KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

OPTIMAL INVESTMENT PORTFOLIO BASED ON MODELS OF RISK AND EXPECTED RETURNS

BY

NYAMEKYE RICHARD KOFI [BSC. (MATHS), PGDE]

(PG 6322411)

A Thesis submitted to the College of Science in partial fulfillment of the requirements for the degree of MSC in Industrial Mathematics at KNUST.

SEPTEMBER, 2013

CANDIDATE'S DECLARATION

I, NYAMEKYE RICHARD KOFI declare that except for reference to other people's work, which have dully been cited. This submission is my own work towards the Master of Science degree and that, it contains no material neither previously published by another person nor prescribed elsewhere.

NYAMEKYE RICHARD KOFI, PG6322411 ……………… ……………….

Students' Name and ID Signature Date

Certified By …………….. ……………….

Prof. S. K Amponsah Signature Date

(Supervisor)

Certified By …………….. ……………….

Prof. S .K Amponsah Signature Date

(Head of Department)

Prof. I. K. Dontwi …………….. ………………

(Dean I.D.L) Signature Date

ABSTRACT

The most prominent problem facing every investor is to maximize returns and at the same time minimize the risk associated with the investment. Unfortunately, all the efforts by the investors to develop a reliable mathematical model to solve this problem optimally for both long term and short term in order to maximize profit, have proved futile. The aim of this thesis is to develop optimization models based on Dynamic Programming (DP) algorithm and Modern Portfolio Theory (MPT), and use them to determine the optimal returns and the risk involved. To achieve these aims, secondary data were collected from six financial institutions in Sunyani municipality from 2006 – 2011. The price series were normalized such that each commodity's price changes have annualized volatility of ten percent (10%). Financial ratio such as coefficient of variation (CV) which measures the relative probability of investing in each of the investments was calculated and used to analyze the data. Based on this empirical data MPT and DP models were formulated and used to find the risk and the corresponding returns involved in various investments. From the analysis, it was found out that, the optimal investment return was GH¢1.51×10^4. Ghana Guinness Limited, StateInsurance Company,Zenith BankBarclays Bankwere the prime investments which contributed most to the optimal expected return. Government of Ghana's Treasury bill and Ghana Commercial Bank were not profitable since their total contribution to the portfolio return were zero. No investor will be interested in investing in these financial institutions. There was a strong positive correlation coefficient between the optimal investments portfolio and the risk associated with investments. This suggests that short term investments in these portfolios are not profitable.

ACKNOWLEGEDGEMENT

I am grateful and most thankful to the Almighty God for keeping me throughout the years stay in the University.

I wish to express my heartfelt gratitude to Professor S. K Amponsah, Head of Mathematics Department KNUST, and Mr. A. K Peprah of Catholic University College, Sunyani, and my lecturers for contributing many useful comments and suggestions as well as demonstrating patience and care in supervising this work.

I am also thankful to the Headmistress of Sunyani Secondary School, who fondly accepted my request to further my education.

I am similarly grateful to Mr. Michael Agyemang of Saint James Seminary Secondary School Abesim, Mr. J. A Marfo of uplink internet café for their wonderful assistance.

To everyone who contributed in diverse ways I say ‘‘Ayekoo’’.

DEDICATION

This project is dedicated to my mother Bertha Boakyewaa Gyabaah, my wife Mary Adubea and all my brothers.

TABLE OF CONTENTS

PAGE

LIST OF TABLES

CHAPTER ONE

1.0 INTRODUCTION

This introductory chapter deals with investor's motivation, background of the study, statement of the problem, objectives, justification, research methodology, significance, limitation, organization and summary of the study.

1.1 Investors motivation

In every investment, there are different motives for investment. The most prominent among them is to earn a return on investment. However, selecting investments on the basis of returns alone is not sufficient. The fact that most investors invest their funds in more than one security suggests that there are other factors besides returns which need to be considered. The investors not only like returns but also dislike risk. The financial market, despite its benefits and rewards, is a complexly volatile industry, which requires critical analysis to adequately evaluate risks relative to returns to aid decisions as regards participation in the industry.

Many possible definitions of risk have been proposed in the literature because different investors adopt different investment strategies in seeking to realize their Investment objectives. In some sense, risk is a subjective concept and this is probably the main characteristic of risk. Thus, even if we can identify some desirable features of an investment risk measure, probably no unique risk measure exists that can be used to solve every investor's problem. One could say that before the publication of the paper by (Artzner et al, 1997-1999) on coherent risk measures, it was hard to discriminate between "good" and "bad" risk measures. The analysis proposed by the author was addressed to point out the value of the risk of future wealth, while most of the portfolio theory was based on the concept of risk in strong connection with the investor's preferences and their "utility function".

Investment activity is central to promotion of economic well being, because it is of the most important economic activity that businesses, consumers and governments can undertake.

Investment refers to the concept of deferred consumption, which involves purchasing an asset, giving a loan or keeping funds in a bank account with the aim of generating future returns. An understanding of the core concepts and a thorough analysis of the options can help an investor create a portfolio that maximizes returns while minimizing risk exposure.

An investment is the current commitment of money or other resources in expectation of reaping future benefits (Brodie et al, 1998). Investment is the flow into stock of capital goods. This flow of investment into the stock of capital are divided into two categories of replacement investment and net investment (Brodie et al 1998).The two make up gross investment, which is the total flow of investment goods into capital stock.

There are many other risk properties that could be used to characterize investor's optimal choices. For this reason, in this thesis, several risk measures and expected return properties have been classified for their financial insight and discussed to show how these properties characterize the different use of a risk measure relative to expected return.

1.2 Background of the study.

In this section, investment goals, portfolio investment, types of investment, insurance and investments are overviewed as part of insurance business as practiced in the portfolio investment.

1.2.1 Investment Goal

People throughout ages have sought to look for avenues to increase their wealth. The best way to maintain and improve once wealth is to invest the little that one has. Individuals, entities, organizations and cooperate bodies make investments towards the future. One difficulty faced by all is where, how, when and for how long to invest, in the mist of all opportunities, to obtain the maximum satisfaction from the investment made. Usually investors might have a certain amount of money to invest. The investor may have various options of investment based on the returns. Any time an investor makes an investment he must decide on the optimal investment strategy. Two very important strategies are active portfolio management and Long term investing. The strategies that one chooses for an optimal strategy will depend on the investors investment goals.

1.2.2 Portfolio

In finance, a portfolio is a collection of investments held by an institution or a private individual. In building up an investment portfolio a financial institution will typically conduct its own investment analysis, whilst a private individual rely on the use of the services of a financial advisor or a financial institution which offers portfolio management services. Holding a portfolio is part of an investment and risk –limiting strategy called diversification.(Markowitz H. 1959) The assets in a portfolio include stocks, bonds, options, warrants, gold certificates, real-estate, future contracts, production facilities, or any other item that is expected to retain its value. Portfolios tend to consist of variety of investment securities in order to minimize investment risk.

1.2.3 Investment portfolio

Investors throughout ages have sought for ways to increase their wealth. The best way to maintain and improve once wealth is to invest the little that one has. Individuals, entities, organizations and cooperate bodies make investments towards the future. The problem faced by all investors is where, how, when and for how long to invest, in the mist of opportunities, to obtain the maximum returns from the investment made. Usually investors might have a certain amount of money to invest. The investor may have various avenues of investment based on optimal returns. From a historical point of view, the optimal investment decision always corresponds to the solution of an "expected utility maximization problem". Therefore, although risk is a subjective and relative concept (Balzer, L A., 2001) we can always state some common risk characteristics in order to identify the optimal choices of some classes of investors, such as non-satiable and/or risk-averse investors. In particular, the link between expected utility theory and the risk of some admissible investments is generally represented by the consistency of the risk measure with a stochastic order.

Portfolio optimization is often called Mean –Variance (MV) optimization. Unfortunately equities with high returns correlate with high risk.

The term mean refers to the mean or expected return of the investment and the variance is the measure of risk associated with the portfolio. The problem can be formulated mathematically in many ways, but the principal problems can be summarized as follows:

(i) minimized risk for a specified expected return

(ii) maximize the expected returns for a specified risk

(iii) minimize the risk and maximize expected returns using a specified risk aversion factor

(iv) minimize the risk regardless of the expected returns

(v) maximize expected returns regardless of the risk

The above problems could have linear or non linear constraints, equality and inequality constraints. The first three (3) problems are essentially mathematically equivalent and the solutions are called Mean-Variance (MV) efficient. The efficient points in the return – risk graph are called the efficient frontier.

The fourth problem gives minimum variance solutions which are for cautious investors. It is also used for comparison and benchmark of other portfolios.

The fifth gives the upper bound of the expected return which can be attained. This is useful for comparisons.

Any of the above problems can be solved in relation to an existing portfolio or benchmark. When market conditions or risk preferences change, it is advisable to rebalance.

1.2.4 Types of Investment

The types of investment according to Brodie et al (1998) are;

(i) Tangible or fixed asset investment

(ii) Inventory investment

(iii) Intangible investment

(iv) Residential investment in housing

(v) Financial investments

1.2.5 Insurance

Insurance is a promise of compensation for specific potential future losses in exchange for a periodic payment (Daykin et al, 1996). Insurance is designed to protect the financial well-being of an individual, company or other entity in the case of unexpected loss. Some forms of insurance are required by law, while others are optional. Agreeing to the terms of an insurance policy creates a contract between the insured and the insurer .The sum of money that is paid to the insurance company (insurer) in order to be insured is called the premium. It is a regular periodic payment from the insured person to the company for the protection of the insured property or person. Premiums on all types of insurance policies are payable in advance. The amount of the premium depends on the type of insurance.

The amount for which the property or the person is insured is called the risk or the face value of the policy. Usually, the insurance company requires that the risk should be a specified percentage of the real value of the property. Below are some of the various types of insurance;

(i) Fire insurance (houses, stores, factories etc.)

(ii) Life assurance (personal insurance, health insurance, etc.)

(iii) Accident insurance (car)

1.2.6 Investment as part of insurance Business

A key feature of insurance is that premiums are received in advanced of the risk being borne with afterwards claim payments. In all this, provision needs to be established in respect of the expected future liabilities. Additional solvency margins should also be maintained. These provisions and reserves should be backed by appropriate assets; having regard to liabilities with reasonable balance between security, liquidity and good investment returns (including interest ,dividend or rental income and capital gains).

Insurance companies often have a predominance of short- term liabilities. Their investments tend to be mainly in cash, short-term deposits and government or corporate bonds of fairly short duration. However, if the cash flow is expected to be positive, more diversified investment policy could be beneficial, including property and equity shares. In some jurisdiction, Life Companies have a greater degree of investment freedom than Non-Life Companies. There are substantial volumes of participation (with profits) in life business. In this case only a part of the sum assured may be guaranteed and the balance depends on the performance of investment (Daykin et al, 1996).

1.3 Statement of the problem

The basic premise of economics is that, due to the scarcity of resources, all economic decisions are made in the face of trade-offs. The trade-off facing every investor is risk versus expected returns. The investment decision is not merely which securities to own, but how to divide the investor's wealth amongst securities. The investor’s problem is really how much and where to invest in order to maximize the expected returns and at the same time how to minimize the risk such as fire outbreaks, theft and terrorist attacks associated with the investment.

As a result, financial institutions such as Banks, Insurance Companies and individual investors are crusading for a device or model that will enable them solve the above problems optimally for both long term and short term in other to maximize their profit.

Let $F_i(x)$, $i = 1, 2, 3, \dots, n$ denote the return from investments i when x units of money are invested in investment, i. x_i $(i = 1, 2, 3, \dots, n)$ is defined as the number of units of money invested in investment, i. The problem of determining how much to invest in each investment in order to maximize total returns can be approached through a multi-stage decision process by modeling a mathematical program to find the optimal policy using Dynamic Programming.

$$
\begin{aligned}
&\textit{Maximize} && \sum F_i(x_i) \\
&\textit{Subject to} && \sum x_i \leq b \\
& && x_i \geq 0 \qquad\qquad (1.0) \\
& && i = 1, 2, 3, \dots, n
\end{aligned}
$$

Where $F_i(x_i)$ functions of a single variable are are, b is a known nonnegative integer and all variables $(x_i's)$ are all positive integrals.

Markowitz, (1952), measured risk of assets using the variance of each asset return. If each component x_i of the n–vector x represents the proportion of an investor's wealth allocated to asset I, then the total return from the portfolio is given by the scalar product of x by the vector of individual asset returns. Therefore, if $R = (R_1, \dots\dots, R_n)$ denotes the n-vector of expected returns of the assets, risk and C can be represented by the expression $\sum_{i=1}^{n} R_i x_i$, and its level of risk by $\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_i x_j$. Markowitz assumes that the aim of the investor is to design a portfolio which minimizes risk while achieving a predetermined expected return, say R_{exp} . Mathematically, the problem can be formulated as follows. For any value of R_{exp} :

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_i x_j \\
\text{s.t} \quad & \sum_{i=1}^{N} R_i x_i = R_{exp} \qquad (1.1) \\
& \sum_{i=1}^{n} x_i \\
& x_i \geq 0 \quad for\ i = 1, \dots\dots. n
\end{aligned}
$$

The first constraint expresses the requirement placed on expected returns .The second constraint, called budget constraint, requires that 100% of the budget be invested in the portfolio. The non-negativity constraints express that no short sales are allowed.

The return (R) on a risky investment is a random variable. It is natural to take the expectation E(R) as the reference value. The variance Var (R) then turns out to be a measure of risk of the investment. In some circumstances the standard deviation

$$\sigma_R = [\mathrm{Var}\,(R)]^{0.5} \tag{1.2}$$

of the return is a more convenient measure of risk. If a quantity is measured in certain units, then the standard deviation will be expressed in the same units, so it can be related directly to the original quantity, in contrast to variance, which will be expressed in squared units. From the above observations, this research work is aimed at developing a Dynamic Programming model to determine optimal returns of investment and also use Markowitz's portfolio model to measure the risk involved in some investments made by some banks in the Sunyani Municipality.

1.4 Objective of the study

The objectives of this study are:

(a) To develop an optimization model based on DP algorithm to determine optimal expected return on investment portfolio

(b) To determine the minimal risk of an investment portfolio using Markowitz portfolio model

(c) To establish the correlation between expected return and risk in investment portfolios option.

1.5 Justification of the study

In real-life situation, many people have a lot of resources and would like to invest, but are not sure of where, when and how to put their resources together in order to accrue the maximum returns. These indecision problems have become a source of misery to them. To justify the products in which to invest, we need to look out for the various forms of investments available, the expected returns from each investment, risk involved and the associated cost. Financial institutions would like to know where to keep their excess cash flows to make maximum returns. All the above can be modeled as a dynamic programming problem. It is known that dynamic programming solves problems in stages and is quicker and less time consuming far less than total enumeration. Also, the return on a risky investment is a random variable. It is natural to take the expectation as the reference value. The variance then turns out to be a measure of risk of the investment. To preserve the unit of a quantity to be measured, the standard deviation of the return is a more convenient measure of risk.

1.6 Research Methodology

This is a quantitative research work and a secondary data, interest rates and yield functions of various investments will be collected from selected Banks and Financial Institutions.

Mathematical methods that will be used in this research work are Dynamic Programming and Modern portfolio theory. Dynamic Programming will be used to determine the optimal returns of the investment and the problem analyzed through a multi-stage decision process. The Modern Portfolio Theory (MPT) which uses the standard deviation of asset returns as the measure of risk, and focuses on maximizing or minimizing it when constructing a portfolio shall be used to measure risk of the investments. The standard deviation simply measures the average distance of the returns from its historical mean. This can be interpreted as a measure of how “volatile” the price of an asset is. The rest of the thesis will focus on attempts to assess this extreme risk and its management. The correlation between expected returns and risk in investment portfolio option will also be established.

1.7 Significance of the study

The problems faced by many investors are; where, when and how to put their resources in order to accrue maximum returns and minimize risk. To select the projects in which to invest, there is the need to look out for various forms of investments available, the expected returns from each investment and its associated risk and cost. Financial institutions would like to know where to keep their excess cash flows to make maximum returns. The dynamic programming model will serve as an efficient tool for solving potential investor’s problems in stages since its quicker, less time consuming and far less than total enumeration. The model will also serve as a tool for operation research specialists for analyzing and solving multiple-objectives optimization problems.

1 .8 Limitation of the study

Readiness of Banks, companies and other financial institutions to make available data for this project was a challenge. Most insurance companies turned the proposal for data down because of competition in the insurance industry .The problem to be considered in this thesis is the Bellman’s Principle of Optimality using Dynamic programming (BPDPP). Only six investments with six returns; will be discussed this is due to time constraints which will not allow the researcher to do total enumeration.

1.9.0 Organization of the study

This chapter deals with the problem statement and objectives of the study. The methodology, significance and limitations of the study are also considered.

Chapter two reviews the related literature of dynamic programming applications and solutions. The Markowitz models for analyzing portfolio management will be reviewed.

Chapter three outlines some algorithm solutions of knapsack problems using total enumeration and Dynamic Programming models. It considers cases where there is total enumeration and compares the time and stages used in solving a problem. The Markowitz modern portfolio selection theory will be analyzed and used to select viable investment and the risk of investment.

Chapter four deals with collection, analysis and interpretation of the data. The chapter also presents the implementation of DP models .The DP will be used to determine optimal investment returns. The maximal or minimal risk and the corresponding investments to be selected will be made using the proposed model for portfolio selection.

Chapter five, the final chapter of the study presents the conclusion, summary and recommendations of the study.

1.9.1 Definitions of Terms and Symbols used

The terms and symbols that have been used in this chapter and those that will be used in the subsequent chapters are summarized below

Utility function - the amount of satisfaction or pleasure that somebody gains from consuming a commodity, product, or service

Tangible or fixed asset investment - This is investment in physical, fixed capital such as plant and machinery.

Inventory investment - It is investment in the 'working capital' of the firm made up of goods produced that are not sold.

Intangible investment - It is the investment in non- physical asset that does not involve the accumulation of tangible assets but will help future productivity capacity, such as good will.

Financial investment – It is the purchase of financial instruments and paper assets.

BPODP – Bellman's Principle of Optimality using Dynamic Programming

MPM - Markowitz Portfolio model

MPT - Modern Portfolio Theory

R - Return of investment

r - risk of investment

DPM - Dynamic Programming Model

CAPM – Capital Asset Pricing Model

CRRA – Constant Relative Risk Aversion

GMD - Gini's Mean Difference

BIS - Bank of International Settlement

CPA - Continuous Probabilistic Analysis

CVaR - Conditional Value at Risk

CML – Capital Market Line

PS – Portfolio Selection

MAD – Mean Absolute Deviation

MV – Mean Variance

KKT – Karush –Kuhn -Tucker

HJB – Hamilton-Jacobi- Bellman

1.9.2 Summary

This chapter considers the motives for investment, background of the study and investment goals portfolio. It discusses types of investments and how investment forms part of the insurance business. It also entails statement of the problem, aims and objectives of the study and justification of the study. Research methodology is highlighted as well as the significance of the study. Limitation of the study, including the organization of the research study is also dealt with.

