∞ n=1,5,... α(n22−16)π22.√√((nπnπ22)22 2−nπ.k22)(cos sinh((nπx√√nπx2 2(()+nπnπ2 2 sin)22(−nπxnπx2 k222)))cosh(√√((nπnπ22)22−k22)y), $|\tau_1 1 1|^2 2 < \alpha$

∞ −2nπ.(cos( )+sin( )) , 1 < α

P n=3,7,...√ α√(n −2 16)22π .2 ((√) −k2)sinh2 (((√) −k))2 cosh2 ((( ) −k)y)1 2 |τ |

$\sum_{n=1}^{\infty} \cosh(2\sqrt{(((n\pi n\pi)2)--kk2).)\sinh(y))(\cos\sqrt{(((n\pi xn\pi)+)2-\sin k2(n\pi x)))}))$, $|\tau_1 \underline{1}|_2 \geq \alpha$

$\sum_{n=1}^{\infty}$ ———2((nπ) −k).sinh((cos(((nπxnπ)+) −sink()))nπxcosh)) (((nπ) −k)y)), $|\tau \underline{1}| < \alpha$ (2.17)

where,

$$\tau_1 = \cosh(\sqrt{((\frac{n\pi}{2})^2 - k^2)y)}.$$

We observe that equation (2.13) together with above boundary condition satisfies both the smoothness requirement in definition (1.2) and data compatibility condition in theorem (1.2),respectively. Therefore, the function (2.17) is a solution to equation (2.13) with above boundary condition.

The proof that equation (2.13) with above boundary conditions has more than one solution is similar to that of equation (2.11). Hence, by SRM, equation (2.13) with above boundary condition is ill-posed in the sense Hadamard.

2.3 Quasi-Reversibility Regularization Method

In this section, we discuss the strength and shortcomings of quasi-reversibility regularization method (Q-RRM). The Q-RRM assumes that a linear operator

$$A : X \to Y$$

where

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is bijective but the inverse operator A^{-1} is not continuous from one Hilbert space H to another Hilbert space H (Lattes and Lions, 1967).

The Q-RRM regularizes only the Helmholtz equation by the subtraction of a product of a square of a regularization parameter α and a mixed fourth-order partial derivative from the Laplace-type operator appearing in the Helmholtz equation. Thus,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \alpha^2 \frac{\partial^4}{\partial x^2 \partial y^2}$$

For example, see [62,63,79]. In [13,14,16,41,58] applied Q-RRM to regularize the solution of Helmholtz equation. In [56], the authors showed that the operator A_* is a unitary from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ to regularize the Helmholtz equation with Cauchy boundary conditions.

We demonstrate a class of boundary conditions which when imposed on Helmholtz equation is regularized by the Q-RRM. Applying Q-RRM to equation (1.22), we obtain the regularized Helmholtz equation as follows:

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) - \alpha^2 \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} = 0, \quad 0 \le x \le 2\pi,\ 0 \le y \le 1$$

$$w(0,y) = w(2\pi,y) = 0, \quad 0 \le y \le 1$$

$$w(x,0) = \frac{1}{n}\sin(nx), \quad 0 \le x \le 2\pi, \qquad (2.18)$$

$$\frac{\partial w(x,0)}{\partial y} = 0, \quad 0 \le x \le 2\pi,$$

By the method of separation of variables, we obtain

$$w(x,y) = X(x)Y(y)$$

$$\frac{Y''(y) + k^2 Y(y)}{Y} = \frac{-X''(x)}{} = \lambda, \quad \text{constant.}$$

$(Y(y) - \alpha^2 Y''(y)) \qquad X(x)$

By separation of variables equation, we obtain

$$X''(x) + \lambda X(x) = 0.$$

When

$$w(0,y) = w(2\pi,y) = 0,$$

$$\Rightarrow X(0) = X(2\pi) = 0,$$

we obtain

$$X_n(x) = \sin(\frac{nx}{2})$$

Also, from separation of variable, we observe that:

$$Y(y) = B_1 \cosh\Big(\sqrt{\Big(\frac{n^2 - 4k^2}{4 + \alpha^2 n^2}\Big)}y\Big) + B_2 \sinh\Big(\sqrt{\Big(\frac{n^2 - 4k^2}{4 + \alpha^2 n^2}\Big)}y\Big)$$

When

$$\frac{\partial w(x,0)}{\partial y} = 0 \Rightarrow Y'(0) = 0,$$

we obtain

$$B_2 = 0$$

$$Y(y) = B_1 \cosh\Big(\sqrt{\Big(\frac{n^2 - 4k^2}{4 + \alpha^2 n^2}\Big)}y\Big)$$

Substituting the expressions for $X(x)$ and $Y(y)$ into the product solution, we obtain

$$w(x,y) = \sum_{n=1}^{\infty} c_n \cosh\Big(\sqrt{\Big(\frac{n^2 - 4k^2}{4 + \alpha^2 n^2}\Big)}y\Big) \sin(\frac{nx}{2})$$

At

$$w(x,0) = \frac{1}{n}\sin(nx),$$

then

$$c_n = \frac{1}{n}.$$

Substituting the expression for c_n back into the above expression, we obtain

$$w(x,y)=\sum_{n=1}^{\infty}\frac{\cosh\left(\sqrt{\left(\frac{n^2-4k^2}{4+\alpha^2n^2}\right)}y\right)\sin(\frac{nx}{2})}{n} \tag{2.19}$$

We observe that equation (2.18) satisfies both the smoothness requirement in definition (1.2) and data compatibility condition in theorem (1.2), respectively. Therefore, the function (2.19) is the solution to equation (2.18).

The proof that equation (2.18) has only one solution, is similar to the proof of uniqueness of equation (2.5).

We show that the regularized equation (2.18) is stable with respect changes in the boundary condition. We observe that if $x_1 = 0$, then $w(x,0) = 0$ and the corresponding solution is:

$$w_1(x,y) = 0.$$

Also, when $x_2 = \delta$, where $0 < \delta \ll \frac{\pi}{12}$ and the corresponding solution is

$$w(x,y)=\sum_{n=odd}^{\infty}\frac{-4\sin(n\delta)\cosh\left(\sqrt{(\frac{n^2-4k^2}{4+\mu^2n^2})}y\right)\sin(\frac{nx}{2})}{n^2\pi}.$$

The change in boundary condition observed as:

$$\begin{aligned}\lim_{n\to\infty}|w(x_1,0)-w(x_2,0)| &= \lim_{n\to\infty}|0-\frac{1}{n}\sin(n\delta)|\\ &\leq \frac{1}{n}\\ \therefore |w(x_1,0)-w(x_2,0)| &\to 0 \;\; as \;\; n\to\infty\end{aligned}$$

is small and the corresponding change in the solution is:

$$\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| = \lim_{n\to\infty}|0-\sum_{n=odd}^{\infty}\frac{-4\sin(n\delta)\cosh\left(\sqrt{(\frac{n^2-4k^2}{4+\alpha^2n^2})}y\right)\sin(\frac{nx}{2})}{n^2\pi}|$$

$$\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| \leq \lim_{n\to\infty}\sum_{n=odd}^{\infty}\frac{|4|\|\sin(n\delta)\|\|\cosh\left(\sqrt{(\frac{n^2-4k^2}{4+\alpha^2n^2})}y\right)\|\|\sin(\frac{nx}{2})\|}{|n^2\pi|}$$

$$\lim_{n\to\infty}|w_1(.,1)-w_2(.,1)| \leq \lim_{n\to\infty}\sum_{n=odd}^{\infty}\frac{4e^{\sqrt{(\frac{n^2-4k^2}{4+\alpha^2n^2})}}}{n^2\pi}.$$

We observe that $4 + (n\mu)^2 > n^2 - 4k^2$ for $\alpha > 1$. This implies that

$$\Rightarrow \lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| = \lim_{n\to\infty} \sum_{n=odd}^{\infty} \frac{4e^{\sqrt{\left(\frac{n^2-4k^2}{4+\alpha^2 n^2}\right)}}}{n^2\pi} < \infty.$$

The $w(x,y)$ is stable for $\alpha > 1$. Hence, by Q-RRM, equation (2.18) is well-posed in the sense of Hadamard.

We regularize Helmholtz equation with both Neumann and Cauchy boundary conditions by Q-RRM as follows:

$$\begin{aligned}
\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) - \alpha^2 \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} &= 0, \quad 0 \le x \le 2\pi,\ 0 \le y \le 1 \\
\frac{\partial w(0,y)}{\partial x} = \frac{\partial w(2\pi,y)}{\partial x} &= 0, \quad 0 \le y \le 1 \\
w(x,0) &= \frac{1}{n}\sin(nx), \quad 0 \le x \le 2\pi \\
\frac{\partial w(x,0)}{\partial y} &= 0, \quad 0 \le x \le 2\pi,
\end{aligned} \tag{2.20}$$

and the corresponding solution is

$$w(x,y) = \sum_{n=odd}^{\infty} \frac{\cosh\left(\sqrt{\left(\frac{n^2-4k^2}{4+\alpha^2 n^2}\right)}\,y\right)\cos\left(\frac{nx}{4}\right)}{n}$$

All the three requirements for well-posedness of equation (2.20) are the same as the one shown above. Hence, equation (2.19) is well-posed in the sense of Hadamard.

Thirdly, we apply Q-RRM to regularize the Helmholtz equation with Cauchy boundary conditions where the boundary deflection is equal to zero. The regularized Helmholtz equation is obtain as:

$$\begin{aligned}
\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) - \alpha^2 \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} &= 0, \quad 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1 \\
\frac{\partial w(x,0)}{\partial y} &= 0, \quad 0 \le x \le \frac{\pi}{2} \qquad (2.21) \\
w(x,0) &= \frac{1}{n}\sin(nx), \quad 0 \le x \le \frac{\pi}{2}
\end{aligned}$$

with the corresponding solution as:

$$w(x,y) = \frac{1}{n}\sin(nx)\cosh\left(\sqrt{\frac{(n^2-k^2)}{(1+\alpha^2 n^2)}}\,y\right) \qquad (2.22)$$

We observe that equation (2.21) satisfies both the smoothness requirement in definition (1.2). On the data compatibility condition, we observe that

$$\int_0^{\frac{\pi}{2}} \frac{\partial w(x,0)}{\partial y}\,dx = \int_0^{\frac{\pi}{2}} 0\,dx = 0$$

which satisfies theorem (1.2). Hence, the function (2.22) is a solution to equation (2.21).

On the uniqueness of equation (2.21) is similar to the proof of uniqueness of equation (2.5). The stability of equation (2.21) is similar to that of equation (2.18). Hence, by Q-RRM, the equation (2.21) is well-posed.

We provide a class of boundary conditions, which when imposed on Helmholtz equation, cannot be regularized by the use of Q-RRM. First, we regularize equation (1.3) by using the Q-RRM, which gives the regularized Helmholtz equation as:

$$\begin{aligned} \frac{\partial^2 w(x,y)}{\partial x^2} &+ \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) - \alpha^2 \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} = 0, \quad 0 \le x \le \frac{\pi}{2}, \; 0 \le y \le 1 \qquad (2.23) \\ \frac{\partial w(x,0)}{\partial y} &= \frac{1}{n}\sin(nx), \quad 0 \le x \le \frac{\pi}{2} \\ w(x,0) &= 0, \quad 0 \le x \le \frac{\pi}{2} \end{aligned}$$

By the method of separation of variables, we obtain the function

$$w(x,y) = \frac{1}{n}\sqrt{\frac{(1+\alpha^2 n^2)}{(n^2-k^2)}}\sin(nx)\sinh\left(\sqrt{\frac{(n^2-k^2)}{(1+\alpha^2 n^2)}}\,y\right) \qquad (2.24)$$

Moreover, we see from the boundary deflection condition of equation (2.23) that:

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx = \frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})]\,.$$

Thus,

$$\frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})] = \begin{cases} \frac{1}{n^2}, & \forall\, n = 1,3,5,\dots \\ \frac{2}{n^2}, & \forall\, n = 2,6,10,\dots \\ 0, & \forall\, n = 4,8,12,\dots \end{cases}$$

We observe that

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx \neq 0, \quad \forall\, n = odd \; or \; n = 2,6,\dots$$

The function (2.23) does not satisfy the data compatibility condition. Thus, theorem 1.2 is not satisfied by function (2.23). This implies that equation (2.23) has no solution. Hence, by Q-RRM, equation (2.23) is ill-posed in the sense of Hadamard.

Also, we take into account of a Neumann problem for the Helmholtz equation in an upper half-plane. By applying Q-RRM to equation (1.4), we obtain the regularized equation as:

∂x^2 ∂y^2 – $\partial x^2 \partial y^2$

$$\frac{\partial^2 w(x,y)}{} + \frac{\partial^2 w(x,y)}{} + k^2 w(x,y) - \alpha^2 \frac{\partial^4 w(x,y)}{} = 0, \quad -1 \le x \le 1, \quad 0 \le y \le 1$$

$$\frac{\partial w(-1,y)}{\partial x} = \frac{\partial w(1,y)}{\partial x} = 0, \quad 0 \le y \le 1$$

$$\frac{\partial w(x,0)}{\partial y} = 0 \quad -1 \le x \le 1 \quad (2.25)$$

$$\frac{\partial w(x,1)}{\partial y} = \cos(2\pi x) \quad -1 \le x \le 1$$

and the corresponding regularized solution as:

$$w(x,y) = \begin{cases} \sum_{n=1,5,\ldots}^{\infty} \dfrac{2n\pi.\cosh\left(\sqrt{\left(\frac{(n\pi)^2-4k^2}{4+\alpha^2(n\pi)^2}\right)}y\right)(\cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2}))}{(n^2-16)\pi^2.\sqrt{\left(\frac{(n\pi)^2-4k^2}{4+\alpha^2(n\pi)^2}\right)}\sinh\left(\sqrt{\left(\frac{(n\pi)^2-4k^2}{4+\alpha^2(n\pi)^2}\right)}y\right)}, \\ \sum_{n=3,7,\ldots}^{\infty} \dfrac{-2n\pi.\cosh\left(\sqrt{\left(\frac{(n\pi)^2-4k^2}{4+\alpha^2(n\pi)^2}\right)}y\right)(\cos(\frac{n\pi x}{2})+\sin(\frac{n\pi x}{2}))}{(n^2-16)\pi^2.\sqrt{\left(\frac{(n\pi)^2-4k^2}{4+\alpha^2(n\pi)^2}\right)}\sinh\left(\sqrt{\left(\frac{(n\pi)^2-4k^2}{4+\alpha^2(n\pi)^2}\right)}y\right)} \\ \sum_{n=1}^{\infty} \dfrac{\cosh\left(\sqrt{\left(\frac{(n\pi)^2-k^2}{1+\alpha^2(n\pi)^2}\right)}y\right)(\cos(n\pi x)+\sin(n\pi x))}{2\sqrt{\left(\frac{(n\pi)^2-k^2}{1+\alpha^2(n\pi)^2}\right)}.\sinh\left(\sqrt{\left(\frac{(n\pi)^2-k^2}{1+\alpha^2(n\pi)^2}\right)}y\right)} \end{cases} \quad (2.26)$$

We observe that equation (2.25) satisfies both the smoothness requirement in definition (1.2) and the data compatibility condition in theorem (1.2), respectively. Hence, the function (2.26) is a solution to equation (2.25).

The proof that equation (2.25) has more than one solution is similar to that of equation (2.11). Hence, by the Q-RRM, the Helmholtz equation with Neumann boundary conditions in an upper half-plane is ill-posed in the sense of Hadamard.

2.4 Quasi-Boundary Value Method

In this section, we discuss the strength and weaknesses of the quasi-boundary value method (Q-BVM) in regularizing Helmholtz equation with above boundary conditions (Clark and Oppenheimer, 1994). We state the definition which is useful in the study of the Q-BVM as below.

Definition 2.5 *Let H be a Hilbert space. Let $A : D(A) \subset H \to H$ be a bounded linear Laplace-type operator. The*

$$A^* : H \to H$$

by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \qquad \forall\, x \in D(A),\, y \in D(A^*).$$

The operator A_* is called the adjoint of A.
The operator $A : H \to H$ is called self-adjoint if,

$$A = A^*,$$

is normal if,

$$A^*A = AA^*$$

and is a unitary if,

$$AA^* = A^*A = I,$$

where I is an identity operator (Chidume, 1989).

The Laplace-type operator in the Helmholtz equation

$$A : X \to Y,$$

where

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is a linear self-adjoint and unbounded from a Hilbert space X into another Hilbert space Y . Thus, in the Q-BVM, the ill-posedness means a unique solution to the Helmholtz exists, but the solution does not depend continuously on the imposed boundary conditions of Helmholtz equation (Clark and Oppenheimer, 1994).

Unlike TRM, SRM and Q-RRM, the Q-BVM regularizes the boundary conditions imposed on the Helmholtz equation by adding a product of a regularization parameter α, where $\alpha \in R^+$, and boundary condition at one of the axes of the spatial variable

[4].

In [14], the authors regularized the same problem by subtracting $\alpha w_0(0)$ instead of adding $\alpha w(0)$, to the initial condition $w(T)$. Using the Q-BVM, [72] and [71] regularized homogeneous linear and nonlinear heat equation, respectively. In [82], the authors applied Q-BVM to regularize unbounded Dirichlet boundary of the Poisson equation in $L^2(R)$ space. In [9,30] applied Q-BVM to regularize the solution of the Helmholtz equation.

We show the class of boundary conditions, which when imposed on Helmholtz equation can be regularized by the use of Q-BVM and the boundary conditions, which when imposed on Helmholtz equation cannot be solved by the Q-BVM. We regularize the boundary conditions imposed on the Helmholtz equation using the QBVM given by (Clark and Oppenheimer, 1994). Firstly, we apply the Q-BVM to regularize equation (1.22) by adding $\alpha w(x,1)$ to the $w(x,0)$, that occurs in equation (1.22). The regularized Helmholtz equation is obtained as:

$$
\begin{aligned}
\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) &= 0, && 0 \leq x \leq 2\pi,\ 0 \leq y \leq 1 \\
w(0,y) = w(2\pi,y) &= 0, && 0 \leq y \leq 1 \\
w(x,0) + \alpha w(x,1) &= \frac{1}{n}\sin(nx), && 0 \leq x \leq 2\pi \\
\frac{\partial w(x,0)}{\partial y} &= 0, && 0 \leq x \leq 2\pi,
\end{aligned}
\qquad (2.27)
$$

By the method of separation of variables, we obtain

$$w(x,y) = X(x)Y(y)$$

$$\frac{Y''(y) + k^2 Y(y)}{Y(y)} = \frac{-X''(x)}{X(x)} = \lambda, \quad constant.$$

At $w(0,y) = w(2\pi,y) = 0$, it implies that:

$$X_n(x) = \sin(\frac{nx}{2})$$

Also, from the separation of variables equation, we observe that

$$Y(y) = B_1 \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right) + B_2 \sinh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)$$

At

$$\partial w(x,0)$$

$$\frac{\quad}{\partial y} = 0$$

$$\Rightarrow Y_0(0) = 0$$

and obtain

$$Y(y) = B_n \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)$$

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Substituting the expressions for $X(x)$ and $Y(y)$ into the product solution yields

$$w(x,y) = \sum_{n=1}^{\infty} c_n \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right) \sin(\frac{nx}{2})$$

At

$$w(x,0) + \alpha w(x,1) = \frac{1}{n}\sin(nx),$$

we obtain

$$c_n = \frac{1}{n\left(1 + \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)}$$

Substituting c_n back into $w(x,y)$ yields

$$w_\alpha(x,y) = \frac{\sum_{n=1}^{\infty} \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n\pi\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)} \tag{2.28}$$

We observe that the function (2.28) satisfies both the smoothness requirement and data compatibility condition in theorem (1.2), respectively. Therefore, the function (2.28) is a solution to equation (2.27).

We show that the function (2.28) is the only solution to equation (2.27).

Proof: By contradiction, Let $u(x,y)$ and $v(x,y)$ be two different solutions of equation (2.27) such that

$$w(x,y) = u(x,y) - v(x,y),$$

Multiplying both sides of equation (2.27) by the solution (2.28) and integrating over $([0,2\pi] \times [0,1])$, we obtain

1 2π 1 2π

$$\int_0^1 \int_0^{2\pi} w_\alpha(x,y)\Delta w_\alpha(x,y)dxdy + k^2 \int_0^1 \int_0^{2\pi} |w_\alpha(x,y)|^2 dxdy = 0$$

Applying the Green's first identity to the first term on the left hand side of above equation, we obtain

$$\int_0^1 \int_0^{2\pi} w_\alpha(x,y)\Delta w(x,y)dxdy = 0 - \int_0^1 \int_0^{2\pi} |\nabla w_\alpha(x,y)|^2 dxdy,$$

which implies that

$$-\int_0^1 \int_0^{2\pi} |\nabla w(x,y)|^2 dxdy + k^2 \int_0^1 \int_0^{2\pi} |w(x,y)|^2 dxdy = 0.$$

For the above to hold, we restrict both

$$\int_0^1 \int_0^{2\pi} |w(x,y)|^2 dxdy = 0$$

and

$$\int_0^1 \int_0^{2\pi} |\nabla w_\alpha(x,y)|^2 dxdy = 0.$$

We see that the above equations hold, if

$$\begin{aligned} w_\alpha(x,y) &= 0 \\ \Rightarrow w_\alpha(x,y) &= 0 \quad in\ ([0,2\pi] \times [0,1]) \end{aligned}$$

and

$$\begin{aligned} \nabla w_\alpha(x,y) &= 0 \\ \Rightarrow w_\alpha(x,y) &= 0 \quad in\ ([0,2\pi] \times [0,1]) \end{aligned}$$

This implies that $w_\alpha(x,y)$ is a function (2.28) in the domain and on the boundary of the domain. Thus,

$$u(x,y) = v(x,y)$$

This implies that solution (2.28) is the only solution to equation (2.27).

We show that the regularized solution (2.27) depends continuously on the boundary conditions. We observe that when $x_1 = \epsilon$, where $0 < \epsilon \ll \frac{\pi}{24}$ in the boundary condition $w(\epsilon, 1) + \alpha w(x, 1) = \frac{1}{n}\sin(nx)$ and the corresponding solution is:

$$w(x,y) = \sum_{n=odd}^{\infty} \frac{4\sin(n\epsilon)\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n\pi\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)}.$$

Again, we perturb *x* from $x_1 = \epsilon$ to $x_2 = \delta$, where $0 < \delta \ll \frac{\pi}{12}$ and $< \delta$. The corresponding solution is

$$w(x,y) = \sum_{n=odd}^{\infty} \frac{4\sin(n\delta)\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n\pi\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)}.$$

The change in boundary condition is observed as:

$$\begin{aligned}
\lim_{n\to\infty} |(w_1(\epsilon,0) + \alpha w_1(\delta,1)) - (w_2(x,0) + \alpha w_2(x,1))| &\leq \frac{2}{n} \\
\lim_{n\to\infty} |(w_1(\epsilon,0) + \alpha w_1(\delta,1)) - (w_2(x,0) + \alpha w_2(x,1))| &\to 0 \;\; as \;\; n \to \infty
\end{aligned}$$

This implies that the change in the boundary condition is small and the corresponding change in the solution is as follows:

$$\begin{aligned}
\lim_{n\to\infty} |w_1(x,y) - w_2(x,y)| &= \lim_{n\to\infty} |\sum_{n=1}^{\infty} \frac{4\sin(n\epsilon)\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n\pi\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)} \\
&- \sum_{n=1}^{\infty} \frac{4\sin(n\delta)\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\sin(\frac{nx}{2})}{n\pi\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)}| \\
\lim_{n\to\infty} |w_1(x,y) - w_2(x,y)| &\leq 8\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{\|\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)\|}{|n\pi|\|\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)\|} \\
\lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| &\leq 8\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{\|\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\|}{|n\pi|\|\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)\|}
\end{aligned}$$

But, we observe that

$$\frac{\|\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\|}{\|\left(1 + \alpha\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)\|} < 1$$

This implies that

$$\lim_{n\to\infty}|w_1(.,1)-w_2(.,1)| \le \frac{8}{n\pi}$$

$$|w_1(.,1)-w_2(.,1)| \to 0 \text{ as } n\to\infty.$$

Hence, using the Q-BVM, equation (2.27) is well-posed in the sense of Hadamard.

