# KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

# ESTIMATING ACTUARIAL PRESENT VALUE FACTORS FOR ANNUITY BUSINESS USING A NON-PARAMETRIC GRADUATION APPROACH

By

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# Declaration

I hereby declare that this submission is my own work towards the award of the PhD degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the University, except where due acknowledgment had been made in the text.

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# Dedication

This work is dedicated to the Almighty God, my parents and family who have been the source of inspiration in my career.

# Abstract

Ensuring income security after retirement has been an issue troubling many countries. Over the years, several attempts have been made to improve upon pensions of retirees. A typical example is the introduction of the new pension scheme in Ghana. Yet, pensioners are still faced with the problem of finding an investment with a guaranteed payout for their pensions. To this end, the use of life annuities is proposed as a solution since life annuities have been touted to ensure a guaranteed payout. However, it is unfortunate to note that not only does Ghana lack a ready annuity market, the country also does not have a standard mortality table to undertake such annuity operations.

Hence, in this study, the modified actuarial exposure method was applied to a pension scheme data in Ghana to estimate the crude mortality rates. The Whittaker-Henderson and the modified Whittaker-Henderson graduation methods were then applied to the estimated crude mortality rates to achieve a smooth and increasing progression of mortality rates with age. The Generalized Cross-Validation (GCV) method was then applied to obtain a suitable smoothing parameter, $h$ at each order of differencing, $z$ that minimized the GCV score. The optimal set of graduated mortality rates with desirable features was produced by the Whittaker-Henderson method at $z = 3$ and $h = 100$. The selected graduated mortality rates were further used to compute life expectancy. Finally, annuity tables were developed from the estimated Actuarial Present Value (APV) factors to be used in pricing the annuity products.

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May God richly bless you all for your support.

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# List of Abbreviations

| | |
|---|---|
| **APV** | Actuarial Present Value |
| **GCV** | Generalised Cross Validation |
| **M W-H** | Modified Whittaker-Henderson |
| **UDD** | Uniform Distribution of Death |
| **W-H** | Whittaker-Henderson |
| **p.a** | per annum |

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# Chapter 1

# Introduction

## 1.1 Background of Study

The dream of every worker is to secure a comfortable retirement where they are assured of a continuous steady stream of income. While some workers are assured of attractive packages at retirement, others are forced into early retirement due to incapacitation mostly without any pension. For many workers, the mere mention of retirement creates a disquieting atmosphere. These workers do not want to be associated with retirement since they do not anticipate the day they would go out of active service and hence do not prepare for it. For this reason, many governments around the world have made basic pension contributions compulsory for workers in active employment. This type of pension plan is known as government-sponsored plan and is also referred to as social security plan. An example of this plan in Ghana is the three-tier pension scheme.

The pension system in Ghana saw a major face-lift in 2008 upon the introduction of the new pension scheme in year 2008, which was backed by the National Pensions Act, ACT 766. This threefold scheme aims at resolving the inequalities and inefficiencies in the pension system. Tier one of the pension scheme is a mandatory contribution scheme with monthly contributions of 13.5% of basic salaries of all employees. This tier is a defined benefit that pays monthly benefits to employees. The tier two and tier three of the new scheme are both defined contribution schemes that pay a lump sum benefit to individuals at retirement. Tier two requires mandatory monthly contributions of 5% on basic salaries of the employees while tier three is an optional contributory scheme with monthly contributions of up to 16.5% of the employee's basic salary. Although most defined benefit plans offer life annuity options to retirees, members of a defined contribution plan are left of their own to spread their pension capital over their retirement period (Poterba, 2001).

In recent times, individuals have been more conscious about their health status. People are living healthier lifestyles by eating right, exercising regularly and receiving better health care. Hence, these individuals are expected to exceed their life expectancy and possibly outlive their retirement income. This risk of longevity has revealed that dependence on government-sponsored plans alone may be woefully inadequate to meet the standards of living most workers hope for at retirement. Therefore, there is the need for both informal and formal workers of the Ghanaian economy to be sensitized on the importance of planning for retirement. Government-sponsored plans, employer-sponsored plans, personal plans and annuities are the main types of retirement plans.

Annuities offer higher expected returns and payouts on pension capital than traditional financial products such as bonds and mortgages (Poterba, 2001; Cannon and Tonks, 2014). Thus, workers who are nearing retirement age and seek to secure a comfortable retirement where they are assured of a regular flow of income tend to fall on annuities to supplement their pensions which us usually are not adequate to meet their basic needs and expenses (Brand, 2017). In addition, Yaari (1965) using life cycle models, proposed that individuals who are not burdened with consumption risk and bequest motives when preparing for an uncertain future should annuitise all pension savings at retirement. Also, Milevsky (2013) highlighted the merits of annuitisation as an optimal product for retirement income.

## 1.2 Problem Statement

In the life of a pensioner, one of the major problems is to find an investment with a guaranteed payout. Unlike pension regulations in the United Kingdom which require individuals to purchase annuities with part of their accumulated pensions as well as in the Netherlands and Australia where pensioners are availed with options such as phased withdrawals and different annuities to decumulate pension accrued, other countries have little to no alternative to do so (Rivera-Rozo, 2009). Ghana falls within the latter bracket where the tier two and tier three of the new pension scheme are defined contribution plans which provide pensioners a lump sum at retirement. Often the investment of this lump sum becomes difficult for retirees because of the unpredictable cash-flow quantum over a long haul due to the type of investment vehicles available in the country. The life annuity market comes as a solution for retirees due to its known cash-flows over a long period. However, Ghana does not have a standard mortality table needed to estimate the actuarial present value factors which serves as a basis for a life annuity business. Also, in most developing countries, lapses in pension record keeping make it difficult to use this existing method to compute exposures leading to the construction of reliable mortality tables.

## 1.3 Objectives

The general objective of this work is to develop annuity tables for pricing life annuity products in Ghana. The specific objectives are outlined as follows:

1. To develop a modified method for computing exposures.

2. To estimate the life expectancy, $e_x$ for $x \geq 60$.

3. To estimate the actuarial present value factors for both single and joint life

annuity.

## 1.4 Methodology

Data for this study were obtained from a pension scheme in Ghana spanning over a period of 10 years (from 2005 to 2015). The exposures needed for the computation of mortality rates were obtained using a modification of the actuarial exposure technique. The crude mortality rates were then smoothened using non-parametric graduation method such as the Whittaker-Henderson and the Modified Whittaker-Henderson methods.
Finally, the optimal graduated mortality rates were used to estimate life expectancies and APV factors which can be used in pricing and reserving of life insurance products at specific interest rates.

## 1.5 Justification

The development of mortality tables of insured lives will serve as a major component in pricing and reserving of life insurance products. The development of actuarial present value factors will also aid in the introduction of life annuities business in Ghana which will help remove the unpredictable cash flows quantum when retirees purchase this product.

## 1.6 Organization of Thesis

This thesis is structured into five main chapters. Chapter one presents the background of the study, problem statement, objectives of the study, an overview of the methodology used in the study and justification of the study. Chapter two reviews the relevant literature and explores the various concepts and tools used in annuity and pension business. In Chapter three, the research methodology employed and their mathematical formulations and assumptions are discussed. Chapter four discusses data collection and preparation, analysis and interpretation of results obtained. Chapter five summarises the findings of the study and concludes with recommendations for further study.

# Chapter 2

# Literature Review

## 2.1 Introduction

This section reviews literature related to the objectives of this study. Items considered include experience studies of mortality and some basic concepts of graduation and annuities.

## 2.2 Experience Studies

An understanding of the historical events of lives in a given population is very fundamental in actuarial work. Actuaries through experience studies are able to analyse the mortality patterns of insured lives which are very key in determining how insurance products are priced. According to Atkinson and McGarry (2016), Edmond Halley designed a life table based on data collected on births and deaths from a city called Breslau now known as Wroclaw in Poland for the valuation of annuities. The life table developed from this study proved valuable in ascertaining the price of annuities upon lives in Breslau. This highlights the importance of such studies in observing past experiences so as to obtain attributes that can aptly be used to make projections for the future. An experience study which is sometimes referred to as mortality study, "is an exercise that evaluates the occurrences of specific events including deaths, incidence of disability lapses within a defined time frame and pertains to a given population", Dunscombe and

Zaidlin (2014). The development of annuity and pension products are strongly hinged on obtaining appropriate mortality rates and other factors as well. These mortality rates are used by actuaries in the development of assumptions for management, dividend setting, valuation of reserves and pricing of insurance products. Before any actuarial work in relation to life insurance is considered, a proper analysis of mortality must be conducted. This is achieved through either a cross-sectional study or longitudinal study. With the cross-sectional study, a sample group of lives to be studied is defined. After, a study is fixed between a start date and end date which is known as the observation period to capture the experiences of the members. Members who are already part of the group at the beginning of the observation period are known as starters while some members join after the start date and others exit before the study ends without dying. Those who remain alive at the end of the observation period are known as enders. The experiences of members before and after the observation period are not counted. These are known as left truncation and right censoring in survival analysis. This type of study is typically used by demographers for population projection and actuaries for pension and life insurance businesses. A cross-sectional study is convenient for large sample data with lives whose time until failure is lengthy. Longitudinal study on the other hand selects a sample group and tracks their experience through subsequent years until all lives in the group have died or generally all have failed. Another name for this type of study is known as a cohort study. A study of this type is convenient for fewer sample data with lives whose time until failure is quiet short. This method is common among clinical studies.

### 2.2.1 Study Periods for Experience Studies

There are two kinds of study periods used for carrying out a mortality study. These are a date-to-date study and an anniversary-to-anniversary study. A date-to-date study (also known as a calendar based study) runs from a fixed calendar date to a later fixed calendar date. On the contrary, an anniversary-to-anniversary study is a type of study where the years of coverage whose policy anniversary dates fall within a certain period" (Mahler, 2013). An anniversary based study has an observation period beginning on the policy anniversary and ending on the policy anniversary in a later year with each life having their own observation period. Such time studies effectively shift the dates of entry forward by a year on the first policy anniversary after the start date of the study.

Another concept of great interest to life insurers when conducting experience studies is the use of insuring ages. Most individuals do not buy insurance policies on their very birthday and hence tend to have fractional ages at their dates of entry into a study. However, for convenience sake, many life insurance companies assign integral ages to a policyholder's true age. This idea of replacing a policyholder's actual birthday with the date at which he or she purchases the policy is known as insuring ages. It is a common practice to find life insurers assign policyholders their age nearest birthday. Few life insurers however prefer to assign policyholders their age last birthday. When an anniversary-to-anniversary study is combined with insuring ages, all lives enter the study at integer ages. According to Mahler (2013), there is an implied underlying mortality assumption in determining the premium of an annuity. Mahler (2013) further pointed out that when the age nearest birthday is applied to a study, the insuring age is considered an unbiased estimator of the actual age. Thus, there can be an absolute error of up to half a year. Though this affects premium by about 2% - 2.5%, an insurer is presumed to willingly accept this in return for the convenience of applying the insuring age. On the contrary, when the age last birthday is applied, the insuring age is considered a biased estimator of the actual age. There can be an absolute error of up to a year which could result in about 5% lower premiums.

## 2.3 Exposure

Exposure refers to the number of units of human life that are subject to the risk of death, incidence of disability, lapse or any other form of decrement within a predetermined observation period (Batten, 1978). In order to estimate $q_x$, the probability of death within a year given that a person aged $x$ is alive at the beginning of the year, the number of insured exposures is computed and compared to the the number of deaths in that period or interval. There are two main techniques taken in calculating exposure, namely the exact exposure method and the actuarial exposure method. Each method of the exposure computation can be analysed under either the individual (ungrouped) data or the interval-based (grouped) data methods for large data set. The individual method measures exposure by each policy. Thus, each life's contribution to each age interval is determined. Interval-based methods on the other hand summarise data into age groups and the contribution to exposure of a life or lives within the age group is determined. Although grouping the events simplifies the calculation of exposure, the interval-based methods lose the granularity of the exact ages events occur. Such loss of accuracy is made considerably small if there are many events in the age intervals.

### 2.3.1 Exact Exposure Method

Under this method, each life contributes to exposures the exact period of time that the life is in the study. Thus, the exact exposure for each life refers to the amount of time spent by the life from entry into the study to exiting from the study. Entry into the study may be at the beginning of the observation period or later during the observation period. Exiting from the study may be due to death, withdrawal or termination of observation period. No special treatment is given to exits due to deaths.

### 2.3.2 Actuarial Exposure Method

The actuarial exposure method differs from the exact exposure method with respect to how deaths during the study are treated. Under this method, each life's contribution to exposure is measured the same way as the exact method except when the life dies in the age interval $(x,x+1]$. For a life that dies in this interval, his or her contribution to exposure is counted as a full year. Thus, it is extended to the end of year (age) of death. In other words, the actual age of death of the life is ignored and the life is assumed to be in the study to the next age even after the termination of the study.

The actuarial exposure method has been used in several studies for over a century. It predates the development of statistical theory and electronic calculators. This method is more preferred due to its simplicity and ease in applying to large data sets since there is less information to keep track of. Unlike the exact exposure method, the actuarial estimator is negatively biased and not consistent. Hence, the exact exposure method is preferred to the actuarial method for its statistical properties.

### 2.3.3 Exposure for Age Interval (Grouped) Data

Let us assume an age interval from $c_j$ to $c_{j+1}$, then we define the following for computing exposures and mortality rates as used by Mahler (2013) .

$P_j$ - the number of lives at the beginning of the age interval carried over from previous interval

$n_j^b$ - the number of new entrants at $c_j$

$n_j^m$ - the number of new entrants during the age interval $[c_j, c_{j+1})$

$w_j^m$ - the number of withdrawals during the age interval $[c_j, c_{j+1})$

$w_j^e$ - the number of withdrawals at $c_{j+1}$

$d_j$ - the number of deaths in the age interval

Then the number of new entrants are added and the number of deaths or withdrawals are subtracted to give;

$$P_j = P_{j-1} + n_{j-1}^m - d_{j-1} - w_{j-1}^m - w_{j-1}^e + n_j^b \tag{2.1}$$

Assume entry, deaths and withdrawals occur uniformly during each interval and not at the beginning or end of the interval. The exact exposure for the interval based method is given by:

$$e_j = P_j + \frac{(n_j^m - w_j^m - d_j)}{2} \tag{2.2}$$

and the mortality rate is computed as

$$\hat{q}_j = 1 - \exp^{-(d_j/e_j)} \tag{2.3}$$

The actuarial exposure for the interval based method is given by:

$$e_j = P_j + \frac{(n_j^m - w_j^m)}{2} \tag{2.4}$$

and the mortality rate is computed as

$$\hat{q}_j = \frac{d_j}{e_j} \tag{2.5}$$

## 2.4 Graduation of Mortality Rates

Crude rates estimated from experience studies tend to exhibit some form of fluctuations. As a result, these rates are revised with the aim of obtaining smoother estimates through a process known as graduation (London, 1985). Graduation of data has been applied in a variety of disciplines including Actuarial Science, Economics, Demography, among others. Graduation in the context of Actuarial Science is usually performed on initial estimates of force of mortality or death probabilities. Graduation involves the use of statistical techniques to produce "better" estimates of crude rates. According to Haberman and Renshaw (1996), graduation refers to the set of principles through which crude probabilities are smoothened to provide a basis for actuarial inferences and calculations. This procedure has become expedient in obtaining quantities that do not exhibit an erratic behavior, thus, reflect a progressive change over a series of ages without sudden and/or huge jumps (Pitacco et al., 2009).

### 2.4.1 Purposes of Graduation

Graduation of mortality rates is one of the essential roles played by actuaries in the analysis of mortality data. Since true mortality rates are assumed to be a smooth function of age, crude mortality estimations are therefore expected to exhibit such smooth progressions from age to age. More practically and as most policyholders would expect, premiums of life insurance products should increase from age to age. This reinforces the need to obtain smoothened mortality rates before computing financial quantities such as premiums and reserves, among others. Graduation should be suitable for the purpose defined. In the case of the life insurance industry, an underestimation of mortality rates when determining premiums could result in financial losses whereas an overestimation of mortality rates could result in similar losses when such rates are applied to pension or annuity work.

Graduation improves crude estimates by making use of information from adjacent data points. It also helps to reduce random sampling errors that are associated with independent estimations of crude rates. In a nutshell, graduation seeks to produce quantities that give a better representation of the underlying unknown true values.

### 2.4.2 Methods of Graduation

Graduation can be done by means of two main methods namely; parametric and non-parametric.
The parametric approach to graduation uses an age-dependent function where parametric models are fitted to initial estimates of mortality with the aim of obtaining the best possible fitting with minimum number of parameters. Examples of the models that fall into this category include the generalized linear models and mortality models such as Gompertz, Makeham, Heligman and Pollard laws (Papaioannou and Sachlas, 2004). The maximum likelihood estimation procedure is used to determine the parameters involved. The Gompertz model assumes a force of mortality that grows exponentially with age while the Makeham model is a modification to the Gompertz model. Heligman and Pollard laws were introduced due to the difficulties in getting a better graduation with the Makeham model since mortality patterns have changed over time (Debon et al., 2005)

The non-parametric methods of graduation include graduation with reference to standard mortality rates, the graphical method, weighted moving averages, kernel methods, the use of splines and Whittaker-Henderson method among others. The reference to standard mortality rates method assumes that the graduated values follow a similar pattern as that of the standard mortality rates. The graphical method involves fitting a smooth curve to the initial estimates of mortality by hand whereas with the kernel methods, kernel functions are employed to obtain graduated values of the death probabilities (Papaioannou and Sachlas, 2004). According to Debon et al. (2006), the kernel method is considered a generalisation of the moving average method. They asserted that modifications to the general equation by Copas and Haberman (1983) and Nadaraya (1964) and Watson (1964) gave graduated values with the Nayadara-Watson estimator providing a better fit for both sexes. Finally, the Whittaker - Henderson method seeks to minimize the function (refer to Equation 3.40 in the thesis) so as to derive the graduated values.

Most of the graduation methods mentioned above involve modeling mortality rates as a function of age as given by;

$$\mu_x = f_\alpha(x) \tag{2.6}$$

where $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_p$ is a vector of parameters estimated using maximum likelihood techniques or minimising the residual sum of squares. In the case of splines, this involves maximising the penalised-log-likelihood or minimising the the penalised sum of squares (Clur et al., 2013). However, there is no best technique in performing graduation. The choice of graduation method is at the discretion of the practitioner and also depends on the setting of the problem, its constraints, mortality data and purpose for which graduation is being done (Papaioannou and Sachlas, 2004). Non-parametric methods are preferred to the parametric methods since they are more flexible; thus, they do not assume an age-dependent function. Non-parametric models incorporate goodness of fit and smoothness into the graduation of observed data. The Whittaker-Henderson graduation methods are described extensively and implemented in this study.

Guerrero et al. (2001) showed that the best linear unbiased estimator of the mortality rates is Whittaker-Henderson's solution to the graduation problem. Chanco (2016) employed the Whittaker-Henderson graduation technique to obtain the smoothness and fitness of the graduated mortality rates relative to the crude rates of mortality. However, Chanco (2016) discovered that both the actual and graduated mortality estimates tend to gradually decrease at older ages which contradicts the natural phenomenon. Hence, the mortality rates were adjusted using the logit function to obtain trends that depicted the natural behavior of mortality at older ages. Partial credibility theory was further applied to these rates to improve the smoothness at older ages.

Debon et al. (2006) in their bid to estimate mortality rates compared various non-parametric graduation methods using mortality data from the Valencia Region of Spain. They pointed out that, unlike parametric methods, the assumption of an age dependent function is not necessary thus, allowing for easy application to a wide range of age groups in any geographical area. The best model for smoothing was achieved from the generalised additive models (GAM) with splines, which provided a better fit for both sexes amongst the alternative methods employed. In an earlier study presented by Debon et al. (2005), the authors proposed a revision of commonly used parametric models by comparing parametric models for mortality graduation. Their analysis was applied to data from the Valencia Region of Spain and concluded that the Gompertz-Makeham function estimated by means of a generalized linear model provided the best fit.

## 2.5 Annuities

The word “annuity” has evolved over the years from its earliest definition by Kopf (1927) as “a yearly payment of a certain sum of money granted to another in fee, for life or year, or in fee, chargeable upon the person of the grantor”, to mean a series of regular payments that can be made monthly, quarterly, semiannually, etc. As defined by Frees et al. (1995), annuities are contractual guarantees issued by financial intermediaries such as insurance companies who promise to provide periodic income over the lifetimes of individuals. Historically, annuities have been used to primarily provide a guaranteed income stream after retirement. Individuals unaware of their survival times, tend to fall on annuities to resolve their consumption risk, (Poterba, 2001).

Annuity contracts are a form of insurance contracts between the insurance company known as the annuity provider and the customer known as the annuity owner where the customer makes a single contribution(s) called premium to the annuity provider in return for future benefit(annuity) payments until death. The person whose life on which the annuity is based is known as the annuitant. In cases where the annuitant dies, the beneficiaries receive the benefits thereof. The annuitant and beneficiary to the annuity contract need not be different from the annuity owner.

Research cited by Kopf (1927) suggests that annuities can be broadly classified under *annuity certain* and *contingent annuity*. An annuity certain is a form of annuity that guarantees payments for a fixed period of time regardless of whether the annuitant survives or dies. This type of annuity can be further classified according to their time of first payment. These include annuity-due where the first payment is made at the beginning of the first payment period, also known as annuity in advance, and annuity-immediate in which the first payment is made at the end of the first payment period. Annuity-immediate is also referred to as annuity in advance. Deferred annuities on the other hand are annuities whose first payment is made after the expiry of a fixed period of time. While annuity payments can be made more frequently than once in a year, continuous annuities have payments which are assumed to be payable every instant by infinitesimally small installments. Though the idea of continuous annuities holds solely in theory, they can be applied to approximate small amounts of incomes received daily. Unlike annuity certain, contingent annuity makes payments depending on the occurrence of an event whose time of occurrence is unknown. As stated by Lombardi (2006), annuity contracts with a purchase rate guarantee for a life contingency qualifies such annuity contracts as life contracts. Hence, contingent annuities whose payments are dependent on the lives of individuals are known as life annuities.

### 2.5.1 Life Annuities

Life annuities enable individuals to transfer their stock of wealth into a steady stream of income that is guaranteed till death of that life. This financial instrument helps individuals to insure against the risk of longevity (Cannon and Tonks, 2014). Life annuity is a form of annuity where payments are made to or from an individual as long as the individual is alive. Thus, the payments are contingent on survival of the annuitant or policyholder. Examples of life annuity include whole life annuity, temporary life annuity, joint life annuity, last survivor life annuity and reversionary life annuity.

A whole life annuity is a form of life annuity where payments are made throughout the lifetime of the annuitant. Temporary life annuity is a form of annuity where payments are made for a fixed period of time during the lifetime of the annuitant and ceases upon the death of the contingent life. Joint life annuity, last survivor annuity and reversionary life annuity are collectively known as multiple life annuities. In the case of joint life annuity, payments are made as long as the contingent lives survive and ceases on the first death amongst the several lives while last survivor life annuity is a form of annuity that makes payments as long as at least one of the contingent lives survives and ceases upon the last death amongst the lives. Reversionary life annuity is a form of life annuity that is payable to survivor when the other contingent life has failed. This is common among couples where payments start on the death of a specified life if his or her spouse is alive and continues throughout the spouse's lifetime. Life annuities like annuity certain can also be payable annually, monthly, quarterly, etc. and can have varying payments.

The benefits received from an annuity contract may be in the form of income or death protection. Usually, the income received comprises of the principal invested and return on principal invested; which is not different from any investment contract. As such, most life insurance companies provide guarantees for their annuity products. These annuity products guarantee to make benefit payments that begin immediately or at a future time.

### 2.5.2 Immediate and Deferred Annuities

Annuity contracts with benefit (annuity) payments commencing immediately after purchase are known as immediate annuities whereas contracts with benefit payments commencing at a future time are known as deferred annuities. The time between the issue of the contract and the start of the benefit payments is known as the accumulation or deferral phase whereas the time during which benefit payments are made is known as the distribution or payout phase.

Immediate annuity basically makes benefit payments beginning not later than twelve months after the annuity is purchased. Such an annuity has a distribution phase and no accumulation phase. Immediate annuities protect against the risk of longevity as they guarantee income for life. Examples of immediate annuities include annuity certain, life annuity, life annuity with a period certain, refund annuity, joint life and survivors life annuity.

Deferred annuity as the name connotes is a type of annuity where benefit payments are postponed to a later date in the future after the purchase of the annuity has been made. This plan is said to have both an accumulation and payout phase. During the accumulation phase, premiums paid by the customer are accumulated in a fund for the customer. Premiums under a deferred annuity plan can either be paid as a lump sum known as a single premium deferred annuity or periodically known as flexible premium deferred annuity.

### 2.5.3 Fixed and Variable Annuities

Fixed annuities guarantee the principal and a minimum rate of return that customers contributions are credited with (Lombardi, 2006). Annuity contributions are invested in the life insurance companys general account with all investment decisions made solely by the insurance company. Thus, the life insurance company assumes risk of the performance of the annuity's securities.

Unlike fixed annuity, a variable annuity has little or no guaranteed returns. The customer has the option to invest in a basket of separate accounts different from the life insurance company's general account. These are accounts that have similar features to that of a mutual fund. Returns on the annuity fund are tied to the investment performance of the accounts selected by the customer. Investment decisions are made by the customer and hence the customer bears the whole investment risk.

Both fixed and variable annuities can either be deferred or immediate, and have single or flexible premium payment. Earnings on these annuities are usually tax-deferred.

# Chapter 3

# Methodology

## 3.1 Introduction

This chapter discusses the assumptions, mathematical and statistical models and formulations which are the necessary building blocks for the data analysis. These include the actuarial estimator approach for computing the exposures and non-parametric models for graduating crude mortality rates (probabilities of death). The study further presents the actuarial present value factors for single and multiple life annuity on traditional actuarial annuity products including whole life annuity, $n$-year temporary life annuity and $n$-year deferred whole life annuity.

## 3.2 Mortality Modelling

Mortality models and distributions of the lifetime of individuals form the building blocks for analysis of mortality data in Actuarial Science. The definitions, notations and related assumptions are discussed in this section.

### 3.2.1 Definitions and Notations

Let $T_0$ be a nonnegative random variable that represents the lifetime of an individual, defined over the interval $[0, +\infty)$ or $[0, \omega)$ where $\omega$ is the limiting age. This implies a survival probability of zero outside the assumed interval. $T_0$ is often referred to as the future lifetime random variable of a newborn.

- **Survival and Cummulative Distribution Functions**

  Let $f(t)$ be the probability density function of $T_0$. Then the survival function (SF) $T_0$ is defined as:

$$S(t) = P(T_0 > t) = \int_t^{+\infty} f(u)du \tag{3.1}$$

The cumulative distribution function (CDF) is given as:

$$F(t) = 1 - S(t) = P(T_0 \leq t) = \int_0^t f(u)du \tag{3.2}$$

For $t > 0$ and $f(t)$ taken as a continuous function, then

$$f(t) = \frac{d}{dt}F(t) = -\frac{d}{dt}S(t) \tag{3.3}$$

In actuarial practice, it is common to define the future lifetime of an individual aged, say $x$, $(x > 0)$ other than a newborn. If $T_x$ denotes the time-until-death or the future lifetime random variable for a life aged $x$ (x), then the following quantities and their actuarial notations describing the distribution of $T_x$ are defined in terms of the distribution of $T_0$.

Analogous to equations (3.1) and (3.2), we have;

$$\begin{aligned} S_x(t) = P(T_x > t) = P(T_0 > x + t | T_0 > x) \\ {}_tp_x = \frac{S(x+t)}{S(x)} \end{aligned} \tag{3.4}$$

$$\begin{aligned} F_x(t) = P(T_x \leq t) = P(x < T_0 \leq x + t | T_0 > x) \\ {}_tq_x = \frac{S(x) - S(x+t)}{S(x)} \end{aligned} \tag{3.5}$$

Similarly,

$$f_x(t) = \frac{d}{dt}F_x(t) = \frac{d}{dt}\frac{F(x+t)}{S(x)} = \frac{f(x+t)}{S(x)} \tag{3.6}$$

The actuarial notations for the SF and CDF of $T_x$ are ${}_tp_x$ and ${}_tq_x$, respectively. From equations (3.4) and (3.5), it can be seen that ${}_tp_x$ and ${}_tq_x$ are conditional probabilities, conditional on a life having survived to age $x$. ${}_tp_x$ represents the probability $(x)$ will live at least $t$ more years whereas ${}_tq_x$ represents the probability $(x)$ will die within the next $t$ years, thus before age $x + t$.

Generally, the probability of death is expressed as

$${}_{h|t}q_x = P(h < T_x \leq h + t) \tag{3.7}$$

When $h = 0$, equation (3.7) reduces to equation (3.5) however when $h = 0$ and $t = 1$, equation 3.7 reduces to $q_x$ Thus, the prefix $t$ in the notations is dropped when $t = 1$ and read as follows

- $q_x$ : the probability that $(x)$ dies within 1 year.
- $p_x$ : the probability that $(x)$ attains age $x + 1$.

When $t = 0$, yields ${}_0p_x = 1$ and hence ${}_0q_x = 0$. $q_x$ plays an imperative role in life insurance with regards to analysing mortality data. It is very useful in the construction of life tables. It is often referred to as the mortality rate at age $(x)$.

### 3.2.2 Force of Mortality:

The instantaneous measure of a life aged $x$ dying before age $x+t$, ${}_tq_x$, is known as the force of mortality. It is referred to as the hazard function in other fields of application. The force of mortality at age $x$ is denoted by $\mu_x$ and defined as:

$$\mu_x = \lim_{t\to 0} \frac{P(x < T_0 \leq x+t | T_0 > x)}{t} = \lim_{t\to 0} \frac{{}_tq_x}{t} \tag{3.8}$$

From equation (3.5), equation (3.8) becomes;

$$\mu_x = \lim_{t\to 0} = \frac{F(x+t) - F(x)}{tS(x)} = \frac{f(x)}{S(x)}$$

$$\mu_x = -\frac{d}{dx} lnS(x)$$

Hence,

$$S(x) = e^{\int_0^x \mu_u du} \tag{3.9}$$

which expresses the survival function in terms of the force of mortality.

### 3.2.3 Life Table

In actuarial practice, actuaries often develop projections of future insured events by studying the incidence of events such as death, sickness, disability, among others so as to estimate the chance of such events occurring in the future. A life table is one popular tool used by actuaries in describing the mortality of individuals. Life tables, also known as mortality tables or actuarial tables, in the actuarial setting defines the distribution of $T_x$ for integer values only. A standard life table shows $q_x$, $l_x$, $d_x$ and other actuarial functions for all integer ages $(x)$ up to a defined $\omega$.

There are two main types of life tables in Actuarial Science, namely; period life table and cohort life table. A period life table defines the mortality rates during a specific period of a certain population. Cohort life table on the other hand defines the mortality rates from an initial cohort or population for their entire lifetime.

Given a survival function ${}_tp_x$, define $\ell_x$ for $x_0 \leq x \leq \omega$. Let $\ell_{x_0}$ be an arbitrary positive number called the radix of the life table. Then

$$\ell_{x_0+t} = \ell_{x_0\,t}p_x \qquad \text{for} \qquad 0 \leq t \leq \omega - x_0 \tag{3.10}$$

From equation (3.10), when $x_0 \leq x \leq x+t \leq \omega$, we have $\ell_{x+t} = \ell_{x_0} \times {}_{x+t-x_0}p_{x_0} = \ell_{x_0} \times {}_{x-x_0}p_{x_0} \times {}_tp_x = \ell_{x\,t}p_x$.
Hence

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x} \tag{3.11}$$

where $\ell_{x+t}$ can be interpreted as the expected number of survivors to age $x+t$ out of $\ell_x$ independent lives aged $x$. Given $\ell_x$ (a positive integer), the number of survivors to age $x+t$ can be viewed as a binomial random variable with parameters $\ell_x$ and ${}_tp_x$. The expected number of deaths within the age interval $(x, x+1]$, $d_x$, is given as

$$d_x = \ell_x - \ell_{x+1} \tag{3.12}$$

Rewriting equation (3.12) in terms of equation (3.11), we have;

$$d_x = \ell_x\left(1 - \frac{\ell_{x+1}}{\ell_x}\right) = \ell_x(1 - p_x) = \ell_x q_x \tag{3.13}$$

### 3.2.4 Assumptions for Fractional Ages

In actuarial practice, certain assumptions are invoked in order to approximate probabilities of deaths and survival for non-integer ages $x$ and durations $t$ from a life table. The frequently used assumptions are discussed below.

- **Uniform Distribution of Death:**

  The Uniform Distribution of Death (UDD) is by far the most common assumption employed in actuarial work. It assumes deaths that are uniformly distributed between integer ages. Then given $x$ is a positive integer and $0 \leq t < 1$; we have

$$\begin{aligned} {}_tq_x &= tq_x \\ {}_tp_x &= 1 - tq_x \\ \mu_{x+t} &= \frac{q_x}{1 - tq_x} \end{aligned} \tag{3.14}$$

- **Constant Force of Mortality:**

  The constant force of mortality assumes force of mortality is constant between integer ages. Then given $x$ is a positive integer, assume $\mu_{x+t} = \mu_x^*$ for $0 < t \leq 1$. Hence,

$$\begin{aligned} {}_tp_x &= e^{\int_0^t \mu_x^* du} = e^{-t\mu_x^*} \\ {}_tp_x &= (p_x)^t \end{aligned} \tag{3.15}$$

  and for $y, t > 0$ and $y + t < 1$, we obtain

$${}_tp_{x+y} = e^{\int_0^t \mu_x^* du} = (p_x)^t \tag{3.16}$$

- **Balducci Assumption:**

  This assumption is given by:

$$\begin{aligned} {}_tq_x &= \frac{tq_x}{1 - (1-t)q_x} \\ \mu_{x+t} &= \frac{q_x}{1 - (1-t)q_x} \end{aligned} \tag{3.17}$$

  Thus, mortality is heavier in the first half of the year than it is in the second half of the year for fractional ages.