Terms and symbols used in the subsequent chapters are explained

The next chapter presents pertinent and relevant literature review on optimal investment portfolio based on models of risk and expected returns using dynamic programming and Markowitz portfolio model.

CHAPTER TWO

Related Literature Review on Portfolio Investment

2.0 Introduction

This section, reviews relevant researches on selection of investment portfolio using dynamic programming and Markowitz portfolio models.

One of the most popular methods to choose optimal investment portfolio is the method that measures the value of risk proposed by Markowitz. This model analyzes different measures of risk such as variance, standard deviation, and conditional value at risk. Different measures of risk are focusing on the different properties of distribution of rate of return. For example the variance measures the dispersion of rate of return and the value-at-risk or conditional value at risk measures the probable loss. In this thesis, expected returns for various investments will be measured using dynamic programming models while Markowitz portfolio models will be used to measure the various risks**.** The optimal portfolio selected on the bases of proposed models will be compared according to level of risk and profitability.

2.1 Dynamic Programming approach

This section reviews the Principle of Optimality, Bellman Equation**,** Ad hoc Methods of Rebalancing, Differential dynamic programming and other literature reviews on dynamic programming**.**

2.2.1 The Principle of Optimality and Bellman Equation

Dynamic programming is both a mathematical optimization method, and a computer programming method. In both contexts, it refers to simplifying a complicated problem by breaking it down into simpler sub problems in a recursive manner. While some decision problems cannot be dealt this way, decisions that span several points in time do often break apart recursively; Bellman calls this the "Principle of Optimality". In computer science, a problem which can be broken down recursively is said to have optimal substructure. If sub problems can be nested recursively inside larger problems, so that dynamic programming methods are applicable, then there is a relation between the value of the larger problem and the values of the sub problems. In the optimization literature this relationship is called the Bellman equation.

2.2.2 Flexibility of Dynamic programming

Based on the programming style, Steffen et al, (2005) introduce a generic product operation of scoring schemes. This leads to a remarkable variety of applications, allowing us to achieve optimizations under multiple objective functions, alternative solutions and back tracing, holistic search space analysis, ambiguity checking, and more, without additional programming effort. The authors demonstrated the method on several applications for RNA secondary structure prediction. The product operation as introduced here adds a significant amount of flexibility to dynamic programming. It provides a versatile text bed for the development of new algorithmic ideas which can immediately be put to practice.

Below are the works of some authors demonstrating the flexibility of DP as applied to problems solving.

Many institutional fund managers generally rebalance using ad hoc methods such as calendar basis or tolerance band triggers. Sun et al, (2005) proposed a different framework that quantifies the cost of a rebalancing strategy in terms of risk-adjusted returns net of transaction costs. The authors then developed an optimal rebalancing strategy that sought to minimize that cost. They used certainty equivalents and the transaction costs associated with a policy to define a cost-to-go function, and they minimized this expected cost-to-go using dynamic programming. The authors applied Monte Carlo simulations to demonstrate that their method outperformed traditional rebalancing. They also showed the robustness of our method to model error by performing sensitivity analyses.

Jacobson's, (2003) differential dynamic programming is a technique, based on dynamic programming rather than the calculus of variations, for determining the optimal control function of a nonlinear system. Unlike conventional dynamic programming where the optimal cost function is considered globally, differential dynamic programming applies the principle of optimality in the neighborhood of a nominal, possibly no optimal, trajectory. This allowed the coefficients of a linear or quadratic expansion of the cost function to be computed in reverse time along the trajectory: these coefficients may then be used to yield a new improved trajectory (i.e. the algorithms are of the "successive sweep" type). A class of nonlinear control problems, linear in the control variables, is studied using differential dynamic programming. It is shown that for the free-end-point problem, the first partial derivatives of the optimal cost function are continuous throughout the state space, and the second partial derivatives experience jumps at switch points of the control function. A control problem that has an analytic solution is used to illustrate these points. The fixed-end-point problem is converted into an equivalent free-end-point problem by adjoining the end-point constraints to the cost functional using Lagrange multipliers: a useful interpretation for Pontryagin's adjoins variables for this type of problem emerges from this treatment. The above results are used to devise new second- and first-order algorithms for determining the optimal bang-bang control by successively improving a nominal guessed control functions.

Herman et al, (2009) developed a multi-period investment portfolio model that includes risky farmland, risky and risk-free nonfarm assets, and debt financing on farmland in the presence of transaction costs and credit constraints. The model is formulated as a stochastic continuous-state dynamic programming problem, and is solved numerically for South-western Minnesota, USA. Results show that optimal investment decisions are dynamic and take into account the future decisions due to uncertainty, partial irreversibility, and the option to wait. The optimal policy includes ranges of inaction, states where the optimal policy in the current year is to wait. The risk-averse farmer makes a lower investment in risky farmland reflecting risk-avoiding behavior. The authors found that, in addition to risk aversion, the length of the planning horizon affects risk-avoiding behavior in investment decisions. Finally, they found that higher debt financing on farmland is optimal when risky nonfarm assets ares included in the optimal investment portfolio and that the probability of exiting farming increases with the risky nonfarm investment.

Ghezzi (1997) considered an immunization problem in which a bond portfolio is to be periodically rebalanced. Max-min optimal control is applied to the problem. The target is to maximize the final portfolio value under the worst possible evolution of interest rates. The optimal control law, obtained by means of dynamic programming, turns out to be different from any duration-based immunization policy.

Jean-Luc Vila et al (1991) used stochastic dynamic programming to study the inter-temporal consumption and portfolio choice of an infinitely lived agent who faces a constant opportunity set and a borrowing constraint. The authors showed that, under general assumptions on the agent's utility function, optimal policies exist and can be expressed as feedback functions of current wealth. They described these policies in detail, when the agent's utility function exhibits constant relative risk aversion.

Optimal asset allocation deals with how to divide the investor's wealth across some asset-classes in order to maximize the investor's gain. Pola et al, (2006) considered the optimal asset allocation in a multi-period investment setting: optimal dynamic asset allocation provides the (optimal) re-balancing policy to accomplish some investment's criteria. Given a sequence of target sets, which represent the portfolio specifications at each re-balancing time, an optimal portfolio allocation is synthesized for maximizing the joint probability for the portfolio to fulfill the target sets requirements. The approach pursued is based on dynamic programming. The optimal solution is shown to conditionally depend on the portfolio realization, thus providing a practical scheme for the dynamic portfolio rebalancing. Finally some case studies are given to show the proposed methodology.

Rudoy et al, (2008) studied the problem of optimal portfolio construction when the trading horizon consists of two consecutive decision intervals and rebalancing is permitted. It is assumed that the log-prices of the underlying assets are non-stationary, and specifically follow a discrete-time integrated vector autoregressive model. The authors extended the classical Markowitz mean-variance optimization approach to a multi-period setting, in which the new objective is to maximize the total expected return, subject to a constraint on the total allowable risk. In contrast to traditional approaches, they adopted a definition for risk which takes into account the non-zero correlations between the inter-stage returns. This portfolio optimization problem amounts not only to determining the relative proportions of the assets to hold during each stage, but also requires one to determine the degree of portfolio leverage. Due to a fixed constraint on the standard deviation of the total return, the leverage decision is equivalent to deciding how to optimally partition the allowed variance, and thus variance can be viewed as a shared resource between the stages. The authors derived the optimal portfolio weights and variance scheduling scheme for a trading strategy based on a dynamic programming approach, which is utilized in order to make the problem computationally tractable. The performance of this method is compared to other trading strategies using both Monte Carlo simulations and real data, and promising results are obtained

Ye, (2007) presented a continuous-time model of optimal life insurance, consumption and portfolio is examined by dynamic programming technique. The Hamilton-Jacobi- Bellman (HJB in short) equation with the absorbing boundary condition is derived. Then explicit solutions for constant relative risk aversion (CRRA in short) utilities with subsistence levels are obtained. Asymptotic analysis is used to analyze the model.

Paolo Battocchio, et al considered a stochastic model for a defined-contribution pension fund in continuous time. In particular, we focus on the portfolio problem of a fund manager who wants to maximize the expected utility of his terminal wealth in a complete financial market with stochastic interest rate. The fund manager must cope with setoff stochastic investment opportunities and two background risks: the salary risk and the inflation risk. We use the stochastic dynamic programming approach. We show that the presence of the inflation risk can solve some problems linked to the use of the stochastic dynamic programming technique, and namely to the stochastic partial differential equation deriving from it. The technique, namely the stochastic partial differential equation deriving from it. We find a closed form solution to the asset allocation problem, without specifying any functional form for the coefficients of the diffusion processes involved in the problem. Finally, the derivation of a closed form solution allows us to analyze in detail the behavior of the optimal portfolio with respect to salary and inflation.

2.3 Portfolio Optimization and Risk Management

This section briefly refers to some of the most outstanding pieces in the literature concerning portfolio optimization optimal returns and risk in portfolio investment and its relation with (Financial) risk measurement. It also presents some recent results and findings on the outcomes of portfolio optimization implementation.

2.3.1 Modern Portfolio theory: Markowitz, Sharpe and Tobin.

Modern portfolio theory (MPT) or portfolio theory was introduced by Harry Markowitz with his paper "Portfolio Selection," which appeared in the 1952 Journal of Finance? Thirty-eight years later, he shared a Nobel Prize with Merton Miller and William Sharpe for what has become a broad theory for portfolio selection. Prior to Markowitz's work, the assessment of the risks and rewards of portfolios was carried out through the analysis of individual securities independently. By formalizing the concept of diversification, he proposed that investors should focus on selecting portfolios based on their joint risk-reward features instead of merely compiling individually attractive securities regardless of their relation to the other assets on their portfolios. Using the historical returns of each asset on a portfolio and statistical measures such as average (return), standard deviation and linear correlation it is possible to estimate the expected return and volatility of any portfolio constructed with those assets. Markowitz used volatility and expected return as proxies for risk and reward. Within the infinite possible alternatives that an investor has to construct a portfolio, Markowitz defined an “optimal” way of doing so by balancing the risk and reward features of the portfolio. The set of portfolios constructed in this optimal manner conforms to what he called the efficient frontier. The author concludes that an investor should select a portfolio that lies on the efficient frontier.

Tobin (1958) expanded on Markowitz's work by adding a risk-free asset to the analysis. The author pointed out that by using leverage or deleverage on the portfolios on the efficient frontier it was possible to outperform them in terms of their risk and reward relation. By doing so the author introduced the notions of “Capital Market Line” and “super-efficient portfolio”.

Sharpe (1964) formalized the Capital Asset Pricing Model (CAPM). Using strong assumptions over investors and market behavior, he created a model that led to interesting conclusions. It was detected that the “market portfolio” sits on the efficient frontier, and is also actually Tobin's super-efficient portfolio. According to CAPM, all investors should hold the market portfolio, leverage or de-leverage with positions in the risk-free asset according to their risk aversion profile. CAPM also introduced the concept of “beta” and relates an asset's expected return to its beta. Portfolio theory provides a broad framework to understand the interactions of risk and reward. It has profoundly influenced the way institutional portfolios are managed, and motivated what is known as “passive management” investment techniques. The mathematics of portfolio theory is widely used in Financial Risk Management and is a theoretical predecessor for more recent risk measures.

2.3.2 Selection of optimal portfolio according to two measures of risk

Selection of an optimal investment portfolio can be made according to three criteria: the expected rate of return (E (x)) and two measures of $(\rho_1(x), \rho_2(x)$. Then the preference relation can be as follows: the random variable R_x $is\ preferred$ to the random variable R_y if and only if $E(R_x) \geq E(R_y)$, $\rho_1(R_x) \leq \rho_1 R_y$ and ρ_2 the un-dominated effective solutions are Pareto effective of multi-objective problem. In the optimization model of this problem the value of the expected rate of return is maximized and both measures of risk ($\rho_1(x), \rho_2\,(x)$) are minimized.

These multi –objective models can be transformed to a single objective problem. This can be done alone by using example method known as, $-$constraint method" . In this method one objective function should be optimized and the remaining objective functions should be transformed into constraints. The scenario approach was used to achieve objective model with linear constraints.

Simple objective models will be used to solve the optimal portfolio problem with two measures of risk. In the first model the variance and CVaR will be used as a risk measure. The mean variance, CVaR model the variance will be minimized and two remaining criteria will be transformed into constraints. The scenario approach was used to achieve a single-objective model with linear constraints. The mean-variance –CVaR model

(MVC model) is following Roman D., et al (2007)

minimize $\quad \sigma^2(R_x)$

$E(R_x) \geq R_0$

$v + \frac{1}{T\,(1-\sigma)} \sum_{j=1}^{n} p_i u_i \ \leq z_c$

$$u_1 + \sum_{j=1}^{n} x_j r_{ij} + v \ \geq 0 \quad for\ i = 1,2, \ldots \ldots, T$$

$$\sum_{j=1}^{n} x_j$$

$$x_j \ \geq 0 \qquad forj = 1,2, \ldots., n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2.0)$$

$$u_i \ \geq 0 \qquad for\ i = 1,2 \ldots \ldots, T$$

Parameter z_c is the value which the Conditional Value –at risk should not exceed the value z_c was fixed according to results from he mean CVaR model. The mean –CVaR model was used for the different assumed level of rate of return of portfolio to obtain a different value of objective function (value of risk). From all these values was selected the minimum (z_{cmin}). In mean CVaR. As in the above model, the variance will be minimized and the Gini's Mean Difference and the expected rate of return will be changed to constraints. In the scenario approach we can apply mean variance GMD model (MVG model) with linear constraints as follows:

$$\min \quad \sigma^2(R_x)$$

$$\frac{1}{2} \sum_{i,k=1}^{T} d_{ik} p_i p_k \ \leq Z_G$$

$$d_{ik} \ \geq \sum_{j=1}^{n} x_j r_{ij} - x_j r_{kj} \quad for\ j, k = 1,2, \ldots., T$$

$$E(R_x) \geq R_0$$

$$\sum_{j=1}^{n} x_j = 1$$

$$x_j \geq 0 \quad for\ j = 1,2,\dots,n$$

$$d_{ik} \geq 0 \quad \text{for i, k = 1,2,….,T……………………} \qquad (2.1)$$

Parameter z_G, which limits the value of Gini's Mean Difference is fixed as the parameter z_c but in this case results from the mean-Gini's model are used.

The optimal portfolio problem with two measures of risk can also be solved by the linear optimization models. Models with the Conditional Value –at-Risk and Gini's Mean Difference can be considered as risk measures. In one of these models the Conditional Value-at-Risk will be minimized and the Gini's Mean Difference will be limited. This linear model can be denoted as the mean –CVaR- GMD model (MCG model).

Minimize $\quad V + \frac{1}{T(1-\alpha)} \sum_{i=1}^{T} p_i u_i$

$$U_{1+} \sum_{j=1}^{n} x_j r_{ij} + v \geq 0 \qquad for\ i = 1, 2, \dots, T$$

$$\frac{1}{2} \sum_{i,k=1}^{T} d_{ik} p_i p_k \leq Z_G$$

$$d_{ik} \geq \sum_{j=1S}^{n} x_i r_{ij} - x_j r_{kj} \quad for\ i, k = 1, 2, \dots, T$$

$$E(R_x) \geq R_0$$

$$\sum_{j=1}^{n} x_j \tag{2.2}$$

$$x_j, d_{ik}, v \geq 0 \qquad for\ j = 1, 2, \dots, T$$

These single objective models prove that the optimal solution of the single objective problem is also a Pareto optimal solution of the original multi-objective problem.

In this thesis, the Markowitz portfolio models swill be used to estimate and evaluate the risk features of a given portfolio and establish the correlation between expected returns and risk in portfolio investment options.

2. 3.3 Extreme Risk Measures

Extreme risk is usually related to the (joint) losses in the value of an asset (or portfolio of assets) bore during "extreme" situations. Extreme situations can be taught as the worst case scenarios that one could expect with respect to the value of the assets held. Various measures of extreme risk are around, but the most renown is the Value-at-Risk (VaR). This measure is used by the capital adequacy directive of the Bank of International Settlement (BIS) in Basle (Basle Committee 1996), that determined that, banks must have a capital cushion sufficient to cover losses on the bank's trading portfolio over a ten-day holding period in 99% of occasions.

However, Artzner, Delbaen, Eber and Heath. (1997, 1999) pointed out some drawbacks of the VaR as a market risk measure. First, they show that VaR is not necessarily 'sub-additive' and explain why this may cause serious problems if the risk-management system of a financial institution is based in VaR-limits for individual books. Furthermore, VaR gives only an upper bound on the losses that occur with a given frequency; VaR tells us nothing about the potential size of the loss given that a loss exceeding this upper bound has occurred. Artzner et al. (1997,

1999) proposed the use of the so-called Expected Shortfall instead of VaR, which addresses the issue of the potential size of a loss in extreme scenarios.

2.3.4 Value at Risk

According to Jorion (2000), the Value at Risk (VaR) estimates the maximum loss (or worst loss) that a portfolio can have within a determined time horizon and a given confidence level. For a horizon of N days and a confidence level 0 %, the VaR is the loss corresponding to the (100- 0) quartile in the distribution of the variations of a portfolio's value during the next N days.

In this thesis, the VaR of a portfolio is estimated using Extreme Value Theory and compared to estimates based on Gaussian assumptions. Portfolio optimization is done using as risk measure the expected shortfall, as suggested by Artzner et. al. (1997,1999).

2.3.5 Expected Shortfall

The expected shortfall (ES) or tail conditional expectation, according to McNeil and Frey (2001) is a risk measure that gives some information about the size of the potential losses given that a loss bigger than VaR has occurred. The tail conditional expectation measures the expected loss given that the loss L exceeds VaR; in mathematical terms it is given by $E[L \mid L > VaR]$. This risk-measure has the advantage over the VaR that is a coherent measure under the Artzner et.al. (1997, 1999). criteria.

CHAPTER TRHEE

METHODOLOGY

3.1Dynamic Programming

Dynamic Programming is a technique that can be used to solve many optimization problems. In most applications, dynamic programming obtains solutions by working backward from the end of a problem toward the beginning, thus breaking up a large, unwieldy problem into a series of smaller, more tractable problems.