Secondly, we apply Q-BVM to regularize both the Neumann and the Dirichlet problems of the Helmholtz equation by adding $\alpha w(x,1)$ to the inhomogeneous boundary condition. We obtain the regularized equation as below:

$$\begin{aligned}
\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}+k^2 w(x,y) &= 0, \quad 0\le x\le 2\pi, \quad 0\le y\le 1\\
\frac{\partial w(0,y)}{\partial x}=\frac{\partial w(2\pi,y)}{\partial x} &= 0, \quad 0\le y\le 1\\
w(x,0)+\alpha w(x,1) &= \frac{1}{n}\sin(nx), \quad 0\le x\le 2\pi\\
\frac{\partial w(x,0)}{\partial y} &= 0, \quad 0\le x\le 2\pi,
\end{aligned} \tag{2.29}$$

with the corresponding solution as:

$$w(x,y)=\frac{\sum_{n=odd}^{\infty}\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}y\right)\cos(\frac{nx}{4})}{n\left(1+\cosh\left(\sqrt{((\frac{n}{2})^2-k^2)}\right)\right)}$$

All the three requirements for well-posedness of equation (2.29) are the same as the one shown above. Hence, equation (2.29) is well-posed.

Thirdly, we regularize, by using the Q-BVM, the Cauchy problem of the Helmholtz equation where boundary deflection is homogeneous. We obtain the regularized equation by adding $\alpha w(x,1)$ to $w(x,0)$ in the boundary conditions as follows:

$$\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}+k^2 w(x,y)=0, \quad 0\le x\le 2\pi, \quad 0\le y\le 1$$

$$\frac{\partial w(x,0)}{\partial y} = 0, \quad 0 \le x \le \frac{\pi}{2} \qquad (2.30)$$

$$w(x,0) + \alpha w(x,1) = \frac{1}{n}\sin(nx), \quad 0 \le x \le \frac{\pi}{2}.$$

$+ k$ $w(x,y) = 0$, 0 x 0 y

The corresponding solution to equation (2.30) is:

$$w(x,y) = \frac{\cosh\left(\sqrt{n^2-k^2}y\right)\sin(nx)}{n\left(1+\alpha\cosh(\sqrt{n^2-k^2})\right)} \qquad (2.31)$$

We observe that equation (2.30) satisfies the smoothness requirement in definition (1.2). On the data compatibility condition, we observe that

$$\frac{\partial w(x,0)}{\partial y} = 0, \qquad 0 \le x \le \frac{\pi}{2}$$

which satisfies theorem (1.2). The proof of uniqueness of equation (2.30) is similar to the one above. The function in (2.31) is a unique solution to equation (2.30).

On the stability of the regularized solution (2.31), we observe that if $x_1 = 0$ then $w(0,0) + \alpha w(0,1) = 0$, and the corresponding solution is

$$w_1(x,y) = 0$$

Also, when $x_2 = \delta$, where $0 < \delta \ll \frac{\pi}{12}$, the corresponding solution is

$$w(x,y) = \frac{\cosh\left(\sqrt{n^2-k^2}y\right)\sin(n\delta)}{n\left(1+\alpha\cosh(\sqrt{n^2-k^2})\right)}$$

The change in the boundary deflection is

$$\therefore |w_1(0,0) - w_2(0,1)| \to 0 \text{ as } n \to \infty.$$

This implies that there is a small change in the boundary deflection. The corresponding change in solution is, therefore as follows:

$$\begin{aligned}\lim_{n\to\infty}|w_1(x,y)-w_2(x,y)| &= \lim_{n\to\infty}|0-\frac{\cosh\left(\sqrt{n^2-k^2}y\right)\sin(n\delta)}{n\left(1+\alpha\cosh(\sqrt{n^2-k^2})\right)}| \\ |w_1(.,1)-w_2(.,1)| &\to 0 \;\; as \;\; n\to\infty.\end{aligned}$$

Hence, using the Q-BVM, equation (2.31) is well-posed.

We present a class of boundary conditions, which when imposed on Helmholtz equation, cannot be regularized by the use of Q-BVM. Using the Q-BVM, we regularize equation (1.3) by adding $\alpha w(x,1)$ to the inhomogeneous boundary deflection $w_y(x,1)$. The regularized equation is obtain as:

$$\begin{aligned}\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}+k^2 w(x,y) &= 0, \quad 0\le x\le\frac{\pi}{2},\; 0\le y\le 1\\ \frac{\partial w(x,0)}{\partial y}+\alpha w(x,1) &= \frac{1}{n}\sin(nx), \quad 0\le x\le\frac{\pi}{2} \qquad (2.32)\\ w(x,0) &= 0, \quad 0\le x\le\frac{\pi}{2}.\end{aligned}$$

By the method of separation of variables, we obtain

$$w(x,y)=\frac{\sinh\left(\sqrt{(n^2-k^2)}y\right)\sin(nx)}{n\left(\sqrt{(n^2-k^2)}+\alpha\sinh(\sqrt{n^2-k^2})\right)} \qquad (2.33)$$

Moreover, we can see from boundary deflection condition of equation (2.32) that

$$\int_0^{\frac{\pi}{2}}\frac{1}{n}\sin(nx)dx=\frac{1}{n^2}[\,1-\cos(\frac{n\pi}{2})]$$

Thus,

$$\frac{1}{n^2}[\ 1-\cos(\frac{n\pi}{2})]\ = \begin{cases} \frac{1}{n^2}, & \forall\ n = 1,3,5,\dots \\ \frac{2}{n^2}, & \forall\ n = 2,6,10,\dots \\ 0, & \forall\ n = 4,8,12,\dots \end{cases}$$

We observe that

$$\int_0^{\frac{\pi}{2}} \frac{1}{n}\sin(nx)dx \neq 0, \quad \forall\ n = odd\ or\ n = 2,6,\dots$$

Equation (2.32) does not satisfy the data compatibility condition. This implies that the function

$$w(x,y) = \frac{\sinh\left(\sqrt{(n^2-k^2)}y\right)\sin(nx)}{n\left(\sqrt{(n^2-k^2)} + \alpha\sinh(\sqrt{n^2-k^2})\right)}$$

is not a solution of equation (2.32). Hence, using the Q-BVM, the equation (2.32) is ill-posed in the sense of Hadamard.

Next, we consider a Neumann problem for the Helmholtz equation in an upper half-plane. Using the Q-BVM, we add $\alpha w(x,1)$ to $w_y(x,1)$ in the boundary conditions of equation (1.4). The regularized Helmholtz equation is obtained as:

$$\begin{aligned} \frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) &= 0\ , && in \quad \Omega \\ \frac{\partial w(-1,y)}{\partial x} = \frac{\partial w(1,y)}{\partial x} &= 0\ , && on \quad 0 \leq y \leq 1 \\ \frac{\partial w(x,0)}{\partial y} &= 0 && -1 \leq x \leq 1 \\ \frac{\partial w(x,1)}{\partial y} + \alpha w(x,1) &= \cos(2\pi x) && -1 \leq x \leq 1 \end{aligned} \qquad (2.34)$$

with the corresponding solution as:

√ ∞ 2nπ.cosh((( ) −k)y)(cos( )+sin( )),

PP n=1,5,... α(n22−16)π2√2. √ √nπ((2nπ2nπ2nπ222)22−2 k222)sinh(nπx √2√nπx2((nπ2nπ2)22−nπx2knπx222))

$$w(x,y) = \sum_{n=3,7,\ldots}^{\infty} -\alpha\sqrt{2(n\pi.n - \cosh(16)^2\pi.2(((()) ^{-}k-k)y))(\sinh \cos((()+\sin)(-k))))}$$

$$\sum_{n=1}^{\infty} \cosh(2\alpha\sqrt{(((n\pi n\pi))^2 - -kk^2)).y\sinh())(\cos\sqrt{(n\pi x((n\pi)+)^2\sin - k(2n\pi x)))}\,)) \quad (2.35)$$

We see that equation (2.34) satisfies both the smoothness requirement in definition (1.2) and the data compatibility condition. The function (2.35) is a solution to equation (2.34).

To see the proof that equation (2.34) has more than one solution.

Proof: By contradiction, Let $u(x,y)$ and $v(x,y)$ be two different solutions of equation (2.34) such that

$$w(x,y) = u(x,y) - v(x,y),$$

Multiplying both sides of equation (2.34) by the solution (2.35) and integrating over ([−1,1] × [0,1]), we obtain

$$\int_0^1\int_{-1}^1 w_\alpha(x,y)\Delta w_\alpha(x,y)dxdy + k^2\int_0^1\int_{-1}^1 |w_\alpha(x,y)|^2 dxdy = 0$$

Applying the Green's first identity to the first term on the left hand side of above equation, we obtain

$$\int_0^1\int_{-1}^1 w_\alpha(x,y)\Delta w_\alpha(x,y)dxdy = 0 - \int_0^1\int_{-1}^1 |\nabla w_\alpha(x,y)|^2 dxdy,$$

which implies that

$$-\int_0^1\int_{-1}^1 |\nabla w(x,y)|^2 dxdy + k^2\int_0^1\int_{-1}^1 |w(x,y)|^2 dxdy = 0 .$$

For the above to hold, we restrict both

$$\int_0^1\int_{-1}^1 |w_\alpha(x,y)|^2 dxdy = 0$$

and

1 1

$$\int_0 \int_{-1} |\nabla w_\alpha(x,y)|^2 dxdy = 0.$$

We see that the above equation that

$$w_\alpha(x,y)$$

has two different values; a zero and a non-zero in $([-1,1]\times[0,1])$. The equation (2.34) has more than one solution. Thus, the second condition of well-posedness according to Hadamard is violated. Hence, using the Q-BVM, the Helmholtz equation with Neumann boundary conditions in an upper half-plane is ill-posed in the sense of Hadamard.

2.5 An Iterative Regularization Method

In this section, we use the iterative regularization method (IRM) to solve Helmholtz equation with above boundary conditions. The IRM assume a unique solution to the Helmholtz equation exists, but the solution does not depend continuously on the imposed boundary conditions. Thus, we regularize neither the Helmholtz equation nor its boundary conditions. We regularize only the solution of the Helmholtz equation. Using the IRM, we restore the stability of the solution by introducing the iterative scheme into the exact solution. We obtain the stabilized solution as the number of iterations increases (Cheng et al., 2014; Zhang and Wei (2014)). Thus, using the IRM, we assume that the Laplace-type operator in the Helmholtz equation

$$A : H \to H, \quad \text{where } A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is bounded $L^2(\mathbb{R})$ space to another $L^2(\mathbb{R})$, but the inverse Laplace-type operator A^{-1} is unbounded.

Using the IRM, we introduce the iterative scheme as:

$$w_m(x,y) = (1-\lambda)^m \hat{w}^{\delta}_{m-1}(x,y) + \lambda \cosh(\sqrt{|y|^2 - k^2}) \sin(\frac{nx}{2}), \quad m = 1,2,3,\ldots$$

where, m is the number of iterations, into equation (1.22) and simplify to obtain the regularize solution as below:

$$w(x,y) = \begin{cases} \cos(\sqrt{k^2 - |y|^2}) \sin(\frac{nx}{2})), & |y| \le k \\ (1-\lambda)^m \hat{u}^{\delta}_0(x,y) + (1-(1-\lambda)^m) \cosh(\sqrt{|y|^2 - k^2}) \sin(\frac{nx}{2}), & |y| \ge k \end{cases} \quad (2.36)$$

where

$$\lambda = e^{-\sqrt{|y|^2 - k^2}} < 1,$$

where λ is the regularization parameter and U_0 is a priori-bounded solution. The proof of the existence and uniqueness of the regularized solution (2.36), follows from the existence and uniqueness of equation (1.22).

On the stability of the regularized solution, we observe that if $x_1 = 0$ then $w(0,0)+ \alpha w(0,1) = 0$, and the corresponding solution is

$$w_1(x,y) = 0$$

Also, we perturb $w(x,y)$ from $x_1 = 0$ and $x_2 = \delta$, where $0 < \delta \ll \frac{\pi}{12}$ that the corresponding solution is

$$w(x,y) = \begin{cases} \cos(\sqrt{k^2 - |y|^2}) \sin(\frac{n\delta}{2}), & |y| \le k \\ (1-\lambda)^m \hat{u}^{\delta}_o(x,y) + (1-(1-\lambda)^m) \cosh(\sqrt{|y|^2 - k^2}) \sin(\frac{n\delta}{2}), & |y| \ge k \end{cases}$$

The change in the boundary deflection is

$$\therefore |w_1(0,0) - w_2(0,1)| \to 0 \; as \; n \to \infty.$$

This is implies that there is a small change in the initial deflection. The corresponding change in solution is as follows:

$_{n\to\infty}$ $|w_1(x,y) - w_2(x,y)| = \lim$

$_{n\to\infty}$ $^m u\hat{}^{\delta}{}_o(x,y) + (1 - (1 - \lambda)^m)\cosh(\text{p}|y|^2 - k^2)\sin(n\delta 2\ )|$

$\lim |0 - (1 - \lambda)$

$|w_1(.,1) - w_2(.,1)| \rightarrow \quad 0 \text{ as } n \rightarrow \infty.$

The above regularized solution meets all the three conditions of well-posedness. Hence, using the iterative regularization method, the Helmholtz equation with Cauchy boundary conditions where boundary deflection is equal to zero is well-posed in the sense of Hadamard.

We show for a number of boundary conditions, which when imposed on Helmholtz equation cannot be solved by the iterative regularization method. Firstly, we impose Cauchy boundary conditions with boundary deflection not equal to zero on Helmhotz equation. The regularized solution is obtained as

$$w(x,y) = \begin{cases} \sin(\sqrt{(k^2-n^2)}y).\frac{\sin(nx)}{n\sqrt{(n^2-k^2)}}, & \forall\ |n| \le k \\ (1-\alpha)^m w_o(x,y) + (1-(1-\alpha)^m)\times & \\ \frac{\sin(nx)}{n\sqrt{(n^2-k^2)}}\sinh(\sqrt{(n^2-k^2)}y), & \forall\ |n| \ge k \end{cases}$$

where α is the regularization parameter and m is the number of iterations. Equation (1.4) has no solution since it faces the same problem as the previous methods discussed above. Hence, by an iterative regularization method, equation (1.4) is ill-posed in the sense of Hadamard. The Helmholtz equation with imposed Neumann boundary conditions in the upper half-plane yields a similar result the one above.

In summary, the Tikhonov regularization method, spectral regularization method, quasi-reversibility regularization method, quasi-boundary value method and iterative regularization method regularize Helmholtz equation which imposed both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous. These methods of regularization cannot solve Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous. In addition, Neumann problem in the upper half-plane for the Helmholtz equation cannot be regularized by any of the above discussed method of regularization.

Chapter 3

Regularized Neumann, Cauchy problems of Helmholtz Equation

In the previous chapter, we showed that Helmholtz equation with Cauchy boundary conditions where boundary deflection is not equal to zero none of the existing methods of regularization can be used to restore the well-posedness of the equation. In addition, these existing methods of regularization cannot be used to solve ill-posed Helmholtz equation with Neumann boundary conditions in the upper halfplane. Therefore, the three requirements of existence, uniqueness and continuous dependence of small changes in the these boundary conditions do not hold. Hence, these existing methods of regularization are insufficient and inefficient for solving illposed Helmholtz equation with imposed Cauchy boundary condition where boundary deflection is inhomogeneous as well as Neumann boundary conditions in the upper half-plane.

In this chapter, we introduce a Divergence Regularization Method (DRM) to solve ill-posed Helmholtz equation with Cauchy boundary conditions where the boundary deflection is not equal to zero. This method enables the solution of Helmholtz equation with Neumann boundary conditions in the upper half-plane. Thirdly, the DRM solves Helmholtz equation with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous. Lastly, we show that our regularized Helmholtz equation meets all the three conditions of well-posedness in the sense of Hadamard.

3.1 Divergence Regularization Method for Regularizing Cauchy Problem for Helmholtz Equation

In this section, we present some background information regarding the propoundment of the DRM in Hilbert space, and the basic ideas of divergence theorem. When the Helmholtz equation is imposed with Cauchy boundary conditions where the boundary deflection is inhomogeneous then the analytic function in a neighbourhood of the hyper-surface is not well-posed. To see this, consider Cauchy problem, in which initial data; $w(x,0)$ and $w_y(x,0)$, are specified on the characteristic curve C in the (x,y)-plane for the Helmholtz equation. Both the Cauchy's boundary data and the coefficients that appears in the Helmholtz equation can be expanded as power series in the neighbourhood of C. We see that the coefficients of second partial derivative and higher order derivatives of $w(x,0)$ (with respect to y) do not vanish on C, on the grounds that higher derivatives can be deduced from the Helmholtz equation and construct their power series solutions (King et al., 2003).

As a consequence, the Cauchy problem of the Helmholtz equation is inconsistent as the boundary data propagate into the (x,y)-plane on the characteristic curve. In addition, any function obtained from the Cauchy problem depends on only the boundary conditions that lie between the two characteristics through (x,y)-plane. Moreover, the discontinuities in the higher order derivatives of $w(x,0)$ can propagate on characteristic curves (King et al., 2003).

Thus, all the three conditions of well-posedness according to Hadamard are violated. Even the Helmholtz equation with $C_\infty(\Omega)$ boundary data and Cauchy's analytic result does not guarantee the existence of the solution as well as uniqueness. These Cauchy data are provided on the arc of the boundary $\partial\Omega$ instead of the entire boundary of the domain Ω.

In order to restore well-posedness of Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous, we introduce the DRM. First, we extend the arc of the boundary (hyper-surface) $\partial\Omega$ to the whole entire boundary of the domain Ω. Thus, we quadraturize the boundary of the domain of Cauchy data to piecewise smooth boundary of two disjoint complementary parts. The following theorems and definitions are useful in proposing the DRM.

Definition 3.1 (Quadrature domain) *A bounded domain in Euclidean space* $\mathbb{R}^2$ *is called a quadrature domain if there is a signed (Borel) measure* μ*, with compact support in* Ω*, such that*

$$\int_{\Omega} h(x)dx = \int h d\mu$$

for every integrable (harmonic) function h on Ω (Gardiner and Sjödin, 2000).

Thus, a bounded domain $\Omega \in \mathbb{R}^2$ is a quadrature domain if there exists finitely many points $x_1,\ldots,x_m$ and the coefficients $c_{kj} \in \mathbb{R}^2$ such that

$$\int_{\Omega} w(x,y)dA = \sum_{k=1}^{m}\sum_{j=0}^{n_k-1} c_{kj} w(x,y)^{j}(a_k), \quad \forall w(x,y) \in \Omega,$$

where dA denotes area measure. The above expression is called a quadrature identity and

$$n = \sum_{k=1}^{m} n_k$$

is the order of the quadrature identity. The a_k are the (partial) derivatives of $w(x,y)$ at a_k (Gardiner and Sjödin, 2000).

Theorem 3.1 *Every positive function on a quadrature domain* Ω *is integrable on* Ω

(Gardiner and Sjödin, 2000).

Corollary 3.1 *If* Ω *is a quadrature domain with respect to a signed measure* μ *with compact support in* Ω*, then it is also a quadrature domain with respect to some positive measure with compact support in* Ω *(Gardiner and Sjödin, 2000).*

Theorem 3.2 (Divergence theorem) *Let* Ω *denote a bounded region in* $\mathbb{R}^2$ *with a smooth boundary* $\partial\Omega$ *with*

$$N = N(x,y)$$

denoting the unit outward-normal vector to $\partial\Omega$*. Then for a smooth-valued function*

$$w(x,y) = \sum_{j=1}^{2} w_j(x,y)e_j,$$

we have

$$\int_{\Omega} \mathbf{div}\, w(x,y)dxdy = \int_{\partial\Omega} w(x,y) \cdot NdS$$

Duchateau and Zachmann (1989).

Proof : See Duchateau and Zachmann (1989).

Theorem 3.3 *A necessary condition for the existence of a solution of the Neumann problem*

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} k^2 w(x,y) = 0 \quad in \quad \Omega$$

$$\frac{\partial w(x)}{\partial n} = g(x) \quad on \quad \partial\Omega$$

is

$$\int_{\partial\Omega} g(x)dx = 0$$

(King et al., 2003).

Proof : We integrate the above homogeneous Helmholtz equation over the boundary of the domain $\partial\Omega$ and apply the divergence theorem to the Laplace operator that occurs in the Helmholtz equation to obtain the result. Thus,

$$\int_{\Omega} \nabla^2 w(x,y)d^2x = \int_{\partial\Omega} \mathbf{n}.\nabla w(x,y)dx = \int_{\partial\Omega} g(x)dx = 0.$$

Definition 3.2 (Scalar) *Let* $w(x,y) \in L^1_{loc}(\mathbb{R}^2)$ *and a fixed* $\eta \in \mathbb{R}^+$, η *is called a scalar with respect to x if*

$$w_{\eta,0}(x,y) = w(\eta x,y)$$

Muthukumar (2013).

DRM entails the introduction of the term

$$(1 + \alpha^{2m})e^m$$

as a regularization term, where $\alpha \in (-\infty,-1)\cup(1,\infty)$ is the regularization parameter and $m \in \mathbb{Z}^+$ is a positive integer, and then, combine with $w(x,y)$ as another variable (unknown function) in the divergence theorem. We show that this term regularizes the Helmholtz equation as well as Cauchy boundary conditions by restoring the stability of the equation. We then apply the Green's first identity to the Laplace operator of $(1 + \alpha^{2m})e^m$ and $w(x,y)$ appearing in the Helmholtz equation. This produces piecewise smooth boundary of two disjoint complementary parts $\partial\Omega_1$ and $\partial\Omega_2$ with $w(x,0)$ and $\frac{\partial w}{\partial y}(x,l)$ are prespecified on $\partial\Omega_1$ and $w(l,y)$ and $\frac{\partial w}{\partial x}(0,y)$ on the rest of the boundary of the domain.