## 3.3 Estimation of Exposures and Mortality Rates

In order to estimate exposures and subsequently the annual probability of death, the exact age approach is used to obtain actual and exact ages of lives at specific times during an observation period so as to determine each life's contribution to each estimation age interval ($x$, $x$+1]. The estimation interval ($x$, $x$+1] is chosen in this study following London (1997). London (1997) makes use of the interval ($x$, $x$+1] since it resonates with the probability that a life dies within one year given that life is alive at age $x$, ($q_x$). The information required on each member to undergo this estimation include; date of observation period, date of joining the sample under observation, date of birth and date of death or withdrawal. Withdrawal here encompasses any random decrement other than death. The exact age at which an event occurs is obtained by deducting the decimal year of birth from the decimal year of the event.

### 3.3.1 Preliminary Concepts

For each life in the observed sample,

- $y_i$ denotes the exact age of life $i$ into the study
- $z_i$ denotes the exact scheduled age of life $i$ out of the study
- $\theta_i$ denotes the exact age at death ( $\theta_i = 0$ if life $i$ did not die during the observation period)
- $\phi_i$ denotes the exact age at withdrawal ( $\phi_i = 0$ if life $i$ did not withdraw

during the observation period)

where $i = 1, 2, 3, \ldots, n$

Furthermore, we denote the age vector by $v_i^{'} = [y_i, z_i, \theta_i, \phi_i]$ which defines the exact ages at which the various events occur. The data provided by the age vector, $v_i^{'}$ is needed to calculate each life's contribution to the age interval ($x$, $x$+1]. In determining life $i$'s contribution, if any, to the age interval ($x$, $x$+1], the following points must be noted.

1. When $y_i \geq x + 1$
2. When $z_i \leq x$
3. When $y_i \leq x$ and $z_i \geq x + 1$
4. When $x < y_i < x + 1$ or $x < z_i < x + 1$ or both
5. When $y_i < x + 1$ and $z_i > x$

In the first instance, life $i$ does not contribute to the interval ($x$, $x$+1]. This is because life $i$ comes under observation after age $x + 1$. In the second instance, life $i$ does not contribute to the interval ($x$, $x$+1] since his/her exit from the study is before age $x$. In the third instance, life $i$ is scheduled to be under observation for the entire interval ($x$,$x$+1]. For the fourth instance, life $i$ can be said to enter the estimation interval ($x$, $x$+1] at age $x + r_i$ and exit at age $x + s_i$ where $r_i$ and $s_i$ represents the interval of age within ($x$, $x$+1] over which life $i$ potentially comes under observation (with $0 \leq r_i < 1$) and potentially exits an observation (with $0 < s_i \leq 1$) respectively. Life $i$ is also scheduled to contribute to the age interval ($x$, $x$+1] in the last instance. However, if life $i$ dies or withdraws from the study before age $x$, his or her contribution to the age interval ($x$, $x$+1] would not be realised. This occurs when $0 < \theta_i < x$ or $0 < \phi_i < x$.

After the above conditions have been duly observed, each age vector is converted into a duration vector for each age interval ($x$, $x$+1]. A duration vector shows the fractional duration within the age interval ($x$, $x$+1] at which point the observation commences, is scheduled to end, or actually ends due to death or withdrawal before the scheduled end date.

Let $u^{'}_{i,x} = [r_i, s_i, l_i, k_i]$ denote the duration vector of life $i$ with subscript $x$ indicating the corresponding age interval ($x$, $x$+1]. Then the components of $u^{'}_{i,x}$ that is $r_i, s_i, l_i, k_i$ are defined as follows;

$$r_i = \begin{cases} 0 & \text{if } y_i \leq x \\ y_i - x & \text{if } x < y_i < x+1 \end{cases} \tag{3.18}$$

$$s_i = \begin{cases} z_i - x & \text{if } x < z_i < x+1 \\ 1 & \text{if } z_i \geq x+1 \end{cases} \tag{3.19}$$

$$l_i = \begin{cases} 0 & \text{if } \theta_i = 0 \\ \theta_i - x & \text{if } x < \theta_i \leq x+1 \\ 0 & \text{if } \theta_i > x+1 \end{cases} \tag{3.20}$$

$$k_i = \begin{cases} 0 & \text{if } \phi_i = 0 \\ \phi_i - x & \text{if } x < \phi_i \leq x+1 \\ 0 & \text{if } \phi_i > x+1 \end{cases} \tag{3.21}$$

### 3.3.2 Actuarial Exposure

The actuarial exposure is used here to calculate the exposure at each age. The actuarial exposure method differs from the other known methods of estimating exposure with regards to how deaths within the age interval are treated. We let $E_{i,x}$ be the actuarial exposure contributed by life $i$ over the age interval

$(x,\ x{+}1]$. Then

$$E_{i,x} = \begin{bmatrix} s_i \\ k_i \\ 1 \end{bmatrix} - r_i \tag{3.22}$$

where $s_i$ is used if person $i$ does not die or withdraw in $(x, x+1]$, $k_i$ is person $i$ withdraws in $(x, x+1]$ and 1 is used if person $i$ dies in $(x, x+1]$. The total actuarial exposure contributions over the entire entire interval $(x, x+1]$ for all persons at age x is given by:

$$E_x = \sum_{i=1}^{n} E_{i,x} \tag{3.23}$$

### 3.3.3 Estimating Mortality Rates, $q_x$

The conventional approach in Survival Analysis for estimating the annual probability of death for age $x$, $q_x$ for the interval $(x,x{+}1]$ is given by the estimator

$$\tilde{Q}_x = \frac{D_x}{n_x} \tag{3.24}$$

where $D_x$ is the random variable for the number of deaths, $n_x$ is the number of lives at exact age $x$. Intuitively, $D_x$ is a binomial random variable with parameters $n_x$ and $q_x$ where mean and variance are given by

$$\begin{aligned} E(D_x) &= n_x q_x \\ Var(D_x) &= n_x q_x (1 - q_x) \end{aligned} \tag{3.25}$$

Hence, $\tilde{Q}_x$ is a binomial proportion random variable with mean and variance given by

$$\begin{aligned} E(\tilde{Q}) &= q_x \\ Var(\tilde{Q}) &= \frac{q_x(1 - q_x)}{n_x} \end{aligned} \tag{3.26}$$

Some properties of the estimator random variable $\tilde{Q}$ include:

1. $\tilde{Q}$ is an unbiased estimator of $q_x$
2. $\tilde{Q}$ is the maximum likelihood estimator of $q_x$
3. $\tilde{Q}$ is also the method of moment estimator of $q_x$
4. Given $n_x$ is sufficiently large, the distribution of the binomial proportion random variable $\tilde{Q}$ can be approximated by a normal distribution.

However, this measure of the mortality rates does not produce an accurate picture of mortality in situations where a life withdraws or enters the study within the estimation interval ($x$, $x$+1]. Such movements known as migration are not captured in the denominator $n_x$. The total exposure, $E_x$, over the age interval ($x$, $x$+1] is used in place of $l_x$ to compensate for such shortcomings. A new estimator which inherently assumes the Balducci assumption is given by:

$$\hat{Q}_x = \frac{D_x}{E_x} \tag{3.27}$$

Due to the presence of migration within the estimation interval, $E_x$ is therefore not precisely a number of independent trials as in the case of $n_x$. As a result the new estimator $\hat{Q}_x$ is not precisely a binomial proportion random variable. That notwithstanding, since $E_x$ is often considerably large, it is safe to assume $D_x$ and $\hat{Q}_x$ are approximately binomial and binomial proportion random variables respectively. The mean and variance of $\hat{Q}_x$ are given by:

$$\begin{aligned} E(\hat{Q}_x) &= q_x \\ Var(\hat{Q}) &= \frac{q_x(1-q_x)}{E_x} \end{aligned} \tag{3.28}$$

The moments of $\hat{Q}_x$ described above are defined in terms of the true and unknown value $q_x$. This value is substituted with its estimate $\hat{q}_x$ to obtain the corresponding estimate of the mean and variance which is given by:

$$\begin{aligned} E(\hat{Q}_x) &= \hat{q}_x \\ Var(\hat{Q}) &= \frac{\hat{q}_x(1-\hat{q}_x)}{E_x} \end{aligned} \tag{3.29}$$

The estimate of the annual probability of death, $\hat{q}_x$, is given by:

$$\hat{q}_x = \frac{d_x}{E_x} \tag{3.30}$$

where

$$d_x = \sum_{i=1}^{n} d_{i,x} \tag{3.31}$$

and $d_{i,x}$ is an indicator variable defined by:

$$d_{i,x} = \begin{cases} 1 & \text{if life } i \text{ dies} \\ 0 & \text{otherwise} \end{cases} \tag{3.32}$$

### 3.3.4 Modified Actuarial Exposure method

The proposed modification to the actuarial estimate of mortality given by equation (3.30) is as discussed below. Define

$A_x$ : number of death but reported inactive

$B_x$: Number of inactive

$d_x^{id}$ : Number of death at age $x$ but reported as inactive

$$\lambda = \frac{\sum_{\forall x} A_x}{\sum_{\forall x} B_x} \tag{3.33}$$

$$E_x^* = E_x + P_x \tag{3.34}$$

$$d_x^* = d_x + d_x^{id} \tag{3.35}$$

Now,

$$E_x^* = \sum E_{x,i} + \sum P_{x,j} \tag{3.36}$$

Where,

$$P_{x,j} = \lambda(1 - k_j) \tag{3.37}$$

$$P_x = \sum_{j=1}^{\alpha} P_{x,j} \tag{3.38}$$

Thus, the new crude probability of death is given by:

$$\hat{q}_x = \frac{d_x^*}{E_x^*} \tag{3.39}$$

## 3.4 Whittaker-Henderson Methods of Graduation

### 3.4.1 Whittaker-Henderson Method

After estimating the crude mortality rates, the Whittaker-Henderson method of graduation is employed to transform the crude mortality rates to obtain graduated mortality rates that are smooth and also fit the crude mortality rates (Whittaker (1923), Henderson (1924) & Henderson (1925)). This method is used frequently in actuarial practice for the construction of mortality tables.

The aim of the Whittaker-Henderson method is to find a sequence of $n$

corresponding graduated values by minimizing the objective function:

$$M(v) = F + hS = \sum_{x=1}^{n} w_x(u_x - v_x)^2 + h\sum_{x=1}^{n-z}(\Delta^z v_x)^2 \tag{3.40}$$

with respect to $v_1, v_2, \cdots, v_n$, where

- $h$ is a positive constant parameter
- $w_x$ are positive weights attributed to the squared deviations
- $z$ is a positive integer that indicates the order of the difference equation ($z < n$)
- $u_x$ are the crude values while $v_x$ are the graduated values
- $\Delta^z v_x$ is the $z$-th forward difference of $v_x$.

The matrix-vector approach proposed by Henderson (1925) is used to minimize the quantity $M$ and obtain the graduated values. Given equation (3.40), we let $I$ and $E$ be an identity operator and a forward shift operator respectively such that

$$\begin{aligned} If(x) &= f(x) \quad \forall x \\ Ef(x) &= f(x+1) \quad \forall x \end{aligned} \tag{3.41}$$

Then the forward difference operator,($\Delta$) is defined in terms of $I$ and $E$ are given as $\Delta = E - I$.

$$\begin{aligned} &\Delta f(x) = (E-I)f(x) = f(x+1) - f(x) \\ &\Delta^2 = (E-I)^2 f(x) = \Delta(\Delta f(x)) = f(x+2) - 2f(x+1) + f(x) \end{aligned} \tag{3.42}$$

from which the general form of the binomial expansion is obtained as follows:

$$\begin{aligned}\Delta^z f(x) &= (E-I)^z f(x) \\ &= \sum_{j=0}^{z} \binom{z}{j} (-1)^{z-j} f(x+j)\end{aligned} \tag{3.43}$$

Hence,

$$\begin{aligned}\Delta^z v_x &= \sum_{r=0}^{z} \binom{z}{r} (-1)^{z-r} v_{x+r} \\ &= \sum_{j=x}^{x+z} \binom{z}{j-x} (-1)^{z+j-x} v_j \\ &= \sum_{j=x}^{n} K_{xj} v_j \quad \text{for} \quad 1 \leq x \leq n-z\end{aligned} \tag{3.44}$$

Let $\boldsymbol{K_z}$ be an $(n-z) \times n$ differencing matrix with elements:

$$K_{xj} = \begin{cases} (-1)^{z+j-x}\binom{z}{j-x} & \text{for } x = 1, 2, ..., z \qquad j = x, x+1, ..., i+z \\ \\ 0 & \text{otherwise} \end{cases} \tag{3.45}$$

For example, if $n = 10$ and $z = 4$,

$$\boldsymbol{K_4} = \begin{pmatrix} 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

If $n = 7$ and $z = 3$

$$\boldsymbol{K_3} = \begin{pmatrix} -1 & 3 & -3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -3 & 1 \end{pmatrix}$$

Now if we let

- $\boldsymbol{u} = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$ denote a vector of observed mortality values
- $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ denote a vector of graduated mortality values
- $\boldsymbol{W} \in \mathbb{R}^{n \times n}$ denote a diagonal matrix containing weights $w_x$
- $\boldsymbol{K_z} \in \mathbb{R}^{(n-z)\times n}$ denote a matrix containing binomial coefficients of order $z$, such that

$$\Delta^z v_x = \begin{pmatrix} \Delta^z v_1 \\ \Delta^z v_2 \\ \vdots \\ \Delta^z v_{n-z} \end{pmatrix} = \boldsymbol{K_z v} \tag{3.46}$$

The optimization problem in equation (3.40) is rewritten in matrix notation as follows:

$$\begin{aligned} \text{Minimise} \quad \boldsymbol{M}(\boldsymbol{v}) &= (\boldsymbol{v} - \boldsymbol{u})^T \boldsymbol{W} (\boldsymbol{v} - \boldsymbol{u}) + h(\boldsymbol{K_z v})^T (\boldsymbol{K_z v}) \\ &= \boldsymbol{v}^T \boldsymbol{W v} + h \boldsymbol{v}^T \boldsymbol{K_z}^T \boldsymbol{K_z v} - 2\boldsymbol{v}^T \boldsymbol{W u} + \boldsymbol{u}^T \boldsymbol{W u} \end{aligned} \tag{3.47}$$

Differentiating equation (3.47)with respect to $\boldsymbol{v}$

$$\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{v}} = 2(\boldsymbol{W} + h\boldsymbol{K_z}^T \boldsymbol{K_z})\boldsymbol{v} - 2\boldsymbol{W u} = 0 \tag{3.48}$$

$$\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{v}} = 2\boldsymbol{B}\boldsymbol{V} - 2\boldsymbol{W}\boldsymbol{U} = 0 \tag{3.49}$$

Where

$$\boldsymbol{B} = \boldsymbol{W} + h{\boldsymbol{K_z}}^T\boldsymbol{K_z} \tag{3.50}$$

$\boldsymbol{B}$ is symmetric, non-singular and positive definite. From equation (3.49), we have

$$\boldsymbol{B}\boldsymbol{v} = \boldsymbol{W}\boldsymbol{u} \tag{3.51}$$

from which we obtain the solution to the minimization problem which occurs at

$$\boldsymbol{v}^* = \boldsymbol{B}^{-1}\boldsymbol{W}\boldsymbol{u} \tag{3.52}$$

### 3.4.2 Variation to Whittaker-Henderson Method

The smooth-measure, S, as defined in the original Whittaker-Henderson in equation (3.40) tends to constrain the graduated values $v_x$ towards a polynomial of degree $z-1$. This is not always the case as the probability of death is generally known to increase more quickly for higher ages. Hence, the form of S should be defined to reflect this general notion of the probability of death. We assume a simple case where the probability of death is taken as a constant plus an exponential term. Thus,

$$q_x = A + Bc^x \tag{3.53}$$

Then we have:

$$\begin{aligned} \Delta q_x &= B(c-1)c^x \\ \Delta^2 q_x &= B(c-1)^2c^x = r\Delta q_x \end{aligned} \tag{3.54}$$

where $r = c - 1$, $r > -1$. Thus

$$\Delta^2 q_x - r\Delta q_x \equiv 0 \tag{3.55}$$

Hence, S is defined as

$$S = \sum_{x=1}^{n-2} (\Delta^2 v_x - r\Delta v_x)^2 \tag{3.56}$$

A generalisation of the above scenario is to assume $v_x$ is constrained towards a polynomial of degree $z - 1$ plus an exponential term. Then, similar to equation (3.55), we have:

$$\Delta^z q_x - r\Delta^{z-1} q_x \equiv 0 \tag{3.57}$$

The smooth-measure, S in equation (3.56) is re-written as

$$S = \sum_{x=1}^{n-z} (\Delta^z v_x - r\Delta^{z-1} v_x)^2 \tag{3.58}$$

Hence, the modified Whittaker-Henderson method is then given as

$$M = F + hS = \sum_{x=1}^{n} W_x (u_x - v_x)^2 + h \sum_{x=1}^{n-z} (\Delta^z v_x - r\Delta^{z-1} v_x)^2 \tag{3.59}$$

for $r > -1$

The minimisation of equation (3.59) is similar to the minimisation of equation (3.40) the original Whittaker-Henderson method. Thus, the graduated values are obtained through a matrix-vector minimisation approach. The definitions of the notations and parameters used in equation (3.59) remain the same as in the original Whittaker-Henderson method.

The second term in equation (3.59) (without parameter $h$) is written in matrix notation as:

$$\boldsymbol{G_z} = \boldsymbol{K_z} - r\boldsymbol{K_{z-1}^{(-1)}} \tag{3.60}$$

where

- $\boldsymbol{K_z}$ is an $(n-z)\times n$ matrix with same definition as before.
- $\boldsymbol{K_{z-1}}$ is an $(n-z+1)\times n$ matrix
- $\boldsymbol{K_{z-1}^{(-1)}}$ is the $\boldsymbol{K_{z-1}}$ matrix without its last row.

For example, if $z=2$ and $n=4$, equation (3.60) becomes;

$$\begin{aligned}\boldsymbol{G_2} &= \boldsymbol{K_2} - r\boldsymbol{K}_1^{(-1)} \\ \boldsymbol{G_2} &= \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} - r\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}\end{aligned} \tag{3.61}$$

Hence, equation (3.59) in matrix notation becomes:

$$\boldsymbol{M} = (\boldsymbol{v}-\boldsymbol{u})^T\boldsymbol{W}(\boldsymbol{v}-\boldsymbol{u}) + h(\boldsymbol{G}_z\boldsymbol{v})^T(\boldsymbol{G}_z\boldsymbol{v}) \tag{3.62}$$

Differentiating equation (3.62) with respect to $\boldsymbol{v}$, we have:

$$\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{v}} = 2(\boldsymbol{W} + h\boldsymbol{G_z}^T\boldsymbol{G}_z)\boldsymbol{v} - 2\boldsymbol{W}\boldsymbol{u} \tag{3.63}$$

$$\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{v}} = 2\boldsymbol{C}\boldsymbol{V} - 2\boldsymbol{W}\boldsymbol{U} = 0 \tag{3.64}$$

where

$$\boldsymbol{C} = \boldsymbol{W} + h\boldsymbol{G_z}^T\boldsymbol{G_z} \tag{3.65}$$

has same properties as $\boldsymbol{B}$ in the original Whittaker-Henderson method, that is equation (3.40).

Hence, the solution to the minimization problem occurs at

$$\boldsymbol{v}^* = \boldsymbol{C}^{-1}\boldsymbol{W}\boldsymbol{u} \tag{3.66}$$

The first term of equations (3.40) and (3.59) is known as the fit-measure. It measures the goodness of fit of the graduated values to the crude values. The second term of equations (3.40) and (3.59) is known as the smooth-measure which measures the degree of smoothness of the graduated values.

The parameter $z$, establishes the degree of polynomial being used as a standard of smoothness; this is often set to 2, 3 or 4 (London, 1985). The appropriate smoothness parameter $h$ to use in the Whittaker-Henderson method has remained a concern for most researchers. No automatic approach to selecting $h$ was proposed by early researchers Weinert (2007). This parameter has often been determined arbitrarily. The smoothness parameter represents a trade-off of emphasis between goodness of fit and smoothness. A low value of $h$ puts more emphasis on the fidelity of graduated values to the crude values and as $h \longrightarrow 0$, the graduated values converge to the crude values. A high value of $h$ indicates a stronger preference for smoothness of the graduated values and as $h \longrightarrow \infty$, the graduated values converge to produce a least-squares fit to a polynomial of degree $(z-1)$.

According to Nocon and Scott (2012), the Generalised Cross Validation (GCV) proposed by Craven and Wahba (1978) is often used to select the optimal smoothness parameter. It was further indicated that the GCV method was applied to Whittaker-Henderson graduation by Brooks et al. (1988) to obtain $h$ that minimizes the GCV score. The GCV score is given as

$$GCV = \frac{1}{n} \frac{(\boldsymbol{u}-\boldsymbol{v})^{\boldsymbol{T}}(\boldsymbol{u}-\boldsymbol{v})}{\left(1-\dfrac{Tr(\boldsymbol{B}^{-1})}{n}\right)^{2}} \tag{3.67}$$

where $Tr$ denotes the trace of a matrix.

The minimisation of equation (3.40) relies partly on the choice of weight $w_x$. The simplest and special case is to assume $w_x = 1$, $\forall\ x$, which is referred to as "Type A" Whittaker graduation method. More generally, the minimisation problem with weights other than $w_x = 1\ \forall x$ is referred to as "Type B" Whittaker graduation method. Theoretically, the weights $w_x$ are chosen to be inversely proportional to $\mathrm{Var}(U_x)$, where $U_x = \hat{Q}_x$ are assumed to be independent unbiased estimators of the crude probabilities $u_x$. Thus,

$$w_x = \frac{E_x}{u_x(1-u_x)} \tag{3.68}$$

In practice, the weights are often taken as $E_x$. As a result, a corresponding sizable value of $h$ must be set so as to normalize the undue emphasis on fit due to the fairly large fit-measure. Alternatively, the weights can be taken as

$$w_x = \frac{E_x}{\bar{E}} \tag{3.69}$$

where $\bar{E}$ is the arithmetic average of $E_x$ $\forall x$

## 3.5 Life Expectancy and Present Value

### 3.5.1 Life Expectancy

After choosing the desired set of graduated mortality rates, they are used to compute the life expectancy at each age. Life expectancy measures the average time (usually in years) $(x)$ is expected to live. Given the curtate future lifetime random variable:

$$K_x = \lfloor T_x \rfloor \tag{3.70}$$

where the distribution of $K_x$ is described in the subsequent sections, the life expectancy at each age $x$ is expressed as:

$$\begin{aligned} e_x = E(K_x) &= \sum_{k=0}^{\infty} kPr(K_x = k) \\ &= \sum_{k=0}^{\infty} k_{k|}q_x \\ &= \sum_{k=0}^{\infty} k({}_kp_x - {}_{k+1}p_x) \\ &= \sum_{k=1}^{\infty} {}_kp_x \end{aligned} \tag{3.71}$$

### 3.5.2 Present Value Concepts

The present value concepts details how to determine the value today of payments to be made in the future. The main features to be considered are interest and discount rates which are frequently used by actuaries to ascertain the value today of future benefits. Interest rate is a measure of interest applied at the end of a period whereas discount rates are applied at the beginning of a period. The compound interest which is the basis of computing interest is assumed here.
The annual effective rate of interest from time t to t+1 is defined by:

$$i_t = \frac{A(t+1) - A(t)}{A(t)} = \frac{a(t+1) - a(t)}{a(t)} \tag{3.72}$$

The annual effective rate of discount from time t to t+1 is given by:

$$d_t = \frac{A(t+1) - A(t)}{A(t+1)} = \frac{a(t+1) - a(t)}{a(t+1)} \tag{3.73}$$

where $a(t)$ and $A(t)$ are the accumulating function and amount function respectively.

Suppose one unit of money is to be received a year from today, then the present value of this single cash-flow is given by

$$v = \frac{1}{1+i} \tag{3.74}$$

also known as the discounting factor. The inverse of this is known as the accumulating factor.

An effective rate of discount ($d$) differs from an effective rate of interest ($i$), but there is a relation between the two rates that allow for easy conversion. Assume a unit amount is invested for a year at an annual effective discount rate $d$. This implies the original principal is $(1-d)$. From equation (3.72)

$$i = \frac{d}{1-d} \tag{3.75}$$

$$d = \frac{i}{1+i} = iv \tag{3.76}$$

The effective rates used above are rates of interest and discount which are payable once per period.

Consider rates of interest and discount that are paid more frequently than once per period. These rates are called nominal rates. The following equivalence formulae are derived for the relation between effective and nominal rates.

$$(1+i) = \left(1 + \frac{i^{(m)}}{m}\right)^m \tag{3.77}$$

$$(1-d) = \left(1 - \frac{d^{(m)}}{m}\right)^m \tag{3.78}$$

where

- $m$ is the number of times the rates are payable in a period, $m > 1$,
- $i^{(m)}$ is the nominal rate of interest payable m times per period,
- $d^{(m)}$ is the nominal rate of discount payable m times per period.

Each side of equation (3.77) represents the accumulation of a unit amount invested for one measurement period while that of equation (3.78) represents the present value of a unit amount paid at the end of one measurement period.
From equations (3.77) and (3.78), the equation (3.79) is derived

$$\left(1+\frac{i^{(m)}}{m}\right)^{m}=\left(1-\frac{d^{(n)}}{n}\right)^{-n} \tag{3.79}$$

where $n$ is the number of times the discount rate is payable in a period.

### 3.5.3 Present Value of Annuity-Certain

Given a total of $n$ payment periods, then the present value of an annuity-due with payments of 1 evaluated at $i$ is given by:

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= 1+v+v^2+\cdots+v^{n-1} \\ \ddot{a}_{\overline{n}|} &= \sum_{t=0}^{n-1} v^t = \frac{1-v^n}{d} \end{aligned} \tag{3.80}$$

The present value of an annuity-immediate with payments of 1 evaluated at $i$ is given by:

$$\begin{aligned} a_{\overline{n}|} &= v+v^2+\cdots+v^{n} \\ a_{\overline{n}|} &= \sum_{t=1}^{n} v^t = \frac{1-v^n}{i} \end{aligned} \tag{3.81}$$

Annuities can be payable more frequently than the interest conversion period.

Consider an annuity-due with payments of 1 at the beginning of each payment period with installments of $\frac{1}{m}$ made at the beginning of each $m$-th of payment period. The present value of such an annuity is given by:

$$\begin{aligned}\ddot{a}_{\overline{n}|}^{(m)} &= \frac{1}{m} + \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{n-\frac{1}{m}} \\ &= \frac{1-v^n}{m[1-(1+i)^{\frac{-1}{m}}]} \\ &= \frac{1-v^n}{d^{(m)}}\end{aligned} \tag{3.82}$$

If an annuity-immediate pays 1 at the end of each payment period in installments of $\frac{1}{m}$ made at the end of each $m$-th of payment period, the present value of such an annuity is given by:

$$\begin{aligned}a_{\overline{n}|}^{(m)} &= \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{n-\frac{1}{m}} + \frac{1}{m}v^{n} \\ &= \frac{1-v^n}{m[(1+i)^{\frac{1}{m}}-1]} \\ &= \frac{1-v^n}{i^{(m)}}\end{aligned} \quad . \tag{3.83}$$

where

- $n$ is the number of payment periods, $n \in \mathbb{N}$
- $m$ is the number of payments per payment period, $m > 1$
- $i$ is the interest rate per payment period
- $\ddot{a}_{\overline{n}|}$, $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}^{(m)}$, $a_{\overline{n}|}^{(m)}$ are the actuarial notations for the respective annuities. These notations are also termed annuity factors.

## 3.6 APV Factors for Single Life Annuity

Before developing the present value for life annuities, a useful concept known as status must be highlighted. A status is an artificially constructed life form for which the notion of life and death can be defined. For example, the status $x$ dies when $(x)$ dies. Another status is $\overline{n}|$, which survives for exactly $n$ time

units and then dies. The time at which death benefits would be paid is certainly oblivious to life insurers as they do not know the exact ages or times at which policyholders will die. Hence, in an attempt to estimate these times of death, actuaries model the future occurrences of deaths by defining a future lifetime random variable of lives at certain ages. This is done using $T_x$, the continuous future lifetime random variable defined in the earlier sections. For convenience and practicability sake, actuaries in the life insurance industry are more concerned with events that happen at integral years or ages than that which happens in the fraction of a year. Hence, the random variable $K_x$, as given by equation (3.70) denotes the curtate future lifetime random variable for (x).
Thus, $K_x$ is seen as the number of whole years completed by $(x)$ which is a discrete random variable with probability function defined by:

$$\begin{aligned} Pr(K_x = k) &= Pr(k \leq T_x < k+1) \\ &= {}_kp_x - {}_{k+1}p_x \\ &= {}_kp_x q_{x+k} \\ &= {}_k|q_x \qquad \text{for k} = 0, 1, 2, \ldots \end{aligned} \tag{3.84}$$

where ${}_kp_x$ is the complement of ${}_kq_x$ indicating the probability that a life aged $x$ survives at least $k$ years

In practice, life annuities are often payable more than once in a year. These payments can be on a monthly, quarterly or semiannual basis. Consider a life annuity with payments of $\frac{1}{m}$ made every $m$-th of a year that $(x)$ survives. A new random variable is defined to describe the number of complete $m$-ths in a year lived by $(x)$. Let $K_x^{(m)}$ be the complete future lifetime rounded down to the lower $\frac{1}{m}$-th of a year. Then

$$K_x^{(m)} = \frac{1}{m}\lfloor mT_x \rfloor \tag{3.85}$$

The probability function of $K_x^{(m)}$, denoting the failure in the $(k+1)^{st}$ $m$-thly time interval is defined by:

$$
\begin{aligned}
Pr(K_x^{(m)}) &= Pr(\frac{k}{m} \leq T_x < \frac{k+1}{m}) \\
&= {}_{\frac{k}{m}}p_x \, {}_{\frac{k}{m}}q_{x+\frac{k}{m}} \qquad\qquad (3.86) \\
&= {}_{\frac{k}{m}|\frac{1}{m}}q_x \qquad\qquad \text{for} \quad k = 0, 1, 2, \ldots
\end{aligned}
$$

The present value of a life annuity is a random variable since it depends on the future lifetime of an individual. The Expected Value of the Present Value (EPV) random variable is known as the Actuarial Present Value (APV). The deterministic approach to finding the present value of a life annuity for both life annuity-due and life annuity-immediate are considered in this section.