In mathematics and computer science, dynamic programming is a method for solving complex problems by breaking them down into simpler sub problems. It is applicable to problems exhibiting the properties of overlapping sub problems which are only slightly smaller and optimal substructure (described below). When applicable, the method takes far less time.

The key idea behind dynamic programming is quite simple. In general, to solve a given problem, we need to solve different parts of the problem (sub problems), then combine the solution of the sub problems to reach an overall solution. Often, many of these sub problems are really the same. The dynamic programming approach seeks to solve each sub problem only once, thus saving a lot of computation. This is especially useful when the number of repeating sub problems is exponentially large.

Top-down dynamic programming simply means storing the results of certain calculations, which are later used again since the completed calculation is a sub-problem of a larger calculation. Bottom-up dynamic programming involves formulating a complex calculation as a recursive series of simpler calculations.

The term dynamic programming was originally was used in the 1940s by Richard Bellman to describe the process of solving problems where one needs to find the best decisions one after the other. By 1953, he refined this to the modern meaning, referring specifically to nesting smaller decision problems inside larger decisions. There after, the field was recognized by the IEEE as a systems analysis and engineering topic. Bellman’s contribution is remembered as Bellman equation, a central result of dynamic programming which restates an optimization problem in recursive form.

The word dynamic was chosen by Bellman because it sounded impressive, not because it described how the method worked. The word programming referred to the use of the method to find an optimal program, in the sense of a military schedule for training or logistics. This usage is the same as that in the phrases linear programming and mathematical programming, a synonym for optimization.

Finding the shortest path in a graph using optimal substructure; a straight line indicates a single edge; a wavy line indicates a shortest path between the two vertices it connects (other nodes on these paths are not shown); the bold line is the overall shortest path from start to goal.

Dynamic programming is both a mathematical optimization method and a computer programming method. In both contexts it refers to simplifying a complicated problem by breaking it down into simpler sub-problems in a recursive manner. While some decision problems cannot be taken apart this way, decisions that span several points in time do often break apart recursively; Bellman called this the “Principle of Optimality”. Likewise, in computer science, a problem which can be broken down recursively is said to have optimal substructure.

If sub-problems can be nested recursively inside larger problems, so that dynamic programming methods are applicable, then there is a relation between the value of the larger problem and the values of the sub-problems. In the optimization literature this relationship is called the Bellman equation.

3.1.1Dynamic Programming in Mathematical Optimization

In terms of mathematical optimization, dynamic programming usually refers to simplifying a decision by breaking it down into a sequence of decision steps over time. This is done by defining sequence of value functions V_1, V_2,…V_n, with an argument y representing the state of the system at times I from 1 to n. The definition of $V_n(y)$ is the value obtained in state y at the last time n. The values V_i at earlier times I = n-1,…, 2,1 can be found by working backwards, using a recursive relationship called the Bellman equation. For i = 2,…n, V_{i-i} for those states. Finally, V_1 at the initial state of the system is the value of the optimal solution. The optimal values of the decision variables can be recovered, one by one, tracking back the calculations already performed.

3.1.2 Dynamic Programming in Computer Programming

There are two key attributes that a problem must have in order for dynamic programming to be applicable. They are optimal substructure and overlapping sub problems which are only slightly smaller.

When the overlapping problems are, half the size of the original problem the strategy is called "divide and conquer" rather than "dynamic programming". This is why merge sort, quick sort, and finding all matches of a regular expression are not classified as dynamic programming problems.

Optimal substructure means that the solution to a given optimization problem can be obtained by the combination of optimal solutions to its sub problems. Consequently, the first step towards devising a dynamic programming solution is to check whether the problem exhibits such optimal substructure. Such optimal substructures are usually described by means of recursion. For example, given a graph G = (V, E), the shortest path p from a vertex u to a vertex v exhibits optimal substructure: take any intermediate vertex w on this shortest path p. If p is truly the shortest path, then the path p_1 from u to w and p_2 from w to v are indeed the shortest paths between the corresponding vertices (by the simple cut-and-paste a argument described in CLRS). Hence, one can easily formulate the solution for finding shortest paths in a recursive manner, which is what the Bellman-Ford algorithm does.

Overlapping sub problems means that the space of sub problems must be small, that is, any recursive algorithm solving the problem should solve the same sub problems over and over, rather than generating new sub problems. Let us, consider for example the recursive formulation for generating the Fibonacci series: $F_i = F_{i-1} + F_{i-2}$, with base case $F_1=F_2=1$. Then $F_{43} = F_{42} + F_{41}$, and $F_{42} = F_{41} + F_{40}$. Now F_{41} is being solved in the recursive sub trees if both F_{43} as well as F_{42}. Even though the total number of sub problems is actually small (only 43 of them), we end up solving the same problems over and over if we adopt a naïve recursive solution such as this. Dynamic programming takes account of this fact and solves each sub-problem only once.

This can be achieved in either of two ways

(i) Top-down approach:

This is the direct fall-out of the recursive formulation of any problem. If the solution to any problem can be formulated recursively using the solution to its sub problems, and if its sub problems are overlapping, then one can easily memorize or store the solutions to the sub problems in a table. Whenever an attempt is made solve a new sub problem, the table must first be examined to see if it is already solved. If a solution has been recorded it can be used directly, otherwise the sub problem can be solved and its solution added to the table.

(ii) Bottom-up approach:

This is the more interesting case. Once the solution to a problem is formulated recursively as in terms of its sub problems, it can be reformulated in a bottom-up fashion: by solving the sub problems first and using their solutions to build-on and arriving at solutions to bigger sub problems. This is also usually done in a tabular form by iteratively generating solutions to bigger sub problems by using the solutions to treat small sub problems. For example, if we already know the values of F_{41} and F_{40}, we can directly calculate the value of F_{42}.

3.1.2 Characteristics of Dynamic Programming Applications

There are a number of characteristics that are common to all problems and all dynamic programming problems.

1. The problem can be divided into stage with a decision required at each stage. In capital budgeting problem the stages were the allocations to a single plant and the decision was how much to spend.

2. Each stage has a number of states associated with it. The states for a capital budgeting problem correspond to the amount spent at that point in time. In the shortest path problem the states were the node reached.

3. The decision at one stage transforms from state into a state in another stage. The decision of much to spend gave a total amount spent for the next stage. The decision of where to go next defined where one arrived in the next stage.

4. Given the current state, the optimal decision for each of the remaining state does not depend on the previous states or decisions. In the budgeting problem, it is not necessary to know how the money was spent in previous stages but to know how much was spent.

In the shortest path problem, it is not necessary to know how you got to a node but to know how you did.

5. There exists a recursive relationship that identifies the optimal decision for stage j, given that stage $j + 1$ has already been solved.

6. The final stage must be solvable by itself. The last time properties are tied up in the recursive relationship given above.

3.1.3 Computational Efficiency of Dynamic Programming

In smaller networks it would be a matter of determining the shortest path form on point to another by enumerating all the possible paths (after all there are only a few paths).

In larger networks however, complete enumeration is practically impossible and the use of dynamic programming is much more efficient in determining the shortest path.

In a network where there are five stages with: stage1 – 1 state, stage 2 – 3 state, stage 3 – 3 state, stage 4 – 2 state and stage 5 – 1 state. Total enumerate will result in 1(3)(3)(2)(1) = 18 paths while DP with result in 1(3)(3)(2)(1) = 18 path.

If in another network there are seven stages with 5 states each. The total enumeration gives $5(5^5)$ paths.

However DP require 4(25) + 5 = 105 addition = DP requires 105 = 0.00 7 times as many additions are explicit enumeration.

3.1.4 Applications of Dynamic Programming

Dynamic programming can be used to solve many types of Integer Programming-consumption and savings problems, shortest path problem, The Knapsack Problem, Network Problems, Inventory Problems, Equipment replacement problems, Resource Allocation Problems etc.

3.1.5 Integer Programming

An Integer Programming problem (IP) is a Linear Programming (LP) problem in which some or all the variables are required to be non- negative integers. An Integer programming in which all variables are required to be integers is a Pure Integer Programming problem.

Many problems can be modeled as an Integer Programming problem. The model is; For a maximization problem

Maximize $\quad Z = \sum_{i=1}^{n} r_i(ax_i + b)$

Subject to $\quad \sum_{i=1}^{n} x_i \leq c_i \qquad (3.0)$

$$x_i \geq 0, 1, 2, 3, 4 \ldots \ldots \ldots . N$$

3.1.6 Optimal Consumption and Saving Problems

A mathematical optimization problem that is often used in dynamic programming to economists concerns a consumer who lives over the periods t = 0, 1, 2, ..., T and must decide how much to consume and how much to save in each period.

Let c_t be consumption in period t, and assume consumption yields utility $u(c_t) = \ln(c_t)$ as long as the consumer lives. Assume the consumer is impatient, so that he discounts future utility by a factor b each period, where $0 < b < 1$. Let k_t be capital in period t. Assume initial capital is a given amount $k_0 > 0$, and suppose that this period's capital and consumption determine next period's capital as $k_t + 1 = Ak_t^a - c_t$, where A is a positive constant and 0 < a < 1. Assume capital cannot be negative. Then the consumer's decision problem can be written as follows:

$$\text{Maximize} \qquad Z = \sum_{t=0}^{T} \mathrm{b}^{\mathrm{t}}\, In\ (c_t)$$

$$\text{Subject to} \qquad k_{t+1} = Ak_t^a - c_t \geq 0 \tag{3.1}$$

for all t = 0,1,2,...,T

Written this way, the problem looks complicated, because it involves solving for all the choice variables $c_0, c_1, c_2, \dots, c_T$ and $k_1, k_2, \dots, k_{T+1}$ simultaneously. (Note that k_0 is not a choice variable—the consumer's initial capital is taken as given.)

The dynamic programming approach to solving this problem involves breaking it apart into a sequence of smaller decisions. A sequence of define a sequence of value functions $V_T(k)$, for $t = 0, 1, 2, \dots, T, T + 1$ which represents the value of any amount of capital k at a time t. Note that $V_{T+1}(k) = 0$, that is, there is (by assumption) no utility from having capital after death.

The value of any quantity of capital at any previous time can be calculated by backward induction using the Bellman equation. In this problem, for each $t = 0, 1, 2, \dots, T$ the Bellman equation is $V_t(k_t) =$ max (1n $(c_t) + bV_{t+1}(k_{t+1})$

$$\text{subject to} \qquad Ak_t^a - c_t \geq 0 \tag{3.3}$$

This problem is much simpler than the one stated earlier because it involves only two decision variables, c_t and k_{t+1}. Intuitively, instead of choosing his whole lifetime plan at birth, the consumer can take things one step at a time. At time t, his current capital k_t is given, and he only needs to choose current consumption c_t and saving k_{t+1}.

To actually solve this problem, one must work backwards. For simplicity sake the current level of capital is denoted as k. $V_{T+1}(k)$ is already known, using the Bellman equation to calculate lifetime. In other words, once $V_{T-j+1}(k)$, is known, $V_{T-j}(k)$, can be calculated which is the maximum of $\ln(c_{T-j}) + bV_{T-j+1}(Ak^a - c_{T-j})$, where c_{T-j} is the choice variable and working backwards, it can be shown that the value function at time t = T − j is where each V_{T-j} is a constant, and the optimal amount to consume at time t = T − j can be simplified to

$$c_T(k) = A\mathrm{k}^a, \text{ and } c_{T-1}(k) = \frac{Ak^a}{1+ab}, \text{ and } c_T - 2(k) = \frac{Ak^a}{1+ab+a^2b^2} \tag{3.4}$$

This is optimal to consume a larger fraction of current wealth as one gets older, finally consuming all the remaining wealth in period T, the last period of life.

3.1.7 The Knapsack Problem

Knapsack problems typically involve an upper limit on some sort of capacity (e.g the weight capacity that can be carried in a knapsack, and a set of discrete items that can be chosen each of which has a value, but also a cost in terms of the capacity (e.g. it has weight) the goal is to find the selection of items that have the greatest total value while still respecting the limit on the capacity. Mathematically, the 0-1-knapsack problem can be formulated as :

Let there be n items, x_i to x_n where x_i has a value v_i and weight w_j. The maximum weight that can be carried in the bag is W. It is common to assume that all values and weights are nonnegative. To simplify the representation, we also assume that the items are listed in increasing order of weight.

$$\text{Maximize} \sum_{i=1}^{n} v_i x_i \tag{3.5}$$

$$Subject\ to\ \sum_{i=1}^{n} w_i x_i \leq W, \quad x_i \in \{0\,,1\}$$

Maximize the sum of the values of the items in the knapsack so that the sum of the weights will be less than the knapsack's capacity. The bounded knapsack problem removes the restriction that there is only one of each item but restricts the number x_i of copies of each kind of item to an integer value c_j .

Mathematically the bounded knapsack problem can be formulated as :

$$\text{Maximize} \sum_{i=1}^{n} v_i x_i$$

$$\text{Subject to } \sum_{i=1}^{n} w_i x_i \leq W,\ x_i \in \{0\,, 1\,, \dots, c_i\} \tag{3.6}$$

The unbounded knapsack problem (UKP) places no upper bound on the number of copies of each kind of item and can be formulated as above except for that the only restriction on x_i is that it is a non-negative integer . If the example with the colored bricks above is viewed as an unbounded knapsack problem, then the solution is to take three yellow boxes and three grey boxes.

Example

Let's consider the knapsack problem which involves just 3 possible items, whose weight (w_j) and value (v_i) are summarized in the table below. The traveler has a traveling bag (knapsack) that takes b(kg) of items. How many pieces of items should be placed in the knapsack so as not exceed the maximum weight of b (kg) and will provide a maximum total value to the traveler?

Item	weight (kg)	value
1. A	w_1	v_1
2. B	w_2	v_2
3. C	w_3	v_3

This problem is formulated as Integer linear programming problem:

Maximize $v_1x_1 + v_2x_2 + v_3x_3$

Subject to $w_1x_1 + w_2x_2 + w_3x_3 \leq b$ (3.7)

Where $x_1, x_2, x_3 \geq 0$ *an integer*

Let's now formulate this knapsack problem for dynamic programming solution.

Step (i)

Stages: The natural breakdown in this case is by item. Since we are practicing backwards recursions, we will work in the order item, C, B, or A. We will first consider just C, then B and C, then A, B and C.

Step (ii) State at a stage: The state at a stage is the amount of carrying capacity remaining in the knapsack.

Step (iii) Decision at a stage: We must decide how many of the items to be considered at the current stage.

Step (iv) Decision update to the state: The number of the items taken reduces the carrying capacity for the future stages in an obvious way.

Step (v) Recursive value relationship: First, let's define a few functions and variables:

(a) is the current stage (1,2,3,indicating A, B, or C)

(b) d_t is the carrying capacity remaining when we reach stage t.

(c) x_t is the decision ,i.e the number of items t that we decide to take

(d) v_t is the value of one of item t

(e) $f_t(d_t)$ is the maximum value obtainable when we enter stage t with remaining carrying capacity of d_t,considering only stages t, t+1,…..,3

(f) The recursive value relationship is

$$f_t(d_t) = \max\{\mathrm{v_t x_t} + \mathrm{f_{t+1}}(\mathrm{d_t} - \mathrm{w_t x_t})\} \text{ where } 0 \leq x_t \leq \frac{d_t}{w_t} \text{ and integer.} \tag{3.8}$$

Note now f(.) appears on both sides of the recursive relationship .The optimum for stage t to stage 3 depends on two things : the value of the current decision at at stage t (i.e. $v_t x_t$) , and the value of the previously found optimum $f_{t+1}(.)$. Note also that

f_{t+1} (.) is calculated with a remaining carrying capacity of $d_{t-} w_t x_t$ meaning that the weight of the items taken at stage t has reduced the carrying capacity d_t with which you entered stage t.

3.2 Dynamic Programming Model for Selection of optimal investment portfolio

We present our proposed DP Model for selecting optimal investment portfolio as follows

Define $M_j(i)$ = the best return beginning in stage j and state i.

$d_j(i) =$ Decisions taken at state that achieves $M_j(i)$

We note that $M_j(0) = 0$ and $d_j(0) = 0$

The model for solving the above is:

Maximize: $$Z = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) + f_6(x)$$

Subject to $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leq w_i \qquad (3.9)$$

This model can be transformed into Integer Linear Programming problem as

$$\text{Max } z = \sum_i^n f_i(x_i) \qquad (3.10)$$

s.t $\sum_i^n x_i \leq w_i$, where

$$x_i \geq 0 \text{ , } an\ integer\ and\ w_i\ is\ the\ maximum\ investmet$$

which can easily be solved by Matlab software.

3.3 Markowitz Portfolio Theory of Portfolio Selection

3.3.1 Introduction

This section reviews the essential parts of the Markowitz's portfolio selection theory and other portfolio selection models.

Markowitz identified the trade-off facing the investor: risk versus expected return. The investment decision is not merely which securities to own, but how to divide the investor's wealth amongst securities. This is the problem of “Portfolio Selection;” hence the title of Markowitz's seminal article published in the March 1952 issue of the Journal of Finance. In that article and subsequent works, Markowitz extends the techniques of linear programming to develop the critical line algorithm. The critical line algorithm identifies all feasible portfolios that minimize risk (as measured by variance or standard deviation) for a given level of expected return and maximize expected return for a given level of risk. When graphed in standard deviation versus expected return space, these portfolios form the efficient frontier. The efficient frontier represents the trade-off between risk and expected return faced by an investor when forming his portfolio. Most of the efficient frontier represents well diversified portfolios. This is because diversification is a powerful means of achieving risk reduction. Markowitz developed mean-variance analysis in the context of selecting a portfolio of common stocks. Over the last two decades, mean-variance analysis has been increasingly applied to asset allocation. Asset allocation is the selection of a portfolio of investments where each component is an asset class rather than an individual security. In many respects, asset allocation is a more suitable application of mean-variance analysis than stock portfolio selection. Mean-variance analysis requires not only knowledge of the expected return and standard deviation on each asset, but also the correlation of returns for each and every pair of assets. Whereas a stock portfolio selection problem might involve hundred of stocks (and hence thousands of correlations), an asset allocation problem typically involves a handful of asset classes (for example stocks, bonds, cash, real estate, and gold). Furthermore, the opportunity to reduce total portfolio risk comes from the lack of correlation across assets. Since stocks generally move together, the benefits of diversification within a stock portfolio are limited. In contrast, the correlation across asset classes is usually low and in some cases negative. Hence, mean-variance is a powerful tool in asset allocation for uncovering large risk reduction opportunities through diversification.

3.3.2 Assumptions

As with any model, it is important to understand the assumptions of mean-variance analysis in order to use it effectively. First of all, mean-variance analysis is based on a single period model of investment. At the beginning of the period, the investor allocates his wealth among various asset classes, assigning a nonnegative weight to each asset.