The DRM ensures that Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous has a solution. Thus, when the inhomogeneous boundary deflection becomes zero, we scale the $x-$ co-ordinate by an even positive integer η. This scalar η ensures that the periodic function at x inhomogeneous boundary deflection becomes zero. Also, homogenization of inhomogeneous boundary deflection in Cauchy boundary conditions by a positive integer scale together with applications of divergence theorem and Green's first identity ensures the uniqueness of solution of the regularized Helmholtz equation. The result is then written as an equation with regularized Cauchy boundary conditions. Hence, the name divergence regularization method. We state the DRM theorem for Helmholtz equation with imposed Cauchy boundary conditions in $\Omega \in H$:

Theorem 3.4 (Divergence Regularization Method) *Let Cauchy boundary conditions be imposed on Helmholtz equation where the boundary deflection is inhomogeneous as:*

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) = 0 \quad in \quad \Omega$$

$$\frac{\partial w(x,0)}{\partial y} = h(x) \qquad on\ \partial\Omega$$

$$w(x,0) = 0 \qquad on \quad \partial\Omega$$

where,

$$\int_{\partial\Omega} h(x)dx \neq 0,$$

then the regularized Helmholtz equation with regularized Cauchy boundary conditions is given below:

$$\partial x^2 \qquad \partial y^2 \qquad \eta,0$$

$$\frac{\partial w_{\eta,0}(x,l)}{\partial y} = (1+\alpha^{2m})^{-1}e^{-m}h(\eta x), \qquad on\ \partial\Omega_1$$

$$w_\eta(x,0) = 0, \qquad on\ \partial\Omega_1 \qquad (3.1)$$

$$\frac{\partial w_{\eta,0}(0,y)}{\partial x} = 0, \qquad on\ \partial\Omega_2$$

$$w_{\eta,0}(l,y) = (1+\alpha^{2m})^{-1}e^{-m}h(y), \qquad on\ \partial\Omega_2,$$

where

$$\frac{\partial^2 w_{\eta,0}(x,y)}{\partial x^2} + \frac{\partial^2 w_{\eta,0}(x,y)}{\partial y^2} + (1+\alpha^{2m})^{-1}e^{-m}k^2 w(x,y) = 0, \quad in\ \Omega$$

$$\int_{\partial\Omega_1} h(\eta x)dx = 0,$$

$h(x) \neq 0$, $h(y) \neq 0$, $\alpha \in (-\infty,-1) \cup (1,\infty)$ *is the regularization parameter,* $m \in \mathbb{Z}^+$ *is a positive integer,* k *as the wave number,* $[0,l]$ *is the square domain with* l *is a radian number and* η *is any even positive integer.*

Proof : We write the linear homogeneous Helmholtz equation as

$$\nabla \cdot (v\nabla w(x,y)) + k^2 w(x,y) = 0$$

By the dot product and product rule, we obtain

$$\nabla \cdot (v\nabla w(x,y)) + k^2 w(x,y) = \nabla w(x,y) \cdot \nabla v + v\Delta w(x,y) + k^2 w(x,y) = 0$$

Integrating both sides over Ω, we obtain

$$\iint_\Omega \nabla \cdot (v\nabla w(x,y))dxdy + \iint_\Omega k^2 w(x,y)dxdy$$

$$= \iint_\Omega \nabla w(x,y) \cdot \nabla v dxdy + \iint_\Omega v\Delta w(x,y)dxdy + \iint_\Omega k^2 w(x,y)dxdy = 0$$

In order to restore the stability of the equation, we substitute $v = (1 + \alpha^{2m})e^m$ into the above equation which yields

$$0 + \iint_\Omega (1 + \alpha^{2m})e^m \Delta w(x,y)dxdy + \iint_\Omega k^2 w(x,y)dxdy = 0 \qquad (3.2)$$

Applying Green's first identity to the first term of equation (3.2), we obtain

$$
\begin{aligned}
&\iint_\Omega (1+\alpha^{2m})e^m \Delta w(x,y)dxdy + k^2 \iint_\Omega w(x,y)dxdy \\
= &\int_{\partial\Omega_1} (1+\alpha^{2m})e^m \left(\frac{\partial w(x,l)}{\partial y} - w(x,0)\right) dx \\
+ &\int_{\partial\Omega_2} (1+\alpha^{2m})e^m \left(w(l,y) - \frac{\partial w(0,y)}{\partial x}\right) dy
\end{aligned}
$$

We then scale the $x-$ co-ordinate of the unknown function $w(x,y)$ of the above equation by a factor η. This scalar makes the trigonometric function at inhomogeneous boundary deflection of Helmholtz equation becomes zero, which when integrated over the boundary of the domain.

$$
\begin{aligned}
&\iint_\Omega (1+\alpha^{2m})e^m \Delta w_{\eta,0}(x,y)dxdy + k^2 \iint_\Omega w_{\eta,0}(x,y)dxdy \\
= &\int_{\partial\Omega_1} (1+\alpha^{2m})e^m \left(\frac{\partial w_{\eta,0}(x,l)}{\partial y} - w_{\eta,0}(x,0)\right) dx \\
+ &\int_{\partial\Omega_2} (1+\alpha^{2m})e^m \left(w_{\eta,0}(l,y) - \frac{\partial w_{\eta,0}(0,y)}{\partial x}\right) dy
\end{aligned}
$$

We see that the above regularized Helmholtz equation together with the regularized Cauchy boundary conditions is written as:

$$
\begin{aligned}
\frac{\partial^2 w_{\eta,0}(x,y)}{\partial x^2} + \frac{\partial^2 w_{\eta,0}(x,y)}{\partial y^2} + (1+\alpha^{2m})^{-1}e^{-m}k^2 w_{\eta,0}(x,y) &= 0, && in\ \Omega \\
\frac{\partial w_{\eta,0}(x,l)}{\partial y} &= (1+\alpha^{2m})^{-1} e^{-m} h(\eta x), && on\ \partial\Omega_1 \\
w_{\eta,0}(x,0) &= 0, && on\ \partial\Omega_1 \\
\frac{\partial w_{\eta,0}(0,y)}{\partial x} &= 0, && on\ \partial\Omega_2 \\
w_{\eta,0}(l,y) &= (1+\alpha^{2m})^{-1}e^{-m}h(y), && on\ \partial\Omega_2,
\end{aligned}
$$

3.1.1 Existence and Uniqueness of Regularized Helmholtz Equation with regularized Cauchy Boundary Conditions

In this section, we show that the regularized Helmholtz equation with regularized Cauchy boundary conditions, equation (3.1), has a solution and also demonstrate that the solution of regularized Helmholtz equation together with regularized boundary conditions is unique.

To prove that there exists a solution to equation (3.1), we show that the inhomogeneous boundary deflection satisfies theorem (3.1). In equation (3.1), we can see that

$$\int_{\partial\Omega} (1+\alpha^{2m})^{-1} e^{-m} h(\eta x)\, dx,$$

since η is any positive integer which makes periodic function $h(\eta x)$ to take ±1 for $\sin(nx)$ values and zero for $\cos(nx)$, depending on the non-zero endpoint of the boundary, which in effect $h(\eta x)$ becomes zero, we obtain

$$\int_{\partial\Omega} (1+\alpha^{2m})^{-1} e^{-m} \times 0\, dx = \int_{\partial\Omega} 0 dx$$

$$= constant = 0$$

By theorem (3.3), equation (3.1) has a solution.

We prove that the DRM provides a unique solution of regularized Helmholtz equation together with regularized Cauchy boundary conditions as follows.

Theorem 3.5 (Uniqueness) *Suppose that Ω denotes a rectangular domain whose boundary consists of two disjoint, complementary parts $\partial\Omega_1$ and $\partial\Omega_2$. Let $h(\eta x)$ and $h(y)$ denote given data functions, then equation* (3.1) *has at most one smooth solution.*

Proof : Suppose that equation (3.1) has two different smooth solutions denoted by $u_{\eta,0}(x,y)$ and $v_{\eta,0}(x,y)$. Also, we let

$$w_{\eta,0}(x,y) = u_{\eta,0}(x,y) - v_{\eta,0}(x,y)$$

and be the solution of equation (3.1).

$$\begin{aligned}
w_{\eta,0}(x,0) &= u_{\eta,0}(x,0) - v_{\eta,0}(x,0)\\
w_{\eta,0}(x,0) &= 0\\
\frac{\partial w_{\eta,0}(x,l)}{\partial y} &= \frac{\partial u_{\eta,0}(x,l)}{\partial y} - \frac{\partial v_{\eta,0}(x,l)}{\partial y}\\
\frac{\partial w_{\eta,0}(x,l)}{\partial y} &= (1+\alpha^{2m})^{-1}e^{-m}(h_1(\eta x) - h_2(\eta x))\\
w_{\eta,0}(l,y) &= u_{\eta,0}(l,y) - v_{\eta,0}(l,y)\\
w_{\eta,0}(l,y) &= (1+\alpha^{2m})^{-1}e^{-m}(h_1(y) - h_2(y))\\
\frac{\partial w_{\eta,0}(0,y)}{\partial x} &= \frac{\partial u_{\eta,0}(0,y)}{\partial x} - \frac{\partial v_{\eta,0}(0,y)}{\partial x}\\
\frac{\partial w_{\eta,0}(0,y)}{\partial x} &= 0.
\end{aligned}$$

Multiplying both sides of equation (3.1) by $w_{\eta,0}(x,y)$ and integrating over a domain Ω, we obtain

$$\iint_\Omega w_{\eta,0}(x,y)\frac{\partial^2 w_{\eta,0}(x,y)}{\partial x^2}dxdy + \iint_\Omega w_{\eta,0}(x,y)\frac{\partial^2 w_{\eta,0}(x,y)}{\partial y^2}dxdy$$
$$+ \iint_\Omega (1+\alpha^{2m})^{-1}e^{-m}k^2|w_{\eta,0}(x,y)|^2 dxdy = 0. \qquad (3.3)$$

Applying the Green's first identity to the first two terms on the left hand side of equation (3.3), we obtain

$$\iint_\Omega w_{\eta,0}(x,y)\Delta w_{\eta,0}(x,y)dxdy = \int_{\partial\Omega_1} w_{\eta,0}(x,0)\frac{\partial w_{\eta,0}(x,l)}{\partial y}dx + \int_{\partial\Omega_2} w_{\eta,0}(l,y)\frac{\partial w_{\eta,0}(0,y)}{\partial x}dy$$
$$- \int_\Omega |\nabla w_{\eta,0}(x,y)|^2 dxdy$$

$$= \int_{\partial\Omega_1} 0 \times (1+\alpha^{2m})^{-1} e^{-m} h(\eta x)\,dx + \int_{\partial\Omega_2} (1+\alpha^{2m})^{-1} e^{-m} h(y) \times 0\,dy - \iint_\Omega |\nabla w_{\eta,0}(x,y)|^2\,dxdy$$

$$= 0 - \iint_\Omega |\nabla w_{\eta,0}(x,y)|^2\,dxdy$$

$$\iint_\Omega w_{\eta,0}(x,y)\Delta w_{\eta,0}(x,y)\,dxdy = -\iint_\Omega |\nabla w_{\eta,0}(x,y)|^2\,dxdy \qquad (3.4)$$

Substituting equation (3.4) into equation (3.3) yields

$$\iint_\Omega w_{\eta,0}(x,y)\frac{\partial^2 w_{\eta,0}(x,y)}{\partial x^2}\,dxdy + \iint_\Omega w_{\eta,0}(x,y)\frac{\partial^2 w_{\eta,0}(x,y)}{\partial y^2}\,dxdy$$

$$+ \iint_\Omega (1+\alpha^{2m})^{-1} e^{-m} k^2 |w_{\eta,0}(x,y)|^2\,dxdy = -\iint_\Omega |\nabla w_{\eta,0}(x,y)|^2\,dxdy$$

$$+ \iint_\Omega (1+\alpha^{2m})^{-1} e^{-m} k^2 |w_{\eta,0}(x,y)|^2\,dxdy$$

$$0 = -\iint_\Omega |\nabla w_{\eta,0}(x,y)|^2\,dxdy + \iint_\Omega (1+\alpha^{2m})^{-1} e^{-m} k^2 |w_{\eta,0}(x,y)|^2\,dxdy$$

In the above equation, it follows that

$$(1+\alpha^{2m})^{-1} e^{-m} k^2 \iint_\Omega |w_{\eta,0}(x,y)|^2\,dxdy = 0$$

$$\Rightarrow w_{\eta,0}(x,y) = 0 \quad in\ \Omega$$

and

$$\iint_\Omega |\nabla w_{\eta,0}(x,y)|^2\,dxdy = 0 \Rightarrow \nabla w_{\eta,0}(x,y) = 0$$

$$\Rightarrow w_{\eta,0}(x,y) = constant = 0 \ in\ \Omega$$

Also, we observe that

$$|\nabla w_{\eta,0}(x,y)| = 0 \text{ on } \partial\Omega_2$$
$$\Rightarrow w_{\eta,0}(x,y) = constant = 0 \text{ on } \partial\Omega_2$$

and

$$h(\eta x) = 0 \text{ on } \partial\Omega_1$$
$$|\nabla w_{\eta,0}(x,y)| = 0$$
$$\Rightarrow w_{\eta,0}(x,y) = constant = 0 \text{ on } \partial\Omega_1$$

Thus, $w_{\eta,0}(x,y)$ is smooth and zero in the domain Ω and its boundary $\partial\Omega$. This implies that

$$u_{\eta,0}(x,y) = v_{\eta,0}(x,y).$$

Hence, equation (3.1) has only one smooth solution.

By applying the Divergence Regularization Method to equation (1.3) by choosing $\eta = 4$ to restore well-posedness of that equation. The regularized form of equation (1.3) together with regularized Cauchy boundary conditions is given as follows:

$$\begin{aligned}
\frac{\partial^2 w_{4,0}(x,y)}{\partial x^2} + \frac{\partial^2 w_{4,0}(x,y)}{\partial y^2} + (1+\alpha^{2m})^{-1}e^{-m}k^2 w(x,y) &= 0, \quad 0 \le x \le 2\pi,\ 0 \le y \le \frac{\pi}{2} \\
w_{4,0}(x,0) &= 0, \quad 0 \le x \le 2\pi \\
\frac{\partial w_{4,0}(x,\frac{\pi}{2})}{\partial y} &= (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(4nx), \quad 0 \le x \le 2\pi \\
\frac{\partial w_{4,0}(0,y)}{\partial x} &= 0, \quad 0 \le y \le \frac{\pi}{2} \\
w_{4,0}(2\pi,y) &= (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(ny), \quad 0 \le y \le \frac{\pi}{2}
\end{aligned} \tag{3.5}$$

We split equation (3.5) into two independent equations as follows:

$$
\begin{aligned}
\frac{\partial^2 w_{4,0}(x,y)}{\partial x^2} + \frac{\partial^2 w_{4,0}(x,y)}{\partial y^2} + (1+\alpha^{2m})^{-1} e^{-m} k^2 w_{\eta,0}(x,y) &= 0, & 0 \le x \le 2\pi,\ 0 \le y \le \frac{\pi}{2} \\
w_{4,0}(x,0) &= 0, & 0 \le x \le 2\pi \\
\frac{\partial w_{4,0}(x,\frac{\pi}{2})}{\partial y} &= 0, & 0 \le x \le 2\pi \\
\frac{\partial w_{4,0}(0,y)}{\partial x} &= 0, & 0 \le y \le \frac{\pi}{2} \\
w_{4,0}(2\pi,y) &= (1+\alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin(ny), & 0 \le y \le \frac{\pi}{2}
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
\frac{\partial^2 w}{\partial x^2} \qquad \frac{\partial}{\partial y^2} \qquad {}_{\eta,0} \qquad \le \qquad \le \qquad \le \qquad \le 2 \\
w_{4,0}(x,0) &= 0, & 0 \le x \le 2\pi \\
\frac{\partial w_{4,0}(x,\frac{\pi}{2})}{\partial y} &= (1+\alpha^{2m})^{-1} e^{-m} \frac{1}{n} \sin(4nx), & 0 \le x \le 2\pi \\
\frac{\partial w_{4,0}(0,y)}{\partial x} &= 0, & 0 \le y \le \frac{\pi}{2} \\
w_{4,0}(2\pi,y) &= 0, & 0 \le y \le \frac{\pi}{2}
\end{aligned}
\tag{3.7}
$$

$_{4,0}(x,y)$ + $\partial^2 w_{\eta,0}(x,y) + (1+\alpha^{2m})^{-1} e^{-m} k^2 w(x,y) =$

0, 0 x 2π, 0 y π

We obtain the classical solution to equation (3.6) by the method of separation of variables as:

$$Y(y) = \sin(ny), \qquad n = 1,3,...$$

and

$$\begin{aligned} X(x) &= B_1 \cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right) \\ &+ B_2 \sinh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right) \end{aligned}$$

When $X_0(0) = 0$, we obtain

$$X(x) = B_1 \cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)$$

81 .

Thus,

$$w_{4,0}(x,y) = \sum_{n=1,3,...}^{\infty} c_n \cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)$$

When $X(2\pi) = (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(ny)$, we obtain

$$\begin{aligned} w_{4,0}(x,y) &= \sum_{n=1,3,...}^{\infty} c_n \cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)\sin(ny) \\ &= (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(ny) \\ \Rightarrow \quad & \sum_{n=1,3,...}^{\infty} c_n \cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)\sin(ny) \\ &= (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(ny). \end{aligned}$$

Using the orthogonality of eigenfunctions, we obtain

$$c_n = \frac{1}{n(1+\alpha^{2m})e^{m}\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)}$$

Substituting the expression for c_n into the above equation yields:

$$w_{4,0}(x,y) = \sum_{n=1,3,...}^{\infty} \frac{\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)}{n(1+\alpha^{2m})e^{m}\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)}$$

In a similar manner, we obtain solution to equation (3.7) with

$$w_{4,0}(x,y) = 0$$

as equation (3.6) since

$$\sin(4nx)\cos(\frac{nx}{4})$$

is orthogonal over $[0,2\pi]$. By the principle of superposition, we obtain solution to equation (3.5) as:

$$w_{4,0}(x,y) = \sum_{n=1,3,\ldots}^{\infty} \frac{\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)}{n(1+\alpha^{2m})e^{m}\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)}$$

We show that equation (3.5) satisfies the three requirements of well-posedness. First, we show that our regularized boundary deflection of equation (3.5) satisfies lemma (3.1). We can see from boundary deflection condition of equation (3.5) that

$$\int_0^{2\pi} \frac{1}{n(1+\alpha^{2m})e^m} .\sin(4nx)dx = \frac{1}{4n^2(1+\alpha^{2m})e^m}\left[-\cos(4nx)\right]_0^{2\pi}$$

$$= \frac{1}{4n^2(1+\alpha^{2m})e^m}[-1+1]$$

$$= 0$$

Thus, lemma (3.1) is satisfied. We conclude that equation (3.5) has at least a solution.

The proof of uniqueness of regularized equation (3.5), is similar to the proof of theorem (3.3) above.

Finally, we demonstrate that regularized equation (3.5) is stable to small changes in the boundary condition. In equation (3.5), we choose $y=\epsilon$ in the boundary condition $w_{4,0}(2\pi,y) = (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(ny)$, where $0<\epsilon \ll \frac{\pi}{36}$. We obtain the regularized Helmholtz equation (3.5) together with new boundary condition as given below:

$$w_{4,0}(2\pi,\epsilon) = (1+\alpha^{2m})^{-1}e^{-m}.\frac{1}{n}\sin(n\epsilon), \qquad 0\le y\le \frac{\pi}{2}$$

and the corresponding solution is as below:

$$w_1(x,y) = \sum_{n=1,3,\ldots}^{\infty} \frac{4\sin(n\epsilon)\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)}{\pi n(1+\alpha^{2m})e^m\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)}.$$

We perturb from

$$w_{4,0}(2\pi,\epsilon) = (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(n\epsilon), \qquad 0 \le y \le \frac{\pi}{2}$$

to

$$w_{4,0}(2\pi,\delta) = (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(n\delta), \qquad 0 \le y \le \frac{\pi}{2}$$

where $0 < \delta \ll \frac{\pi}{36},\ \delta > \epsilon$ with the corresponding solution as:

$$w_2(x,y) = \sum_{n=1,3,\ldots}^{\infty} \frac{4\sin(n\delta)\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)}{\pi n(1+\alpha^{2m})e^m\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)}$$

We observe the following inequalities:

$$\|\sin(n\epsilon)\| \le 1, \quad \|\sin(n\delta)\| \le 1 \;\; and \;\; \|\cosh(y)\| \le e^y$$

The change in the boundary condition is:

$$\begin{aligned}
\lim_{m,n\to\infty} |w_{4,0}(2\pi,\epsilon) - w_{4,0}(2\pi,\delta)| &= \lim_{m,n\to\infty} |(1+\alpha^{2m})^{-1}e^{-m}.\frac{1}{n}\sin(n\epsilon) \\
&- (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\sin(n\delta)| \\
&\le \lim_{m,n\to\infty}(1+\alpha^{2m})^{-1}e^{-m}.\frac{1}{n}(\|\sin(n\epsilon)\| + \|\sin(n\delta)\|) \\
\lim_{m,n\to\infty} |w_{4,0}(2\pi,\epsilon) - w_{4,0}(2\pi,\delta)| &\le 2\lim_{m,n\to\infty}(1+\alpha^{2m})^{-1}e^{-m}.\frac{1}{n} \\
\lim_{m,n\to\infty} |w_{4,0}(2\pi,\epsilon) - w_{4,0}(2\pi,\delta)| &\to 0 \;\; as \;\; m,n\to\infty.
\end{aligned}$$

This implies that there is a small change in the boundary condition. Moreover, we observe the corresponding change in the solution $w(x,y)$ as:

$$\begin{aligned}
&\lim_{m,n\to\infty} |w_1(x,y) - w_2(x,y)| \\
&= \lim_{m,n\to\infty} \Big| \sum_{n=1,3,\ldots}^{\infty} \frac{4\sin(n\epsilon)\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)}{\pi n(1+\alpha^{2m})e^m\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)} \\
&- \sum_{n=1,3,\ldots}^{\infty} \frac{4\sin(n\delta)\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)}{\pi n(1+\alpha^{2m})e^m\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)}\Big| \\
&\le \lim_{m,n\to\infty} \frac{4}{\pi(1+\alpha^{2m})e^m}\Big\| \sum_{n=1,3,\ldots}^{\infty} \frac{\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)}{\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)} \\
&\times \Big\|(\|\sin(n\epsilon)\|\|\sin(ny)\| + \|\sin(n\delta)\|\|\sin(ny)\|) \\
&\lim_{m,n\to\infty} |w_1(x,y) - w_2(x,y)| \le \lim_{m,n\to\infty} \frac{8}{\pi(1+\alpha^{2m})e^m}.\frac{e^{\sqrt{(n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2)}.x}}{e^{2\pi\sqrt{(n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2)}}}
\end{aligned}$$

We observe that

$$\sup_{0\le x\le 2\pi} |x_i| \le 2\pi,$$

which yields

$$\lim_{m,n\to\infty} |w_1(x,y) - w_2(x,y)| \le \lim_{m\to\infty} \frac{8}{\pi(1+\alpha^{2m})e^m}$$

We can see that

$$(1+\alpha^{2m})^{-1}e^{-m} \to 0 \text{ as } m \to \infty$$

$$\Rightarrow \lim_{m,n\to\infty} |w_1(x,y) - w_2(x,y)| \to 0 \text{ as } m,n \to \infty.$$

This implies that a small change in the boundary condition $w_{4,0}(2\pi,y)$ from $x_1 = \epsilon$ to $x_2 = \delta$ results in a small change in solution

$$\sum_{n=1,3,\dots}^{\infty} \frac{\cosh\left(\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}x\right)\sin(ny)}{n(1+\alpha^{2m})e^m\cosh\left(2\pi\sqrt{[n^2-(1+\alpha^{2m})^{-1}e^{-m}k^2]}\right)}.$$

Thus, the regularized equation (3.5) is stable. Hence, the regularized Cauchy problem for the regularized Helmholtz equation is well-posed.

