SOLVING PARTIAL DIFFERENTIAL EQUATIONS RELATED TO OPTION PRICING WITH NUMERICAL METHOD

BY

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DECLARATION

I hereby declare that this submission is my own work towards the award of Masters of Science (M.Sc.) and that, to the best of my knowledge, it contains no material previously published by another person nor materials which has been accepted for the award of degree of the university, except where due acknowledgement has been made in the text.

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ABSTRACT

In this thesis, we solve the special differential partial equation, Black-Scholes Equation for valuing option pricing through numerical methods. Options considered include European style, American style, and the exotic option with major reference to the European style option. The aim is to find accurately the value of the various option styles by determining whether a grid point not greater than 60 can be used to determine the value of options with reliable accuracy, by setting a higher order discretization in space and time as well as grid stretching around the interesting region. Whether a highly accurate scheme will also work for the exotic options and finally whether implied volatility can be calculated using iterative methods in less iteration. The fourth order difference scheme and the grid stretching in space by means of an analytic coordinate transformation are employed. By experiment, we showed numerically that a grid size or space of 20 to 40 is all that is needed to achieve accuracy in the value of option. A fourth order Backward Difference Scheme is sufficient enough to yield accuracy in the exotic options and also it is possible to use few iteration to obtain implied volatility. The numerical experiment thus confirms the proposed methods.

DEDICATION

I dedicate this piece of work to my parents, guardians and all who in diverse ways have selflessly and willing supported me with love up to this stage.

ACKNOWLEDGEMENT

In life we always meet people who in various ways, no matter how small it is help us to achieve our dreams, complete a task or help us move from one level of life to another. I therefore wish to extend my sincere gratitude to all those who have help me come out with this piece of work.

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CHAPTER ONE

INTRODUCTION

An important way to always solve problems happened in the finance and economics has been the involvement of mathematical techniques. Options, a financial instrument that provide the individual investor with the flexibility needed in almost any investment situation one may encounter are widely used on market and exchanges. In calculating the price of an option the famous **Black-Scholes Model** is a convenient model to use.

A numerical method will be proposed though the exact solution of **Black-Scholes Equation** is known. This is because for the different types of options, example European, American etc. there is the need to create a general numerical model for the different types of the options. For instance it is very difficult or impossible to solve American options by means of analytics. However if numerical methods works for European style option, then it will be the basis to get American option as well as other complex options.

Although numerical methods based on partial differential equations (PDE's) is not that popular in finance, the possibility of efficiently discretizing the partial differential equations (PDE's), yields an algorithm that's more efficient.

Another issue to look at is the **implied volatility,** which is the natural tendency of the underlying security market price to fluctuate either up or down. From stock exchange trends, the volatility of assets prices in the future is not known, thus it has to be estimated. Once the value of the option price is known the only parameter not known in the **Black-Scholes equation** is volatility.

It is of no thought that the sole aim of investors is to either maximum profit, minimize loss or invest in a risk free investment. As such for a developing country like Ghana in other for us to maintain and increase the growth rate of investment, options which are traded exchanges must be considered looking at the numerous advantages.

This thesis, as part of it objective seeks to propose the consideration of options in the stock or exchange market in Ghana.

1.1 Background

The **Black-Scholes model** is a mathematical model employed in a financial market containing certain derivative instruments. From the model a formula called **Black-Scholes formula** is generated. This formula gives the price of European style options which of course form the basis of American style options as well as other options by using numerical methods. The formula gave a boost to the options market across the globe especially Chicago Board Options Exchange. Many empirical tests have shown that the Black-Scholes price is fairly close to the observed prices despite well-known discrepancies such as **in-the-money** and **out-of-the-money.**

The model was first articulated by **Fischer Black** and **Myron Scholes** in their 1973 paper. A partial differential equation now called the **Black-Scholes equation** was derived. This equation governs the price of option overtime. The main aim behind the derivative was to hedge perfectly the option by buying and selling the underlying security in just right way, thus eliminating risk. This hedge forms the basis of conservative strategy used to limit investment loss and implies that there is only one right price for the option and it is given by **Black-Scholes formula**.

1.2 Problem Statement

Give the this background, it is surprising and interesting to know that options have not yet be considered in traded exchanges in Ghana. Currently for an investor to invest in shares or securities in Ghana, the investor must go through the consideration of companies' performance, existing sector conditions in which these companies operate as well as profit growth and dividend.

Moreover, currently the price of share is determine by bids and offers and so the fluctuation of share price is by whether bid exceed offer or offer exceed bid.

As indicated early, options provides the investor with orderly, efficient and liquid market, flexibility, leverage, limited risk for buy and guaranteed contract performance. Also despite the fact that both option investor and stock investor have the ability to follow trading volume, price movement etc., and option investor can quickly and easily learn the price at which his order has been executed. This paper thus seeks to contribute to existing literature by providing evidence and supports regarding Black-Scholes equation in determine option pricing.

1.3 Objective of the Study

The objective of this study is to accurately determine the option price of the various option styles using numerical method such as solving partial differential equation now known as the **Black-Scholes equation,** specifically, we evaluate

- The possibility of using grid points not greater than 60 to determine the value of an option with reliable accuracy, by setting a higher order discretization in space and time and also by using grid stretching around interesting region?
- Whether a highly accurate numerical scheme will also work for other complex options with continues final conditions?
- Can the unknown parameter, **implied volatility** be calculated using some iterative methods in less iteration?

1.4 Justification

Since there are other options aside the European style options, like the American style options and other more complex options which cannot be solved analytically, there is therefore the need to consider a numerical method to solve these complex option styles by solving partial differential equation. It is the hope that this study will give an indication as to how option pricing is determined. In addition, the study contributes to existing literature by providing evidence and supports regarding Black-Scholes equation in determine option pricing. It is therefore the hope of the researcher that readers' especially traders in the investment market, understanding of this subject will be enhanced and thus the need to consider options in the exchange trade in Ghana.

1.5 Methodology

The methods to be considered in this study involve Black-Scholes Analysis which includes stochastic model, partial differential equation, types of options. Discretization of partial differential equation in the area of space discretization, coordinate transformation and numerical time integration. Validation of discrete systems, Special features of the Black-Scholes partial differential equation, and pricing experiment of numerical options.

1.6 Organization of The Study

The paper is organized into five chapters. Chapter one presents the background of the study, problem statement, and objective of study, justification and methodology. This is followed by a discussion of the pertinent literature on the subject matter. The methods employed in conducting the study is covered in chapter three while chapter four systematically shows the numerical result of option pricing. The last session of this research, chapter five, details the conclusions and recommendations made. The researcher is confident that traders in the exchange trade market and other stakeholders will find this report informative and helpful for making decisions, increasing the growth rate of investors and strengthening investment in the exchange market.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter focuses on the review of relevant literature on solving partial differential equation related option pricing with numerical methods. Thus the concepts, theories and principles relevant to the study are captured in this chapter. It must be noted that the researcher did not come across any literature which explicitly studied option pricing in Ghana. Even in the exchange market, not much was seen and therefore the review for the purposes of this study and it proposal was basically done based on theory and studies done on other exchange trade elsewhere apart from Ghana. Areas such as stable numerical methods for **PDE** models of Asian options, numerical methods for American option pricing, numerical simulation of American options, European put-call options pricing and Black-Scholes equation, partial differential equation in finance, Stochastic volatility model in option pricing, and option pricing and partial differential equation are captured in this chapter. Thus, the chapter presents the conceptual and theoretical basis for the study.

2.2 Stable Numerical Methods for PDE Models of Asian Options

Asian options, also known as average options are path-dependent option contract which payoff depends on the average value of the asset price over some predefined period of time. Through these exotic financial instruments, the option holder is provided with enough and suitable protection against possible harm caused by implied volatility in the price of the underlying security. This especially is the case when movement price is speculatively attempted near the expiry date. For issuer, Asian options simply represent the ability to attain a better forecasting of the long and short position and therefore are more relaxed in dealing with the maturity situation.

The inexistence of general analytical solution for the price of the Asian option has led to the development of variety of techniques to analyze the arithmetic average Asian options. In general solving partial differential equation in two space dimensions can lead to the price of Asian option being found.

Jan Vecer (2000,2002) in his article or paper unified pricing of Asian option, employed one dimensional PDE which is a reduction of two dimension PDE for a floating strike Asian option, to provide a simpler and unifying approach for pricing Asian options, for both discrete and continuous average. He therefore established that the result of one dimension PDE for the Asian price option can be easily implemented to give extremely fast and accurate results. His approach also incorporates the cases of discrete and continuous dividends.

Adam Reherek (2011) in his paper, stable numerical methods for PDE model of Asian option, concluded that the Van Leer flux limiter provides more accurate results and thus the most convenient method for pricing options related to Asian option, especially with stability as concern. This he concluded on after considering either discrete or continuous sampling of price of the underlying security, difference between applying arithmetic or geometric averaging, period over which average price is calculated and other methods.

2.3 American Option Pricing

Options in recent times have become extremely important and attractive to investors, both for hedging and speculation and also the fact that there is now a systematic way of determining it worth.

Numerical methods have and always form an important part of pricing of financial derivatives or options, especially in the case where there isn't closed form analytical formula solution to the derivatives. This mathematical tool led to the derivation of **PDE** now called the **Black-Scholes Equation.**

America option unlike the European option, exercise is permitted at any time during the life span of the option.

S.C. Benbow (2005), in his book, numerical methods of American options, did identify the following conditions associated with a European call option as a problem. With these he focused his report on the valuation of American options by producing an accurate method for the valuation. S.C. Benbow made use of the Black-Scholes model and later it modification based on dividend paying asset in other to achieve his aim. He thus concluded that the finite difference scheme method applied to the American call problem yielded an undervaluing option. This was as a result of the instability of the algorithm by directly applying the derivative to the boundary conditions. As a result for stability to be achieved in the valuation of option an approximation of the derivative was necessary.

David Bundi Niwiga (2005), also in his book, numerical method for the evaluation of financial derivatives, aimed to introduce a concept of financial derivatives, definitions and mathematical tools important in the valuation of options by considering methods such as Black-Scholes model, binomial model, finite difference methods and monte Carlo simulation method. He however concerns himself with the pricing of options, forwards and futures using the above mention mathematical tools.

2.4 Numerical Simulation of American Options

Options, which belong to a group known as financial derivatives are characterize by terms such as call option, put option, underlying asset, strike price, maturity, premium, holder and a writer. It must be noted that European and American options are just styles of options or exercise right and not a geographical classification. Options are put into two main groups; the standard options also known as vanillas, these options are traded actively at an exchange and their values may be determine "market to market". While as the second group, exotic options which do not have active market and so have their values gained by "market to model", are specially designed to fit the needs of their writers clients. Aside the groups of options, options are also categorized into two main purposes. One of such purpose is speculation where asset values are expected to fall in the future and so money is made available to buy the asset. The other purpose is hedging.

Pauly Oliver (2004), published in his book, Numerical simulation of American options, stated that a general closed-form analytical solution does not exist for the evaluation of American options. This is because the partial differential equation, known as Black-Scholes Equation, needs to be solved with a free boundary value. Despite the categorization of approximations into Analytical approximations, Stochastically simulation methods and Numerical solutions of the Black-Scholes Equation due to the non-existence of a general closed-form solution for American options, Pauly Oliver (2004), stated that to deal with this problem is to apply numerical methods. He considered in his research, two numerical approaches, which are the **Method of Finites Differences** and **Method of Finite Elements** for solving constraint partial differential equation and also theoretical background of numerical analysis, Practical use of an adopted grid and the exploration of B-Spline Finite Element. He therefore concluded that the linear Finite Element and the Finite Difference are capable tools to price American options. However he considered the Finite Element as a preferable approach to the Finite Difference due to its lower relative errors.

2.5 European Put-Call Option and Black-Scholes Equation

Option pricing problem has been in existence for some time now, in recent times however quite a number of methods, theory and algorithms have been looked at or considered in solving this option pricing problem, especially with the creation of more exotic financial derivatives. Black-Scholes Equation is without thought one of the most significant mathematical tools for the financial market for governing the value of financial derivatives precisely options. The Black-Scholes Equation seeks to solve this problem by constructing a portfolio. The equation also aims at completely eliminating risk in a portfolio and derives a precise value for the option, to excellently hedge a stock. Manjari Govada (2012), in his work, European put-call option pricing and Black-Scholes Equation, he did discuss the derivation of Black-Scholes Equation for European put and call options. This he did by employing both analytical and numerical methods. The importance of Black-Scholes Equation in pricing options cannot be understated was Manjari Govada final statement after his research.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

The chapter discusses the method used in accomplishing the aims of this research work. The chapter consists of the analysis of the Black-Scholes equation, discretization of the PDE, validation of the discrete systems and also some special features of the Black-Scholes PDE is considered. These features help us choose the appropriate difference scheme, transformation as well as choice of grid.

3.2 Black-Scholes Analysis

Under this topic the Black-Scholes equation will be derived to obtain the value of an option based on the definitions of the options discussed in the previous chapter prior to this chapter 3. Conditions such as the boundary and final conditions which distinguish the different types of options will be stated for the options.

3.2.1 Geometric Brownian motion

Research, development of Brownian motion as well as its application to stock price and it use to model market behavior has led to the assumption of Black-Scholes model, that security prices follow a geometric Brownian motion diffusion process. This assumption implies the following

$$dS = vSdt + \sigma SdX \tag{3.0}$$

where

S = security price at time t

dt = time change

v = drift parameter

σ = volatility parameter

dS = change in S over time dt

dX = Weiner process or standard normal distribution with mean being 0 and variance being 1 thus $dX \sim N(0,1)$.

Geometric Brownian motion therefore assumes that stock price returns are defined by a constant drift v plus the random element by the Weiner process multiply by a constant volatility parameter. Rearranging equation (3.0) we have

$$\frac{dS}{S} = vdt + \sigma dX \tag{3.1}$$

Thus returns, which is the change in stock price divided by its original value is given by the equation (3.1). It is therefore very clear from equation (3.1) that stock returns follows a normal distribution with mean vdt and volatility parameter σdX when assuming Geometric Brownian motion.

3.2.2 Stochastic model

We first describe the price process of stock. And this is model by a stochastic differential equation based on the Geometric Brownian motion as describe above. The stock price is the price of the underlying of an option. The stock price change is assumed to be a Markov process. There are two contributions to the return according to the Samuelson model. These are the deterministic contribution and stochastic contribution. If v is the average rate of growth or the drift parameter, then the deterministic contribution in time change dt is found to be vdt. The other contribution that is the stochastic contribution relate to the random change in the stock prices. With the volatility related to the standard deviation of the returns and dX the Wieners process, the contribution is assumed to be σdX thus results in the equation (3.1). This is a stochastic differential equation. The Wiener process has the following properties

$E(dX) = 0$ And $E(dX)^2 = dt$. To know that σ is proportional to $\text{var}(dS)$, the expectation and variances are calculated as follows

$$payoff = \begin{cases} 0, S < E \\ Q/d, E < S < E + d \\ 0, S > E + d \end{cases}$$ since $E(dX) = 0$.

$$\text{var}(dS) = E(dS^2) - (E(dS))^2$$

$$\text{var}(dS) = E((vSdt + \sigma SdX)^2) - (vSdX)^2 = \sigma^2 S^2 dX^2$$ since $E(S^2 dXdt) = 0$

The square root of the variance σ which is a standard deviation is given by $\sigma = \frac{\sqrt{\text{var}(dS)}}{S}$.

3.2.3 Partial differential equation

From Taylor's equation, the general option value say $V(S,t)$ can be express as follows

$$dV = \frac{dV}{dS} dS + \frac{dV}{dt} dt + \frac{1}{2} \frac{d^2V}{dS^2} dS^2 + \ldots\ldots\ldots \tag{3.2}$$

where the dots indicate the terms which is neglected for $dt \to 0$. From equation 3.1, the drift term vdt and the volatility term σdX are the dominant terms since for $dt \to 0$ they are of the size dt and $\sqrt{dt}$ respectively. It thus follows that

$$dS^2 = (vSdt + \sigma SdX)^2 = v^2 S^2 dt^2 + 2\sigma v S^2 dXdt + \sigma^2 S^2 dX^2 \tag{3.3}$$

By applying ito's lemma process and the assumption that $dX^2 \rightarrow dt$ as $dt \rightarrow 0$, with probability 1, the equation (3.3) reads to a leading order $dS^2 \rightarrow \sigma^2 S^2 dt$ (3.4)

Inserting equation (3.4) into (3.2) follows that

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (vS \frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt \tag{3.5}$$

Definition 1 Portfolio: A portfolio is the collection of all options, shares and other derivatives own by a trader.

Now, setting up a portfolio consisting of one option with value V and a certain number $-\Delta$ of the underlying asset, the portfolio value will be

$$\Pi = V - \Delta S \tag{3.6}$$

Whereas the change in the portfolio will be

$$d\Pi = d(V - \Delta S) = dV - \Delta dS \tag{3.7}$$

Combining equations (3.1), (3.5), (3.6) and (3.7), we have

$$d\Pi = \sigma(\frac{\partial V}{\partial S} - V)dX + (vS \frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S^2} + \frac{\partial V}{\partial t} - v\Delta S)dt \tag{3.8}$$

We therefore chooses $\Delta = \dfrac{\partial V}{\partial S}$ (3.9)

to eliminate the main randomness contribution. The portfolio in equation (3.6) is deterministic meaning it is instantaneously risk free, by choose of Δ in equation (3.9). The change in an instantaneously risk free portfolio should equal the exponential growth of placing money in the bank. Thus,

$$d\Pi = r\Pi dt = (\tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt \tag{3.10}$$

Finally after substituting equation (3.6) into that of (3.10) and dividing by dt . The famous Black-Scholes equation for valuing an option with value V is obtained

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.11}$$

This can be rewritten as $\frac{\partial V}{\partial t} + L_{BS}V = 0$ (3.12)

This equation has some quite a number of properties and also in it analysis, certain assumptions are made. This includes the following

Trading of the underlying stock can take place continuously.

It is possible to short sell, since asset may be sold without processing them.

There is no dividend payment for basic model during the options life.

It is also assumed that there are no arbitrage possibilities.

Transaction cost associated with hedging is not included.

The stock price follows the log normed distribution which rises from equation (3.1).

The risk-free interest rate r and the volatility σ are both known functions of time over life of the option.

Like every model there are some sorts of drawbacks, and so is this model. One of the essential drawbacks of this model is that the volatility is assumed to be a constant function. In reality however this is not the case, but for many options the Black-Scholes model can still be used successfully. There are a lot of researches currently on more accurate modeling of stock price processes, see from example chapter 2 [2.5]. These models handle the aspect of non-constant volatility more accurately. The improved stock models which are beyond the scope of this thesis have an important impact on the equations for option prices. There will be only a slight change to the general Black-Scholes equation when a constant dividend payment is assumed. Therefore the function Π in equation (3.6) and considering a constant dividend payment δS , we change the definition of $d\Pi$ as $d\Pi = dV - \Delta dS - \delta S \Delta dt$ (3.13)

And so, the Black-Scholes PDE in this case reads

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S\frac{\partial V}{\partial S} - rV = 0 \tag{3.14}$$

This dividend payment which is some ratio of the stock price can be interpreted as some kind of interest rate. This dividend payment is mainly useful for options on an index. In that case continuous dividend payment can be assumed. Otherwise in the case options in regular stocks, it makes sense to include a discrete dividend payment which takes place only once or twice a year.

3.3 Option Types

In equations (3.11) or (3.14), the parabolic equations are of second order partial differential equation in the stock price space say (S) and first order in time space say (t). One final or initial condition is a necessity. From the literature of partial differential equation (see in chapter 2[2.6]), a diffusion equation of this type is (ill-posed) if it comes with an initial condition. However with option problems final conditions exist. The difference between the American and European style as well as between put and call and the other types of options is as a result of the final boundary conditions. The Black-Scholes equation as seen in equation (3.11) is a convection diffusion reaction equation of special form. For the purpose of this research work, American options style will not be that much considered. And so the call and put option values will be denoted as C and P respectively.

Definition 2 Arbitrage: Is a technique of simultaneously buying at a low price in one market and selling at a higher price in another market of a commodity, security, or monies to make a profit of the spread between the prices.

Definition 3 Exotic options: These are derivatives instrument which have features making them more complex than commonly traded vanilla, usually relating to determination of payoffs.

Definition 4 Spread options: These are commonly used within commodity markets as well as foreign exchange options to provide a payoff based on the difference between 2 or 3 assets.