During the period, each asset generates a random rate of return so that at the end of the period, wealth is influenced by the weighted average of the returns. In selecting asset weights, the investor faces a set of linear constraints, one of which is that the weights must sum up as one. Based on the game theory work of Von Neumann and Morgenstern, economic theory postulates that individuals make decisions under uncertainty by maximizing the expected value of an increasing concave utility function of consumption. In a one period model, consumption is end of period wealth. In general, maximizing expected utility of ending period wealth by choosing portfolio weights is a complicated stochastic nonlinear programming problem. To summarize the assumptions:

(i) Investors seek to maximize the expected return of total wealth.

(ii) All investors have the same expected single period investment horizon.

(iii) All investors are risk-adverse, that is they will only accept greater risk if they are compensated with a higher expected return.

(iv) Investors base their investment decisions on the expected return and risk.

(v). All markets are perfectly efficient (e.g. no taxes and no transaction cost).

The utility function is assumed to increase and concave. In terms of the approximating utility function, this translates into expected utility increasing in expected return (more is better than less) and decreasing in variance (the less risk the better). Hence, of all feasible portfolios, the investor should only consider those that maximize expected return for a given level of variance, or minimize variance for a given level of expected return. These portfolios form the mean-variance efficient set.

3.3.3 Konno -Yamazaki Model

Konno and Yamazaki (1991) proposed a new model using mean absolute deviation (MAD) as risk measure to overcome the weaknesses of the mean-variance model proposed by Markowitz. One of the most significant problems being the computational difficulty associated with solving a large scale quadratic problem associated with a dense covariance matrix**.** They introduced the risk function

$$w(x) = E\left[\left|\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right|\right] \qquad (3.11)$$

Where,

R_j = Random variable representing the rate of return on asset *Sj*

x_j = Amount invested in *Sj*

M_0 = Total fund amount

E [.] = Expected value of random variable in bracket

They then go on to state and prove the following theorem:

If $(R_1,\ldots\ldots..R_n)$ are multivariate normally distributed , then

$$w(x) = \sqrt{\frac{2}{\pi}}\,\sigma(x) \qquad (3.12)$$

Where $\sigma(x)$ = Standard deviation

They proved that these two measures *(w(x)* and R_i) are the same if $(R_1\ldots R_n)$ are multivariate normally distributed.

So the Model becomes the following;

Min w(x) $E[\ \left|\sum_{j=1}^{n} R_j x_j - E[\sum_{j=1}^{n} R_j x_j]\right|]$

s.t

$$\left|\sum_{j=1}^{n} E[R_j] x_j\right| \geq \rho M_0$$

$$\sum_{j=1}^{n} E[R_j] x_j \ \geq \ M_0 \tag{3.13}$$

$$\sum_{j=1}^{n} x_j = M$$

$$0 \ \leq j \ \leq u_j \ \ , j = 1, \dots . n$$

Konno and Yamazaki assumed that the expected value of the random variable can be approximated by the average from the data.

Therefore,

$$r_j = E\left[\ R_j\right] = \sum_{t=1}^{T} r_{jt}/T$$

Now,

$$E[\left|\sum_{j=1}^{n} R_j\ x_j - E[\sum_{j=1}^{n} R_j x_j]\right|] = \frac{1}{T}\sum_{j=1}^{T}\left|(r_{jt} - r_j) x_j\right| \tag{3.14}$$

Let

$a_{jt} = r_{jt} - r_j$, j= 1,…….,n; t=1,…..,T .

Model in 3.11 can be stated as,

$$\text{Min } \sum_{t=1}^{T}\left|\sum_{j=1}^{n} a_{jt} x_j\right|/T$$

$$\text{st} \quad \begin{cases} \sum_{j=1}^{n} r_j x_j \geq \rho M_0, \\ \sum_{j=1}^{n} x_j = M_0 \\ 0 \leq x_j \leq u_j, j = 1, \dots \dots ., n \end{cases} \tag{3.15}$$

which is equivalent to the following linear program:

$$\text{Min } \sum_{t=1}^{T} \frac{y_t}{T}$$

$$y_t + \sum_{j=1}^{n} a_{jt} x_j \geq 0, t = 1, \dots \dots ., T,$$

$$y_t - \sum_{j=1}^{n} a_{jt} x_j \geq 0, t = 1, \dots \dots, T$$

$$\sum_{j=1}^{n} r_j x_j \geq \rho M_0,$$

$$\sum_{j=1}^{n} x_j = M_0 \tag{3.16}$$

$$0 \leq x_j \leq u_j, j = 1, \dots \dots \dots, n$$

Konno-Yamazaki states the following advantages over Markowitz's model:

(1). No need to calculate the covariance matrix.

(2). Solving their linear program is much easier compared to solving a quadratic program.

(3) The optimal solution size is smaller

(4). T can be used as a control variable to restrict the number of assets in the portfolio.

3.3.4 Young Portfolio Selection Model

Young (1998) proposed a principle for choosing portfolios based on historical returns. Young used minimum return rather than variance as a measure of risk. He defined the optimal portfolio as one that would minimize the maximum loss over all past historical periods, subject to a restriction on the minimum acceptable average return across all observed time mean-variance analysis. If returns data are skewed, or if the portfolio optimization problem involves a large number of decision variables, his model would be advantageous to use. The model is**:**

$$Max_{MP,W}\ M_P$$

$$\text{st} \begin{cases} \sum_{j=1}^{N} w_j\, y_{jt} - M_p \geq 0\ , t = 1, \dots . . T \\ \sum_{j=1}^{N} w_j\, \overline{y_j} \geq G, \\ \sum_{j=1}^{N} w_j \leq W, \end{cases} \tag{3.17}$$

$w_j \geq 0\ , j = 1, \dots \dots, N.$

Where,

y_{jt} = Return on one dollar invested in security j in time period t

w_j= Portfolio allocation to security j

$\overline{y_J}$ = Average return on security j $= \frac{1}{T} \sum_{t=1}^{T} y_{tj}$

y_{pt} = Return on portfolio in time period t = $\sum_{j=1}^{N} x_j y_{jt}$

E_p = Average return on portfolio = $\sum_{j=1}^{N} x_j \overline{y_J}$

M_p = Minimum return on portfolio = $min_t y_{pt}$

The optimum portfolio maximizes Mp under imposed restrictions,

(i). Ep (average return) exceeds a minimum level G

(ii). Net asset allocations does not exceed total budget allocation W

Thus, Mp represents the portfolio's minimum return at the end of each time period and since Mp is being maximized, the portfolio will take on the maximum value of the minimum returns. According to Young, this model presents logical advantages over other portfolio optimization models if asset prices are not normally distributed and similar results when they are.

He states an equivalent model that seeks to maximize expected return, subject to a restriction that the portfolio return exceeds some threshold H in each observation period:

$$max_w \qquad E = \sum_{j=1}^{N} w_j \, \overline{y_J}$$

$$\text{st} \quad \begin{cases} \sum_{j=1}^{N} w_j y_{jt} \geq H\ , t = 1, \dots\dots, T \\ \sum_{j=1}^{N} w_j \leq W\ , \\ w_j \geq 0, j = 1, \dots\dots N \end{cases} \qquad (3.18)$$

This model has considerable advantage as it is a linear program and allows the model to treat additional complexities such as:

(a). transaction costs

(b). logical side constraints like

(i). inclusion or exclusion of both assets *a* and *b*

(ii). holding more than ¢*d* worth of asset *a*

Thus the minimum or maximum model is capable of incorporating a large number of modeling complexities and variations.

3.3.5 Problem of selection of optimal investment portfolio

To determine the optimal investment portfolios we use the mean-risk models. Let's consider a set of *n* securities denoted by R_j the rate of return of j security at the end of the investment period. R_j is a random variable because the future price of security is unknown. By the vector $x=(x_1, x_2, \dots, x_n)$ the portfolio where x_j expresses the weights defining portfolio *x*. The rate of return of portfolio x is a random variable defined as $R_X=x_1R_1+x_2R_2+\dots+x_n R_n$ The distribution of this random variable is defined by the function $F(r)=P(R_X \leq r)$ and is dependent on the choice of $X=(x_1, x_2, .., x_n)$. The component s of the vector *x* (weights) must satisfy two of the following conditions: all components should sum up to

$$\sum_{i=1}^{n} x_j = 1$$

and all weights should be non negative

$$x_j \geq 0 \, ,$$

Which means the short sales are not allowed?

Let's consider two portfolios $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$, with the rates of return

$R_X = x_1R_1 + x_2R_2 + \ldots + x_nR_n$ and $R_Y = y_1R_1 + y_2R_2 + \ldots, y_nR_n$ respectively . To decide which portfolio of x or y is better we can use the appropriate preference relation. If $\rho(\cdot)$ is a measure of risk and $E(\cdot)$ is the expected rate of return of portfolio (mean) then the portfolio x is preferred to y if and only if $E(R_X) \geq E(R_Y)$ and $\rho(R_X) \leq \rho(R_y)$ with at least one strict inequality . To choose an optimal portfolio according to this relation we can use the bi-criteria optimization model where risk is minimized and expected rate of return of portfolio is maximized. Usually a single objective mean risk model is used where the value of risk is minimized and the expected rate of return is constant:

$$\text{minimize} \quad \rho(R_x)$$

$$E(R_x) \geq R_0 \tag{3.19}$$

$$\sum_{j=1}^{n} x_j = 1$$

$$x_j \geq 0 \; for \; j = 1,2, \ldots, n$$

R_0 denote the required level of rate of return of portfolio and is defined by the investor.

It is also possible to maximize the expected rate of return and the value of the risk imposed restrictions.

In the model, such as the above, many different measures of risk were used. The most important measure is standard deviation or variance defined as σ^2(x) = $E[E(R_x) - R_x]^2$. It is a standard risk measure and it measures the dispersion of rate of return. The mean-variance model (MV model) is as follows:

$$\text{minimize} \qquad \sigma^2(R_x)$$

$$E(R_x) \geq R_0$$

$$\sum_{j=1}^{n} x_j = 1 \qquad 3.20)$$

$$x_j \geq 0 \; for \; j = 1,2, \dots, n$$

The other measures of risk which can be applied in model like the above are the conditional value at Risk and the Gini's mean difference. Using the scenario approach we can receive the mean-risk model with both measures in the linear form. In practice very often we assume that the rates of return of portfolio are discrete variables. These variables can be described by the realizations for T periods. For this purpose we can generate scenario or use the historical data. Let p_i denote the probability of scenario I (for *i=1,2,….,T* and *j=1,2,…..,T*) and $\sum_{i=1}^{T} p_i = 1$.

Random rates of return can be defined in discrete probability space. Let r_{ij} is the rate of return of security j in scenario I (for *i=1,2 …T* and *j=1,2,….,n*). This random variable R_j represents rate of return of security j by finite distribution $\{ r_{1j}, r_{2j},\ldots r_{Tj} \}$ with probability $p_1, p_2, \ldots, p_T$. Random variable $R_x = \{ R_{x1}, R_{x2}, \ldots, R_{xT} \}$ represents rate of return of portfolio $x = (x_1, x_2, \ldots, x_n)$. R_{xi} is a rate of return of portfolio in period i and is defined in the following way : $R_{Xi} = x_1 r_{i1} + x_2 r_{i2} + \ldots + x_n r_{in}$.

The conditional Value –at-Risk measures the expected loss corresponding to a number of worst cases, depending on the chosen confidence level α. The conditional Value at-Risk for portfolio x can be defined as

$$\text{CVaR } R_\alpha(R_X) = \min F_\alpha(x, v), \qquad V \epsilon R$$

Where $F_\alpha(x, v) = v + \frac{1}{T(1-\alpha)} E\{[-R_x - v]^+\}$

$$[u]^+ = \begin{cases} u \; for \; u \geq 0 \\ 0 \; for \; u < 0 \end{cases} \tag{3.21}$$

In the case when R_X is discrete random variable, function F_α (x,v) can be written as follows

$$F_\alpha(x, v) = v + \frac{1}{T(1-\alpha)} \sum_{i=1}^{T} p_i \left[-\sum_{j=1}^{n} x_j r_{ij} - v \right]^+ \tag{3.22}$$

By introducing the additional variables u_i defined by the condition $u_i + \sum_{i,j=1}^{n} x_j r_{ij} + v \geq 0$,

We obtain the mean-CVaR model (MCmodel) in the linear form:

$$minimize \quad v + \frac{1}{T(1-\alpha)} \sum_{i=1}^{T} p_i u_i$$

$$E(R_X) \geq R_{\mathbf{0}}$$

$$u_i + \sum_{j=1}^{n} x_j r_{ij} + v \geq 0 \; for \; i = 1, 2, \ldots, T$$

$u_i \geq 0 \quad for\ i = 1,2 \dots \dots T$

$$\sum_{j=1}^{n} x_j = 1 \qquad 3.23$$

$x_j \geq 0 \quad for\ j = 1,2, \dots \dots n$

where $x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_T$ are decision variables.

Mean-risk model in which risk is expressed by the Gini's Mean Difference can also be presented in the linear form. The Gini's Mean Difference (T) for security j is defined as**:**

$$\Gamma = \frac{1}{2} \sum_{i,k=1}^{T} \left| r_{ij} - r_{kj} \right| p_i\, p_k \qquad (3.23)$$

To calculate the Gini's Mean Difference (GMD) for portfolio x the following formula is used

$$\Gamma = \frac{1}{2} \sum_{i=1}^{T} \sum_{jk}^{n} \left| x_j r_{ij} - r_{ik} \right| p_j\, p_k$$

Let's introduce the additional variables defined as $d_{ik} \geq x_j r_{ij} - x_j r_{kj}$ (for *I*, *k=1,2,….,T*).For the scenario data the model with the Gini's Mean Difference (MG model) is the following :

Minimize $\Gamma p = \frac{1}{2} \sum_{i,k=1}^{T} d_{ik} p_i\, p_k$

$d_{ik} \geq \sum_{j=1}^{n} x_j r_{ij} - x_j r_{jk} \quad for\ i, k = 1,2\ , \dots .\ , T$

$E(R_x) \geq R_0$

$$\sum_{j=1}^{n} x_J$$

$$x_j \geq 0 \ for \ i, k, = 1,2, \dots., n$$

$$d_{ik} \geq 0 \ \ for \ i, k = 1,2, \dots., T \tag{3.25}$$

Using both models for different value of assumed level of rate of return we can receive the effective optimal solution.

3.3.6 Market equilibrium returns

The Black and Lettermen (1998), model uses the market equilibrium weights or capital asset pricing model (CAPM) as the basis. CAPM is developed by forming the efficient frontier of the market portfolios and tracing the capital market line (CML). The CML is tangent to the efficient frontier at the market portfolio. Therefore; there is no other combination of risky and riskless assets that can provide better returns for a given level of risk. CAPM:

$$\text{E}\ (r_i) = r_j + \frac{\sigma_i}{\sigma_m}\ (r_m - r_j)$$

$$= r_j + \beta_i \left(r_m - r_j\right), \tag{3.26}$$

where E (r_i) = Expected return on asset i

r_f = Risk free asset return

r_m = Return on market portfolio

σ_i = Standard deviation of returns on asset i

σ_m = Standard deviation of returns on market portfolio

$$\beta_i = \frac{\sigma_i}{\sigma_m} \tag{3.27}$$

The model uses CAPM in reverse. It assumes market portfolio is held by mean-variance investors and it uses optimization to back out the optimal expected returns. They define market equilibrium returns as:

$$\pi = \lambda \sum \omega$$

where, N = Number of assets

π= Vector of implied excess returns (N,1)

Σ = Covariance matrix of returns (N, N)

ω = Vector of market capitalization weights of the assets (N,1)

$$\lambda = \text{Risk aversion coefficient} = \frac{(r_m - r_j)}{\sigma_m^2} \tag{3.28}$$

3.3.6.1 Investor views

The views of the investors are incorporated into the model in the following form:

$$Q+\varepsilon = \begin{pmatrix} Q_1 \\ \vdots \\ Q_K \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_A \end{pmatrix} \tag{3.29}$$

Where,

K= Number of investor views

Q = Vector of investor views

ε = Error term

If $\varepsilon = 0$ that means the investor has 100% confidence in his views. ω denotes the variance of each error term. Assuming that each error term is independent of each other the covariance matrix Ω is a diagonal matrix with the following form:

$$\Omega=\begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots 0 & \omega_k \end{bmatrix} \tag{3.30}$$

Using the above formulation the model incorporates both absolute as well as relative views of the investors.

3.4 Proposed Mathematical Models for Portfolio Selection.

We present the mathematical model for portfolio selection based on expected return and risk. First, we give the necessary definitions and propositions.

3.4.0 Definitions and Propositions

Definition 3.4.1

A portfolio is a vector $(x1(n), \ldots, x_{m(}n), y(n))$ indicating the number of shares and bonds held by an investor between times $n - 1$ and n. A sequence of portfolios indexed by $n = 1, 2, \ldots$ is called an *investment strategy*. The *wealth* of an investor or the value of the strategy at time $n \geq 1$ is $V(n) = \sum_{j=1}^{m} x_j(n)s_j(n) + y(n)A(n)$ (3.31)

At time $n = 0$ the *initial wealth* is given by $V(0) == \sum_{j=1}^{m} x_j(1)s_j(0) + y(1)A(0)$ (3.32)

This means that the contents of a portfolio can be adjusted by buying or selling some assets at any time step, as long as the current value of the portfolio remains unaltered.

Definition 3.4.2

An investment strategy is called self-financing if the portfolio constructed at time $n \geq 1$ to be held over the next time step $n + 1$ is financed entirely by the current wealth $V(n)$, that is ,

$$\sum_{j=1}^{m} x_j(n+1)s_j + y(n+1)A(n) = V(n) \tag{3.33}$$

Definition 3.4.3

An investment strategy is called predictable if for each $n = 0, 1, 2, \dots$ the portfolio

$(x_1(n + 1), \dots, xm(n + 1), y(n + 1))$ constructed at time n depends only on the nodes of the tree of market scenarios reached up to and including time n.

The next proposition shows that the position taken in the risk-free asset is always determined by the current wealth and the positions in risky assets.

Proposition 3.4.4

Given the initial wealth $V(0)$ and a predictable sequence $(x_1(n), \dots, xm(n))$, $n = 1, 2, \dots$ of positions in risky assets, it is always possible to find a sequence $y(n)$ of risk-free positions such that $(x1(n), \dots, xm(n), y(n))$ is a predictable self-financing investment strategy.