3.2 Regularized Helmholtz Equation with Neumann Boundary Conditions

In this section, we regularize equation (1.4) together with its boundary conditions. We shift the *x* spatial variable to the right hand side by one.

Let $w(x,y) \in L^1_{loc}(\mathbb{R}^2)$ and for fixed numbers $1,0 \in \mathbb{R}^2$, we introduce the shift operator:

$$\tau_{1,0}w(x,y) = w((x+1),y),$$

substitute $v = (1+\alpha^{2m})e^m$ into equation (3.2) and applying Green's first identity, we obtain

$$(1+\alpha^{2m})e^m \int_{\partial\Omega} \frac{\partial \tau_{1,0}w(0,y)}{\partial x}dy + \int_{\partial\Omega} (1+\alpha^{2m})e^m \frac{\partial \tau_{1,0}w(2,y)}{\partial x}dy$$

$$+ \int (1+\alpha^{2m})e^m \, \partial \tau_{1,0}w(x,0)dx + \int (1+\alpha^{2m})e^m \, \partial\tau_{1,0}w(x,1)dx$$

$\partial\Omega$ ∂y $\partial\Omega$ ∂y

$$= 0 + \iint_\Omega (1+\alpha^{2m})e^{m}\Delta\tau_{1,0}w(x,y)\,dxdy + k^2\iint_\Omega \tau_{1,0}w(x,y)\,dxdy$$

We write the above equation in a form:

$$\frac{\partial^2\tau_{1,0}w(x,y)}{\partial x^2} + \frac{\partial^2\tau_{1,0}w(x,y)}{\partial y^2} + (1+\alpha^{2m})^{-1}e^{-m}k^2\tau_{1,0}w(x,y)=0, \quad in\ \Omega$$

$$\frac{\partial w(0,y)}{\partial x} = \frac{\partial w(2,y)}{\partial x} = 0, \quad on\ \partial\Omega_1$$

$$\frac{\partial w(x,0)}{\partial y} = (1+\alpha^{2m})^{-1}e^{-m}, \quad on\ \partial\Omega_2$$

$$\frac{\partial w(x,1)}{\partial y} = (1+\alpha^{2m})^{-1}e^{-m}h(x), \quad on\ \partial\Omega_2$$

By divergence regularization method with shifting of $x-$ spatial variable, the regularized form of equation (1.5) is as follows:

$$\frac{\partial^2\tau_{1,0}w(x,y)}{\partial x^2} + \frac{\partial^2\tau_{1,0}w(x,y)}{\partial y^2} + (1+\alpha^{2m})^{-1}e^{-m}k^2\tau_{1,0}w(x,y)=0, \quad 0\le x\le 2,\ 0\le y\le 1$$

$$\frac{\partial\tau_{1,0}w(0,y)}{\partial x} = \frac{\partial\tau_{1,0}w(2,y)}{\partial x} = 0, \quad 0\le y\le 1$$

$$\frac{\partial\tau_{1,0}w(x,0)}{\partial y} = (1+\alpha^{2m})^{-1}e^{-m} \quad 0\le x\le 2 \tag{3.8}$$

$$\frac{\partial\tau_{1,0}w(x,1)}{\partial y} = (1+\alpha^{2m})^{-1}e^{-m}\cos(2\pi x) \quad 0\le x\le 2$$

where

$$\cos(2\pi(x+1)) = \cos(2\pi x).$$

We split equation (3.7) into two independent equations as follows:

∂x^2 ∂y^2 $_{1,0}$ $\leq \leq \leq \leq$

$\dfrac{\partial\tau_{1,0}w(0,y)}{\partial x} = \dfrac{\partial\tau_{1,0}w(2,y)}{\partial x} = 0, \qquad 0 \leq y \leq 1$

$\partial\tau_{1,0}w(x,0)$

∂y $\leq \leq \partial^2\tau_{1,0}w(x,y) +$

$\partial\tau_{1,0}w(x,1)$ $2m\ -1\ -m$ $\partial^2\tau_{1,0}w(x,y)$

$_{2m})^{-1}e^{-m}k^2\tau \quad w(x,y) =$

$0, \ 0 \quad x \quad 2, 0 \quad y$

$1 + (1 + \alpha$

$= 0 \qquad 0 \quad x \quad 2 \qquad (3.9)$

$= (1 + \alpha \quad) \quad e \quad \cos(2\pi x) \qquad 0 \leq x \leq 2$

∂y

and

∂x^2 ∂y^2 $_{1,0}$ $\leq \leq \leq \leq$

$\partial\tau_{1,0}w(0,y)$ $\partial\tau_{1,0}w(2,y)$

∂x ∂x $\leq \leq$

$\partial\tau_{1,0}w(x,0)$ $2m\ -1\ -m$ $\partial^2\tau_{1,0}w(x,y) +$

$\partial^2\tau_{1,0}w(x,y)$

$+ (1 + \alpha_{2m})^{-1}e^{-m}k^2\tau \qquad w(x,y) = 0, 0 \quad x \quad 2, 0 \quad y \quad 1$

$= \quad = 0, \qquad 0 \quad y \quad 1$

$= (1 + \alpha \quad) \quad e \qquad 0 \leq x \leq 2 \qquad (3.10)$

∂y

$\dfrac{\partial\tau_{1,0}w(x,1)}{\partial y} = 0. \quad 0 \leq x \leq 2$

We obtain classical solution to equation (3.9) by the method of separation of variables as:

$$X_n(x) = A_n \cos(\frac{n\pi x}{2}), \qquad n = 0, 1, 2, \ldots,$$

and

$$Y(y) = B_1 \cosh\left(\sqrt{[(\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2]}y\right) + B_2 \sinh\left(\sqrt{[(\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2]}y\right)$$

When $Y_0(0) = 0$, we obtain

$$Y(y) = B_n \cosh\left(\sqrt{[(\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2]}y\right)$$

When

$$n = 0 \qquad \Rightarrow \lambda = 0$$

and

$$Y_o(y) = c_o \cos\left(\sqrt{((1+\alpha^{2m})^{-1}e^{-m}k^2)}y\right)$$

Thus,
For consistent system, we observe that

$$c_o = 0$$

and

$$\sum_{n=1}^{\infty} \sqrt{((\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2)}c_n \sinh\left(\sqrt{((\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2)}\right)\cos(\frac{n\pi x}{2})$$
$$= \quad (1+\alpha^{2m})^{-1}\cos(2\pi x)$$

We obtain c_n in the above equation in a similar manner. Thus,

$$\begin{aligned}
w(x,y) &= c_o \cos\left(\sqrt{((1+\alpha^{2m})^{-1}e^{-m}k^2)}y\right) \\
&+ \sum_{n=1}^{\infty} c_n \cosh\left(\sqrt{\left((\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2\right)}y\right)\cos(\frac{n\pi x}{2}) \\
\Rightarrow \frac{\partial w(x,1)}{\partial y} &= -\sqrt{((1+\alpha^{2m})^{-1}e^{-m}k^2)}.c_o \sin(\sqrt{((1+\alpha^{2m})^{-1}e^{-m}k^2)}) \\
&+ \sum_{n=1}^{\infty} \sqrt{[\ (\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2]}\ c_n \sinh\left(\sqrt{[\ (\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2]}\right) \\
&\times \cos(\frac{n\pi x}{2}) = (1+\alpha^{2m})^{-1}e^{-m}\cos(2\pi x)
\end{aligned}$$

$$w(x,y) = \sum_{n=1}^{\infty} \frac{\cosh(\tau_1 y)\cos(\frac{n\pi x}{2})}{(1+\alpha^{2m})e^m \tau_1 \sinh(\tau_1)} \qquad (3.11)$$

where

$$\tau_1 = \sqrt{[\ (\frac{n\pi}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2]}$$

In a similar manner, we obtain solution to equation (3.9) as follows:

$$w(x,y) \ = \ \sum_{n=1}^{\infty} \frac{2\sin(n\pi)\cosh(\tau_1(1-y))\cos(\frac{n\pi x}{2})}{n\pi(1+\alpha^{2m})e^m \sinh(\tau_1)}$$

By the principle of superposition,

$w(x,y)$= X$_{n\infty}$ ______ (1 +cosh(α $\tau_1 y$)cos($_{n\pi x 2}$)+ X$_\infty$
______ 2sin($n\pi$)cosh($_{2\tau m1}$)(1e_m−sinh(y))cosτ_1)($_{n\pi x2}$)

$_{2m}$)$e_m\tau$

$_{=1}$ $_1$ sinh(τ_1) $_{n=1}$ $n\pi$(1 + α

$w(x,y)$= X$_{n\infty=1}$ ______ (1 +cosh($\alpha_{2m}\tau_1$)ye)$_m$cosτ_1 sinh(($_{n\pi x2}$)τ_1) (3.12)

We show that equation (3.12) satisfies the three requirements of well-posedness. We first discuss the existence of solution to equation (3.8). We see from equation (3.8) that its coefficients are continuously differentiable. On the data compatibility condition, we observe that all the boundary conditions are zero except $\frac{\partial \tau_{1,0} w(x,0)}{\partial y} =$ $(1 + \alpha^{2m})^{-1}e^{-m}$ and $\frac{\partial \tau_{1,0} w(x,1)}{\partial y} = (1+\alpha^{2m})^{-1}e^{-m}\cos(2\pi x)$. We can see that

$$\int_0^2 (1+\alpha^{2m})^{-1}e^{-m}dx \ = \ (1+\alpha^{2m})^{-1}e^{-m}[\ x]\ _0^2$$

$$= \ \frac{2}{(1+\alpha^{2m})e^m}$$

But we observe that

$$\frac{2}{(1+\alpha^{2m})e^m} \to 0 \ as \ m \to \infty.$$

$$\therefore \int_0^2 (1+\alpha^{2m})^{-1}e^{-m}dx = 0$$

Also,

$$\int_0^2 (1+\alpha^{2m})^{-1}\cos(2\pi x)\,dx = (1+\alpha^{2m})^{-1}\,\frac{1}{2\pi}\left[\sin(2\pi x)\right]_0^2 = 0.$$

Since the boundary conditions of equation (3.8) satisfy lemma (3.1), we conclude that the regularized equation (3.8) has at least a solution.

We now show that the regularized equation (3.8) has only one solution. To see this, we suppose that equation (3.8) has two different smooth solutions denoted by $\tau_{1,0}u(x,y)$ and $\tau_{1,0}v(x,y)$. Also, we let

$$w_{\eta,0}(x,y) = \tau_{1,0}u(x,y) - \tau_{1,0}v(x,y)$$

and be the solution of equation (3.8).

$$\frac{\partial w(0,y)}{\partial x} = \frac{\partial u(0,y)}{\partial x} - \frac{\partial v(0,y)}{\partial x} = 0, \quad 0 \le y \le 1$$

$$\frac{\partial w(2,y)}{\partial x} = \frac{\partial u(2,y)}{\partial x} - \frac{\partial v(2,y)}{\partial x} = 0, \quad 0 \le y \le 1$$

$$\frac{\partial w(x,0)}{\partial y} = \frac{\partial u(x,0)}{\partial y} - \frac{\partial v(x,0)}{\partial y} = 0, \quad 0 \le x \le 2$$

$$\frac{\partial w(x,1)}{\partial y} = \frac{\partial u(x,1)}{\partial y} - \frac{\partial v(x,1)}{\partial y} = (1+\alpha^{2m})^{-1}e^{-m}(\cos(2\pi x_1) - \cos(2\pi x_2)), \quad 0 \le x \le 2.$$

Shifting the $x-$ co-ordinate of equation (3.8) by 1, multiplying by $\tau_{1,0}w(x,y)$ and integrating it over a domain $[0,1]\times[0,2]$, we obtain

$$\int_0^1\int_0^2 \frac{\partial^2 \tau_{1,0}w(x,y)}{\partial x^2}\tau_{1,0}w(x,y)\,dxdy + \int_0^1\int_0^2 \frac{\partial^2 \tau_{1,0}w(x,y)}{\partial y^2}\tau_{1,0}w(x,y)\,dxdy$$

$$+\int_0^1\int_0^2 (1+\alpha^{2m})^{-1}e^{-m}k^2|\tau_{1,0}w(x,y)|^2\,dxdy = 0$$

Applying the Green's first identity to the first two terms on the left hand side of above equation, we have

$$\begin{aligned}\int_0^1\int_0^2 \tau_{1,0}w(x,y)\Delta\tau_{1,0}w(x,y)\,dxdy &= \int_0^2 (1+\alpha^{2m})e^m\frac{\partial w(0,y)}{\partial x}\,dy + \int_0^2 (1+\alpha^{2m})e^m\frac{\partial w(2,y)}{\partial x}\,dy\\ &+ \int_0^1 (1+\alpha^{2m})e^m\frac{\partial w(x,0)}{\partial y}\,dx + \int_0^1 (1+\alpha^{2m})e^m\frac{\partial w(x,1)}{\partial y}\,dx\\ &- \int_0^1\int_0^2 |\nabla\tau_{1,0}w(x,y)|^2\,dxdy\\ &= 0 - \int_0^1\int_0^2 |\nabla\tau_{1,0}w(x,y)|^2\,dxdy\end{aligned}$$

$$\int_0^1\int_0^2 \tau_{1,0}w(x,y)\Delta\tau_{1,0}w(x,y)\,dxdy = -\int_0^1\int_0^2 |\nabla\tau_{1,0}w(x,y)|^2\,dxdy$$

Thus,

$$\int_0^1\int_0^2 \tau_{1,0}w(x,y)\frac{\partial^2\tau_{1,0}w(x,y)}{\partial x^2}\,dxdy + \int_0^1\int_0^2 \tau_{1,0}w(x,y)\frac{\partial^2\tau_{1,0}w(x,y)}{\partial y^2}\,dxdy$$

$$+\int_0^1\int_0^2 (1+\alpha^{2m})^{-1}e^{-mk^2}|\tau_{1,0}w(x,y)|^2\,dxdy = -\int_0^1\int_0^2 |\nabla\tau_{1,0}w(x,y)|^2\,dxdy$$

$$+\int_0^1\int_0^2 (1+\alpha^{2m})^{-1}e^{-mk^2}|\tau_{1,0}w(x,y)|^2\,dxdy$$

$$0 = -\int_0^1\int_0^2 |\nabla\tau_{1,0}w(x,y)|^2\,dxdy + \int_0^1\int_0^2 (1+\alpha^{2m})^{-1}e^{-mk^2}|\tau_{1,0}w(x,y)|^2\,dxdy$$

In the above equation, it follows that

$$(1+\alpha^{2m})^{-1}e^{-mk^2}\int_0^1\int_0^2 |\tau_{1,0}w(x,y)|^2\,dxdy = 0$$

$$\Rightarrow \tau_{1,0}w(x,y) = 0 \quad in\ [0,2]\times[0,1]$$

and

$$\int_0^1\int_0^2 |\nabla\tau_{1,0}w(x,y)|^2\,dxdy = 0$$

$$\Rightarrow \nabla\tau_{1,0}w(x,y) = 0$$

$$\Rightarrow \tau_{1,0}w(x,y) \quad = \quad constant = 0 \; in \; [0,2] \times [0,1] \quad Z \quad Z$$

Also, we observe that

$$|\nabla \tau_{1,0}w(x,y)| \quad = \quad 0 \; on \; \partial\Omega_2$$

$$\Rightarrow \tau_{1,0}w(x,y) \quad = \quad 0 \; on \; \partial[0,2]$$

$$|\nabla \tau_{1,0}w(x,y)|^2 dxdy \quad = \quad 0$$

0 0

Thus, $\tau_{1,0}w(x,y)$ is smooth and zero in the domain Ω and its boundary $\partial\Omega$. This implies that

$$\tau_{1,0}u(x,y) = \tau_{1,0}v(x,y)$$

Hence, equation (3.8) has only one smooth solution. We conclude that equation (3.12) is the only solution to the regularized equation (3.8)

Lastly, we show that equation (3.8) is stable to small changes in the boundary deflection $\frac{\partial \tau_{1,0} w(x,1)}{\partial y}$. In equation (3.8), we choose $x = \epsilon$, in $\frac{\partial \tau_{1,0} w(x,1)}{\partial y}$, where $0 < \epsilon \ll \frac{1}{18}$, ϵ is any value in the interval [0,2]. We obtain regularized equation (3.8) together with new initial deflection

$$\frac{\partial \tau_{1,0} w(\epsilon,1)}{\partial y} = (1+\alpha^{2m})^{-1} e^{-m} \cos(2\pi\epsilon) \qquad 0 \leq x \leq 2$$

and the corresponding solution is

$$w_1(x,y) = \sum_{n=1}^{\infty} \frac{-2\sin(n\pi)\cosh\left(\tau_1(1-y)\right)\cos(\frac{n\pi x}{2})}{n\pi(1+\alpha^{2m})\tau_1 \sinh(\tau_1)}$$

We perturb the boundary deflection from

$$\frac{\partial \tau_{1,0} w(\epsilon,1)}{\partial y} = (1+\alpha^{2m})^{-1} e^{-m} \cos(2\pi\epsilon)$$

to

$$\frac{\partial \tau_{1,0} w(\delta,1)}{\partial y} = (1+\alpha^{2m})^{-1} e^{-m} \cos(2\pi\delta),$$

$0 < \delta \ll \frac{1}{36}$, $\delta > \epsilon$ and the corresponding solution is

$$w_2(x,y) = \sum_{n=1}^{\infty} \frac{-2\sin(n\pi)\cosh(\tau_1(1-y))\cos(\frac{n\pi x}{2})}{n\pi(1+\alpha^{2m})e^m \tau_1 \sinh(\tau_1)}$$

The change in the boundary deflection is:

$$\begin{aligned}
\lim_{m\to\infty} |\frac{\partial w_1(\epsilon,1)}{\partial y} - \frac{\partial w_2(\delta,1)}{\partial y}| &= \lim_{m\to\infty} |(1+\alpha^{2m})^{-1}e^{-m}\cos(2\pi\epsilon) - (1+\alpha^{2m})^{-1}e^{-m}\cos(2\pi\delta)| \\
\lim_{m\to\infty} |\frac{\partial w_1(\epsilon,1)}{\partial y} - \frac{\partial w_2(\delta,1)}{\partial y}| &\leq \lim_{m\to\infty} \frac{1}{(1+\alpha^{2m})e^m}.
\end{aligned}$$

But

$$\frac{1}{(1+\alpha^{2m})e^m} \to 0 \ \ as \ \ m \to 0$$

,

which in turn, implies that

$$\lim_{m\to\infty} |\frac{\partial w_1(\epsilon,1)}{\partial y} - \frac{\partial w_2(\delta,1)}{\partial y}| \to 0 \ \ as \ \ m \to \infty$$

.

Hence, there is a small change in boundary deflection.

We discuss corresponding change in the solution *w*(*x,y*) as

$$\begin{aligned}
\lim_{m,n\to\infty} |w_1(x,y) - w_2(x,y)| = \lim_{m,n\to\infty} \Big| \sum_{n=1}^{\infty} \frac{2\sin(n\pi)\cosh(\tau_1(1-y))\cos(\frac{n\pi x}{2})}{n\pi(1+\alpha^{2m})e^m\tau_1\sinh(\tau_1)} \\
- \sum_{n=1}^{\infty} \frac{2\sin(n\pi)\cosh(\tau_1(1-y))\cos(\frac{n\pi x}{2})}{n\pi(1+\alpha^{2m})e^m\tau_1\sinh(\tau_1)} \Big| \\
\Rightarrow \lim_{m,n\to\infty} |w_1(x,y) - w_2(x,y)| \to 0 \quad as \ \ m,n \to \infty.
\end{aligned}$$

Thus, the regularized equation (3.8) is stable to the small changes in the initial deflection. Hence, the regularized Neumann problem for the Helmholtz equation is well-posed in the sense of Hadamard.

3.3 Regularized Helmholtz Equation with Cauchy and Dirichlet Boundary Conditions where the Boundary Deflection is equal to zero

In this section, we regularize equation (1.22), as well as its boundary conditions. We obtain boundary conditions in a different form. Applying Green's first identity to equation (3.2), we obtain the following boundary conditions:

$$w(0,y) = w(2\pi,y) = 0, \qquad 0 \leq y \leq 1$$

$$w(x,0) = \frac{1}{n(1+\alpha^{2m})e^{m}}\sin(nx), 0 \leq x \leq 2\pi$$

$$\frac{\partial w(x,0)}{\partial y} = 0, \quad 0 \leq x \leq 2\pi,$$

We obtain classical solution to equation (3.5) together with the above boundary conditions by the method of separation of variables as:

$$w(x,y) = \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{[\ (\frac{n}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2]}\ y\right)\sin(\frac{nx}{2})}{n(1+\alpha^{2m})e^{m}} \tag{3.13}$$

We show the existence of solution to regularize equation (3.5) with the above boundary conditions. We see that equation (3.5) satisfies definition (1.3) On the data compatibility condition, we observe that

$$\frac{\partial w(x,0)}{\partial y} = 0,$$

Thus, lemma (3.1) is satisfied. Therefore there exists a solution to regularize equation (3.5) together with the above boundary conditions.

The proof that equation (3.5) together with above boundary conditions has only one solution, follows the proof of theorem (3.3). We conclude equation (3.13) is the only solution to the regularized equation (3.5) with the above boundary conditions.

Lastly, we show that the regularized equation (3.5) is stable to small changes in boundary condition $w(x,0)$. In equation (3.5), we choose $x = \epsilon$, in $w(x,0)$, where $0 \leq \epsilon \leq \frac{1}{38\pi}$. We obtain a new boundary condition

$$w_1(x,0) = \frac{1}{n(1+\alpha^{2m})e^{m}}\sin(n\epsilon)$$

with the corresponding solution

$$w(x,y) = \sum_{n=1,3}^{\infty} \frac{4\sin(n\epsilon)\cosh\left(\sqrt{((\frac{n}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2)}y\right)\sin(\frac{nx}{2})}{n^2(1+\alpha^{2m})e^{m}\pi}.$$

We perturb from

$$w_1(x,0) = \frac{1}{n(1+\alpha^{2m})e^m}\sin(n\epsilon)$$

to

$$w_2(x,0) = \frac{1}{n(1+\alpha^{2m})e^m}\sin(n\delta)$$

where $0 \leq \delta \leq \frac{1}{38\pi},\ \delta > \epsilon$ and the corresponding solution is

$$w(x,y) = \sum_{n=1,3}^{\infty} \frac{4\sin(n\delta)\cosh\left(\sqrt{((\frac{n}{2})^2 - (1+\alpha^{2m})^{-1}e^{-m}k^2)}y\right)\sin(\frac{nx}{2})}{n^2(1+\alpha^{2m})e^m\pi}.$$

We observe the change in the boundary condition as

$$\begin{aligned}\lim_{m,n\to\infty}|w_1(x,0)-w_2(x,0)| &= \lim_{m,n\to\infty}|\frac{1}{n(1+\alpha^{2m})e^m}\sin(n\epsilon) - \frac{1}{n(1+\alpha^{2m})e^m}\sin(n\delta)| \\ &\leq \frac{2}{n(1+\alpha^{2m})e^m}\end{aligned}$$

$$\frac{1}{n(1+\alpha^{2m})} \to 0 \ \ as \ \ m,n\to\infty.$$

Thus, there is a small change in boundary condition.