3.3.1 European call option

A call option gives the holder the right to exercise his option at maturity time (T). It makes sense to buy the underlying asset at maturity time (T), if the stock/asset price(s) is higher than the exercise price (E), thus (S > E). An asset can be bought for E and immediately sell it on the market for S. Option is very much worthless if this isn't the case. The value of the option is thus known at maturity time, and it's either zero (0) or S-E, which is the net amount of profit. So, when the final condition of a call is known, the then problem is well-posed problem. The boundary conditions it follows from economic arguments. If $S = 0$, then the value of the call option equals zero (0). For $S \longrightarrow \infty$, the holder will exercise and the value of his option will be simply the asset price (S) corrected by the dividend minus the exercise price (E) corrected by the case if the holder invest his money in the bank.

$$C(S,t) = Se^{-\delta(T-t)} - Ee^{-r(T-t)}$$

To summarize the problem we have

$$\frac{\partial C}{\partial t} + L_{BS} = 0$$

$$C(0,t)=0$$

$$\lim_{S\to\infty} C(S,t)=Se^{-\delta(T-t)}-Ee^{-(T-t)} \tag{3.15}$$

$$C(S,T)=\max(S-E,0)=\begin{cases} S-E, S>E \\ 0, S\le E \end{cases}$$

The known analytic solution of the Call option reads

$$C(S,t)=Se^{-\delta(T-t)}N(d_1)-Ee^{-r(T-t)}N(d_2) \tag{3.16}$$

Where

$$d_1=\frac{\ln S-\ln E+(r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{3.17}$$

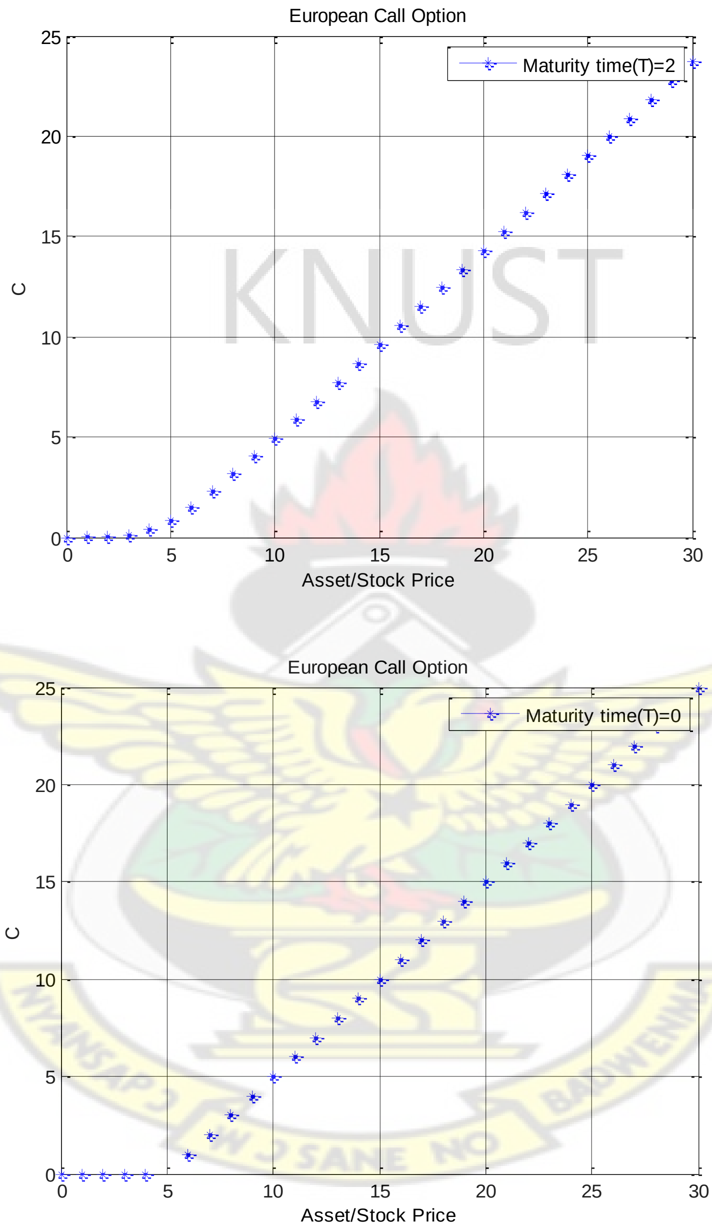
$$d_2=d_1-\sigma\sqrt{T-t} \tag{3.18}$$

$$N(y)=\frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}x^2}dx \tag{3.19}$$

The probability that the asset price will be above the exercise price is given by $N(d_2)$.

A graphical representation of the Call option solution is shown in figure 3.1. Next to the final condition at $t=T$, C at $t=0$ is presented for an option $E=GH¢5.00$, $\sigma=0.3$, $r=0.05$, $\delta=0.03$ and $T=2.0$

Figure 3.1: Showing the final value and solution for European call option with final value, and parameters as follows: E=GH¢ 5.00, σ=0.3, δ=0.03, r=0.05 and T=[0,2]



3.3.2 European put option

The put option gives the holder the right to sell the underlying asset for the strike/exercise price E at maturity time (T). Unlike call option here the option is worthless if the asset price is more than the exercise price, thus $\forall$ S > E P(S,T)=0. On the other hand net profit E-S is realized if the asset price is below the exercise price. Once again boundary conditions follow from economic arguments.

$$\frac{\partial P}{\partial t} + L_{BS} P = 0$$

$$P(0,t) = Ee^{-r(T-t)}$$

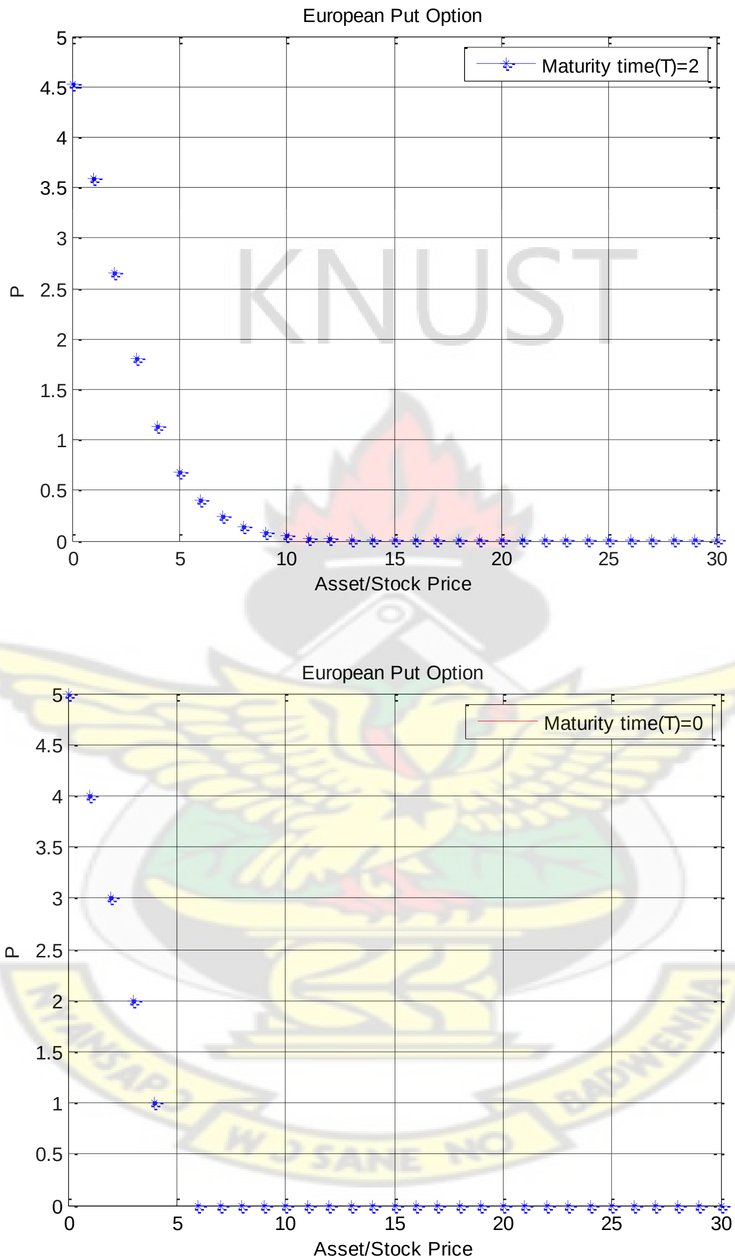
$$\lim_{S \to \infty} P(S,t) = 0 \tag{3.20}$$

With d_1, d_2, N from equation (3.17), (3.18) and (3.19) the analytic solution of the put option reads

$$P(S,t) = Ee^{-r(T-t)} N(-d_2) - Se^{-\delta(T-t)} N(-d_1) \tag{3.21}$$

The graphical representation of the put option solution is seen in figure 3.2 with the same parameters as in figure 3.1.

Figure 3.2: Showing the final value and solution for European put option. Parameters as in Figure 3.1



The put-call parity establishes a relationship between the put and call option prices, thus

$$C(S,t)+Ee^{-r(T-t)}N(-d_2)=P(S,t)+Se^{-\delta(T-t)}N(-d_1) \tag{3.22}$$

The arbitrage principle is the bases for this relationship. At $t = T$ the value of both side of the equation are certain.

$$\max(S - E, 0) + E = \max(E - S, 0) + S \qquad (*)$$

Both side of the equation becomes S if S>E, and E if E>S. with this an arbitrage risk free portfolio can be composed.

3.3.3 Digital call option

This option also known as binary option is a type of exotic option, which offers fixed payout. There exist a large variety of exotic options now a days and are usually characterized by the different boundary or final conditions from the standard and vanilla European option. The discontinuity of the digital option payoff is the reason for it studies. The principle here is quite simple and can find a place in the tool box of trade average retailer. For digital call the option pays, if the value of the stock price is greater or equal the exercise price. If the stock price (S) is higher than the exercise price (E), the holder receives a fixed amount Q, which is equal to one (1) for pure digital. Example if the holder pay GH¢ 0.40 for a digital call and the asset price is above the exercise price, the holder makes a net profit of GH¢0.60. These types of options are not useful in the America style option since the options are immediately exercised. The final condition for digital call option is the Heaviside function,

$H(S): C = 1$ if $S > E$ and $C = 0$ if $S < E$. The left boundary is the same as the normal call, $C(0,t) = 0$ and the right boundary is on the payoff amount corrected by interest rate. To summarize

$$\frac{\partial C}{\partial t} + L_{BS} C = 0$$

$$C(0,t) = 0$$

$$C(S,t) = Qe^{-r(T-t)} \quad \text{as} \quad S \longrightarrow \infty \tag{3.23}$$

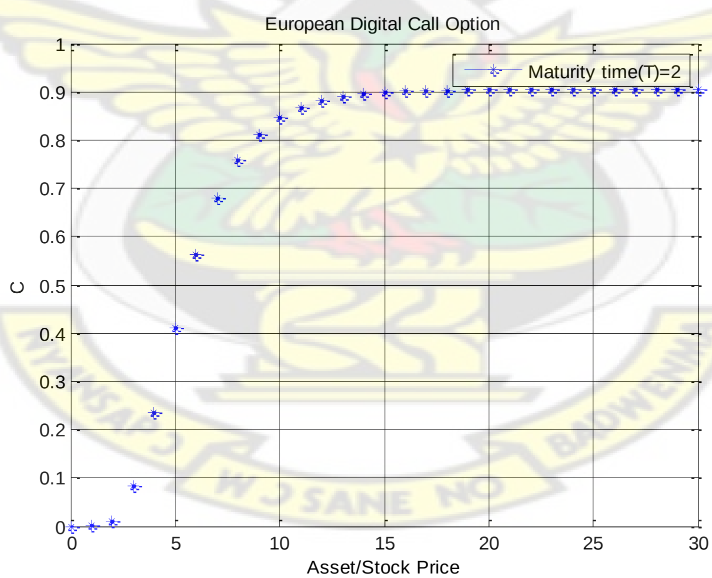
$$C(S,T) = QH(S-E) = \begin{cases} Q, S > E \\ 0, S < E \end{cases}$$

With d_1, N from equation (3.17), (3.19), the analytic solution of the digital call option is

$$C(S,t) - Qe^{-r(T-t)}N(d_2) \tag{3.24}$$

Figure 3.3 represent the final value and solution of the European digital call option. There is however another type of the digital call option which is available in the American style option and is called the asset or nothing call.

Figure 3.3: Showing the final value and solution for European digital call option with final value, and parameters as follows: *E*=GH¢ 5.00, σ=0.3, δ=0.03, r=0.05 and T=[0,2]



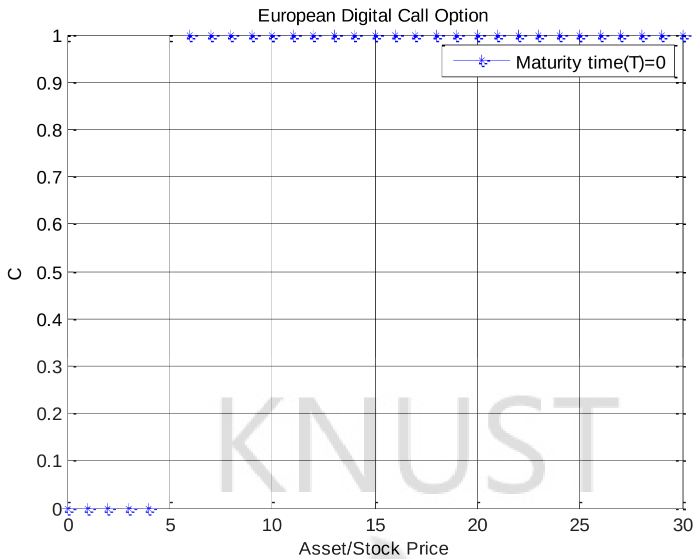
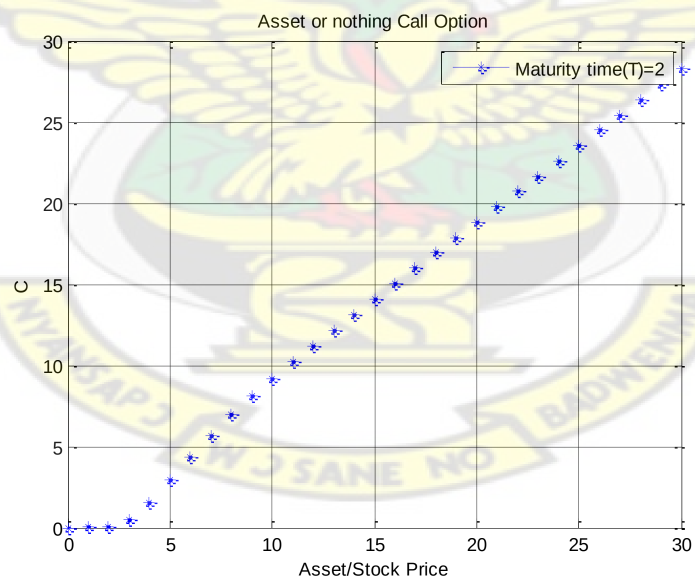
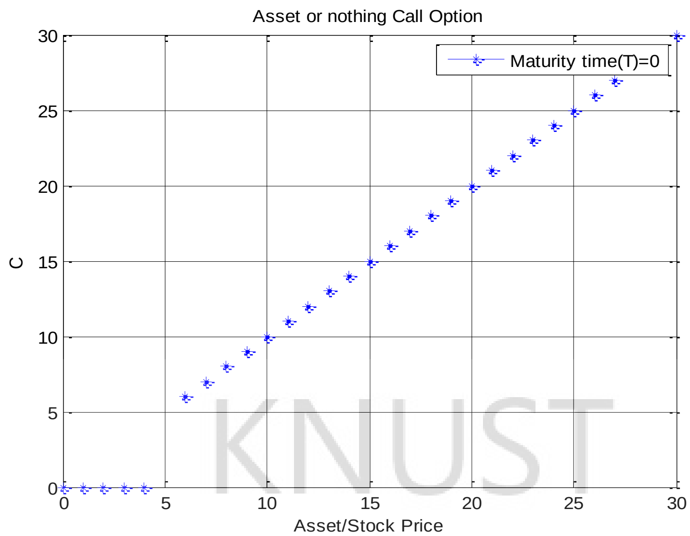


Figure 3.4: Showing the final value and solution for asset or nothing call option. Parameter

Figure 3.3





3.3.4 Digital put option

A digital put option gives the holder a specific amount of the underlying asset if it is below the strike price expectation. The summarize equation is as follows

$$\frac{\partial P}{\partial t} + L_{BS} P = 0$$

$$P(0,t) = Qe^{-r(T-t)}$$

$$\lim_{S \to \infty} P(S,t) = 0 \tag{3.25}$$

$$P(S,T) = Q(1 - H(S-E)) = \begin{cases} 0, S > E \\ Q, S < E \end{cases}$$

With d_2, N from equation (3.18), (3.19) the analytical solution of the digital put option reads

$$P(S,t) = Qe^{-r(T-t)} N(-d_2) \tag{3.26}$$

Figure 3.5, the final condition and the solution at $t = 0$ is shown. Just as in digital call, in digital put there exists also asset or nothing put a variant of the digital put option type.

Figure 3.5: Showing the final value and solution for European digital put option. Parameters as in Figure 3.3

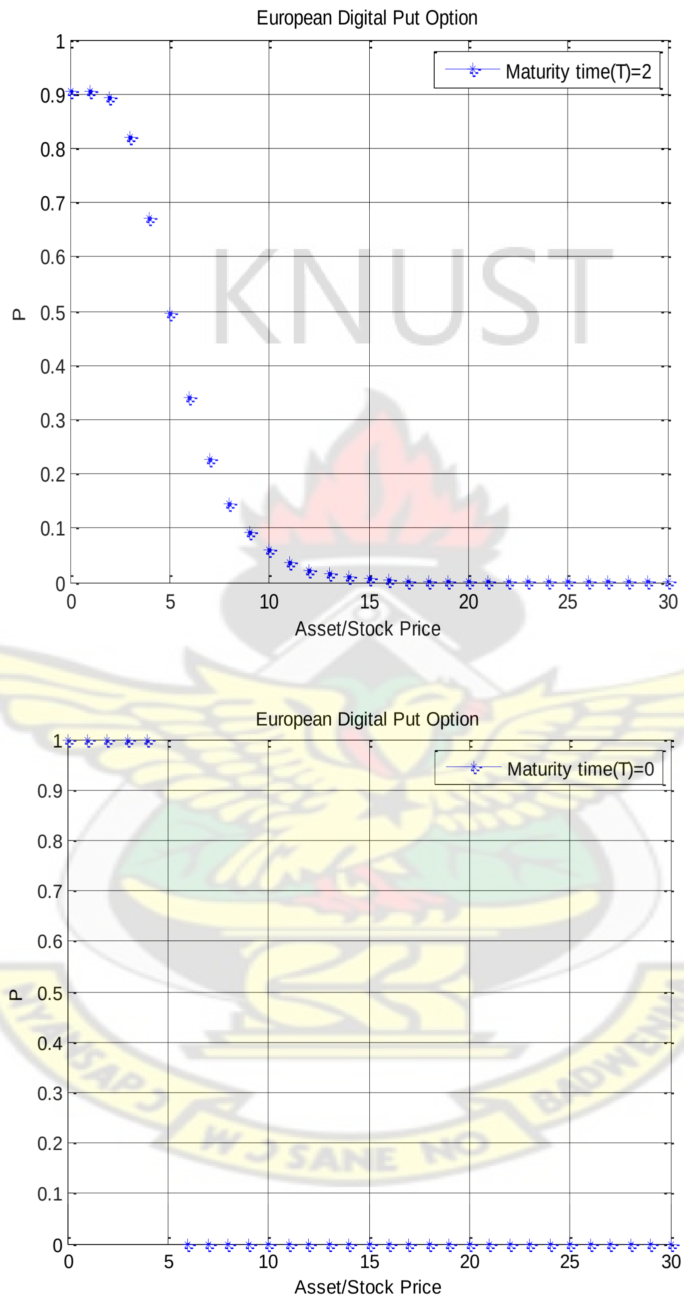
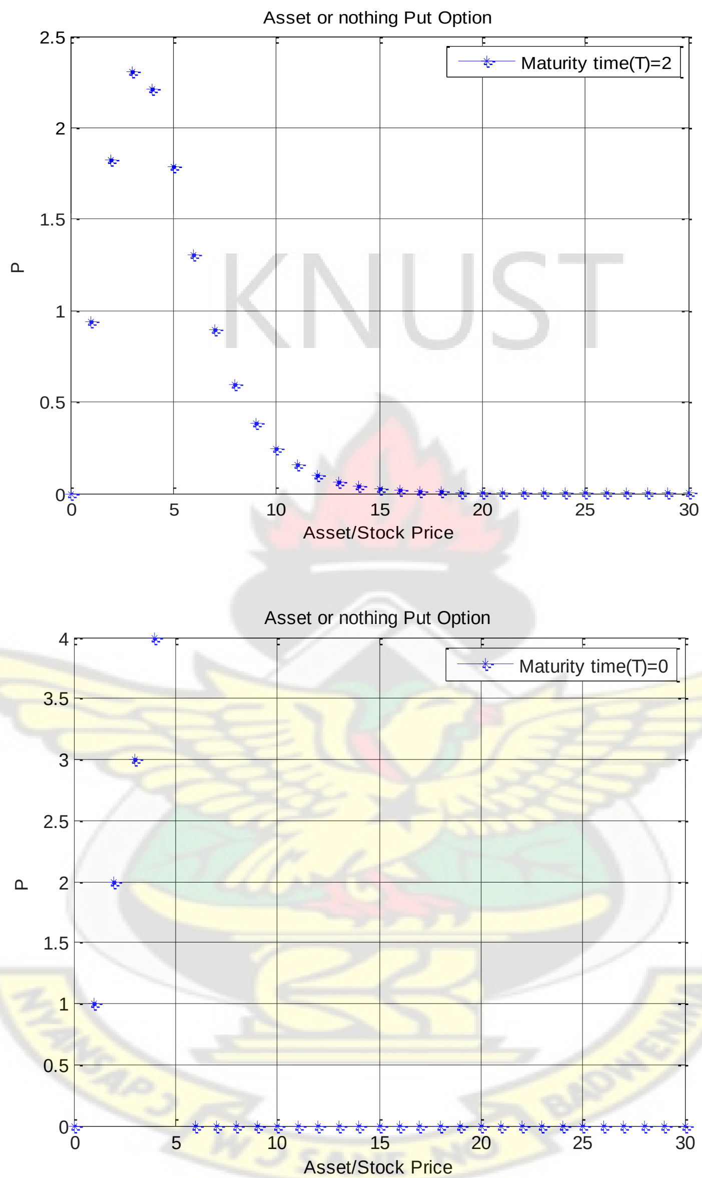


Figure 3.6: Showing the final value and solution for asset or nothing put option. Parameters as in Figure 3.3



3.3.5 Linear combinations

Trading options is usually done with a variety of options and not a single option. Trading a combination of options on the same underlying asset is a very common strategy in practice. See the definition of spread in 3.2 (definition 4). In this situation a trader acts as a holder and as a writer. All options have the same maturity time. For example a bull spread, is a long position call with an exercise price E_1, and a short position with exercise price E_2, with $E_1 < E_2$. With a bull spread the profit of the holder and at the same time, the losses for the writer are reduce as compared to a single option figure (3.7).