Proof

Put $y(1) = \frac{V(0) - X_1(1)s_1(0) - \dots - x_m s_m(0)}{A(0)}$ and then compute

$V(1) = X_1(1)S_1(1) + \cdots + x_m(1)S_m(1) + y(1)A(1).$

Next, $y(2) = \frac{V(1) - x_1(2)s_1(1) - \cdots - x_m(2)s_m(1)A(1)}{A(1)}$

$V(2) = x_1(2)s_1(2) + \cdot\cdot\cdot + x_m(2)S_m(2) + y(2)A(2)$,and so on. This clearly defines a self-financing strategy. The strategy is predictable because $y(n + 1)$ can be expressed in terms of stock and bond prices up to time n.

Proposition 3.4.5

The return RS_v on a portfolio consisting of two securities is the weighted average

$$R_v = w_1R_1 + w_2R_2 \tag{3.34}$$

where w_1 and w_2 are the weights and R_1 and R_2 the returns on the two components.

3.5 Risk and Expected Return on a Portfolio

The expected return on a portfolio consisting of two securities can easily be expressed in terms of the weights and the expected returns on the components,

$$E(K_v) = w_1 E(R_1) + w_2 E(R_2) \tag{3.35}$$

This follows at once from (3.32 by the additive of mathematical expectation

Proof

Suppose that the portfolio consists of x_1 shares of security 1 and x_2 shares of security 2. Then the initial and final values of the portfolio are

$V(0) = x_1S_1(0) + x_2S_2(0)$, $V(1) = x_1S_1(0)(1 + R_1 + x_2S_2(0)(1 + R_2)$

$= V(0)(w_1(1 + R_1 + w_2(1 + R_2))$.

As a result, the return on the portfolio is

$$R_v = \frac{v(1)-v(0)}{v(0)} = w_1 r_1 + w_2 r_2$$

Theorem 3.5.1

The variance of the return on a portfolio is given by

$$Var(R_v) = w_1{}^2 Var(R_1) + w_2{}^2\, Var(R_2) + 2w_1 w_2\, Cov(R_1, R_2). \quad (3.36)$$

Proof

Substituting $R_v = w_1 R_1 + w_2 R_2$ and collecting the terms with $w_1{}^2$, $w_2{}^2$ and $w_1 w_2$, we compute

$$Var(R_v) = E(R_v)^2 - E(R_v)^2$$

$$= w_1^2 [E(R_1^2) - E(R_1)^2] + w_2^2 [E(R_2)^2) - E(R_2)^2] + 2w_1 w_2 [E(R_1 R_2) - E(K_1)E(K_2)]$$

$$= w_1^2\, Var(R_1) + w_2{}^2\, Var(R_2) + 2w_1 w_2\, Cov(R_1 R_2).$$

To avoid clutter, we introduce the following notation for the expectation and variance of a portfolio and its components:

$\mu V = E\, R_1\ )$, $\sigma_v = \sqrt{Var(r_v)}$

$\mu_1 = E(R_1\ )$, $\sigma_1 = \sqrt{Var(r_1})$

$\mu_2 = E(R_2)$, $\sigma_2 = \sqrt{var(r_2)}$

We shall also use the correlation coefficient

$$\rho_{12} = \frac{cov(\sigma_1 \sigma_2)}{\sigma_1 \sigma_2} . \quad (3.37)$$

Formulae (5.4) and (5.5) can be written as

$$\mu V = w_1 \mu_1 + w_1 \mu_2 \quad (5.7)$$

$$\sigma_v^2 = w_1^2 \rho\ _1^2 + w_2 \sigma_1^2 + 2w_1 w_2 \rho_{12} \sigma_1 \sigma_2. \quad (3.38)$$

3.6 Mathematical Formulation of the Proposed Model

The basic formulation which can solve the optimal portfolio selection problem is

Min $\{C^T X - X^T C X\}$

$$\text{st } \begin{cases} AX = b \\ x \geq 0 \end{cases} \tag{3.39}$$

Consider the Karush – Kuhn – Tucker conditions. KKT conditions provide the necessary conditions for optimality.

For,

Minimize *f(x)*

$$st. \begin{cases} g_i(x) \geq 0 \ , \ i = 1, \dots \dots, m \\ x \geq 0 \ , \end{cases}$$

The necessary conditions for optimality of x^*:

$$-\nabla f(x^*) + \sum_{i=1}^{m} \big(\lambda_i \nabla g_i(x^*)\big) \leq 0$$

$$[-\nabla f(x^*) + \textstyle\sum_{i=1}^{m} \big(\lambda_i \nabla g_i(x^*)\big) \leq 0]]x^* = 0$$

$$+\textstyle\sum_{i=1}^{m} \big(\lambda_i \nabla g_i(x^*)\big) = 0 \tag{3.40}$$

$$\lambda^* \geq 0$$

The above stated conditions are sufficient too, if *f(x)* is convex and, $g_{1,\dots\dots} ., g_m$, are concave.

Using the KKT conditions, we get the following conditions for x to be optimal in the set of equations in (1) was obtained

$c - 2Cx + A^T \lambda + v = 0$

$v^T x = 0$

$v >= 0$

For the sake of conversion into a standard form λ can be expressed

as a sum of two non –negative integers:

$\lambda = \lambda^{+} - \lambda^{-}$

$\lambda^{+}, \lambda^{-} >= 0$

Substituting the above equations in the KKT necessary conditions,

$c - 2Cx + A^{T} \lambda^{-} - A^{T} \lambda^{-} - + v = 0$

Now, the problem is to find λ, x, V which satisfies the following formulation,

$Min \sum_{i=1}^{n} u_i u_i$

$Ax = b$

$st \; - 2Cx + A^{T} \lambda^{+} - A^{T} \lambda^{-} + v + Fu = - c \qquad (3.41)$

$x, \lambda^{T}, \lambda^{-}, v, u >= 0$

Where F is a diagonal (n x n) matrix $F(i,i) = \begin{cases} 1, if c(i) \geq 0 \\ -1, if c(i) < 0 \end{cases}$

Markowitz's portfolio selection model based on risk was presented. From the theory of portfolio selection based on risk, the standard deviation or the variance measures the dispersion of the rate of returns. Hence, this can be used to measure the risk of portfolio investment.

Let's consider the portfolio

$X = (x_1, x_2, \ldots \ldots \ldots \ldots \ldots . x_n)$ and the return from x be

$R = (R_1, R_2, \ldots\ldots\ldots\ldots, R_n)$

The $Var(x)$ subject to the expected returns was minimized as follows:

$$Min \; Var(x)$$

$$\text{s.t} \quad E(R_x) \geq R_0$$

$$\sum_{j=1}^{n} x_j = 1$$

$$x_j \geq 0 \; for \; j = 1,2, \ldots \ldots, n$$

Where $Var(x) = E[E(R_x) - R_x]^{2}$ and R_0 denote the required level of rate of returns of portfolio

3.6 Summary

This chapter discusses the research methodology of the study.

The next chapter shall put forward the data collection and the analysis of the study and implement the proposed model.

CHAPTER FOUR

DATA COLLECTION, ANALYSIS AND IMPLEMENTATION OF MODEL

4.0 Introduction

This chapter presents the description and collection of data and analysis of the data. The chapter also presents the proposed model used on the data and its implementation.

4.1 Data description and collection

In this chapter, six investment features are considered in order to test the performance of the trading strategy and returns. Data was collected on the six investments –Government of Ghana's Treasury Bills, Barclays Bank Ghana, Ghana Commercial Bank, Zenith Bank Limited, Guinness Ghana Limited and State Insurance Company, Sunyani. Sample period ranges from 2006 to 2011. We normalize the price series such that each commodity's price changes have annualized volatility of 10%. Each commodity characteristic is its past returns at various time horizons. As such, in order to predict the one-year return factors for the commodities, pooled panel regression on the data set is run to obtain the annual returns. Based on the data collected, DP model and MPM were developed in chapter three. The DPM is then used to determine the optimal investment returns and the corresponding investments to be made, while the MPM is used to compute the risk of each investment.

Table 4.1: Average amount invested and corresponding returns in GH₵1$\times$ $\mathbf{10^2}$

Investment	**Returns**									
$f_1(x)$	0	10	20	25	33	35	40	45	55	60
$f_2(x)$	0	15	30	40	45	50	55	70	80	90
$f_3(x)$	0	12	22	32	35	40	50	60	65	70
$f_4(x)$	0	18	40	50	55	65	70	70	85	95
$f_5(x)$	0	20	28	38	50	75	75	80	80	85
$f_6(x)$	0	21	25	35	41	53	60	70	76	84
Amount Invested	0	100	200	300	400	500	600	700	800	900

Key:

Government of Ghana’s Treasury Bill, [$f_1(x)$]

Barclays Bank Ghana, Sunyani, [$f_2(x)$]

Ghana Commercial Bank, [$f_3(x)$]

Zenith Bank, Sunyani, [$f_4(x)$]

Ghana Guinness Limited, Sunyani,[$f_5(x)$]

State Insurance Company, Sunyani, [$f_6(x)$]

4.2 Preliminary analysis of data

The continuous Probabilistic Analysis (CPA) was used for the analysis of the optimal returns of the various investments. The variability of the project outcomes which result from the variability of the individual optimal investment portfolio enabled the researcher to make a probability assessment of the likelihood of the various optimal returns and the variability (risk) of using D.P and Markowitz models.

The most useful measure for statistical purposes is the standard deviation. Initially the mean of the returns over the period 2006 – 2011 was computed. It was afterwards followed by the dispersion and variance of the investment returns. The S.D is obtained by combining S.D's of individual investment returns, using what is known as the statistical sum. Having calculated the means and S.D's of various investment returns, the relative variability of the distribution of investment returns were computed, using the formula;

Coefficient of variation = $\frac{\sigma_i}{x} \times 100\%, where$ x is the mean of investment returns.

The results were compared and recorded in table 4.2

Table 4.2: Distribution of investments

Investment	M	Var	S.D	CV
$f_1(x)$	32.3	325.44	18.04	55.9
$f_2(x)$	47.5	711.29	26.67	56.1
$f_3(x)$	38.6	480.05	21.91	56.8
$f_4(x)$	54.8	769.51	27.74	50.6
$f_5(x)$	53.1	820.82	28.65	54.0
$f_6(x)$	46.5	649.23	25.48	54.8

M = Mean, S. D = Standard Deviation , Var = Variance and CV= Coefficient of variation

The result in Table 4.2 enabled the researcher to make probability statements about the investment portfolio returns, which reflect the variability expected in each investment and the distribution of the competing investment to be compared favorably in terms of risk.

4.3 Implementation of the proposed models

Chapter three, we develop the DP model and MPM based on the data collected from various investments. The models are implemented in the subsequent subsections.

4.3.1 Dynamic Programming Model (DPM)

The model for solving optimal investment portfolio based on expected returns using DPM is stated below:

Define $M_j(i) =$ the best return begining in stage j and state i

$$d_j(i) = \text{decisions taken at state that achieves } \mathrm{M_j(i)}$$

$$\mathrm{W_i} = \text{maximum investment available}$$

It is noted that $\mathrm{M_j}(0) = 0$ and $\mathrm{d_j}(0) = 0$

$$\sum_{i=1}^{n} f_i\, x_i \quad \text{is then maximized}$$

$$\text{s.t} \quad \sum_{i=1}^{n} \mathrm{x_i} \le \mathrm{W_i} \quad \text{, where } \mathrm{x_i} \ge 0, \quad \text{an integer.} \qquad (4.1)$$

4.3.2 Solution of DPM using DP algorithm

The solution is provided at last stage of the process, stage 6. It is assumed that the previous stages have been completed and the allocation of money to the investment 6 completed since it is not known how much was allocated to the previous investment (investment 5), the available units for investment 6 are unknown. As such many possibilities can be considered**.**

Iteration I: (Bottom up approach)

From investment 6; using model (4.1), we have

$M_6(9) = \max\,[f_6(0), f_6(1), f_6(2), f_6(3), f_6(4), f_6(5), f_6(6), f_6(7), f_6(8), f_6(9)]$

$= \max\,[\,0,\ 21, 25, 35, 41, 53, 60, 70, 76, 84]$ $\quad M_6(9) = 84$ With $d_6(9) = 9$

$M_6(8) = \max\,[f_6(0), f_6(1), f_6(2), f_6(3), f_6(4), f_6(5), f_6(6), f_6(7), f_6(8)]$

$= \max\,[\,0,\ 21, 25, 35, 41, 535, 60, 70, 76]$ $\quad M_6(8) = 76$ With $d_6\,(8) = 8$

$M_6(7) = \max\,[f_6(0), f_6(1), f_6(2), f_6(3), f_6(4), f_6(5), f_6(6), f_6(7)]$

$max = [0, 21, 25, 35, 41, 53, 60, 70]$ $\quad M_6(7\,) = 70$ With $d_6 = 7$

$M_6(6) = \max\,[f_6(0), f_6(1), f_6(2), f_6(3), f_6(4), f_6(5), f_6(6)]$

$= \max\,[0, 21, 25, 35, 41, 53, 60]$ $\quad M_6(6) = 60$ with $d_6(6) = 6$

$M_6(5) = \max\,[f_6(0), f_6(1), f_6(2), f_6(3), f_6(4), f_6(5)]$

$= \max\,[0, 21, 25, 35, 41, 53]$ $\quad M_6(5) = 53$ with $d_6(5) = 5$

$M_6(4) = \max\,[f_6(0), f_6(1), f_6(2), f_6(3), f_6(4)]$

$= \max [0, 21, 25, 35, 41]$ $\quad\quad$ $M_6(4) = 41$ with $d_6(4) = 4$

$M_6(3) = \max [f_6(0), f_6(1), f_6(2), f_6(3)]$

$= \max [0, 21, 25, 35\ ]$ $\quad\quad$ $M_6(3) = 35$ with $d_6(3) = 3$

$M_6(2) = \max [f_6(0), f_6(1), f_6(2)]$

$= \max [0, 21{,}25\ ]$ $\quad\quad$ $M_6(2\ ) = 25\ with\ d_6 = 2$

$M_6(1) = \max [f_6(0), f_6(1)]$

$= \max [0, 21]$ $\quad\quad$ $M_6(1) = 21\ with\ d_6(1) = 1$

$M_6(0) = \max [f_6(0)] \ = \max [0]$ $\quad\quad$ $M_6(0) = 0$ with $d_6(0) = 0$

Iteration II

From investment 5:

$$M_5(9) = \max [f_5(0) + M_6(9), \quad f_5(1) + M_6(8), \quad f_5(2) + M_6(7), \quad f_5(3) + M_6(6),$$
$$f_5(4) + M_6(5), \quad f_5(5) + M_6(4), \quad f_5(6) + M_6(3), \quad f_5(7) + M_6(2),$$
$$f_5(8) + M_6(1), \quad f_5(9) + M_6(0)]$$

$$M_5(9) = \max [0 + 84, \quad 20 + 76, \quad 28 + 70, \quad 38 + 63, \quad 50 + 53,$$
$$75 + 41, 75 + 35, \quad 80 + 25, \quad 805 + 21, \quad 85 + 0]$$

$$M_5(9) = \max [84, \ 96, \ 98, \ 101, \ 103, \ 116, \ 110, \ 105, \ 101, \ 85]$$

$$M_5(9) = 116 \text{ with } d_5(9) = 5$$

$M_5(8) = \max [f_5(0) + M_6(8), \quad f_5(1) + M_6(7), \quad f_5(2) + M_6(6), \quad f_5(3) + M_6(5),$

$f_5(4) + M_6(4), \quad f_5(5) + M_6(3), \quad f_5(6) + M_6(2), \quad f_5(7) + M_6(1),$

$f_5(8) + M_6(0)]$

$M_5(8) = \max [0 + 76, \quad 20 + 70, \quad 28 + 63, \quad 38 + 53, \quad 50 + 41, \quad 75 + 35,$

$75 + 25, \quad 80 + 21, \quad 80 + 0]$

$M_5(8) = \max [76, \quad 90, \quad 91, \quad 91, \quad 91, \quad 110, \quad 100, \quad 101, \quad 80]$

$M_5(8) = 110$ with $d_5(8) = 5$

$M_5(7) = \max [f_5(0) + M_6(7), \quad f_5(1) + M_6(6), \quad f_5(2) + M_6(2), \quad f_5(3) + M_6(4),$

$f_5(4) + M_6(3), \quad f_5(5) + M_6(2), \quad f_5(6) + M_6(1), \quad f_5(7) + M_6(0)]$

$M_5(7) = \max [0 + 70, \quad 20 + 63, \quad 28 + 53, \quad 38 + 41, \quad 50 + 35, \quad 75 + 25,$

$75 + 21, \quad 80 + 0]$

$M_5(7) = \max [70 \quad 83, \quad 81, \quad 79, \quad 85, \quad 100, \quad 96, \quad 80]$, $M_5(7) = 100$ with $d_5(7) = 5$

$M_5(6) = \max [f_5(0) + M_6(6), \quad f_5(1) + M_6(5), \quad f_5(2) + M_6(4), \quad f_5(3) + M_6(3),$

$f_5(4) + M_6(2), \quad f_5(5) + M_6(1), \quad f_5(6) + M_6(0)]$

$M_5(6) = \max [(0 + 63), (20 + 53), (28 + 41), (38 + 35), (50 + 25), (75 + 21),$