The corresponding change in the solution *w*(*x*,*y*) is given below.

$$\begin{aligned}&\lim_{m,n\to\infty}|w_1(x,y)-w_2(x,y)| \\ &= \lim_{m,n\to\infty}\Big|\sum_{n=1}^{\infty}\frac{4\sin(n\epsilon)\cosh\left(\sqrt{((\frac{n}{2})^2-(1+\alpha^{2m})^{-1}e^{-m}k^2)}y\right)\sin(\frac{nx}{2})}{n^2(1+\alpha^{2m})e^m\pi} \\ &- \sum_{n=1}^{\infty}\frac{4\sin(n\delta)\cosh\left(\sqrt{((\frac{n}{2})^2-(1+\alpha^{2m})^{-1}e^{-m}k^2)}y\right)\sin(\frac{nx}{2})}{n^2(1+\alpha^{2m})e^m\pi}\Big| \\ &\leq \frac{e^{(\sqrt{((\frac{n}{2})^2-(1+\alpha^{2m})^{-1}e^{-m}k^2)}y)}}{n^2(1+\alpha^{2m})e^m\pi}\end{aligned}$$

We observe that

$$(1+\alpha^{2m})^{-1}e^{-m} \to 0 \ as\ m\to\infty,$$

so that the numerator

$$e^{(\sqrt{((\frac{n}{2})^2-(1+\alpha^{2m})^{-1}e^{-m}k^2)}y)} \to e^{\frac{n}{2}y} \ \ as \ \ m\to\infty.$$

Also, we observe that

$$\sup_{0\leq y\leq 1} |y_i| \leq 1$$
,

the expression

$$\frac{1}{n^2(1+\alpha^{2m})e^{m\pi}}$$

decays faster than the growth of $e^{\frac{n}{2}y}$,

$$\lim_{m,n\to\infty} \frac{1}{n^2(1+\alpha^{2m})e^{m\pi}}.e^{\frac{n}{2}y} \to 0 \ \ as \ \ m,n\to\infty$$
.

This implies that a small change in the boundary condition $w_1(x,0)$ to $w_2(x,0)$ brings a small change in the solution. Thus, equation (3.5), together with the above boundary conditions is stable. The equation (3.5) together with the above boundary conditions satisfies all the three conditions of well-posedness. Hence, equation (3.5) together with above the boundary conditions is well-posed in the sense of Hadamard.

In summary, the divergence regularization method DRM regularizes Cauchy problem of Helmholtz equation by introducing a regularization term $(1 + \alpha^{2m})e^m$, which restores the stability of the equation, and then, applies Green's first identity to Laplace operator of $(1 + \alpha^{2m})e^m$ and $w(x,y)$ appearing in the Helmholtz equation. This produces piecewise smooth boundary of two disjoint complementary parts $\partial\Omega_1$ and $\partial\Omega_2$ where, $w(x,0)$ and $\frac{\partial w}{\partial y}(x,l)$ are prespecified on $\partial\Omega_1$ and $w(l,y)$ and $\frac{\partial w(0,y)}{\partial x}$ on $\partial\Omega_2$.

Finally, this method incorporates an even positive integer scale η in x-coordinate of the unknown function $w(x,y)$ which makes the inhomogeneous boundary deflection in Cauchy boundary conditions zero. This η depends on the kind of periodic function imposed on inhomogeneous boundary deflection in Cauchy boundary conditions, as well as, non-zero endpoint of the boundary of the domain.

We observe that when $\frac{1}{n}\cos(nx)$ is imposed at boundary deflection in the Cauchy problem of Helmholtz equation, we obtain

$$\begin{aligned}
\frac{\partial^2 w(x,y)}{\partial x^2} &+ \frac{\partial^2 w(x,y)}{\partial y^2} + k^2 w(x,y) = 0, \quad 0 \le y \le \frac{\pi}{2}, \quad 0 \le y \le \frac{\pi}{2} \\
w(x,0) &= 0, \qquad 0 \le x \le \frac{\pi}{2} \\
\frac{\partial w(x,0)}{\partial y} &= \frac{1}{n}\cos(nx), \qquad 0 \le x \le \frac{\pi}{2}
\end{aligned}$$

then the regularized Helmholtz equation is obtained by a similar procedure, except that, we choose scaling factor $\eta = 2$. We see from the above that, non-zero endpoint of boundary of the domain is $\pi/2$, so we choose $\eta = 2$. That is, the denominator of the non-zero endpoint of the boundary of the domain. The regularized equation is given below:

$$\begin{aligned}
\frac{\partial^2 w_{2,0}(x,y)}{\partial x^2} &+ \frac{\partial^2 w_{2,0}(2x,y)}{\partial y^2} + (1+\alpha^{2m})^{-1}e^{-m}k^2 w_{2,0}(x,y) = 0, \quad 0 \le x \le \pi, \; 0 \le y \le \frac{\pi}{2} \\
w_{2,0}(x,0) &= 0, \qquad 0 \le x \le \pi \\
\frac{\partial w_{2,0}(x,\frac{\pi}{2})}{\partial y} &= (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\cos(2nx), \qquad 0 \le x \le \pi \\
\frac{\partial w_{2,0}(0,y)}{\partial x} &= 0, \qquad 0 \le y \le \frac{\pi}{2} \\
w_{2,0}(\pi,y) &= (1+\alpha^{2m})^{-1}e^{-m}\frac{1}{n}\cos(2ny), \qquad 0 \le y \le \frac{\pi}{2}
\end{aligned}$$

But if, $\frac{1}{n}\sin(nx)$, is imposed at boundary deflection in the Cauchy problem of Helmholtz equation, then we choose $\eta = 4$ instead of 2.

In conclusion, we observe that when $\frac{1}{n}\sin(nx)$ is imposed at boundary deflection in Cauchy problem of Helmholtz equation, then we choose $\eta = 2m$, whereas when $\frac{1}{n}\cos(nx)$ is imposed then we choose $\eta = m$, where m is denominator of the non-zero endpoint of the boundary of the domain in the Cauchy boundary conditions. The DRM regularizes ill-posed Helmholtz equation with periodic function such as $\sin(nx)$ and $\cos(nx)$ imposed at boundary deflection in the Cauchy problem.

In regularizing Neumann problem of Helmholtz equation we make use of a shifting operator $\tau_{1,0}$ of x coordinate of unknown function $w(x,y)$, regularization term $(1 + \alpha^{2m})e^m$ and then apply Green's first identity. That is, the DRM solves

$$Aw(x,y) = f$$

where the null space

$$N(A) = \{w : Aw(x,y) = 0\}$$

is not trivial.

Unlike other methods of regularization discussed in the previous chapter, the new DRM regularizes Cauchy problem of the Helmholtz equation where the boundary deflection is inhomogeneous, as well as Neumann problem for Helmholtz equation in the upper half-plane. Moreover, the DRM regularizes both Dirichlet and Cauchy problems of the Helmholtz equation where boundary deflection is homogeneous.

Chapter 4

Adaptive Wavelet Spectral Finite Difference Method for Regularized Helmholtz Equation

In the previous chapter, we introduced a DRM for solving ill-posed Helmholtz equation with Cauchy boundary where the boundary deflection is not equal to zero, Neumann boundary conditions in the upper half-plane and both Dirichlet and Cauchy boundary conditions, where boundary deflection is equal to zero. The analytic solutions provided for problems in the previous chapter are not easy to compute and require a lot of computational time owing to the complexity of the functions. Also, the Fourier transform or series method analyzes the global regularity of the solutions to the regularized Helmholtz equation with regularized boundary conditions only in the frequency domain. The solution functions are made of infinite number of terms.

We seek solutions to the regularized Helmholtz equation that are fast and require less computational effort to execute as compared to other methods. The well-known quantitative methods such as finite-difference method, finite element method, finite volume method, Euler method, do not give time-domain information on the regularized equation but rather fail to include boundary information when the domain is irregularized by partition. Irregular partitioning of the domain enables us to assess the regularity of the solutions of the equation being sought.

In this chapter, we introduce Adaptive Wavelet Spectral Finite Difference (AWSFD) method to obtain the approximate solutions to equations (3.5), (3.8) and (3.5) with both Dirichlet and Cauchy boundary conditions where the boundary deflection is equal to zero. In addition, we give the interpolation scheme in the AWSFD method which approximates the solution of the regularized Helmholtz equation.

This chapter is divided into three main sections; section 4.1 gives the brief account on the wavelets and their properties, section 4.2 contains the Adaptive Wavelet Finite Difference Method for solving the regularized Neumann, Cauchy and both Cauchy and Dirichlet problems of Helmholtz equation. In section 4.3, we provide the analysis of solutions of regularized Helmholtz equation by DRM and by AWSFD method. We also compare the results by DRM with other regularization methods.

4.1 Overview of wavelets

The wavelet method for solving the type of equation considered earlier is efficient, fast and above all provides time-frequency information on the regularized Helmholtz equation. This method analyzes the pointwise regularity of the solutions being sought for the regularized Helmholtz equation with Cauchy or Neumann boundary conditions. Moreover, in the wavelet method, the partial sums of the series converge, irrespective of the order of terms that are taken for approximation. These series are unconditional bases for $\Omega \subset H$ subspace. Moreover, wavelet methods of approximation includes the boundary of the domain in approximating the solution of the regularized Helmholtz equation.

Definition 4.1 (Wavelet) *A family of functions constructed from the translation and dilation of a single function $\psi(x)$, is called the mother wavelet*

$$\psi_{j,k}(x) = \frac{1}{\sqrt{|j|}}\psi(\frac{x-k}{j}), j, k \in \mathbb{R}, \quad j \neq 0$$
,

where j is the dilation/scaling parameter which measures the degree of compression or scale, and k is the translation parameter which determines the time location of the wavelet

(Debnath and Shah, 2015).

The compressed versions and high frequencies of the mother wavelet are observed when the modulus of the scaling parameter is less than one. As the scale decreases, the resolution in the time domain decreases (finer), whereas that in the frequency domain increases (coarser). On the other hand, if the modulus of the scaling parameter is equal or greater than one, then the mother wavelet is stretched and low frequecies are observed. The function or the signal becomes coarser in the time domain and finer in the frequency domain as the scaling parameter increases. These variations in the two domains are determined by the Heisenberg uncertainty principle. These wavelet functions are called first generation wavelets. The construction of any of these wavelet functions satisfies the so-called multiresolution analysis MRA.

Definition 4.2 (Multiresolution Analysis) *Let* $\{\varphi_{ok}\}$ *be an orthonormal system in* $L^2(\mathrm{R})$. *The sequence of spaces* $\{V_j, j \in \mathrm{Z}\}$, *generated by* $\varphi(x)$ *is called a multiresolution analysis MRA of* $\Omega \subset H$ *if it satisfies the following properties:*

1. $V_j \subset V_{j+1}, \quad j \in \mathrm{Z}$
2. $\bigcup_{j \in Z} V_j = L^2(\mathrm{R})$
3. $\cap_{j \in Z} V_j = \{0\}$
4. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
5. $f(x) \in V_j \Leftrightarrow f(x - k) \in V_j, \forall\, k \in Z$
6. *there exists a function* $\varphi(x)$ *(called scaling function or father wavelet) such that*

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k),\ k \in Z$$

constitute an orthonormal basis for corresponding subspace V_j
(Urban, 2009).

The following theorem is key to our results which will follow.

.

First generation wavelet methods have been applied successfully over the last two decades to obtain closed form solutions to partial differential equations. See for example, [1,2,50,61,83]. Irregular sampled data points on the regularized Helmholtz equation cannot be approximated by any of the first generation wavelet methods. Recently, the second generation wavelets have been introduced by (Sweldens, 1996). These wavelets are a generalization of biorthogonal wavelets, which are desirable for applications to regularized Helmholtz equation whose solutions are sought on a general domain than R^n. Thus, they are suitable for irregular grids or intervals. All the properties of the first generation wavelets are maintained in the second generation wavelets with the exception of the translation and dilation properties. These wavelets can generally be constructed by the use of lifting scheme which facilitates the calculation of wavelet filters; high pass and low pass, in turn, gives wavelet algebraic equation. Unlike the first generation wavelets, second generation wavelets are endowed with dual multiresolution analysis

$$\tilde{M} = \{\tilde{V}_j \subset L | j \in J\}.$$

The spaces $\tilde{V}_j$, are spanned by dual scaling functions $\tilde{\varphi}^j_k$, which are biorthogonal to the primal scaling functions. The scaling functions φ^j_k are expressed as

$$\varphi_{jk} = \sum_{l \in K_{j+1}} h_{jk,j}\varphi_{jl+1},$$

where, $h^j_{k,j}$ are the filter coefficients [59]. Studies such as [75,76,77,80] have made use of second generation wavelets to obtain approximate solutions of variants of partial differential equations. Apart from the MRA, the wavelet functions must meet the following conditions.

A wavelet is be said to admissible if,

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$$

where $\hat{\omega}$ is the Fourier transform of $\omega(x)$. If $\psi(x) \in L^2(\mathbb{R})$, then $\psi_{j,k}(x) \in L^2(\mathbb{R})$ for all $j,k \in \mathbb{Z}$. The corresponding norm of the wavelet is

$$\begin{aligned} \|\psi_{j,k}(x)\|^2 &= |j|^{-1} \int_{-\infty}^{\infty} \left|\psi\left(\frac{x-k}{j}\right)\right|^2 dx \\ &= \int_{-\infty}^{\infty} |\psi(x)|^2 dx \\ &= \|\psi\| \end{aligned}$$

and its Fourier transform is given by

$$\begin{aligned} \hat{\psi}_{j,k}(\omega) &= |j|^{-1/2} \int_{-\infty}^{\infty} e^{-i\omega x}\psi\left(\frac{x-k}{j}\right) dx \\ &= |j|^{1/2} e^{-ik\omega} \hat{\psi}(j\omega) \end{aligned}$$

(Stark, 2005).

Secondly, the wavelet must satisfy the vanishing moment.

Definition 4.3 *The k^{th} vanishing moment of a wavelet is defined as*

$$\int_{\mathbb{R}} x^k \psi(x) dx = 0,$$

where $k \geq 0$. For example, see (Beylkin, 1992).

The vanishing moments indicate the flatness at both ends of the wavelet function on the defined domain. The greater the number of vanishing moments, the fewer the wavelet coefficients needed for the approximation of the regularized Helmholtz equation, and faster the convergence of the approximated solution (wavelet series) in $\Omega \subset H$. The regularity of the wavelet function, as well as, localization property cannot be overemphasized. The admissibility condition of the wavelet function together with smoothness and localization properties gives rise to another desirable property called bandpass filters. These bandpass filters also determine the rapid decay of the frequency response as ω approaches infinity.

4.2 Adaptive Wavelet Spectral Finite Difference Method for solving the Regularized Neumann, Cauchy and both Cauchy and Dirichlet Problems of Helmholtz Equation

A number of wavelet functions have been introduced for obtaining approximate solutions of equations. Traditional wavelet functions such as morlet, mallat, maxican hat, etc are not supported compactly. Another undesirable property of these wavelet functions is the complexity of the calculation of wavelet coefficients needed for the construction of approximate solution to a regularized Helmholtz equation. The wavelet functions with compact support are simple and easier in usage for the construction of orthonormal solutions in $\Omega \subset H$.

The compactly supported wavelet functions like B-splines

$$B_n(x) = B_{n-1} * B_o(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} B_{n-1}(t)dt, \qquad n \in \mathbb{N}$$

where

$$B_o = \chi_{[\frac{-1}{2},\frac{1}{2}]}(x);$$

for $n \in Z^+$

$$\tilde{B}_n = B_n\big(x - \frac{(n+1)}{2}\big)$$

are $C^{n-1}(\Omega)$ functions. This implies that B-spline wavelets have $n - 1$ degrees of regularity which are proportional to the number of their vanishing moments. The Franklin wavelet and the Battle-Lemari´e wavelet are second and n^{th} order of Bspline wavelets, respectively. Haar wavelet function is not continuously differentiable on [0,1) [7,11,18,36,40,49]. Constructions of approximated solution with compact support involve construction of connection coefficients.

In order to obtain approximated solutions for the regularized Helmholtz equation, together with the above regularized boundary conditions, we make use of wavelet function with high degree of regularity defined on compact support and many vanishing moments. The desirable property of vanishing moments depend on the smoothness and support of the wavelet function. The smoother the wavelet function, the greater the number of vanishing moments, the lesser the number of wavelet coefficients needed for the construction of orthonormal solutions, and the faster the convergence of approximated solutions in $\Omega \subset H$.

A Daubechies wavelet function of order N has the largest number of vanishing moments which are compactly supported on $[0,2N - 1]$. Moreover, the high number of vanishing moments lead to high compressibility of orthonormal solution in $\Omega \subset H$. With the use of Daubechies wavelet function, we can include the boundary information of the domain in obtaining the approximate solution of the regularized Helmholtz equation.

Owing to the properties of the Daubechies wavelet function, we use it in the Adaptive Wavelet Spectral Finite Difference (AWSFD) method to obtain approximate solutions of regularized Helmholtz equation. In addition, we give the interpolation scheme for the AWSFD method, which approximates Daubechies scaling function.

The unknown function $w(x,y)$ of regularized Helmholtz equation is approximated as a linear combination of integer shifts of the Daubechies scaling functions. Thus, $w(x,y)$ is approximated as piecewise polynomials of degree $(N - 1)$ in $\Omega \subset H$. The function $w(x,y)$ is represented as a limit of successive approximations, each of which is a finer version of the previous approximations. Each successive approximation corresponds to a different level of resolution (scale).

Definition 4.4 (Daubechies wavelets) *For $N \in \mathbb{N}$, a Daubechies wavelet of class $D2N$ is a function $\psi = \psi_N \in L^2(\mathbb{R})$ defined by*

$$\psi(x) = \sqrt{2}\sum_{k=0}^{2N-1}(-1)^k h_{2N-1-k}\phi(2x-k)$$
,

where $h_0,\ldots,h_{2N-1}$ are constant filter coefficients satisfying the condition

$$\sum_{k=0}^{N-1} h_{2k} = \frac{1}{\sqrt{2}} = \sum_{k=0}^{N-1} h_{2k+1}$$

as well as, for $l = 0,1,\ldots,N-1$,

$$\sum_{k=2l}^{2N-1+2l} h_k h_{k-2l} = \begin{cases} 1, & if\ l = 0 \\ 0, & if l \neq 0 \end{cases}$$

and where $\varphi =_N \varphi : \mathbb{R} \to \mathbb{R}$ is the (Daubechies) scaling functions, given by

$$\phi(x) = \sqrt{2}\sum_{k=0}^{2N-1} h_k \varphi(2x-k)$$

and satisfying

$$\varphi(x) = 0, \qquad for x \in \mathbb{R}\backslash[0,2N-1]$$

as well as

$$Z_{\mathbb{R}}\, \varphi(2x-k)\varphi(2x-l)dx = \delta_{k,l},$$

where

$$1, \quad k = l$$

$\delta_{k,l} =$

$$\begin{cases} 0, & k \neq l \end{cases}$$

(Andreas de Vries, 2006).

Thus, Daubechies wavelets are defined in terms of their scaling functions. These (scaling) functions determine the nature of the wavelet function. The area under the scaling functions is normalized to be one. That is,

$$\int_{\Omega} \phi(x)dx = 1.$$

The scaling function $\varphi(x)$ and its translates are orthonormal and the wavelet function $\psi(x)$ has N vanishing moments.

From the definition of Daubechies wavelet, we need additional formulae for the calculation of constant filter coefficients for the implementation of the AWSFD method to obtain the approximated solution of regularized equation with above regularized boundary conditions. We state the following definitions, theorems and lemmas which would be incorporated in the AWSFD method to obtain closed form solutions to regularized Helmholtz equation with above regularized boundary conditions.

Theorem 4.1 *For any scaling function* $\varphi(x) \in L^2(\mathbb{R})$ *the following conditions are equivalent:*

(i) The system

$$\{\varphi_{0,n} = \varphi(x - n), n \in \mathbb{Z}\}$$

is orthonormal.

(ii)

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1, \qquad (4.1)$$

almost everywhere a. e.(Debnath and Shah, 2015). **Proof**: By

Fourier transform, we have

$$\varphi_{0,n}(x) = \varphi(x-n)$$

$$= e^{-in\omega}\hat{\varphi}(\omega)$$

Applying Parseval relation for the Fourier of the inner product of two different scaling functions, we have

$$\begin{aligned}
\langle \varphi_{0,n}, \varphi_{0,m} \rangle &= \langle \varphi_{0,0}, \varphi_{0,m-n} \rangle \\
&= \frac{1}{2\pi} \langle \hat{\varphi}_{0,0}, \hat{\varphi}_{0,m-n} \rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(m-n)\omega} \cdot |\hat{\varphi}(\omega)|^2 d\omega \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} e^{-i(m-n)\omega} \cdot |\hat{\varphi}(\omega)|^2 d\omega \\
&= \frac{1}{2\pi} \int_{0}^{2\pi} e^{-i(m-n)\omega} \cdot \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi k)|^2 d\omega
\end{aligned}$$

Thus, it follows from the completeness of

$$\{e^{-in\omega}, n \in \mathbb{Z}\}$$

in $L^2(0,2\pi)$ that

$$\langle \varphi_{0,n}, \varphi_{0,m} \rangle = \delta_{n,m},$$

if and only if

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = 1$$

Theorem 4.2 (Biorthogonality) *For any two functions* $\varphi(x)$*,* $\psi(x) \in L^2(\mathbb{R})$*, the set of functions*

$$\{\varphi_{0,n} = \varphi(x - n),\ n \in \mathbb{Z}\}$$

and

$$\{\psi_{0,n} = \psi(x - n),\ n \in \mathbb{Z}\}$$

are biorthogonal, that is,

$$\langle \varphi_{0,n}, \varphi_{0,m} \rangle = 0, \qquad \forall n,m \in \mathbb{Z}$$

if and only if

$$\sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi k)\overline{\hat{\psi}}(\omega + 2\pi k) = 0 \quad ,a.e$$

Proof: See Daubechies (1992).

Lemma 1 *Suppose that equation* (4.1) *holds then the Fourier transform of the scaling function* $\varphi(x)$ *satisfies the following condition:*

$$\phi(\omega) = \hat{m}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}),$$

where

$$\hat{m}(\omega) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} c_k e^{-ik\omega} \qquad (4.2)$$

is a 2π*−periodic function and satisfies the orthogonality condition*

$$|\hat{m}(\omega)|^2 + |\hat{m}(\omega + \pi)|^2 = 1, \quad a.e \qquad (4.3)$$

(Burrus et al., 1998).