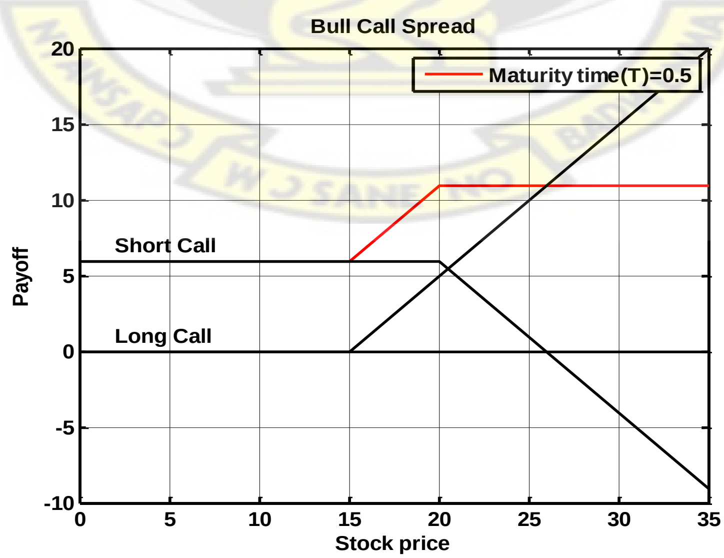
Suppose a long option call bought for GH¢ 3 with $E_1 = GH¢15$ and a short position call sold for GH¢ 1 with exercise price $E_2 = GH¢20$. The profit at t=T, defined as the payoff minus the cost of the bull spread reads

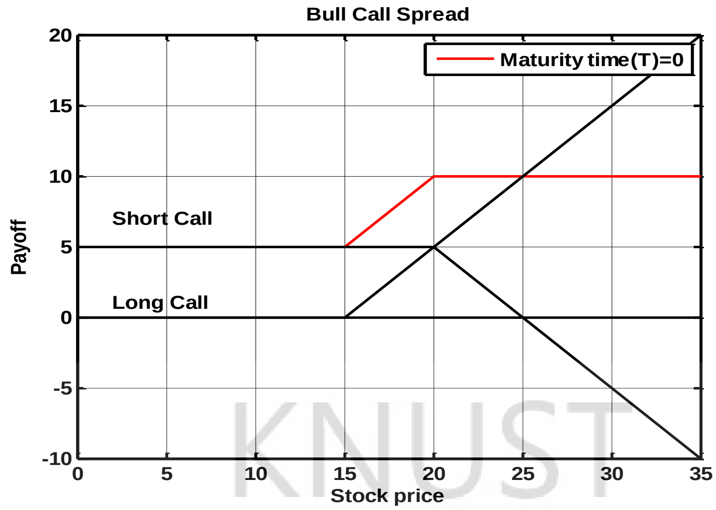
$$profit = \begin{cases} -2, S < E_1 \\ S - 17, E_1 < S < E_2 \\ 3, S_2 > E_2 \end{cases}$$

Figure 3.7 shows the solution and payoff for a bull spread.

Figure 3.7: Solution and final value of bull spread with parameters:

$$E_1 = 15, E_2 = 20, \sigma = 0.3, r = 0.05, \delta = 0.03, T = [0, 0.5].$$





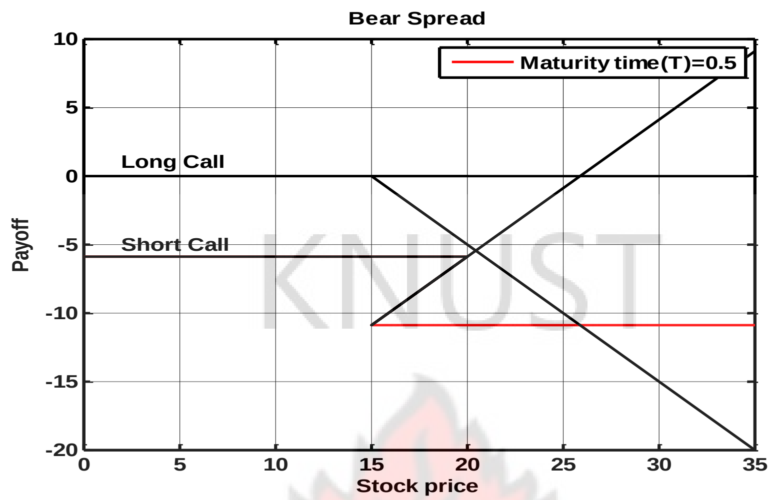
The bear spread happens to be the reverse of a bull spread, in that it is a combination of a long position with a higher exercise price E_2 and a short position with a lower exercise price E_1, See figure 3.8. Consider as an example a long position call bought for GH¢1 with exercise price $E_1 = GH¢20$ and a short position call sold of GH¢3 with exercise price $E_2 = GH¢15$. The profit thus is

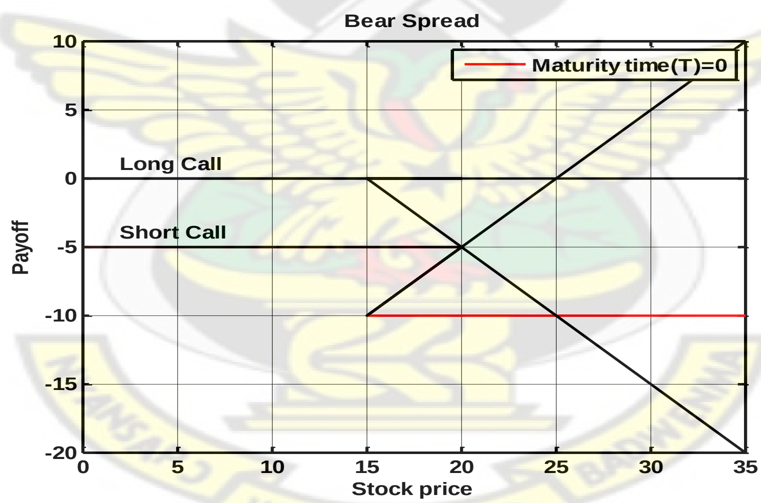
$$profit = \begin{cases} 2, S < E_1 \\ 15 - S, E_1 < S < E_2 \\ -3, S > E_2 \end{cases}$$

As compare to a single option loses of the holder are reduce as well as the profit of the writer for bear spread. In figure 3.8 the payoff and the solution at $t = 0$ for a bear spread are presented.

Figure 3.8: Solution and final value of bear spread with parameters:

$$E_1 = 15, E_2 = 20, \sigma = 0.3, r = 0.05, \delta = 0.03, T = [0, 0.5].$$





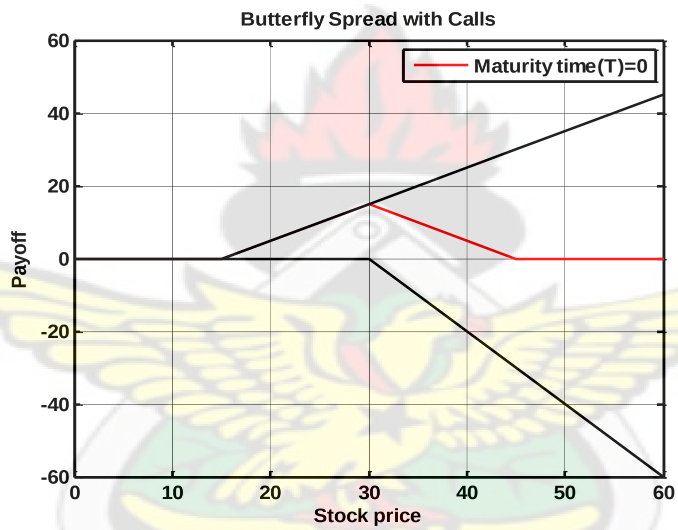
For a spread like butterfly spread there is a combination of four options. Two long position calls with exercise price E_1 and E_3, and two short position calls with exercise prices $E_2 = \frac{1}{2}(E_1 + E_3)$. Here the payoff reads

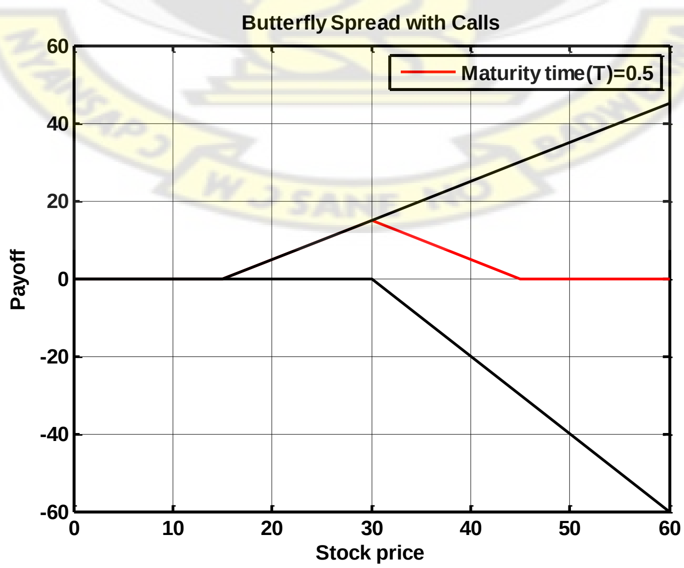
$$payoff = \begin{cases} 0, S < E_1 \\ S - E, E_1 < S < E_2 \\ E_3 - S, E_2 < S > E_3 \\ 0, S > E_3 \end{cases}$$

In figure 3.9 the payoff function and solution of the butterfly spread is represented. The usual European options are more risky than all these spreads.

Figure 3.9: Solution and final value of Butterfly spread with parameters:

$$E_1 = 15, E_3 = 45, \sigma = 0.3, r = 0.05, \delta = 0.03, T = [0, 0.5].$$





For a digital spread often called super share. An example is a combination of a long position cash-or-nothing call with amount Q/d and exercise price E and a short position cash-or-nothing call with Q/d and exercise price $E+d$. The payoff thus is

$$payoff = \begin{cases} 0, S < E \\ Q/d, E < S < E+d \\ 0, S > E+d \end{cases}$$

Note that the values of all spreads are described by the Black-Scholes equation, because the spread strategies can be seen as linear combinations of single options, and the equation is linear. It must also be noted that this research work considers options on only one underlying asset.

3.3.6 Barrier option

This option belongs to the exotic options as well. The down and out barrier option will be discussed in this research work. A down and out option is worthless if $S < B$ (barrier amount) and therefore the left side boundary condition will change to $V(B,t) = 0$ instead of $V(0,t) = 0$. Here the differential equation has to be solved in the region of

$S \in [B, S\max]$

The exact solution reads

$$V(S,t) = C(S,t) - (S/B)^{-(k-1)} C(\frac{B^2}{S}, t) \quad (3.27)$$

Where $C(S,t)$ denote the solution of the standard European call and $k = 2r/\sigma^2$

3.4 The Greeks

In the derivation of the Black-Scholes equation, the elimination of the randomness in the option pricing process is employed to derive the deterministic Black-Scholes equation. One of the important parameters in option pricing is $\Delta = \dfrac{\partial V}{\partial S}$ which is the quantities that eliminate the main contribution to randomness in the model. It is the rate of change of the option price with respect to the price of the underlying asset. It indicates the number of shares that should be kept with each option issued in order to cope with a loss in the case of exercise.

Another important parameter is the derivation of Δ, that is Γ. This is defined as the rate of change of the portfolio Δ with respect to the price of the underlying asset. Γ is an indication of the sensitivity of Δ. If Γ is low, it is only necessary to change sometimes the portfolio. However, when it is high, the portfolio under consideration results only for a very short period of time in a risk-less scenario. These parameters are known as Greeks and are given by

$$\Delta = \frac{\partial V}{\partial S} \tag{3.28}$$

$$\Gamma = \frac{\partial^2 V}{\partial S^2} \tag{3.29}$$

$$\Theta = \frac{\partial V}{\partial t} \tag{3.30}$$

$$\nu = \frac{\partial V}{\partial \sigma} \tag{3.31}$$

$$\rho = \frac{\partial V}{\partial r} \tag{3.32}$$

Θ is defined as the rate of change of the option price with respect to time when all other parameters are kept fixed. ν is known as vega and define as the rate of change of the option price with respect to volatility of the underlying asset while ρ is defined as the rate of change of the option price with respect to the interest rate in the market. The value of Δ and Γ are investigated in this thesis.

By working down the analytic solution of V , the exact solution of the Greeks for European options can be determined.

$$\Delta = e^{-\delta(T-t)} N(d_1)$$

$$\Gamma = e^{-\delta(T-t)} \frac{N(d_1)}{\sigma S \sqrt{T-t}}$$

$$\Theta = \frac{-SN^1(d_1)\sigma e^{-\delta(T-t)}}{2(T-t)} + \delta SN(d_1)e^{-\delta(T-t)} - rSe^{-r(T-t)}N(d_2)$$

$$\nu = e^{-\delta(T-t)} S\sqrt{T-t} N^1(d_1)$$

$$\rho = S(T-t)e^{-r(T-t)} N(d_2)$$

With $N^1(x)$, the derivative of equation 3.19 which is the normal distribution. In figure 3.10 to 3.11 several Greeks are plotted for the parameter set used in figure 3.1. It is challenging to find an accurate approximation of the Greeks numerically, since numerical differentiation usually reduces the order of accuracy. Often in numerical experiments, one sees that the accuracy of the Greeks is better than expected, probably due to the smoothness of the solutions. With the highly accurate discretization's, however, we expect reasonable accuracy of the hedging parameters.

Figure 3.11: Δ of a European call. Parameter as in figure 3.1

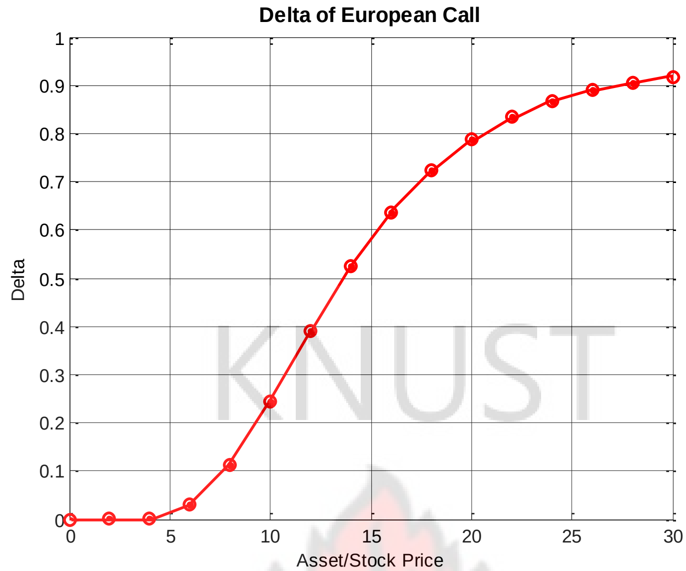
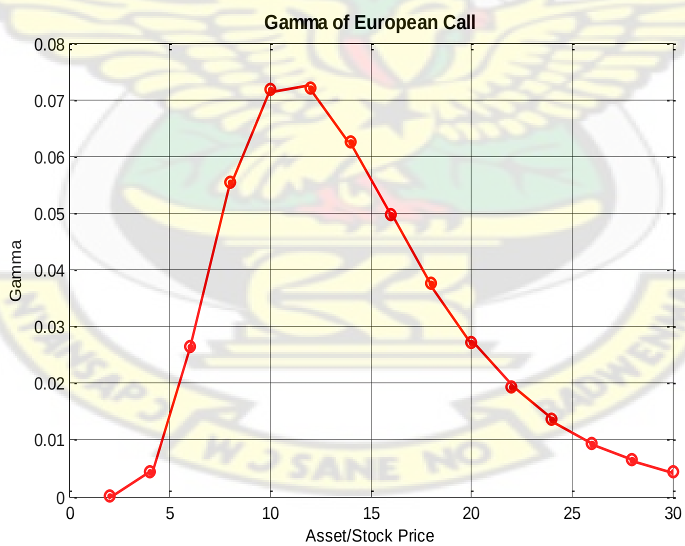


Figure 3.11: Γ of a European call. Parameter as in figure 3.1



3.5 Discretization of the Partial Differential Equation

To get a fast solution of the Black –Scholes equation a numerical method will be developed. This will be the aim of this section. In many practical situations, a second order accurate solution which leads to decreasing error quadratically is used. However when needed, a fourth order accurate solution will be more preferable as the numerical solution is often obtained faster as fewer grid points are necessary for the same accuracy. A powerful technique to get many grid points in the region of interest is grid stretching. After the resulting linear algebraic system has been solved for each time step, the aim will be to discretize the equation in space and in time. A direct method will be used to solve the matrix equation, since a sparse matrix results from a one dimensional space discretization. For other exotic operations as well as American options style an iterative methods are mandatory. However this will not be considered in this research work. Nevertheless later in this chapter and chapter 4, some properties of the matrix will be given, to determine which discretization is beneficial for iterative methods.

3.5.1 Space discretization

A parabolic partial differential equation is a type of second order PDE which describes wide range of family of problems. Boundary conditions for parabolic problems are analogous to the elliptic case: Eg. Dirichlet.

Considering the general form of a parabolic PDE with non-constant coefficient, Dirichlet boundary conditions and initial conditions, we have

$$\frac{\partial u}{\partial t} = \alpha(x)\frac{\partial^2 u}{\partial x^2} + \beta(x)\frac{\partial u}{\partial x} + \gamma(x)u(x,t) + f(x,t) \tag{3.32}$$

$$u(a,t) = b(t) \tag{3.33}$$

$$u(b,t) = R(t) \tag{3.34}$$

$$u(x,0) = \Phi(x) \tag{3.35}$$

To solve this equation, a numerical method needs to be employed on a grid with N points and a constant step size h. Such a grid is called equidistant grid. For an interval $[a,b]$, the step size h is $h = \frac{(b-a)}{N}$. Let each point $a+ib$ be denoted by, thus $x_i = a + ih$.

3.5.1.1 Second order accuracy

To numerically solve PDEs, the solution intervals need to be discretized into a set of discrete points with u as a function of one or more discrete points in a given neighborhood t_j.

To obtain the second order central difference scheme, approximation, let's consider the Taylor's expansions of the given point say $x_i = a + ih$

$$x = x + x_i - x_i$$
$$x = x_i + (x - x_i)$$
$$x = x_i + h$$

$$u(x) = u(x_i + h)$$

$$u(x_i+h) = u(x_i) + hu^{'}(x_i) + \frac{1}{2!}h^2u^{"}(x_i) + \ldots + \frac{1}{n!}h^n u^n(x_i) + \frac{1}{(n+1)!}h^{(n+1)}u^{(n+1)}x_{(in)} \tag{3.36}$$

If $h = d$, then as h increases it approaches zero

$$\begin{aligned} u(x_i+h) &= u(x_i) + hu^{'}(x_i) + 0(h)^2 \\ u^{'}(x_i) &= \frac{u(x_i+h) - u(x_i)}{h} + 0(h)^2 \end{aligned} \tag{3.37}$$

$$u(x_i+h) = u(x_i) + hu^{'}(x_i) + \frac{h^2u^{"}}{2!}(x_i) + \ldots.. + \tag{*}$$

$$u(x_i - h) = u(x_i) - hu^{'}(x_i) + \frac{h^2 u^{"}}{2!}(x_i) + ... + \qquad (**)$$

$$u(x_i - h) = u(x_i) - hu^{'}(x_i) + 0(h)^2$$
$$u^{'}(x_i) = \frac{u(x_i) - u(x_i - h)}{h} + 0(h)^2 \qquad (3.38)$$

With linear combinations of u at the points x_j, it is possible to get a second order approximation of the first and second derivatives.

$(*) - (**)$ yields $u^{'}(x_i) = \frac{u(x_i + h) - u(x_i - h)}{2h} + 0(h)^2 \qquad (3.39)$

$(*) + (**)$ yields $u^{"}(x_i) = \frac{u(x_i + h) + u(x_i - h) - 2u(x_i)}{h^2} + 0(h)^2 \qquad (3.40)$

$$u^{'}(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} + 0(h)^2 \qquad (3.41)$$

Hence u_i is the abbreviation for $u(x_i)$. It is therefore possible to discretize differential for (3.33). In each point x_i the factors in front of the differential operator can be evaluated. For the second order approximation, the discretized system reads

$$\frac{\partial u_i}{\partial t} = \alpha_i \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \beta_i \frac{u_{i+1} - u_{i-1}}{2h} + \gamma_i u_i + f_i(t) \qquad (3.42)$$

This equation is true for $1 \le i \le N \le N - 1$. There is a need for a special treatment of the first and last points. In a matrix form, systems of equation (3.42) read

$$\frac{du}{dt} = Au + b(t) + f(t) \qquad (3.43)$$

Where f is the discretized source function, A the matrix coefficient and u the discrete solution. The vector b contains the boundary values and may be time dependent function. The equation of the first point in the second order accuracy, thus reads

$$\frac{\partial u_1}{\partial t} = \alpha_1 \frac{u_2 - 2u_1 + u_0}{h} + \beta_i \frac{u_2 - u_0}{2h} + \gamma_1 u_1 + f_1(t) \tag{3.44}$$

From (3.33), $u_0 = L(t)$

In the same way, the right boundary in the second order reads

$$\frac{\partial u_{N-1}}{\partial t} = \alpha_{N-1} \frac{u_N - 2u_{N-1} + u_{N-2}}{h^2} + \beta_{N-1} \frac{u_N - u_{N-2}}{2h} + \gamma_{N-1} u_{N-1} + f_{N-1} \tag{3.45}$$

With $u_N = R(t)$. Vector b reads

$$b_i = \begin{cases} \left(\alpha \frac{(a+h)}{h^2} - \beta \frac{(a+h)}{2h} \right) L(t), i = 1 \\ 0, 2 \le i \le N-2 \\ \left(\alpha \frac{(b-h)}{h^2} + \beta \frac{(b-h)}{2h} \right) R(t), i = N-1 \end{cases} \tag{3.46}$$

And the matrix elements are

$$\begin{aligned} a_{ii} &= \frac{-2}{h^2} \alpha_i + \gamma_i \\ a_{ii} &= \frac{1}{h^2} \alpha_i + \frac{1}{2h} \beta_i \\ a_{ii} &= \frac{1}{h^2} \alpha_i - \frac{1}{2h} \beta_i \end{aligned} \tag{3.47}$$

3.5.1.2 High order accuracy

For fourth order accurate discretization, more neighboring points are required or needed for

Taylor's expansion. Just as in (3.36), it follows that for the points $\begin{matrix} x_{i+2} = x_i + 2h \\ x_{i-2} = x_i - 2h \end{matrix}$ we have

$$u(x_i + h) = u(x_i) + hu^{'}(x_i) + \frac{1}{2!} h^2 u^{"}(x_i) + \dots + \frac{1}{n!} h^n u^n (x_i) + \frac{1}{(n+1)!} h^{(n+1)} u^{(n+1)} x_{(in)}$$

(3.48)

With the assumption that all derivatives exist and also with the same abbreviation $u_i = u(x_i)$,

the fourth order approximations of the derivatives are

$$\frac{1}{12h^2}(-u_{i+2} + 16u_{i+1} - 30u_i - u_{i-2}) + 0(h)^4 = u_i'' \tag{3.49}$$

$$\frac{1}{12h}(-u_{i+2} + 8u_{i+1} - 8u_{i-2}) + 0(h)^4 = u_i' \tag{3.50}$$

Combining (3.49) and (3.50), then (3.33) reads

$$\frac{\partial u_i}{\partial t} = \frac{1}{12h^2}\alpha_i(-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}) + \tag{3.51}$$
$$\frac{1}{12h}\beta_i(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}) + \gamma_i u_i + f_i(t)$$

At the left boundary, the point x_1 and x_2 needs a special treatment and the points x_{N-1} and x_{N-2} at the right boundary of the fourth order approximation.