### 3.6.1 APV Factor: Whole Life Annuity

Consider an annuity-due that pays a unit amount each year that annuitant $(x)$ survives. The present value random variable for this annuity is expressed as:

$$
\ddot{Y}_x = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{for } K_x \geq 0 \\ 0 & \text{for } K_x < 0 \end{cases} \qquad (3.87)
$$

The APV is denoted by $\ddot{a}_x$:

$$
\begin{aligned}
\ddot{a}_x = E(\ddot{Y}_x) &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{K_x+1}|}\,{}_{k|}q_x \\
&= \sum_{k=0}^{\infty} v^k \, {}_kp_x
\end{aligned} \qquad (3.88)
$$

Consider an annuity-immediate that pays a unit amount each year that annuitant $(x)$ survives. The present value random variable for this annuity is expressed as:

$$
Y_x = \begin{cases} a_{\overline{K_x}|} & \text{for } K_x \geq 0 \\ 0 & \text{for } K_x < 0 \end{cases} \qquad (3.89)
$$

The APV is denoted by $a_x$:

$$\begin{aligned} a_x = E(Y_x) &= \sum_{k=0}^{\infty} a_{\overline{K_x}|}\,{}_{k|}q_x \\ &= \sum_{k=1}^{\infty} v^k\,{}_kp_x \end{aligned} \tag{3.90}$$

Consider a whole life annuity-due that makes payments $m$ times a year. The present value random variable is expressed as:

$$\ddot{Y}_x^m = \begin{cases} \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} & \text{if } K_x^{(m)} \geq 0 \\ 0 & \text{for } K_x^{(m)} < 0 \end{cases} \tag{3.91}$$

The APV is denoted by $\ddot{a}_x^{(m)}$:

$$\begin{aligned} \ddot{a}_x^{(m)} = E(\ddot{Y}_x^m) &= \sum_{k=0}^{\infty} \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|}\,{}_{\frac{k}{m}|\frac{1}{m}}q_x \\ &= \frac{1}{m}\sum_{k=0}^{\infty} v^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x \end{aligned} \tag{3.92}$$

Consider an whole life annuity-immediate that makes payments $m$ times in a year. The present value random variable for this annuity is expressed as:

$$Y_x^m = \begin{cases} a^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} & \text{for } K_x^{(m)} \geq 0 \\ 0 & \text{for } K_x^{(m)} < 0 \end{cases} \tag{3.93}$$

The APV is denoted by $a_x^{(m)}$:

$$\begin{aligned} a_x^{(m)} = E(Y_x^m) &= \sum_{k=0}^{\infty} a^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|}\,{}_{\frac{k}{m}|\frac{1}{m}}q_x \\ &= \frac{1}{m}\sum_{k=1}^{\infty} v^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x \end{aligned} \tag{3.94}$$

### 3.6.2 APV Factor: N-Year Temporary Life Annuity

Consider an annuity-due that pays a unit amount each year that annuitant $(x)$ survives, for a maximum of $n$ years. The present value random variable for this annuity is expressed as:

$$\ddot{Y}_{x:\overline{n}|} = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{for } K_x \leq n-1 \\ \ddot{a}_{\overline{n}|} & \text{for } K_x \geq n \end{cases} \tag{3.95}$$

The APV is denoted by $\ddot{a}_{x:\overline{n}|}$

$$\begin{aligned} \ddot{a}_{x:\overline{n}|} = E(\ddot{Y}_{x:\overline{n}|}) &= \sum_{k=0}^{n-1} \ddot{a}_{\overline{K_x+1}|\,k|}q_x + \ddot{a}_{\overline{n}|\,n}p_x \\ &= \sum_{k=0}^{n-1} v^k\,{}_kp_x \end{aligned} \tag{3.96}$$

Consider an annuity-immediate that pays a unit amount each year that annuitant $(x)$ survives, for a maximum of n years. The present value random variable for this annuity is expressed as:

$$Y_{x:\overline{n}|} = \begin{cases} a_{\overline{K_x}|} & \text{for } K_x \leq n-1 \\ a_{\overline{n}|} & \text{for } K_x \geq n \end{cases} \tag{3.97}$$

The APV is denoted by $a_{x:\overline{n}|}$

$$\begin{aligned} a_{x:\overline{n}|} = E(Y_{x:\overline{n}|}) &= \sum_{k=0}^{n-1} a_{\overline{K_x}|\,k|}q_x + a_{\overline{n}|\,n}p_x \\ &= \sum_{k=1}^{n} v^k\,{}_kp_x \end{aligned} \tag{3.98}$$

Consider an n-year temporary life annuity-due that pays $m$ times a year. The present value random variable for this annuity is expressed as:

$$\ddot{Y}^{m}_{x:\overline{n}|} = \begin{cases} \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} & \text{for } K_x^{(m)} \leq mn-1 \\ \ddot{a}^{(m)}_{\overline{n}|} & \text{for } K_x^{(m)} \geq n \end{cases} \tag{3.99}$$

The APV is denoted by $\ddot{a}^{(m)}_{x:\overline{n}|}$

$$\begin{aligned} \ddot{a}^{(m)}_{x:\overline{n}|} &= E(\ddot{Y}^{m}_{x:\overline{n}|}) = \sum_{k=0}^{mn-1} \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|}\, {}_{\frac{k}{m}|\frac{1}{m}}q_x + \ddot{a}^{(m)}_{\overline{n}|}\, {}_np_x \\ &= \frac{1}{m}\sum_{k=0}^{mn-1} v^{\frac{k}{m}}\, {}_{\frac{k}{m}}p_x \end{aligned} \tag{3.100}$$

Consider an n-year temporary annuity-immediate that pays $m$ times a year. The present value random variable for this annuity is expressed as:

$$Y^{m}_{x:\overline{n}|} = \begin{cases} a^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} & \text{for } K_x^{(m)} < mn-1 \\ a^{(m)}_{\overline{n}|} & \text{for } K_x^{(m)} \geq n \end{cases} \tag{3.101}$$

The APV is denoted by $a^{(m)}_{x:\overline{n}|}$

$$\begin{aligned} a^{(m)}_{x:\overline{n}|} &= E(Y^{m}_{x:\overline{n}|}) = \sum_{k=0}^{nm} a^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|}\, {}_{\frac{k}{m}|\frac{1}{m}}q_x + a^{(m)}_{\overline{n}|}\, {}_np_x \\ &= \frac{1}{m}\sum_{k=1}^{mn} v^{\frac{k}{m}}\, {}_{\frac{k}{m}}p_x \end{aligned} \tag{3.102}$$

### 3.6.3 APV Factor: N-Year Deferred Whole Life Annuity

Consider an annuity-due that pays a unit amount each year that annuitant $(x)$ survives, but commences at age $x+n$. The present value random variable for this annuity is expressed as:

$${}_n|\ddot{Y}_x = \begin{cases} 0 & \text{for } K_x \leq n-1 \\ v^n \ddot{a}_{\overline{K_x+1-n}|} & \text{for } K_x \geq n \end{cases} \tag{3.103}$$

The APV is denoted by ${}_{n|}\ddot{a}_x$

$$\begin{aligned} {}_{n|}\ddot{a}_x = E({}_{n}|\ddot{Y}_x) &= \sum_{k=n}^{\infty} v^n \ddot{a}_{\overline{K_x+1-n}|} {}_{k|}q_x \\ &= \sum_{k=n}^{\infty} v^k {}_kp_x \end{aligned} \tag{3.104}$$

Consider an annuity-immediate that pays a unit amount each year that annuitant $(x)$ survives, but commences at age $x+n$. The present value random variable for this annuity is expressed as:

$$ {}_{n}|Y_x = \begin{cases} 0 & \text{for } K_x \leq n-1 \\ v^n a_{\overline{K_x-n}|} & \text{for } K_x \geq n \end{cases} \tag{3.105}$$

The APV is denoted by ${}_{n|}a_x$

$$\begin{aligned} {}_{n|}a_x = E({}_{n}|Y_x) &= \sum_{k=n}^{\infty} v^n a_{\overline{K_x-n}|} {}_{k|}q_x \\ &= \sum_{k=n+1}^{\infty} v^k {}_kp_x \end{aligned} \tag{3.106}$$

Consider an $n$-year deferred annuity-due that pays $m$ times a year. The present value random variable for this annuity is expressed as:

$$ {}_{n}|\ddot{Y}_x^m = \begin{cases} 0 & \text{for } K_x^{(m)} \leq n-1 \\ v^n \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}-n}|} & \text{for } K_x^{(m)} \geq mn \end{cases} \tag{3.107}$$

The APV is denoted by ${}_{n|}\ddot{a}_x^{(m)}$

$$\begin{aligned} {}_{n|}\ddot{a}_x^{(m)} = E(\ddot{Y}_x^m) &= \sum_{k=mn}^{\infty} v^n \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}-n}|} {}_{\frac{k}{m}|\frac{1}{m}}q_x \\ &= \frac{1}{m} \sum_{k=mn}^{\infty} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_x \end{aligned} \tag{3.108}$$

Consider an $n$-year deferred annuity-immediate that pays $m$ times a year. The present value random variable for this annuity is expressed as:

$$
{}_{n}|Y_x^m = \begin{cases} 0 & \text{for } K_x^{(m)} \le n-1 \\ v^n a^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}-n|}} & \text{for } K_x^{(m)} \ge mn+1 \end{cases} \tag{3.109}
$$

The APV is denoted by ${}_{n|}a_x^{(m)}$

$$
\begin{aligned}
{}_{n|}a_x^{(m)} &= E({}_{n}|Y_x^m) = \sum_{k=n}^{\infty} v^n a^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}-n|}} \, {}_{\frac{k}{m}|\frac{1}{m}}q_x \\
&= \frac{1}{m} \sum_{k=mn+1}^{\infty} v^{\frac{k}{m}} \, {}_{\frac{k}{m}}p_x
\end{aligned} \tag{3.110}
$$

## 3.7 APV Factors for Multiple Life Annuity

Actuarial present value factors for life annuities as discussed earlier have been developed for a single life only. The ideas described in the single life cases can be extended to the case of several lives. Specifically, the bivariate case of two lives, $(x)$ and $(y)$ is determined. The actuarial present values for multiple life annuities are analogous to the single life cases. They can be obtained through the current payment technique by substituting the respective status of multiple lives for the generic status $x$ in the single life case. Three kinds of multiple life annuity, namely joint life annuity, last survivor annuity and reversionary annuity are considered in this section.

### 3.7.1 Joint Life Annuity

The joint life status for two lives $(x)$ and $(y)$ is represented by $(xy)$. This status survives as long as both $(x)$ and $(y)$ are alive and fails upon first death of either lives. Let $T_{xy}$ denote the joint future lifetime of $(x)$ and $(y)$.

Then

$$T_{xy} = min(T_x, T_y)$$

The random variable $T_{xy}$ defined here is characterized by the joint life status $xy$. The distribution of $T_{xy}$, for $t > 0$ with actuarial notation ${}_tq_{xy}$ is given by:

$$\begin{aligned} {}_tq_{xy} &= F_{T_{xy}}(t) = Pr(min(T_x, T_y) \leq t) \\ &= 1 - Pr((T_x \cap T_y) > t) \\ &= 1 - S_{T_x,T_y}(t, t) \\ &= 1 - {}_tp_{xy} \end{aligned} \tag{3.111}$$

Assuming $T_x$ and $T_y$ are independent

$$\begin{aligned} {}_tq_{xy} &= 1 - {}_tp_{x\,t}p_y \\ &= {}_tq_x + {}_tq_y - {}_tq_{x\,t}q_y \end{aligned} \tag{3.112}$$

Let $K_{xy}$ denote the curtate joint future lifetime of $(x)$ and $(y)$ where $K_{xy} = \lfloor T_{xy} \rfloor$

$$\begin{aligned} Pr(K_{xy} = k) &= Pr(k \leq T_{xy} < k + 1) \\ &= {}_kp_{xy} - {}_{k+1}p_{xy} \\ &= {}_{k|}q_{xy} \qquad \text{for } k = 0, 1, 2, \ldots \end{aligned} \tag{3.113}$$

Assuming independent lifetimes, we obtain:

$$Pr(K_{xy} = k) = {}_kp_{x\,k}p_y(q_{x+k} + q_{y+k} - q_{x+k}q_{y+k}) \tag{3.114}$$

- **APV Factor: Joint Life Whole Life Annuity**

  If a whole life annuity due pays a unit amount so long as the status $(xy)$

survives, then the APV, denoted by $\ddot{a}_{xy}$ is given by:

$$\begin{aligned}\ddot{a}_{xy} &= \sum_{k=0}^{\infty} v^k {}_k p_{xy} \\ &= \sum_{k=0}^{\infty} v^k {}_k p_x {}_k p_y \quad \text{assuming independent lives}\end{aligned} \tag{3.115}$$

Consider a whole life annuity immediate that pays a unit amount so long as the status $(xy)$ survives. The APV, denoted by $a_{xy}$ is given by:

$$\begin{aligned}a_{xy} &= \sum_{k=1}^{\infty} v^k {}_k p_{xy} \\ &= \sum_{k=1}^{\infty} v^k {}_k p_x {}_k p_y \quad \text{assuming independent lives}\end{aligned} \tag{3.116}$$

- **APV Factor: Joint Life N-Year Temporary Life Annuity**

  Now let us consider an n-year temporary life annuity due that pays a unit amount so long as the status $(xy)$ survives. The APV, denoted by $\ddot{a}_{xy:\overline{n}|}$ is given by:

$$\begin{aligned}\ddot{a}_{xy:\overline{n}|} &= \sum_{k=0}^{n-1} v^k {}_k p_{xy} \\ &= \sum_{k=0}^{n-1} v^k {}_k p_x {}_k p_y \quad \text{assuming independent lives}\end{aligned} \tag{3.117}$$

  Consider an n-year temporary life annuity due that pays a unit amount so long as the status $(xy)$ survives. The APV, denoted by $a_{xy:\overline{n}|}$ is given by:

$$\begin{aligned}a_{xy:\overline{n}|} &= \sum_{k=1}^{n} v^k {}_k p_{xy} \\ &= \sum_{k=1}^{n} v^k {}_k p_x {}_k p_y \quad \text{assuming independent lives}\end{aligned} \tag{3.118}$$

### 3.7.2 Last Survivor Annuity

The last survivor status for two lives $(x)$ and $(y)$ is represented by $(\overline{xy})$. This status survives as long as at least one of the lives $(x)$ and $(y)$ are alive and fails upon the second death. Let $T_{\overline{xy}}$ denote the joint future lifetime of $(x)$ and $(y)$. Then

$$T_{\overline{xy}} = max[T_x, T_y] \tag{3.119}$$

The random variable $T_{\overline{xy}}$ defined here is characterized by the joint life status $\overline{xy}$. The relationship between $T_{xy}$ and $T_{\overline{xy}}$ is as given below; Since $T_{xy}$ represents the time of first failure between $(x)$ and $(y)$ and $T_{\overline{xy}}$ represents the time of second failure between $(x)$ and $(y)$, it follows that $T_{xy}$ is one of $T_x$ or $T_y$ and $T_{\overline{xy}}$ is necessarily the other one (Cunningham et al. (2012)). Hence the following relations and identities are derived.

$$\begin{aligned} T_{xy} + T_{\overline{xy}} &= T_x + T_y \\ T_{xy} T_{\overline{xy}} &= T_x T_y \\ {}_tq_{xy} + {}_tq_{\overline{xy}} &= {}_tq_x + {}_tq_y \end{aligned} \tag{3.120}$$

$$\begin{aligned} K_{xy} + K_{\overline{xy}} &= K_x + K_y \\ K_{xy} K_{\overline{xy}} &= K_x K_y \\ {}_kq_{xy} + {}_kq_{\overline{xy}} &= {}_kq_x + {}_kq_y \end{aligned} \tag{3.121}$$

The distribution of $T_{\overline{xy}}$, for $t > 0$ with actuarial notation ${}_tq_{\overline{xy}}$ is given by:

$$\begin{aligned} {}_tq_{\overline{xy}} &= F_{T(\overline{xy})}(t) = Pr(max(T_x, T_y) \leq t) \\ &= Pr((T_x \cap T_y) \leq t) \\ &= 1 - S_{T_x,T_y}(t,t) \\ &= 1 - {}_tp_{\overline{xy}} \\ &= 1 - ({}_tp_x + {}_tp_y - {}_tp_{xy}) \end{aligned} \tag{3.122}$$

Assuming $T_x$ and $T_y$ are independent:

$$\begin{aligned} {}_tq_{\overline{xy}} &= 1 - ({}_tp_x + {}_tp_y - {}_tp_{x\,t}p_y) \\ {}_tq_{\overline{xy}} &= {}_tq_{x\,t}q_y \end{aligned} \tag{3.123}$$

Let $K_{\overline{xy}}$ denote the curtate joint future lifetime of $(x)$ and $(y)$ where $K_{\overline{xy}} = \lfloor T_{\overline{xy}} \rfloor$.

$$\begin{aligned} Pr(K_{\overline{xy}} = k) &= Pr(k \leq T_{\overline{xy}} < k+1) \\ &= {}_kp_{\overline{xy}} - {}_{k+1}p_{\overline{xy}} \\ &= {}_{k|}q_{\overline{xy}} \qquad \text{for } k = 0, 1, 2, \ldots \end{aligned} \tag{3.124}$$

The APV factor for last survivor whole life annuity is derived. Consider a whole life annuity due that pays a unit amount so long as the status $(\overline{xy})$ survives. The APV is denoted by $\ddot{a}_{\overline{xy}}$ and is expressed as:

$$\begin{aligned} \ddot{a}_{\overline{xy}} &= \sum_{k=0}^{\infty} v^k {}_kp_{\overline{xy}} \\ &= \sum_{k=0}^{\infty} v^k ({}_kp_x + {}_kp_y - {}_kp_{xy}) \\ &= \sum_{k=0}^{\infty} v^k ({}_kp_x + {}_kp_y - {}_kp_{x\,k}p_y) \text{ assuming independent lives} \end{aligned} \tag{3.125}$$

Consider a whole life annuity immediate that pays a unit amount so long as the status $(\overline{xy})$ survives. The APV is denoted by $a_{\overline{xy}}$ and expressed as:

$$\begin{aligned} a_{\overline{xy}} &= \sum_{k=1}^{\infty} v^k {}_kp_{\overline{xy}} \\ &= \sum_{k=1}^{\infty} v^k ({}_kp_x + {}_kp_y - {}_kp_{xy}) \\ &= \sum_{k=1}^{\infty} v^k ({}_kp_x + {}_kp_y - {}_kp_{x\,k}p_y) \text{ assuming independent lives} \end{aligned} \tag{3.126}$$

Also, for APV factor for last survivor N-Year Temporary Life Annuity, we consider an n-year temporary life annuity due that pays a unit amount so long as the status $(\overline{xy})$ survives. The APV is denoted by $\ddot{a}_{\overline{xy}:\overline{n}|}$ and expressed as:

$$\begin{aligned}
\ddot{a}_{\overline{xy}:\overline{n}|} &= \sum_{k=0}^{n-1} v^k {}_kp_{\overline{xy}} \\
&= \sum_{k=0}^{n-1} v^k ({}_kp_x + {}_kp_y - {}_kp_{xy}) \qquad (3.127) \\
&= \sum_{k=0}^{n-1} v^k ({}_kp_x + {}_kp_y - {}_kp_{x\,k}p_y) \text{ assuming independent lives}
\end{aligned}$$

Consider an n-year temporary life annuity immediate that pays a unit amount so long as the status $(\overline{xy})$ survives. The APV is denoted by $a_{\overline{xy}:\overline{n}|}$ and expressed as:

$$\begin{aligned}
a_{\overline{xy}:\overline{n}|} &= \sum_{k=1}^{n} v^k {}_kp_{\overline{xy}} \\
&= \sum_{k=1}^{n} v^k ({}_kp_x + {}_kp_y - {}_kp_{xy}) \qquad (3.128) \\
&= \sum_{k=1}^{n} v^k ({}_kp_x + {}_kp_y - {}_kp_{x\,k}p_y) \text{ assuming independent lives}
\end{aligned}$$

### 3.7.3 Reversionary Annuity

This is a type of annuity that is payable to $(y)$ as long as $(y)$ is alive and $(x)$ has failed. This annuity is common among couples. In the simple case of two lives, a husband and a wife, this annuity starts payments on the death of a specified life (either the husband or wife), if his or her spouse is alive, and continues throughout the spouse's lifetime.

Consider an annuity payable to $(y)$ after the death of $(x)$. The present value random variable is given by:

$$Z = \begin{cases} {}_{K(x)+1|}\ddot{a}_{\overline{K(y)-K(x)}|} & \text{if } K(x) \leq K(y) \\ 0 & \text{if } K(x) > K(y) \end{cases} \tag{3.129}$$

$$Z = \begin{cases} \ddot{a}_{\overline{K(y)}|} - \ddot{a}_{\overline{K(x)}|} & \text{if } K(x) \leq K(y) \\ \ddot{a}_{\overline{K(y)}|} - \ddot{a}_{\overline{K(y)}|} & \text{if } K(x) > K(y) \end{cases} \tag{3.130}$$

$$Z = \ddot{a}_{\overline{K(y)}|} - \ddot{a}_{\overline{K(xy)}|} \tag{3.131}$$

The actuarial present value is denoted by $\ddot{a}_{x|y}$

$$\begin{aligned} \ddot{a}_{x|y} &= E(Z) \\ \ddot{a}_{x|y} &= \sum_{k=0}^{\infty} v^k \,{}_kp_y\,{}_kq_x \\ \ddot{a}_{x|y} &= \ddot{a}_y - \ddot{a}_{xy} \\ \ddot{a}_{x|y} &= (1 + a_y) - (1 + a_{xy}) \\ \ddot{a}_{x|y} &= a_y - a_{xy} \\ \text{which implies} \qquad \ddot{a}_{x|y} &= a_{x|y} \end{aligned} \tag{3.132}$$

This kind of annuity is intuitively a deferred annuity with a random deferral period equal to the time until failure of the second life. Essentially, an n-year deferred life annuity is a special case of the reversionary annuity defined above if status (y) is an n-year term certain.

# Chapter 4

# Results and Discussion

## 4.1 Introduction

The various statistical and actuarial methods described in the previous chapters are applied in this chapter in line with the objectives of the study. The pension data obtained for the analysis spanned the 2005 - 2015 period. The variables of interest considered included birth date, policy issue date, withdrawal date and date of death of pensioners. Statistical tests and graphical diagnostics are done to access the validity of the methods used. The R statistical software and Microsoft Excel are applied to perform the various computations and obtain the results of the analyses.

## 4.2 Estimation of Crude Mortality Rates

The number of deaths for each age $x$ was recorded after which the actuarial exposure method per equation (3.23) was used to compute the exposure for each member age at $x$. Once these quantities had been computed, the annual crude mortality rates ($q_x$) were easily obtained per equation (3.30) as the ratio of the number of deaths ($d_x$) to the exposure ($E_x$) at each age $x$. These values were obtained for all ages of the members of the pension scheme ranging from 18 to 110 years. Table (4.1) gives a snippet of the exposures, number of deaths and crude mortality rates while the summary statistics of the completed table are given in

Table 4.1: Exposure, number of deaths and crude rates at each age

| Age ($x$) | $E_x$ | $d_x$ | $\hat{q}_x$ |
|---|---|---|---|
| 18 | 53202.00 | 101 | 0.00190 |
| 19 | 79457.00 | 203 | 0.00255 |
| 20 | 109951.00 | 311 | 0.00283 |
| 21 | 136191.00 | 68 | 0.00050 |
| 22 | 157411.00 | 127 | 0.00081 |
| 23 | 178241.00 | 452 | 0.00254 |
| 24 | 197029.00 | 597 | 0.00303 |
| 25 | 208165.00 | 683 | 0.00328 |
| 26 | 219348.00 | 770 | 0.00351 |
| 27 | 231056.00 | 810 | 0.00351 |
| 28 | 237765.00 | 877 | 0.00369 |
| 29 | 238302.00 | 920 | 0.00386 |
| 30 | 241460.00 | 978 | 0.00405 |
| ⋮ | ⋮ | ⋮ | ⋮ |

Table (4.2). The graphical distributions of the computed exposures and crude mortality rates as well as the number of deaths are presented in figures (4.1) and (4.2).

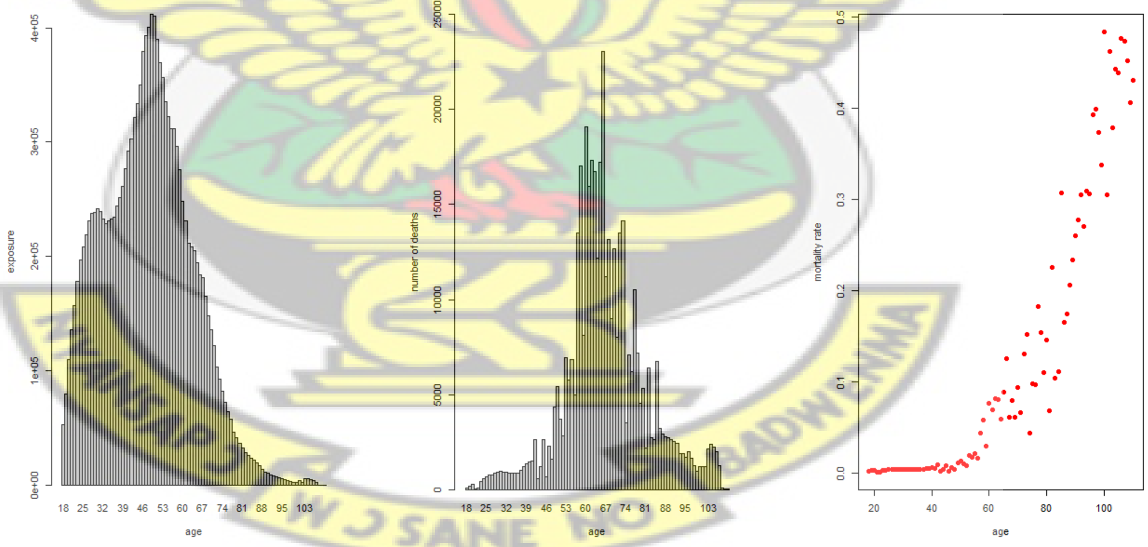

Figure 4.1: The distribution of exposures, number of deaths and crude mortality rates respectively per age of pension scheme members

The bulk of the number of exposures are captured between ages 46 and 53 and gradually decreases thereafter. The plot of the number of deaths is low between ages 18 and 46 relative to the exposures at the other ages. The number of deaths also drops significantly after age 85. This drop can be attributed to the lack of mortality data and misreporting of deaths at such ages.

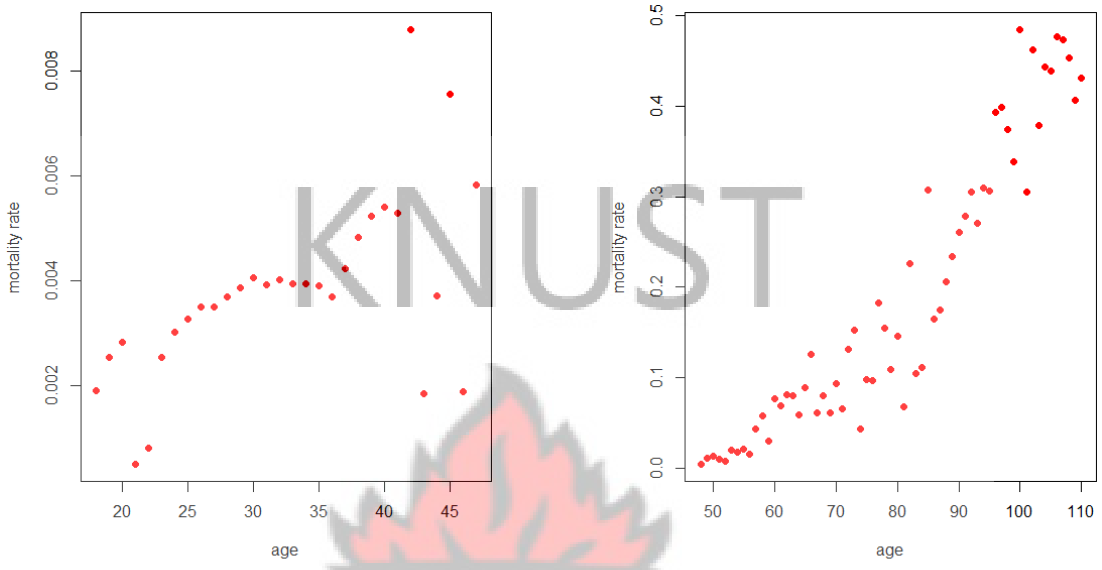

Figure 4.2: Crude mortality rates of 18-49 year-old (left) and 50-110 year-old (right) pension scheme members

The crude mortality rates are divided into the young ages (18 - 49 years) and adult ages (50 - 110 years) as seen in Figure 4.2. It can be observed that the crude mortality rates are dispersed at ages 18-23 and 41-47 years and remain relatively stable in between these age groups. However, the mortality rates observed in the adult ages are evenly dispersed. Overall, the crude mortality rates reveal an upward increasing trend with significant fluctuations observed throughout. This emphasizes the need to smooth the crude rates to make them suitable for pricing life insurance products.

Table (4.2) presents the summary statistics of ages, exposures, number of deaths and crude mortality rates of members of the pension scheme. The ages of the members of the pension schemes observed were distributed between 18 and 110 years with the mean and median having the same value of 64 years, where the distribution appears symmetrical. The computed exposures appear more variable, ranging from 130 to 411,953 per age with a mean of 157,117 while the number of deaths recorded for each age also ranges from 56 to 23,017 with a mean of 4,556. Most of these deaths occurred among members aged between 55 and 75 years. The crude mortality rates also varied from 0.0005 to 0.48429 with a mean value of 0.12904, exceeding the median (0.06573) by about half.

Table 4.2: Summary statistics of age, exposures, number of deaths and crude mortality rates

| | Minimum | 1st Quartile | Median | Mean | 3rd Quartile | Maximum |
|---|---|---|---|---|---|---|
| Age, $x$ | 18 | 41 | 64 | 64 | 87 | 110 |
| $E_x$ | 130.1 | 16924.0 | 157411.0 | 157116.8 | 244398.0 | 411953.0 |
| $d_x$ | 56 | 978 | 2291 | 4556 | 6394 | 23017 |
| $q_x$ | 0.00050 | 0.00423 | 0.06573 | 0.12904 | 0.22578 | 0.48429 |

## 4.3 Graduation of Mortality Rates

The Whittaker-Henderson (W-H) and Modified Whittaker-Henderson (MWH) methods as well detailed in Chapter 3 were applied to obtain the graduated mortality rates of members of the pension scheme. The weights, $w_x$ used in each graduation process were computed per equation (3.69). The order of differences was selected at $z = 2, 3, 4$ and 5 while the smoothness parameter was arbitrarily chosen at $h = 100, 150, 200, 300$ and 400. The assumed growth rate $r$ for the Modified Whittaker-Henderson graduation was computed as the average change in the crude mortality rates over all ages. The graduation of crude mortality rates is performed at each combination of $z$ and $h$ for both graduation methods. The snippets of the graduated mortality rates at each combination of $z$ and $h$ are shown in Tables (4.3) - (4.10) while the complete computations for all ages are provided in the Appendix section.

Table 4.3: Graduated mortality rates using W-H at z=2

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00145 | 0.00145 | 0.00147 | 0.00154 | 0.00164 |
| 19 | 0.00255 | 0.00164 | 0.00165 | 0.00168 | 0.00175 | 0.00183 |
| 20 | 0.00283 | 0.00183 | 0.00185 | 0.00188 | 0.00195 | 0.00203 |
| 21 | 0.00050 | 0.00203 | 0.00206 | 0.00209 | 0.00215 | 0.00222 |
| 22 | 0.00081 | 0.00225 | 0.00228 | 0.00231 | 0.00236 | 0.00242 |
| 23 | 0.00254 | 0.00249 | 0.00251 | 0.00253 | 0.00258 | 0.00262 |
| 24 | 0.00303 | 0.00274 | 0.00275 | 0.00276 | 0.00279 | 0.00282 |
| 25 | 0.00328 | 0.00298 | 0.00298 | 0.00298 | 0.00300 | 0.00301 |

Table 4.4: Graduated mortality rates using W-H at z=3

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00186 | 0.00174 | 0.00165 | 0.00156 | 0.00151 |
| 19 | 0.00255 | 0.00175 | 0.00171 | 0.00168 | 0.00165 | 0.00164 |
| 20 | 0.00283 | 0.00175 | 0.00176 | 0.00177 | 0.00179 | 0.00180 |
| 21 | 0.00050 | 0.00185 | 0.00190 | 0.00193 | 0.00197 | 0.00199 |
| 22 | 0.00081 | 0.00205 | 0.00211 | 0.00214 | 0.00219 | 0.00222 |
| 23 | 0.00254 | 0.00233 | 0.00237 | 0.00240 | 0.00244 | 0.00246 |
| 24 | 0.00303 | 0.00266 | 0.00268 | 0.00269 | 0.00271 | 0.00271 |
| 25 | 0.00328 | 0.00299 | 0.00298 | 0.00297 | 0.00297 | 0.00296 |

Table 4.5: Graduated mortality rates at W-H z=4

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00263 | 0.00257 | 0.00252 | 0.00242 | 0.00233 |
| 19 | 0.00255 | 0.00191 | 0.00189 | 0.00188 | 0.00185 | 0.00184 |
| 20 | 0.00283 | 0.00158 | 0.00159 | 0.00160 | 0.00162 | 0.00164 |
| 21 | 0.00050 | 0.00156 | 0.00158 | 0.00161 | 0.00166 | 0.00169 |
| 22 | 0.00081 | 0.00178 | 0.00181 | 0.00184 | 0.00188 | 0.00192 |
| 23 | 0.00254 | 0.00216 | 0.00218 | 0.00220 | 0.00223 | 0.00225 |
| 24 | 0.00303 | 0.00261 | 0.00262 | 0.00263 | 0.00263 | 0.00264 |
| 25 | 0.00328 | 0.00306 | 0.00305 | 0.00305 | 0.00303 | 0.00301 |

Table 4.6: Graduated mortality rates at W-H at z=5

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00269 | 0.00271 | 0.00274 | 0.00277 | 0.00280 |
| 19 | 0.00255 | 0.00195 | 0.00194 | 0.00193 | 0.00192 | 0.00190 |
| 20 | 0.00283 | 0.00158 | 0.00157 | 0.00155 | 0.00153 | 0.00151 |
| 21 | 0.00050 | 0.00153 | 0.00153 | 0.00152 | 0.00150 | 0.00149 |
| 22 | 0.00081 | 0.00174 | 0.00174 | 0.00174 | 0.00174 | 0.00174 |
| 23 | 0.00254 | 0.00213 | 0.00213 | 0.00213 | 0.00214 | 0.00216 |
| 24 | 0.00303 | 0.00259 | 0.00259 | 0.00260 | 0.00262 | 0.00264 |
| 25 | 0.00328 | 0.00305 | 0.00305 | 0.00306 | 0.00308 | 0.00310 |

Table 4.7: Graduated mortality rates using Modified W-H at z=2

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00146 | 0.00147 | 0.00149 | 0.00157 | 0.00167 |
| 19 | 0.00255 | 0.00164 | 0.00166 | 0.00169 | 0.00176 | 0.00185 |
| 20 | 0.00283 | 0.00183 | 0.00186 | 0.00189 | 0.00196 | 0.00204 |
| 21 | 0.00050 | 0.00203 | 0.00206 | 0.00209 | 0.00216 | 0.00223 |
| 22 | 0.00081 | 0.00225 | 0.00228 | 0.00231 | 0.00237 | 0.00243 |
| 23 | 0.00254 | 0.00249 | 0.00251 | 0.00253 | 0.00258 | 0.00263 |
| 24 | 0.00303 | 0.00273 | 0.00275 | 0.00276 | 0.00279 | 0.00282 |
| 25 | 0.00328 | 0.00297 | 0.00297 | 0.00298 | 0.00300 | 0.00301 |

Table 4.8: Graduated mortality rates using Modified W-H at z=3

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00186 | 0.00173 | 0.00165 | 0.00155 | 0.00150 |
| 19 | 0.00255 | 0.00175 | 0.00171 | 0.00168 | 0.00165 | 0.00163 |
| 20 | 0.00283 | 0.00175 | 0.00176 | 0.00177 | 0.00179 | 0.00180 |
| 21 | 0.00050 | 0.00185 | 0.00190 | 0.00193 | 0.00197 | 0.00199 |
| 22 | 0.00081 | 0.00205 | 0.00211 | 0.00214 | 0.00219 | 0.00222 |
| 23 | 0.00254 | 0.00233 | 0.00238 | 0.00241 | 0.00244 | 0.00246 |
| 24 | 0.00303 | 0.00266 | 0.00268 | 0.00269 | 0.00271 | 0.00271 |
| 25 | 0.00328 | 0.00299 | 0.00298 | 0.00297 | 0.00297 | 0.00296 |

Table 4.9: Graduated mortality rate using Modified W-H at z=4

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00263 | 0.00257 | 0.00251 | 0.00241 | 0.00233 |
| 19 | 0.00255 | 0.00191 | 0.00189 | 0.00188 | 0.00185 | 0.00184 |
| 20 | 0.00283 | 0.00158 | 0.00159 | 0.00160 | 0.00162 | 0.00165 |
| 21 | 0.00050 | 0.00156 | 0.00159 | 0.00161 | 0.00166 | 0.00170 |
| 22 | 0.00081 | 0.00179 | 0.00181 | 0.00184 | 0.00188 | 0.00192 |
| 23 | 0.00254 | 0.00217 | 0.00219 | 0.00220 | 0.00223 | 0.00225 |
| 24 | 0.00303 | 0.00261 | 0.00262 | 0.00263 | 0.00263 | 0.00264 |
| 25 | 0.00328 | 0.00306 | 0.00305 | 0.00304 | 0.00303 | 0.00301 |

Table 4.10: Graduated mortality rate using Modified W-H at z=5

| $x$ | $\hat{q}_x$ | Graduated rates | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00270 | 0.00272 | 0.00274 | 0.00278 | 0.00280 |
| 19 | 0.00255 | 0.00194 | 0.00194 | 0.00193 | 0.00191 | 0.00190 |
| 20 | 0.00283 | 0.00157 | 0.00156 | 0.00155 | 0.00152 | 0.00150 |
| 21 | 0.00050 | 0.00153 | 0.00152 | 0.00152 | 0.00150 | 0.00149 |
| 22 | 0.00081 | 0.00174 | 0.00174 | 0.00174 | 0.00174 | 0.00175 |
| 23 | 0.00254 | 0.00213 | 0.00213 | 0.00213 | 0.00215 | 0.00216 |
| 24 | 0.00303 | 0.00259 | 0.00259 | 0.00260 | 0.00262 | 0.00264 |
| 25 | 0.00328 | 0.00305 | 0.00305 | 0.00306 | 0.00308 | 0.00310 |

The GCV criterion (equation (3.67) was then applied to select the value of $h$ that minimized the GCV score for the graduated mortality rates. The GCV scores for the various values of $h$ are given in Table (4.11)

Table 4.11: GCV Scores

| | h=100 | | h=150 | | h=200 | |
|---|---|---|---|---|---|---|
| | W-H | M W-H | W-H | M W-H | W-H | M W-H |
| z=2 | 0.0604 | 0.0751 | 0.0246 | 0.0285 | 0.0161 | 0.0182 |
| z=3 | 0.000551 | 0.000532 | 0.000781 | 0.00075 | 0.00101 | 0.000971 |
| z=4 | 0.0000549 | 0.000054 | 0.0000654 | 0.0000643 | 0.000074 | 0.0000727 |
| z=5 | 0.0000143 | 0.0000132 | 0.0000163 | 0.0000149 | 0.0000179 | 0.0000163 |
| | h=300 | | h=400 | | | |
| | W-H | M W-H | W-H | M W-H | | |
| z=2 | 0.0104 | 0.0115 | 0.0082 | 0.00892 | | |
| z=3 | 0.0015 | 0.00143 | 0.00203 | 0.00192 | | |
| z=4 | 0.0000881 | 0.0000865 | 0.0004 | 0.000098 | | |
| z=5 | 0.0000206 | 0.0000186 | 0.0000227 | 0.0000205 | | |

The GCV minimum scores were obtained at $h = 400; z = 2$ and $h = 100; z = 3, 4$ and 5 for both W-H and MW-H methods of graduation. The GCV criterion selects $h = 400$ at $z = 2$ and $h = 100$ at $z = 3, 4, 5$ for both Whittaker-Henderson and Modified Whittaker-Henderson. The plots of the graduated mortality rates are displayed in Figures 4.3 - 4.5. The shapes of the graduated curves of the mortality rates for both W-H and MW-H as shown in Figures (4.3) and (4.4 ) were found to be very identical though the actual graduated values are different.