$(75 + 0)]$

$M_5(6) = \max [63, \quad 73, \quad 69, \quad 73, \quad 75, \quad 96, \quad 75]$

$M_5(6) = 96$ with $d_5(6) = 5$

$$M_5(5) = \max\,[f_5(0) + M_6(5),\quad f_5(1) + M_6(4),\quad f_5(2) + M_6(3),\quad f_5(3) + M_6(2),$$
$$f_5(4) + M_6(1),\quad f_5(5) + M_6(0)\,]$$

$$M_5(5) = \max\,[0 + 53,\quad 20 + 41,\quad 28 + 35,\quad 38 + 25,\quad 50 + 21,\quad 75 + 0]$$

$$M_5(5) = \max\,[53,\ 61,\ 63,\ 63,\ 71,\ 75] \qquad M_5(5) = 75 \text{ with } d_5(5) = 5$$

$$M_5(4) = \max\,[f_5(0) + M_6(4),\quad f_5(1) + M_6(3),\quad f_5(2) + M_6(2),\quad f_5(3) + M_6(1),$$
$$f_5(4) + M_6(0)]$$

$$M_5(4) = \max\,[0 + 41,\ 20 + 35,\ 28 + 25,\ 38 + 21,\ 50 + 0]$$

$$M_5(4) = \max\,[41,\ 55,\ 53,\ 59,\ 50],\quad M_5(4) = 59 \text{ with } d_5(4) = 3$$

$$M_5(3) = \max\,[f_5(0) + M_6(3),\quad f_5(1) + M_6(2),\quad f_5(2) + M_6(1),\quad f_5(3) + M_6(0)]$$

$$M_5(3) = \max\,[0 + 35,\ 20 + 25,\ 28 + 21,\ 38 + 0]$$

$$M_5(3) = \max\,[35\,,\ 45,\ 49,\ 38] \qquad M_5(3) = 49 \text{ with } d_5(3) = 2$$

$$M_5(2) = \max\,[f_5(0) + M_6(2),\ f_5(1) + M_6(1),\ f_5(2) + M_6(0)]$$

$$M_5(2) = \max\,[0 + 25,\ 20 + 21,\ 28 + 0]$$

$$M_5(2) = \max\,[25,\ 41,\ 28] \qquad M_5(2) = 41 \text{ with } d_5(2) = 1$$

$$M_5(1) = \max\,[f_5(0) + M_6(1),\ f_5(1) + M_6(0)]$$

$$M_5(1) = \max\,[0 + 21,\ 20 + 0] \qquad M_5(1) = \max\,[21,\ 20]$$

$$M_5(1) = 25 \text{ with } d_5(1) = 1$$

$$M_5(0) = \max\,[f_5(0) + M_6(0)]$$

$M_5(0) = \max[0 + 0]$

$M_5(0) = \max[0]$ $\quad\quad$ $M_5(0) = 0$ with $d_5(0) = 0$

Using the algorithm above through iterations I- VI, the optimal returns from the various investments are shown in Table 4.3 below. The computations for the optimal returns for investment - 4 to investment – 1 can be found in Appendix A

4.3.3 Allocations of investment

The optimal return from the investment is GH¢1.51 x 10^4 which is obtained by starting the allocation from stage 1, and later to stage 2 and then to stage 6 as follows:

(i) With GH¢9.0 x 10^4, available allocate to stage 1, $d_1(\text{GH¢}9.0 \times 10^4) = 0$ leaving GH¢9.0 x 10^4 – 0 = GH¢9.0 x 10^4

(ii) With GH¢9.0 x 10^4, available allocate to stage 2, $d_2(\text{GH¢}9.0 \times 10^4) = GH¢1.0 \times 10^4$ leaving GH¢9.0 x 10^4 – GH¢1.0 x 10^4 = GH¢8.0 x 10^4

(iii) With GH¢8.0 x 10^4, available allocate to stage 3, $d_3(\text{GH¢}8.0 \times 10^4) = 0$, leaving GH¢8.0 x 10^4 – 0 = GH¢8.0 x 10^4

(iv) With GH¢8.0 x 10^4, available allocate to stage 4, $d_4(\text{GH¢}8.0 \times 10^4) = GH¢2.0 \times 10^4$ leaving GH¢8.0 x 10^4 – GH¢2.0 x 10^4 =GH¢6.0 x 10^4.

(v) With GH¢6.0 x 10^4, available allocate to stage 5, $d_5 = \text{GH¢}6.0 \times 10^4) = \text{GH¢}5.0 \times 10^4$, leaving = GH¢6.0 x 10^4 – GH¢5.0 x 10^4 =GH¢1.0 x 10^4

(vi) With GH¢1.0 x 10^4, available allocate to stage 6, d_6(GH¢1.0 x 10^4) $= GH$¢1.0 x 10^4, leaving GH¢1.0 x 10^4 – GH¢1.0 x 10^4 = 0

Table 4.3; Optimal Returns

Investment $f_i x_i$	1	2	3	4	5	6
Amount invested (x10^2)	0	100	0	200	500	100
Optimal returns (x10^2)	0	15	0	40	75	21

4.3.4 Interpretation of results

Table 4.3 shows that with GH¢9.0 x 10^4 available for investment and given its corresponding annual returns from various financial institutions it becomes unattractive to invest in Government of Ghana's Treasury Bills and Ghana Commercial Bank. One should however invest GH¢1.0 x 10^4 in Barclays Bank to get GH¢1.5 x 10^3, GH¢2.0 x 10^4 in Zenith Bank, Sunyani to get GH¢4.0 x 10^3, GH¢5.0x10^4 in Ghana Guinness Limited, Sunyani for a return of GH¢7.5 x 10^3 and GH¢1.0 x 10^4 in State Insurance Cooperation Limited, Sunyani for a return of GH¢2.1 x 10^3.

4.3.5 Further analysis of optimal investment

In order to carry out post optimal analysis on optimal investment the optimal returns was used to formulate the objective function on account of the various investments returns, and thus formulate the entire process as integer linear programming. Matlab package was used for the analysis as follows

Max Z= $0x_1 + 15x_2 + 0x_3 + 40x_4 + 75x_5 + 21x_6$

s.t $\quad 10x_1 + 15x_2 + 12x_3 + 18x_4 + 20x_5 + 21x_6 \le 100$

$$20x_1 + 30x_2 + 22x_3 + 40x_4 + 28x_5 + 25x_6 \le 200$$

$$25x_1 + 40x_2 + 32x_3 + 50x_4 + 38x_5 + 35x_6 \le 300$$

$$33x_1 + 45x_2 + 35x_3 + 55x_4 + 50x_5 + 41x_6 \le 400$$

$$350x_1 + 50x_2 + 40x_3 + 65x_4 + 75x_5 + 53x_6 \le 500$$

$$405x_1 + 55x_2 + 50x_3 + 70x_4 + 75x_5 + 60x_6 \le 600$$

$$45x_1 + 70x_2 + 60x_3 + 70x_4 + 80x_5 + 70x_6 \le 700$$

$$55x_1 + 80x_2 + 65x_3 + 85x_4 + 80x_5 + 76x_6 \le 800$$

$$60x_1 + 90x_2 + 70x_3 + 95x_4 + 80x_5 + 84x_6 \le 900$$

$$x_i \ge 0\,, integer \;\;, where\ i = 1\,, 2\,, 3\,, \dots\dots, 6$$

Using matlab package we have;

f = [0 -15 0 -40 -75 -21]

$$A = \begin{bmatrix} 10 & 15 & 12 & 18 & 20 & 21 \\ 20 & 30 & 22 & 40 & 28 & 25 \\ 25 & 40 & 32 & 50 & 38 & 35 \\ 33 & 45 & 35 & 55 & 50 & 41 \\ 35 & 50 & 40 & 65 & 75 & 53 \\ 40 & 55 & 50 & 70 & 75 & 60 \\ 45 & 70 & 60 & 70 & 80 & 70 \\ 55 & 80 & 65 & 85 & 80 & 76 \\ 60 & 90 & 70 & 95 & 80 & 84 \end{bmatrix}$$

b = [100 200 300 400 500 600 700 800 900]

The syntax is: [*x*, *fval*, *exit flag*, *output*] = bintprog (*f*, *A*, *b*)

Table 4.4: The solution to the ILP problem

Decision Variables	Solution Variables	Unit Cost	Total Contribution	Reduced Cost
x_1	0	0	0	0
x_2	1	15	15	
x_3	0	0	0	0
x_4	1	40	40	
x_5	1	75	75	
x_6	1	21	21	

Optimal value = GH¢1.51 x 10^4

Number of iterations = 4 and time = 1.35724sec

4.3.6 Interpretation of the results

From table 4.4 above, it can be inferred that returns from investments 1 (f_1x_1) and investment 3 (f_3x_3) have an optimal portfolio value of zero since the solution variables $x_1 = x_3 = 0$. On the other hand, returns from investment 5 (f_5x_5) contributed most to the optimal value. This is followed by returns from investment 4, investment 6 and investment 2 respectively since the solution variables $x_2 = x_4 = x_6 = x_5 = 1$.

The results from ILP clearly shows that DP is very efficient in allocating resources for optimal investment returns from a portfolio since the results from both ILP and DP gave a maximum optimal value of GH¢1.51x10^4.

4.3.7 Implementation of proposed model for portfolio selection based on risk.

The proposed model for portfolio selection based on risk was formulated in chapter three as:

Min Var(x)

s.t $E(R_x) \geq R_0$

$\sum_{j=1}^{n} x_j = 1$

$x_j \geq 0 \; for \; j = 1,2,\ldots\ldots,n$

where $Var\ (x) = E[E(R_x) - R_x]^{2}$ and R_0 denotes the required level of rate of returns of portfolio.

Substituting the values of tables 4.1 and 4.2 into this model, we have;

min Var(x) = $325.44x_1 + 711.29x_2 + 480.05x_3 + 769.51x_4 + 820.82x_5 + 649.23x_6$

s.t $10x_1 + 15x_2 + 12x_3 + 18x_4 + 20x_5 + 21x_6 \leq 100$

$20x_1 + 30x_2 + 22x_3 + 40x_4 + 28x_5 + 25x_6 \leq 200$

$25x_1 + 40x_2 + 32x_3 + 50x_4 + 38x_5 + 35x_6 \leq 300$

$33x_1 + 45x_2 + 35x_3 + 55x_4 + 50x_5 + 41x_6 \leq 400$

$350x_1 + 50x_2 + 40x_3 + 65x_4 + 75x_5 + 53x_6 \leq 500$

$405x_1 + 55x_2 + 50x_3 + 70x_4 + 75x_5 + 60x_6 \leq 600$

$45x_1 + 70x_2 + 60x_3 + 70x_4 + 80x_5 + 70x_6 \leq 700$

$55x_1 + 80x_2 + 65x_3 + 85x_4 + 80x_5 + 76x_6 \leq 800$

$60x_1 + 90x_2 + 70x_3 + 95x_4 + 80x_5 + 84x_6 \leq 900$

$x_i \geq 0 , integer \; , where \; i = 1 , 2 ,3 , \ldots\ldots ,6$

4.3.8 Solution of the proposed model

Table 4.4 : The solution to the ILP problem

Decision Variables	Solution Variables	Unit Cost	Total Contribution	Reduced Cost
x_1	1	325.44	325.44	0
x_2	1	711.29	711.29	
x_3	1	480.05	480.05	0
x_4	1	769.51	769.51	
x_5	1	820.82	820.82	
x_6	1	649.23	649.23	

Optimal risk value = GH¢3.75630 x 10^5

Number of iterations = 6 and time = 1.52884se

4.3.9 Interpretation of the results

The optimal solution $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$ and the objective function value Z= GH¢3.75630 x 10^5. Since variance is a measure of risk, it follows that the investor will invest where there is minimum risk and highest return. However, from table 4.4, the maximum return from the investment portfolio is associated with the highest risk. All investors are risk-adverse. That is, they will only accept greater risk if they are compensated with a higher expected return.

The correlation coefficient of portfolio investment always satisfies $-1 \leq \rho_{ij} \leq 1$ the next proposition is usually concerned with two typical cases when ρ_{ij} assumes one of the extreme values 1 or -1, which means perfect or negative correlation between the investments.

From DP analysis, investments $f_2,\ f_4,\ f_5\ and\ f_6$ were selected to be invested. The correlations coefficient between the expected returns of these investments and risk involved is then computed as shown in the table 4.5

Table 4.5 Correlations between investment Returns and Risks

Investment	Returns	Risks	correlation coefficient
f_1	0	325.04	
f_2	15	711.29	
f_3	0	480.82	
f_4	40	769.51	} 0.9286
f_5	75	820.82	
f_6	21	649.23	

From table 4.5, correlation between expected returns and risk of investment is 0.9286.This suggested that there is a strong positive correlation between expected returns and the risk of investment. High expected returns can be attributed to high risk of investment. This means short term investments should be adopted in any of the investments. That is portfolio with long term investment is strictly not allowed.

4.4.0 Findings

From the DP, MPT algorithms and correlation coefficient analysis, it was found out that, the optimal investment return was GH¢$1.51\text{x}10^4$. Investments five, six, four and two were the prime investments which contributed most to the optimal expected return. Investment one and three were not profitable since their total contribution to the portfolio return was zero. No investor will be interested in investing in these financial institutions.

One obvious observation about these investments is that they all have high risk of investment. Also, the returns from these investments are very high. There is a strong positive correlation between expected returns and risk of the investments. This seems to suggest that, any rational investor will be interested in a situation where there is a higher expected return with relative high risk of investment. .

4.5 Summary

In this chapter, the proposed model formulated was implemented and output results were discussed. The last chapter of this project presents the summary, conclusions of the whole work, and recommendations of the study.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

This final chapter, present the summary, discussions, conclusions and recommendations of the study.

5.2 Summary

The primary aims of this thesis are; to develop optimization models based on DP algorithm and MPT, and use them to determine the optimal returns of investments and the risks involved. The thesis also seeks to find the correlation between expected returns and risk of investments.

To achieve these aims, secondary data was collected from six financial institutions in the Sunyani municipality from 2006 – 2011. The price series were normalized such that each commodity's price changes had annualized volatility of ten percent (10%). Financial ratio such as coefficient of variation (CV) which measures the relative probability of investing in each of the investment was calculated and used to analyze the data. Based on this empirical data, MPT and DP models were formulated and used to find the risk and corresponding returns involved in various investments.

From the literature, DP was found to be more efficient algorithm for determining how much to invest in each investment portfolio than the Knapsack algorithm.

Through the analysis of the investments, Barclays Bank, Zenith Bank, Ghana Guinness Limited and State Insurance Company were respectively selected because they had high optimal returns [see table 4.3].

One obvious observation about these investments is that they all have high risk of investment. Also, the returns from these investments are very high. There is a strong positive correlation between expected returns and risk of the investments. This seems to suggest that, any rational investor will be interested in a situation where there is a higher expected return with relative high risk of investment

5.3 Discussions and Conclusions

The use of DP in investment portfolios analysis helps to decide whether to accept or reject an investment with more realism. There are two main points to note. One is the proposition that DP in investment portfolios allows to relax the low-before-high fare order of arrival bookings. The problem can be solved from any direction without any particular arrangement of the investments. However, the optimal solution was obtained by a backward substitution. The DP provided the optimal policy for the portfolio problem by evaluating the whole tree of possibilities and making at each point in time the decision that would imply higher future expected revenues, and processing backward recursion. The darker side was the increase in the computation difficulties according to the dimension of the problem.

Through DP algorithm a maximum return of ¢1.51 x 10^4 was obtained from an investment of GH¢9.0 x 10^4. This optimal maximum return of GH¢1.51 x 10^4, did not include the contributions from Government of Ghana Treasury bill and Ghana Commercial Bank. This is because the DP algorithm rejected these investments. Any attempt to invest in these investments will seriously affect returns and as such no investor would be ready to risk his hard won resources.

The MPT based on measure of risk, helped the researcher to take realistic decisions based on account of expected returns from the DP algorithm. Normally, an investor will invest where there is minimum risk and very high returns. In this case, the investor is interested in investing in a portfolio where there is a higher risk with the highest returns. This confirms the fact that all investors are risk-adverse, That is, they will only accept greater risk if they are compensated with higher expected return, [see tables 4.4 and 4.5]. There is a strong positive correlation between expected returns and risk of the investments. The higher the risk the better the expected returns.

5.4 Recommendations

It has emerged from the conclusion that the use of scientific methods such as DP and MPT to solve the investor problem of where and how much to invest should adopt these models in their allocations of funds for optimum investment portfolio in areas where they are sure to get optimum returns with minimal risk.

Secondly, it is recommend that investment companies and other financial institutions be educated to use scientific methods such as the mathematical models to help them select viable investment instead of relying on ad hoc or judgmental approach to investment..

Lastly, it is recommended that investors should not invest too much money in a single investment. One should always divide the resources available in bits to invest in different investments.

REFRENCE

1. Artzner, P. Delbaen, F. , Heath, J-M., Eber, D., and H. Ku (1997-1999). Coherent multi-period measures of risk.

2. Balzer, L.A., (2001). Investment risk: a unified approach to upside and downside returns, in (Sortion F, and Satechell eds). Managing downside risk in financial markets: theory and implementation Orth-Heinermann, Oxford 103-156.

3. Basle Committee (1996). Overview of the Amendment of the Capital Accord to Incorporate Market Risk, Basle Committee on Banking Supervision.

4. Bellman, R., (1957).Dynamic Programming. Princeton N.J: Princeton University Press.

5. Black and Letterman, R. (1998). “Global portfolio optimization,” Financial Analysts] Black, F., Journal, September-October.

6. Brodie D M, Daubechies G, Loris (1998). Sparse and stable Markowitz portfolios. Proc. Natl. Acad. Sci., pp. 106.

7. Daykin and Tonny, (1996). Model Risk, 9(5): 34-35.

8. Ghezzi (1997). Influence of mathematical models in finance on practice: past, present and future.

9. Herman, E.J. Value at Risk (2009). The benchmark for Controlling Market Risk. McGraw-Hill, Chicago future. Financial Practice and Education, 5(1) 7-15.

10. Jacobson. (2003). A multi dimensional framework for financial economic decisions. Journal of Multi-Criteria Decision Analysis, 11(3):111-124, 23.

11. Jean Luc Vila and Elton E. J (1991). Strategic financial management in multinational financial conglomerate: A multiple goal stochastic programming approach. European Journal of Operational Research, 128(20): 418 – 434.

12. Jorion, P. (2000). Value at Risk: the New Benchmark for Managing Financial Risk. McGraw-Hill.

13. Konno, H. Shirakawa, H. Yamazaki, (1991). A mean- absolute deviation- skewness portfolio optimization model, Analysis of Operational. Research, 45 205-220.

14. Markowitz, H.(1952). Theories of uncertainty and financial behavior, conometrica,19:325-326.

14. Markowitz, H. (1959). Portfolio selection; efficient diversification of investment, New York Wiley.

15. Merton Miller and William Shape (1995) Influence of mathematical models in finance on practice: past, present, and future. Financial Practice and Education, 5(1):7-15.

16. McNeil, A.J. (1999) Extreme Value Theory for Risk Managers. Risk Special Volume, to appear. ETH Preprint (www.math.ethz.ch/ mcneil/pub list.html).

17. McNeil, A. and Frey, R. (2001), Estimation of tail related risk measures for heteroscedastic Financial time series: An extreme value approach, Journal of Empirical Finance 7,271-300 different distributional hypothesis on returns in asset allocation to appear in Finance Letters.

18. Paolo Battocchio M., Giacometti R., Ortobelli S. and S., Rachev (2005). The impact of 15.

19. Pola P. M (2006). Multidimensional measure of risk: The prospect ranking vector.

20. Roman, D., Darby - Dorman, K., Mitra G (2007). Mean-risk models using two risk

Measures: a multi-objective approach. Quantitative Finance, vol.7, 443-458.

21. Rudoy and Booth., (2001). The classification of parametric choices under uncertainty: analysis of the portfolio choice problem, Theory and Decision 51, 297-327.