At the point x_2 the equation reads

$$\frac{\partial u_2}{\partial t} = \frac{1}{12h^2}\alpha_2(-u_4 + 16u_3 - 30u_2 + 16u_1 - u_0) + \tag{3.52}$$
$$\frac{1}{12h}\beta_2(-u_4 + 8u_3 - 8u_1 + u_0) + \gamma_2 u_2 + f_2(t)$$

And that of x_1 reads

$$\frac{\partial u_1}{\partial t} = \frac{1}{12h^2}\alpha_2(-u_3 + 16u_2 - 30u_1 + 16u_0 - u_{-1}) + \tag{3.53}$$
$$\frac{1}{12h}\beta_2(-u_3 + 8u_2 - 8u_0 + u_{-1}) + \gamma_1 u_1 + f_1(t)$$

The value u_{-1} can be obtained by extrapolation. The different possibilities are

$$u_{-1} = 2u_0 - u_1 + 0(h)^2 \tag{3.54}$$

$$u_{-1} = 3u_0 - 3u_1 + u_2 + 0(h)^3 \tag{3.55}$$

$$u_{-1} = 4u_0 - 6u_1 + 4u_2 - u_3 + 0(h)^4 \tag{3.56}$$

$$u_{-1} = 5u_0 - 10u_1 + 10u_2 - 5u_3 + u_4 + 0(h)^5 \tag{3.57}$$

For the right boundary the derivation is omitted since the points x_{N-2} and x_{N-1} goes in a similar way as in the left boundary.

The first and last row of the matrix system will change just as the vector b changes. From (3.56) and as mentioned earlier on in equation 3.46, here vector b reads

$$b_i = \begin{cases} \left(\alpha \dfrac{(a+h)}{h^2} - \beta \dfrac{(a+h)}{3h^2} \right) L(t), i = 1 \\ \left(-\alpha \dfrac{(a+2h)}{12h^2} - \beta \dfrac{(a+2h)}{12h} \right) L(t), i = 2 \\ 0, 3 \leq i \leq N-3 \\ \left(-\alpha \dfrac{(b-2h)}{12h^2} - \beta \dfrac{(b-2h)}{12h} \right) R(t), i = N-2 \\ \left(\alpha \dfrac{(b-h)}{h^2} + \beta \dfrac{(b-h)}{3h} \right) R(t), i = N-1 \end{cases} \tag{3.58}$$

The matrix element in equation 3.43 reads

$$\begin{aligned} a_{ii} &= \frac{-15}{4h^2} \alpha_i + \gamma_i \\ a_{ii+1} &= \frac{4}{3h^2} \alpha_i + \frac{4}{h} \beta_i \\ a_{ii-1} &= \frac{4}{3h^2} \alpha_i - \frac{4}{h} \beta_i \\ a_{ii+2} &= \frac{-1}{12h^2} \alpha_i - \frac{1}{12h} \beta_i \\ a_{ii-2} &= \frac{-1}{12h^2} \alpha_i + \frac{1}{12h} \beta_i \end{aligned} \tag{3.59}$$

The corrected first row and second row of the equation (3.59) reads

$$a_{11} = \frac{-2}{h^2}\alpha(a+h) - \frac{1}{2h}\beta(a+h) + \gamma(a+h)$$

$$a_{12} = \frac{1}{h^2}\alpha(a+h) + \frac{1}{h}\beta(a+h)$$

$$a_{13} = \frac{-1}{6h}\beta(a+h) \tag{3.60}$$

$$a_{N-1,N-1} = \frac{-2}{h^2}\alpha(b-h) + \frac{1}{2h}\beta(b-h) + \gamma(b-h)$$

$$a_{N-1,N-2} = \frac{1}{h^2}\alpha(b-h) - \frac{1}{h}\beta(b-h)$$

$$a_{N-1,N-3} = \frac{1}{6h}\beta(b-h)$$

Another approach perhaps the second one for the correction of the first and last grid points is to use the backward difference scheme at the first and last grid points with different scheme

$$\frac{\partial u_i}{\partial x} = \frac{-3u_0 - 10u_1 + 18u_2 - 6u_3 + 4u_4}{12h^2} + 0(h)^4 \tag{3.61}$$

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{10u_0 - 15u_1 - 4u_2 + 14u_3 - 6u_4 + u_5}{12h^2} + 0(h)^4 \tag{3.62}$$

and similarly for the last point

$$\frac{\partial u_{N-1}}{\partial x} = \frac{3u_N + 10u_{N-1} - 18u_{N-2} - 6u_{N-3} - u_{N-4}}{12h^2} + 0(h)^4 \tag{3.63}$$

$$\frac{\partial u^2{}_{N-1}}{\partial x^2} = \frac{10u_N - 15u_{N-1} - 4u_{N-2} + 14u_{N-3} - 6u_{N-4} + u_{N-5}}{12h^2} + 0(h)^4 \tag{3.64}$$

the corrected first and last row from the matrix element in (3.59) is now

$$
\begin{aligned}
a_{11} &= \frac{-5}{4h^2}\alpha(a+h) - \frac{5}{6h}\beta(a+h) + \gamma(a+h) \\
a_{12} &= \frac{-1}{3h^2}\alpha(a+h) + \frac{3}{2h}\beta(a+h) \\
a_{13} &= \frac{7}{6h^2}\alpha(a+h) - \frac{1}{2h}\beta(a+h) \\
a_{14} &= \frac{-1}{2h^2}\alpha(a+h) + \frac{1}{12h}\beta(a+h) \\
a_{15} &= \frac{1}{12h^2}\alpha(a+h) \\
a_{N-1,N-1} &= \frac{-5}{4h^2}\alpha(b-h) + \frac{5}{6h}\beta(b-h) + \gamma(b-h) \\
a_{N-1,N-2} &= \frac{-1}{3h^2}\alpha(b-h) - \frac{31}{2h}\beta(b-h) \\
a_{N-1,N-3} &= \frac{7}{6h^2}\alpha(a+h) - \frac{1}{2h}\beta(b-h) \\
a_{N-1,N-4} &= \frac{-1}{2h^2}\alpha(a+h) - \frac{1}{2h}\beta(b-h) \\
a_{N-1,N-5} &= \frac{1}{12h^2}\alpha(a+h)
\end{aligned}
\tag{3.65}
$$

The only change on vector b is on the first and last elements, thus these element in the vector b from (3.58), now reads

$$
\begin{aligned}
b_1 &= (\frac{5}{6h^2}\alpha(a+h) - \frac{1}{4h}\beta(a+h))L(t) \\
b_{N-1} &= (\frac{5}{6h^2}\alpha(a+h) + \frac{1}{4h}\beta(b-h))R(t)
\end{aligned}
\tag{3.66}
$$

3.5.2 Coordinate transformation

As defined in [12] Coordinate transformation is defined as a process of establishing relationship between coordinate systems in order to transform points from one system to the other. In numerical solution of PDEs a discrete domain is chosen where algebraic analogues of PDEs are solved. The introduction of grid and estimation of values of the unknowns at the grid points though the solutions of these algebraic equations are one standard method used. The local error and hence the accuracy of the solution is determined by spacing of the grid points. The spacing also determines the number of calculations to be made to cover the domain of the problem and thus the cost of the computation. For a well behaved problem, a grid of uniform mesh spacing in each of the coordinate directions gives satisfactory results. There are however classes of problems where the solution is more difficult to estimate in some regions perhaps due to discontinuity or non-differentiation as in option pricing than in others. The reason for using a coordinate transformation is to simplify the payoff function. One could use a uniform grid having a spacing time enough so that the local errors estimated in these difficult regions are acceptable. This approach however is computationally costly. The local refinement near the discontinuous payoff conditions seems logically a choice to retain a satisfactory accuracy. The local refinement principle is simple, in that one needs to get more points in the neighborhood of the grid point where the non-differentiable conditions or discontinuity occurs. This can be done by adaptive grid refinement for some regions, based on the error indicator or by a coordinate transformation. The most elegant way in our application is the analytic coordinate transformation. The derived space discretization in (section 3.5.1) is based on equidistant grid. After the analytic transformation, the discretization can still be used as the only coefficients in front of the derivatives change. Consider the coordinate transformation $y_i = \varphi_i(x_i)$ with inverse $x_i = \varphi_i^{-1}(y_i)$. From the principle of chain rule, then first order derivatives of the function $u(x)$ will read

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}\left(\frac{dx}{dy}\right)^{-1} = \varphi^{-1}(y)\frac{du}{dy} \tag{3.67}$$

A second order derivative on the equation (4.36) reads

$$\frac{d^2u}{dx^2} = \left(\frac{dx}{dy}\right)^{-1} \frac{d}{dy}\left(\left(\frac{dx}{dy}\right)^{-1}\left(\frac{du}{dy}\right)\right) = \left(\frac{dx}{dy}\right)^{-2} \frac{d^2u}{dy^2} - \left(\frac{dx}{dy}\right)^{-3} \frac{d^2x}{dy^2}\frac{du}{dy} \tag{3.68}$$

Appling (3.67) and (3.68) to (3.32) the factors α, β and γ changes into

$$\alpha = \frac{\alpha(\varphi(y))}{(\varphi'(y))^2} \tag{3.69}$$

$$\beta = \frac{\beta(\varphi(y))}{\varphi'(y)} - \alpha(\varphi(y))\frac{\varphi''(y)}{(\varphi(y))^3} \tag{3.70}$$

$$\gamma = \gamma(\varphi(y)) \tag{3.71}$$

Here if $\beta = 0$ in the original equation, the transformed equation will contain extra convection term. The standard diffusion equation thus turns into a convection-diffusion equation with non-constant coefficients. PDE is the target transform equation to solve on an equidistant grid. The left and right boundary conditions are transformed into $\varphi(a)$ and $\varphi(b)$. And thus the new step size will be

$h_{new} = \dfrac{\varphi(b) - \varphi(a)}{N}$ With the assumption that φ is a monotonically increasing function.

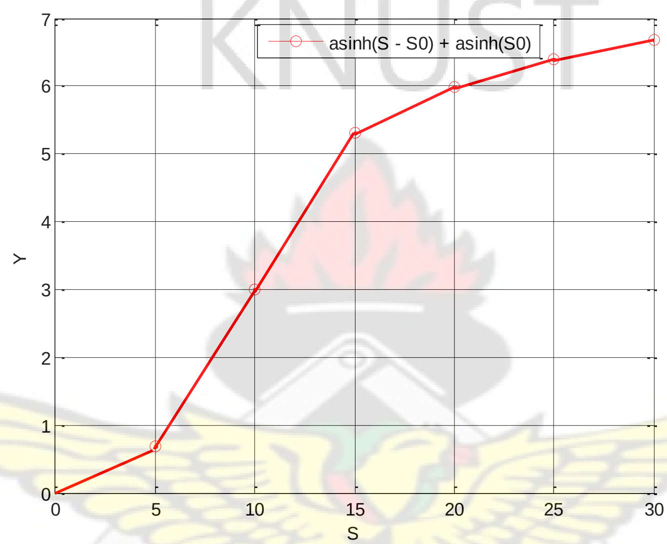
3.5.2.1 Type of transformations

Considering the two basic transformation types; linear transformation and nonlinear transformation and applying the technique of one of the two, a non-differentiable payoff function remains only along the independent variable (x) path. An analytic grid stretching in this coordinate direction represents a technique which may cluster grid points in the region of interest and this can improve the accuracy of the solution in the case of a payoff function that is non-differentiable. It preserves the order of functions. It will be of convenience, to use monotonically increasing functions for our transformation. For an option pricing an interesting transformation function reads

$$y = \varphi(s) = \sinh^{-1}(S - S_0) + \sinh^{-1} S_0 \tag{3.72}$$

For stock price / asset price $S_0 = GH¢10$

Figure 3.12: Representation of transformation function of (3.72)



The transformation considered per the aim of this research paper reads,

$$y = \varphi(s) = \sinh^{-1}(\mu(S - S_0)) + \sinh^{-1}(\mu S_0) \tag{3.73}$$

μ = rate of stretching

Before monotonic: The coordinate of the independent variable x can be written as a function of the dependent variable y via the stretching function φ .

In many cases the satisfactory value for μ is $\mu = 5$. With (3.73), the grid is refined around $S = S_0$

In figure 3.13, the stretching function $\mu = 1, 5, 10$ has been plotted

For the transformation 3.73, the inverse and the derivatives read

$$\varphi(y) = \frac{1}{\mu}\sinh(y - \sinh^{-1}(\mu S_0)) + S_0 \tag{3.74}$$

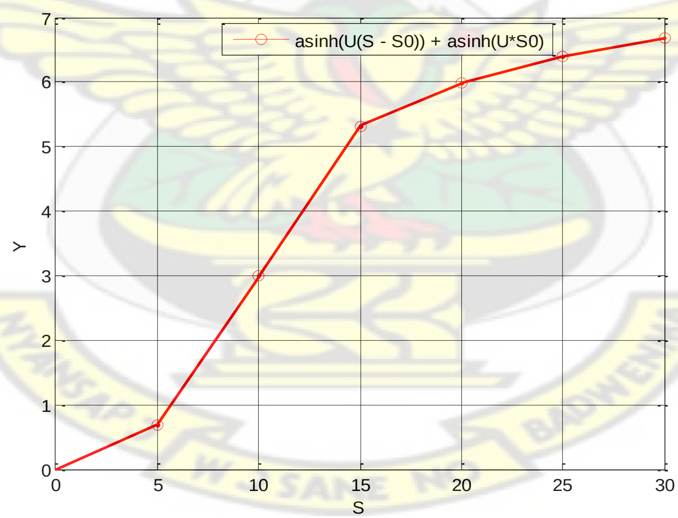
$$\varphi'(y) = \frac{1}{\mu}\cosh(y - \sinh^{-1}(\mu S_0)) \tag{3.75}$$

$$\varphi''(y) = \frac{1}{\mu}\sinh(y - \sinh^{-1}(\mu S_0)) \tag{3.76}$$

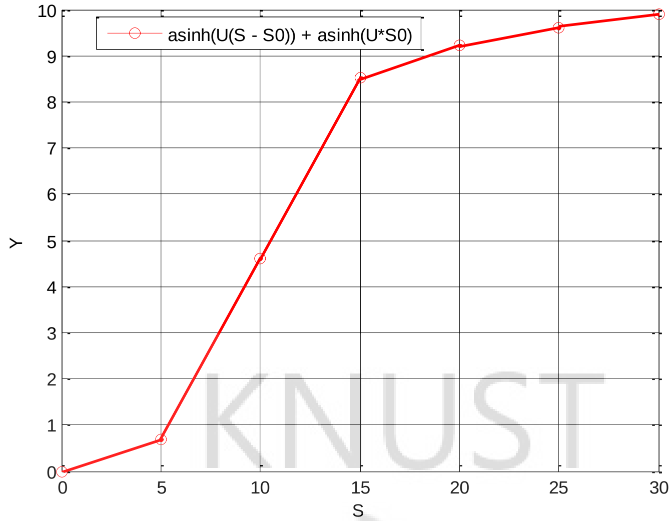
We can therefore form a structure of transformation if stretching around more than one point is needed

$$\begin{aligned} y &= \varphi_1(x)\ \ z = \varphi_2(y) \Leftrightarrow z = \varphi_2(\varphi_1(x)) \\ x &= \varphi_1(y)\ \ y = \varphi_2(z) \Leftrightarrow x = \varphi_1(\varphi_2(z)) \end{aligned} \tag{3.77}$$

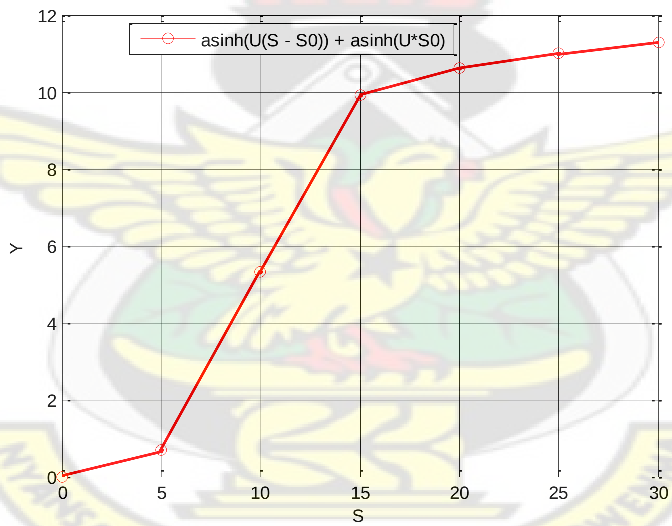
Figure 3.13: Representation of transformation function of (3.73)



$\mu = 1$



$\mu = 5$



$\mu = 10$

3.5.3 Numerical time integration

A system of ordinary time differential equation, which may have been transformed after discretization of the equation reads

$$\frac{du}{dt} = Au + b(t) + f(t)$$
$$u(0) = \phi \tag{3.78}$$

Where A = generated matrix by the second or fourth order scheme. The vector b contains boundary conditions, $f(t)$ being the source function and ϕ being the initial transformed condition in the equation (3.35). It is known that PDEs of different types which occur in areas such as chemical kinetics control theory; electronic solid mechanics etc. are stiff due to the presence of different scales. To solve such a problem however, there is the need to use implicit methods which is unconditionally stable. A second and fourth order accurate scheme in time is thus employed. Given M intervals the time intervals on M divided and the time step defined as $k = \frac{T}{M}$. To get the second $(0(h^2 + k^2))$ or fourth $(0(h^4 + k^4))$ order approximation of the solution, a time integration of $0(k^2)$ or $0(k^4)$ is required.

3.5.3.1 Crank Nicolson method

Crank Nicolson implicit finite difference method is an average of implicit and explicit methods. This method is obtained by taking the forward difference from equation (3.37) and backward difference from equation (3.38) and averaging. This method has become a very popular finite scheme for approximating the Black-Scholes equation.

In deriving Crank Nicholson scheme we have from [1] the scheme reads

$$(I - \frac{1}{2}kA)u^{j+1} = (I + \frac{1}{2}kA)u^j + \frac{1}{2}k(b^j + b^{j+1} + f^j + f^{j+1}) \tag{3.79}$$

With I being the identity matrix and u^j the vector evaluation at time $t = jk$.

One draw-back of the Crank Nicholson scheme is that it responds to damp discontinuous in the initial condition with oscillations which are weakly damped and therefore may persist for a long time. To resolve this however a special damping initialization steps are necessary.

3.5.3.2 Backward difference scheme

This is another well-known implicit scheme in the sense that the unknown variable cannot be easily expressed in terms of the known. This scheme is also $0(k^2)$ that is second order as seen in [8]

$$(\frac{3}{2}I - kA)u^{j+1} = 2u^j - \frac{1}{2}u^{j+1} + k(b^{j+1} + f^{j+1}) \tag{3.80}$$

Due to its two step method, an initialization step is always necessary. In (section 3.4.3.5) the approximate initialization step is described. The fourth order scheme however reads

$$(\frac{25}{12}I - kA)u^{j+1} = 4u^j + 3u^{j-1} - \frac{4}{3}u^{j-2} + k(b^{j+1} + f^{j+1}) \tag{3.81}$$

Three steps are needed for this method

3.4.3.3 Implicit Runge-Kutta methods

Unlike the fourth order scheme described in (3.81) of (section 3.5.3.2) which is a multi-step method. Runge-Kutta methods are single-step but multi-stage methods, this simply means that more calculations are performed for one time step, but all with explicitly known time value. Runge-Kutta method seeks to improve upon the low accuracy level of the Euler Method. The most readily used Runge-Kutta method is the four function evaluations per time step and this depends on Simpsons Quadrature rule of integration. The use of the implicit method here is because of it more stability and accuracy as compared to the explicit method. The equation (3.78) can therefore be rewritten as

$$\frac{du}{dt} = z(t,u) \tag{3.82}$$

$$u(0) = u_0 = \phi_0 \tag{3.83}$$

There are several schemes of Runge-Kutta, however as can be seen in [2], the dynamics of Runge-Kutta methods, the general expression of the Runge-Kutta scheme is written as

$$\begin{aligned} y_i &= u^j + k\sum_{m=1}^{q} p_{im} z(t^j + c_m k, y_m) \\ u^{j+1} &= u^j + k\sum_{m=1}^{q} w_m z(t^j + c_m k, y_m) \end{aligned} \tag{3.84}$$

In the equation (3.84) we have to solve a system of equations with unknown vectors of size $N-1$. However LU decomposition can be applied to solve this problem if necessary. In this research work, the scheme to be used is the $0(k^4)$ Gauss-Legendre Scheme with parameters.

$$p = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{1}{6}\sqrt{3} \\ \frac{1}{4} + \frac{1}{6}\sqrt{3} & \frac{1}{4} \end{pmatrix} \tag{3.85}$$

$$c = \left(\frac{1}{2} - \frac{1}{6}\sqrt{3} \ldots\ldots \frac{1}{2} + \frac{1}{6}\sqrt{3}\right)^T$$

$$w = \left(\frac{1}{2} \ldots \frac{1}{2}\right)^T$$

3.5.3.4 Padé approximation

This is equations of an M^{th} degree polynomial over N^{th} degree polynomial that matches the Taylor series of a function of highest order accuracy.