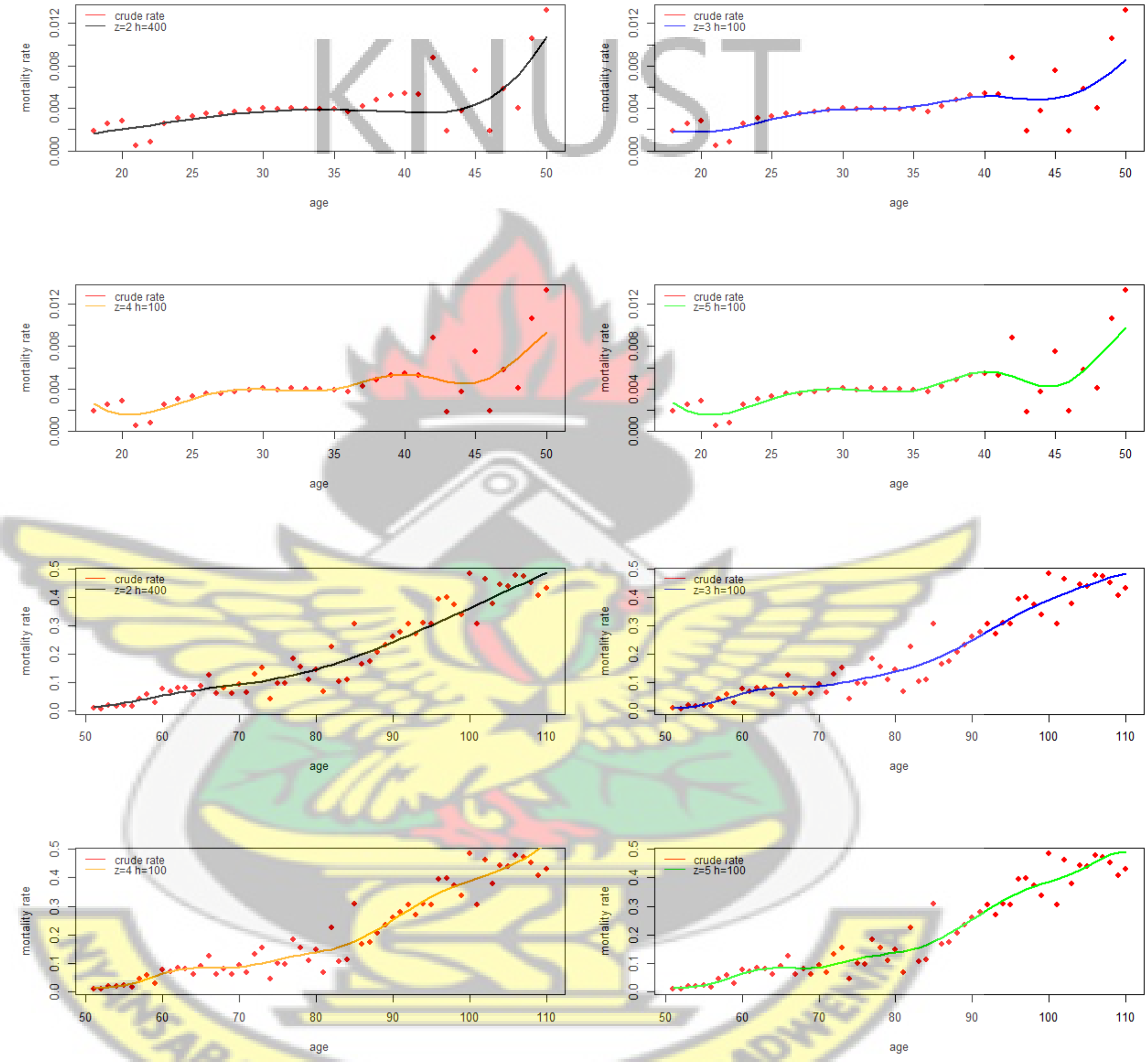

Figure 4.3: Plot of Whittaker-Henderson graduated mortality rates (solid line) of 18-49 year-olds (top two rows) and 50-110 year-olds (bottom two rows) at h=100 for z = 3,4,5 and h=400 for z = 2, and crude mortality rates (red dots) at various ages

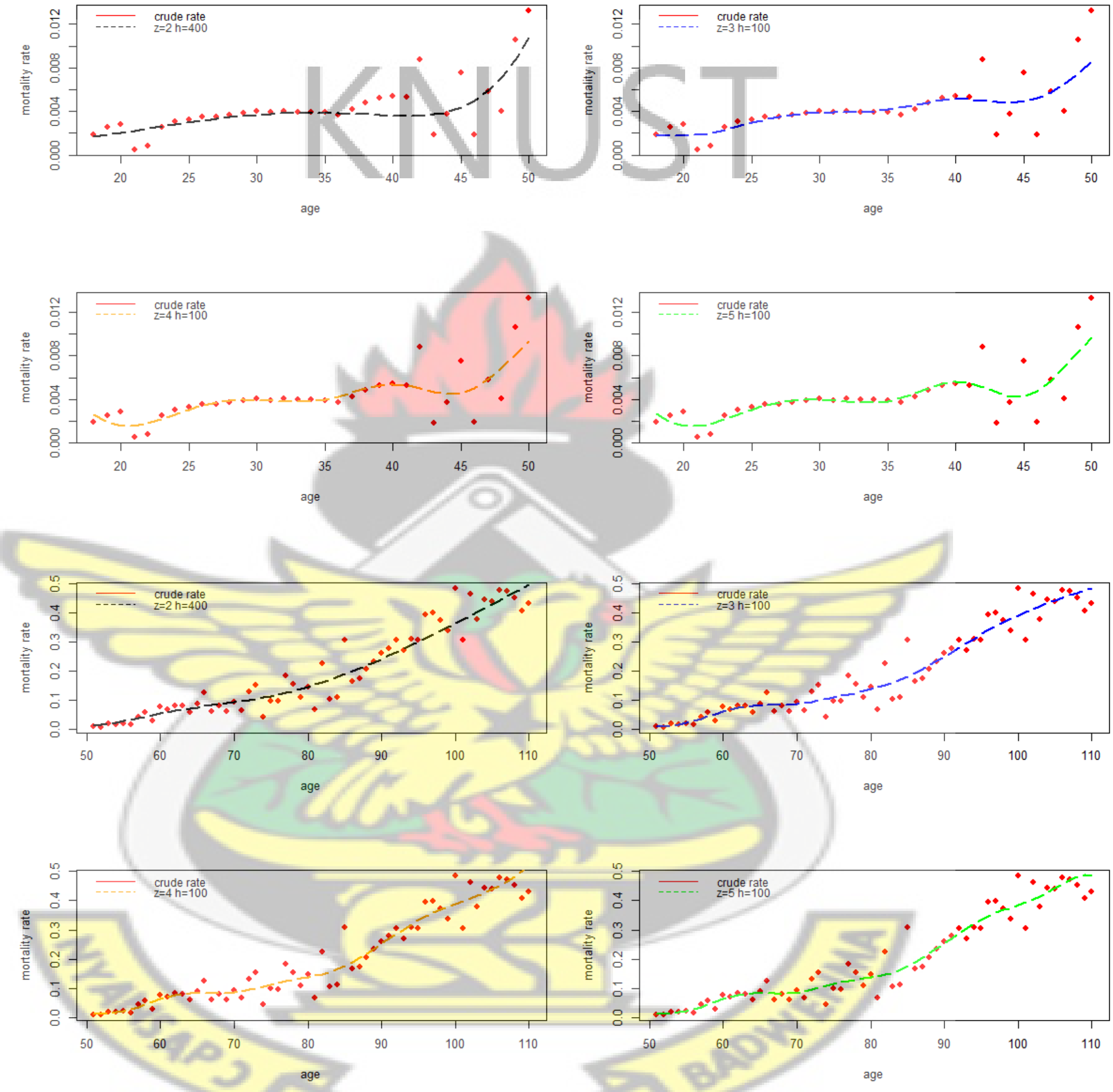

Figure 4.4: Plot of Modified Whittaker-Henderson graduated mortality rates (dashed line) of 18-49 year-olds (top two rows) and 50-110 year-olds (bottom two rows) at h=100 for z = 3,4,5 and h=400 for z = 2, and crude mortality rates (red dots) at various ages

For both graduation methods, the graduated mortality rates generally exhibit a wiggly mortality pattern and increase gradually at young ages (18 - 49 years) after which there is a continuous rise at older ages (50 -110 years). The graduated mortality rates at $z = 2$; $h = 400$ and $z = 3$; $h = 100$, provided a good fit to the crude mortality for both the young and adult ages with less significant wiggling, unlike the other selected pairs of $z$ and $h$ (see Figures 4.3 and (4.4)). However, a very smooth curve is observed for adults at $h = 100$ and $z = 3$ while the assumption of a continuous monotonically increasing mortality rates (Miller, 1946) is violated for young ages at $z = 4$ and 5; $h = 100$. Apart from the graduated mortality rates produced at $z = 3$; $h = 100$, the other pairs tend to rise sharply between ages 45 and 50.

The fitness and smoothness of the graduated mortality rates were further assessed by various statistical tests including mean absolute percentage error (MAPE), $R^2$, MSE and Sign tests used in Debon et al. (2006) and Tomas (2013). Thus, they were applied to compare the crude mortality rates to the graduated rates produced by the W-H and MW-H methods. The best graduated mortality rates were produced at $z = 3$, $h = 100$ for both W-H and MW-H methods. These graduated sets were chosen not only due to their fit to the crude rates but also because they show a commensurate degree smoothness. The two sets of graduated values obtained by the two methods at $z = 3$, $h = 100$ were evaluated to ascertain the optimal graduated rates to be used in the annuity pricing model. The results of the various significance tests performed are shown in Table (4.12).

Table 4.12: Comparison of W-H and M W-H graduated rates at $z = 3$, $h = 100$

| Test | | W-H | M W-H |
|---|---|---|---|
| MAPE | (%) | 27.32% | 28.34% |
| $R^2$ | value | 0.9826 | 0.9724 |
| MSE | value | 0.00102 | 0.00152 |
| Sign test; | +(-) | 47(46) | 48(45) |
| | p-value | 1 | 0.8357 |

The two methods produced results which make it difficult to choose one of them because the p-values for the sign test show no significant difference between the

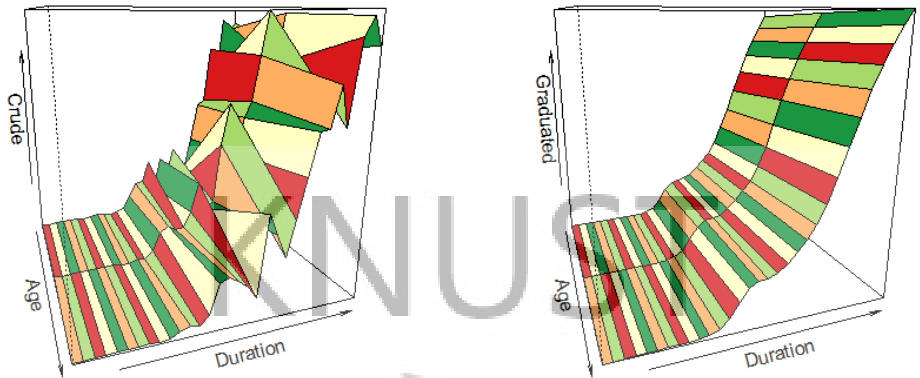

Figure 4.5: Three-dimensional plots of crude and estimated graduated mortality rates at $h = 100; z = 3$ via M-W method

crude and graduated mortality rates. However, the W-H method was selected due to its least value of MAPE and MSE, and highest $R_2$ (0.9826) and p-value (1). Figure (4.5) illustrates a three-dimensional plot of the crude and graduated mortality rates at $h = 100; z = 3$ via M-W method. The crude mortality rates are indicated by the rough surface (left) and the graduated mortality rates are indicated by the smooth surface (right) of Figure (4.5). Table (4.13) shows the complete table.

Table 4.13: Crude and graduated mortality rates at $z = 3; h = 100$ using W-H

| $x$ | Crude rates | Graduated rates | $x$ | Crude rates | Graduated rates |
|---|---|---|---|---|---|
| 18 | 0.00190 | 0.00186 | 65 | 0.08862 | 0.08273 |
| 19 | 0.00255 | 0.00175 | 66 | 0.12521 | 0.08344 |
| 20 | 0.00283 | 0.00175 | 67 | 0.06149 | 0.08360 |
| 21 | 0.00050 | 0.00185 | 68 | 0.07967 | 0.08392 |
| 22 | 0.00081 | 0.00205 | 69 | 0.06078 | 0.08500 |
| 23 | 0.00254 | 0.00233 | 70 | 0.09321 | 0.08720 |
| 24 | 0.00303 | 0.00266 | 71 | 0.06573 | 0.09049 |
| 25 | 0.00328 | 0.00299 | 72 | 0.13084 | 0.09468 |
| 26 | 0.00351 | 0.00328 | 73 | 0.15214 | 0.09941 |
| 27 | 0.00351 | 0.00353 | 74 | 0.04316 | 0.10447 |
| 28 | 0.00369 | 0.00371 | 75 | 0.09818 | 0.10986 |
| 29 | 0.00386 | 0.00382 | 76 | 0.09658 | 0.11546 |
| 30 | 0.00405 | 0.00388 | 77 | 0.18268 | 0.12109 |
| 31 | 0.00392 | 0.00391 | 78 | 0.15416 | 0.12664 |
| 32 | 0.00402 | 0.00393 | 79 | 0.10955 | 0.13221 |
| 33 | 0.00394 | 0.00396 | 80 | 0.14568 | 0.13808 |
| 34 | 0.00394 | 0.00404 | 81 | 0.06808 | 0.14458 |
| 35 | 0.00390 | 0.00418 | 82 | 0.22578 | 0.15198 |
| 36 | 0.00370 | 0.00437 | 83 | 0.10421 | 0.16046 |
| 37 | 0.00423 | 0.00460 | 84 | 0.11099 | 0.17012 |
| 38 | 0.00483 | 0.00483 | 85 | 0.30733 | 0.18096 |
| 39 | 0.00523 | 0.00502 | 86 | 0.16512 | 0.19287 |
| 40 | 0.00541 | 0.00512 | 87 | 0.17431 | 0.20579 |
| 41 | 0.00529 | 0.00511 | 88 | 0.20576 | 0.21962 |
| 42 | 0.00878 | 0.00501 | 89 | 0.23325 | 0.23420 |
| 43 | 0.00185 | 0.00488 | 90 | 0.26033 | 0.24934 |
| 44 | 0.00372 | 0.00482 | 91 | 0.27787 | 0.26482 |
| 45 | 0.00755 | 0.00493 | 92 | 0.30544 | 0.28042 |
| 46 | 0.00188 | 0.00524 | 93 | 0.27072 | 0.29594 |
| 47 | 0.00582 | 0.00578 | 94 | 0.30918 | 0.31121 |
| 48 | 0.00403 | 0.00655 | 95 | 0.30627 | 0.32608 |
| 49 | 0.01058 | 0.00750 | 96 | 0.39349 | 0.34042 |
| 50 | 0.01327 | 0.00860 | 97 | 0.39879 | 0.35414 |
| 51 | 0.00952 | 0.00996 | 98 | 0.37378 | 0.36721 |
| 52 | 0.00774 | 0.01184 | 99 | 0.33804 | 0.37962 |
| 53 | 0.01952 | 0.01451 | 100 | 0.48429 | 0.39143 |
| 54 | 0.01720 | 0.01824 | 101 | 0.30538 | 0.40267 |
| 55 | 0.02129 | 0.02322 | 102 | 0.46266 | 0.41343 |
| 56 | 0.01596 | 0.02950 | 103 | 0.37844 | 0.42374 |
| 57 | 0.04338 | 0.03692 | 104 | 0.44348 | 0.43363 |
| 58 | 0.05745 | 0.04500 | 105 | 0.43904 | 0.44312 |
| 59 | 0.02949 | 0.05324 | 106 | 0.47714 | 0.45219 |
| 60 | 0.07676 | 0.06115 | 107 | 0.47350 | 0.46086 |
| 61 | 0.06915 | 0.06817 | 108 | 0.45281 | 0.46910 |
| 62 | 0.08164 | 0.07390 | 109 | 0.40650 | 0.47692 |
| 63 | 0.08041 | 0.07818 | 110 | 0.43044 | 0.48432 |
| 64 | 0.05937 | 0.08105 | | | |

## 4.4 Estimation of Life Expectancy and APV factors

### 4.4.1 Estimated Life Expectancy

The life expectancy of insured lives provides life insurers a fair idea of how many more years annuitants or policyholders are expected to live. This guides life insurers in their investment decisions and aids in making proper provision for payments and other liabilities as and when they are due. The life expectancy of members of the pension scheme for each age $x$ was computed per equation (3.71) from age 60 since the retirement age in Ghana is 60 years. Table (4.14) presents the probability that (x) survives the next birthday, $(p_x)$ and their corresponding life expectancies, $(e_x)$ while the plot of life expectancy against age is shown in figure (4.6).

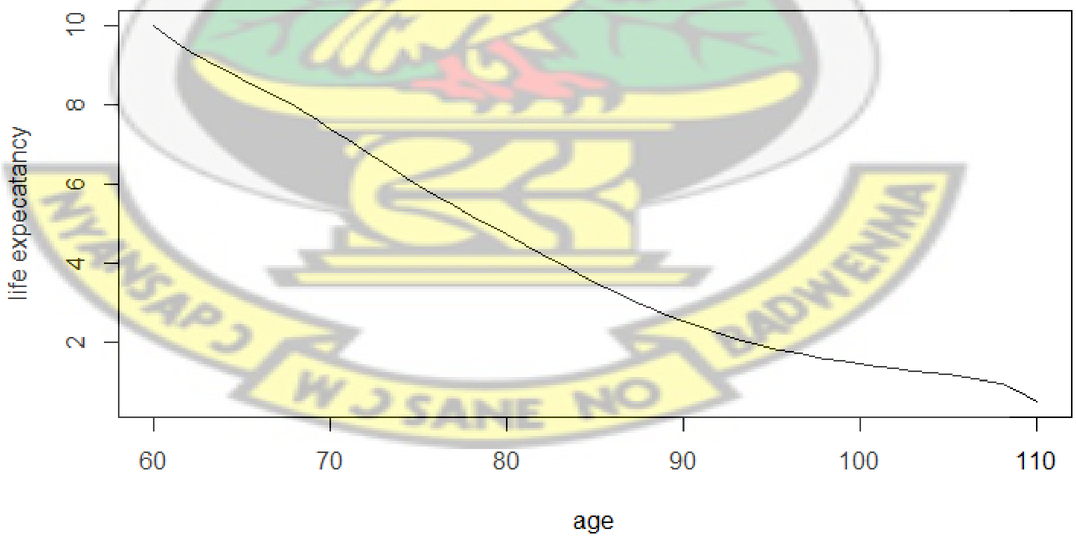

Figure 4.6: Plot of estimated life expectancy versus age of the pension scheme member.

Table 4.14: Estimated life expectancy, $e_x$ and probability of survival, $p_x$ at age x

| $x$ | $p_x$ | $e_x$ | $x$ | $p_x$ | $e_x$ |
|---|---|---|---|---|---|
| 60 | 0.93885 | 10.00 | 86 | 0.80713 | 3.30 |
| 61 | 0.93183 | 9.65 | 87 | 0.79421 | 3.09 |
| 62 | 0.92610 | 9.36 | 88 | 0.78038 | 2.89 |
| 63 | 0.92182 | 9.10 | 89 | 0.76580 | 2.70 |
| 64 | 0.91895 | 8.88 | 90 | 0.75066 | 2.53 |
| 65 | 0.91727 | 8.66 | 91 | 0.73518 | 2.37 |
| 66 | 0.91656 | 8.44 | 92 | 0.71958 | 2.23 |
| 67 | 0.91640 | 8.21 | 93 | 0.70406 | 2.09 |
| 68 | 0.91608 | 7.96 | 94 | 0.68879 | 1.97 |
| 69 | 0.91500 | 7.68 | 95 | 0.67392 | 1.86 |
| 70 | 0.91280 | 7.40 | 96 | 0.65958 | 1.77 |
| 71 | 0.90951 | 7.11 | 97 | 0.64586 | 1.68 |
| 72 | 0.90532 | 6.81 | 98 | 0.63279 | 1.60 |
| 73 | 0.90059 | 6.52 | 99 | 0.62038 | 1.53 |
| 74 | 0.89553 | 6.25 | 100 | 0.60857 | 1.46 |
| 75 | 0.89014 | 5.97 | 101 | 0.59733 | 1.40 |
| 76 | 0.88454 | 5.71 | 102 | 0.58657 | 1.34 |
| 77 | 0.87891 | 5.46 | 103 | 0.57626 | 1.29 |
| 78 | 0.87336 | 5.21 | 104 | 0.56637 | 1.24 |
| 79 | 0.86779 | 4.96 | 105 | 0.55688 | 1.18 |
| 80 | 0.86192 | 4.72 | 106 | 0.54781 | 1.12 |
| 81 | 0.85542 | 4.48 | 107 | 0.53914 | 1.05 |
| 82 | 0.84802 | 4.23 | 108 | 0.53090 | 0.95 |
| 83 | 0.83954 | 3.99 | 109 | 0.52308 | 0.79 |
| 84 | 0.82988 | 3.75 | 110 | 0.51568 | 0.52 |
| 85 | 0.81904 | 3.52 | | | |

It is inferred from Table (4.14) that policyholders after retirement are expected to live 10 more years on the average. The plot of the estimated life expectancy against age as given in Figure 4.6 shows a steep decline of expected number of years to be lived by policyholders between ages 60 and 90 after which the curve then flattens, declining gradually until age 110. Expectedly, $p_x$ decreases gradually from 0.93885 at age 60 to 0.51568 at age 110.

### 4.4.2 Estimated APV Factors

After selecting the desired graduated mortality rates (graduated $\hat{q}_x$) for all ages, a life and actuarial table is developed for all the ages. The selected graduated mortality rates are then used to estimate the whole life APV factors only at specific interest rates of 4.55%, 10% and 15% per annum (p.a). The estimated APV factors were tabulated to develop annuity tables for selected ranges of ages and at each interest rate. The pure endowment factors (${}_nE_x = v^n {}_np_x$) also known as actuarial discounting factors were computed at each of the interest rates to allow for easy computation of products such as $n$-year temporary life annuity and $n$-year deferred whole life annuity. The multiple life annuities for $(x)$ and $(y)$ were assumed to follow the same mortality table while their APV factors were also computed by assuming independence.

The various tables constructed for estimating APV factors at single and joint lives at varying interest rates $(4.55\%, 10\%$ and $15\%)$ are shown in Tables (4.15)-(4.32).

Table 4.15: Estimated $a_x$ at $i = 4.55\%$

| $x$ | $a_x$ | $x$ | $a_x$ |
|---|---|---|---|
| 60 | 7.06696 | 86 | 2.82703 |
| 61 | 6.86974 | 87 | 2.66193 |
| 62 | 6.70775 | 88 | 2.50418 |
| 63 | 6.57257 | 89 | 2.35492 |
| 64 | 6.45441 | 90 | 2.21503 |
| 65 | 6.34325 | 91 | 2.08504 |
| 66 | 6.23001 | 92 | 1.96514 |
| 67 | 6.10644 | 93 | 1.85521 |
| 68 | 5.96669 | 94 | 1.75491 |
| 69 | 5.80964 | 95 | 1.66374 |
| 70 | 5.63823 | 96 | 1.58108 |
| 71 | 5.45790 | 97 | 1.50618 |
| 72 | 5.27397 | 98 | 1.43816 |
| 73 | 5.09059 | 99 | 1.37613 |
| 74 | 4.90970 | 100 | 1.31914 |
| 75 | 4.73190 | 101 | 1.26622 |
| 76 | 4.55778 | 102 | 1.21626 |
| 77 | 4.38716 | 103 | 1.16785 |
| 78 | 4.21870 | 104 | 1.11882 |
| 79 | 4.05021 | 105 | 1.06530 |
| 80 | 3.87964 | 106 | 1.00002 |
| 81 | 3.70596 | 107 | 0.90855 |
| 82 | 3.52945 | 108 | 0.76185 |
| 83 | 3.35136 | 109 | 0.50032 |
| 84 | 3.17353 | 110 | 0.00000 |
| 85 | 2.99808 | | |

Table 4.16: Estimated ${}_nE_x$ at $i = 4.55\%$

| $x$ | ${}_1E_x$ | ${}_2E_x$ | ${}_3E_x$ | ${}_4E_x$ | ${}_5E_x$ | ${}_{10}E_x$ | ${}_{20}E_x$ |
|---|---|---|---|---|---|---|---|
| 18 | 0.95470 | 0.91155 | 0.87036 | 0.83094 | 0.79315 | 0.62561 | 0.38502 |
| 19 | 0.95481 | 0.91166 | 0.87037 | 0.83078 | 0.79278 | 0.62445 | 0.38387 |
| 20 | 0.95481 | 0.91156 | 0.87011 | 0.83030 | 0.79205 | 0.62315 | 0.38261 |
| 21 | 0.95471 | 0.91129 | 0.86960 | 0.82954 | 0.79107 | 0.62182 | 0.38132 |
| 22 | 0.95452 | 0.91085 | 0.86889 | 0.82859 | 0.78994 | 0.62054 | 0.38008 |
| 23 | 0.95425 | 0.91029 | 0.86808 | 0.82757 | 0.78876 | 0.61937 | 0.37895 |
| 24 | 0.95394 | 0.90969 | 0.86725 | 0.82658 | 0.78767 | 0.61836 | 0.37798 |
| 25 | 0.95362 | 0.90913 | 0.86649 | 0.82571 | 0.78676 | 0.61750 | 0.37716 |
| 26 | 0.95334 | 0.90863 | 0.86587 | 0.82502 | 0.78605 | 0.61677 | 0.37643 |
| 27 | 0.95310 | 0.90824 | 0.86540 | 0.82452 | 0.78556 | 0.61609 | 0.37569 |
| 28 | 0.95293 | 0.90798 | 0.86509 | 0.82421 | 0.78524 | 0.61543 | 0.37484 |
| 29 | 0.95283 | 0.90782 | 0.86492 | 0.82403 | 0.78504 | 0.61474 | 0.37377 |
| 30 | 0.95277 | 0.90774 | 0.86482 | 0.82391 | 0.78487 | 0.61400 | 0.37239 |
| 31 | 0.95274 | 0.90770 | 0.86476 | 0.82378 | 0.78464 | 0.61323 | 0.37063 |
| 32 | 0.95272 | 0.90765 | 0.86464 | 0.82356 | 0.78427 | 0.61249 | 0.36837 |
| 33 | 0.95269 | 0.90755 | 0.86443 | 0.82319 | 0.78375 | 0.61183 | 0.36545 |
| 34 | 0.95262 | 0.90735 | 0.86407 | 0.82266 | 0.78306 | 0.61127 | 0.36158 |
| 35 | 0.95248 | 0.90705 | 0.86358 | 0.82201 | 0.78229 | 0.61079 | 0.35642 |
| 36 | 0.95230 | 0.90667 | 0.86302 | 0.82132 | 0.78155 | 0.61033 | 0.34961 |
| 37 | 0.95208 | 0.90625 | 0.86246 | 0.82070 | 0.78097 | 0.60979 | 0.34078 |
| 38 | 0.95186 | 0.90587 | 0.86201 | 0.82028 | 0.78065 | 0.60907 | 0.32972 |
| 39 | 0.95168 | 0.90560 | 0.86176 | 0.82013 | 0.78061 | 0.60802 | 0.31641 |
| 40 | 0.95158 | 0.90552 | 0.86177 | 0.82025 | 0.78077 | 0.60650 | 0.30108 |
| 41 | 0.95159 | 0.90562 | 0.86198 | 0.82049 | 0.78092 | 0.60438 | 0.28412 |
| 42 | 0.95169 | 0.90583 | 0.86223 | 0.82064 | 0.78081 | 0.60143 | 0.26611 |
| 43 | 0.95181 | 0.90600 | 0.86230 | 0.82045 | 0.78021 | 0.59731 | 0.24769 |
| 44 | 0.95187 | 0.90596 | 0.86199 | 0.81971 | 0.77890 | 0.59153 | 0.22944 |
| 45 | 0.95176 | 0.90557 | 0.86116 | 0.81828 | 0.77680 | 0.58355 | 0.21187 |
| 46 | 0.95147 | 0.90480 | 0.85976 | 0.81617 | 0.77394 | 0.57282 | 0.19530 |
| 47 | 0.95095 | 0.90361 | 0.85780 | 0.81341 | 0.77027 | 0.55885 | 0.17995 |
| 48 | 0.95022 | 0.90205 | 0.85537 | 0.80999 | 0.76557 | 0.54135 | 0.16586 |
| 49 | 0.94931 | 0.90018 | 0.85243 | 0.80568 | 0.75944 | 0.52040 | 0.15295 |
| 50 | 0.94825 | 0.89795 | 0.84871 | 0.79999 | 0.75122 | 0.49641 | 0.14100 |
| 51 | 0.94695 | 0.89502 | 0.84365 | 0.79221 | 0.74014 | 0.47010 | 0.12982 |
| 52 | 0.94516 | 0.89091 | 0.83659 | 0.78160 | 0.72553 | 0.44246 | 0.11926 |
| 53 | 0.94260 | 0.88513 | 0.82696 | 0.76763 | 0.70712 | 0.41467 | 0.10927 |
| 54 | 0.93903 | 0.87731 | 0.81438 | 0.75018 | 0.68524 | 0.38788 | 0.09985 |
| 55 | 0.93427 | 0.86725 | 0.79888 | 0.72973 | 0.66081 | 0.36307 | 0.09108 |
| 56 | 0.92826 | 0.85509 | 0.78107 | 0.70730 | 0.63515 | 0.34095 | 0.08300 |
| 57 | 0.92117 | 0.84143 | 0.76196 | 0.68424 | 0.60984 | 0.32200 | 0.07565 |
| 58 | 0.91344 | 0.82717 | 0.74279 | 0.66203 | 0.58643 | 0.30639 | 0.06904 |
| 59 | 0.90556 | 0.81318 | 0.72477 | 0.64200 | 0.56605 | 0.29390 | 0.06314 |
| 60 | 0.89799 | 0.80036 | 0.70896 | 0.62509 | 0.54943 | 0.28404 | 0.05787 |