Proof: Let $\varphi(x) \in v_1$ and define

$$\phi_{1,k}(x) = \sqrt{2}\phi(2x - k)$$

is an orthonormal basis for v_1. The scaling function $\varphi(x)$ can be written as

$$\phi(x) = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k \phi(2x - k),$$

where

$$c_k = \langle \varphi(x), \varphi_{1,k} \rangle$$

and

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty.$$

Applying the Fourier transform of the above equation yields

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} c_k e^{-ik\frac{\omega}{2}} \hat{\varphi}(\frac{\omega}{2})$$

$$\hat{\varphi}(\omega) = \hat{m}(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}).$$

We have obtained result for the first part of our claim. To prove for result in equation (4.3), we substitute the above equation into equation (4.3) which yields

$$\sum_{k=-\infty}^{\infty} |\hat{m}(\frac{\omega + 2\pi k}{2})\hat{\phi}(\frac{\omega + 2\pi k}{2})|^2 = 1$$

$$\sum_{k=-\infty}^{\infty} |\hat{m}(\frac{\omega}{2} + \pi k)\hat{\phi}(\frac{\omega}{2} + \pi k)|^2 = 1$$

Since the functions $\hat{m}(\omega)$ and $\hat{\varphi}(\omega)$ are 2π-periodic functions, the results also hold for multiples of ω. Substituting ω by 2ω into the above equation, we obtain

$$\sum_{k=-\infty}^{\infty} |\hat{m}(\omega + \pi k)\hat{\phi}(\omega + \pi k)|^2 = 1.$$

Separating the above infinity sum into even and odd integers and applying $2\pi-$periodic property of the function $\hat{m}(\omega)$, we obtain

$$\sum_{k=-\infty}^{\infty} \left|\hat{m}(\omega+2\pi k)\right|^2 \left|\hat{\phi}(\omega+2\pi k)\right|^2 + \sum_{k=-\infty}^{\infty} |\hat{m}(\omega+(2k+1)\pi)|^2 |\hat{\phi}(\omega+(2k+1)\pi)|^2 = 1$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \left|\hat{m}(\omega)\right|^2 \left|\hat{\phi}(\omega+2\pi k)\right|^2 + \sum_{k=-\infty}^{\infty} \left|\hat{m}(\omega+\pi)\right|^2 \left|\hat{\phi}(\omega+\pi+2\pi k)\right|^2 = 1$$

$$\Rightarrow |\hat{m}(\omega)|^2 + |\hat{m}(\omega+\pi)|^2 = 1 \tag{4.4}$$

Theorem 4.3 *If* $\varphi(x) \in C^m$, *support* $\varphi(x) \subset [0, N-1]$, *and*

$$\phi(x) = \sum_{k=0}^{N-1} c_n \phi(2x-k),$$

then $(N-1) \geq m+1$.

Proof: See Daubechies (1992). We can see from equation (4.2) that

$$\hat{m}(\pi) = 0,$$

implies that $\hat{m}(\omega)$ has a factor at $\omega = \pi$ and

$$\hat{m}(0) = 1.$$

Also, we can see that the infimum of the inequality in theorem (4.3) implies that $\psi(x) \in C^N(\Omega)$. It follows from definition (4.3) that Daubechies wavelet of order N has N vanishing moments. We can write equation (4.2) as a generating function

$\hat{m}_o(\omega) = \left(\frac{1+e^{-i\omega}}{2} \right)^N \hat{L}(\omega)$, (4.5) where, $L\hat{}(\omega)$ is $2\pi-$periodic function in $C^N(\Omega)$. We then derive a polynomial equation of order $N - 1$ to generate Daubechies scaling functions.

Using equation (4.3), we obtain

$$
\begin{aligned}
1 &= |\hat{m}_o(\omega)|^2 + |\hat{m}_o(\omega+\pi)|^2 \\
1 &= \hat{m}_o(\omega)\hat{m}_o(-\omega) + \hat{m}_o(\omega+\pi)\hat{m}_o(-(\omega+\pi)) \\
&= \left(\frac{1+e^{-i\omega}}{2} \right)^N \left(\frac{1+e^{i\omega}}{2} \right)^N |\hat{L}(\omega)|^2 + \left(\frac{1+e^{-i(\omega+\pi)}}{2} \right)^N \left(\frac{1+e^{i(\omega+\pi)}}{2} \right)^N |\hat{L}(\omega+\pi)|^2 \\
1 &= \left(\cos^2(\frac{\omega}{2})\right)^N R(\cos\omega) + \left(\cos^2(\frac{\omega+\pi}{2})\right)^N R(-\cos\omega)
\end{aligned}
$$

where, $|L\hat{}(\omega)|^2$ is a polynomial of the form

$$|L\hat{}(\omega)|^2 = R(\cos\omega)$$

and

$$|L\hat{}(\omega+\pi)|^2 = R(-\cos\omega).$$

We can see from above equation that

$$
\begin{aligned}
1 &= \left(1-\sin^2(\frac{\omega}{2})\right)^N R(1-2\sin^2(\frac{\omega}{2})) + \left(\sin^2(\frac{\omega}{2})\right)^N R(-1+2\sin^2(\frac{\omega}{2})) \\
1 &= (1-y)^N R(1-2y) + y^N R(1-2(1-y)) \\
1 &= (1-y)^N P_{N-1}(y) + y^N P_{N-1}(1-y) \qquad (4.6)
\end{aligned}
$$

where

$$y = \sin^2(\frac{\omega}{2}).$$

and

$$0 \le \sin^2(\frac{\omega}{2}) \le 1.$$

Hence,

$$P(y) \ge 0.$$

The polynomial $P_{N-1}(y)$ of order $N-1$ is insufficient on the grounds that there is no formula to generate its values. In order to evaluate the values of polynomial $P_{N-1}(y)$, we derive a algebraic polynomial of degree $N-1$. Thus,

$$
\begin{aligned}
1 &= \left((1-y)+y\right)^N \\
1 &= \sum_{k=0}^{2N-1} \binom{2N-1}{k} (1-y)^k y^{2N-1-k} \\
1 &= \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^k y^{2N-1-k} + \sum_{k=N}^{2N-1} \binom{2N-1}{k} (1-y)^k y^{2N-1-k}
\end{aligned}
$$

Substituting $n = 2N-1-k$ as an index in the second term of the right hand side, we obtain

$$
\begin{aligned}
1 &= \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^k y^{2N-1-k} + \sum_{n=0}^{N-1} \binom{2N-1}{n} (1-y)^{2N-1-n} y^n \\
1 &= y^N \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^k y^{N-1-k} + (1-y)^N \sum_{n=0}^{N-1} \binom{2N-1}{n} (1-y)^{N-1-n} y^n \\
1 &= y^N P_{N-1}(1-y) + (1-y)^N P_{N-1}(y),
\end{aligned}
$$

where

$$P_{N-1}(y) = \sum_{k=0}^{N-1} \begin{pmatrix} 2N-1 \\ k \end{pmatrix} (1-y)^k y^{N-1-k} \tag{4.7}$$

$P_{N-1}(y)$ is a polynomial of degree $N - 1$ (Walnut, 2002).

The equations (4.6) and (4.7) generate Daubechies scaling coefficient in manner that

$$(-1)^k h_{2N-1-k} = 2\delta_{k,l}$$

We provide the main procedure of AWSFD method for obtaining the approximated solutions to equations (3.5), (3.8) and (3.5) with both Dirichlet and Cauchy boundary conditions where the boundary deflection is zero. The AWSFD method, is a composite method comprising finite difference, Daubechies wavelet function and spectral analysis. We begin with equation (3.5) as follows:

The AWSFD method involves two different ways of discretization; the discretization of one of the independent variables in the Helmholtz equation followed by the discretization of the other independent spatial variable using suitable wavelet basis. Algebraic equations (4.12) and (4.14), are obtained after the Helmholtz equation has been discretized twice. Both equations are used to obtain the approximated solution of the Helmholtz equation. The coarser equation (4.12) is obtained when i =6 k, which acts as an initial approximation for the equation (4.14) at a finer scale $j + 1$. The equation (4.14) is performed recursively to obtain the approximated solution of the Helmholtz equation.

Thus, the current approximated solution $w_{j,k}(x,y)$ is a scaled version of the previous approximations. By MRA,

$$\mathbf{V}_{m+1} = \mathbf{V}_0 \oplus \left(\bigoplus_{m=0}^{m} \mathbf{W}_m \right)$$

At each level of scale j, we calculate the Daubechies scaling filter coefficients to a desirable order N using equations (4.6) − (4.7) and then substitute these values into equation (4.10) to obtain approximated solution at that scale.

Moreover, unlike other adaptive wavelet approximations, we introduce interpolation scheme in the AWSFD method to mimic the approximation errors. We introduce AWSFD method to obtain the numerical solution of equation (3.5). By this method, we obtain

$$w(x,y) = w_{4,0}(x,y)$$
$$and\ \beta = (1+\alpha)^{-2m}e^{-m}k^2$$

in equation (3.5) which yields:

$$\frac{\partial^2 w(x,y)}{\partial y^2} + \frac{\partial^2 w(x,y)}{\partial x^2} + \beta w(x,y) = 0$$

We discretized $w(x,y)$ at n points of $y-$ spatial window $[0,\frac{\pi}{2}]$ into $M+1$ equally spaced samples with

$$0 = y_o < y_1 < \ldots < y_M = \frac{\pi}{2}$$
$$\Delta y = y_{j+1} - y_j = \frac{\pi}{2M}.$$

using difference quotient, we obtain

$$w^{j+1}(x) - \quad 2w\Delta^j y(2x) + w^{j-1}(x) + d^2dxw(2x) \quad {}_j(x) = 0$$
$$+\beta w$$

$$w^{j+1}(x) + (\eta \quad 2)w^j(x) + w^{j-1}(x) + \_\_ = 0, \qquad (4.8)$$

$\Rightarrow$ – β dx^2 $\eta\, d^2w^j(x)$

where,

$$\vartheta = \beta\Delta y^2.$$

Let

$$\xi = 0,1,...,n-1$$

be the sampling points, then

$$x = \Delta x\xi,$$

where Δx is the spatial interval between two sampling points. We then project $w^j(x)$ in a $\mathbf{V}_0$ space as

$$w^j(x) = w^j(\xi) = \sum_k w_k^j(\xi)\phi(\xi - k), \quad k \in \mathbb{Z} \qquad (4.9)$$

where $w_k^j(\xi)$ are approximation coefficients. Substituting $w^j(\xi)$ and $x = \Delta x\xi$ into equation (4.8), we obtain

$$\sum_{k=-\infty}^{\infty} w_k^{j+1}(\xi)\phi(\xi - k) + (\vartheta - 2)\sum_{k=-\infty}^{\infty} w_k^j(\xi)\phi(\xi - k) + \sum_{k=-\infty}^{\infty} w_k^{j-1}(\xi)\phi(\xi - k)$$

$$+ \quad \gamma \sum_{k=-\infty}^{\infty} w_k^j(\xi)\phi''(\xi - k) = 0,$$

where,

$$\gamma = \frac{\Delta y^2}{\Delta x^2}.$$

Multiplying the above equation by $\varphi(\xi - i)$, where $i = 0,1,...,n-1$ and taking the inner product with $\varphi(\xi - k)$, we obtain

∞ $j+1$ $2\pi/\Delta x$

X = $w_k(\xi)$Z0 $\varphi(\xi - k)\varphi(\xi - i)d\xi$ $k-\infty$

∞ j $2\pi/\Delta x$

$$+ (\vartheta - 2) \sum_{k=-\infty}^{\infty} w_k^j(\xi) \int_0^{2\pi/\Delta x} \varphi(\xi - k)\varphi(\xi - i)d\xi$$

$$+ \sum_{k=-\infty}^{\infty} w_k^{j-1}(\xi) \int_0^{2\pi/\Delta x} \varphi(\xi - k)\varphi(\xi - i)d\xi$$

$$+ \gamma \sum_{k=-\infty}^{\infty} w_k^j(\xi) \int_0^{2\pi/\Delta x} \varphi''(\xi - k)\varphi(\xi - i)d\xi = 0.$$

By definition (4.4), we can see from the above equation that

$$\gamma \sum_{k=i-N+2}^{i+N-2} \Omega^2_{i-k} w_k^j = 0, \quad for \;\; k \neq i$$

or

$$w_i^{j+1} + (\vartheta - 2)w_i^j + w_i^{j-1} + \gamma \sum_{k=i-N+2}^{i+N-2} \Omega^2_{i-k} w_k^j = 0, \quad for \;\; k = i, \quad i = 0, 1, 2, \ldots, n-1$$

where, Ω^2_{i-k} are second-order connection coefficients defined as

$$\Omega^2_{i-k} = \int_0^{2\pi/\Delta x} \phi''(\xi - k)\phi(\xi - i)d\xi$$

and its $n \times n$ second-order derivative circulant connection coefficients matrix is:

$$\Omega^2_{i-k} = \begin{pmatrix} \Omega^2_0 & \Omega^2_{-1} & \cdots & \cdots & \cdots & \Omega^2_1 \\ & \Omega^2_0 & \cdots & \Omega^2_{-N+2} & \Omega^2_{N-2} & \cdots \\ \vdots & \cdots & \Omega^2_{-N+3} & \cdots & 0 & \cdots & \Omega^2_2 \\ & \vdots & & \vdots & \vdots & \ddots \\ \Omega^2_{-1} & \Omega^2_{-2} & \cdots & 0 \cdots & \Omega^2_{N-3} & \cdots & \Omega^2_0 \end{pmatrix} \quad (4.10)$$

from the above equation. For the evaluation of second-order connection coefficients see appendix 3.

For the sake of consistency, we change the subscript of w_k^j to w_{i-k}^j to match Ω_{i-k}. Thus,

$$\gamma \sum_{k=i-N+2}^{i+N-2} \Omega_{i-k}^2 w_{i-k}^j = 0 \quad for \;\; k \neq i \tag{4.11}$$

or

$$w_i^{j+1} + (\vartheta - 2) w_i^j + w_i^{j-1} + \gamma \sum_{k=i-N+2}^{i+N-2} \Omega_{i-k}^2 w_{i-k}^j = 0, \quad for \;\; k = i,$$

We see from equation (4.11) that

$$\gamma = 06$$

$$\Rightarrow \sum_{k=i-N+2}^{i+N-2} \Omega_{i-k}^2 w_{i-k}^j = 0$$

Thus, we have

$$\det\left(\sum_{r=N-2}^{-N+2} \gamma \Omega_r^2 - \lambda I\right) = 0 \tag{4.12}$$

$$and \; w_r^j \; {=}6 \;\; 0,$$

where $r = i - k$ which is an initial approximation for equation (4.14). where λ are the eigenvalues of a matrix containing elements Ω_r^2 and w_r^j are associated eigenvectors. I is an $n \times n$ identity matrix.

Several methods like circular convolution method, penalty function method have been proposed for treating the boundary conditions. For example, see [25,34,38].

In this work, the ghost point w_r^{j-1} is approximated from the y–spatial boundary condition as

$$\frac{w_r^{j+1} - w_r^{j-1}}{2\Delta y} = \frac{\beta}{k^2 n}\sin(4nx)$$

$$\Rightarrow w_r^{j-1} = w_r^{j+1} - \frac{\beta \Delta y}{k^2 n}\sin(4nx)$$

$$\Rightarrow w_r^{j+1} = w_r^{j-1}, \qquad \forall \sin(4nx) = 0.$$

Substituting w_r^{j-1} into the above equation yields

$$2\sum_{r=N-2}^{-N+2} w_r^{j+1} + (\vartheta - 2)\sum_{r=N-2}^{-N+2} w_r^{j} + \gamma \sum_{r=N-2}^{-N+2} \Omega_r^2 w_r^j = 0, \quad for \;\; k = i,$$

$$\sum_{r=N-2}^{-N+2} w_r^{j+1} = -\frac{1}{2}\Big[(\vartheta - 2)\sum_{r=N-2}^{-N+2} w_r^{j} + \gamma \sum_{r=N-2}^{-N+2} \Omega_r^2 w_r^j\Big] \quad for \;\; k = i, \;\; and \;\; j = 0 .$$

For $j \geq 1$, we obtain

$$\sum_{r=N-2}^{-N+2} w_r^{j+1} = -\frac{1}{2^{j+1}}\Big[(\vartheta - 2)\sum_{r=N-2}^{-N+2} w_r^{j} + \gamma \sum_{r=N-2}^{-N+2} \Omega_r^2 w_r^j\Big] \quad for \;\; k = i \tag{4.13}$$

For second and subsequent approximations, we use the expression on the right hand side of equation (4.13) without the negative sign which gives

$$\sum_{r=N-2}^{-N+2} w_r^{j+1} = \frac{1}{2^{j+1}}\Big[(\vartheta - 2)\sum_{r=N-2}^{-N+2} w_r^{j} + \gamma \sum_{r=N-2}^{-N+2} \Omega_r^2 w_r^j\Big], \quad for \;\; j \geq 1 \tag{4.14}$$

where,

$$0 < \gamma < 1$$

$$\sum_{r=N-2}^{-N+2} w_r^j > 0$$

We use equation (4.14) to approximate the solution of the regularized Helmholtz equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous. The approximated solution of the regularized Helmholtz equation is obtained by increasing the level of resolution j.

Also, we implement an interpolation scheme into AWSFD method as follows. We write equation (3.5) as a polynomial of degree $p-1$ for $w^j(\xi)$ in the neighbourhood of $\xi = 0$. Thus, we rewrite equation for $w^j(\xi)$ as

$$w^j(\xi) = \sum_{j=0}^{p-1} c_j \xi^j \qquad (4.15)$$

Applying inner product of the expressions on both left and right hand sides of equation (4.15) with $\varphi(\xi - i)$, we obtain

$$\int_0^{2\pi/\Delta x} w^j(\xi)\varphi^j(\xi - i)\,d\xi = \sum_{j=0}^{p-1} c_j \int_0^{2\pi/\Delta x} \xi^j \varphi^j(\xi - i)\,d\xi$$

$$w_i^j(\xi) = \sum_{j=0}^{p-1} c_j \mu_i^j(\xi), \qquad i = -1, -2, \ldots, -N+2 \qquad (4.16)$$

where,

$$\mu_i^j(\xi) = \int_0^{2\pi/\Delta x} \xi^j \phi^j(\xi - i)\,d\xi$$

are the moments of the scaling function. Since there is no explicit formula for calculating Daubechies scaling functions, we use an interpolation scheme obtained from equation (4.16). The values of the interpolant at location $\varphi^j(\xi - i)$ would be constructed as a polynomial $P_{2N-1}(\tau)$ of order $(2N-1)$ on the values of the function at location ξ_{k+1}^j and evaluate them as $w(\xi_{k+1}^j)$. Thus,

$$\mu_i^j(\xi) = \sum_{l=-N+1}^{N} B_{k,l}^j w(\xi_{k+l}^j), \qquad (4.17)$$

where $B_{k,l}^j$ is the Daubechies scaling coefficients and N is the order of Daubechies scaling function.

In order to obtain adaptive wavelet solution of equation (3.5), we solve equation (4.16) recursively. First, we approximate the left hand side equation (4.16) by the boundary conditions of equation (3.5) with respect to ξ, and the right hand side by the substitution the values of $\mu_i^j(\xi)$, given by the equation (4.17), into equation (4.16), which gives the constant coefficients $c_{0j}s$. Thus,

$$\begin{pmatrix} w_{-1}^j \\ w_{-2}^j \\ \vdots \\ w_{-N+2}^j \end{pmatrix} = \begin{pmatrix} \mu_{0,-1} & \mu_{1,-1} & \cdots & \mu_{p-1,-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{0,-N+2} & \mu_{1,-N+2} & \cdots & \mu_{p-1,-N+2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{pmatrix}$$

We can solve the above matrix as

$$c = A^{-1} w_i^j(\xi),$$

where A^{-1} is an $n \times n$ inverse matrix of

$$A = \begin{bmatrix} \mu^0_{-1} & \mu^1_{-1} & \cdots & \mu^{p-1}_{-1} \\ \mu^0_{-2} & \mu^1_{-2} & \cdots & \mu^{p-1}_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu^0_{-N+2} & \mu^1_{-N+2} & \cdots & \mu^{p-1}_{-N+2} \end{bmatrix}$$

We then compute $w_i^j(\xi)$ as

$$w_i^j(\xi) = \sum_{j=0}^{p-1} c_j \mu^j_{i-n}(\xi), \quad i = (n-1)-p+1, (n-1)-p+2, \ldots, n-1 \quad . \tag{4.18}$$

as a second iteration. The associated matrix is of the formula

$$\begin{pmatrix} w^j_{(n-1)-p+1} \\ w^j_{(n-1)-p+2} \\ \vdots \\ w^j_{(n-1)} \end{pmatrix} = \begin{bmatrix} \mu^0_{-p} & \mu^1_{-p} & \cdots & \mu^{p-1}_{-p} \\ \mu^0_{-p+1} & \mu^1_{-p+1} & \cdots & \mu^{p-1}_{-p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu^0_{-1} & \mu^1_{-1} & \cdots & \mu^{p-1}_{-1} \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{pmatrix}$$

The left hand side of the above matrix is obtained using the finite difference scheme of $w_i^{j+1}(\tau)$. The $c_j^{\prime} s$ are then substituted into equation (4.15) for

$$i = n, n+1, \ldots, n+N-2$$

to derive the matrix for

$$w^j_{(n-1)-p+1}, w^j_{(n-1)-p+2}, \ldots, w^j_{n-1}$$

as

$$\begin{pmatrix} w^j_{(n)} \\ w^j_{(n+1)-p+2} \\ \vdots \\ w^j_{(n-1)+N-2} \end{pmatrix} = \begin{bmatrix} \mu^0_{0} & \mu^1_{0} & \cdots & \mu^{p-1}_{0} \\ \mu^0_{1} & \mu^1_{1} & \cdots & \mu^{p-1}_{1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu^0_{-N+2} & \mu^1_{-N+2} & \cdots & \mu^{p-1}_{-N+2} \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{pmatrix}$$

The approximation coefficients are substituted into equation (4.12) to obtain the connection coefficients.

In equations (3.8) and (3.5) with Cauchy boundary conditions, where the boundary deflection is equal to zero, we discretize x spatial variable by difference quotient instead of y spatial variable, project the result orthogonally in V_0 space and then find the inner product with $\varphi(\xi - k)$ to obtain two algebraic equations for k =6 i and $k = i$. The equations are solved spectrally.

In a nutshell, the AWSFD method approximates regularized Helmholtz equation together with either Cauchy boundary conditions or Neumann boundary conditions in the upper half-plane or both Cauchy and Dirichlet boundary conditions (where boundary deflection is zero) with minimum error. Unlike other wavelet methods, this method approximates the regularized equation by first discretizing the one of the spatial variables which obtains of ODEs and then project the discretized equation orthogonally in appropriate space V_j using Daubechies scaling function. This provides two sets of system of algebraic equations. We solve the first equation spectrally and the result is substituted into the second equation.

4.3 Numerical Results

In this section, we provide the numerical solution of equation (3.5), (3.8) and (3.5) together with both regularized Cauchy and Dirichlet boundary conditions where the boundary deflection is homogeneous. We compare the analytic solutions of the regularized Helmholtz equation by DRM with wavelet solutions by the AWSFD method.
All plots of the solutions by DRM, as well as solutions by AWSFD method are obtained using matlab.

In figure 4.1, we display the solution of the regularized Helmholtz equation with regularized Cauchy boundary conditions where the boundary deflection is inhomogeneous in two dimensions co-ordinates. The regularized solution $w(x,y)$ of the regularized Helmholtz equation increases slowly as n increases with other parameters held constant. In figure 4.2, we display a similar graph of solution of the regularized Helmholtz equation by increasing n = 5, and finally perturbing α from 2 to 50 in figure 4.3.