From [2], the Padé method which starts from the Crank-Nicholson scheme reads

$$\sum_{k=0}^{N}(-1)^k \binom{N}{k}\frac{(M+N+k)!}{(M+N)!}h^k z_k(u^{j+1}) = \sum \binom{M}{k}\frac{(M+N-k)!}{(M+N)}h^k z_k(u^j) \qquad (3.86)$$

Is of order $M+N$. Thus the so called Padé (1,1) method reads

$$\frac{u^{j+1}-u}{k} = \frac{1}{2}(A(u^{j+1}+u^j)+b^j+b^{j+1}+f^j+f^{j+1})+0(k^2) \qquad (3.87)$$

That of Padé (1,2) reads

$$\frac{u^{j+1}-u}{k} = \frac{1}{3}(A(u^j+b^j+f^j)+\frac{2}{3}(Au^{j+1}+b^{j+1}f^{j+1})-\frac{1}{6}k(A(Au^{j+1}+b^{j+1})+f^{j+1}+\frac{db^{j+1}}{dt}+\frac{df^{j+1}}{dt})+0(k^3) \qquad (3.88)$$

And the scheme Padé (2,2) reads

$$\frac{u^{j+1}-u}{k} = \frac{1}{2}(Au^j+b^j+f^j+Au^{j+1}+b^{j+1}f^{j+1})+\frac{1}{12}k(A(Au^j+b^j+f^j)+\frac{db^j}{dt}+\frac{df^j}{dt})$$
$$-\frac{1}{12}k(A(Au^{j+1}+b^{j+1}+f^{j+1})+\frac{db^{j+1}}{dt}+\frac{df^{j+1}}{dt})+0(k^4) \qquad (3.89)$$

Here the derivative of vector b and the source vector f can easily be computed. The Padé methods are implicit and also another multi-stage formula for ordinary differential equation as it is seen in the form of (3.82).

3.5.3.5 Initialization

An initialization step is needed for the backward formula for the second order accuracy. To do this, one step of Crank Nicolson method should be taken. In the case of non-smooth initial conditions, such a condition should be treated by another initialization step, which thus exhibit damping. For that method, two step of the Backward Euler scheme of the first order can apply as seen below.

$$(I - kA)u^{j+1} = u^j + k(b^{j+1} + f^{j+1}) \tag{3.90}$$

Several possibilities for the initialization of backward difference formula of the fourth order can be given. However the multi-staging of the implicit Padé method and the Runge-Kutta method makes this single step method complicated. To handle the Runge-Kutta method, matrix equations have to be inverted. While with Padé approximation both matrix multiplication as well as inversion must be performed. Combining the fourth order method describe in the previous sections is the best way to handle this methods. First we take the first step of the Guass Lagendra method and continue with the fourth order backward difference formula. Another approach will be to start with the second order backward difference scheme and continue by the third order and finally the fourth order. The third order backward difference formula reads

$$(\frac{11}{6}I - kA)u^{j+1} = 3u^j + \frac{3}{2}u^{j-1} - \frac{1}{3}u^{j-3} + k(b^{j+1} + f^{j+1}) \tag{3.91}$$

For this research work, we prefer the combination of the Backward Euler (3.90) and Crank-Nicolson (3.79) for the second order solution and the combination of Runge-Kutta (3.84) and with the fourth order backward difference formula for the fourth order solution.

3.5.4 Numerical differentiation: The Greeks

The Greeks are computed by numerical differentiation as mentioned earlier on in (section 3.4). This is done by using the different scheme for discretization of the Black-Scholes equation. Given the solution of the problem $u(x_i, t)$, then the derivative with the second order accurate scheme is given by

$$\Delta_i = \frac{\partial u}{\partial x} = \frac{u_{i+1} - u_{i-1}}{2h} \tag{3.92}$$

$$\Gamma_i = \frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{3.93}$$

and the fourth order accurate scheme reads

$$\Delta_i = \frac{\partial u}{\partial x} = \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} - u_{i-2}}{12h} \tag{3.94}$$

$$\Gamma_i = \frac{\partial^2 u}{\partial x^2} = \frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12h^2} \tag{3.95}$$

The derivatives at the boundaries are known for Black-Scholes, but with the fourth order accurate scheme, one need the backward difference formula to find the derivatives in the points u_1 and u_{N-1}. The derivative in those points reads

$$\Delta_1 = \frac{\partial u}{\partial x} = \frac{-3u_0 - 10u_1 + 18u_2 - 6u_3 + u_4}{12h} \tag{3.96}$$

$$\Gamma_1 = \frac{\partial^2 u}{\partial x^2} = \frac{10u_0 - 15u_1 - 4u_2 + 14u_3 - 6u_4 + u_5}{12h^2} \tag{3.97}$$

$$\Delta_{N-1} = \frac{\partial u}{\partial x} = \frac{-3u_N - 10u_{N-1} + 18u_{N-2} - 6u_{N-3} + u_{N-4}}{12h} \tag{3.98}$$

$$\Gamma_{N-1} = \frac{\partial^2 u}{\partial x^2} = \frac{10u_N - 15u_{N-1} - 4u_{N-2} + 14u_{N-3} - 6u_{N-4} + u_{N-5}}{12h^2} \tag{3.99}$$

$$\Delta_{T,i} = \frac{d\varphi}{dy_i}\Delta_i \tag{4.0}$$

$$\Gamma_{T,i} = \left(\frac{d\varphi}{dy}\right)_i^{-2}\Gamma_i - \frac{d^2\varphi}{dy^2}\left(\frac{d\varphi}{dy}\right)_i^{-3}\Delta_i \tag{4.1}$$

When the transformation is used, the derivative of the same scheme will now read as seen in (4.0) and (4.1), then these derivatives of the transformation are known in each point.

3.6 Validation of discrete systems

In validating our discrete system, some reference tests are performed with diffusion type of equation and an analytic solution in a polynomial form. Here the fourth order scheme is tested, where the numerical error is given the maximum norm as $\|.\|_\infty$ or L^2 - norm as seen below.

$$\varepsilon_\infty = \|u - u_{ex}\|_\infty = \max\left\{\left|u_i - u_{ex,i}\right| : i = 1 \ldots N\right\} \tag{4.2}$$

$$\varepsilon_2 = \|u - u_{ex}\|_2 = \frac{1}{N}\sqrt{\sum_{i=1}^{N}(u_i - u_{ex,i})^2} \tag{4.3}$$

where u is the numerical solution and u_{ex} is the exact solution.

NB: $\left|u_i - u_{ex,i}\right|$ is the error made at the point $x = a + ih$ on the grid with N point. The step size h is given by $h = \frac{(b-a)}{N}$ with the interval $[a,b]$. In applying the discretization describe earlier in (section 3.5) to the Black-Scholes equation, the value a will be set to zero and all calculation shall be performed in the interval $[0,b]$.

3.6.1 Constant coefficients test problem

Parabolic differential equation with constant coefficient should be the first problem to test the fourth order discretization and space and time.

$$\begin{aligned}
&\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}-u(x,t)+f(x,t)\\
&u(0,t)=-t^5\\
&u(b,t)=(b-t)^5\\
&u(x,0)=x^5\\
&f(x,t)=(x-t)^5-10(x-t)^4-10(x-t)^3
\end{aligned} \tag{4.4}$$

The exact solution is given by

$$u(x,t)=(x-t)^5 \tag{4.5}$$

For $T=1, b=1, N=10, M=10$ with extrapolation at the boundaries.

The matrix for (4.4) reads

$$\begin{pmatrix}
-106 & 60 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
60 & -126 & 73.33 & -5 & 0 & 0 & 0 & 0 & 0\\
-3.33 & 60 & -126 & 73.33 & -5 & 0 & 0 & 0 & 0\\
0 & -3.33 & 60 & -126 & 73.33 & -5 & 0 & 0 & 0\\
0 & 0 & -3.33 & 60 & -126 & 73.33 & -5 & 0 & 0\\
0 & 0 & 0 & -3.33 & 60 & -126 & 73.33 & -5 & 0\\
0 & 0 & 0 & 0 & -3.33 & 60 & -126 & 73.33 & -5\\
0 & 0 & 0 & 0 & 0 & -3.33 & 60 & -126 & 73.33\\
0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 40 & -96
\end{pmatrix}$$

A test with different numbers of space and time steps was performed for (4.4). The convergence order in both infinite norm $\|.\|_\infty$ and L^2 -norm as well as the errors are presented in table 3.1 below.

Table 3.1: Results of test problem (4.4) with b = 1; T = 1 and extrapolation at the boundaries.

$Grid(NXM)$	$\|u-u_{ex}\|\infty$	$Conv_\infty$	$\|u-u_{ex}\|2$	$Conv_2$
$10\ X\ 10$	$8.05\ X\ 10^{-4}$		$5.49\ X\ 10^{-4}$	
$20\ X\ 20$	$5.91\ X\ 10^{-5}$	13.64	$3.79\ X\ 10^{-5}$	14.47
$40\ X\ 40$	$4.06\ X\ 10^{-6}$	14.56	$2.50\ X\ 10^{-6}$	15.16
$80\ X\ 80$	$2.67\ X\ 10^{-7}$	15.19	$1.61\ X\ 10^{-7}$	15.54

By applying backward difference scheme at the boundaries in the first and last row the matrix form changes as shown below and the result in table 3.2

$$\begin{pmatrix}
-71.83 & -1.67 & 53.33 & -24.17 & 4.17 & 0 & 0 & 0 & 0 \\
60 & -126 & 73.33 & -5 & 0 & 0 & 0 & 0 & 0 \\
-3.33 & 60 & -126 & 73.33 & -5 & 0 & 0 & 0 & 0 \\
0 & -3.33 & 60 & -126 & 73.33 & -5 & 0 & 0 & 0 \\
0 & 0 & -3.33 & 60 & -126 & 73.33 & -5 & 0 & 0 \\
0 & 0 & 0 & -3.33 & 60 & -126 & 73.33 & -5 & 0 \\
0 & 0 & 0 & 0 & -3.33 & 60 & -126 & 73.33 & -5 \\
0 & 0 & 0 & 0 & 0 & -3.33 & 60 & -126 & 73.33
\end{pmatrix}$$

0 0 0 0 4.17 -25.83 63.33 -31.67 -55.17

Table 3.2: Results of test problem (4.4) with b = 1; T = 1 and backward difference at the boundaries.

$Grid(NXM)$	$\|u-u_{ex}\|\infty$	$Conv_\infty$	$\|u-u_{ex}\|2$	$Conv_2$
$10\ X\ 10$	$5.20\ X\ 10^{-4}$		$3.59\ X\ 10^{-4}$	
$20\ X\ 20$	$3.42\ X\ 10^{-5}$	15.19	$2.43\ X\ 10^{-5}$	14.77
$40\ X\ 40$	$2.16\ X\ 10^{-6}$	15.82	$1.56\ X\ 10^{-6}$	15.63
$80\ X\ 80$	$1.35\ X\ 10^{-7}$	15.96	$9.82\ X\ 10^{-8}$	15.85

From table 3.1 and 3.2 it could be notice that the errors with backward difference scheme at the boundaries is smaller than that with extrapolation.

3.6.2 Non-constant coefficients test problem

Due to the non-constant coefficient of the Black-Scholes equation, there is a need for a test to be performed with the non-constant coefficients.

The parabolic equation used for the test is as below

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} - u(x,t) + f(x,t) \\
&u(0,t) = -t^5 \\
&u(b,t) = (b-t)^5 \\
&u(x,0) = x^5 \\
&f(x,t) = (x-t)^5 - 5(x-t)^4 - 5x(x-t)^4 - 10(x-t)^3
\end{aligned} \tag{4.6}$$

$u(x,t) = (x-t)^5$ gives the solution of (4.6). In both infinite norm $\|.\|_\infty$ and L^2 -norm, the order of convergence and errors are presented with backward difference at the boundaries as seen in the table 3.3

Table 3.3: Results of test problem (4.6) with b = 1; T = 1 and backward differencing at boundaries.

$Grid(NXM)$	$\|\|u-u_{ex}\|\|\infty$	$Conv_\infty$	$\|\|u-u_{ex}\|\|2$	$Conv_2$
$10\ X\ 10$	$1.17\ X\ 10^{-3}$		$7.97\ X\ 10^{-4}$	
$20\ X\ 20$	$8.58\ X\ 10^{-5}$	13.59	$5.98\ X\ 10^{-5}$	13.32
$40\ X\ 40$	$5.71\ X\ 10^{-6}$	15.03	$4.02\ X\ 10^{-6}$	14.86
$80\ X\ 80$	$3.67\ X\ 10^{-7}$	15.56	$2.60\ X\ 10^{-7}$	15.46

Just like the previous result in table 3.1 and 3.2, the convergence behavior is again asymptotically of fourth order. However the errors are larger than previous equations.

3.6.3 Transformed test problem

With grid transformation as describe earlier on in (section 3.4.2)**,** the matrix for $N = 10$ space and $M = 10$ time steps, with the given parameters $T = 1, b = 1, \mu = 5, x_0 = 0.5$, the extrapolation at the boundaries reads

$$\begin{pmatrix} -3.03 & 2.03 & -0.23 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.37 & -9.88 & 7.10 & -0.59 & 0 & 0 & 0 & 0 & 0 \\ -0.30 & 8.94 & -25.66 & 17.36 & -1.35 & 0 & 0 & 0 & 0 \\ 0 & -0.93 & 33.33 & -49.50 & 31.38 & -2.31 & 0 & 0 & 0 \\ 0 & 0 & -2.38 & 33.33 & -72.98 & 43.45 & -3.03 & 0 & 0 \\ 0 & 0 & 0 & -2.38 & 41.27 & -84.23 & 47.50 & -3.16 & 0 \\ 0 & 0 & 0 & 0 & -2.47 & 40.82 & -79.96 & 43.40 & -2.79 \\ 0 & 0 & 0 & 0 & 0 & -2.14 & 34.66 & -66.85 & 35.58 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.09 & 20.08 & -41.96 \end{pmatrix}$$

and that of the Backward difference reads

$$\begin{pmatrix} -2.97 & 1.81 & 0.11 & -0.22 & 0.06 & 0 & 0 & 0 & 0 \\ 2.37 & -9.88 & 7.10 & -0.59 & 0 & 0 & 0 & 0 & 0 \\ -0.30 & 8.94 & -25.66 & 17.36 & -1.35 & 0 & 0 & 0 & 0 \\ 0 & -0.93 & 33.33 & -49.50 & 31.38 & -2.31 & 0 & 0 & 0 \\ 0 & 0 & -2.38 & 33.33 & -72.98 & 43.45 & -3.03 & 0 & 0 \\ 0 & 0 & 0 & -2.38 & 41.27 & -84.23 & 47.50 & -3.16 & 0 \\ 0 & 0 & 0 & 0 & -2.47 & 40.82 & -79.96 & 43.40 & -2.79 \\ 0 & 0 & 0 & 0 & 0 & -2.14 & 34.66 & -66.85 & 35.58 \\ 0 & 0 & 0 & 0 & 1.72 & -10.35 & 24.32 & -7.68 & -26.32 \end{pmatrix}$$

Table 3.4 and 3.5 gives the result for the errors and order of convergence.

Table 3.4: Results of transformed test problem (4.6) with $T=1, b=1, \mu=5, x_0=0.5$ **and extrapolation at the boundaries**

$Grid(NXM)$	$\|u-u_{ex}\|\infty$	$Conv_\infty$	$\|u-u_{ex}\|2$	$Conv_2$
$10\ X\ 10$	$5.48\ X\ 10^{-4}$		$3.64\ X\ 10^{-4}$	
$20\ X\ 20$	$1.30\ X\ 10^{-4}$	4.21	$9.09\ X\ 10^{-5}$	4.00
$40\ X\ 40$	$1.46\ X\ 10^{-5}$	8.88	$1.04\ X\ 10^{-5}$	8.72
$80\ X\ 80$	$1.35\ X\ 10^{-6}$	10.81	$8.69\ X\ 10^{-7}$	12.01

Table 3.5 Results of transformed test problem (4.6) with $T=1, b=1, \mu=5, x_0=0.5$ **and backward differences**

$Grid(NXM)$	$\|u-u_{ex}\|\infty$	$Conv_\infty$	$\|u-u_{ex}\|2$	$Conv_2$
$10\ X\ 10$	$1.51\ X\ 10^{-3}$		$5.88\ X\ 10^{-4}$	
$20\ X\ 20$	$2.16\ X\ 10^{-4}$	6.97	$8.47\ X\ 10^{-5}$	6.93
$40\ X\ 40$	$1.82\ X\ 10^{-5}$	11.86	$8.16\ X\ 10^{-6}$	10.38
$80\ X\ 80$	$1.21\ X\ 10^{-6}$	5.01	$6.10\ X\ 10^{-7}$	13.38

For further test the influence of μ in our transformation is checked. The only parameter we vary in the same test problem is μ by choosing $\mu=1$ and $\mu=10$ since it has already been established in (section 3.4.2.1) that $\mu = 5$ in many cases yields a satisfactory value. Table 3.6 and 3.7 represents the convergence results.

Table 3.6 Results of transformed test problem (4.6) with $T = 1, b = 1, \mu = 1, x_0 = 0.5$ and backward differences

$Grid(NXM)$	$\|u - u_{ex}\|\infty$	$Conv_\infty$	$\|u - u_{ex}\|2$	$Conv_2$
$10\ X\ 10$	$1.22\ X\ 10^{-3}$		$7.99\ X\ 10^{-4}$	
$20\ X\ 20$	$9.18\ X\ 10^{-5}$	13.28	$6.20\ X\ 10^{-5}$	12.90
$40\ X\ 40$	$6.12\ X\ 10^{-6}$	15.00	$4.22\ X\ 10^{-6}$	14.66
$80\ X\ 80$	$3.92\ X\ 10^{-7}$	15.60	$2.74\ X\ 10^{-7}$	15.41

Table 3.7 Results of transformed test problem (4.6) with $T = 1, b = 1, \mu = 10, x_0 = 0.5$ and backward differences

$Grid(NXM)$	$\|u - u_{ex}\|\infty$	$Conv_\infty$	$\|u - u_{ex}\|2$	$Conv_2$
$10\ X\ 10$	$1.45\ X\ 10^{-3}$		$6.15\ X\ 10^{-4}$	
$20\ X\ 20$	$4.07\ X\ 10^{-4}$	3.55	$1.27\ X\ 10^{-4}$	4.85
$40\ X\ 40$	$4.48\ X\ 10^{-5}$	9.09	$1.51\ X\ 10^{-5}$	8.40
$80\ X\ 80$	$3.33\ X\ 10^{-6}$	13.47	$1.31\ X\ 10^{-6}$	11.49

As can been seen from the test experiment, it follows that equidistant grid converges in fourth order but for transformed grid this is observed mainly asymptotically. However errors made are fewer when a proper μ is chosen. From our result in table 5.5, 5.6 and 5.7 it could be seen that $\mu = 10$ have a lower convergence which is probably due to severe stretching.

For Black-Scholes, x_0 will be chosen close to E and the parameter μ will play an import role in determining the error and convergence. The absent of singularities makes the transformation not that useful in the test problems discussed in this chapter.

3.7 Special features of Black-Scholes PDE

Discussion of some topics of Black-Scholes will be looked at under this section in order to choose the proper difference scheme, transformation as well as choice of grid. In this discussion, the techniques to smoothen the final condition will also be discussed.

3.7.1 Difference method and time integration

Local refinement (see section 3.4.2) can be use near singularities in the final conditions. In doing this the time direction must first of all be transformed to obtain a forward difference problem with an initial condition. Letting τ be the new time, we have $\tau = T - t$. From (3.15) we have a new equation with our new time $\tau = T - t$

$$
\begin{aligned}
&\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta) S \frac{\partial C}{\partial S} - rC(S,\tau) \\
&C(0,\tau) = 0 \\
&C(S,\tau) = Se^{-\delta\tau} - Ee^{-r\tau} \\
&S(S,0) = \max(S-E,0) = \begin{cases} S-E, S \succ E \\ 0, S \prec E \end{cases}
\end{aligned}
\tag{4.7}
$$

This is a well posed system which can easily be solved by numerical schemes as described in (section 3.5). For easy work, the time τ will be replaced with t. Now applying the grid stretching transformation as in (3.73) to the initial condition, with x_0 which is chosen close to E for Black-Scholes, $x_0 = E$, thus $C(y,0) = \max(\frac{1}{\mu}\sinh(y - \sinh^{-1}\mu E), 0)$ (4.8)

In doing this the sharp edge in the initial condition of Vanilla options disappears while that of the exotic options shows that a discontinuous payoff remains a discontinuity in the transformed case.

3.7.2 Smoothing initial conditions

In many financial problems non-smooth initial conditions or data such as time, which can cause convergence problems for numerical method, exist. Some type of smoothing of that data must however be performed. In doing so the discontinuous initial condition must be treated carefully when applying a time integration scheme.