Table 4.17: Estimated $a_{xy}$ at $i = 4.55\%$, age difference $d = (x - y)$

| $d$ | 0 | 1 | 2 | 3 | $d$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 4.41692 | 4.52870 | 4.66137 | 4.81245 | 86 | 1.47257 | 1.52780 | 1.58125 | 1.63210 |
| 61 | 4.23904 | 4.32673 | 4.43548 | 4.56457 | 87 | 1.36327 | 1.41625 | 1.46811 | 1.51817 |
| 62 | 4.10408 | 4.17072 | 4.25640 | 4.36270 | 88 | 1.25961 | 1.30986 | 1.35962 | 1.40821 |
| 63 | 4.00293 | 4.05290 | 4.11814 | 4.20214 | 89 | 1.16246 | 1.20958 | 1.25680 | 1.30346 |
| 64 | 3.92504 | 3.96348 | 4.01236 | 4.07635 | 90 | 1.07239 | 1.11610 | 1.16042 | 1.20477 |
| 65 | 3.85942 | 3.89173 | 3.92917 | 3.97700 | 91 | 0.98972 | 1.02987 | 1.07105 | 1.11275 |
| 66 | 3.79570 | 3.82700 | 3.85827 | 3.89468 | 92 | 0.91447 | 0.95105 | 0.98896 | 1.02779 |
| 67 | 3.72382 | 3.75910 | 3.78921 | 3.81935 | 93 | 0.84645 | 0.87955 | 0.91417 | 0.95000 |
| 68 | 3.63599 | 3.67910 | 3.71293 | 3.74173 | 94 | 0.78527 | 0.81507 | 0.84648 | 0.87930 |
| 69 | 3.52981 | 3.58191 | 3.62323 | 3.65549 | 95 | 0.73049 | 0.75721 | 0.78557 | 0.81542 |
| 70 | 3.40792 | 3.46771 | 3.51767 | 3.55709 | 96 | 0.68160 | 0.70548 | 0.73098 | 0.75801 |
| 71 | 3.27624 | 3.34080 | 3.39815 | 3.44589 | 97 | 0.63803 | 0.65934 | 0.68218 | 0.70655 |
| 72 | 3.14081 | 3.20718 | 3.26911 | 3.32399 | 98 | 0.59914 | 0.61818 | 0.63862 | 0.66052 |
| 73 | 3.00646 | 3.07228 | 3.13596 | 3.19527 | 99 | 0.56434 | 0.58140 | 0.59971 | 0.61934 |
| 74 | 2.87548 | 2.93963 | 3.00277 | 3.06377 | 100 | 0.53303 | 0.54840 | 0.56483 | 0.58245 |
| 75 | 2.74864 | 2.81075 | 2.87225 | 2.93272 | 101 | 0.50473 | 0.51863 | 0.53346 | 0.54931 |
| 76 | 2.62682 | 2.68644 | 2.74594 | 2.80482 | 102 | 0.47895 | 0.49161 | 0.50505 | 0.51937 |
| 77 | 2.51009 | 2.56718 | 2.62424 | 2.68116 | 103 | 0.45537 | 0.46694 | 0.47920 | 0.49219 |
| 78 | 2.39723 | 2.45238 | 2.50691 | 2.56141 | 104 | 0.43368 | 0.44428 | 0.45548 | 0.46734 |
| 79 | 2.28585 | 2.34021 | 2.39276 | 2.44470 | 105 | 0.41348 | 0.42318 | 0.43343 | 0.44425 |
| 80 | 2.17353 | 2.22828 | 2.27992 | 2.32980 | 106 | 0.39398 | 0.40278 | 0.41208 | 0.42190 |
| 81 | 2.05883 | 2.11468 | 2.16653 | 2.21537 | 107 | 0.37259 | 0.38037 | 0.38860 | 0.39729 |
| 82 | 1.94162 | 1.99862 | 2.05137 | 2.10025 | 108 | 0.34014 | 0.34649 | 0.35321 | 0.36029 |
| 83 | 1.82277 | 1.88051 | 1.93424 | 1.98384 | 109 | 0.26171 | 0.26562 | 0.26974 | 0.27408 |
| 84 | 1.70379 | 1.76154 | 1.81587 | 1.86630 | 110 | - | 0.00000 | 0.00000 | 0.00000 |
| 85 | 1.58648 | 1.64339 | 1.69766 | 1.74860 | | | | | |

Table 4.18: Estimated $a_{xy}$ at $i = 4.55\%$, age difference $d(= x - y)$

| $d$ | 4 | 5 | 10 | 20 | $d$ | 4 | 5 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 4.97759 | 5.15063 | 5.94202 | 6.68759 | 86 | 1.67969 | 1.72365 | 1.90230 | 2.14751 |
| 61 | 4.71152 | 4.87207 | 5.67441 | 6.48561 | 87 | 1.56569 | 1.61006 | 1.78575 | 2.03498 |
| 62 | 4.48891 | 4.63256 | 5.43066 | 6.31413 | 88 | 1.45501 | 1.49934 | 1.67463 | 1.92323 |
| 63 | 4.30645 | 4.43031 | 5.20428 | 6.16409 | 89 | 1.34893 | 1.39263 | 1.56888 | 1.81265 |
| 64 | 4.15890 | 4.26149 | 4.98945 | 6.02541 | 90 | 1.24850 | 1.29103 | 1.46822 | 1.70460 |
| 65 | 4.03981 | 4.12099 | 4.78201 | 5.88838 | 91 | 1.15440 | 1.19539 | 1.37250 | 1.60092 |
| 66 | 3.94141 | 4.00302 | 4.58009 | 5.74480 | 92 | 1.06705 | 1.10618 | 1.28172 | 1.50318 |
| 67 | 3.85464 | 3.90020 | 4.38321 | 5.58749 | 93 | 0.98665 | 1.02365 | 1.19600 | 1.41243 |
| 68 | 3.77064 | 3.80472 | 4.19241 | 5.41170 | 94 | 0.91321 | 0.94786 | 1.11546 | 1.32891 |
| 69 | 3.68289 | 3.71050 | 4.01015 | 5.21696 | 95 | 0.84657 | 0.87874 | 1.04022 | 1.25244 |
| 70 | 3.58770 | 3.61364 | 3.83976 | 5.00606 | 96 | 0.78643 | 0.81606 | 0.97031 | 1.18280 |
| 71 | 3.48336 | 3.51228 | 3.68442 | 4.78357 | 97 | 0.73237 | 0.75949 | 0.90557 | 1.11957 |
| 72 | 3.36948 | 3.40499 | 3.54516 | 4.55413 | 98 | 0.68385 | 0.70856 | 0.84575 | 1.06203 |
| 73 | 3.24768 | 3.29094 | 3.42079 | 4.32231 | 99 | 0.64037 | 0.66276 | 0.79062 | 1.00912 |
| 74 | 3.12049 | 3.17045 | 3.30801 | 4.09167 | 100 | 0.60135 | 0.62157 | 0.73987 | 0.95973 |
| 75 | 2.99109 | 3.04524 | 3.20261 | 3.86563 | 101 | 0.56630 | 0.58451 | 0.69327 | 0.91291 |
| 76 | 2.86266 | 2.91842 | 3.10084 | 3.64825 | 102 | 0.53468 | 0.55108 | 0.65051 | 0.86793 |
| 77 | 2.73745 | 2.79269 | 2.99876 | 3.44319 | 103 | 0.50604 | 0.52083 | 0.61131 | 0.82424 |
| 78 | 2.61575 | 2.66944 | 2.89257 | 3.25284 | 104 | 0.47991 | 0.49329 | 0.57527 | 0.78122 |
| 79 | 2.49660 | 2.54833 | 2.77990 | 3.07746 | 105 | 0.45570 | 0.46782 | 0.54173 | 0.73773 |
| 80 | 2.37910 | 2.42835 | 2.66031 | 2.91617 | 106 | 0.43226 | 0.44321 | 0.50917 | 0.69105 |
| 81 | 2.26251 | 2.30911 | 2.53520 | 2.76768 | 107 | 0.40644 | 0.41607 | 0.47329 | 0.63403 |
| 82 | 2.14623 | 2.19059 | 2.40682 | 2.63014 | 108 | 0.36773 | 0.37554 | 0.42106 | 0.54810 |
| 83 | 2.02971 | 2.07280 | 2.27763 | 2.50138 | 109 | 0.27862 | 0.28336 | 0.31039 | 0.38314 |
| 84 | 1.91274 | 1.95561 | 2.14948 | 2.37922 | 110 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 85 | 1.79575 | 1.83908 | 2.02386 | 2.26176 | | | | | |

Table 4.19: Estimated $a_{\overline{xy}}$ at $i = 4.55\%$, age difference $d(= x - y)$

| $d$ | 0 | 1 | 2 | 3 | $d$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 9.71700 | 9.84336 | 9.99179 | 10.16383 | 86 | 4.18148 | 4.29730 | 4.41931 | 4.54628 |
| 61 | 9.50045 | 9.60997 | 9.73936 | 9.89137 | 87 | 3.96060 | 4.07270 | 4.19190 | 4.31729 |
| 62 | 9.31142 | 9.40677 | 9.51832 | 9.65015 | 88 | 3.74874 | 3.85624 | 3.97158 | 4.09404 |
| 63 | 9.14221 | 9.22742 | 9.32417 | 9.43739 | 89 | 3.54738 | 3.64952 | 3.76006 | 3.87849 |
| 64 | 8.98377 | 9.06350 | 9.14980 | 9.24780 | 90 | 3.35767 | 3.45386 | 3.55879 | 3.67220 |
| 65 | 8.82709 | 8.90593 | 8.98665 | 9.07401 | 91 | 3.18036 | 3.27021 | 3.36891 | 3.47646 |
| 66 | 8.66432 | 8.74626 | 8.82615 | 8.90790 | 92 | 3.01581 | 3.09913 | 3.19121 | 3.29227 |
| 67 | 8.48905 | 8.57734 | 8.66048 | 8.74149 | 93 | 2.86398 | 2.94080 | 3.02609 | 3.12024 |
| 68 | 8.29740 | 8.39403 | 8.48378 | 8.56822 | 94 | 2.72455 | 2.79505 | 2.87357 | 2.96066 |
| 69 | 8.08948 | 8.19443 | 8.29285 | 8.38416 | 95 | 2.59699 | 2.66144 | 2.73338 | 2.81346 |
| 70 | 7.86855 | 7.98017 | 8.08726 | 8.18758 | 96 | 2.48057 | 2.53934 | 2.60502 | 2.67829 |
| 71 | 7.63956 | 7.75534 | 7.86940 | 7.97871 | 97 | 2.37433 | 2.42792 | 2.48774 | 2.55454 |
| 72 | 7.40713 | 7.52469 | 7.64309 | 7.75962 | 98 | 2.27717 | 2.32615 | 2.38062 | 2.44138 |
| 73 | 7.17472 | 7.29228 | 7.41253 | 7.53355 | 99 | 2.18792 | 2.23289 | 2.28260 | 2.33787 |
| 74 | 6.94391 | 7.06065 | 7.18090 | 7.30382 | 100 | 2.10524 | 2.14687 | 2.19246 | 2.24286 |
| 75 | 6.71515 | 6.83085 | 6.95024 | 7.07315 | 101 | 2.02772 | 2.06673 | 2.10890 | 2.15507 |
| 76 | 6.48874 | 6.60323 | 6.72153 | 6.84355 | 102 | 1.95357 | 1.99087 | 2.03034 | 2.07302 |
| 77 | 6.26422 | 6.37775 | 6.49481 | 6.61569 | 103 | 1.88034 | 1.91717 | 1.95488 | 1.99479 |
| 78 | 6.04018 | 6.15348 | 6.26957 | 6.38919 | 104 | 1.80396 | 1.84239 | 1.87959 | 1.91770 |
| 79 | 5.81458 | 5.92871 | 6.04461 | 6.16329 | 105 | 1.71712 | 1.76094 | 1.79973 | 1.83731 |
| 80 | 5.58574 | 5.70157 | 5.81842 | 5.93700 | 106 | 1.60606 | 1.66254 | 1.70676 | 1.74597 |
| 81 | 5.35309 | 5.47092 | 5.58964 | 5.70930 | 107 | 1.44450 | 1.52819 | 1.58525 | 1.63007 |
| 82 | 5.11728 | 5.23678 | 5.35771 | 5.47941 | 108 | 1.18357 | 1.32391 | 1.40866 | 1.46687 |
| 83 | 4.87994 | 5.00030 | 5.12308 | 5.24716 | 109 | 0.73893 | 0.99655 | 1.13912 | 1.22626 |
| 84 | 4.64327 | 4.76334 | 4.88710 | 5.01319 | 110 | - | 0.50032 | 0.76185 | 0.90855 |
| 85 | 4.40967 | 4.52822 | 4.65177 | 4.77893 | | | | | |

Table 4.20: Estimated $a_{\overline{xy}}$ at $i = 4.55\%$, age difference $d(= x - y)$

| $d$ | 4 | 5 | 10 | 20 | $d$ | 4 | 5 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 10.35956 | 10.57720 | 11.87727 | 14.58847 | 86 | 4.67678 | 4.80933 | 5.48250 | 6.90952 |
| 61 | 10.06754 | 10.26787 | 11.53441 | 14.31620 | 87 | 4.44760 | 4.58132 | 5.26334 | 6.73339 |
| 62 | 9.80504 | 9.98451 | 11.19534 | 14.03441 | 88 | 4.22269 | 4.35619 | 5.04825 | 6.54764 |
| 63 | 9.57122 | 9.72845 | 10.86207 | 13.74173 | 89 | 4.00407 | 4.13582 | 4.83626 | 6.35191 |
| 64 | 9.36247 | 9.49801 | 10.53685 | 13.43727 | 90 | 3.79356 | 3.92208 | 4.62645 | 6.14867 |
| 65 | 9.17318 | 9.28923 | 10.22211 | 13.12090 | 91 | 3.59258 | 3.71668 | 4.41850 | 5.94202 |
| 66 | 8.99635 | 9.09674 | 9.92012 | 12.79315 | 92 | 3.40227 | 3.52089 | 4.21286 | 5.73593 |
| 67 | 8.82436 | 8.91399 | 9.63255 | 12.45468 | 93 | 3.22348 | 3.33574 | 4.01056 | 5.53338 |
| 68 | 8.65046 | 8.73455 | 9.36048 | 12.10635 | 94 | 3.05673 | 3.16198 | 3.81298 | 5.33570 |
| 69 | 8.47000 | 8.55355 | 9.10459 | 11.74925 | 95 | 2.90221 | 3.00004 | 3.62160 | 5.14321 |
| 70 | 8.28054 | 8.36785 | 8.86544 | 11.38450 | 96 | 2.75979 | 2.85006 | 3.43780 | 4.95607 |
| 71 | 8.08098 | 8.17563 | 8.64322 | 11.01341 | 97 | 2.62902 | 2.71182 | 3.26254 | 4.77376 |
| 72 | 7.87118 | 7.97542 | 8.43656 | 10.63809 | 98 | 2.50921 | 2.58481 | 3.09658 | 4.59483 |
| 73 | 7.65256 | 7.76635 | 8.24237 | 10.26206 | 99 | 2.39951 | 2.46828 | 2.94044 | 4.41722 |
| 74 | 7.42744 | 7.54889 | 8.05609 | 9.88992 | 100 | 2.29887 | 2.36131 | 2.79430 | 4.23904 |
| 75 | 7.19871 | 7.32489 | 7.87254 | 9.52714 | 101 | 2.20610 | 2.26280 | 2.65800 | 4.05927 |
| 76 | 6.96908 | 7.09725 | 7.68694 | 9.17973 | 102 | 2.11974 | 2.17135 | 2.53089 | 3.87778 |
| 77 | 6.74030 | 6.86843 | 7.49483 | 8.85328 | 103 | 2.03794 | 2.08517 | 2.41176 | 3.69497 |
| 78 | 6.51265 | 6.63985 | 7.29283 | 8.55206 | 104 | 1.95805 | 2.00165 | 2.29846 | 3.51112 |
| 79 | 6.28551 | 6.41158 | 7.07995 | 8.27785 | 105 | 1.87583 | 1.91662 | 2.18731 | 3.32564 |
| 80 | 6.05832 | 6.18318 | 6.85756 | 8.03043 | 106 | 1.78402 | 1.82304 | 2.07194 | 3.13599 |
| 81 | 5.83061 | 5.95463 | 6.62866 | 7.80802 | 107 | 1.66996 | 1.70873 | 1.94143 | 2.93645 |
| 82 | 5.60192 | 5.72601 | 6.39659 | 7.60706 | 108 | 1.51294 | 1.55417 | 1.77895 | 2.71793 |
| 83 | 5.37186 | 5.49726 | 6.16431 | 7.42254 | 109 | 1.28700 | 1.33577 | 1.56606 | 2.47210 |
| 84 | 5.14042 | 5.26813 | 5.93374 | 7.24871 | 110 | 1.00002 | 1.06530 | 1.31914 | 2.21503 |
| 85 | 4.90828 | 5.03864 | 5.70611 | 7.07957 | | | | | |

Table 4.21: Estimated $a_x$ at $i = 10\%$

| $x$ | $a_x$ | $x$ | $a_x$ |
|---|---|---|---|
| 60 | 5.13620 | 86 | 2.40432 |
| 61 | 5.01781 | 87 | 2.27674 |
| 62 | 4.92339 | 88 | 2.15334 |
| 63 | 4.84788 | 89 | 2.03528 |
| 64 | 4.78494 | 90 | 1.92349 |
| 65 | 4.72766 | 91 | 1.81864 |
| 66 | 4.66946 | 92 | 1.72111 |
| 67 | 4.60400 | 93 | 1.63101 |
| 68 | 4.52641 | 94 | 1.54824 |
| 69 | 4.43517 | 95 | 1.47255 |
| 70 | 4.33190 | 96 | 1.40355 |
| 71 | 4.22030 | 97 | 1.34074 |
| 72 | 4.10421 | 98 | 1.28349 |
| 73 | 3.98678 | 99 | 1.23113 |
| 74 | 3.86954 | 100 | 1.18292 |
| 75 | 3.75304 | 101 | 1.13815 |
| 76 | 3.63787 | 102 | 1.09594 |
| 77 | 3.52399 | 103 | 1.05523 |
| 78 | 3.41045 | 104 | 1.01429 |
| 79 | 3.29548 | 105 | 0.96994 |
| 80 | 3.17731 | 106 | 0.91592 |
| 81 | 3.05494 | 107 | 0.83917 |
| 82 | 2.92841 | 108 | 0.71214 |
| 83 | 2.79855 | 109 | 0.47553 |
| 84 | 2.66678 | 110 | 0.00000 |
| 85 | 2.53480 | | |

Table 4.22: Estimated ${}_nE_x$ at $i = 10\%$

| $x$ | ${}_1E_x$ | ${}_2E_x$ | ${}_3E_x$ | ${}_4E_x$ | ${}_5E_x$ | ${}_{10}E_x$ | ${}_{20}E_x$ |
|---|---|---|---|---|---|---|---|
| 18 | 0.90740 | 0.82346 | 0.74729 | 0.67810 | 0.61520 | 0.37637 | 0.13935 |
| 19 | 0.90750 | 0.82356 | 0.74731 | 0.67798 | 0.61491 | 0.37568 | 0.13894 |
| 20 | 0.90750 | 0.82348 | 0.74709 | 0.67759 | 0.61435 | 0.37490 | 0.13848 |
| 21 | 0.90741 | 0.82323 | 0.74665 | 0.67697 | 0.61359 | 0.37410 | 0.13802 |
| 22 | 0.90723 | 0.82283 | 0.74604 | 0.67619 | 0.61270 | 0.37333 | 0.13756 |
| 23 | 0.90697 | 0.82233 | 0.74534 | 0.67536 | 0.61180 | 0.37262 | 0.13716 |
| 24 | 0.90668 | 0.82179 | 0.74463 | 0.67455 | 0.61095 | 0.37201 | 0.13681 |
| 25 | 0.90638 | 0.82127 | 0.74398 | 0.67384 | 0.61024 | 0.37150 | 0.13651 |
| 26 | 0.90611 | 0.82083 | 0.74344 | 0.67327 | 0.60969 | 0.37105 | 0.13624 |
| 27 | 0.90588 | 0.82048 | 0.74304 | 0.67287 | 0.60931 | 0.37065 | 0.13598 |
| 28 | 0.90572 | 0.82024 | 0.74278 | 0.67261 | 0.60906 | 0.37025 | 0.13567 |
| 29 | 0.90562 | 0.82009 | 0.74263 | 0.67246 | 0.60891 | 0.36983 | 0.13528 |
| 30 | 0.90556 | 0.82002 | 0.74255 | 0.67237 | 0.60877 | 0.36938 | 0.13478 |
| 31 | 0.90554 | 0.81998 | 0.74249 | 0.67226 | 0.60859 | 0.36893 | 0.13414 |
| 32 | 0.90552 | 0.81994 | 0.74239 | 0.67208 | 0.60831 | 0.36848 | 0.13333 |
| 33 | 0.90549 | 0.81984 | 0.74220 | 0.67178 | 0.60790 | 0.36808 | 0.13227 |
| 34 | 0.90542 | 0.81967 | 0.74190 | 0.67135 | 0.60737 | 0.36775 | 0.13087 |
| 35 | 0.90529 | 0.81940 | 0.74148 | 0.67081 | 0.60677 | 0.36746 | 0.12900 |
| 36 | 0.90512 | 0.81905 | 0.74099 | 0.67025 | 0.60620 | 0.36718 | 0.12654 |
| 37 | 0.90491 | 0.81867 | 0.74051 | 0.66974 | 0.60575 | 0.36686 | 0.12334 |
| 38 | 0.90470 | 0.81832 | 0.74012 | 0.66941 | 0.60550 | 0.36642 | 0.11934 |
| 39 | 0.90453 | 0.81809 | 0.73992 | 0.66929 | 0.60548 | 0.36579 | 0.11452 |
| 40 | 0.90444 | 0.81802 | 0.73993 | 0.66938 | 0.60559 | 0.36488 | 0.10897 |
| 41 | 0.90445 | 0.81811 | 0.74011 | 0.66958 | 0.60571 | 0.36360 | 0.10283 |
| 42 | 0.90454 | 0.81830 | 0.74032 | 0.66970 | 0.60563 | 0.36183 | 0.09632 |
| 43 | 0.90466 | 0.81845 | 0.74038 | 0.66955 | 0.60516 | 0.35935 | 0.08965 |
| 44 | 0.90470 | 0.81841 | 0.74011 | 0.66894 | 0.60414 | 0.35587 | 0.08304 |
| 45 | 0.90461 | 0.81807 | 0.73940 | 0.66778 | 0.60252 | 0.35107 | 0.07668 |
| 46 | 0.90433 | 0.81736 | 0.73819 | 0.66605 | 0.60030 | 0.34462 | 0.07069 |
| 47 | 0.90383 | 0.81628 | 0.73651 | 0.66380 | 0.59744 | 0.33621 | 0.06513 |
| 48 | 0.90314 | 0.81488 | 0.73443 | 0.66101 | 0.59381 | 0.32568 | 0.06003 |
| 49 | 0.90228 | 0.81320 | 0.73191 | 0.65750 | 0.58905 | 0.31308 | 0.05536 |
| 50 | 0.90127 | 0.81118 | 0.72871 | 0.65285 | 0.58267 | 0.29865 | 0.05103 |
| 51 | 0.90004 | 0.80853 | 0.72436 | 0.64650 | 0.57408 | 0.28282 | 0.04699 |
| 52 | 0.89833 | 0.80481 | 0.71831 | 0.63784 | 0.56275 | 0.26619 | 0.04317 |
| 53 | 0.89590 | 0.79960 | 0.71003 | 0.62644 | 0.54846 | 0.24947 | 0.03955 |
| 54 | 0.89251 | 0.79254 | 0.69923 | 0.61220 | 0.53149 | 0.23335 | 0.03614 |
| 55 | 0.88799 | 0.78344 | 0.68592 | 0.59550 | 0.51255 | 0.21842 | 0.03297 |
| 56 | 0.88227 | 0.77245 | 0.67062 | 0.57720 | 0.49264 | 0.20512 | 0.03004 |
| 57 | 0.87553 | 0.76011 | 0.65422 | 0.55838 | 0.47301 | 0.19372 | 0.02738 |
| 58 | 0.86818 | 0.74723 | 0.63776 | 0.54026 | 0.45485 | 0.18433 | 0.02499 |
| 59 | 0.86069 | 0.73460 | 0.62229 | 0.52391 | 0.43905 | 0.17682 | 0.02285 |
| 60 | 0.85350 | 0.72302 | 0.60871 | 0.51011 | 0.42615 | 0.17088 | 0.02095 |

Table 4.23: Estimated $a_{xy}$ at $i = 10\%$, age difference $d(= x - y)$

| $d$ | 0 | 1 | 2 | 3 | $d$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 3.51476 | 3.59566 | 3.69132 | 3.79932 | 86 | 1.31775 | 1.36427 | 1.40902 | 1.45136 |
| 61 | 3.38629 | 3.44975 | 3.52874 | 3.62212 | 87 | 1.22505 | 1.27008 | 1.31393 | 1.35603 |
| 62 | 3.28985 | 3.33758 | 3.39981 | 3.47730 | 88 | 1.13637 | 1.17944 | 1.22189 | 1.26314 |
| 63 | 3.21943 | 3.25431 | 3.30125 | 3.36252 | 89 | 1.05259 | 1.09329 | 1.13390 | 1.17385 |
| 64 | 3.16749 | 3.19321 | 3.22752 | 3.27379 | 90 | 0.97435 | 1.01237 | 1.05079 | 1.08906 |
| 65 | 3.12596 | 3.14648 | 3.17169 | 3.20547 | 91 | 0.90205 | 0.93721 | 0.97315 | 1.00940 |
| 66 | 3.08676 | 3.10608 | 3.12608 | 3.15077 | 92 | 0.83585 | 0.86807 | 0.90136 | 0.93534 |
| 67 | 3.04175 | 3.06391 | 3.08261 | 3.10203 | 93 | 0.77568 | 0.80499 | 0.83556 | 0.86711 |
| 68 | 2.98421 | 3.01253 | 3.03390 | 3.05191 | 94 | 0.72131 | 0.74783 | 0.77570 | 0.80474 |
| 69 | 2.91159 | 2.94732 | 2.97463 | 2.99514 | 95 | 0.67242 | 0.69629 | 0.72157 | 0.74812 |
| 70 | 2.82547 | 2.86783 | 2.90229 | 2.92850 | 96 | 0.62861 | 0.65003 | 0.67285 | 0.69699 |
| 71 | 2.73017 | 2.77702 | 2.81788 | 2.85100 | 97 | 0.58944 | 0.60861 | 0.62913 | 0.65098 |
| 72 | 2.63055 | 2.67950 | 2.72469 | 2.76401 | 98 | 0.55438 | 0.57156 | 0.58997 | 0.60967 |
| 73 | 2.53050 | 2.57965 | 2.62687 | 2.67039 | 99 | 0.52291 | 0.53835 | 0.55488 | 0.57260 |
| 74 | 2.43197 | 2.48035 | 2.52775 | 2.57324 | 100 | 0.49454 | 0.50847 | 0.52336 | 0.53930 |
| 75 | 2.33576 | 2.38299 | 2.42963 | 2.47528 | 101 | 0.46882 | 0.48146 | 0.49492 | 0.50930 |
| 76 | 2.24270 | 2.28836 | 2.33386 | 2.37877 | 102 | 0.44537 | 0.45690 | 0.46913 | 0.48214 |
| 77 | 2.15300 | 2.19699 | 2.24094 | 2.28471 | 103 | 0.42388 | 0.43444 | 0.44561 | 0.45745 |
| 78 | 2.06581 | 2.10853 | 2.15079 | 2.19301 | 104 | 0.40408 | 0.41378 | 0.42401 | 0.43483 |
| 79 | 1.97915 | 2.02157 | 2.06250 | 2.10299 | 105 | 0.38567 | 0.39456 | 0.40394 | 0.41385 |
| 80 | 1.89095 | 1.93406 | 1.97458 | 2.01367 | 106 | 0.36797 | 0.37606 | 0.38461 | 0.39364 |
| 81 | 1.79988 | 1.84434 | 1.88540 | 1.92394 | 107 | 0.34880 | 0.35600 | 0.36361 | 0.37164 |
| 82 | 1.70566 | 1.75160 | 1.79382 | 1.83275 | 108 | 0.31996 | 0.32590 | 0.33217 | 0.33878 |
| 83 | 1.60899 | 1.65607 | 1.69959 | 1.73951 | 109 | 0.24873 | 0.25245 | 0.25637 | 0.26049 |
| 84 | 1.51107 | 1.55871 | 1.60323 | 1.64429 | 110 | - | 0.00000 | 0.00000 | 0.00000 |
| 85 | 1.41348 | 1.46092 | 1.50591 | 1.54785 | | | | | |

Table 4.24: Estimated $a_{xy}$ at $i = 10\%$, age difference $d(= x - y)$

| $d$ | 4 | 5 | 10 | 20 | $d$ | 4 | 5 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 3.91588 | 4.03602 | 4.54895 | 4.94541 | 86 | 1.49075 | 1.52692 | 1.67264 | 1.86896 |
| 61 | 3.72751 | 3.84119 | 4.37599 | 4.82407 | 87 | 1.39578 | 1.43268 | 1.57710 | 1.77899 |
| 62 | 3.56891 | 3.67226 | 4.21770 | 4.72390 | 88 | 1.30267 | 1.33991 | 1.48525 | 1.68871 |
| 63 | 3.43883 | 3.52905 | 4.06970 | 4.63918 | 89 | 1.21261 | 1.24968 | 1.39713 | 1.59841 |
| 64 | 3.33426 | 3.40961 | 3.92782 | 4.56301 | 90 | 1.12663 | 1.16302 | 1.31260 | 1.50927 |
| 65 | 3.25114 | 3.31088 | 3.78902 | 4.48849 | 91 | 1.04546 | 1.08080 | 1.23161 | 1.42294 |
| 66 | 3.18400 | 3.22904 | 3.65182 | 4.40928 | 92 | 0.96957 | 1.00355 | 1.15422 | 1.34089 |
| 67 | 3.12616 | 3.15877 | 3.51591 | 4.31957 | 93 | 0.89926 | 0.93161 | 1.08064 | 1.26414 |
| 68 | 3.07069 | 3.09418 | 3.38208 | 4.21513 | 94 | 0.83467 | 0.86514 | 1.01104 | 1.19307 |
| 69 | 3.01239 | 3.03047 | 3.25247 | 4.09477 | 95 | 0.77575 | 0.80419 | 0.94563 | 1.12764 |
| 70 | 2.94808 | 2.96453 | 3.12996 | 3.95987 | 96 | 0.72232 | 0.74865 | 0.88451 | 1.06774 |
| 71 | 2.87607 | 2.89469 | 3.01753 | 3.81334 | 97 | 0.67407 | 0.69829 | 0.82761 | 1.01314 |
| 72 | 2.79577 | 2.81967 | 2.91653 | 3.65845 | 98 | 0.63063 | 0.65276 | 0.77479 | 0.96326 |
| 73 | 2.70816 | 2.73854 | 2.82649 | 3.49841 | 99 | 0.59154 | 0.61168 | 0.72588 | 0.91726 |
| 74 | 2.61509 | 2.65133 | 2.74511 | 3.33594 | 100 | 0.55638 | 0.57461 | 0.68068 | 0.87420 |
| 75 | 2.51905 | 2.55925 | 2.66930 | 3.17376 | 101 | 0.52469 | 0.54116 | 0.63902 | 0.83326 |
| 76 | 2.42268 | 2.46474 | 2.59612 | 3.01523 | 102 | 0.49604 | 0.51092 | 0.60067 | 0.79382 |
| 77 | 2.32788 | 2.37008 | 2.52237 | 2.86367 | 103 | 0.47005 | 0.48350 | 0.56541 | 0.75545 |
| 78 | 2.23505 | 2.27648 | 2.44483 | 2.72142 | 104 | 0.44630 | 0.45850 | 0.53294 | 0.71765 |
| 79 | 2.14345 | 2.18371 | 2.36134 | 2.58914 | 105 | 0.42431 | 0.43540 | 0.50275 | 0.67953 |
| 80 | 2.05234 | 2.09097 | 2.27127 | 2.46651 | 106 | 0.40315 | 0.41318 | 0.47358 | 0.63881 |
| 81 | 1.96112 | 1.99790 | 2.17554 | 2.35285 | 107 | 0.38009 | 0.38899 | 0.44175 | 0.58927 |
| 82 | 1.86925 | 1.90446 | 2.07590 | 2.24689 | 108 | 0.34574 | 0.35303 | 0.39550 | 0.51379 |
| 83 | 1.77626 | 1.81069 | 1.97434 | 2.14709 | 109 | 0.26481 | 0.26932 | 0.29500 | 0.36415 |
| 84 | 1.68188 | 1.71642 | 1.87240 | 2.05177 | 110 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 85 | 1.58645 | 1.62171 | 1.77138 | 1.95947 | | | | | |