22. Sharpe, William F (1964). Capital asset prices: A theory of market equilibrium under conditions of risk, Journal of Finance, 19 (3), 425-442.

23. Steffen and Aitken, (2005). Valuing models and modeling value. The Journal of Portfolio Management, pages 106-114, spring.

24. Sun and Zeleny. M (200): An application of goal programming to the management of a country bank. Journal of Bank Research, 7(4): 311-319.

25. Ye, (2008). "An Empirical Bayes Approach to Efficient Portfolio Selection," Journal of Financial and Quantitative Analysis, 21, No. 3, 293–305). "For better performance; Constrain portfolio weights," Journal of Portfolio Management, 14, 29–34.

26. Young, M.R. (1998). A minimax portfolio selection rule with linear programming solution, Management Science 44, 673-683.

Appendix A

DP Iterations

Iteration I (Bottom up approach)

From investment 6; using model (4.1), we have

$\mathrm{M}_6(9) = \max\,[\mathrm{f}_6(0), \mathrm{f}_6(1), \mathrm{f}_6(2), \mathrm{f}_6(3), \mathrm{f}_6(4), \mathrm{f}_6(5), \mathrm{f}_6(6), \mathrm{f}_6(7), \mathrm{f}_6(8), \mathrm{f}_6(9)]$

$= \max\,[\,0,\ 21, 25, 35, 41, 53, 60, 70{,}76, 84]$ $\qquad M_6(9) = 84$ with $d_6(9) = 9$

$\mathrm{M}_6(8) = \max\,[\mathrm{f}_6(0), \mathrm{f}_6(1), \mathrm{f}_6(2), \mathrm{f}_6(3), \mathrm{f}_6(4), \mathrm{f}_6(5), \mathrm{f}_6(6), \mathrm{f}_6(7), \mathrm{f}_6(8)]$

$= \max\,[\,0,\ 21, 25, 35, 41, 535, 60, 70, 76]$ $\qquad M_6(8) = 76$ with $d_6\,(8) = 8$

$max = [0, 21, 25, 35, 41, 53, 60, 70]$ $\qquad M_6(7\,) = 70\ with\ \ d_6 = 7$

$\mathrm{M}_6(6) = \max\,[\mathrm{f}_6(0), \mathrm{f}_6(1), \mathrm{f}_6(2), \mathrm{f}_6(3), \mathrm{f}_6(4), \mathrm{f}_6(5), \mathrm{f}_6(6)]$

$= \max\,[0, 21, 25, 35, 41, 53, 60]$ $\qquad M_6(6) = 60$ with $d_6(6) = 6$

$\mathrm{M}_6(5) = \max\,[\mathrm{f}_6(0), \mathrm{f}_6(1), \mathrm{f}_6(2), \mathrm{f}_6(3), \mathrm{f}_6(4), \mathrm{f}_6(5)]$

$= \max\,[0, 21, 25, 35, 41, 53]$ $\qquad M_6(5) = 53$ with $d_6(5) = 5$

$M_6(4) = \max\,[f_6(0), f_6(1), f_6(2), f_6(3), f_6(4)]\ = \max\,[0, 21, 25, 35, 41]\ \ M_6(4) = 41$ with $d_6(4) = 4$

$\mathrm{M}_6(3) = \max\,[\mathrm{f}_6(0), \mathrm{f}_6(1), \mathrm{f}_6(2), \mathrm{f}_6(3)]\ \ = \max\,[0, 21, 25, 35\ \ ]\ \ \mathrm{M}_6(3) = 35$ with $\mathrm{d}_6(3) = 3$

$\mathrm{M}_6(2) = \max\,[\mathrm{f}_6(0), \mathrm{f}_6(1), \mathrm{f}_6(2)]\ \ \ = \max\,[0, 21{,}25\,]\ \ \ \ \mathrm{M}_6(2\,) = 25$ with $\mathrm{d}_6 = 2$

$\mathrm{M}_6(1) = \max\,[\mathrm{f}_6(0), \mathrm{f}_6(1)] \;\; = \max\,[0, 21] \quad \mathrm{M}_6(1) = 21$ with $\mathrm{d}_6(1) = 1$

$\mathrm{M}_6(0) = \max\,[\mathrm{f}_6(0)] \;\; = \max\,[0] \quad \mathrm{M}_6(0) = 0$ with $\mathrm{d}_6(0) = 0$

Iteration II

From investment 5:

$$\begin{aligned}\mathrm{M}_5(9) = \max\,[&\mathrm{f}_5(0) + \mathrm{M}_6(9), \quad \mathrm{f}_5(1) + \mathrm{M}_6(8), \quad \mathrm{f}_5(2) + \mathrm{M}_6(7), \quad \mathrm{f}_5(3) + \mathrm{M}_6(6),\\ &\mathrm{f}_5(4) + \mathrm{M}_6(5), \quad \mathrm{f}_5(5) + \mathrm{M}_6(4), \quad \mathrm{f}_5(6) + \mathrm{M}_6(3), \quad \mathrm{f}_5(7) + \mathrm{M}_6(2),\\ &\mathrm{f}_5(8) + \mathrm{M}_6(1), \quad \mathrm{f}_5(9) + \mathrm{M}_6(0)]\end{aligned}$$

$$\begin{aligned}\mathrm{M}_5(9) = \max\,[&0 + 84, \quad 20 + 76, \quad 28 + 70, \quad 38 + 63, \quad 50 + 53,\\ &75 + 41, 75 + 35, \quad 80 + 25, \quad 805 + 21, \quad 85 + 0]\end{aligned}$$

$$M_5(9) = \max\,[84, \; 96, \; 98, \; 101, \; 103, \; 116, \; 110, \; 105, \; 101, \; 85]$$

$M_5(9) = 116$ with $d_5(9) = 5$

$$\begin{aligned}\mathrm{M}_5(8) = \max\,[&\mathrm{f}_5(0) + \mathrm{M}_6(8), \quad \mathrm{f}_5(1) + \mathrm{M}_6(7), \quad \mathrm{f}_5(2) + \mathrm{M}_6(6), \quad \mathrm{f}_5(3) + \mathrm{M}_6(5),\\ &\mathrm{f}_5(4) + \mathrm{M}_6(4), \quad \mathrm{f}_5(5) + \mathrm{M}_6(3), \quad \mathrm{f}_5(6) + \mathrm{M}_6(2), \quad \mathrm{f}_5(7) + \mathrm{M}_6(1),\\ &\mathrm{f}_5(8) + \mathrm{M}_6(0)]\end{aligned}$$

$$\begin{aligned}M_5(8) = \max\,[&0 + 76, \quad 20 + 70, \quad 28 + 63, \quad 38 + 53, \quad 50 + 41, \quad 75 + 35,\\ &75 + 25, \quad 80 + 21, \quad 80 + 0]\end{aligned}$$

$\mathrm{M}_5(8) = 110 \qquad \mathrm{M}_5(8) = \max\,[76, \; 90, \; 91, \; 91, \; 91, \; 110, \; 100, \; 101, \; 80]$

with $d_5(8) = 5$

$$\mathrm{M}_5(7) = \max\,[\mathrm{f}_5(0) + \mathrm{M}_6(7), \quad \mathrm{f}_5(1) + \mathrm{M}_6(6), \quad \mathrm{f}_5(2) + \mathrm{M}_6(2), \quad \mathrm{f}_5(3) + \mathrm{M}_6(4),$$
$$\mathrm{f}_5(4) + \mathrm{M}_6(3), \quad \mathrm{f}_5(5) + \mathrm{M}_6(2), \quad \mathrm{f}_5(6) + \mathrm{M}_6(1), \quad \mathrm{f}_5(7) + \mathrm{M}_6(0)]$$

$$\mathrm{M}_5(7) = \max\,[0 + 70, \quad 20 + 63, \quad 28 + 53, \quad 38 + 41, \quad 50 + 35, \quad 75 + 25,$$
$$75 + 21, \quad 80 + 0\,]$$

$$M_5(7) = \max[70 \quad 83, \quad 81, \quad 79, \quad 85, \quad 100, \quad 96, \quad 80] \quad M_5(7) = 100 \text{ with } d_5(7) = 5$$

$$\mathrm{M}_5(6) = \max\,[\mathrm{f}_5(0) + \mathrm{M}_6(6), \quad \mathrm{f}_5(1) + \mathrm{M}_6(5), \quad \mathrm{f}_5(2) + \mathrm{M}_6(4), \quad \mathrm{f}_5(3) + \mathrm{M}_6(3),$$
$$\mathrm{f}_5(4) + \mathrm{M}_6(2), \quad \mathrm{f}_5(5) + \mathrm{M}_6(1), \quad \mathrm{f}_5(6) + \mathrm{M}_6(0)]$$

$$M_5(6) = \max\,[(0 + 63), (20 + 53), (28 + 41), (38 + 35), (50 + 25), (\,75 + 21),$$
$$(75 + 0)]$$

$$M_5(6) = \max\,[63, \quad 73, \quad 69, \quad 73, \quad 75, \quad 96, \quad 75] \quad M_5(6) = 96 \text{ with } d_5(6) = 5$$

$$\mathrm{M}_5(5) = \max\,[\mathrm{f}_5(0) + \mathrm{M}_6(5), \quad \mathrm{f}_5(1) + \mathrm{M}_6(4), \quad \mathrm{f}_5(2) + \mathrm{M}_6(3), \quad \mathrm{f}_5(3) + \mathrm{M}_6(2),$$
$$\mathrm{f}_5(4) + \mathrm{M}_6(1), \quad \mathrm{f}_5(5) + \mathrm{M}_6(0)\,]$$

$$M_5(5) = \max\,[0 + 53, \quad 20 + 41, \quad 28 + 35, \quad 38 + 25, \quad 50 + 21, \quad 75 + 0]$$

$$M_5(5) = \max\,[53, \quad 61, \quad 63, \quad 63, \quad 71, \quad 75] \qquad M_5(5) = 75 \text{ with } d_5(5) = 5$$

$$\mathrm{M}_5(4) = \max\,[\mathrm{f}_5(0) + \mathrm{M}_6(4), \quad \mathrm{f}_5(1) + \mathrm{M}_6(3), \quad \mathrm{f}_5(2) + \mathrm{M}_6(2), \quad \mathrm{f}_5(3) + \mathrm{M}_6(1),$$
$$\mathrm{f}_5(4) + \mathrm{M}_6(0)]$$

$$M_5(4) = max\,[0 + 41, \; 20 + 35, \; 28 + 25, \quad 38 + 21, \; 50 + 0]$$

$$M_5(4) = max\,[41, \quad 55, \quad 53, \quad 59, \quad 50] \quad M_5(4) = 59 \; with \; d_5(4) = 3$$

$$\mathrm{M}_5(3) = \max\,[\mathrm{f}_5(0) + \mathrm{M}_6(3), \quad \mathrm{f}_5(1) + \mathrm{M}_6(2), \quad \mathrm{f}_5(2) + \mathrm{M}_6(1), \quad \mathrm{f}_5(3) + \mathrm{M}_6(0)]$$

$M_5(3) = max\ [0 + 35,\ 20 + 25,\ 28 + 21,\ \ 38 + 0]$

$M_5(3) = max\ [35\ \ 45,\ \ 49,\ \ 38]$ $\quad$ $M_5(3) = 49\ \ with\ d_5(3) = 2$

$\mathrm{M}_5(2) = \max\ [\mathrm{f}_5(0) + \mathrm{M}\ _6(2),\ \mathrm{f}_5(1) + \mathrm{M}_6(1),\ \mathrm{f}_5(2) + \mathrm{M}_6(0)]$

$\mathrm{M}_5(2) = \max\ [0 + 25,\ 20 + 21,\ 28 + 0]$ $\mathrm{M}_5(2) = \max\ [25,\ \ 41,\ \ 28]$ $\mathrm{M}_5(2) = 41$ with $\mathrm{d}_5(2) = 1$

$\mathrm{M}_5(1) = \max\ [\mathrm{f}_5(0) + \mathrm{M}_6(1),\ \mathrm{f}_5(1) + \mathrm{M}_6(0)]$

$\mathrm{M}_5(1) = \max\ [0 + 21,\ 20 + 0]$ $\quad$ $\mathrm{M}_5(1) = \max\ [21,\ \ 20]$ $\quad$ $\mathrm{M}_5(1) = 25$ with $\mathrm{d}_5(1) = 1$

$\mathrm{M}_5(0) = \max\ [\mathrm{f}_5(0) + \mathrm{M}_6(0)]$

$M_5(0) = max\ [0 + 0]$ $\quad$ $M_5(0) = max\ [0]$ $\quad$ $M_5(0) = 0\ with\ \ d_5(0) = 0$

$\mathrm{M}_5(2) = \max\ [\mathrm{f}_5(0) + \mathrm{M}_6(2),\ \mathrm{f}_5(1) + \mathrm{M}_6(1),\ \mathrm{f}_5(2) + \mathrm{M}_6(0)]$

$M_5(2) = max\ [0 + 29,\ 25 + 24,\ 33 + 0]$ $\quad$ $M_5(2) = max\ [29,\ \ 49,\ \ 33]$ $\quad$ $M_5(2) = 49$ with $\mathrm{d}_5(2) = 1$

$\mathrm{M}_5(1) = \max\ [\mathrm{f}_5(0) + \mathrm{M}_6(1),\ \mathrm{f}_5(1) + \mathrm{M}_6(0)]$ $\quad$ $\mathrm{M}_5(1) = \max\ [0 + 24,\ 25 + 0]$

$\mathrm{M}_5(1) = \max\ [24,\ \ 25]$ $\quad$ $\mathrm{M}_5(1) = 25$ with $\mathrm{d}_5(1) = 1$

$\mathrm{M}_5(0) = \max\ [\mathrm{f}_5(0) + \mathrm{M}_6(0)]$ $\quad$ $\mathrm{M}_5(0) = \max\ [0 + 0]$ $\quad$ $\mathrm{M}_5(0) = \max\ [0]$ $\quad$ $\mathrm{M}_5(0) = 0$ with $\mathrm{d}_5(0) = 0$

Iteration III

From investment 4;

$$\mathrm{M}_4(9) = \max\,[\mathrm{f}_4(0) + \mathrm{M}_5(9), \quad \mathrm{f}_4(1) + \mathrm{M}_5(8), \quad \mathrm{f}_4(2) + \mathrm{M}_5(7), \quad \mathrm{f}_4(3) + \mathrm{M}_5(6),$$
$$\mathrm{f}_4(4) + \mathrm{M}_5(5), \quad \mathrm{f}_4(5) + \mathrm{M}_5(4), \quad \mathrm{f}_4(6) + \mathrm{M}_5(3), \quad \mathrm{f}_4(7) + \mathrm{M}_5(2),$$
$$\mathrm{f}_4(8) + \mathrm{M}_5(1), \quad \mathrm{f}_4(9) + \mathrm{M}_5(0)]$$

$$\mathrm{M}_4(9) = \max\,[0 + 116, \quad 18 + 110, \quad 40 + 100, \quad 50 + 96, \quad 55 + 75,$$
$$65 + 59, 74 + 57, \quad 70 + 49, \quad 85 + 21, \quad 95 + 0]$$

$$\mathrm{M}_4(9) = \max\,[116, \quad 128, \quad 140, \quad 146, \quad 130, \quad 124, \quad 131, \quad 119, \quad 106, \quad 95]$$

$$\mathrm{M}_4(9) = 146 \quad \text{with} \quad \mathrm{d}_4(9) = 3$$

$$\mathrm{M}_4(8) = \max\,[\mathrm{f}_4(0) + \mathrm{M}_5(8), \quad \mathrm{f}_4(1) + \mathrm{M}_5(7), \quad \mathrm{f}_4(2) + \mathrm{M}_5(6), \quad \mathrm{f}_4(3) + \mathrm{M}_5(5),$$
$$\mathrm{f}_4(4) + \mathrm{M}_5(4), \quad \mathrm{f}_4(5) + \mathrm{M}_5(3), \quad \mathrm{f}_4(6) + \mathrm{M}_5(2), \quad \mathrm{f}_4(7) + \mathrm{M}_5(1),$$
$$\mathrm{f}_4(8) + \mathrm{M}_5(0)]$$

$$\mathrm{M}_4(8) = \max\,[0 + 110, \quad 18 + 100, \quad 40 + 96, \quad 50 + 75, \quad 55 + 59, \quad 65 + 49,$$
$$70 + 41, \quad 70 + 21, \quad 85 + 0]$$

$$M_4(8) = max\,[110, \quad 118, \quad 136, \quad 125, \quad 114, \quad 114, \quad 111, \quad 91, \; 85] \qquad M_4(8) = 136$$
$$with \;\; d_4(8) = 2$$

$$\mathrm{M}_4(7) = \max\,[\mathrm{f}_4(0) + \mathrm{M}_5(7), \quad \mathrm{f}_4(1) + \mathrm{M}_5(6), \quad \mathrm{f}_4(2) + \mathrm{M}_5(5), \quad \mathrm{f}_4(3) + \mathrm{M}_5(4),$$
$$\mathrm{f}_4(4) + \mathrm{M}_5(3), \quad \mathrm{f}_4(5) + \mathrm{M}_5(2), \quad \mathrm{f}_4(6) + \mathrm{M}_5(1), \quad \mathrm{f}_4(7) + \mathrm{M}_5(0)]$$

$$\mathrm{M}_4(7) = \max\,[0 + 100, 18 + 96, 40 + 75, \quad 50 + 59, 55 + 49,65 + 41,70 + 21, 70 + 0]$$

$$\mathrm{M}_4(7) = \max\,[100,\ 114, 115, 109, 104, 106, 91, 70] \quad \mathrm{M}_4(7) = 115 \text{ with } \ \mathrm{d}_4(7) = 2$$

$$M_4(6) = max\,[f_4(0) + M_5(6), \quad f_4(1) + M_5(5), \quad f_4(2) + M_5(4), \quad f_4(3) + M_5(3),$$
$$f_4(4) + M_5(2), \quad f_4(5) + M_5(1), \quad f_4(6) + M_5(0)]$$

$$M_4(6) = max\,[0 + 96, 18 + 75, 40 + 59, \ \ 50 + 49, 55 + 41, 65 + 21, 70 + 0]$$

$$M_4(6) = max\,[96,\ 92,\ \ 99,\ \ 99,\ 96,\ \ 86,\ \ 70] \quad M_4(6) = 99\ with \ \ d_4(6) = 2$$

$$M_4(5) = max\,[f_4(0) + M_5(5), f_4(1) + M_5(4), f_4(2) + M_5(3),\ \ f_4(3) + M_5(2),$$
$$f_4(4) + M_5(1), \quad f_4(5) + M_5(0)]$$

$$M_4(5) = max\,[0 + 75, \quad 18 + 59, \quad 40 + 49,\ \ 50 + 41, \quad 55 + 21, \quad 65 + 0]$$

$$M_4(5) = max\,[75,\ \ 77,\ \ 89,\ \ 91,\ \ 76,\ 65] \qquad M_4(5) = 91\ with\ \ d_4(5) = 3$$

$$\mathrm{M}_4(4) = \max\,[\mathrm{f}_4(0) + \mathrm{M}_5(4), \quad \mathrm{f}_4(1) + \mathrm{M}_5(3), \quad \mathrm{f}_4(2) + \mathrm{M}_5(2), \quad \mathrm{f}_4(3) + \mathrm{M}_5(1),$$
$$\mathrm{f}_4(4) + \mathrm{M}_5(0)]$$

$$M_4(4) = max\,[0 + 59,\ 18 + 49,\ 40 + 41,\ \ 50 + 21,\ 55 + 0]$$

$$M_4(4) = max\,[59,\ \ 67,\ 81,\ \ 71,\ \ 55] \quad M_4(4) = 81 with \ \ d_4(4) = 2$$

$$\mathrm{M}_4(3) = \max\,[\mathrm{f}_4(0) + \mathrm{M}_5(3), \quad \mathrm{f}_4(1) + \mathrm{M}_5(2), \quad \mathrm{f}_4(2) + \mathrm{M}_5(1), \quad \mathrm{f}_4(3) + \mathrm{M}_5(0)]$$

$$M_4(3) = max\,[0 + 49,\ \ 18 + 41,\ \ 40 + 21,\ \ 50 + 0]$$

$$M_4(3) = max\,[49,\ \ 59,\ 61,\ 50] \quad M_4(3) = 61\ with \quad d_4(3) = 2$$

$$\mathrm{M}_4(2) = \max\,[\mathrm{f}_4(0) + \mathrm{M}_5(2),\ \mathrm{f}_4(1) + \mathrm{M}_5(1),\ \mathrm{f}_4(2) + \mathrm{M}_5(0)]$$