In [7] it was suggested that when discontinuities occur midway between grid points, by shifting the grid, accuracy increases. Using Crank-Nicolson method type, which can theoretically attain second order convergence for any volatility structure, it means that by applying one or more first order time steps as initialization for Crank-Nicolson, initial condition is smoothen. A possibility for initialization for fourth order time scheme is to start with Runge-Kutta method as seen in (section 3.5.3.3).

For our interest we will use this for the discontinuous initial condition.

3.6.3 Distance field boundary

Unlike numerical solution which has fixed boundaries, for Black-Scholes there is a boundary condition at infinity. What it means is that $S_{\max}$ must be chosen as large as possible, though this may be difficult for numeric scheme as the number of grid points may grow excessively.

In [8], the proper size of the domain is proposed after a careful analysis. The distance field boundary reads

$$S_{\max} = \max(2E, Exp(\sqrt{2\sigma^2 T \ln 100})) \tag{4.9}$$

σ Plays an important role in the relationship between E and $S_{\max}$. With (4.9), a solution can be computed with an accuracy of at least $100^{-1}E$ and the minimal size of the domain is $2E$. In practical cases however, brokers would like to have an accuracy of at least GH¢0.01, so experimental results from [12] yields the formula

$$S_{\max} = \max(RE, EExp(\sqrt{2\sigma^2 T \ln 100})) \tag{4.10}$$

In the case that volatility is high, equation (4.78) is also valid and $R > 2$ is necessary. In an equidistant grid, there will be many points without any financial interest in the region $S \in [2E, RE]$. As the number of point in $S \in [2E, RE]$ is minimized, the strength of the transformation is noticed.

3.7.4 Choice of grid

From Pooly,D. in [8], it was known that the exact position of a discontinuity in the initial condition related to the position of the grid point have an influence in increase accuracy. Test result in [8] shows that if E is not exactly between two grid points when applying a numerical scheme to a digital option described in (3.23), a satisfactory accuracy will not be obtained. There two algorithm that can be applied to get E on the right position.

3.7.4.1 E on the grid

A proposal method to get E on the grid point after transformation is made. Assume that the maximal value on the grid will be related to E, according to (4.9), in the sense that $S_{\max} = RE$, with $R \geq 2$ and assume a grid of N points. To satisfy both properties, $S_{\max}$ must be translated to the right side. In the equidistant grid, there are about $\frac{N}{R}$ points on the left side of E and the others are at the right side of E. The neighboring grid point to E reads

$$\eta = \frac{N}{R} \tag{4.11}$$

Thus the step size will be

$$h = \frac{E}{\eta} \tag{4.12}$$

The new maximal value of the grid is the $S_{\max} = Nh$

The transformation (3.73), which is not linear, requires another treatment. In the case of stretch grid, the property $S_{\max} = RE$ transform into $y_{\max} = q\varphi(E)$ with

$$q = \frac{\varphi(RE)}{\varphi(E)} \tag{4.13}$$

Combining (4.13) with (4.11), the nearest grid points reads

$$\eta T = \frac{N}{q} \tag{4.14}$$

And new step size will be $h = \varphi \frac{E}{\eta T}$ (4.15)

3.6.4.2 E between two grid points

Just a slight modification in the step size definition, is enough to get E in the middle between two grid points say (x_n and x_{n+1}). The points now on the grid are $E-\frac{1}{2}h$ and $E+\frac{1}{2}h$ from (4.12)

$$h=\frac{E+\frac{1}{2}}{\eta}\Leftrightarrow h=\frac{E}{\eta}(1-\frac{1}{2\eta})^{-1} \tag{4.16}$$

And with transformation and (4.15), we have

$$h=\frac{\varphi(E)+\frac{1}{2}h}{\eta T}\Leftrightarrow h=\frac{\varphi(E)}{\eta T}(1-\frac{1}{2\eta T})^{-1} \tag{4.17}$$

3.7.5 Lagrange interpolation

With the transformation, many points are in the region around E, but sometimes values in the neighborhood of E which are not on the grid must be calculated. A typical example is the requirement of option values near the present asset price S_0. An appropriate way to do this is through Lagrange interpolation. The interpolation polynomial for calculating the point x given the set $[x_i : i=1.....n]$ and $x \neq x_i : \forall_i$ is given by

$$p(x)=\sum_{i=1}^{n}x_i\prod_{\substack{i=1\\i\neq k}}^{n}\frac{x-x_k}{x_i-x_k} \tag{4.18}$$

For a second order interpolation, only two points $x \in [x_1, x_2]$ are necessary

$$p(x)=x_1\frac{x-x_2}{x_1-x_2}+x_2\frac{x-x_1}{x_2-x_1} \tag{4.19}$$

And for fourth order

$$p(x) = x_1 \frac{x-x_2}{x_1-x_2}\frac{x-x_3}{x_1-x_3}\frac{x-x_4}{x_1-x_4} + x_2 \frac{x-x_1}{x_2-x_1}\frac{x-x_3}{x_2-x_3}\frac{x-x_4}{x_2-x_4} + x_3 \frac{x-x_1}{x_3-x_1}\frac{x-x_2}{x_3-x_2}\frac{x-x_4}{x_3-x_4} + x_4 \frac{x-x_1}{x_4-x_1}\frac{x-x_2}{x_4-x_2}\frac{x-x_3}{x_4-x_3} \tag{4.20}$$

We can only apply this formula to the unknown $y(x)$ and is also necessary for volatility search.

3.7.6 Implied volatility

Option prices obtained from the Black-Scholes model are functions of the parameters: time t, the strike price E, the risk-free rate r, the current underlying price S and the market volatility σ. The only unknown parameter that is not directly observed from the market is the volatility of the market and it has to be estimated. Estimation of volatility from the historical data of the underlying is called the ***historical volatility***. Black-Scholes model namely the volatility, there is a one-to-one correspondence between the value of any financial derivative contract, such as an option, and the volatility of its underlying asset. In general, the more volatile the asset, the more the derivative contract is worth. Thus, when a market has set the price for a contract, it is often the case that this price corresponds to a unique implied volatility. Suppose we use the Black-Scholes model to infer the volatility used by option traders to price the option. We search for volatility such that the model represents an option price that corresponds to the market price. The volatility obtained this way is called the ***implied volatility***. The Black-Scholes model assumes that the implied volatility is constant and homogeneous for options on the same underlying with different strikes and maturities. However, in practice the implied volatility of call or put options at a given time t is a function of the strike price and the time to maturity T ¡t.

often deeply out of the money or deeply in the money options have significantly higher implied volatilities than options at the money.

3.7.6.1 The bisection method

Below are the procedures to calculate the solution of σ_{imp}

1. Take two values of σ : σ_{high} high and σ_{low} low in such a way that if C_{market} = $C(\sigma_{imp})$

then $C(\sigma_{imp})$ < C(σ_{high}) and C(σ_{imp}) > C(σ_{low}) due to the monotonicity of C(σ), we can setup a root finding procedure. It is trivial that $\sigma_{imp} \in (\sigma_{low}; \sigma_{high})$.

2. Take $\sigma_{mid} = \frac{1}{2}(\sigma_{low} + \sigma_{high})$

3. Calculate $C(\sigma_{mid})$

4. Calculate Q = (C_{market} - $C(\sigma_{mid})$) X (C_{market} - C (σ_{high})).

5. If Q < 0 then $\sigma_{low} = \sigma_{mid}$, otherwise, if Q > 0 then $\sigma_{high} = \sigma_{mid}$ and repeat from point 2 until the desired accuracy is reached

This slowly converging method for practical application is very robust

3.7.6.2 Inverse quadratic interpolation method

Considering the general Black-Scholes equation solution as a function of σ and subtracting the unknown market price, $V(\sigma) - V_{market}$. Where $V(\sigma)$ is nonlinear and the derivative is generally not known. The inverse quadratic interpolation method to determine σ is as follows

1. Choose three σ's , called σ_a, σ_b and σ_c

2. Calculate $V_a = V(\sigma_a) - V_{market}$, $V_b = V(\sigma_b) - V_{market}$ and $V_c = V(\sigma_c) - V_{market}$

3. Define $u = \frac{V_b}{V_c}, v = \frac{V_b}{V_a}$ and $w = \frac{V_a}{V_c}$

4. Define $p = V(W(u-w)(\sigma_c - \sigma_b) - (1-u)(\sigma_b - \sigma_a)$ and $q = (u-1)(v-1)(w-1)$

5. Then $\sigma_c = \sigma_a, \sigma_a = \sigma_b, \sigma_b = \sigma_b + \frac{p}{q}$ and $V_c = V_a, V_a = V_b$ and compute the new iterant

$V_b = V(\sigma_b) - V_{market}$

6. If $|V_b| \prec \varepsilon$ then $\sigma_{market} = \sigma_b$. Otherwise repeat from 3

The advantage here is that only one calculation of the option price must be done.

CHAPTER FOUR

NUMERICAL RESULT OF OPTION PRICING

4.1 Introduction

In this chapter, the numerical experiment will be related to option pricing problem talk about in chapter 3 and also the search for volatility. This experiment or better still numerical solution is heavily based on the European option style as the reference option. The values assign to the option pricing parameters that will be used is as below

Strike price (E) = 15

Volatility (σ) = 0.3

Interest rate (r) = 0.04

Dividend payment (δ) = 0.02

Maturity time in half year (T) = 0.5

Any change in parameter along the line will be explicitly stated.

4.2 European Vallina Options

The computation of the plain vanilla call gives some insight in the properties of the numerical scheme. The availability of analytic solution makes it possible for the properties of the numerical scheme to be investigated. As can be seen in the previous section 3.6 of chapter 3, the use of backward difference scheme will be a better option since it gives a better accuracy at the boundaries as compare to extrapolation. The available analytical solution is the basis for comparison as the numerical solutions are computed. These numerical solutions are performing on equidistant grids, transformed grids, and the European put option. The error reduction factors are defined in the tables below. The aim for accuracy is for small number of grid point not greater than 80 x 80 points.

4.2.1 Equidistant grid

Here a test result base on Crank Nicolson scheme for both second and fourth order accuracy is performed. Table 4.1, 4.2, 4.3, 4.4 shows the two test result for both second and fourth order accuracy.

Table 4.1: Crank Nicolson solution of the European calls with $S_{\max}$ and E on a grid point with $R = 2$ using equation (4.10)

$Grid(NXM)$	$\|\|C - C_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$1.68\ X\ 10^{-1}$		$3.03\ X\ 10^{-2}$		$3.22\ X\ 10^{-2}$	
$20\ X\ 20$	$3.55\ X\ 10^{-2}$	4.73	$1.01\ X\ 10^{-2}$	2.99	$6.19\ X\ 10^{-3}$	5.21
$40\ X\ 40$	$8.57\ X\ 10^{-3}$	4.15	$2.78\ X\ 10^{-3}$	3.64	$1.55\ X\ 10^{-3}$	3.98
$80\ X\ 80$	$2.13\ X\ 10^{-3}$	4.02	$7.05\ X\ 10^{-4}$	3.95	$3.80\ X\ 10^{-4}$	4.09

Table 4.2: Crank Nicolson solution of the European calls with $S_{\max}$ and E between two grid points with $R = 2$ using equation (4.10)

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_{\infty}$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_{\infty}$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$4.46\ X\ 10^{-2}$		$1.53\ X\ 10^{-2}$		$2.12\ X\ 10^{-3}$	
$20\ X\ 20$	$1.10\ X\ 10^{-2}$	4.04	$2.95\ X\ 10^{-3}$	5.19	$3.13\ X\ 10^{-3}$	0.68
$40\ X\ 40$	$2.60\ X\ 10^{-3}$	4.24	$8.07\ X\ 10^{-4}$	3.65	$7.82\ X\ 10^{-4}$	4.01
$80\ X\ 80$	$6.33\ X\ 10^{-4}$	4.11	$2.47\ X\ 10^{-4}$	3.27	$1.91\ X\ 10^{-4}$	4.09

Table 4.3: Fourth order solution of European call with E on a grid point using equation (4.10)

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_{\infty}$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_{\infty}$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$3.30\ X\ 10^{-1}$		$3.72\ X\ 10^{-2}$		$2.27\ X\ 10^{-2}$	
$20\ X\ 20$	$7.46\ X\ 10^{-2}$	4.42	$1.15\ X\ 10^{-2}$	3.22	$1.06\ X\ 10^{-2}$	2.14
$40\ X\ 40$	$1.39\ X\ 10^{-2}$	5.35	$3.04\ X\ 10^{-3}$	3.80	$1.49\ X\ 10^{-3}$	7.11
$80\ X\ 80$	$3.41\ X\ 10^{-3}$	4.09	$7.88\ X\ 10^{-4}$	3.86	$3.58\ X\ 10^{-4}$	4.15

Table 4.4: Fourth order solution of European call with E between two grid points using equation (4.10)

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_\infty$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_\infty$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_\infty$
$10\ X\ 10$	$1.60\ X\ 10^{-1}$		$9.51\ X\ 10^{-2}$		$2.23\ X\ 10^{-2}$	
$20\ X\ 20$	$3.70\ X\ 10^{-2}$	4.32	$1.71\ X\ 10^{-2}$	5.57	$3.65\ X\ 10^{-3}$	6.12
$40\ X\ 40$	$7.12\ X\ 10^{-3}$	5.19	$2.01\ X\ 10^{-3}$	8.48	$6.45\ X\ 10^{-4}$	5.66
$80\ X\ 80$	$1.75\ X\ 10^{-3}$	4.07	$4.34\ X\ 10^{-4}$	4.64	$1.74\ X\ 10^{-4}$	3.70

The test result for the second order accuracy indicates in table 4.2 where E is place exactly between two grid points that, there is a loss of convergence. Therefore apply $R=3$ just as in that of the fourth order accuracy, the table shows the result of retained convergence.

Table 4.5: Crank Nicolson solution of European calls with E between two grid points with $R=3$ using equation (4.10)

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_\infty$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_\infty$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_\infty$
$10\ X\ 10$	$1.33\ X\ 10^{-1}$		$9.74\ X\ 10^{-2}$		$2.13\ X\ 10^{-2}$	
$20\ X\ 20$	$3.31\ X\ 10^{-2}$	4.02	$9.60\ X\ 10^{-3}$	10.14	$6.22\ X\ 10^{-3}$	3.42
$40\ X\ 40$	$6.38\ X\ 10^{-3}$	5.18	$1.73\ X\ 10^{-3}$	5.55	$1.88\ X\ 10^{-3}$	3.30
$80\ X\ 80$	$1.53\ X\ 10^{-3}$	4.18	$4.94\ X\ 10^{-4}$	3.50	$4.59\ X\ 10^{-4}$	4.10

Therefore the result in table 4.5 should be preferred. Also the result for the fourth order accuracy as seen in table 4.3 and table 4.4 is not of any highest accuracy as compare to the second order accuracy on the equidistant grid. It follows that the fourth order accuracy is not performing better than the second order. At the point of non-differentiability, where $S = E$, the error and convergence error as shown in table 4.6 indicate a better accuracy if E is place between two grid point.

Table 4.6: Crank Nicolson solution of a call point $S = E$ with $R = 2$ using equation (4.10)

$Grid(NXM)$	$error\ in\ E$ $E\ on\ a\ grid\ point$	$Conv_{\infty}$	$error\ in\ E$ $E\ on\ a\ grid\ point$	$Conv_{\infty}$
$10\ X\ 10$	$1.68\ X\ 10^{-1}$		$4.32\ X\ 10^{-2}$	
$20\ X\ 20$	$3.55\ X\ 10^{-2}$	4.73	$1.94\ X\ 10^{-3}$	22.31
$40\ X\ 40$	$8.57\ X\ 10^{-3}$	4.15	$1.87\ X\ 10^{-4}$	10.34
$80\ X\ 80$	$2.13\ X\ 10^{-3}$	4.03	$3.12\ X\ 10^{-5}$	6.02

4.2.2 Grid transformation

Under this section only fourth order scheme will be considered since second order just remains second order accurate after transformation. In table 4.7 to table 4.9 the transformation different values of μ can be seen. Just as it was with the equidistant grid, the $S_{\max}$ is determine from (4.10) with $R = 3$. The values for μ is given as $\mu = 1, 5, and 10$.

Table 4.7: Fourth order solution of the European call with $R = 3$ and $\mu = 1$ using (4.10)

$Grid(NXM)$	$\|\|C - C_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_\infty$
$10\ X\ 10$	$1.05\ X\ 10^{-2}$		$1.94\ X\ 10^{-2}$		$6.30\ X\ 10^{-3}$	
$20\ X\ 20$	$1.05\ X\ 10^{-3}$	10.04	$3.14\ X\ 10^{-3}$	6.16	$1.32\ X\ 10^{-3}$	4.78
$40\ X\ 40$	$9.33\ X\ 10^{-5}$	11.24	$2.92\ X\ 10^{-4}$	10.78	$9.69\ X\ 10^{-5}$	13.61
$80\ X\ 80$	$2.52\ X\ 10^{-5}$	3.70	$2.55\ X\ 10^{-5}$	11.42	$8.89\ X\ 10^{-6}$	10.90

Table 4.8: Fourth order solution of the European call with $R = 3$ and $\mu = 5$ using (4.10)

$Grid(NXM)$	$\|\|C - C_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_\infty$
$10\ X\ 10$	$1.08\ X\ 10^{-1}$		$7.77\ X\ 10^{-2}$		$2.67\ X\ 10^{-2}$	
$20\ X\ 20$	$6.44\ X\ 10^{-3}$	16.77	$8.76\ X\ 10^{-3}$	8.86	$2.75\ X\ 10^{-3}$	9.70
$40\ X\ 40$	$4.03\ X\ 10^{-4}$	15.96	$8.49\ X\ 10^{-4}$	10.32	$3.71\ X\ 10^{-4}$	7.42
$80\ X\ 80$	$2.79\ X\ 10^{-5}$	14.44	$8.24\ X\ 10^{-5}$	10.30	$3.34\ X\ 10^{-5}$	11.13

Table 4.9: Fourth order solution of the European call with $R=3$ and $\mu=10$ using (4.10)

$Grid(NXM)$	$\|\|C-C_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Delta-\Delta_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Gamma-\Gamma_{ex}\|\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$2.28\ X\ 10^{-1}$		$1.48\ X\ 10^{-1}$		$3.92\ X\ 10^{-2}$	
$20\ X\ 20$	$1.28\ X\ 10^{-2}$	17.85	$9.09\ X\ 10^{-3}$	16.25	$4.20\ X\ 10^{-3}$	9.34
$40\ X\ 40$	$7.76\ X\ 10^{-4}$	16.44	$1.56\ X\ 10^{-3}$	5.82	$4.49\ X\ 10^{-4}$	9.36
$80\ X\ 80$	$4.90\ X\ 10^{-5}$	15.85	$1.41\ X\ 10^{-4}$	11.03	$4.00\ X\ 10^{-5}$	11.21

From table 4.7 to 4.9 the error for 20 x 20 points is less than 0.01 with the transformation for $\mu=1,5$. This result is satisfactory. With $\mu=5$ and E place on a grid and between exactly two grid points, table 4.10 and 4.11 shows the result of the next test.

Table 4.10: Fourth order solution of the European call with E on a grid point

$Grid(NXM)$	$\|\|C-C_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Delta-\Delta_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Gamma-\Gamma_{ex}\|\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$2.23\ X\ 10^{-1}$		$1.38\ X\ 10^{-1}$		$4.01\ X\ 10^{-2}$	
$20\ X\ 20$	$7.32\ X\ 10^{-3}$	30.42	$8.82\ X\ 10^{-3}$	15.68	$3.41\ X\ 10^{-3}$	11.78
$40\ X\ 40$	$4.33\ X\ 10^{-4}$	16.92	$1.08\ X\ 10^{-3}$	8.17	$3.72\ X\ 10^{-4}$	9.15
$80\ X\ 80$	$1.87\ X\ 10^{-5}$	23.11	$8.89\ X\ 10^{-5}$	12.15	$3.52\ X\ 10^{-5}$	10.58

Table 4.11: Fourth order solution of the European call with E between two grid points.

$Grid(NXM)$	$\|\|C - C_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$4.01\ X\ 10^{-1}$		$2.62\ X\ 10^{-1}$		$6.16\ X\ 10^{-2}$	
$20\ X\ 20$	$9.34\ X\ 10^{-3}$	42.99	$7.68\ X\ 10^{-3}$	34.12	$4.13\ X\ 10^{-3}$	14.91
$40\ X\ 40$	$5.34\ X\ 10^{-4}$	17.48	$1.26\ X\ 10^{-3}$	6.10	$3.43\ X\ 10^{-4}$	12.02
$80\ X\ 80$	$3.10\ X\ 10^{-5}$	17.22	$9.75\ X\ 10^{-5}$	12.92	$3.75\ X\ 10^{-5}$	9.15

For completeness, table 4.12 below presents the error and the convergence at the point of non-differentiability. With the available 3 outcome concerning the position of E on the grid points, only 20 x 20 points are sufficient for the required accuracy as seen in table 4.12. The number of grid points is highly influence by a higher μ. In figure 4.1 the stretch grid and solutions are displayed. It thus follows that the fourth order accuracy in combination with the transformation will be a preferred choice. With this accession the result of the next in the thesis will be based on fourth order accuracy.

Table 4.12: Fourth order solution of a call the point of non-differentiability and $\mu = 5$.