Table 4.25: Estimated $a_{\overline{xy}}$ at $i = 10\%$, age difference $d(= x - y)$

| $d$ | 0 | 1 | 2 | 3 | $d$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 6.75769 | 6.82197 | 6.89828 | 6.98701 | 86 | 3.49087 | 3.57485 | 3.66209 | 3.75153 |
| 61 | 6.64941 | 6.70433 | 6.77051 | 6.84911 | 87 | 3.32841 | 3.41096 | 3.49760 | 3.58749 |
| 62 | 6.55700 | 6.60370 | 6.65984 | 6.72752 | 88 | 3.17028 | 3.25061 | 3.33575 | 3.42499 |
| 63 | 6.47644 | 6.51705 | 6.56453 | 6.62164 | 89 | 3.01795 | 3.09531 | 3.17809 | 3.26572 |
| 64 | 6.40245 | 6.43970 | 6.48088 | 6.52903 | 90 | 2.87262 | 2.94638 | 3.02602 | 3.11115 |
| 65 | 6.32944 | 6.36619 | 6.40394 | 6.44565 | 91 | 2.73522 | 2.80491 | 2.88076 | 2.96256 |
| 66 | 6.25223 | 6.29112 | 6.32839 | 6.36666 | 92 | 2.60636 | 2.67167 | 2.74323 | 2.82103 |
| 67 | 6.16628 | 6.20960 | 6.24911 | 6.28696 | 93 | 2.48632 | 2.54711 | 2.61407 | 2.68738 |
| 68 | 6.06860 | 6.11789 | 6.16200 | 6.20219 | 94 | 2.37515 | 2.43141 | 2.49363 | 2.56212 |
| 69 | 5.95871 | 6.01423 | 6.06454 | 6.10951 | 95 | 2.27266 | 2.32448 | 2.38197 | 2.44553 |
| 70 | 5.83833 | 5.89922 | 5.95601 | 6.00741 | 96 | 2.17848 | 2.22606 | 2.27893 | 2.33756 |
| 71 | 5.71039 | 5.77516 | 5.83755 | 5.89568 | 97 | 2.09203 | 2.13567 | 2.18414 | 2.23799 |
| 72 | 5.57788 | 5.64499 | 5.71142 | 5.77535 | 98 | 2.01261 | 2.05267 | 2.09706 | 2.14636 |
| 73 | 5.44308 | 5.51136 | 5.58020 | 5.64830 | 99 | 1.93934 | 1.97627 | 2.01698 | 2.06207 |
| 74 | 5.30712 | 5.37598 | 5.44601 | 5.51659 | 100 | 1.87131 | 1.90558 | 1.94306 | 1.98436 |
| 75 | 5.17039 | 5.23963 | 5.31023 | 5.38201 | 101 | 1.80747 | 1.83961 | 1.87435 | 1.91234 |
| 76 | 5.03313 | 5.10262 | 5.17359 | 5.24594 | 102 | 1.74652 | 1.77720 | 1.80975 | 1.84493 |
| 77 | 4.89506 | 4.96495 | 5.03617 | 5.10886 | 103 | 1.68660 | 1.71675 | 1.74778 | 1.78072 |
| 78 | 4.75516 | 4.82599 | 4.89761 | 4.97055 | 104 | 1.62450 | 1.65575 | 1.68623 | 1.71761 |
| 79 | 4.61185 | 4.68442 | 4.75703 | 4.83042 | 105 | 1.55423 | 1.58968 | 1.62124 | 1.65205 |
| 80 | 4.46371 | 4.53877 | 4.61324 | 4.68769 | 106 | 1.46386 | 1.50980 | 1.54559 | 1.57752 |
| 81 | 4.31007 | 4.38797 | 4.46508 | 4.54152 | 107 | 1.32954 | 1.39909 | 1.44551 | 1.48182 |
| 82 | 4.15120 | 4.23181 | 4.31194 | 4.39118 | 108 | 1.10431 | 1.22541 | 1.29588 | 1.34330 |
| 83 | 3.98816 | 4.07094 | 4.15396 | 4.23639 | 109 | 0.70231 | 0.93521 | 1.05832 | 1.13094 |
| 84 | 3.82252 | 3.90666 | 3.99199 | 4.07748 | 110 | - | 0.47553 | 0.71214 | 0.83917 |
| 85 | 3.65613 | 3.74067 | 3.82747 | 3.91538 | | | | | |

Table 4.26: Estimated $a_{\overline{xy}}$ at $i = 10\%$, age difference $d(= x - y)$

| $d$ | 4 | 5 | 10 | 20 | $d$ | 4 | 5 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 7.08753 | 7.19816 | 7.81708 | 8.82652 | 86 | 3.84199 | 3.93237 | 4.36959 | 5.20485 |
| 61 | 6.94044 | 7.04384 | 7.66361 | 8.74189 | 87 | 3.67952 | 3.77248 | 4.22366 | 5.10176 |
| 62 | 6.80789 | 6.90127 | 7.50737 | 8.65070 | 88 | 3.51745 | 3.61199 | 4.07856 | 4.99102 |
| 63 | 6.69050 | 6.77226 | 7.34966 | 8.55183 | 89 | 3.35746 | 3.45238 | 3.93364 | 4.87201 |
| 64 | 6.58694 | 6.65676 | 7.19215 | 8.44434 | 90 | 3.20116 | 3.29527 | 3.78821 | 4.74610 |
| 65 | 6.49441 | 6.55305 | 7.03664 | 8.32768 | 91 | 3.04991 | 3.14215 | 3.64200 | 4.61598 |
| 66 | 6.40892 | 6.45831 | 6.88486 | 8.20158 | 92 | 2.90487 | 2.99428 | 3.49531 | 4.48443 |
| 67 | 6.32580 | 6.36867 | 6.73822 | 8.06592 | 93 | 2.76700 | 2.85272 | 3.34894 | 4.35365 |
| 68 | 6.24068 | 6.28016 | 6.59769 | 7.92069 | 94 | 2.63704 | 2.71835 | 3.20398 | 4.22470 |
| 69 | 6.15047 | 6.18966 | 6.46408 | 7.76594 | 95 | 2.51543 | 2.59183 | 3.06172 | 4.09798 |
| 70 | 6.05331 | 6.09507 | 6.33816 | 7.60183 | 96 | 2.40233 | 2.47353 | 2.92335 | 3.97372 |
| 71 | 5.94823 | 5.99509 | 6.22060 | 7.42869 | 97 | 2.29767 | 2.36355 | 2.78985 | 3.85163 |
| 72 | 5.83485 | 5.88856 | 6.11110 | 7.24741 | 98 | 2.20110 | 2.26173 | 2.66203 | 3.73071 |
| 73 | 5.71378 | 5.77465 | 6.00823 | 7.05980 | 99 | 2.11213 | 2.16768 | 2.54052 | 3.60936 |
| 74 | 5.58635 | 5.65337 | 5.90941 | 6.86861 | 100 | 2.03010 | 2.08086 | 2.42572 | 3.48606 |
| 75 | 5.45431 | 5.52572 | 5.81148 | 6.67727 | 101 | 1.95420 | 2.00053 | 2.31777 | 3.35987 |
| 76 | 5.31944 | 5.39345 | 5.71129 | 6.48986 | 102 | 1.88339 | 1.92576 | 2.21638 | 3.23055 |
| 77 | 5.18294 | 5.25817 | 5.60568 | 6.31046 | 103 | 1.81632 | 1.85523 | 2.12083 | 3.09836 |
| 78 | 5.04498 | 5.12079 | 5.49206 | 6.14245 | 104 | 1.75092 | 1.78692 | 2.02958 | 2.96343 |
| 79 | 4.90513 | 4.98133 | 5.36932 | 5.98776 | 105 | 1.68378 | 1.71748 | 1.93974 | 2.82522 |
| 80 | 4.76290 | 4.83943 | 5.23796 | 5.84705 | 106 | 1.60872 | 1.64088 | 1.84588 | 2.68142 |
| 81 | 4.61789 | 4.69499 | 5.09972 | 5.71998 | 107 | 1.51431 | 1.54613 | 1.73815 | 2.52663 |
| 82 | 4.46966 | 4.54800 | 4.95674 | 5.60497 | 108 | 1.38069 | 1.41435 | 1.60012 | 2.35167 |
| 83 | 4.31782 | 4.39838 | 4.81102 | 5.49942 | 109 | 1.18066 | 1.22049 | 1.41165 | 2.14664 |
| 84 | 4.16224 | 4.24587 | 4.66394 | 5.39999 | 110 | 0.91592 | 0.96994 | 1.18292 | 1.92349 |
| 85 | 4.00333 | 4.09042 | 4.51650 | 5.30304 | | | | | |

Table 4.27: Estimated $a_x$ at $i = 15\%$

| $x$ | $a_x$ | $x$ | $a_x$ |
|---|---|---|---|
| 60 | 4.07699 | 86 | 2.10999 |
| 61 | 3.99392 | 87 | 2.00631 |
| 62 | 3.92902 | 88 | 1.90510 |
| 63 | 3.87893 | 89 | 1.80744 |
| 64 | 3.83908 | 90 | 1.71422 |
| 65 | 3.80434 | 91 | 1.62617 |
| 66 | 3.76958 | 92 | 1.54372 |
| 67 | 3.72966 | 93 | 1.46710 |
| 68 | 3.68038 | 94 | 1.39634 |
| 69 | 3.62017 | 95 | 1.33131 |
| 70 | 3.54994 | 96 | 1.27180 |
| 71 | 3.47242 | 97 | 1.21742 |
| 72 | 3.39059 | 98 | 1.16771 |
| 73 | 3.30696 | 99 | 1.12214 |
| 74 | 3.22279 | 100 | 1.08011 |
| 75 | 3.13857 | 101 | 1.04105 |
| 76 | 3.05482 | 102 | 1.00427 |
| 77 | 2.97160 | 103 | 0.96893 |
| 78 | 2.88816 | 104 | 0.93362 |
| 79 | 2.80299 | 105 | 0.89569 |
| 80 | 2.71454 | 106 | 0.84967 |
| 81 | 2.62182 | 107 | 0.78369 |
| 82 | 2.52469 | 108 | 0.67164 |
| 83 | 2.42374 | 109 | 0.45485 |
| 84 | 2.32003 | 110 | 0.00000 |
| 85 | 2.21496 | | |

Table 4.28: Estimated ${}_nE_x$ at $i = 15\%$

| $x$ | ${}_1E_x$ | ${}_2E_x$ | ${}_3E_x$ | ${}_4E_x$ | ${}_5E_x$ | ${}_{10}E_x$ | ${}_{20}E_x$ |
|---|---|---|---|---|---|---|---|
| 18 | 0.86795 | 0.75342 | 0.65400 | 0.56764 | 0.49259 | 0.24130 | 0.05728 |
| 19 | 0.86804 | 0.75350 | 0.65400 | 0.56753 | 0.49236 | 0.24086 | 0.05711 |
| 20 | 0.86804 | 0.75342 | 0.65381 | 0.56720 | 0.49191 | 0.24036 | 0.05692 |
| 21 | 0.86796 | 0.75320 | 0.65343 | 0.56669 | 0.49130 | 0.23984 | 0.05673 |
| 22 | 0.86778 | 0.75284 | 0.65290 | 0.56604 | 0.49059 | 0.23935 | 0.05655 |
| 23 | 0.86754 | 0.75238 | 0.65228 | 0.56534 | 0.48987 | 0.23890 | 0.05638 |
| 24 | 0.86725 | 0.75188 | 0.65166 | 0.56466 | 0.48919 | 0.23851 | 0.05623 |
| 25 | 0.86697 | 0.75141 | 0.65109 | 0.56407 | 0.48862 | 0.23818 | 0.05611 |
| 26 | 0.86671 | 0.75100 | 0.65062 | 0.56360 | 0.48818 | 0.23789 | 0.05600 |
| 27 | 0.86650 | 0.75068 | 0.65027 | 0.56326 | 0.48788 | 0.23763 | 0.05589 |
| 28 | 0.86634 | 0.75046 | 0.65004 | 0.56304 | 0.48768 | 0.23738 | 0.05577 |
| 29 | 0.86624 | 0.75033 | 0.64991 | 0.56292 | 0.48756 | 0.23711 | 0.05561 |
| 30 | 0.86619 | 0.75026 | 0.64984 | 0.56284 | 0.48745 | 0.23683 | 0.05540 |
| 31 | 0.86617 | 0.75023 | 0.64979 | 0.56275 | 0.48730 | 0.23653 | 0.05514 |
| 32 | 0.86615 | 0.75019 | 0.64970 | 0.56260 | 0.48708 | 0.23625 | 0.05480 |
| 33 | 0.86612 | 0.75011 | 0.64954 | 0.56235 | 0.48675 | 0.23599 | 0.05437 |
| 34 | 0.86605 | 0.74994 | 0.64927 | 0.56199 | 0.48632 | 0.23577 | 0.05379 |
| 35 | 0.86593 | 0.74969 | 0.64891 | 0.56154 | 0.48585 | 0.23559 | 0.05303 |
| 36 | 0.86577 | 0.74938 | 0.64848 | 0.56107 | 0.48539 | 0.23541 | 0.05201 |
| 37 | 0.86557 | 0.74903 | 0.64806 | 0.56065 | 0.48503 | 0.23520 | 0.05070 |
| 38 | 0.86537 | 0.74871 | 0.64772 | 0.56036 | 0.48483 | 0.23493 | 0.04905 |
| 39 | 0.86520 | 0.74850 | 0.64754 | 0.56026 | 0.48480 | 0.23452 | 0.04707 |
| 40 | 0.86511 | 0.74843 | 0.64755 | 0.56034 | 0.48490 | 0.23393 | 0.04479 |
| 41 | 0.86512 | 0.74851 | 0.64770 | 0.56051 | 0.48499 | 0.23312 | 0.04227 |
| 42 | 0.86521 | 0.74868 | 0.64789 | 0.56061 | 0.48493 | 0.23198 | 0.03959 |
| 43 | 0.86532 | 0.74883 | 0.64794 | 0.56048 | 0.48455 | 0.23039 | 0.03685 |
| 44 | 0.86537 | 0.74879 | 0.64771 | 0.55997 | 0.48374 | 0.22816 | 0.03413 |
| 45 | 0.86528 | 0.74847 | 0.64708 | 0.55900 | 0.48244 | 0.22508 | 0.03152 |
| 46 | 0.86501 | 0.74783 | 0.64603 | 0.55755 | 0.48066 | 0.22094 | 0.02906 |
| 47 | 0.86454 | 0.74685 | 0.64456 | 0.55567 | 0.47838 | 0.21556 | 0.02677 |
| 48 | 0.86387 | 0.74556 | 0.64273 | 0.55333 | 0.47546 | 0.20880 | 0.02468 |
| 49 | 0.86304 | 0.74402 | 0.64053 | 0.55039 | 0.47165 | 0.20072 | 0.02275 |
| 50 | 0.86209 | 0.74217 | 0.63773 | 0.54650 | 0.46655 | 0.19147 | 0.02098 |
| 51 | 0.86090 | 0.73975 | 0.63393 | 0.54119 | 0.45967 | 0.18132 | 0.01931 |
| 52 | 0.85927 | 0.73635 | 0.62862 | 0.53394 | 0.45060 | 0.17066 | 0.01774 |
| 53 | 0.85695 | 0.73158 | 0.62139 | 0.52439 | 0.43916 | 0.15994 | 0.01626 |
| 54 | 0.85370 | 0.72511 | 0.61193 | 0.51247 | 0.42557 | 0.14961 | 0.01486 |
| 55 | 0.84937 | 0.71680 | 0.60029 | 0.49850 | 0.41040 | 0.14004 | 0.01355 |
| 56 | 0.84391 | 0.70674 | 0.58690 | 0.48318 | 0.39446 | 0.13151 | 0.01235 |
| 57 | 0.83746 | 0.69546 | 0.57255 | 0.46742 | 0.37875 | 0.12420 | 0.01125 |
| 58 | 0.83043 | 0.68367 | 0.55814 | 0.45226 | 0.36420 | 0.11818 | 0.01027 |
| 59 | 0.82327 | 0.67211 | 0.54460 | 0.43857 | 0.35155 | 0.11336 | 0.00939 |
| 60 | 0.81639 | 0.66151 | 0.53272 | 0.42702 | 0.34122 | 0.10956 | 0.00861 |

Table 4.29: Estimated $a_{xy}$ at $i = 15\%$, age difference $d(= x - y)$

| $d$ | 0 | 1 | 2 | 3 | $d$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 2.95138 | 3.01476 | 3.08934 | 3.17284 | 86 | 1.20106 | 1.24140 | 1.28005 | 1.31645 |
| 61 | 2.85061 | 2.90044 | 2.96245 | 3.03541 | 87 | 1.12019 | 1.15953 | 1.19768 | 1.23417 |
| 62 | 2.77540 | 2.81267 | 2.86165 | 2.92261 | 88 | 1.04230 | 1.08018 | 1.11738 | 1.15338 |
| 63 | 2.72141 | 2.74820 | 2.78495 | 2.83328 | 89 | 0.96823 | 1.00426 | 1.04009 | 1.07520 |
| 64 | 2.68298 | 2.70204 | 2.72849 | 2.76482 | 90 | 0.89866 | 0.93251 | 0.96661 | 1.00046 |
| 65 | 2.65369 | 2.66819 | 2.68697 | 2.71310 | 91 | 0.83404 | 0.86550 | 0.89757 | 0.92982 |
| 66 | 2.62705 | 2.64021 | 2.65441 | 2.67290 | 92 | 0.77458 | 0.80355 | 0.83340 | 0.86379 |
| 67 | 2.59620 | 2.61142 | 2.62421 | 2.63809 | 93 | 0.72030 | 0.74677 | 0.77431 | 0.80265 |
| 68 | 2.55521 | 2.57543 | 2.59017 | 2.60255 | 94 | 0.67107 | 0.69510 | 0.72031 | 0.74652 |
| 69 | 2.50153 | 2.52800 | 2.54757 | 2.56177 | 95 | 0.62664 | 0.64835 | 0.67129 | 0.69535 |
| 70 | 2.43607 | 2.46833 | 2.49396 | 2.51283 | 96 | 0.58671 | 0.60624 | 0.62702 | 0.64897 |
| 71 | 2.36230 | 2.39863 | 2.42988 | 2.45462 | 97 | 0.55090 | 0.56844 | 0.58718 | 0.60710 |
| 72 | 2.28411 | 2.32260 | 2.35779 | 2.38798 | 98 | 0.51877 | 0.53452 | 0.55139 | 0.56940 |
| 73 | 2.20487 | 2.24387 | 2.28114 | 2.31516 | 99 | 0.48988 | 0.50406 | 0.51924 | 0.53548 |
| 74 | 2.12627 | 2.16495 | 2.20269 | 2.23874 | 100 | 0.46378 | 0.47660 | 0.49029 | 0.50494 |
| 75 | 2.04900 | 2.08701 | 2.12442 | 2.16093 | 101 | 0.44009 | 0.45174 | 0.46414 | 0.47737 |
| 76 | 1.97387 | 2.01081 | 2.04757 | 2.08374 | 102 | 0.41843 | 0.42908 | 0.44036 | 0.45237 |
| 77 | 1.90123 | 1.93693 | 1.97262 | 2.00813 | 103 | 0.39856 | 0.40833 | 0.41866 | 0.42959 |
| 78 | 1.83037 | 1.86516 | 1.89960 | 1.93403 | 104 | 0.38025 | 0.38923 | 0.39871 | 0.40873 |
| 79 | 1.75962 | 1.79432 | 1.82780 | 1.86094 | 105 | 0.36322 | 0.37147 | 0.38018 | 0.38937 |
| 80 | 1.68713 | 1.72264 | 1.75592 | 1.78804 | 106 | 0.34692 | 0.35445 | 0.36241 | 0.37080 |
| 81 | 1.61163 | 1.64856 | 1.68252 | 1.71433 | 107 | 0.32945 | 0.33618 | 0.34329 | 0.35080 |
| 82 | 1.53283 | 1.57133 | 1.60654 | 1.63888 | 108 | 0.30340 | 0.30900 | 0.31492 | 0.32114 |
| 83 | 1.45120 | 1.49103 | 1.52765 | 1.56109 | 109 | 0.23792 | 0.24148 | 0.24523 | 0.24917 |
| 84 | 1.36778 | 1.40844 | 1.44625 | 1.48094 | 110 | - | 0.00000 | 0.00000 | 0.00000 |
| 85 | 1.28394 | 1.32477 | 1.36330 | 1.39906 | | | | | |

Table 4.30: Estimated $a_{xy}$ at $i = 15\%$, age difference $d(= x - y)$

| $d$ | 4 | 5 | 10 | 20 | $d$ | 4 | 5 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 3.26196 | 3.35252 | 3.71822 | 3.95987 | 86 | 1.35015 | 1.38097 | 1.50433 | 1.66816 |
| 61 | 3.11704 | 3.20413 | 3.59397 | 3.87542 | 87 | 1.26846 | 1.30016 | 1.42314 | 1.59317 |
| 62 | 2.99431 | 3.07452 | 3.48005 | 3.80734 | 88 | 1.18776 | 1.22000 | 1.34459 | 1.51733 |
| 63 | 2.89344 | 2.96419 | 3.37319 | 3.75164 | 89 | 1.10913 | 1.14147 | 1.26876 | 1.44084 |
| 64 | 2.81264 | 2.87217 | 3.27012 | 3.70324 | 90 | 1.03358 | 1.06553 | 1.19556 | 1.36470 |
| 65 | 2.74907 | 2.79644 | 3.16835 | 3.65679 | 91 | 0.96180 | 0.99303 | 1.12500 | 1.29043 |
| 66 | 2.69871 | 2.73430 | 3.06665 | 3.60730 | 92 | 0.89431 | 0.92453 | 1.05721 | 1.21937 |
| 67 | 2.65624 | 2.68169 | 2.96465 | 3.54989 | 93 | 0.83148 | 0.86038 | 0.99239 | 1.15255 |
| 68 | 2.61605 | 2.63382 | 2.86300 | 3.48070 | 94 | 0.77347 | 0.80084 | 0.93076 | 1.09035 |
| 69 | 2.57370 | 2.58677 | 2.76342 | 3.39830 | 95 | 0.72032 | 0.74598 | 0.87256 | 1.03282 |
| 70 | 2.52645 | 2.53788 | 2.66845 | 3.30336 | 96 | 0.67195 | 0.69579 | 0.81793 | 0.97995 |
| 71 | 2.47274 | 2.48575 | 2.58085 | 3.19787 | 97 | 0.62813 | 0.65013 | 0.76687 | 0.93160 |
| 72 | 2.41180 | 2.42915 | 2.50200 | 3.08412 | 98 | 0.58853 | 0.60871 | 0.71928 | 0.88731 |
| 73 | 2.34428 | 2.36717 | 2.43182 | 2.96461 | 99 | 0.55282 | 0.57123 | 0.67506 | 0.84637 |
| 74 | 2.27159 | 2.29965 | 2.36865 | 2.84136 | 100 | 0.52061 | 0.53732 | 0.63406 | 0.80794 |
| 75 | 2.19575 | 2.22744 | 2.31000 | 2.71654 | 101 | 0.49152 | 0.50666 | 0.59615 | 0.77134 |
| 76 | 2.11900 | 2.15260 | 2.25352 | 2.59302 | 102 | 0.46517 | 0.47886 | 0.56116 | 0.73599 |
| 77 | 2.04305 | 2.07707 | 2.19654 | 2.47368 | 103 | 0.44123 | 0.45363 | 0.52892 | 0.70155 |
| 78 | 1.96828 | 2.00194 | 2.13623 | 2.36074 | 104 | 0.41933 | 0.43060 | 0.49919 | 0.66762 |
| 79 | 1.89408 | 1.92703 | 2.07058 | 2.25499 | 105 | 0.39907 | 0.40932 | 0.47155 | 0.63346 |
| 80 | 1.81983 | 1.85162 | 1.99884 | 2.15637 | 106 | 0.37964 | 0.38897 | 0.44496 | 0.59718 |
| 81 | 1.74501 | 1.77540 | 1.92168 | 2.06449 | 107 | 0.35869 | 0.36700 | 0.41620 | 0.55320 |
| 82 | 1.66915 | 1.69834 | 1.84049 | 1.97848 | 108 | 0.32769 | 0.33457 | 0.37456 | 0.48575 |
| 83 | 1.59175 | 1.62043 | 1.75691 | 1.89712 | 109 | 0.25330 | 0.25761 | 0.28218 | 0.34833 |
| 84 | 1.51256 | 1.54152 | 1.67226 | 1.81906 | 110 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 85 | 1.43180 | 1.46160 | 1.58766 | 1.74308 | | | | | |

Table 4.31: Estimated $a_{\overline{xy}}$ at $i = 15\%$, age difference $d(= x - y)$

| $d$ | 0 | 1 | 2 | 3 | $d$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 5.20261 | 5.24197 | 5.28908 | 5.34390 | 86 | 3.01892 | 3.08355 | 3.14997 | 3.21728 |
| 61 | 5.13723 | 5.17047 | 5.21120 | 5.25995 | 87 | 2.89244 | 2.95677 | 3.02359 | 3.09217 |
| 62 | 5.08264 | 5.11027 | 5.14436 | 5.18615 | 88 | 2.76791 | 2.83123 | 2.89771 | 2.96668 |
| 63 | 5.03644 | 5.05975 | 5.08790 | 5.12264 | 89 | 2.64664 | 2.70828 | 2.77366 | 2.84222 |
| 64 | 4.99519 | 5.01597 | 5.03962 | 5.06818 | 90 | 2.52979 | 2.58915 | 2.65272 | 2.72008 |
| 65 | 4.95499 | 4.97523 | 4.99629 | 5.02026 | 91 | 2.41829 | 2.47489 | 2.53603 | 2.60145 |
| 66 | 4.91211 | 4.93371 | 4.95425 | 4.97561 | 92 | 2.31286 | 2.36634 | 2.42454 | 2.48736 |
| 67 | 4.86311 | 4.88782 | 4.90978 | 4.93065 | 93 | 2.21389 | 2.26405 | 2.31896 | 2.37867 |
| 68 | 4.80556 | 4.83461 | 4.85980 | 4.88217 | 94 | 2.12160 | 2.16833 | 2.21974 | 2.27598 |
| 69 | 4.73880 | 4.77255 | 4.80225 | 4.82797 | 95 | 2.03599 | 2.07930 | 2.12712 | 2.17968 |
| 70 | 4.66380 | 4.70177 | 4.73636 | 4.76677 | 96 | 1.95689 | 1.99687 | 2.04111 | 2.08993 |
| 71 | 4.58254 | 4.62373 | 4.66271 | 4.69819 | 97 | 1.88395 | 1.92079 | 1.96156 | 2.00666 |
| 72 | 4.49707 | 4.54041 | 4.58274 | 4.62278 | 98 | 1.81665 | 1.85061 | 1.88812 | 1.92963 |
| 73 | 4.40905 | 4.45368 | 4.49824 | 4.54174 | 99 | 1.75439 | 1.78579 | 1.82032 | 1.85845 |
| 74 | 4.31931 | 4.36481 | 4.41069 | 4.45647 | 100 | 1.69644 | 1.72564 | 1.75753 | 1.79259 |
| 75 | 4.22814 | 4.27436 | 4.32111 | 4.36823 | 101 | 1.64202 | 1.66943 | 1.69905 | 1.73140 |
| 76 | 4.13576 | 4.18258 | 4.23004 | 4.27804 | 102 | 1.59012 | 1.61624 | 1.64402 | 1.67404 |
| 77 | 4.04197 | 4.08949 | 4.13755 | 4.18627 | 103 | 1.53929 | 1.56487 | 1.59132 | 1.61944 |
| 78 | 3.94594 | 3.99460 | 4.04338 | 4.09270 | 104 | 1.48699 | 1.51332 | 1.53918 | 1.56595 |
| 79 | 3.84636 | 3.89683 | 3.94679 | 3.99687 | 105 | 1.42817 | 1.45784 | 1.48444 | 1.51060 |
| 80 | 3.74195 | 3.79489 | 3.84677 | 3.89810 | 106 | 1.35242 | 1.39091 | 1.42088 | 1.44780 |
| 81 | 3.63201 | 3.68779 | 3.74229 | 3.79565 | 107 | 1.23794 | 1.29718 | 1.33609 | 1.36652 |
| 82 | 3.51656 | 3.57519 | 3.63269 | 3.68880 | 108 | 1.03987 | 1.14633 | 1.20639 | 1.24618 |
| 83 | 3.39627 | 3.45740 | 3.51791 | 3.57719 | 109 | 0.67178 | 0.88501 | 0.99332 | 1.05535 |
| 84 | 3.27227 | 3.33532 | 3.39847 | 3.46091 | 110 | - | 0.45485 | 0.67164 | 0.78369 |
| 85 | 3.14598 | 3.21022 | 3.27539 | 3.34060 | | | | | |

Table 4.32: Estimated $a_{\overline{xy}}$ at $i = 15\%$, age difference $d(= x - y)$

| $d$ | 4 | 5 | 10 | 20 | $d$ | 4 | 5 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | | | | $x$ | | | | |
| 60 | 5.40570 | 5.47304 | 5.82925 | 6.30445 | 86 | 3.28453 | 3.35084 | 3.66048 | 4.21140 |
| 61 | 5.31662 | 5.38046 | 5.74557 | 6.27054 | 87 | 3.16159 | 3.23084 | 3.55477 | 4.14280 |
| 62 | 5.23614 | 5.29424 | 5.65828 | 6.23287 | 88 | 3.03737 | 3.10883 | 3.44867 | 4.06816 |
| 63 | 5.16522 | 5.21617 | 5.56823 | 6.19054 | 89 | 2.91327 | 2.98600 | 3.34167 | 3.98677 |
| 64 | 5.10344 | 5.14665 | 5.47655 | 6.14278 | 90 | 2.79064 | 2.86366 | 3.23320 | 3.89946 |
| 65 | 5.04919 | 5.08490 | 5.38456 | 6.08899 | 91 | 2.67068 | 2.74312 | 3.12298 | 3.80816 |
| 66 | 4.99989 | 5.02919 | 5.29360 | 6.02878 | 92 | 2.55451 | 2.62551 | 3.01120 | 3.71494 |
| 67 | 4.95234 | 4.97699 | 5.20474 | 5.96178 | 93 | 2.44306 | 2.51182 | 2.89844 | 3.62151 |
| 68 | 4.90342 | 4.92549 | 5.11882 | 5.88767 | 94 | 2.33709 | 2.40294 | 2.78560 | 3.52878 |
| 69 | 4.85081 | 4.87248 | 5.03649 | 5.80617 | 95 | 2.23716 | 2.29956 | 2.67372 | 3.43707 |
| 70 | 4.79306 | 4.81639 | 4.95848 | 5.71705 | 96 | 2.14357 | 2.20217 | 2.56385 | 3.34667 |
| 71 | 4.72934 | 4.75625 | 4.88550 | 5.62017 | 97 | 2.05640 | 2.11101 | 2.45687 | 3.25743 |
| 72 | 4.65918 | 4.69109 | 4.81761 | 5.51578 | 98 | 1.97551 | 2.02610 | 2.35353 | 3.16856 |
| 73 | 4.58285 | 4.62017 | 4.75407 | 5.40484 | 99 | 1.90063 | 1.94724 | 2.25451 | 3.07876 |
| 74 | 4.50114 | 4.54331 | 4.69322 | 5.28901 | 100 | 1.83130 | 1.87410 | 2.16027 | 2.98670 |
| 75 | 4.41524 | 4.46107 | 4.63291 | 5.17059 | 101 | 1.76696 | 1.80620 | 2.07107 | 2.89154 |
| 76 | 4.32641 | 4.37464 | 4.57087 | 5.05247 | 102 | 1.70681 | 1.74283 | 1.98683 | 2.79297 |
| 77 | 4.23552 | 4.28512 | 4.50471 | 4.93766 | 103 | 1.64984 | 1.68301 | 1.90711 | 2.69111 |
| 78 | 4.14267 | 4.19318 | 4.43231 | 4.82884 | 104 | 1.59440 | 1.62516 | 1.83077 | 2.58603 |
| 79 | 4.04748 | 4.09875 | 4.35258 | 4.72773 | 105 | 1.53768 | 1.56648 | 1.75545 | 2.47719 |
| 80 | 3.94952 | 4.00148 | 4.26563 | 4.63516 | 106 | 1.47431 | 1.50176 | 1.67651 | 2.36248 |
| 81 | 3.84841 | 3.90124 | 4.17256 | 4.55125 | 107 | 1.39393 | 1.42097 | 1.58491 | 2.23680 |
| 82 | 3.74370 | 3.79795 | 4.07479 | 4.47523 | 108 | 1.27756 | 1.30600 | 1.46478 | 2.09098 |
| 83 | 3.63498 | 3.69146 | 3.97379 | 4.40554 | 109 | 1.09725 | 1.13086 | 1.29481 | 1.91396 |
| 84 | 3.52200 | 3.58150 | 3.87056 | 4.34005 | 110 | 0.84967 | 0.89569 | 1.08011 | 1.71422 |
| 85 | 3.40498 | 3.46790 | 3.76587 | 4.27622 | | | | | |

## 4.5 Application APV Factors

The annuity tables developed in the previous section are applied to a case study in order to compute either the annual income or single premium under a number of scenarios.

### 4.5.1 Applications

Consider two applications as follows:

**Application A**

John, a member of the pension scheme, turns 50 today and wishes to purchase a deferred payout annuity. Using the annuity tables developed at 15% per annum (p.a) interest (see Tables (4.26) - (4.32)), how much single premium beyond any expense consideration would John have to pay today for each of the following scenarios?

1. An annuity that pays an annual income of Ghc 60,000 starting at the end age 60.
2. An annuity that pays an annual income of Ghc 60,000 for the next 20 years starting at the end of age 60.
3. An annuity that pays an annual income with initial payment of Ghc 60,000 and subsequent payment geometrically increasing by a ratio of 10% p.a starting at the end of age 60.
4. A last survivor annuity that pays an annual income of Ghc 60,000 to John and his spouse aged 45 starting at the end of his birthday at age 60.

**Application B**

Mary, another member of the scheme, aged 62 wishes to purchase a life annuity by paying a single premium of Ghc 35000 (a lump sum payment received from the Tier two option of the Pension system in Ghana). Using the annuity tables developed at 10% p.a interest, what annual income should Mary expect to receive under each of the following scenarios? Assume there are no expenses in the premium.

5. An annuity that pays an annual income starting at the end of age 62.

6. A joint annuity that pays an annual income to Mary and her spouse aged 67 starting at the end of her birthday at age 62.

7. A last survivor annuity that pays an annual income to Mary and her spouse aged 67 starting at the end of her birthday at age 62.