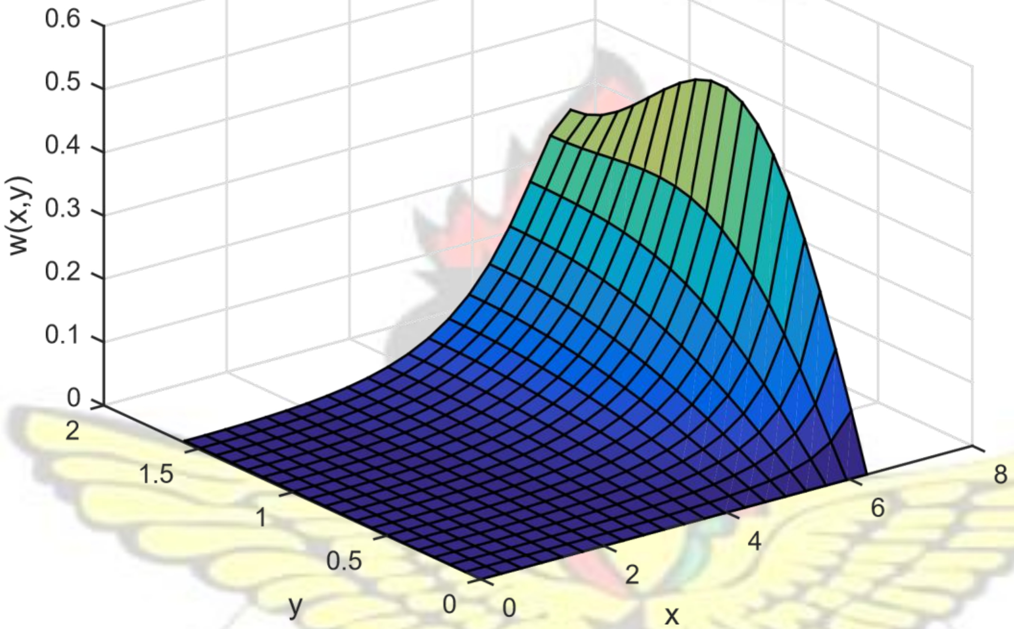

Figure 4.1: Regularized Helmholtz equation with regularized Cauchy boundary conditions for n = 3, α = m = k = 2 in two dimensions co-ordinates

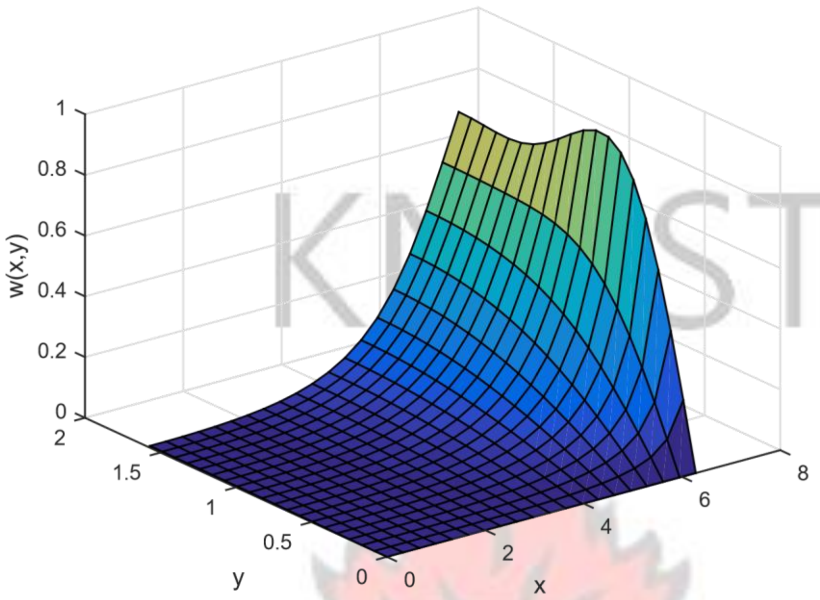

Figure 4.2: regularized Helmholtz equation with regularized Cauchy boundary conditions for n = 5, $\alpha = m = k = 2$ in two dimensions co-ordinates

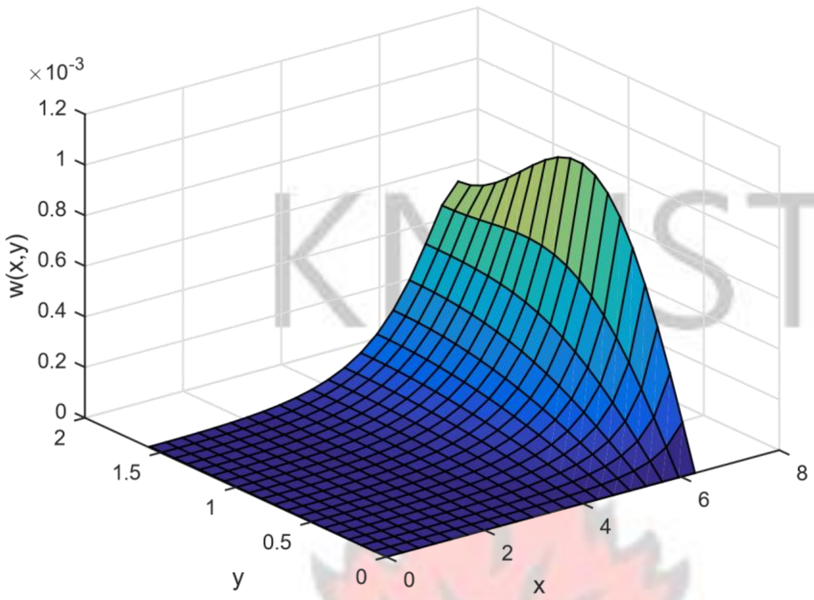

Figure 4.3: regularized Helmholtz equation with regularized Cauchy boundary conditions for n = 3, α = 50, m = k = 2 in two dimensions co-ordinates

We perform similar analysis of equation (3.5) in one dimension co-ordinate, we plot $w(.,y)$ against y with spatial variable x at equilibrium as shown in figure 4.4. We observe that the solution of regularized Helmholtz equation $w(.,y)$ grows slowly as the spatial variable y increases with other parameters held constant. These solutions become asymptotically to x as m becomes large.

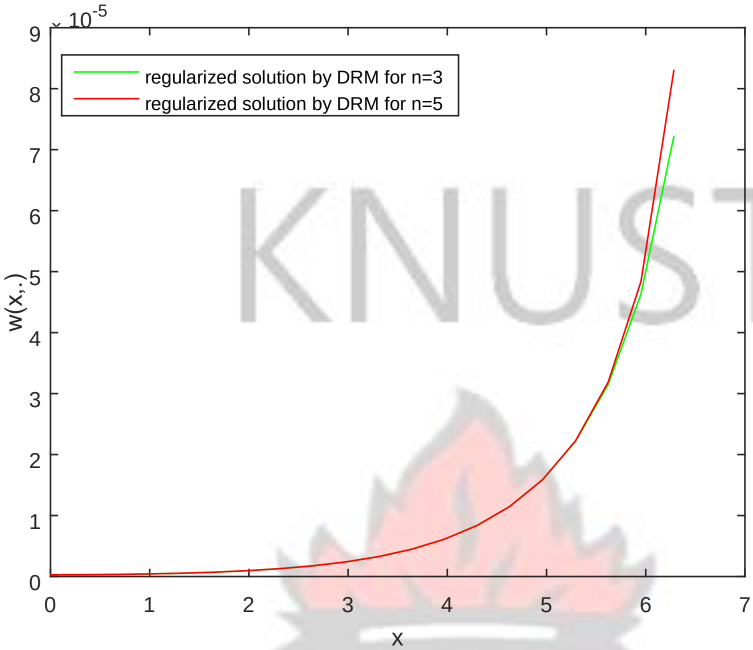

Figure 4.4: The solution of regularized Helmholtz equation with regularized Cauchy boundary conditions for n = 3 α = m = k = 2 in one dimension co-ordinate

We perform the quantitative analysis of equation (3.8). In figures (4.5) and (4.6), we analysis the regularized Helmholtz equation in two dimensions co-ordinates and one dimension co-ordinate, respectively. We see from figure 4.5 that changes in both spatial variables x and y result in small changes in solution $w(x,y)$. Similar trends are shown in figure 4.6, which confirms the stability of the regularized solution. Also, we observed that varying the values of m, of the regularization parameter α, leads to the stability of the regularized solution of the regularized Helmholtz equation, see figure 4.7.

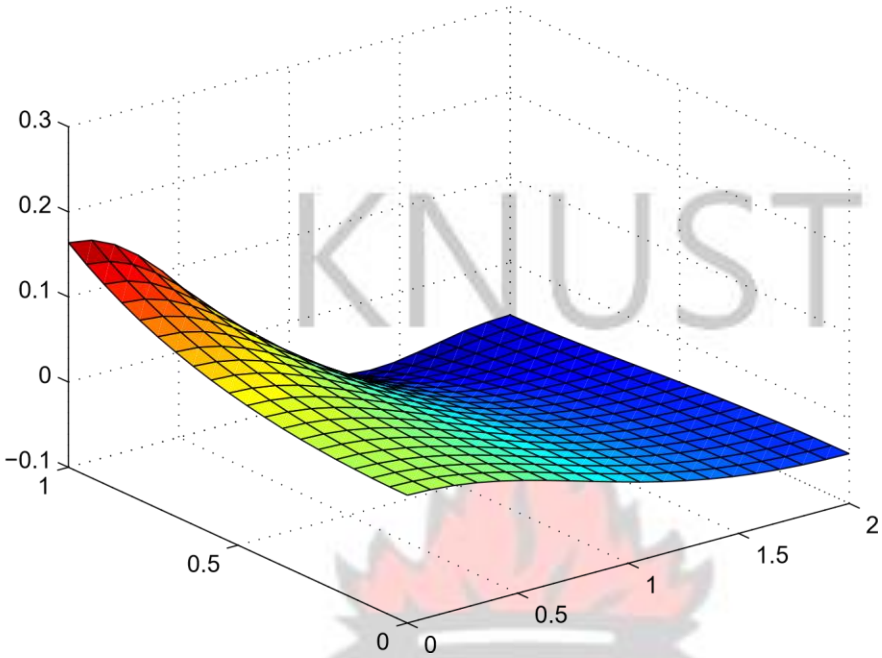

Figure 4.5: Solution of regularized Helmholtz equation with regularized Neumann boundary conditions for n = 3 in two dimensions co-ordinates

The solutions of equation (3.5) with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous in two dimensions co-ordinates are shown in figure 4.8. We display similar solution in one dimension co-ordinate by plotting $w(.,y)$ against y with x held at equilibrium, see figure 4.9. The regularized solution $w(.,y)$ starts at approximately zero increases slowly to approximately 0.03.

The growth of the regularized solution becomes very slow as n increases.

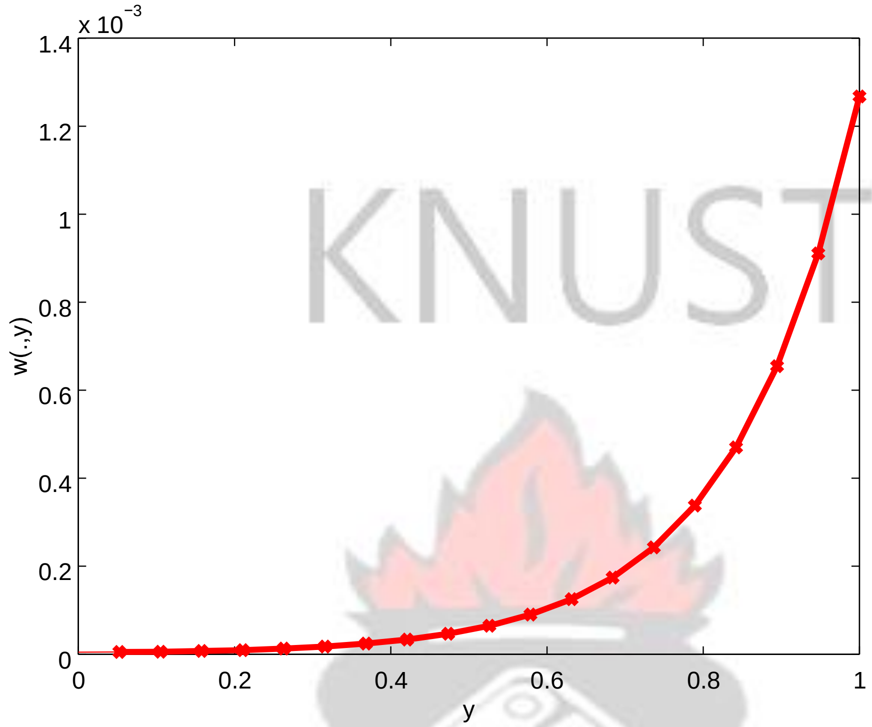

Figure 4.6: Solution of regularized Helmholtz equation with regularized Neumann boundary conditions in one dimension co-ordinate for $n = 3$

In order to confirm the analytic solutions by DRM, we give the quantitative solutions of equations (3.5), (3.8) and equation (3.5) together with both Dirichlet and Cauchy boundary where boundary deflection is homogeneous by ADSFD method. Firstly, we display solutions by AWSFD method of equation (3.5). In figure 4.10, we display the initial approximated solutions by equation (4.12) and equation (4.14), respectively. We observe that solution given by equation (4.12) starts at approximately 2.5 and ends 4.4, whereas solution given by equation (4.14) starts increases steady from 0.2 and grows steadily to about 1.2. In figure 4.11, we compare the solutions by AWSFD method at different resolution levels (j = 10,11,12) with regularized solution by DRM for $k = m = \alpha = 2$ and $n = 3$.

By AWSFD method, the approximated solution gets better as the resolution level

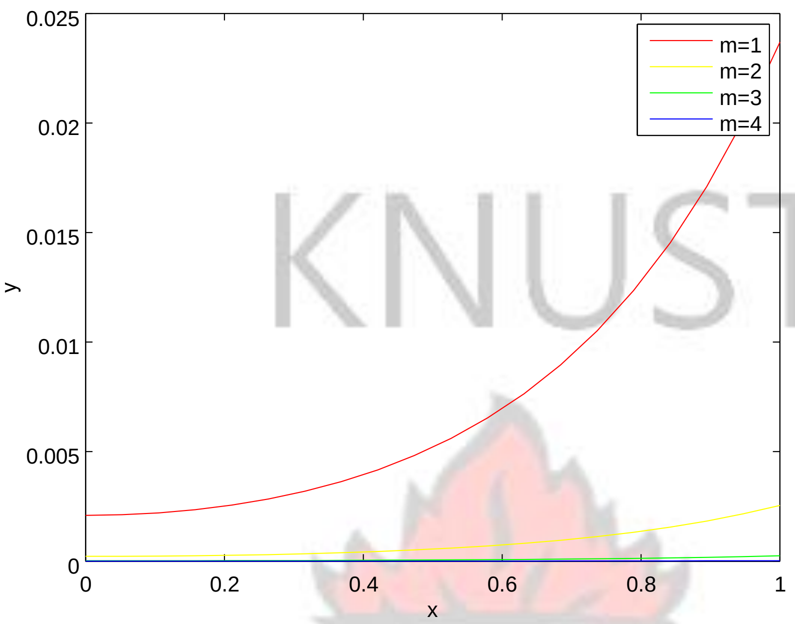

Figure 4.7: Comparison of solutions of regularized Helmholtz equation with regularized Neumann boundary conditions for different values of m

increases. Thus, solution at j = 11 is better than solution at j = 10 and so on. At each j + 1 resolution level, equation (4.14) is updated by the current resolution j. Comparatively, we observe that end points of solutions by AWSFD method do not grow sharply as that of the solution given by DRM. This is due to additive of the regularization term $(1 + \alpha)^{-2m} e^{-m}$ in the equation (4.14) of the AWSFD method. Notwithstanding, the γ reduces the growth in the determination of eigenvalues in equation (4.12).

Also, we perform similar analysis on equation (3.8) as follows. In figure 4.12, we display solution given by equation (4.12), which is the initial approximated solution for equation (4.14) in the space V_0. We can see that the solution by equation (4.12) is slightly deviated from the solution given by equation (4.14) on the grounds that equation (4.12) does not involve the regularization term $(1 + \alpha)^{-2m} e^{-m}$. Secondly,

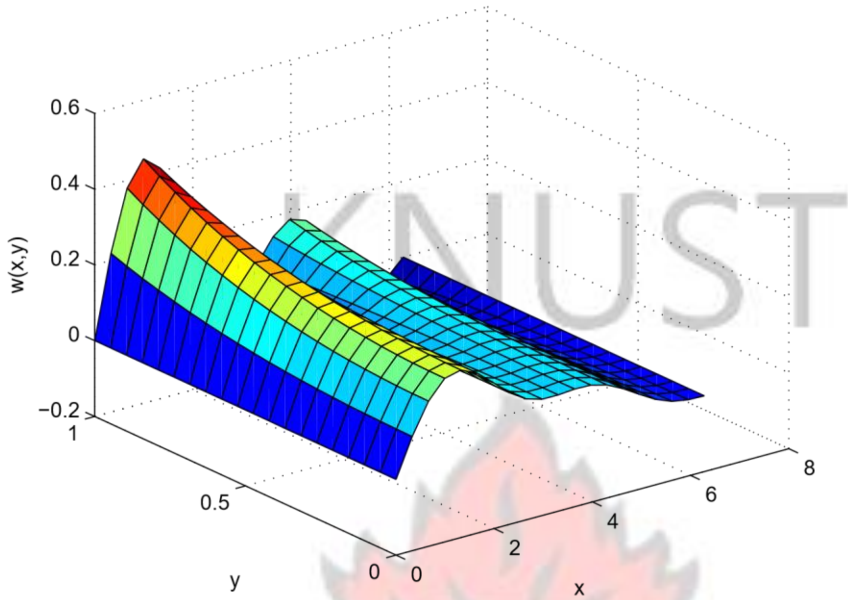

Figure 4.8: Solution of equation (3.5) together with both regularized Dirichlet and Cauchy boundary conditions for $n = 4$ in two dimensions co-ordinates

equation (4.12) provides only approximated solution of the regularized Helmholtz at resolution level $j = 0$. For the sake of comparison, the solution by DRM and the solutions by AWSFD method, we display solutions by AWSFD method for resolution $j = 10,11,12$ and by DRM for $m = n = k = \alpha = 2$, see figure 4.13. We observe that approximated solution by AWSFD method is better for a low level of resolution.

Last but not least, we display the solutions of equation (3.5) together with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous by AWSFD method. In figure 4.14, we display quantitative solutions of equation (4.12) and of equation (4.14) by AWSFD method and then compared with solution by DRM. We can see that an initial approximated solution given by equation (4.12) is farther away from the other two solutions. The first approximated solution by equation (4.14) of AWSFD method at resolution $j = 1$ (magenta) is close to so-

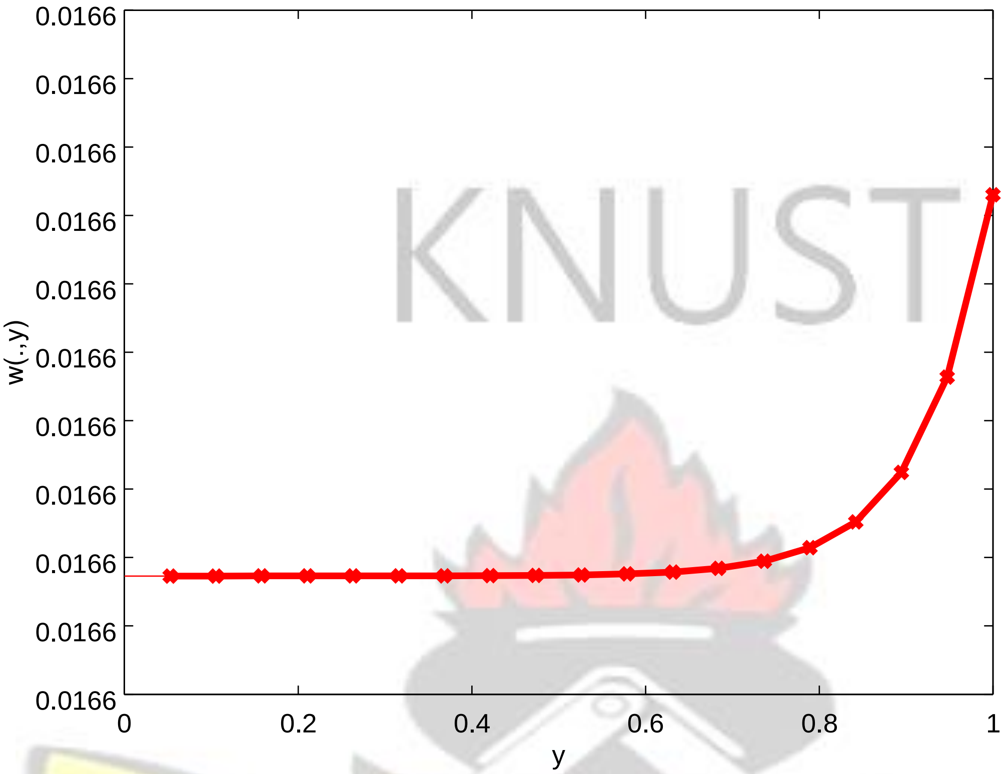

Figure 4.9: Solution of equation (3.5) together with both regularized Dirichlet and Cauchy boundary conditions for $n = 4$ in one dimension co-ordinate

lution by DRM (red). The approximation by AWSFD is sharp. We then increase the resolution level of the AWSFD method from $j = 1$ to 2 and so, see figure 4.15. Comparatively, the solutions by AWSFD method draw close to solution by DRM as the resolution level increases from zero to one, two, three and four.