$Grid(NXM)$	$error\ in\ E$ $place\ E\ N.N.$	$Conv_{\infty}$	$error\ in\ E$ $E\ on\ a\ grid\ point$	$Conv_{\infty}$	$error\ in\ E$ $E\ between\ points$	$Conv_{\infty}$
$10\ X\ 10$	$8.47\ X\ 10^{-2}$		$1.78\ X\ 10^{-1}$		$3.27\ X\ 10^{-1}$	
$20\ X\ 20$	$5.10\ X\ 10^{-3}$	16.62	$5.75\ X\ 10^{-3}$	31.04	$7.44\ X\ 10^{-3}$	44.02
$40\ X\ 40$	$3.22\ X\ 10^{-4}$	15.83	$3.36\ X\ 10^{-4}$	17.11	$4.28\ X\ 10^{-4}$	17.40
$80\ X\ 80$	$2.29\ X\ 10^{-5}$	14.04	$1.31\ X\ 10^{-5}$	25.71	$2.55\ X\ 10^{-5}$	16.77

4.2.2.1 Unknown E position

Unlike the previous test where the position of the parameter E of the reference option is either on the grid or between two grids, here the position of the parameter E is unknown. Using (4.10), the parameter assumes an initial value of 0.15 and then increase with a constant value of 10 as μ decreases with the same value. The test result is seen in table 7.13 to 7.16

Table 4.13: European call with $E = 0.15$, and $\mu = 500, R = 3$.

$Grid(NXM)$	$\|\|C - C_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$1.08\ X\ 10^{-3}$		$7.77\ X\ 10^{-2}$		2.67	
$20\ X\ 20$	$6.44\ X\ 10^{-5}$	16.77	$8.76\ X\ 10^{-3}$	8.86	$2.75\ X\ 10^{-1}$	9.70
$40\ X\ 40$	$4.03\ X\ 10^{-6}$	15.96	$8.49\ X\ 10^{-4}$	10.32	$3.71\ X\ 10^{-2}$	7.42
$80\ X\ 80$	$2.79\ X\ 10^{-7}$	14.44	$8.24\ X\ 10^{-5}$	10.30	$3.34\ X\ 10^{-3}$	11.13

Table 4.14: European call with $E = 1.5$, and $\mu = 50, R = 3$.

$Grid(NXM)$	$\|\|C - C_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_{\infty}$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$1.08\ X\ 10^{-2}$		$7.77\ X\ 10^{-2}$		$2.67\ X\ 10^{-1}$	
$20\ X\ 20$	$6.44\ X\ 10^{-4}$	16.77	$8.76\ X\ 10^{-3}$	8.86	$2.75\ X\ 10^{-2}$	9.70
$40\ X\ 40$	$4.03\ X\ 10^{-5}$	15.96	$8.49\ X\ 10^{-4}$	10.32	$3.71\ X\ 10^{-3}$	7.42
$80\ X\ 80$	$2.79\ X\ 10^{-6}$	14.44	$8.24\ X\ 10^{-5}$	10.30	$3.44\ X\ 10^{-4}$	11.13

Table 4.15: European call with $E = 15$, and $\mu = 5, R = 3$.

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_\infty$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_\infty$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_\infty$
$10\ X\ 10$	$1.08\ X\ 10^{-1}$		$7.77\ X\ 10^{-2}$		$2.67\ X\ 10^{-2}$	
$20\ X\ 20$	$6.44\ X\ 10^{-3}$	16.77	$8.76\ X\ 10^{-3}$	8.86	$2.75\ X\ 10^{-3}$	9.70
$40\ X\ 40$	$4.03\ X\ 10^{-4}$	15.96	$8.49\ X\ 10^{-4}$	10.32	$3.71\ X\ 10^{-4}$	7.42
$80\ X\ 80$	$2.79\ X\ 10^{-5}$	14.44	$8.24\ X\ 10^{-5}$	10.30	$3.34\ X\ 10^{-5}$	11.13

Table 4.16: European call with $E = 150$, and $\mu = 0.5, R = 3$.

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_\infty$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_\infty$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_\infty$
$10\ X\ 10$	1.08		$7.77\ X\ 10^{-2}$		$2.67\ X\ 10^{-1}$	
$20\ X\ 20$	$6.44\ X\ 10^{-2}$	16.77	$8.76\ X\ 10^{-3}$	8.86	$2.75\ X\ 10^{-2}$	9.70
$40\ X\ 40$	$4.03\ X\ 10^{-3}$	15.96	$8.49\ X\ 10^{-4}$	10.32	$3.71\ X\ 10^{-3}$	7.42
$80\ X\ 80$	$2.79\ X\ 10^{-4}$	14.44	$8.24\ X\ 10^{-5}$	10.30	$3.44\ X\ 10^{-4}$	11.13

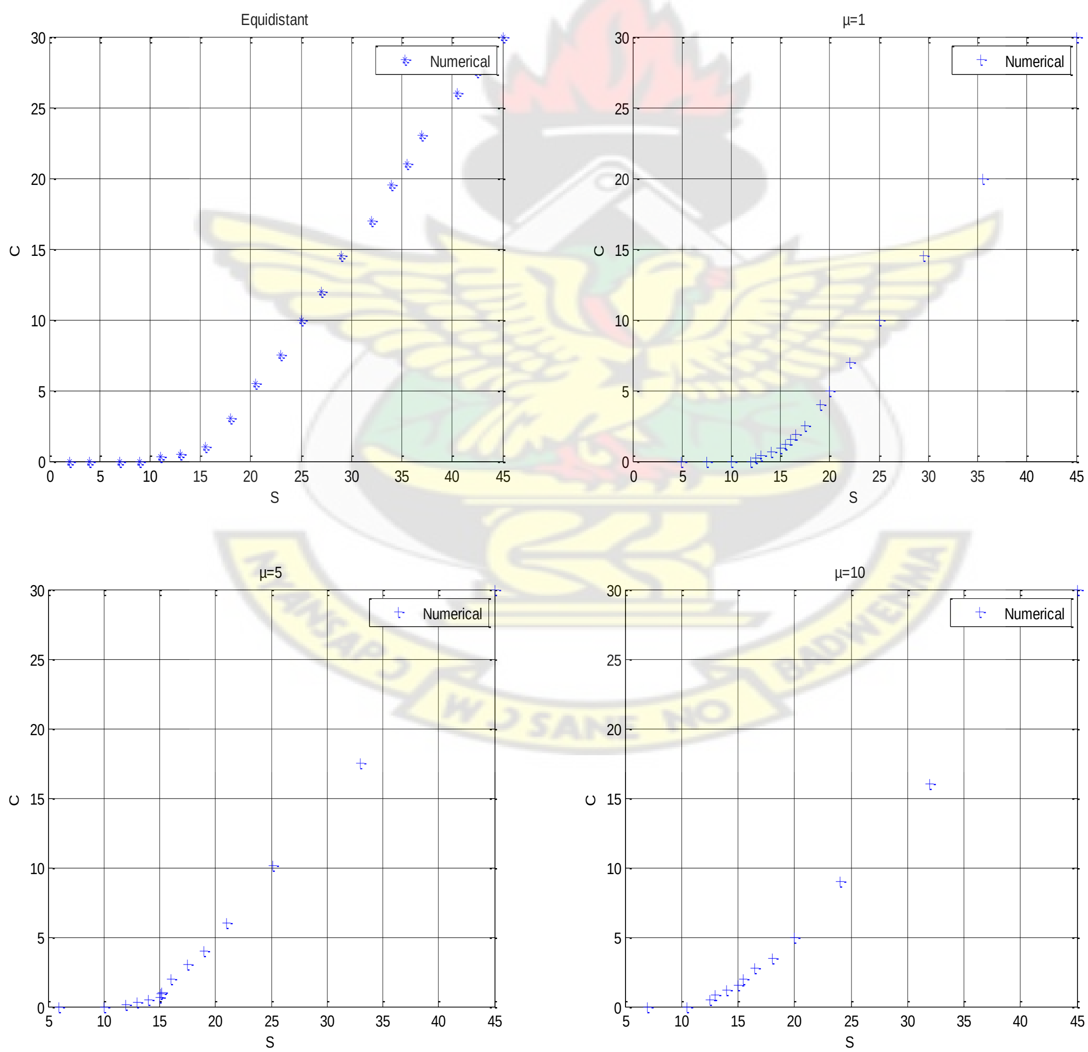
4.2.3 European put

For the relation put-call parity to be authenticated, result for the call option must be valid for the put option. From (4.10) a test for the put option is performed with outer boundaries, with the position of E unknown. These test results is seen in table 4.16

Table 4.17: European put with $E = 15$, and $\mu = 5, R = 3$.

$Grid(NXM)$	$\|P - P_{ex}\|\infty$	$Conv_{\infty}$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_{\infty}$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_{\infty}$
$10\ X\ 10$	$9.65\ X\ 10^{-2}$		$8.35\ X\ 10^{-2}$		$2.83\ X\ 10^{-2}$	
$20\ X\ 20$	$6.13\ X\ 10^{-3}$	15.74	$8.69\ X\ 10^{-3}$	9.61	$2.75\ X\ 10^{-3}$	10.32
$40\ X\ 40$	$3.95\ X\ 10^{-4}$	15.53	$1.02\ X\ 10^{-3}$	8.50	$3.42\ X\ 10^{-4}$	8.04
$80\ X\ 80$	$2.74\ X\ 10^{-5}$	14.42	$9.40\ X\ 10^{-5}$	10.88	$3.45\ X\ 10^{-5}$	9.89

Figure 4.1: Plots of option price of a call with the stretched grids



4.3 Digital Option

Digital option as discussed earlier on is characterized by it different boundary or final condition. It is also known to have a discontinuous payoff, and thus proper time integration is needed. Here the parameters in which the numerical oscillations may occur with improper numerical time integration are the Greeks. Using Crank Nicolson scheme with $N = 100$ and $M = 10$ and option parameters being $E = 40, \sigma = 0.3, r = 0.05, T = 0.5$. The numerical solution of these options shows that

1: Figure 4.2, where the numerical solution of the option C presented is pure Crank Nicolson discretization and a fourth order discretization with backward difference formula and grid stretching.

2. The numerical solution of the option Γ also has the Crank Nicolson scheme preceded by two steps of backward Euler.

3. A test with E on a grid and between two grids is shown in table 4.18, 4.19, 4.20, 4.21, 4.22. It must be noted that with this option the position of E is very important due to it discontinuity. It is also noted that only Crank Nicolson discretization gives numerical oscillation in Γ.

Table 4.18: European digital call with E on a grid, $\mu = 1.875$.

$Grid(NXM)$	$\|\|C - C_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_\infty$
$10\ X\ 10$	$2.23\ X\ 10^{-2}$		$9.40\ X\ 10^{-3}$		$1.58\ X\ 10^{-3}$	
$20\ X\ 20$	$6.74\ X\ 10^{-3}$	3.31	$1.40\ X\ 10^{-3}$	6.70	$4.33\ X\ 10^{-4}$	3.65
$40\ X\ 40$	$3.39\ X\ 10^{-3}$	1.99	$4.81\ X\ 10^{-4}$	2.19	$5.54\ X\ 10^{-5}$	7.80
$80\ X\ 80$	$1.65\ X\ 10^{-3}$	2.05	$1.51\ X\ 10^{-4}$	3.19	$2.47\ X\ 10^{-5}$	2.24

Table 4.19: European digital call with E between two grid point, $\mu = 1.875$.

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_\infty$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_\infty$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_\infty$
$10\ X\ 10$	$3.08\ X\ 10^{-2}$		$2.22\ X\ 10^{-2}$		$1.17\ X\ 10^{-3}$	
$20\ X\ 20$	$5.05\ X\ 10^{-3}$	6.09	$3.47\ X\ 10^{-3}$	6.40	$4.19\ X\ 10^{-4}$	2.80
$40\ X\ 40$	$3.34\ X\ 10^{-4}$	15.11	$4.57\ X\ 10^{-4}$	7.60	$8.02\ X\ 10^{-5}$	5.22
$80\ X\ 80$	$1.98\ X\ 10^{-5}$	16.92	$3.54\ X\ 10^{-5}$	12.89	$6.17\ X\ 10^{-6}$	13.00

Table 4.20: European assert-or-nothing call, $\mu = 1.875$.

$Grid(NXM)$	$\|C - C_{ex}\|\infty$	$Conv_\infty$	$\|\Delta - \Delta_{ex}\|\infty$	$Conv_\infty$	$\|\Gamma - \Gamma_{ex}\|\infty$	$Conv_\infty$
$10\ X\ 10$	1.95		1.09		$5.77\ X\ 10^{-2}$	
$20\ X\ 20$	$2.19\ X\ 10^{-1}$	8.91	$1.47\ X\ 10^{-1}$	7.38	$1.90\ X\ 10^{-2}$	3.04
$40\ X\ 40$	$1.45\ X\ 10^{-2}$	15.06	$1.93\ X\ 10^{-2}$	7.61	$3.34\ X\ 10^{-3}$	5.68
$80\ X\ 80$	$8.47\ X\ 10^{-4}$	17.17	$1.49\ X\ 10^{-3}$	12.96	$2.57\ X\ 10^{-4}$	13.02

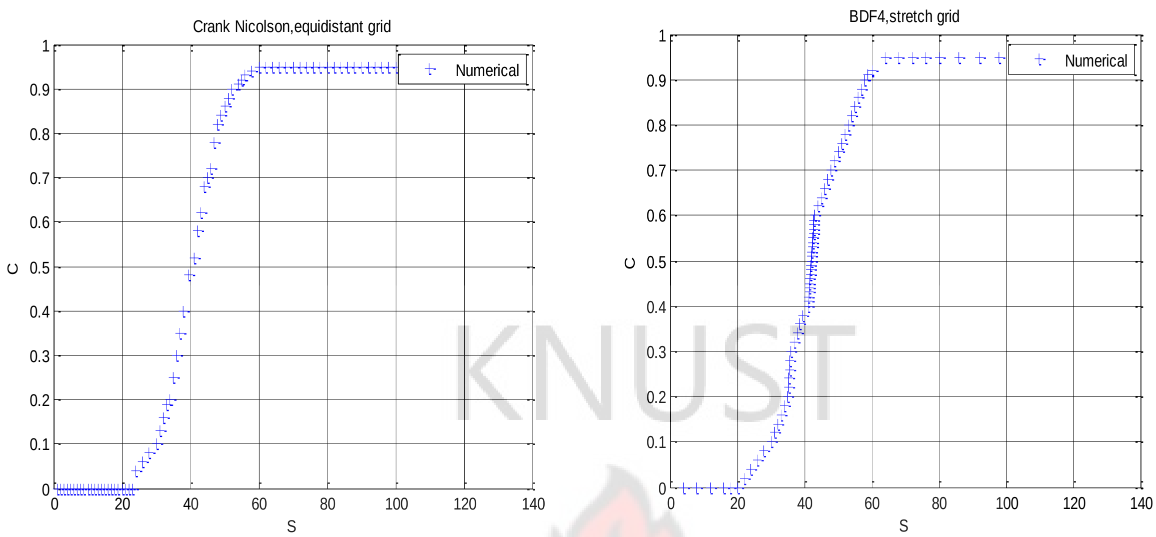
Table 4.21: European put, $\mu = 1.875$.

$Grid(NXM)$	$\|\|P - P_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_\infty$
$10\ X\ 10$	$3.08\ X\ 10^{-2}$		$2.22\ X\ 10^{-2}$		$1.17\ X\ 10^{-3}$	
$20\ X\ 20$	$5.05\ X\ 10^{-3}$	6.09	$3.47\ X\ 10^{-3}$	6.40	$4.19\ X\ 10^{-4}$	2.80
$40\ X\ 40$	$3.34\ X\ 10^{-4}$	15.11	$4.57\ X\ 10^{-4}$	7.60	$8.02\ X\ 10^{-5}$	5.22
$80\ X\ 80$	$1.98\ X\ 10^{-5}$	16.92	$3.54\ X\ 10^{-5}$	12.89	$6.17\ X\ 10^{-6}$	13.00

Table 4.22: European assert-or-nothing put, $\mu = 1.875$.

$Grid(NXM)$	$\|\|P - P_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Delta - \Delta_{ex}\|\|\infty$	$Conv_\infty$	$\|\|\Gamma - \Gamma_{ex}\|\|\infty$	$Conv_\infty$
$10\ X\ 10$	2.29		1.28		$5.83\ X\ 10^{-2}$	
$20\ X\ 20$	$2.04\ X\ 10^{-1}$	11.21	$1.38\ X\ 10^{-1}$	9.26	$1.92\ X\ 10^{-2}$	3.04
$40\ X\ 40$	$1.40\ X\ 10^{-2}$	14.60	$1.90\ X\ 10^{-2}$	7.28	$3.32\ X\ 10^{-3}$	5.76
$80\ X\ 80$	$8.20\ X\ 10^{-4}$	17.06	$1.51\ X\ 10^{-3}$	12.59	$2.56\ X\ 10^{-4}$	13.00

Figure 4.2: Plots of solution C of a digital call option (N = 100;M = 10)



4.4 Linear Combination

In providing numerical solution for the various option types, equation (4.10) together with grid transformation has been used. Difficulties however will be encountered if same approach is used for spread option, and grid stretching will not be the same for the separate options. This means that the S coordinates for the separate options are not the same and thus the need to adjust the vectors to each other.

4.4.1 Butterfly spread

Due to the difficulties explain in section 4.4.1, Lagrange interpolation of the fourth order accuracy will be used for the test of the spread type. Parameters

$E_1 = 15, E_2 = 20, E_3 = 25, R = 3.$ result for this test is seen in table 4.23

Table 4.23: Solution of butterfly spread.

$Grid(NXM)$	$\|\|V-V_{ex}\|\|\infty$ $adj.to\ E_1$	$Conv_\infty$	$\|\|V-V_{ex}\|\|\infty$ $adj.to\ E_2$	$Conv_\infty$	$\|\|V-V_{ex}\|\|\infty$ $adj.to\ E_3$	$Conv_\infty$
$10\ X\ 10$	$2.62\ X\ 10^{-1}$		$2.45\ X\ 10^{-1}$		$2.50\ X\ 10^{-1}$	
$20\ X\ 20$	$6.38\ X\ 10^{-2}$	4.10	$3.36\ X\ 10^{-2}$	7.30	$6.20\ X\ 10^{-2}$	4.02
$40\ X\ 40$	$4.22\ X\ 10^{-3}$	15.11	$2.76\ X\ 10^{-3}$	12.18	$3.77\ X\ 10^{-3}$	16.45
$80\ X\ 80$	$2.49\ X\ 10^{-4}$	16.95	$1.85\ X\ 10^{-4}$	14.88	$2.82\ X\ 10^{-4}$	13.38

4.4.2 Bull spread

Once again Lagrange interpolation of fourth order is used for this test with parameters

$E_1 = 15, E_2 = 20, R = 3.$ Result seen in table 4.24.

Table 4.24: Solution of bull spread.

$Grid(NXM)$	$\|\|V-V_{ex}\|\|\infty$ $adj.to\ E_1$	$Conv_\infty$	$\|\|V-V_{ex}\|\|\infty$ $adj.to\ E_2$	$Conv_\infty$
$10\ X\ 10$	$4.04\ X\ 10^{-1}$		$3.39\ X\ 10^{-1}$	
$20\ X\ 20$	$1.16\ X\ 10^{-1}$	3.49	$2.50\ X\ 10^{-2}$	15.69
$40\ X\ 40$	$3.26\ X\ 10^{-3}$	35.43	$1.46\ X\ 10^{-3}$	17.17
$80\ X\ 80$	$2.62\ X\ 10^{-4}$	12.45	$1.32\ X\ 10^{-4}$	11.02

In both test for butterfly and bull spread 40 x 40 grid points yield a satisfactory result. The Lagrange interpolation mapping of regions with many grid points to regions with a few grid points is but one of the reasons for the irregular convergence.

4.5 Volatility Search

The Black Scholes discretization on a stretch grid with fourth order accuracy is used in the search for volatility. In cases where the underlying asset price is not too far from the exercise price, a few grid points will be enough to cover the option price. The stopping criteria for the implied volatility method are defined by an error tolerance.

$$\left|C(\sigma_{imp}) - C_{market}\right| < Error\ Tolerance$$

For both bisection and quadratic inverse interpolation, the error tolerance for the search methods comparison is $1X10^{-3}$ and $1X10^{-5}$. The convergence result for the bisection method as well as the quadratic inverse interpolation is shown in the table 4.25 and 4.26 respectively. Parameters are

$$E = 15$$
$$r = 0.04$$
$$\delta = 0.02$$
$$T = 0.5$$

Table 4.25: Volatility search with bisection method.

$Grid(NXM)$	$Iteration\ Number$	$Volatility$	$\lvert C(\sigma) - C_{market}\rvert$
$20\ X\ 20$	1	0.5000	0.8284
	2	0.2750	0.0985
	3	0.3875	0.3661
	10	0.2987	0.0004

$Grid(NXM)$	$Iteration\ Number$	$Volatility$	$\lvert C(\sigma) - C_{market}\rvert$
$40\ X\ 40$	1	0.5000	0.8132
	2	0.2750	0.1024
	3	0.3875	0.3591
	10	0.3005	0.0023

Table 4.26: Volatility search with quadratic interpolation and parameters

$$\sigma_a = 0.2, \sigma_b = 0.4, \sigma_c = 0.6$$

$Grid(NXM)$	$Iteration\ Number$	$Volatility$	$\lvert C(\sigma) - C_{market}\rvert$
$20\ X\ 20$	1	0.4000	$4.03\ X\ 10^{-1}$
	2	0.3026	$1.38\ X\ 10^{-2}$
	3	0.2988	$4.87\ X\ 10^{-5}$
	4	0.2988	$6.80\ X\ 10^{-7}$

$Grid(NXM)$	$Iteration\ Number$	$Volatility$	$\lvert C(\sigma) - C_{market} \rvert$
$40\ X\ 40$	1	0.4000	$3.96\ X\ 10^{-1}$
	2	0.3035	$1.45\ X\ 10^{-2}$
	3	0.2999	$7.27\ X\ 10^{-5}$
	4	0.2999	$1.04\ X\ 10^{-7}$

Comparing the test result of the bisection method to the inverse quadratic interpolation, the inverse quadratic interpolation shows a fast convergence reaching the error tolerance of $1X10^{-3}$ in just 3 iterations and $1X10^{-5}$ in 4 iterations. It is also observed that to accurately recover the volatility, more grid points will be needed, that is grid point beyond 20. With 40 grid points the numerical convergence is found to be 0.3

While with 20 grid points is 0.2988. The convergences of both methods are plotted in figure 4.3 and 4.4.