Table (4.33) summarizes the scenarios in Application A and B while their solutions are summarized in Table (4.34).

Table 4.33: Summary of scenarios

| Scenario | Age at purchase | Annuity and Payment Type | Payment Amount Ghc | Increment | Interest rate | Duration of Annuity | Premium Ghc |
|---|---|---|---|---|---|---|---|
| 1 | 50 | Deferred Whole Life Annuity-Immediate | 60000 | - | 15% p.a | for life | ? |
| 2 | 50 | Deferred Temporary Annuity-Immediate | 60000 | - | 15% p.a | 20 years | ? |
| 3 | 50 | Deferred Increasing Whole Life Annuity-Immediate | 60000 | 10% p.a | 15% p.a | for life | ? |
| 4 | 50 & 45 | Deferred Last Survivor Life Annuity-Immediate | 60000 | - | 15% p.a | for life | ? |
| 5 | 62 | Whole Life Annuity-Immediate | ? | - | 10% p.a | for life | 35000 |
| 6 | 62 & 67 | Joint Life Annuity-Immediate | ? | - | 10% p.a | for life | 35000 |
| 7 | 62 & 67 | Last Survivor Annuity-Immediate | ? | - | 10% p.a | for life | 35000 |

### 4.5.2 Solutions to the Aplications

Let P be the single premium and X be the annual income. Then the solutions to the scenarios A1 - A4 are given as follows;

- Scenario A1:

$$P = 60000\,{}_{10}E_{50}a_{60}$$

$$P = 60000 \times 0.19147 \times 4.07699$$

$$P = 46837.28$$

- Scenario A2:

$$P = 60000\,{}_{10}E_{50}a_{60:\overline{20|}}$$

$$P = 60000\,{}_{10}E_{50}(a_{60} - {}_{20}E_{60}a_{80})$$

$$P = 60000 \times 0.19147(4.07699 - 0.00861 \times 2.71454)$$

$$P = 46565.33$$

- Scenario A3:

$$P = 60000\,{}_{10}E_{50}(1.1vp_{60} + 1.1^2v^2{}_2p_{60} + 1.1^3v^3{}_3p_{60} + 1.1^4v^4{}_4p_{60} + \dots)$$

$$P = 60000\,{}_{10}E_{50}\left(\frac{1.1p_{60}}{1.15} + \frac{1.1^2{}_2p_{60}}{1.15^2} + \frac{1.1^3{}_3p_{60}}{1.15^3} + \frac{1.1^4{}_4p_{60}}{1.15^4} + \dots\right)$$

$$P = 60000\,{}_{10}E_{50}a_{60}*$$

$$P = 60000 \times 0.19147 \times 7.066961$$

$$P = 81186.65$$

where $a_{60}*$ is valued at 4.55% p.a interest rate

- Scenario A4:

$$
\begin{aligned}
P &= 60000v^{10}\,{}_{10}p_{\overline{50:45}}\,a_{\overline{60:55}} \\
P &= 60000v^{10}({}_{10}p_{50} + {}_{10}p_{45} - {}_{10}p_{50} \times {}_{10}p_{45})a_{\overline{60:55}} \\
P &= 60000({}_{10}E_{50} + {}_{10}E_{45} - {}_{10}E_{50} \times {}_{10}p_{45})a_{\overline{60:55}} \\
P &= 60000\left({}_{10}E_{50} + {}_{10}E_{45} - {}_{10}E_{50} \times {}_{10}p_{45} \times \frac{v^{10}}{v^{10}}\right) a_{\overline{60:55}} \\
P &= 60000\left({}_{10}E_{50} + {}_{10}E_{45} - {}_{10}E_{50} \times \frac{{}_{10}E_{45}}{v^{10}}\right) a_{\overline{60:55}} \\
P &= 60000\left(0.19147 + 0.22508 - 0.19147 \times \frac{0.22508}{1.15^{-10}}\right) \times 5.47304 \\
P &= 60000 \times 0.24220 \times 5.47304 \\
P &= 79534.21
\end{aligned}
$$

The solutions to scenarios B1 - B3 are as follows:

- Scenario B1:

$$
\begin{aligned}
35000 &= X a_{62} \\
X &= \frac{35000}{a_{62}} \\
X &= \frac{35000}{4.92342} \\
X &= 7108.88
\end{aligned}
$$

- Scenario B2:

$$
\begin{aligned}
35000 &= X a_{62:67} \\
X &= \frac{35000}{a_{62:67}} \\
X &= \frac{35000}{3.15877} \\
X &= 11080.26
\end{aligned}
$$

- Scenario B3:

$$\begin{aligned} 35000 &= X a_{\overline{62:67}} \\ X &= \frac{35000}{a_{\overline{62:67}}} \\ X &= \frac{35000}{6.36865} \\ X &= 5495.67 \end{aligned}$$

Table 4.34: Summary of answered scenarios

| Scenario | Age at purchase | Annuity and Payment Type | Payment Amount Ghc | Increment | Interest rate | Duration of Annuity | Premium Ghc |
|---|---|---|---|---|---|---|---|
| 1 | 50 | Deferred Whole Life Annuity-Immediate | 60000 | - | 15% p.a | for life | 46837.28 |
| 2 | 50 | Deferred Temporary Annuity-Immediate | 60000 | - | 15% p.a | 20 years | 46565.33 |
| 3 | 50 | Deferred Increasing Whole Life Annuity-Immediate | 60000 | 10% p.a | 15% p.a | for life | 81186.65 |
| 4 | 50 & 45 | Deferred Last Survivor Life Annuity-Immediate | 60000 | - | 15% p.a | for life | 79534.21 |
| 5 | 62 | Whole Life Annuity-Immediate | 7108.88 | - | 10% p.a | for life | 35000 |
| 6 | 62 & 67 | Joint Life Annuity-Immediate | 11080.26 | - | 10% p.a | for life | 35000 |
| 7 | 62 & 67 | Last Survivor Annuity-Immediate | 5495.67 | - | 10% p.a | for life | 35000 |

# Chapter 5

# Conclusion and Recommendations

## 5.1 Summary of Findings

The increasing life expectancy in Ghana poses a severe challenge for the management of public pension schemes in the country. Knowledge of mortality patterns of insured lives serves as a major component in pricing and reserving life insurance and pension products such as annuities, which do not exist in Ghana. The study sought to employ non-parametric methods of graduation to smooth mortality rates of a pension scheme with ultimate aim of developing life tables for establishment of annuity business in Ghana.

1. It was observed that the crude mortality rates computed from the data exhibited some random fluctuations over the entire age range of 18 - 110 years of members of the pension scheme.

2. In order to achieve mortality rates that progressed smoothly as age increased for actuarial purposes of annuity pricing, the W-H and the modified W-H methods of graduation were applied to the crude mortality rates. A comparison of the output from the techniques indicated that, there was no significant difference between the two methods since both methods produced identical smooth curves for the same choice of order of differencing,($z$) and smoothing parameter, ($h$). However, the final graduated mortality rates obtained for further analysis were selected at $z = 3$ and $h = 100$ via the W-H method which provided a good fit to the crude mortality rates at most

ages. It was also observed that the graduated mortality rates at ages 41 to 45 decreased gradually and then increased. This could be as a result of the significant change in the number of deaths at those ages.

3. The graduated mortality rates at $z = 3$ and $h = 100$ were further used to compute life expectancies at each age. A graph of the life expectancy ($e_x$) as a function of age ($x$) decreases with increasing $x$, which is consistent with literature that as age increases, the average number of years more a life has to live decreases.

4. To serve as basis for annuity markets in Ghana, the graduated mortality rates were also used to develop actuarial present value (APV) factors. The factors decreased as interest rates were increased from 4.55%, 10% and finally to 15%. This is consistent with the notion that price (APV) is inversely proportional to interest rates. Also, age increased with the APV factors as expected since the crude mortality rates were smoothed to reduce the fluctuations in the curve.

5. Finally, the gradual decrease in graduated mortality rates observed from ages 41 to 45 did not affect the values of the APV factors and the life expectancies.

## 5.2 Conclusion

1. The actuarial method for computing exposures has been modified by adjusting the withdrawal component in the duration vector for a unit period. The modified method was applied to a national pension scheme data to obtain crude mortality rates for graduation.

2. A mortality table have been developed from graduation of crude rates using non parametric methods. The average number of years more to be lived by

$(x)$ that is the life expectancy have been computed for all the ages of the members of the pension scheme.

3. The graduated rates were used to develop the actuarial present value factors tables for single and joint lives for annuity pricing at different interest rates at a unit amount.

## 5.3 Limitations

Gender for a greater number of pensioners were not indicated in the pension data obtained. Also, there were no data on smokers and non-smokers. Hence the computation of crude mortality rates and the ensuing analysis have been carried out without these information.

## 5.4 Recommendation

The tables developed in this study can be adapted for pricing annuity products in Ghana.

## 5.5 Further studies

- Developing independent tables for males and females, and also considering their smoking habits.

- Other graduation methods including those that constrain graduated mortality rates to certain desirable features of the underlying mortality rates through parametric methods should be explored.

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# Appendix A - Graduated mortality rates by W-H

Table 5.1: Graduated mortality rates using W-H at z=2

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00145 | 0.00145 | 0.00147 | 0.00154 | 0.00164 |
| 19 | 0.00255 | 0.00164 | 0.00165 | 0.00168 | 0.00175 | 0.00183 |
| 20 | 0.00283 | 0.00183 | 0.00185 | 0.00188 | 0.00195 | 0.00203 |
| 21 | 0.00050 | 0.00203 | 0.00206 | 0.00209 | 0.00215 | 0.00222 |
| 22 | 0.00081 | 0.00225 | 0.00228 | 0.00231 | 0.00236 | 0.00242 |
| 23 | 0.00254 | 0.00249 | 0.00251 | 0.00253 | 0.00258 | 0.00262 |
| 24 | 0.00303 | 0.00274 | 0.00275 | 0.00276 | 0.00279 | 0.00282 |
| 25 | 0.00328 | 0.00298 | 0.00298 | 0.00298 | 0.00300 | 0.00301 |
| 26 | 0.00351 | 0.00320 | 0.00319 | 0.00319 | 0.00319 | 0.00319 |
| 27 | 0.00351 | 0.00339 | 0.00338 | 0.00338 | 0.00337 | 0.00335 |
| 28 | 0.00369 | 0.00357 | 0.00355 | 0.00355 | 0.00353 | 0.00349 |
| 29 | 0.00386 | 0.00371 | 0.00370 | 0.00370 | 0.00366 | 0.00361 |
| 30 | 0.00405 | 0.00383 | 0.00383 | 0.00382 | 0.00378 | 0.00370 |
| 31 | 0.00392 | 0.00393 | 0.00394 | 0.00393 | 0.00386 | 0.00377 |
| 32 | 0.00402 | 0.00402 | 0.00403 | 0.00401 | 0.00393 | 0.00382 |
| 33 | 0.00394 | 0.00411 | 0.00411 | 0.00408 | 0.00398 | 0.00385 |
| 34 | 0.00394 | 0.00419 | 0.00419 | 0.00414 | 0.00401 | 0.00385 |
| 35 | 0.00390 | 0.00428 | 0.00426 | 0.00419 | 0.00402 | 0.00384 |
| 36 | 0.00370 | 0.00438 | 0.00432 | 0.00423 | 0.00402 | 0.00381 |
| 37 | 0.00423 | 0.00449 | 0.00438 | 0.00426 | 0.00401 | 0.00378 |
| 38 | 0.00483 | 0.00458 | 0.00442 | 0.00427 | 0.00398 | 0.00373 |
| 39 | 0.00523 | 0.00464 | 0.00444 | 0.00425 | 0.00394 | 0.00368 |
| 40 | 0.00541 | 0.00465 | 0.00442 | 0.00421 | 0.00388 | 0.00363 |
| 41 | 0.00529 | 0.00461 | 0.00436 | 0.00415 | 0.00383 | 0.00361 |
| 42 | 0.00878 | 0.00453 | 0.00428 | 0.00409 | 0.00381 | 0.00363 |
| 43 | 0.00185 | 0.00442 | 0.00421 | 0.00405 | 0.00384 | 0.00372 |
| 44 | 0.00372 | 0.00440 | 0.00424 | 0.00412 | 0.00398 | 0.00394 |
| 45 | 0.00755 | 0.00451 | 0.00440 | 0.00433 | 0.00430 | 0.00434 |
| 46 | 0.00188 | 0.00480 | 0.00474 | 0.00474 | 0.00483 | 0.00497 |
| 47 | 0.00582 | 0.00536 | 0.00537 | 0.00543 | 0.00564 | 0.00588 |
| 48 | 0.00403 | 0.00623 | 0.00631 | 0.00646 | 0.00679 | 0.00712 |
| 49 | 0.01058 | 0.00745 | 0.00763 | 0.00786 | 0.00832 | 0.00874 |
| 50 | 0.01327 | 0.00902 | 0.00934 | 0.00966 | 0.01026 | 0.01076 |
| 51 | 0.00952 | 0.01100 | 0.01149 | 0.01193 | 0.01265 | 0.01321 |
| 52 | 0.00774 | 0.01358 | 0.01424 | 0.01477 | 0.01557 | 0.01616 |
| 53 | 0.01952 | 0.01692 | 0.01767 | 0.01824 | 0.01906 | 0.01963 |
| 54 | 0.01720 | 0.02101 | 0.02179 | 0.02235 | 0.02311 | 0.02360 |
| 55 | 0.02129 | 0.02592 | 0.02663 | 0.02710 | 0.02769 | 0.02805 |
| 56 | 0.01596 | 0.03164 | 0.03214 | 0.03244 | 0.03277 | 0.03293 |
| 57 | 0.04338 | 0.03806 | 0.03821 | 0.03826 | 0.03823 | 0.03814 |
| 58 | 0.05745 | 0.04475 | 0.04451 | 0.04428 | 0.04386 | 0.04351 |
| 59 | 0.02949 | 0.05140 | 0.05078 | 0.05028 | 0.04949 | 0.04888 |
| 60 | 0.07676 | 0.05792 | 0.05692 | 0.05616 | 0.05502 | 0.05419 |
| 61 | 0.06915 | 0.06385 | 0.06259 | 0.06164 | 0.06025 | 0.05925 |
| 62 | 0.08164 | 0.06903 | 0.06763 | 0.06659 | 0.06507 | 0.06398 |
| 63 | 0.08041 | 0.07336 | 0.07199 | 0.07096 | 0.06945 | 0.06836 |

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.07693 | 0.07572 | 0.07478 | 0.07339 | 0.07238 |
| 65 | 0.08862 | 0.07993 | 0.07894 | 0.07817 | 0.07699 | 0.07612 |
| 66 | 0.12521 | 0.08230 | 0.08165 | 0.08111 | 0.08024 | 0.07959 |
| 67 | 0.06149 | 0.08410 | 0.08391 | 0.08367 | 0.08322 | 0.08284 |
| 68 | 0.07967 | 0.08589 | 0.08613 | 0.08619 | 0.08616 | 0.08607 |
| 69 | 0.06078 | 0.08798 | 0.08855 | 0.08887 | 0.08921 | 0.08940 |
| 70 | 0.09321 | 0.09060 | 0.09136 | 0.09186 | 0.09251 | 0.09294 |
| 71 | 0.06573 | 0.09373 | 0.09458 | 0.09520 | 0.09609 | 0.09674 |
| 72 | 0.13084 | 0.09735 | 0.09822 | 0.09892 | 0.10001 | 0.10083 |
| 73 | 0.15214 | 0.10126 | 0.10216 | 0.10294 | 0.10421 | 0.10520 |
| 74 | 0.04316 | 0.10545 | 0.10642 | 0.10729 | 0.10873 | 0.10988 |
| 75 | 0.09818 | 0.11023 | 0.11122 | 0.11214 | 0.11371 | 0.11498 |
| 76 | 0.09658 | 0.11557 | 0.11654 | 0.11749 | 0.11914 | 0.12050 |
| 77 | 0.18268 | 0.12139 | 0.12236 | 0.12332 | 0.12502 | 0.12644 |
| 78 | 0.15416 | 0.12754 | 0.12855 | 0.12955 | 0.13131 | 0.13278 |
| 79 | 0.10955 | 0.13408 | 0.13518 | 0.13621 | 0.13803 | 0.13952 |
| 80 | 0.14568 | 0.14115 | 0.14233 | 0.14340 | 0.14522 | 0.14672 |
| 81 | 0.06808 | 0.14885 | 0.15007 | 0.15114 | 0.15293 | 0.15438 |
| 82 | 0.22578 | 0.15726 | 0.15846 | 0.15947 | 0.16116 | 0.16253 |
| 83 | 0.10421 | 0.16631 | 0.16743 | 0.16835 | 0.16989 | 0.17113 |
| 84 | 0.11099 | 0.17605 | 0.17701 | 0.17780 | 0.17912 | 0.18019 |
| 85 | 0.30733 | 0.18643 | 0.18716 | 0.18777 | 0.18882 | 0.18968 |
| 86 | 0.16512 | 0.19728 | 0.19776 | 0.19818 | 0.19892 | 0.19954 |
| 87 | 0.17431 | 0.20863 | 0.20883 | 0.20903 | 0.20942 | 0.20977 |
| 88 | 0.20576 | 0.22044 | 0.22032 | 0.22027 | 0.22028 | 0.22033 |
| 89 | 0.23325 | 0.23266 | 0.23219 | 0.23188 | 0.23148 | 0.23121 |
| 90 | 0.26033 | 0.24522 | 0.24439 | 0.24380 | 0.24296 | 0.24235 |
| 91 | 0.27787 | 0.25803 | 0.25684 | 0.25597 | 0.25469 | 0.25372 |
| 92 | 0.30544 | 0.27104 | 0.26950 | 0.26835 | 0.26663 | 0.26530 |
| 93 | 0.27072 | 0.28419 | 0.28233 | 0.28091 | 0.27875 | 0.27705 |
| 94 | 0.30918 | 0.29745 | 0.29528 | 0.29361 | 0.29101 | 0.28895 |
| 95 | 0.30627 | 0.31077 | 0.30834 | 0.30643 | 0.30340 | 0.30097 |
| 96 | 0.39349 | 0.32412 | 0.32146 | 0.31932 | 0.31588 | 0.31309 |
| 97 | 0.39879 | 0.33746 | 0.33461 | 0.33227 | 0.32844 | 0.32529 |
| 98 | 0.37378 | 0.35077 | 0.34778 | 0.34526 | 0.34105 | 0.33756 |
| 99 | 0.33804 | 0.36407 | 0.36096 | 0.35828 | 0.35371 | 0.34987 |
| 100 | 0.48429 | 0.37735 | 0.37415 | 0.37132 | 0.36641 | 0.36223 |
| 101 | 0.30538 | 0.39061 | 0.38735 | 0.38439 | 0.37914 | 0.37462 |
| 102 | 0.46266 | 0.40387 | 0.40057 | 0.39747 | 0.39190 | 0.38705 |
| 103 | 0.37844 | 0.41714 | 0.41380 | 0.41058 | 0.40467 | 0.39949 |
| 104 | 0.44348 | 0.43042 | 0.42704 | 0.42369 | 0.41746 | 0.41195 |
| 105 | 0.43904 | 0.44369 | 0.44028 | 0.43681 | 0.43026 | 0.42442 |
| 106 | 0.47714 | 0.45697 | 0.45353 | 0.44994 | 0.44306 | 0.43689 |
| 107 | 0.47350 | 0.47024 | 0.46677 | 0.46306 | 0.45587 | 0.44936 |
| 108 | 0.45281 | 0.48351 | 0.48001 | 0.47618 | 0.46867 | 0.46183 |
| 109 | 0.40650 | 0.49677 | 0.49325 | 0.48930 | 0.48147 | 0.47430 |
| 110 | 0.43044 | 0.51003 | 0.50649 | 0.50241 | 0.49427 | 0.48677 |

Table 5.2: Graduated mortality rates using W-H at z=3

| Age | qx | graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00186 | 0.00174 | 0.00165 | 0.00156 | 0.00151 |
| 19 | 0.00255 | 0.00175 | 0.00171 | 0.00168 | 0.00165 | 0.00164 |
| 20 | 0.00283 | 0.00175 | 0.00176 | 0.00177 | 0.00179 | 0.00180 |
| 21 | 0.00050 | 0.00185 | 0.00190 | 0.00193 | 0.00197 | 0.00199 |
| 22 | 0.00081 | 0.00205 | 0.00211 | 0.00214 | 0.00219 | 0.00222 |
| 23 | 0.00254 | 0.00233 | 0.00237 | 0.00240 | 0.00244 | 0.00246 |
| 24 | 0.00303 | 0.00266 | 0.00268 | 0.00269 | 0.00271 | 0.00271 |
| 25 | 0.00328 | 0.00299 | 0.00298 | 0.00297 | 0.00297 | 0.00296 |
| 26 | 0.00351 | 0.00328 | 0.00325 | 0.00323 | 0.00321 | 0.00319 |
| 27 | 0.00351 | 0.00353 | 0.00348 | 0.00346 | 0.00342 | 0.00339 |
| 28 | 0.00369 | 0.00371 | 0.00366 | 0.00363 | 0.00359 | 0.00356 |
| 29 | 0.00386 | 0.00382 | 0.00378 | 0.00376 | 0.00372 | 0.00370 |
| 30 | 0.00405 | 0.00388 | 0.00386 | 0.00385 | 0.00383 | 0.00381 |
| 31 | 0.00392 | 0.00391 | 0.00391 | 0.00391 | 0.00391 | 0.00391 |
| 32 | 0.00402 | 0.00393 | 0.00395 | 0.00397 | 0.00399 | 0.00401 |
| 33 | 0.00394 | 0.00396 | 0.00401 | 0.00404 | 0.00408 | 0.00411 |
| 34 | 0.00394 | 0.00404 | 0.00410 | 0.00413 | 0.00419 | 0.00424 |
| 35 | 0.00390 | 0.00418 | 0.00423 | 0.00427 | 0.00433 | 0.00438 |
| 36 | 0.00370 | 0.00437 | 0.00441 | 0.00443 | 0.00449 | 0.00454 |
| 37 | 0.00423 | 0.00460 | 0.00461 | 0.00462 | 0.00466 | 0.00470 |
| 38 | 0.00483 | 0.00483 | 0.00481 | 0.00481 | 0.00482 | 0.00485 |
| 39 | 0.00523 | 0.00502 | 0.00497 | 0.00495 | 0.00496 | 0.00497 |
| 40 | 0.00541 | 0.00512 | 0.00506 | 0.00504 | 0.00504 | 0.00504 |
| 41 | 0.00529 | 0.00511 | 0.00507 | 0.00506 | 0.00506 | 0.00504 |
| 42 | 0.00878 | 0.00501 | 0.00502 | 0.00503 | 0.00503 | 0.00500 |
| 43 | 0.00185 | 0.00488 | 0.00494 | 0.00497 | 0.00497 | 0.00493 |
| 44 | 0.00372 | 0.00482 | 0.00491 | 0.00494 | 0.00493 | 0.00487 |
| 45 | 0.00755 | 0.00493 | 0.00501 | 0.00502 | 0.00497 | 0.00488 |
| 46 | 0.00188 | 0.00524 | 0.00526 | 0.00523 | 0.00513 | 0.00501 |
| 47 | 0.00582 | 0.00578 | 0.00571 | 0.00563 | 0.00546 | 0.00532 |
| 48 | 0.00403 | 0.00655 | 0.00638 | 0.00623 | 0.00602 | 0.00588 |
| 49 | 0.01058 | 0.00750 | 0.00725 | 0.00707 | 0.00686 | 0.00675 |
| 50 | 0.01327 | 0.00860 | 0.00835 | 0.00821 | 0.00807 | 0.00802 |
| 51 | 0.00952 | 0.00996 | 0.00982 | 0.00976 | 0.00976 | 0.00983 |
| 52 | 0.00774 | 0.01184 | 0.01187 | 0.01194 | 0.01213 | 0.01232 |
| 53 | 0.01952 | 0.01451 | 0.01475 | 0.01496 | 0.01534 | 0.01565 |
| 54 | 0.01720 | 0.01824 | 0.01866 | 0.01899 | 0.01951 | 0.01991 |
| 55 | 0.02129 | 0.02322 | 0.02374 | 0.02413 | 0.02471 | 0.02512 |
| 56 | 0.01596 | 0.02950 | 0.03001 | 0.03037 | 0.03087 | 0.03121 |
| 57 | 0.04338 | 0.03692 | 0.03726 | 0.03750 | 0.03781 | 0.03800 |
| 58 | 0.05745 | 0.04500 | 0.04510 | 0.04515 | 0.04519 | 0.04518 |
| 59 | 0.02949 | 0.05324 | 0.05306 | 0.05291 | 0.05264 | 0.05242 |
| 60 | 0.07676 | 0.06115 | 0.06070 | 0.06034 | 0.05978 | 0.05936 |
| 61 | 0.06915 | 0.06817 | 0.06751 | 0.06700 | 0.06625 | 0.06569 |
| 62 | 0.08164 | 0.07390 | 0.07316 | 0.07259 | 0.07176 | 0.07115 |
| 63 | 0.08041 | 0.07818 | 0.07748 | 0.07696 | 0.07619 | 0.07562 |

| Age | qx | | | Graduated | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.08105 | 0.08052 | 0.08013 | 0.07954 | 0.07910 |
| 65 | 0.08862 | 0.08273 | 0.08246 | 0.08226 | 0.08196 | 0.08171 |
| 66 | 0.12521 | 0.08344 | 0.08351 | 0.08357 | 0.08362 | 0.08361 |
| 67 | 0.06149 | 0.08360 | 0.08407 | 0.08440 | 0.08482 | 0.08506 |
| 68 | 0.07967 | 0.08392 | 0.08471 | 0.08525 | 0.08595 | 0.08640 |
| 69 | 0.06078 | 0.08500 | 0.08591 | 0.08654 | 0.08739 | 0.08794 |
| 70 | 0.09321 | 0.08720 | 0.08799 | 0.08857 | 0.08937 | 0.08992 |
| 71 | 0.06573 | 0.09049 | 0.09100 | 0.09139 | 0.09198 | 0.09243 |
| 72 | 0.13084 | 0.09468 | 0.09479 | 0.09494 | 0.09523 | 0.09551 |
| 73 | 0.15214 | 0.09941 | 0.09916 | 0.09905 | 0.09903 | 0.09911 |
| 74 | 0.04316 | 0.10447 | 0.10394 | 0.10362 | 0.10331 | 0.10322 |
| 75 | 0.09818 | 0.10986 | 0.10910 | 0.10862 | 0.10808 | 0.10783 |
| 76 | 0.09658 | 0.11546 | 0.11457 | 0.11398 | 0.11329 | 0.11293 |
| 77 | 0.18268 | 0.12109 | 0.12020 | 0.11961 | 0.11887 | 0.11847 |
| 78 | 0.15416 | 0.12664 | 0.12593 | 0.12543 | 0.12480 | 0.12444 |
| 79 | 0.10955 | 0.13221 | 0.13183 | 0.13153 | 0.13113 | 0.13088 |
| 80 | 0.14568 | 0.13808 | 0.13811 | 0.13807 | 0.13796 | 0.13787 |
| 81 | 0.06808 | 0.14458 | 0.14500 | 0.14522 | 0.14543 | 0.14552 |
| 82 | 0.22578 | 0.15198 | 0.15273 | 0.15317 | 0.15366 | 0.15391 |
| 83 | 0.10421 | 0.16046 | 0.16142 | 0.16202 | 0.16272 | 0.16311 |
| 84 | 0.11099 | 0.17012 | 0.17116 | 0.17184 | 0.17267 | 0.17313 |
| 85 | 0.30733 | 0.18096 | 0.18195 | 0.18264 | 0.18349 | 0.18397 |
| 86 | 0.16512 | 0.19287 | 0.19373 | 0.19435 | 0.19514 | 0.19559 |
| 87 | 0.17431 | 0.20579 | 0.20644 | 0.20693 | 0.20757 | 0.20793 |
| 88 | 0.20576 | 0.21962 | 0.21998 | 0.22029 | 0.22070 | 0.22092 |
| 89 | 0.23325 | 0.23420 | 0.23423 | 0.23431 | 0.23443 | 0.23446 |
| 90 | 0.26033 | 0.24934 | 0.24901 | 0.24884 | 0.24864 | 0.24846 |
| 91 | 0.27787 | 0.26482 | 0.26414 | 0.26372 | 0.26317 | 0.26277 |
| 92 | 0.30544 | 0.28042 | 0.27942 | 0.27877 | 0.27790 | 0.27728 |
| 93 | 0.27072 | 0.29594 | 0.29470 | 0.29385 | 0.29269 | 0.29186 |
| 94 | 0.30918 | 0.31121 | 0.30980 | 0.30880 | 0.30741 | 0.30641 |
| 95 | 0.30627 | 0.32608 | 0.32460 | 0.32350 | 0.32194 | 0.32083 |
| 96 | 0.39349 | 0.34042 | 0.33896 | 0.33784 | 0.33620 | 0.33502 |
| 97 | 0.39879 | 0.35414 | 0.35281 | 0.35173 | 0.35010 | 0.34892 |
| 98 | 0.37378 | 0.36721 | 0.36609 | 0.36512 | 0.36360 | 0.36246 |
| 99 | 0.33804 | 0.37962 | 0.37878 | 0.37797 | 0.37664 | 0.37563 |
| 100 | 0.48429 | 0.39143 | 0.39090 | 0.39030 | 0.38923 | 0.38840 |
| 101 | 0.30538 | 0.40267 | 0.40248 | 0.40211 | 0.40137 | 0.40076 |
| 102 | 0.46266 | 0.41343 | 0.41354 | 0.41341 | 0.41304 | 0.41271 |
| 103 | 0.37844 | 0.42374 | 0.42412 | 0.42424 | 0.42427 | 0.42426 |
| 104 | 0.44348 | 0.43363 | 0.43422 | 0.43459 | 0.43506 | 0.43541 |
| 105 | 0.43904 | 0.44312 | 0.44386 | 0.44446 | 0.44541 | 0.44615 |
| 106 | 0.47714 | 0.45219 | 0.45304 | 0.45387 | 0.45531 | 0.45649 |
| 107 | 0.47350 | 0.46086 | 0.46174 | 0.46280 | 0.46476 | 0.46642 |
| 108 | 0.45281 | 0.46910 | 0.46997 | 0.47124 | 0.47376 | 0.47593 |
| 109 | 0.40650 | 0.47692 | 0.47773 | 0.47921 | 0.48231 | 0.48504 |
| 110 | 0.43044 | 0.48432 | 0.48501 | 0.48670 | 0.49042 | 0.49373 |

Table 5.3: Graduated mortality rate using W-H at z=4

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00263 | 0.00257 | 0.00252 | 0.00242 | 0.00233 |
| 19 | 0.00255 | 0.00191 | 0.00189 | 0.00188 | 0.00185 | 0.00184 |
| 20 | 0.00283 | 0.00158 | 0.00159 | 0.00160 | 0.00162 | 0.00164 |
| 21 | 0.00050 | 0.00156 | 0.00158 | 0.00161 | 0.00166 | 0.00169 |
| 22 | 0.00081 | 0.00178 | 0.00181 | 0.00184 | 0.00188 | 0.00192 |
| 23 | 0.00254 | 0.00216 | 0.00218 | 0.00220 | 0.00223 | 0.00225 |
| 24 | 0.00303 | 0.00261 | 0.00262 | 0.00263 | 0.00263 | 0.00264 |
| 25 | 0.00328 | 0.00306 | 0.00305 | 0.00305 | 0.00303 | 0.00301 |
| 26 | 0.00351 | 0.00343 | 0.00342 | 0.00340 | 0.00337 | 0.00334 |
| 27 | 0.00351 | 0.00371 | 0.00369 | 0.00367 | 0.00362 | 0.00359 |
| 28 | 0.00369 | 0.00388 | 0.00385 | 0.00383 | 0.00379 | 0.00376 |
| 29 | 0.00386 | 0.00395 | 0.00392 | 0.00390 | 0.00387 | 0.00385 |
| 30 | 0.00405 | 0.00393 | 0.00391 | 0.00390 | 0.00389 | 0.00389 |
| 31 | 0.00392 | 0.00387 | 0.00386 | 0.00386 | 0.00388 | 0.00390 |
| 32 | 0.00402 | 0.00380 | 0.00381 | 0.00383 | 0.00387 | 0.00391 |
| 33 | 0.00394 | 0.00377 | 0.00381 | 0.00385 | 0.00391 | 0.00395 |
| 34 | 0.00394 | 0.00383 | 0.00389 | 0.00394 | 0.00400 | 0.00404 |
| 35 | 0.00390 | 0.00399 | 0.00406 | 0.00411 | 0.00417 | 0.00419 |
| 36 | 0.00370 | 0.00427 | 0.00433 | 0.00436 | 0.00439 | 0.00439 |
| 37 | 0.00423 | 0.00463 | 0.00465 | 0.00465 | 0.00463 | 0.00460 |
| 38 | 0.00483 | 0.00499 | 0.00496 | 0.00492 | 0.00484 | 0.00479 |
| 39 | 0.00523 | 0.00526 | 0.00517 | 0.00510 | 0.00499 | 0.00493 |
| 40 | 0.00541 | 0.00536 | 0.00524 | 0.00516 | 0.00505 | 0.00499 |
| 41 | 0.00529 | 0.00526 | 0.00515 | 0.00508 | 0.00501 | 0.00498 |
| 42 | 0.00878 | 0.00499 | 0.00493 | 0.00491 | 0.00491 | 0.00494 |
| 43 | 0.00185 | 0.00468 | 0.00470 | 0.00475 | 0.00483 | 0.00491 |
| 44 | 0.00372 | 0.00449 | 0.00460 | 0.00470 | 0.00486 | 0.00497 |
| 45 | 0.00755 | 0.00457 | 0.00475 | 0.00487 | 0.00505 | 0.00516 |
| 46 | 0.00188 | 0.00503 | 0.00520 | 0.00531 | 0.00545 | 0.00552 |
| 47 | 0.00582 | 0.00585 | 0.00595 | 0.00600 | 0.00604 | 0.00603 |
| 48 | 0.00403 | 0.00694 | 0.00691 | 0.00687 | 0.00678 | 0.00668 |
| 49 | 0.01058 | 0.00813 | 0.00797 | 0.00784 | 0.00763 | 0.00747 |
| 50 | 0.01327 | 0.00930 | 0.00907 | 0.00889 | 0.00862 | 0.00843 |
| 51 | 0.00952 | 0.01049 | 0.01027 | 0.01011 | 0.00987 | 0.00972 |
| 52 | 0.00774 | 0.01196 | 0.01183 | 0.01174 | 0.01162 | 0.01156 |
| 53 | 0.01952 | 0.01410 | 0.01410 | 0.01412 | 0.01417 | 0.01423 |
| 54 | 0.01720 | 0.01731 | 0.01746 | 0.01759 | 0.01782 | 0.01801 |
| 55 | 0.02129 | 0.02195 | 0.02221 | 0.02243 | 0.02278 | 0.02305 |
| 56 | 0.01596 | 0.02818 | 0.02847 | 0.02871 | 0.02910 | 0.02940 |
| 57 | 0.04338 | 0.03585 | 0.03608 | 0.03629 | 0.03661 | 0.03685 |
| 58 | 0.05745 | 0.04450 | 0.04463 | 0.04474 | 0.04491 | 0.04502 |
| 59 | 0.02949 | 0.05350 | 0.05350 | 0.05349 | 0.05345 | 0.05341 |
| 60 | 0.07676 | 0.06218 | 0.06204 | 0.06190 | 0.06166 | 0.06145 |
| 61 | 0.06915 | 0.06985 | 0.06958 | 0.06934 | 0.06892 | 0.06858 |
| 62 | 0.08164 | 0.07596 | 0.07562 | 0.07532 | 0.07481 | 0.07440 |
| 63 | 0.08041 | 0.08024 | 0.07988 | 0.07957 | 0.07907 | 0.07867 |