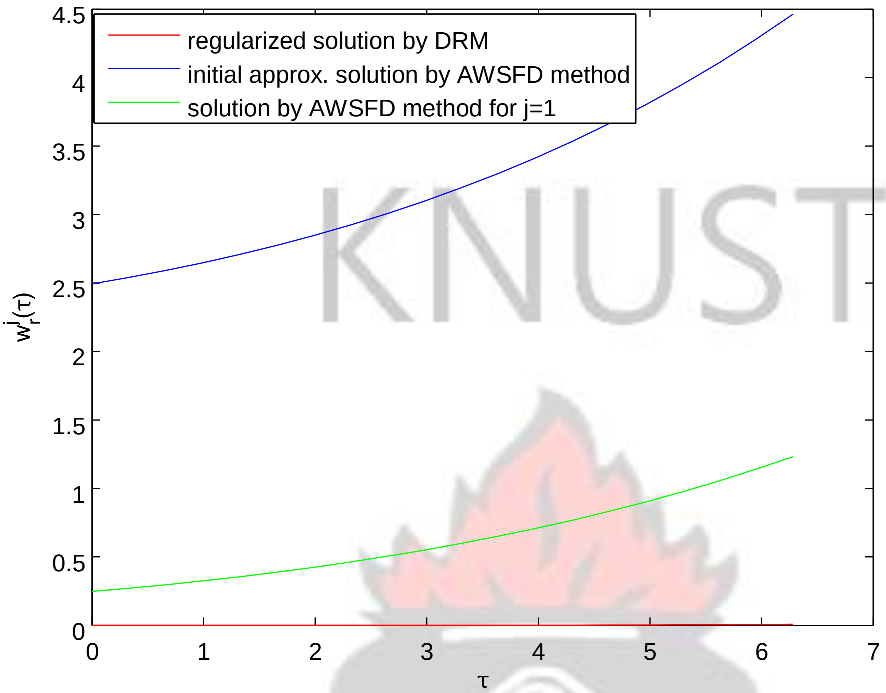

Figure 4.10: The regularized solution by DRM, initial approximated solution of equation (4.12) and solution of equation (4.14) with AWSFD method at $j = 1$

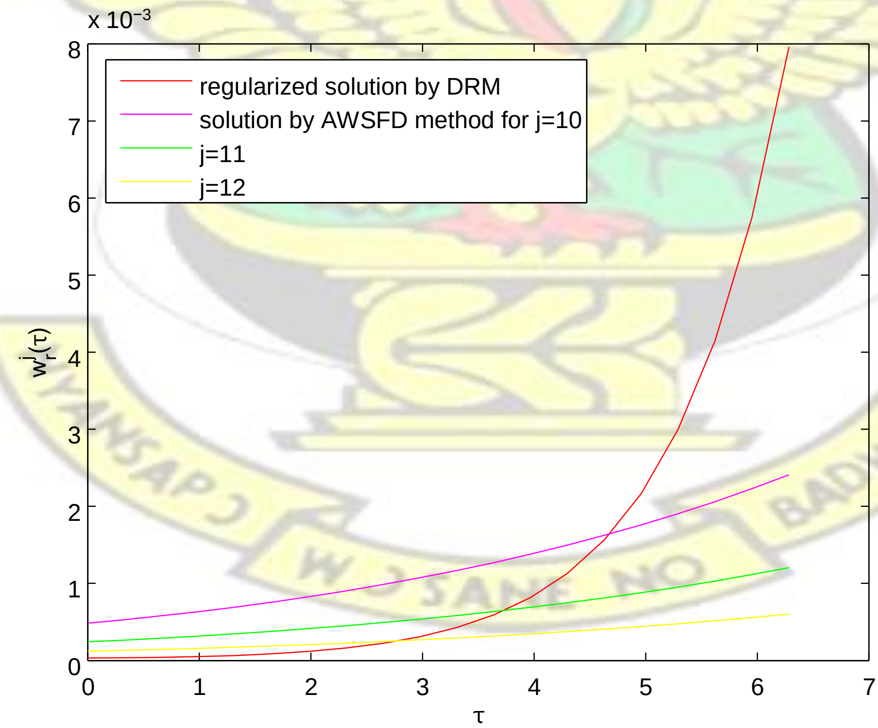

Figure 4.11: Comparison of the regularized solution by DRM and solutions using AWSFD method for j = 10,11,12

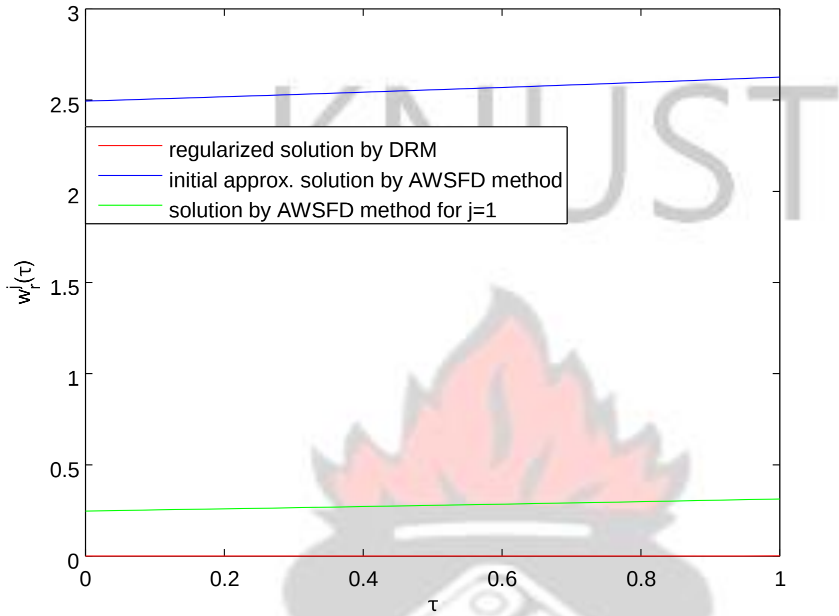

Figure 4.12: Comparison of regularized solution by DRM, initial approximated solution by equation (4.12) and solution by equation (4.14) of AWSFD method at j = 1

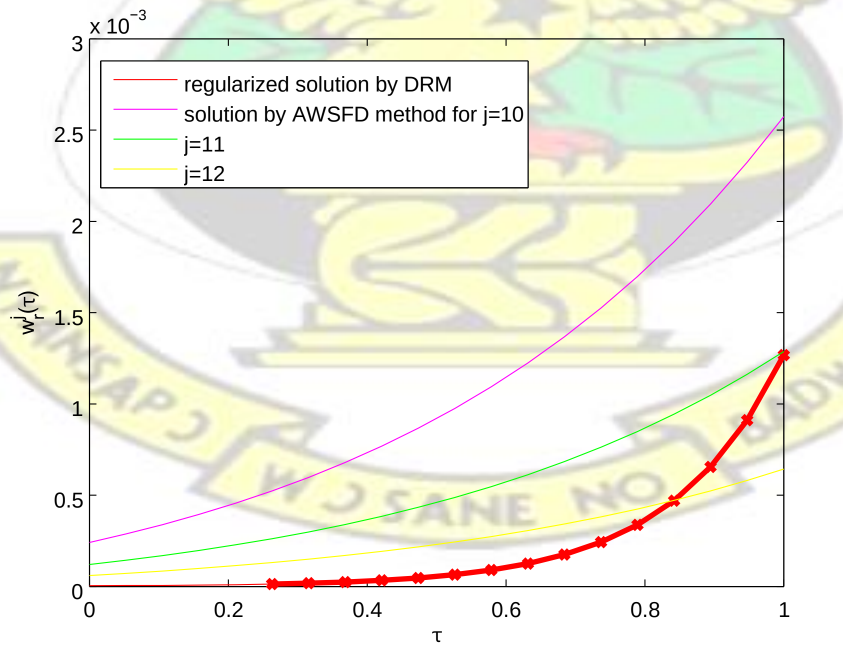

Figure 4.13: Comparison of solution by DRM and solutions by AWSFD method for 6,7,8

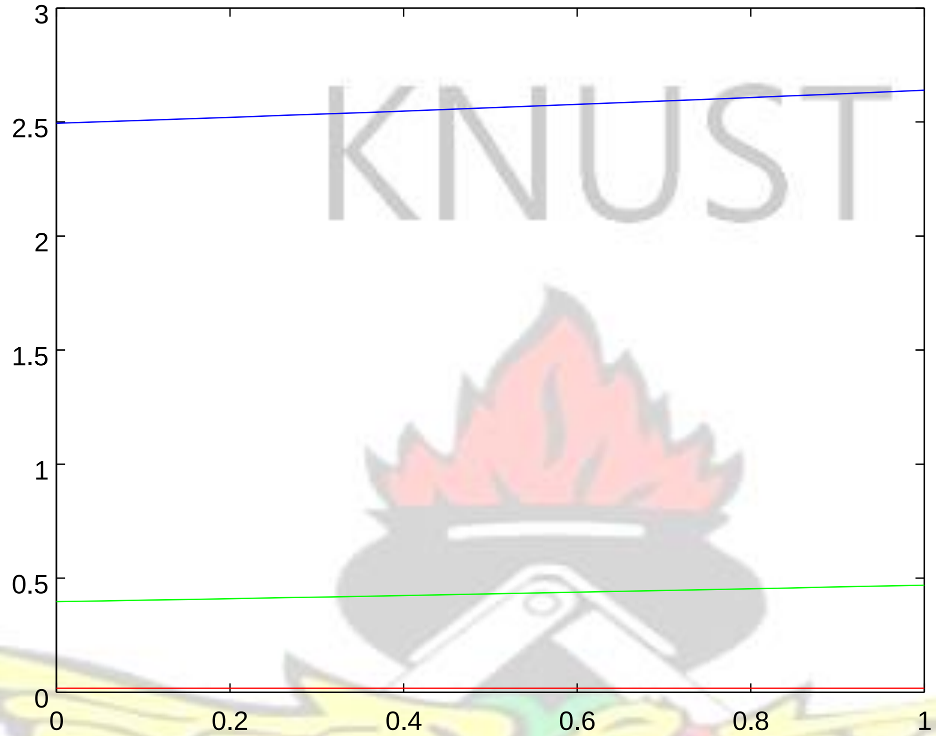

Figure 4.14: Comparison of the regularized solution by DRM, initial approximated solution given by equation (4.12) and solution given by equation (4.14) of AWSFD method at j = 1

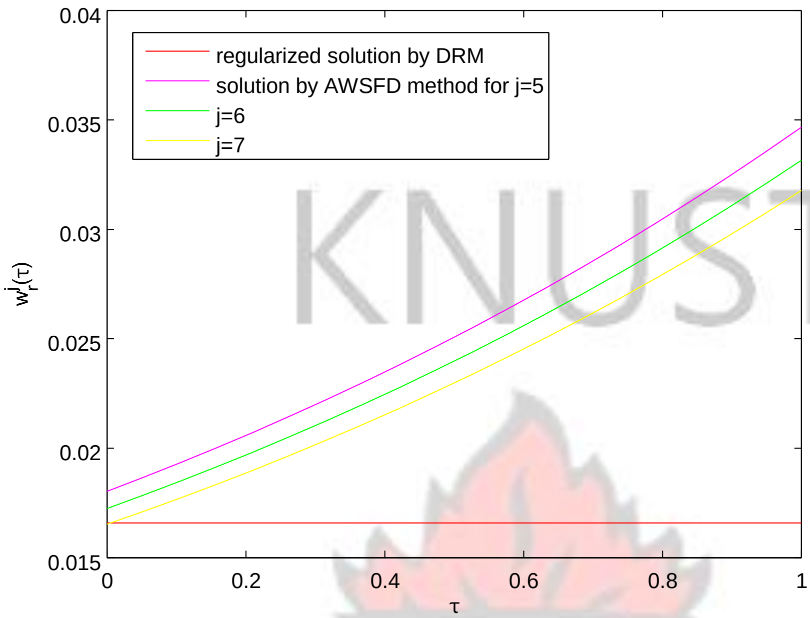

Figure 4.15: Comparison of the regularized solution by DRM, initial approximated solution given by equation (4.12) and solution given by equation (4.14)

Chapter 5

Summary, Concluding Remarks and Suggestions for future work

In this chapter, we summary the main findings of the thesis. We discuss the strengths of the DRM for solving ill-posed Helmholtz equations with Cauchy boundary conditions where the boundary deflection is inhomogeneous, then with Neumann boundary conditions in the upper half-plane and finally with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous, and the AWSFD method for the above equations. We give a brief summary of the previous chapter whilst the most siginificant conclusions of each chapter are highlighted.

5.1 Outline of the Previous Chapters

In this section, we summarize each of the four previous chapters. In chapter one, we showed the three kinds of boundary conditions, which when imposed on Helmholtz equation lead to ill-posedness. Thus, we proved that the Helmholtz equation with Cauchy boundary conditions has no solution. None of the three conditions of wellposedness according to Hadamard is satisfied. Secondly, we showed that Neumann problem of the Helmholtz equation has no unique solution. We then proved that when Helmholtz equation is imposed with both Cauchy and Dirichlet boundary conditions, a unique solution exists, but the solution does not depend continuously on the small changes in the boundary conditions.

In the chapter two, we showed that the TRM, SRM, Q-RRM, Q-BVM and IRM cannot regularize the Cauchy problem of the Helmholtz equation where the boundary deflection is inhomogeneous, as well as Helmholtz equation with Neumann boundary conditions in the upper half-plane. We observed that neither the regularized Helmholtz equation nor the regularized solution by the use of these existing methods of regularization satisfy the Riesz representation theorem. In the case of both Cauchy and Dirichlet problems of the Helmholtz equation all the existing methods of regularization ensured that the inverse operator is continuous. Thus, these methods of regularization do not satisfy bounded inverse theorem.

To circumvent the regularization of Cauchy problem of the Helmholtz equation where the boundary deflection is inhomogeneous, as well as Neumann problem, we showed in chapter three of the thesis that the (introduced) DRM regularizes both the Helmholtz equation and the imposed boundary conditions. We gave the DRM in theorem 3.3. The alternative characterization of the DRM theorem, theorem 3.3, in terms of the Laplace operator in the Hilbert space is as follows:

Theorem 5.1 *Let $A : \Omega \subset H \to H$ be a (linear) Laplace operator in the Helmholtz equation from the subspace of a Hilbert space Ω into a Hilbert space H, where*

$$Aw(x,y) = 0, \quad in \quad \Omega$$

$$\text{with } \frac{\partial w(x,0)}{\partial y} = h(x) \quad \text{on } \partial\Omega$$

$$w(x,0) = 0 \quad \text{on} \quad \partial\Omega,$$

and

$$\int_{\partial\Omega} h(x)dx \neq 0,$$

then the regularized Helmholtz equation with regularized Cauchy boundary conditions is:

$$\begin{aligned}
Aw_{\eta,0}(x,y) &= 0, \quad \text{in} \quad \Omega \\
\frac{\partial w_{\eta,0}(x,l)}{\partial y} &= (1+\alpha^{2m})^{-1} e^{-m} h(\eta x), \quad \text{on } \partial\Omega_1 \\
w_{\eta}(x,0) &= 0, \quad \text{on } \partial\Omega_1 \\
\frac{\partial w_{\eta,0}(0,y)}{\partial x} &= 0, \quad \text{on } \partial\Omega_2 \\
w_{\eta,0}(l,y) &= (1+\alpha^{2m})^{-1} e^{-m} h(y), \quad \text{on } \partial\Omega_2,
\end{aligned}$$

where

$$\int_{\partial\Omega_1} h(\eta x)dx = 0,$$

$h(x) \neq 0$, $h(y) \neq 0$, $\alpha \in (-\infty,-1) \cup (1,\infty)$ *is the regularization parameter,* $m \in \mathbb{Z}^+$ *is a positive integer,* k *as the wave number,* $[0,l]$ *is the square domain with* l *is a radian number and* η *is any even positive integer.*

Thus, using the DRM, we showed that the range of the operator $R(A)$ is closed, the null space of the operator $N(A)$ is trivial and the inverse operator

$$A^{-1} : \mathbb{R}^2 \rightarrow \Omega$$

is continuous. In other words, the operator A is a homeomorphism of $\Omega \subset \mathbb{R}^2$ onto $\mathbb{R}^2$ in the case of Cauchy problem of the Helmholtz equation. In the case of the Neumann problem of the Helmholtz equation, we established that

$$N(A) = 0$$

and $A^{-1} : R^2 \rightarrow \Omega$ is continuous. Finally, we showed that when Helmholtz equation is imposed with both Cauchy and Dirichlet boundary conditions, $A^{-1} : R^2 \rightarrow \Omega$ is continuous with use of DRM. This satisfies closed graph theorem.

The AWSFD method is shown chapter four to confirm the analytic solutions given by DRM. The solutions obtained using the AWSFD method were compared with the regularized solutions by DRM. At lower wavelet resolution, the solution by AWSFD method not close to the regularized solution, but as the wavelet resolution increases the solution becomes close to the regularized solution by DRM.

5.2 Concluding Remarks

We observed that the DRM solves the Cauchy problem of the Helmholtz equation where the boundary deflection is not equal to zero. The DRM employs a positive integer scaler η in x spatial variable of the unknown function $w(x,y)$, which homogenizes the inhomogeneous boundary deflection in the Cauchy boundary conditions. The integer scaler η ensures the existence of solution to Helmholtz equation. The value of η depends on the periodic function such as $\sin(nx)$, $\cos(nx)$, as well as scalar multiplier of spatial variable (angle) imposed at inhomogeneous boundary deflection in Cauchy problem of the Helmholtz equation. The regularization term $(1+\alpha)^{-2m}e^{-m}$ restores the stability of the Cauchy problem of Helmholtz equation. Homogenization of inhomogeneous boundary deflection in Cauchy boundary conditions by a positive integer scale together with applications of divergence theorem and Green's first identity ensures the uniqueness of the solution of the regularized Helmholtz equation.

In order to establish uniqueness of solution to the Neumann problem for the Helmholtz equation in the upper half-plane, the DRM employs shifting operator instead of a positive integer scale to shift the x spatial variable to a suitable fixed number. On the other hand, the methods of regularization: TRM, SRM, Q-RRM, Q-BVM and IRM cannot solve Cauchy problem (where the boundary deflection is inhomogeneous) of Helmholtz equation, as well as Neumann problem in the upper half-plane of the Helmholtz equation.

In the case of both Dirichlet and Cauchy boundary conditions (where the boundary deflection is equal to zero) of Helmholtz equation, all the above methods of regularization solve the problem. We observed that DRM provides the best approximated solution as compared with these existing methods of regularization. This is observed when a positive integer m of the regularization term $(1 + \alpha)^{-2m}e^{-m}$ increases. The solution becomes consistent and stable.

We observed that the coarser scale equation (4.12) of the AWSFD method does not give good approximated solution of equation (3.5). This coarser scale equation does not contain the regularization term $(1+\alpha)^{-2m}e^{-m}$. It leads to approximated solution only in resolution level $j = 0$. Equation (4.12) acts as an initial approximation for finer scaler equation (4.14), which results in the approximated solution of equation (3.5) by performing a series of iterations. Thus, equation (4.14) gives approximation at a finer scale $j + 1$.

Using the AWSFD method, we solved the equations (3.5), (3.8) and equation (3.5) together with regularized Cauchy boundary conditions where the boundary deflection is equal to zero for Daubechies wavelet of order six (6) numerically. We observed that the solution by AWSFD method approximates the solution by DRM when the resolution level is increased, as well as, the regularization term $(1 + \alpha)^{-2m}e^{-m}$. At resolution $j = 0$, the solution by AWSFD method has a greater estimated error compared to the DRM.

Our results show that the AWSFD method provides best approximation at a low wavelet resolution level for solving the Helmholtz equation with both Dirichlet and Cauchy boundary conditions where the boundary deflection is homogeneous and the Neumann boundary conditions in the upper half-plane and then by Cauchy boundary conditions where the boundary deflection is inhomogeneous. This is due to the fact that in the case of both Dirichlet and Cauchy problems, where the boundary deflection is zero for the Helmholtz equation, we restored only the stability of the solution as compared with other discussed problems for the Helmholtz equation.

Similarly, AWSFD method provides best approximation at a lower wavelet resolution level of Neumann problem for the Helmholtz equation in the upper halfplane than Cauchy problem where the boundary deflection is inhomogeneous for the Helmholtz equation on the grounds that we restored uniqueness and stability in the Neumann problem for the Helmholtz equation as compared with the non-regularized Cauchy problem for the Helmholtz equation.

5.3 Suggestions for future work

Most of the stationary problems in acoustic are nonlinear in nature. These nonlinear problems have linear part which plays the central role in optimizing their solutions. We are of the view that the extension this work will help in the full understanding of the Helmholtz equation as a stationary process.

Firstly, We intend to investigate the Cauchy problem of the Helmholtz equation where the boundary deflection is imposed with either transcendental function $\ln(x)$ or inverse trigonometric functions like $\sec(nx)$,$cosec(nx)$, $0 \leq x \leq \pi$.

The well-posedness of the Cauchy problem of Poisson-Boltzmann equation in unbounded domain could be investigated by modifying the DRM.

We intend to investigate the regularization of $p(x)$-Laplace equation with Cauchy boundary conditions where the boundary deflection is inhomogeneous.

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Appendix 1

matlab code solving Helmholtz equation.

script file α =input('α'); k=input('k'); n=input('n'); r=linspace(a,b,20) t = linspace(c,d,20) [x,y] = meshgrid(r,t) p = (1/(1+ $\alpha^{(2_* m)}$))*exp(-m); p1= sqrt(n-p*k); f=(1+ $\alpha^{(2_* m)}$)*exp(m); d = (1/d).*cosh(p1*x)*sin(n*y); surf(x,y,z)

Appendix 2

matlab code for calculating connection coefficients The

function of $[cc2] = concoeff(L,o,\gamma,e)$

M=L/2; a=zeros(1,L-1);

$D = (factorial(2 * M - 1)/(factorial(M - 1) * (4^(M - 1))))^2$; for

i=1:2:L-1 m=(i+1)/2;

$a(i) = ((-1)^(m - 1)) * D/(factorial(M - m) * factorial(M + m - 1) * i)$; end

b2=zeros(L-1,L-1); for l=0:L-2 if

2*l¡=L-2

$i = 2 * l;b2(l + 1,i + 1) = 2^n$; end

for k=1:L/2 i l=2*l-2*k+1;

if i l < = L-2 and i 1 > = 0 i=i l;

$b2(l + 1,i + 1) = b2(l + 1,i + 1) + (2^(n - 1)) * a(2 * k - 1)$; elseif

i 1 > = - L+2 and i 1 < 0 i=-i l;

$b2(l + 1,i + 1) = b2(l + 1,i + 1) + ((-1)^n) * (2^(n - 1)) * a(2 * k - 1)$ $b2(l$

$+ 1,i + 1) = b2(l + 1,i + 1) + (2^(n - 1)) * a(2 * k - 1)$; elseif il > = -L+2

and $il < 0$

i=-i l;

$b2(l + 1,i + 1) = b2(l + 1,i + 1) + ((-1)n) * (2(n - 1)) * a(2 * k - 1)$; end

i 2 =(2*l+2*k-1);

if i 2 < = L-2 and i 2 > = 0
i = i 2;

$b2(l + 1,i + 1) = b2(l + 1,i + 1) + (2^(n - 1)) * a(2 * k - 1)$; else

if i 2 > = -L+2 and i 2 < 0 i=-i 2;

$b2(l+1,i+1) = b2(l+1,i+1) + ((-1)^n) * (2^{(n-1)}) * a(2*k-1)$; end

end end for g=1:L-1 b2(g,g)=b2(g,g)-1; end r=null(b2);

$NM = sqrt2$; for i=2:L-1 $[a,b] = eig(\gamma b2)$;

m=length(b) ; for j=1:m g(j)=g(j) d

=abs(sum(a(:,e))) ; y= a1*exp(g(j)*x) $k = k$

$+ ((i-1)^n) * r(i)$; $cc2 = ((-1)^n) * r *$

$factorial(n)/(2 * NM)$; end end

Appendix 3

Connection coefficients of order two

Let the $2nd$ order connection coefficients be

$$\Delta_k^2 = \int_0^x \phi^2(y-k)\phi(y)dy$$

By the definition of $2nd$ order derivative of the scaling function

$$\phi^2(x) = 4\sum_{k=0}^{N-1} a_k \phi^2(2x-k)$$

Substituting the second-order derivative of the scaling function into equation for second order connection coefficients, we obtain

$$\Delta_k^2 = 4\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i a_j \int_0^x \phi^2(2y-2k-j)\phi(2y-j)dy$$

Integrating by substituting $u = 2y$, we obtain

$$\Delta_k^2 = 2\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i a_j \int_0^{2x-i} \varphi^2(u-2k-j+i)\varphi(u)du$$

$$\Delta_k^2 = 2\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i a_j \Delta^2_{2k+j-i}(2x-1)$$