The number of iteration increases if the initial underlying asset (S_0) happens not to be in the neighborhood of E . Suppose an option price $C = 4.05$ and the initial underlying asset price (S_0)=19.23, for 40 grid point, the convergence result is seen in table 4.27 and 4.28 below.

Table 4.27: Volatility search of second test with bisection

$Iteration\ Number$	$Volatility$	$\lvert C(\sigma) - C_{market} \rvert$
11	0.3001	$1.41\ X\ 10^{-4}$
16	0.3000	$4.37\ X\ 10^{-6}$

Table 4.28: Volatility search of second test inverse quadratic interpolation

$Iteration\ Number$	$Volatility$	$\lvert C(\sigma) - C_{market} \rvert$
7	0.3000	$1.84\ X\ 10^{-5}$
8	0.3000	$2.48\ X\ 10^{-9}$

Figure 4.3: Convergence of the bisection method

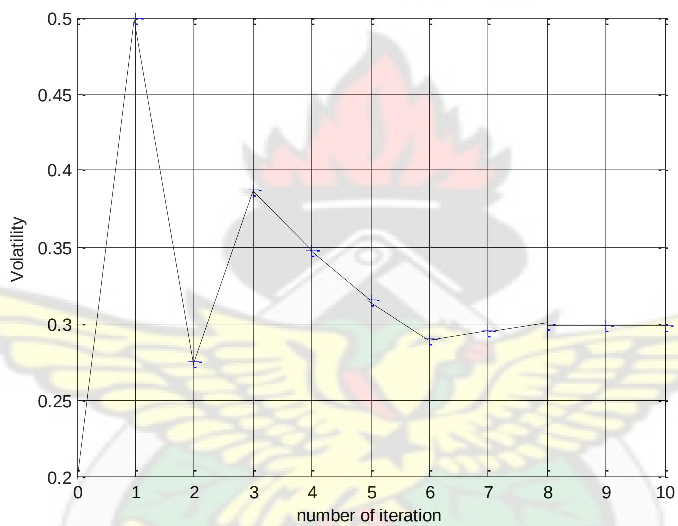
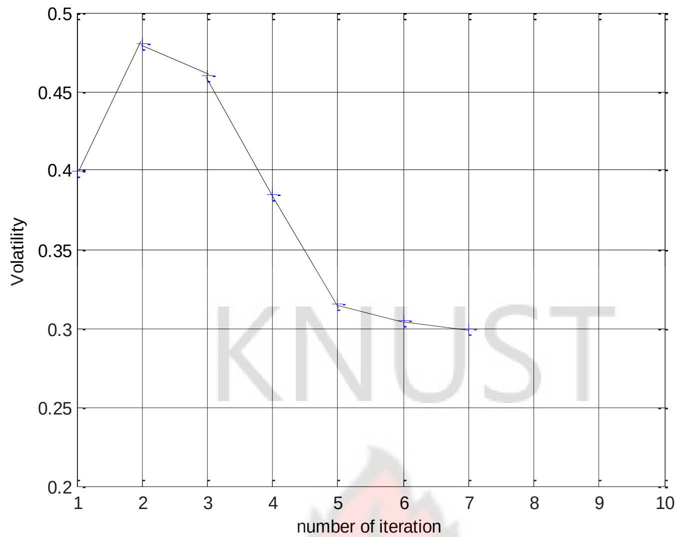


Figure 4.4: Convergence of the inverse quadratic interpolation method



CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.1 Introduction

In this thesis, the special partial differential equation is solved to evaluate the financial derivative option contract. The various option styles such as European style, American style and many of the exotic style were considered, with major reference given to the European style. In determining the accuracy of the value of the mention option styles, unlike the European style with a known exact solution, for the American and exotic styles a numerical experiment is necessary. Numerical experiment of the options covered the areas of European Vanilla option, Digital option and implied volatility.

5.2 Conclusion

European Vanilla forms the basis of the reference option. In this thesis the fourth order scheme was preferred for a further test to the second order scheme. This preference is due to the facts that second order scheme remains second order after transformation. Thus test beyond equidistant grid is done with fourth order scheme. This thesis thus propose the fourth order accurate space and time discretization using grid stretching by means of analytical coordinate transformation. In urgency, a small discretization error with a few grids stretching parameter μ is needed. In order to achieve this, a proper grid stretching parameter and 20 to 40 grid space will be sufficient enough to achieve an accuracy of (0.01). Thus for our reference option, a sufficient accuracy of the hedge parameter is observed.

Digital options, which has a non-differentiable final condition is but one of the exotic options. Here Crank Nicolson scheme and the fourth order backward difference scheme have been employed. Apart from Crank Nicolson scheme which gives oscillations in the hedge parameter Γ, all other results are satisfactory. For the accuracy of the numerical solution, the position of E with respect to the grid is very important due to it non differentiability at the final condition. Implied volatility, with its importance in valuing an option can be found in less than 10 iterations by using the inverse quadratic interpolation method.

One can therefore say that to obtain an accuracy of (0.01), by using grid stretching and highly discretize scheme only 20 to 40 grid size and time step is needed as well as the great influence of μ.The scheme also works well for the exotic options. It is also observed that volatility can be obtained in less than 10 iterations.

5.3 Recommendation

The proposed scheme can be generalize to a higher dimensional problem, as in options of more assets. Based on our findings we therefore do recommend with all reasons the finite difference scheme on stretch grid for solving partial differential equation related to option pricing using numerical methods. This is so because from experiment it is proven that this proposed scheme yields high level accuracy in determining the value of options by reducing errors, thus the reliability of the proposed scheme.

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APPENDIX: Matlab Code for plotting figures

European Call and Put: Figure 3.1 and 3.2

Data source: Virtual data

```
function [X] = BlackScholesEuro(CallPut, P, K, R,T,D, Volatility)
clc
clear
Volatility =input('Enter Volatility:');
%T=input('Enter exercise time:');
R=input('Enter interest rate :');
D=input('Enter dividend value:');
K =input('Enter strike price:');
T=input('Enter maturity time:');
callput=input('enter 1 for call option and 0 for put option:');
if (callput>1|| callput<0)
    disp('you have entered wrong entry:');
callput=input('enter 1 for call option and 0 for put option:');
end
for S=0:1:30
dt = Volatility * sqrt(T);
df = R - D + 0.5 * Volatility ^ 2;

d1 = (log( S / K ) + df *T ) / dt;
d2 = d1 - dt;
nd1 = normcdf(d1);
nd2 = normcdf(d2);
nnd1 = normcdf(-d1);
nnd2 = normcdf(-d2);

if (callput==1)
    callprice = (S * exp(-D*T) * nd1) - (K * exp(-R * T) * nd2);
    X=callprice;
end

if (callput==0)
    putprice= (K * exp(-R * T) * nnd2) - (S * exp(-D * T) * nnd1);
    X=putprice;
end
plot(S,X,'-*');
hold on
grid on
legend(['Maturity time(T)=' num2str(T)])
end
if(callput==1)
%title(['Volatility=' num2str(Volatility),', Interest Rate=' num2str(R),',
Strike Price=' num2str(K),', Dividend=' num2str(D),', Maturity Time='
num2str(T) ])
title('European Call Option')
xlabel('Asset/Stock Price')
ylabel('C')
end
```

```
if(callput==0)
%title(['Volatility=' num2str(Volatility),', Interest Rate=' num2str(R),',
Strike Price=' num2str(K),', Dividend=' num2str(D),',  Maturity Time='
num2str(T) ])
title('European Put Option')
xlabel('Asset/Stock Price')
ylabel('P')

end
```

Binary and Digital Options: Figure 3.3 and 3.5

Data source: Virtual data

```
function [X] = BlackScholesEuro(CallPut, P, K, R,T,D, Volatility)
clc
clear
Volatility =input('Enter Volatility:');
%T=input('Enter exercise time:');
R=input('Enter interest rate :');
D=input('Enter dividend value:');
K =input('Enter strike price:');
T=input('Enter maturity time:');
callput=input('enter 1 for call option and 0 for put option:');
if (callput>1|| callput<0)
    disp('you have entered wrong entry:');
callput=input('enter 1 for call option and 0 for put option:');
end
for S=0:1:30
dt = Volatility * sqrt(T);
df = R - D + 0.5 * Volatility ^ 2;

d1 = (log( S / K ) + df *T ) / dt;
d2 = d1 - dt;
%nd1 = normcdf(d1);
nd2 = normcdf(d2,0,1);
%nnd1 = normcdf(-d1);
nnd2 = normcdf(-d2,0,1);

if (callput==1)
    callprice = exp(-R * T) * nd2;
    X=callprice;
end

if (callput==0)
    putprice= (exp(-R * T) * nnd2) ;
    X=putprice;
end
plot(S,X,'-*');
hold on
grid on
```

```
legend(['Maturity time(T)=' num2str(T)])
end
if(callput==1)
%title(['Volatility=' num2str(Volatility),', Interest Rate=' num2str(R),',
Strike Price=' num2str(K),', Dividend=' num2str(D),', Maturity Time='
num2str(T) ])
title('European Call Option')
xlabel('Asset/Stock Price')
ylabel('C')
end
if(callput==0)
%title(['Volatility=' num2str(Volatility),', Interest Rate=' num2str(R),',
Strike Price=' num2str(K),', Dividend=' num2str(D),',  Maturity Time='
num2str(T) ])
title('European Put Option')
xlabel('Asset/Stock Price')
ylabel('P')

end
```

Asset or nothing call and put options: Figure 3.4 and 3.6

Data source: Virtual data

```
function [X] = BlackScholesEuro(CallPut, P, K, R,T,D, Volatility)
clc
clear
Volatility =input('Enter Volatility:');
%T=input('Enter exercise time:');
R=input('Enter interest rate :');
D=input('Enter dividend value:');
K =input('Enter strike price:');
T=input('Enter maturity time:');
callput=input('enter 1 for call option and 0 for put option:');
if (callput>1|| callput<0)
    disp('you have entered wrong entry:');
callput=input('enter 1 for call option and 0 for put option:');
end
for S=0:1:30
dt = Volatility * sqrt(T);
df = R - D + 0.5 * Volatility ^ 2;

d1 = (log( S / K ) + df *T ) / dt;
d2 = d1 - dt;
nd1 = normcdf(d1);
%nd2 = normcdf(d2,0,S);
nnd1 = normcdf(-d1);
%nnd2 = normcdf(-d2,0,S);

if (callput==1)
    callprice = (S * exp(-D*T)) * nd1;
    X=callprice;
```

```
end

if (callput==0)
    putprice= (S * exp(-D * T) * nnd1);
    X=putprice;
end
plot(S,X,'-*');
hold on
grid on
legend(['Maturity time(T)=' num2str(T)])
end
if(callput==1)
%title(['Volatility=' num2str(Volatility),', Interest Rate=' num2str(R),',
Strike Price=' num2str(K),', Dividend=' num2str(D),', Maturity Time='
num2str(T) ])
title('Asset or nothing Call Option')
xlabel('Asset/Stock Price')
ylabel('C')
end
if(callput==0)
%title(['Volatility=' num2str(Volatility),', Interest Rate=' num2str(R),',
Strike Price=' num2str(K),', Dividend=' num2str(D),',  Maturity Time='
num2str(T) ])
title('Asset or nothing Put Option')
xlabel('Asset/Stock Price')
ylabel('P')

end
```

Butterfly Spread Option: Figure 3.9

Data source: [4]

```
function [X] = BlackScholesEuro(P, K1,K3, R,T,D, Volatility)
clear all
close all
clc
Volatility =input('Enter Volatility:');
R=input('Enter interest rate :');
D=input('Enter dividend value:');
K1 =input('Enter strike price for long position call E1:');
K3 =input('Enter strike price for long position call E3:');
T=input('Enter maturity time:');
K2=0.5*(K1+K3);
%clc
%St    = 92;     % Stock price
%K1    = 15;    % Exercise price for long call
%k2    = 20;    % Exercise price for short call, K1<K2
%T     = 1;      % Time to expiration, i.e. 0.25 for quarter of year
%sigma = 0.3;    % Volatility
%r     = 0.05;   % Interest rate
    K = [K1,K2,K3]';
```

```
%Calculate the terms for the BS option prices

for St=0:2:30
    dt = Volatility * sqrt(T);
    df = R - D + 0.5 * Volatility ^ 2;
    d1 = (log( St / K ) + df *T ) / dt;
    d2 = d1 - dt;

    %d1  = (log(St./K) + (r+Volatility.^2/2).*T)/(Volatility.*sqrt(T));
    %d2  = d1-Volatility.*sqrt(T);
    %Set the coordinates
    x=[0;K1;K2;K3;K3+K1];
    cal = [];
    nd1 = normcdf(d1);
    nd2 = normcdf(d2);

    %Calculate to plain vanilla call option prices
    for i = 1:3
        cal(i) = St.*nd1(i) - K(i).*exp(-R.*T).*nd2(i);
        %cal(i) = St.*normcdf(d1(i)) - K(i).*exp(-r.*T).*normcdf(d2(i));
    end
    %Value of plain vanilla options at time T
    cal_T  = cal.*exp(R*T);
    %Calculate the payoff at each coordinate

    y1=[(-cal_T(1))*exp(R*T);(-cal_T(1))*exp(R*T);(-
cal_T(1))*exp(R*T)+(K2-K1);(-cal_T(1))*exp(R*T)+(K3-K1);(-
cal_T(1))*exp(R*T)+K3];

y2=2*[cal_T(2)*exp(R*T);cal_T(2)*exp(R*T);cal_T(2)*exp(R*T);cal_T(2)*exp(R*T)
-(K3-K2); cal_T(2)*exp(R*T)-(K1+K3-K2)];
    y3=[(-cal_T(3))*exp(R*T);(-cal_T(3))*exp(R*T);(-cal_T(3))*exp(R*T);(-
cal_T(3))*exp(R*T);(-cal_T(3))*exp(R*T)+K1];
     %Determine the strategy payoff
    y=y1+y2+y3;
  end
%Plot strategy
plot(x,y,'-r','LineWidth',2)
hold on
grid on
legend(['Maturity time(T)=' num2str(T)])
xlabel('Stock price','FontSize',11,'FontWeight','Bold')
ylabel('Payoff','FontSize',11,'FontWeight','Bold')
%xlim([0,40])
%ylim([0,6])
title('Butterfly Spread Calls','FontSize',11,'FontWeight','Bold')
set(gca,'LineWidth',1.6,'FontSize',11,'FontWeight','Bold');
box on
%Plot plain vanilla option payoff profiles
plot(x,y1,'-k','LineWidth',2)
plot(x,y2,'-k','LineWidth',2)
plot(x,[0,0,0,0]','-k','LineWidth',2)
text(2,10,'Long Call E1','FontSize',11,'FontWeight','Bold')
text(2,1,'Long Call E3','FontSize',11,'FontWeight','Bold')
hold off
```

Bull Spread and Bear Spread (reverse of bull spread): Figure 3.7 and 3.8

Data source: [4]

```
function [X] = BlackScholesEuro(P, K1,K2, R,T,D, Volatility)
clear all
close all
clc
Volatility =input('Enter Volatility:');
%T=input('Enter exercise time:');
R=input('Enter interest rate :');
D=input('Enter dividend value:');
K1 =input('Enter strike price for long call:');
K2 =input('Enter strike price for short call:');
T=input('Enter maturity time:');
%clc
%St    = 92;     % Stock price
%K1    = 15;    % Exercise price for long call
%k2    = 20;    % Exercise price for short call, K1<K2
%T     = 1;      % Time to expiration, i.e. 0.25 for quarter of year
%sigma = 0.3;    % Volatility
%r     = 0.05;   % Interest rate
    K = [K1,K2]';
    %Calculate the terms for the BS option prices

    for St=0:1:25
        dt = Volatility * sqrt(T);
        df = R - D + 0.5 * Volatility ^ 2;
        d1 = (log( St / K ) + df *T ) / dt;
        d2 = d1 - dt;

        %d1  = (log(St./K) + (r+Volatility.^2/2).*T)/(Volatility.*sqrt(T));
        %d2  = d1-Volatility.*sqrt(T);
        %Set the coordinates
        x   = [0;K1;K2;K1+K2];
        cal = [];
        nd1 = normcdf(d1);
        nd2 = normcdf(d2);

        %Calculate to plain vanilla call option prices
        for i = 1:2
            cal(i) = St.*nd1(i) - K(i).*exp(-R.*T).*nd2(i);
            %cal(i) = St.*normcdf(d1(i)) - K(i).*exp(-r.*T).*normcdf(d2(i));
        end
        %Value of plain vanilla options at time T
        cal_T  = cal.*exp(R*T);
        %Calculate the payoff at each coordinate
        y1 = [-cal_T(1);-cal_T(1);K(2)-K(1)-cal_T(1);K(2)-cal_T(1)];
        y2 = [cal_T(2);cal_T(2);cal_T(2);-K(1)+cal_T(2)];
        %Determine the strategy payoff
        y = y1+y2;
    end
    %Plot strategy
    plot(x,y,'-r','LineWidth',2)
```

```
    hold on
    grid on
    legend(['Maturity time(T)=' num2str(T)])
    xlabel('Stock price','FontSize',11,'FontWeight','Bold')
    ylabel('Payoff','FontSize',11,'FontWeight','Bold')
    %xlim([0,40])
    %ylim([0,6])
    title('Bull Call Spread','FontSize',11,'FontWeight','Bold')
    set(gca,'LineWidth',1.6,'FontSize',11,'FontWeight','Bold');
    box on
    %Plot plain vanilla option payoff profiles
    plot(x,y1,'-k','LineWidth',2)
    plot(x,y2,'-k','LineWidth',2)
    plot(x,[0,0,0,0]','-k','LineWidth',2)
    text(2,10,'Short Call','FontSize',11,'FontWeight','Bold')
    text(2,1,'Long Call','FontSize',11,'FontWeight','Bold')
    hold off
```

Greek Delta: Figure 3.10

Data source: Virtual data

```
function [X] = BlackScholesEuro(CallPut, P, K, R,T,D, Volatility)
clc
clear
Volatility =input('Enter Volatility:');
%T=input('Enter exercise time:');
R=input('Enter interest rate :');
D=input('Enter dividend value:');
K =input('Enter strike price:');
T=input('Enter maturity time:');
%callput=input('enter 1 for call option and 0 for put option:');
%if (callput>1|| callput<0)
%    disp('you have entered wrong entry:');
%callput=input('enter 1 for call option and 0 for put option:');
%end
for S=0:2:30
dt = Volatility * sqrt(T);
df = R - D + 0.5 * Volatility ^ 2;

d1 = (log(S / K) + df *T ) / dt;
d2 = d1 - dt;
nd1 = normcdf(d1);
nd2 = normcdf(d2);
%nnd1 = normcdf(-d1);
%nnd2 = normcdf(-d2);

%if (callput==1)
    nnd1 = exp(-d1 * d1 / 2) / sqrt(2 * 3.1429);
    callprice = exp(-D*T)*nnd1/(S*Volatility*sqrt(T));
    X=callprice;
%end
```

```
%if (callput==0)
%    putprice= K * exp(-R * T) * nnd2 - S * exp(-D * T) * nnd1;
%    X=putprice;
%end
plot(S,X,'R--','LineWidth',2);
hold on
grid on
end
%if(callput==1)
title('Gamma of European Call','FontSize',11,'FontWeight','Bold')
xlabel('Asset/Stock Price')
ylabel('Gamma')
```


Greek Gamma: Figure 3.11

Data source: Virtual data

```
function [X] = BlackScholesEuro(CallPut, P, K, R,T,D, Volatility)
clc
clear
Volatility =input('Enter Volatility:');
%T=input('Enter exercise time:');
R=input('Enter interest rate :');
D=input('Enter dividend value:');
K =input('Enter strike price:');
T=input('Enter maturity time:');
%callput=input('enter 1 for call option and 0 for put option:');
%if (callput>1|| callput<0)
%    disp('you have entered wrong entry:');
%callput=input('enter 1 for call option and 0 for put option:');
%end
for S=0:2:30
dt = Volatility * sqrt(T);
df = R - D + 0.5 * Volatility ^ 2;

d1 = (log(S / K) + df *T ) / dt;
d2 = d1 - dt;
nd1 = normcdf(d1);
nd2 = normcdf(d2);
%nnd1 = normcdf(-d1);
%nnd2 = normcdf(-d2);

%if (callput==1)
    nnd1 = exp(-d1 * d1 / 2) / sqrt(2 * 3.1429);
    callprice = exp(-D*T)*nnd1/(S*Volatility*sqrt(T));
    X=callprice;
%end

%if (callput==0)
%    putprice= K * exp(-R * T) * nnd2 - S * exp(-D * T) * nnd1;
%    X=putprice;
```

```
%end
plot(S,X,'ro-','LineWidth',2);
hold on
grid on
end
%if(callput==1)
title('Gamma of European Call','FontSize',11,'FontWeight','Bold')
xlabel('Asset/Stock Price')
ylabel('Gamma')
```

Transformation Function without stretch parameter: Figure 3.12

Data source: Virtual data

```
function [Y] = TransformationStretch(S0)
clc
clear
S0=input('Enter assets prices:');
U=input('Enter stretch parameter:');
for S=0:5:30
    callprice = asinh(S - S0) + asinh(S0);
    Y=callprice;
plot(S,Y,'ro-');
hold on
grid on
legend('asinh(S - S0) + asinh(S0)')
end
xlabel('S')
ylabel('Y')
```

Transformation Function with stretch parameter: Figure 3.13

Data source: Virtual data

```
function [Y] = TransformationStretch(S0,U)
clc
clear
S0=input('Enter assets prices:');
U=input('Enter stretch parameter:');
for S=0:5:30
    callprice = asinh(U*(S - S0)) + asinh(U*S0);
    Y=callprice;
plot(S,Y,'ro-');
hold on
grid on
legend('asinh(U(S - S0)) + asinh(U*S0)')
end
xlabel('S')
ylabel('Y')
```