$M_4(2) = \max[0 + 41, \; 18 + 21, \quad 40 + 0] \;\; M_4(2) = \max[41, \; 29, \; 40]$

$M_4(2) = 41 \;\text{ with }\; d_4(2) = 0$

$M_4(1) = \max[f_4(0) + M_5(1), \; f_4(1) + M_5(0)]$

$M_4(1) = \max[0 + 21, \; 18 + 0]$

$M_4(1) = max\,[21, \; 18] \quad M_4(1) = 21\; with \;\; d_4(1) = 0$

$M_4(0) = \max[f_4(0) + M_5(0) \;] \;\; M_4(0) = \max[\; 0 \;] \;\; M_4(0) = 0 \;\text{ with }\; d_4(0) = 0$

Iteration IV

From investment 3**;**

$$M_3(9) = \max[f_3(0) + M_4(9), \quad f_3(1) + M_4(8), \quad f_3(2) + M_4(7), \quad f_3(3) + M_4(6),$$
$$f_3(4) + M_4(5), \quad f_3(5) + M_4(4), \quad f_3(6) + M_4(3), \quad f_3(7) + M_4(2),$$
$$f_3(8) + M_4(1), \quad f_3(9) + M_4(0)]$$

$$M_3(9) = \max[0 + 146, \quad 12 + 136, \quad 22 + 115, \quad 32 + 99, \quad 35 + 91, \quad 40 + 81,$$
$$50 + 61, \quad 60 + 41, \quad 65 + 21, \quad 70 + 0]$$

$$M_3(9) = max\,[146, \; 148, \; 137, \; 131, \; 126, \; 121, \; 111, \; 101, \; 86, \; 70]$$

$$M_3(9) = 148 \qquad with \qquad d_3(9) = 1$$

$$M_3(8) = \max[f_3(0) + M_4(8), \quad f_3(1) + M_4(7), \quad f_3(2) + M_4(6), \quad f_3(3) + M_4(5),$$
$$f_3(4) + M_4(4), \quad f_3(5) + M_4(3), \quad f_3(6) + M_4(2), \quad f_3(7) + M_4(1),$$
$$f_3(8) + M_4(0)]$$

$M_3(8) = [0 + 136, 12 + 115, 22 + 99, 32 + 91, 35 + 81, 40 + 61, 50 + 41, 60 + 21, 65 + 0]$

$M_3(8) = max\,[136, \quad 127, \quad 121, \quad 123, \quad 116, \quad 101, \quad 91, \quad 81, \quad 65] \quad (8) = 136\; w\; ith$

$sd_3(8) = 0$

$\mathrm{M_3(7) = max\,[f_3(0) + M_4(7), \quad f_3(1) + M_4(6), \quad f_3(2) + M_4(5), \quad f_3(3) + M_4(4), \quad f_3(4) + M_4(3), \quad f_3(5) + M_4(2), \quad f_3(6) + M_4(1), \quad f_3(7) + M_4(0)]}$

$\mathrm{M_3(7) = max\,[0 + 115, \quad 12 + 99, \quad 22 + 91, \quad 32 + 81, \quad 35 + 61 \quad 40 + 41, 50 + 21, \quad 60 + 0]}$

$\mathrm{M_3(7) = max\,[115, \quad 111, \quad 113, \quad 113, \quad 96, \quad 81, \quad 71, \quad 60]}$

$\mathrm{M_3(7) = 115\ with\ \ d_3(7) = 0}$

$M_3(6) = \mathrm{max\,[f_3(0) + M_4(6), \quad f_3(1) + M_4(5), \quad f_3(2) + M_4(4), \quad f_3(3) + M_4(3), \quad f_3(4) + M_4(2), \quad f_3(5) + M_4(1), \quad f_3(6) + M_4(0)]}$

$\mathrm{M_3(6) = max\,[0 + 99, \quad 12 + 91, \quad 22 + 81, \quad 32 + 61, \quad 35 + 41, \quad 40 + 21, \quad 50 + 0]}$

$\mathrm{M_3(6) = max\,[99, \quad 103, \quad 103, \quad 93, \quad 76, \quad 61, \quad 50] \quad M_3(6) = 103 \quad with}$

$\mathrm{d_3(6) = 1}$

$M_3(5) = max\,[f_3(0) + M_4(5), \quad f_3(1) + M_4(4), \quad f_3(2) + M_4(3), \quad f_3(3) + M_4(2), \quad f_3(4) + M_4(1), \quad f_3(5) + M_4(0)]$

$M_3(5) = max\ [0 + 91, \quad 12 + 81, \quad 22 + 61, \quad 32 + 41, \quad 35 + 21, \quad 40 + 0]$

$M_3(5) = max\ [91, \quad 93, \quad 83, \quad 73, \quad 56, \quad 40]$ $M_3(5) = 93\ with\ \ d_3(5) = 1$

$\mathrm{M}_3(4) = \max\ [\mathrm{f}_3(0) + \mathrm{M}_4(4), \quad \mathrm{f}_3(1) + \mathrm{M}_4(3), \quad \mathrm{f}_3(2) + \mathrm{M}_4(2), \quad \mathrm{f}_3(3) + \mathrm{M}_4(1),$

$\mathrm{f}_3(4) + \mathrm{M}_4(0)]$

$\mathrm{M}_3(4) = \max\ [0 + 81, \quad 12 + 61, \quad 22 + 41, \quad 32 + 21, \quad 35 + 0]$

$M_3(4) = max\ [\ 81, \quad 73, \quad 63, \quad 53, \quad 35]$ $\quad M_3(4) = 80\ with\ \ d_3(4) = 0$

$\mathrm{M}_3(3) = \max\ [\mathrm{f}_3(0) + \mathrm{M}_4(3), \quad \mathrm{f}_3(1) + \mathrm{M}_4(2), \quad \mathrm{f}_3(2) + \mathrm{M}_4(1), \quad \mathrm{f}_3(3) + \mathrm{M}_4(0)]$

$\mathrm{M}_3(3) = \max\ [\ 0 + 61, \quad 12 + 41, \quad 22 + 21, \quad 32 + 0]$

$\mathrm{M}_3(3) = \max\ [61, \quad 53, \quad 43, \quad 32]$ $\quad \mathrm{M}_3(3) = 61 \text{ with } \mathrm{d}_3(3) = 0$

$\mathrm{M}_3(2) = \max\ [\mathrm{f}_3(0) + \mathrm{M}_4(2),\ \ \mathrm{f}_3(1) + \mathrm{M}_4(1),\ \ \mathrm{f}_3(2) + \mathrm{M}_4(0)]$

$\mathrm{M}_3(2) = \max\ [0 + 41,\ \ 12 + 21,\ \ 22 + 0]$ $\qquad \mathrm{M}_3(2) = \max\ [41, \quad 33, \quad 22]$

$\mathrm{M}_3(2) = 41 \text{ with } \quad \mathrm{d}_3(2) = 0$

$\mathrm{M}_3(1) = \max\ [\mathrm{f}_3(0) + \mathrm{M}_4(1),\ \ \mathrm{f}_3(1) + \mathrm{M}_4(0)]$

$M_3(1) = max[0 + 21,\ \ 12 + 0]$ $\ M_3(1) = max\ [21\ , \quad 12]$ $M_3(1) = 21\ with\ d_3(1) = 0$

$\mathrm{M}_3(0) = \max\ [\mathrm{f}_3(0) + \mathrm{M}_4(0)]$ $\ \mathrm{M}_3(0) = \max\ [0 + 0]$ $\mathrm{M}_3(0) = \max\ [0]$

$\mathrm{M}_3(0) = 0 \ \text{ with } \quad \mathrm{d}_3(1) = 0$

Iteration V

From investment 2:

$$\mathrm{M_2(9) = max\,[f_2(0) + M_3(9), \quad f_2(1) + M_3(8), \quad f_2(2) + M_3(7), \quad f_2(3) + M_3(6),}$$

$$\mathrm{f_2(4) + M_3(5), \quad f_2(5) + M_3(4), \quad f_2(6) + M_3(3), \quad f_2(7) + M_3(2),}$$

$$\mathrm{2(8) + M_3(1), \quad f_2(9) + M_3(0)]}$$

$$M_2(9) = max\,[0 + 148, \quad 15 + 136, \quad 30 + 115, \quad 40 + 103, \quad 45 + 93,$$

$$50 + 81{,}55 + 61, \quad 70 + 41, \quad 80 + 21, \quad 90 + 0]$$

$$M_2(9) = max\,[148, \quad 151, \quad 145, \quad 143, \quad 138, \quad 131, \quad 116, \quad 111, \quad 121, \quad 90]$$

$$M_2(9) = 151\, with \quad d_2(9) = 1$$

$$\mathrm{M_2(8) = max\,[f_2(0) + M_3(8), \quad f_2(1) + M_3(7), \quad f_2(2) + M_3(6), \quad f_2(3) + M_3(5),}$$

$$\mathrm{f_2(4) + M_3(4), \quad f_2(5) + M_3(3), \quad f_2(6) + M_3(2), \quad f_2(7) + M_3(1),}$$

$$\mathrm{2(8) + M_3(0)]}$$

$$M_2(8) = max\,[0 + 136, \quad 15 + 115, \quad 30 + 103, \quad 40 + 93, \quad 45 + 81,$$

$$50 + 61{,}55 + 41, \quad 70 + 21, \quad 80 + 0]$$

$$M_2(8) = max\,[136, \quad 130, \quad 133, \quad 133, \quad 126, \quad 111, \quad 96, \quad 91, \quad 80]$$

$$M_2(8) = 136\, with \quad d_2(8) = 0$$

$$\mathrm{M_2(7) = max\,[f_2(0) + M_3(7), \quad f_2(1) + M_3(6), \quad f_2(2) + M_3(5), \quad f_2(3) + M_3(4),}$$

$$\mathrm{f_2(4) + M_3(3), \quad f_2(5) + M_3(2), \quad f_2(6) + M_3(1), \quad f_2(7) + M_3(0)]}$$

$$M_2(7) = max\ [0 + 115, \quad 15 + 103, \quad 30 + 93, \quad 40 + 81, \quad 45 + 61, \quad 50 + 41, \quad 55 + 21, \quad 70 + 0]$$

$$M_2(7) = max[115, \ 118, \ 123, \ 121, \ 106, \ 91, \ 76, \ 70]\ M_2(7) = 123\ with\ \ d_2(7) = 2$$

$$\mathrm{M}_2(6) = \max\ [\mathrm{f}_2(0) + \mathrm{M}_3(6), \quad \mathrm{f}_2(1) + \mathrm{M}_3(5), \quad \mathrm{f}_2(2) + \mathrm{M}_3(4), \quad \mathrm{f}_2(3) + \mathrm{M}_3(3), \quad \mathrm{f}_2(4) + \mathrm{M}_3(2), \quad \mathrm{f}_2(5) + \mathrm{M}_3(1), \quad \mathrm{f}_2(6) + \mathrm{M}_3(0)]$$

$$M_2(6) = max\ [0 + 103, 15 + 93, 30 + 81, \ 40 + 61, 45 + 41, 50 + 21, 55 + 0]$$

$$M_2(6) = max\ [103, \ 108, \ 111, \ 101, \ 96, \ 71, \ 55]\ \ M_2(6) = 111 with\ \ d_2(6) = 2$$

$$\mathrm{M}_2(5) = \max\ [\mathrm{f}_2(0) + \mathrm{M}_3(5), \quad \mathrm{f}_2(1) + \mathrm{M}_3(4), \quad \mathrm{f}_2(2) + \mathrm{M}_3(3), \quad \mathrm{f}_2(3) + \mathrm{M}_3(2), \quad \mathrm{f}_2(4) + \mathrm{M}_3(1), \quad \mathrm{f}_2(5) + \mathrm{M}_3(0)]$$

$$\mathrm{M}_2(5) = \max\ [0 + 93, \ 15 + 81, \ 30 + 61, \ \ 40 + 41, \ 45 + 21, \ 50 + 0]$$

$$M_2(5) = max\ [93, \ 96, \quad 91 \quad 81, \quad 66, \quad 50]\quad M_2(5) = 96\ with\ \ d_2(5) = 1$$

$$\mathrm{M}_2(4) = \max\ [\mathrm{f}_2(0) + \mathrm{M}_3(4), \quad \mathrm{f}_2(1) + \mathrm{M}_3(3), \quad \mathrm{f}_2(2) + \mathrm{M}_3(2), \quad \mathrm{f}_2(3) + \mathrm{M}_3(1), \quad \mathrm{f}_2(4) + \mathrm{M}_3(0)]$$

$$M_2(4) = max\ [0 + 81, \quad 15 + 61, \quad 30 + 41\ , \ 40 + 21, \quad 45 + 0]$$

$$M_2(4) = max\ [81, \ 76, \quad 71, \quad 61, \quad 45]\quad M_2(4) = 81\ with\ \ d_2(4) = 0$$

$$\mathrm{M}_2(3) = \max\ [\mathrm{f}_2(0) + \mathrm{M}_3(3), \quad \mathrm{f}_2(1) + \mathrm{M}_3(2), \quad \mathrm{f}_2(2) + \mathrm{M}_3(1), \quad \mathrm{f}_2(3) + \mathrm{M}_3(0)]$$

$$\mathrm{M}_2(3) = \max\ [0 + 61, \ 15 + 41 \ \ 30 + 21, \ \ 40 + 0]$$

$$M_2(3) = max\ [61, \quad 56, \quad 51, \quad 40]\quad M_2(3) = 61\ with\quad d_2(3) = 0$$

$\mathrm{M_2(2) = max\ [f_2(0) + M_3(2),\ f_2(1) + M_3(1),\ f_2(2) + M_3(0)]}$

$\mathrm{M_2(2) = max\ [0 + 41,\ 15 + 21,\ 30 + 0]}$

$\mathrm{M_2(2) = max\ [41, \quad 36, \quad 30] \quad M_2(3) = 41\ with \quad d_2(2) = 0}$

$\mathrm{M_2(1) = max\ [f_2(0) + M_3(1),\ f_2(1) + M_3(0)]}$

$\mathrm{M_2(1) = max\ [0 + 21,\ 15 + 0]}$

$\mathrm{M_2(1) = max\ [21, \quad 15] \quad M_2(1) = 21\ with\ \ d_2(1) = 0}$

$\mathrm{M_2(0) = max\ [f_2(0) + M_3(0)]}$

$\mathrm{M_2(0) = max\ [0 + 0] \quad M_2(0) = max\ [0]\ , \quad M_2(0) = 0\ with \quad d_2(0) = 0}$

Iteration VI

From investment 1*:*

$$\mathrm{M_1(9) = max\ [f_1(0) + M_2(9), \quad f_1(1) + M_2(8), \quad f_1(2) + M_2(7), \quad f_1(3) + M_2(6),}$$
$$\mathrm{f_1(4) + M_2(5), \quad f_1(5) + M_2(4), \quad f_1(6) + M_2(3), \quad f_1(7) + M_2(2),}$$
$$\mathrm{f_1(8) + M_2(1), \quad f_1(9) + M_2(0)]}$$

$$M_1(9) = max\ [0 + 151, \quad 10 + 136, \quad 20 + 123, \quad 25 + 111, \quad 33 + 96\ , 35 + 81,$$
$$40 + 61, \quad 45 + 41, \quad 55 + 21, \quad 60 + 0]$$

$\mathrm{M_1(9) = max\ [151,\ 146, 143,\ \ 136, 129,116, 101, 86, 76, 60] \ \ M_1(9) = 151\ \ with\ d_1(9) = 0}$

Hence the optimal investment is GH ₵1.51x10^4.

Appendix B

Matlab code for DP Algorithm

Input: values v and weights w for item 1 to n ; number of distinct items n; knapsack capacity W

```
For w from 0 to w do
  T[0,w] :=0
end for
for i from 1 to n do
    for j from 0 to W do
      if j >= w[i] then
        T[i,j] : = max (T[i-1,j] ,T[i,j-w[i]] +v[i])
     else
           T [ i , j]  : = T[i-1,j]
        end if
   end for
end for
```

The maximum of the empty set is taken to be zero. Tabulating the results from m [0] up through m[W] gives the solution. Since the calculation of each m[w] involves examining n items, and there are W values of m[w] to calculate, the running time of dynamic programming solution is

O(nW). Dividing w_1 , w_2 ,. . ., w_n , W by their greatest common divisor is an obvious way to improve the running time.

The O(nW) complexity does not contradict the fact that the knapsack problem is NP –complete, since W ,unlike n ,is not polynomial in the length of the input to the problem. The length of W input to the problem is proportional to the number of bits in Wlog, not to W itself