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.08268 | 0.08236 | 0.08211 | 0.08171 | 0.08141 |
| 65 | 0.08862 | 0.08349 | 0.08328 | 0.08315 | 0.08296 | 0.08283 |
| 66 | 0.12521 | 0.08308 | 0.08309 | 0.08313 | 0.08322 | 0.08330 |
| 67 | 0.06149 | 0.08215 | 0.08244 | 0.08268 | 0.08307 | 0.08338 |
| 68 | 0.07967 | 0.08164 | 0.08215 | 0.08255 | 0.08317 | 0.08364 |
| 69 | 0.06078 | 0.08236 | 0.08295 | 0.08341 | 0.08410 | 0.08463 |
| 70 | 0.09321 | 0.08478 | 0.08525 | 0.08563 | 0.08622 | 0.08667 |
| 71 | 0.06573 | 0.08887 | 0.08909 | 0.08928 | 0.08960 | 0.08987 |
| 72 | 0.13084 | 0.09424 | 0.09415 | 0.09411 | 0.09409 | 0.09412 |
| 73 | 0.15214 | 0.10033 | 0.09998 | 0.09973 | 0.09939 | 0.09916 |
| 74 | 0.04316 | 0.10666 | 0.10615 | 0.10575 | 0.10515 | 0.10471 |
| 75 | 0.09818 | 0.11293 | 0.11234 | 0.11185 | 0.11109 | 0.11051 |
| 76 | 0.09658 | 0.11887 | 0.11827 | 0.11777 | 0.11696 | 0.11633 |
| 77 | 0.18268 | 0.12424 | 0.12374 | 0.12330 | 0.12259 | 0.12201 |
| 78 | 0.15416 | 0.12896 | 0.12867 | 0.12840 | 0.12792 | 0.12750 |
| 79 | 0.10955 | 0.13324 | 0.13326 | 0.13322 | 0.13308 | 0.13291 |
| 80 | 0.14568 | 0.13756 | 0.13791 | 0.13812 | 0.13836 | 0.13847 |
| 81 | 0.06808 | 0.14250 | 0.14311 | 0.14354 | 0.14413 | 0.14451 |
| 82 | 0.22578 | 0.14861 | 0.14936 | 0.14992 | 0.15075 | 0.15134 |
| 83 | 0.10421 | 0.15627 | 0.15700 | 0.15760 | 0.15854 | 0.15925 |
| 84 | 0.11099 | 0.16568 | 0.16627 | 0.16680 | 0.16771 | 0.16844 |
| 85 | 0.30733 | 0.17682 | 0.17719 | 0.17760 | 0.17835 | 0.17901 |
| 86 | 0.16512 | 0.18954 | 0.18969 | 0.18993 | 0.19045 | 0.19095 |
| 87 | 0.17431 | 0.20368 | 0.20362 | 0.20367 | 0.20391 | 0.20420 |
| 88 | 0.20576 | 0.21899 | 0.21875 | 0.21863 | 0.21856 | 0.21860 |
| 89 | 0.23325 | 0.23521 | 0.23483 | 0.23455 | 0.23417 | 0.23395 |
| 90 | 0.26033 | 0.25202 | 0.25153 | 0.25110 | 0.25044 | 0.24996 |
| 91 | 0.27787 | 0.26906 | 0.26848 | 0.26794 | 0.26704 | 0.26635 |
| 92 | 0.30544 | 0.28596 | 0.28532 | 0.28471 | 0.28366 | 0.28281 |
| 93 | 0.27072 | 0.30238 | 0.30174 | 0.30110 | 0.29997 | 0.29904 |
| 94 | 0.30918 | 0.31804 | 0.31742 | 0.31682 | 0.31571 | 0.31478 |
| 95 | 0.30627 | 0.33269 | 0.33216 | 0.33163 | 0.33067 | 0.32983 |
| 96 | 0.39349 | 0.34617 | 0.34579 | 0.34541 | 0.34468 | 0.34402 |
| 97 | 0.39879 | 0.35844 | 0.35827 | 0.35808 | 0.35768 | 0.35727 |
| 98 | 0.37378 | 0.36956 | 0.36965 | 0.36970 | 0.36968 | 0.36959 |
| 99 | 0.33804 | 0.37973 | 0.38010 | 0.38040 | 0.38079 | 0.38104 |
| 100 | 0.48429 | 0.38925 | 0.38988 | 0.39039 | 0.39118 | 0.39173 |
| 101 | 0.30538 | 0.39849 | 0.39929 | 0.39997 | 0.40104 | 0.40186 |
| 102 | 0.46266 | 0.40781 | 0.40867 | 0.40942 | 0.41065 | 0.41161 |
| 103 | 0.37844 | 0.41759 | 0.41836 | 0.41905 | 0.42025 | 0.42121 |
| 104 | 0.44348 | 0.42816 | 0.42867 | 0.42918 | 0.43009 | 0.43087 |
| 105 | 0.43904 | 0.43978 | 0.43989 | 0.44006 | 0.44044 | 0.44079 |
| 106 | 0.47714 | 0.45270 | 0.45229 | 0.45197 | 0.45151 | 0.45120 |
| 107 | 0.47350 | 0.46714 | 0.46610 | 0.46516 | 0.46356 | 0.46228 |
| 108 | 0.45281 | 0.48329 | 0.48158 | 0.47987 | 0.47679 | 0.47423 |
| 109 | 0.40650 | 0.50136 | 0.49895 | 0.49635 | 0.49144 | 0.48725 |
| 110 | 0.43044 | 0.52154 | 0.51845 | 0.51483 | 0.50773 | 0.50153 |

Table 5.4: Graduated mortality rates using W-H at z=5

| Age | qx | graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00269 | 0.00271 | 0.00274 | 0.00277 | 0.00280 |
| 19 | 0.00255 | 0.00195 | 0.00194 | 0.00193 | 0.00192 | 0.00190 |
| 20 | 0.00283 | 0.00158 | 0.00157 | 0.00155 | 0.00153 | 0.00151 |
| 21 | 0.00050 | 0.00153 | 0.00153 | 0.00152 | 0.00150 | 0.00149 |
| 22 | 0.00081 | 0.00174 | 0.00174 | 0.00174 | 0.00174 | 0.00174 |
| 23 | 0.00254 | 0.00213 | 0.00213 | 0.00213 | 0.00214 | 0.00216 |
| 24 | 0.00303 | 0.00259 | 0.00259 | 0.00260 | 0.00262 | 0.00264 |
| 25 | 0.00328 | 0.00305 | 0.00305 | 0.00306 | 0.00308 | 0.00310 |
| 26 | 0.00351 | 0.00344 | 0.00345 | 0.00346 | 0.00348 | 0.00348 |
| 27 | 0.00351 | 0.00373 | 0.00375 | 0.00376 | 0.00376 | 0.00375 |
| 28 | 0.00369 | 0.00392 | 0.00394 | 0.00393 | 0.00392 | 0.00390 |
| 29 | 0.00386 | 0.00401 | 0.00401 | 0.00399 | 0.00396 | 0.00394 |
| 30 | 0.00405 | 0.00400 | 0.00398 | 0.00396 | 0.00392 | 0.00389 |
| 31 | 0.00392 | 0.00393 | 0.00389 | 0.00386 | 0.00382 | 0.00380 |
| 32 | 0.00402 | 0.00381 | 0.00377 | 0.00375 | 0.00373 | 0.00373 |
| 33 | 0.00394 | 0.00371 | 0.00369 | 0.00369 | 0.00371 | 0.00373 |
| 34 | 0.00394 | 0.00370 | 0.00371 | 0.00374 | 0.00379 | 0.00384 |
| 35 | 0.00390 | 0.00382 | 0.00387 | 0.00392 | 0.00400 | 0.00406 |
| 36 | 0.00370 | 0.00411 | 0.00419 | 0.00425 | 0.00433 | 0.00438 |
| 37 | 0.00423 | 0.00455 | 0.00462 | 0.00467 | 0.00472 | 0.00474 |
| 38 | 0.00483 | 0.00503 | 0.00507 | 0.00508 | 0.00508 | 0.00505 |
| 39 | 0.00523 | 0.00544 | 0.00541 | 0.00538 | 0.00531 | 0.00525 |
| 40 | 0.00541 | 0.00563 | 0.00554 | 0.00547 | 0.00535 | 0.00526 |
| 41 | 0.00529 | 0.00552 | 0.00540 | 0.00532 | 0.00519 | 0.00511 |
| 42 | 0.00878 | 0.00514 | 0.00504 | 0.00498 | 0.00489 | 0.00484 |
| 43 | 0.00185 | 0.00464 | 0.00461 | 0.00459 | 0.00458 | 0.00459 |
| 44 | 0.00372 | 0.00426 | 0.00431 | 0.00435 | 0.00443 | 0.00450 |
| 45 | 0.00755 | 0.00423 | 0.00434 | 0.00443 | 0.00458 | 0.00469 |
| 46 | 0.00188 | 0.00469 | 0.00483 | 0.00494 | 0.00509 | 0.00521 |
| 47 | 0.00582 | 0.00566 | 0.00576 | 0.00584 | 0.00594 | 0.00602 |
| 48 | 0.00403 | 0.00698 | 0.00699 | 0.00700 | 0.00702 | 0.00702 |
| 49 | 0.01058 | 0.00840 | 0.00832 | 0.00826 | 0.00817 | 0.00809 |
| 50 | 0.01327 | 0.00971 | 0.00958 | 0.00947 | 0.00931 | 0.00918 |
| 51 | 0.00952 | 0.01092 | 0.01078 | 0.01068 | 0.01050 | 0.01036 |
| 52 | 0.00774 | 0.01224 | 0.01216 | 0.01209 | 0.01197 | 0.01188 |
| 53 | 0.01952 | 0.01411 | 0.01412 | 0.01411 | 0.01409 | 0.01407 |
| 54 | 0.01720 | 0.01705 | 0.01714 | 0.01719 | 0.01728 | 0.01734 |
| 55 | 0.02129 | 0.02150 | 0.02162 | 0.02171 | 0.02187 | 0.02200 |
| 56 | 0.01596 | 0.02765 | 0.02775 | 0.02785 | 0.02804 | 0.02820 |
| 57 | 0.04338 | 0.03536 | 0.03542 | 0.03550 | 0.03566 | 0.03580 |
| 58 | 0.05745 | 0.04418 | 0.04420 | 0.04424 | 0.04433 | 0.04442 |
| 59 | 0.02949 | 0.05344 | 0.05343 | 0.05343 | 0.05345 | 0.05347 |
| 60 | 0.07676 | 0.06239 | 0.06237 | 0.06234 | 0.06228 | 0.06221 |
| 61 | 0.06915 | 0.07032 | 0.07029 | 0.07023 | 0.07010 | 0.06996 |
| 62 | 0.08164 | 0.07666 | 0.07660 | 0.07652 | 0.07632 | 0.07613 |
| 63 | 0.08041 | 0.08106 | 0.08095 | 0.08084 | 0.08061 | 0.08041 |

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.08342 | 0.08326 | 0.08313 | 0.08292 | 0.08275 |
| 65 | 0.08862 | 0.08392 | 0.08375 | 0.08365 | 0.08351 | 0.08341 |
| 66 | 0.12521 | 0.08302 | 0.08295 | 0.08292 | 0.08292 | 0.08295 |
| 67 | 0.06149 | 0.08157 | 0.08166 | 0.08176 | 0.08194 | 0.08210 |
| 68 | 0.07967 | 0.08063 | 0.08089 | 0.08109 | 0.08142 | 0.08168 |
| 69 | 0.06078 | 0.08117 | 0.08149 | 0.08173 | 0.08210 | 0.08240 |
| 70 | 0.09321 | 0.08374 | 0.08398 | 0.08416 | 0.08444 | 0.08467 |
| 71 | 0.06573 | 0.08828 | 0.08834 | 0.08839 | 0.08849 | 0.08859 |
| 72 | 0.13084 | 0.09429 | 0.09416 | 0.09407 | 0.09396 | 0.09389 |
| 73 | 0.15214 | 0.10105 | 0.10080 | 0.10061 | 0.10032 | 0.10011 |
| 74 | 0.04316 | 0.10793 | 0.10765 | 0.10742 | 0.10704 | 0.10673 |
| 75 | 0.09818 | 0.11448 | 0.11422 | 0.11399 | 0.11359 | 0.11326 |
| 76 | 0.09658 | 0.12035 | 0.12014 | 0.11995 | 0.11962 | 0.11932 |
| 77 | 0.18268 | 0.12533 | 0.12521 | 0.12510 | 0.12489 | 0.12469 |
| 78 | 0.15416 | 0.12941 | 0.12943 | 0.12944 | 0.12942 | 0.12937 |
| 79 | 0.10955 | 0.13294 | 0.13314 | 0.13328 | 0.13348 | 0.13360 |
| 80 | 0.14568 | 0.13652 | 0.13689 | 0.13715 | 0.13754 | 0.13782 |
| 81 | 0.06808 | 0.14094 | 0.14137 | 0.14169 | 0.14219 | 0.14257 |
| 82 | 0.22578 | 0.14686 | 0.14722 | 0.14751 | 0.14799 | 0.14839 |
| 83 | 0.10421 | 0.15469 | 0.15486 | 0.15504 | 0.15539 | 0.15572 |
| 84 | 0.11099 | 0.16453 | 0.16446 | 0.16448 | 0.16463 | 0.16483 |
| 85 | 0.30733 | 0.17624 | 0.17596 | 0.17583 | 0.17577 | 0.17582 |
| 86 | 0.16512 | 0.18952 | 0.18914 | 0.18893 | 0.18871 | 0.18861 |
| 87 | 0.17431 | 0.20407 | 0.20374 | 0.20352 | 0.20322 | 0.20302 |
| 88 | 0.20576 | 0.21963 | 0.21947 | 0.21931 | 0.21901 | 0.21875 |
| 89 | 0.23325 | 0.23600 | 0.23605 | 0.23599 | 0.23574 | 0.23546 |
| 90 | 0.26033 | 0.25293 | 0.25316 | 0.25319 | 0.25302 | 0.25275 |
| 91 | 0.27787 | 0.27016 | 0.27047 | 0.27054 | 0.27043 | 0.27020 |
| 92 | 0.30544 | 0.28734 | 0.28759 | 0.28766 | 0.28757 | 0.28737 |
| 93 | 0.27072 | 0.30406 | 0.30415 | 0.30415 | 0.30405 | 0.30390 |
| 94 | 0.30918 | 0.31991 | 0.31976 | 0.31967 | 0.31955 | 0.31945 |
| 95 | 0.30627 | 0.33449 | 0.33411 | 0.33394 | 0.33382 | 0.33379 |
| 96 | 0.39349 | 0.34750 | 0.34700 | 0.34681 | 0.34673 | 0.34680 |
| 97 | 0.39879 | 0.35886 | 0.35839 | 0.35825 | 0.35829 | 0.35848 |
| 98 | 0.37378 | 0.36871 | 0.36846 | 0.36845 | 0.36867 | 0.36899 |
| 99 | 0.33804 | 0.37749 | 0.37757 | 0.37774 | 0.37817 | 0.37861 |
| 100 | 0.48429 | 0.38581 | 0.38626 | 0.38661 | 0.38721 | 0.38772 |
| 101 | 0.30538 | 0.39440 | 0.39512 | 0.39560 | 0.39626 | 0.39676 |
| 102 | 0.46266 | 0.40395 | 0.40475 | 0.40522 | 0.40579 | 0.40616 |
| 103 | 0.37844 | 0.41497 | 0.41556 | 0.41587 | 0.41618 | 0.41634 |
| 104 | 0.44348 | 0.42762 | 0.42777 | 0.42779 | 0.42771 | 0.42758 |
| 105 | 0.43904 | 0.44166 | 0.44125 | 0.44094 | 0.44046 | 0.44008 |
| 106 | 0.47714 | 0.45637 | 0.45554 | 0.45502 | 0.45435 | 0.45388 |
| 107 | 0.47350 | 0.47050 | 0.46980 | 0.46944 | 0.46907 | 0.46887 |
| 108 | 0.45281 | 0.48229 | 0.48281 | 0.48327 | 0.48409 | 0.48477 |
| 109 | 0.40650 | 0.48945 | 0.49294 | 0.49532 | 0.49870 | 0.50115 |
| 110 | 0.43044 | 0.48916 | 0.49819 | 0.50405 | 0.51194 | 0.51743 |

# Appendix B - Graduated mortality rates by MW-H

Table 5.5: Graduated mortality rates using MW-H at z=2

| Age | qx | graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00146 | 0.00147 | 0.00149 | 0.00157 | 0.00167 |
| 19 | 0.00255 | 0.00164 | 0.00166 | 0.00169 | 0.00176 | 0.00185 |
| 20 | 0.00283 | 0.00183 | 0.00186 | 0.00189 | 0.00196 | 0.00204 |
| 21 | 0.00050 | 0.00203 | 0.00206 | 0.00209 | 0.00216 | 0.00223 |
| 22 | 0.00081 | 0.00225 | 0.00228 | 0.00231 | 0.00237 | 0.00243 |
| 23 | 0.00254 | 0.00249 | 0.00251 | 0.00253 | 0.00258 | 0.00263 |
| 24 | 0.00303 | 0.00273 | 0.00275 | 0.00276 | 0.00279 | 0.00282 |
| 25 | 0.00328 | 0.00297 | 0.00297 | 0.00298 | 0.00300 | 0.00301 |
| 26 | 0.00351 | 0.00320 | 0.00319 | 0.00319 | 0.00319 | 0.00319 |
| 27 | 0.00351 | 0.00339 | 0.00338 | 0.00338 | 0.00337 | 0.00335 |
| 28 | 0.00369 | 0.00356 | 0.00355 | 0.00355 | 0.00352 | 0.00349 |
| 29 | 0.00386 | 0.00371 | 0.00370 | 0.00369 | 0.00366 | 0.00360 |
| 30 | 0.00405 | 0.00383 | 0.00383 | 0.00382 | 0.00377 | 0.00370 |
| 31 | 0.00392 | 0.00393 | 0.00394 | 0.00392 | 0.00386 | 0.00377 |
| 32 | 0.00402 | 0.00402 | 0.00403 | 0.00401 | 0.00393 | 0.00382 |
| 33 | 0.00394 | 0.00411 | 0.00411 | 0.00408 | 0.00398 | 0.00384 |
| 34 | 0.00394 | 0.00419 | 0.00419 | 0.00414 | 0.00401 | 0.00385 |
| 35 | 0.00390 | 0.00428 | 0.00426 | 0.00419 | 0.00402 | 0.00384 |
| 36 | 0.00370 | 0.00438 | 0.00432 | 0.00423 | 0.00402 | 0.00381 |
| 37 | 0.00423 | 0.00449 | 0.00438 | 0.00426 | 0.00400 | 0.00377 |
| 38 | 0.00483 | 0.00458 | 0.00442 | 0.00427 | 0.00398 | 0.00373 |
| 39 | 0.00523 | 0.00463 | 0.00443 | 0.00425 | 0.00393 | 0.00368 |
| 40 | 0.00541 | 0.00465 | 0.00441 | 0.00421 | 0.00388 | 0.00363 |
| 41 | 0.00529 | 0.00461 | 0.00436 | 0.00415 | 0.00383 | 0.00360 |
| 42 | 0.00878 | 0.00452 | 0.00428 | 0.00409 | 0.00380 | 0.00362 |
| 43 | 0.00185 | 0.00442 | 0.00421 | 0.00405 | 0.00384 | 0.00372 |
| 44 | 0.00372 | 0.00440 | 0.00423 | 0.00411 | 0.00398 | 0.00394 |
| 45 | 0.00755 | 0.00451 | 0.00440 | 0.00433 | 0.00430 | 0.00434 |
| 46 | 0.00188 | 0.00480 | 0.00474 | 0.00474 | 0.00483 | 0.00497 |
| 47 | 0.00582 | 0.00536 | 0.00537 | 0.00544 | 0.00565 | 0.00588 |
| 48 | 0.00403 | 0.00623 | 0.00631 | 0.00646 | 0.00680 | 0.00713 |
| 49 | 0.01058 | 0.00745 | 0.00763 | 0.00786 | 0.00833 | 0.00875 |
| 50 | 0.01327 | 0.00902 | 0.00934 | 0.00967 | 0.01027 | 0.01077 |
| 51 | 0.00952 | 0.01101 | 0.01150 | 0.01194 | 0.01266 | 0.01323 |
| 52 | 0.00774 | 0.01359 | 0.01425 | 0.01478 | 0.01558 | 0.01617 |
| 53 | 0.01952 | 0.01693 | 0.01768 | 0.01825 | 0.01907 | 0.01964 |
| 54 | 0.01720 | 0.02102 | 0.02181 | 0.02236 | 0.02312 | 0.02361 |
| 55 | 0.02129 | 0.02593 | 0.02664 | 0.02711 | 0.02770 | 0.02806 |
| 56 | 0.01596 | 0.03165 | 0.03215 | 0.03245 | 0.03277 | 0.03293 |
| 57 | 0.04338 | 0.03806 | 0.03821 | 0.03826 | 0.03823 | 0.03814 |
| 58 | 0.05745 | 0.04475 | 0.04451 | 0.04428 | 0.04386 | 0.04351 |
| 59 | 0.02949 | 0.05139 | 0.05077 | 0.05027 | 0.04949 | 0.04888 |
| 60 | 0.07676 | 0.05791 | 0.05691 | 0.05615 | 0.05502 | 0.05418 |
| 61 | 0.06915 | 0.06384 | 0.06257 | 0.06162 | 0.06024 | 0.05924 |
| 62 | 0.08164 | 0.06901 | 0.06762 | 0.06658 | 0.06506 | 0.06397 |
| 63 | 0.08041 | 0.07334 | 0.07198 | 0.07095 | 0.06944 | 0.06835 |

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.07692 | 0.07571 | 0.07477 | 0.07339 | 0.07237 |
| 65 | 0.08862 | 0.07992 | 0.07893 | 0.07816 | 0.07698 | 0.07611 |
| 66 | 0.12521 | 0.08229 | 0.08165 | 0.08110 | 0.08024 | 0.07958 |
| 67 | 0.06149 | 0.08410 | 0.08391 | 0.08368 | 0.08322 | 0.08284 |
| 68 | 0.07967 | 0.08590 | 0.08614 | 0.08620 | 0.08616 | 0.08606 |
| 69 | 0.06078 | 0.08799 | 0.08857 | 0.08888 | 0.08921 | 0.08938 |
| 70 | 0.09321 | 0.09062 | 0.09138 | 0.09187 | 0.09251 | 0.09292 |
| 71 | 0.06573 | 0.09374 | 0.09459 | 0.09521 | 0.09608 | 0.09670 |
| 72 | 0.13084 | 0.09737 | 0.09823 | 0.09893 | 0.09998 | 0.10078 |
| 73 | 0.15214 | 0.10128 | 0.10217 | 0.10294 | 0.10417 | 0.10513 |
| 74 | 0.04316 | 0.10547 | 0.10642 | 0.10728 | 0.10868 | 0.10979 |
| 75 | 0.09818 | 0.11024 | 0.11121 | 0.11211 | 0.11363 | 0.11485 |
| 76 | 0.09658 | 0.11558 | 0.11652 | 0.11744 | 0.11904 | 0.12035 |
| 77 | 0.18268 | 0.12139 | 0.12232 | 0.12324 | 0.12489 | 0.12625 |
| 78 | 0.15416 | 0.12752 | 0.12849 | 0.12945 | 0.13114 | 0.13255 |
| 79 | 0.10955 | 0.13404 | 0.13509 | 0.13608 | 0.13782 | 0.13926 |
| 80 | 0.14568 | 0.14110 | 0.14222 | 0.14323 | 0.14498 | 0.14642 |
| 81 | 0.06808 | 0.14877 | 0.14993 | 0.15093 | 0.15264 | 0.15404 |
| 82 | 0.22578 | 0.15715 | 0.15827 | 0.15922 | 0.16083 | 0.16216 |
| 83 | 0.10421 | 0.16617 | 0.16720 | 0.16806 | 0.16952 | 0.17073 |
| 84 | 0.11099 | 0.17587 | 0.17674 | 0.17747 | 0.17872 | 0.17976 |
| 85 | 0.30733 | 0.18620 | 0.18684 | 0.18740 | 0.18839 | 0.18923 |
| 86 | 0.16512 | 0.19701 | 0.19741 | 0.19777 | 0.19847 | 0.19909 |
| 87 | 0.17431 | 0.20831 | 0.20843 | 0.20858 | 0.20895 | 0.20932 |
| 88 | 0.20576 | 0.22008 | 0.21989 | 0.21981 | 0.21982 | 0.21991 |
| 89 | 0.23325 | 0.23226 | 0.23173 | 0.23140 | 0.23103 | 0.23082 |
| 90 | 0.26033 | 0.24479 | 0.24391 | 0.24332 | 0.24254 | 0.24203 |
| 91 | 0.27787 | 0.25758 | 0.25636 | 0.25551 | 0.25433 | 0.25349 |
| 92 | 0.30544 | 0.27058 | 0.26904 | 0.26794 | 0.26635 | 0.26518 |
| 93 | 0.27072 | 0.28373 | 0.28190 | 0.28056 | 0.27857 | 0.27707 |
| 94 | 0.30918 | 0.29701 | 0.29492 | 0.29335 | 0.29098 | 0.28915 |
| 95 | 0.30627 | 0.31038 | 0.30806 | 0.30628 | 0.30354 | 0.30139 |
| 96 | 0.39349 | 0.32381 | 0.32130 | 0.31934 | 0.31623 | 0.31376 |
| 97 | 0.39879 | 0.33726 | 0.33461 | 0.33248 | 0.32904 | 0.32626 |
| 98 | 0.37378 | 0.35072 | 0.34797 | 0.34571 | 0.34195 | 0.33886 |
| 99 | 0.33804 | 0.36420 | 0.36140 | 0.35901 | 0.35496 | 0.35157 |
| 100 | 0.48429 | 0.37770 | 0.37488 | 0.37238 | 0.36805 | 0.36437 |
| 101 | 0.30538 | 0.39124 | 0.38842 | 0.38583 | 0.38123 | 0.37726 |
| 102 | 0.46266 | 0.40484 | 0.40203 | 0.39936 | 0.39450 | 0.39023 |
| 103 | 0.37844 | 0.41850 | 0.41571 | 0.41296 | 0.40784 | 0.40329 |
| 104 | 0.44348 | 0.43222 | 0.42946 | 0.42664 | 0.42126 | 0.41642 |
| 105 | 0.43904 | 0.44601 | 0.44329 | 0.44039 | 0.43475 | 0.42963 |
| 106 | 0.47714 | 0.45986 | 0.45718 | 0.45420 | 0.44831 | 0.44290 |
| 107 | 0.47350 | 0.47378 | 0.47114 | 0.46809 | 0.46194 | 0.45623 |
| 108 | 0.45281 | 0.48776 | 0.48516 | 0.48204 | 0.47563 | 0.46964 |
| 109 | 0.40650 | 0.50180 | 0.49925 | 0.49606 | 0.48939 | 0.48311 |
| 110 | 0.43044 | 0.51592 | 0.51341 | 0.51015 | 0.50322 | 0.49664 |

Table 5.6: Graduated mortality rates using MW-H at z=3

| Age | qx | | | graduated | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00186 | 0.00173 | 0.00165 | 0.00155 | 0.00150 |
| 19 | 0.00255 | 0.00175 | 0.00171 | 0.00168 | 0.00165 | 0.00163 |
| 20 | 0.00283 | 0.00175 | 0.00176 | 0.00177 | 0.00179 | 0.00180 |
| 21 | 0.00050 | 0.00185 | 0.00190 | 0.00193 | 0.00197 | 0.00199 |
| 22 | 0.00081 | 0.00205 | 0.00211 | 0.00214 | 0.00219 | 0.00222 |
| 23 | 0.00254 | 0.00233 | 0.00238 | 0.00241 | 0.00244 | 0.00246 |
| 24 | 0.00303 | 0.00266 | 0.00268 | 0.00269 | 0.00271 | 0.00271 |
| 25 | 0.00328 | 0.00299 | 0.00298 | 0.00297 | 0.00297 | 0.00296 |
| 26 | 0.00351 | 0.00328 | 0.00325 | 0.00323 | 0.00321 | 0.00319 |
| 27 | 0.00351 | 0.00353 | 0.00348 | 0.00345 | 0.00342 | 0.00339 |
| 28 | 0.00369 | 0.00370 | 0.00366 | 0.00363 | 0.00359 | 0.00356 |
| 29 | 0.00386 | 0.00382 | 0.00378 | 0.00376 | 0.00372 | 0.00370 |
| 30 | 0.00405 | 0.00388 | 0.00386 | 0.00385 | 0.00383 | 0.00381 |
| 31 | 0.00392 | 0.00391 | 0.00391 | 0.00391 | 0.00391 | 0.00391 |
| 32 | 0.00402 | 0.00393 | 0.00395 | 0.00397 | 0.00399 | 0.00401 |
| 33 | 0.00394 | 0.00396 | 0.00401 | 0.00404 | 0.00408 | 0.00411 |
| 34 | 0.00394 | 0.00404 | 0.00410 | 0.00414 | 0.00419 | 0.00424 |
| 35 | 0.00390 | 0.00418 | 0.00423 | 0.00427 | 0.00433 | 0.00438 |
| 36 | 0.00370 | 0.00437 | 0.00441 | 0.00444 | 0.00449 | 0.00454 |
| 37 | 0.00423 | 0.00460 | 0.00461 | 0.00462 | 0.00466 | 0.00470 |
| 38 | 0.00483 | 0.00483 | 0.00481 | 0.00481 | 0.00482 | 0.00485 |
| 39 | 0.00523 | 0.00502 | 0.00497 | 0.00495 | 0.00496 | 0.00497 |
| 40 | 0.00541 | 0.00511 | 0.00506 | 0.00504 | 0.00504 | 0.00504 |
| 41 | 0.00529 | 0.00511 | 0.00507 | 0.00506 | 0.00506 | 0.00504 |
| 42 | 0.00878 | 0.00501 | 0.00502 | 0.00503 | 0.00502 | 0.00500 |
| 43 | 0.00185 | 0.00488 | 0.00494 | 0.00497 | 0.00497 | 0.00493 |
| 44 | 0.00372 | 0.00483 | 0.00491 | 0.00494 | 0.00493 | 0.00487 |
| 45 | 0.00755 | 0.00493 | 0.00501 | 0.00502 | 0.00497 | 0.00488 |
| 46 | 0.00188 | 0.00524 | 0.00526 | 0.00523 | 0.00513 | 0.00501 |
| 47 | 0.00582 | 0.00578 | 0.00571 | 0.00563 | 0.00546 | 0.00532 |
| 48 | 0.00403 | 0.00655 | 0.00637 | 0.00623 | 0.00602 | 0.00587 |
| 49 | 0.01058 | 0.00749 | 0.00724 | 0.00707 | 0.00686 | 0.00674 |
| 50 | 0.01327 | 0.00859 | 0.00835 | 0.00821 | 0.00807 | 0.00802 |
| 51 | 0.00952 | 0.00996 | 0.00982 | 0.00976 | 0.00977 | 0.00983 |
| 52 | 0.00774 | 0.01184 | 0.01187 | 0.01195 | 0.01213 | 0.01233 |
| 53 | 0.01952 | 0.01451 | 0.01475 | 0.01497 | 0.01535 | 0.01566 |
| 54 | 0.01720 | 0.01824 | 0.01866 | 0.01900 | 0.01952 | 0.01991 |
| 55 | 0.02129 | 0.02322 | 0.02375 | 0.02414 | 0.02471 | 0.02512 |
| 56 | 0.01596 | 0.02951 | 0.03001 | 0.03037 | 0.03087 | 0.03121 |
| 57 | 0.04338 | 0.03692 | 0.03727 | 0.03750 | 0.03781 | 0.03800 |
| 58 | 0.05745 | 0.04501 | 0.04511 | 0.04516 | 0.04519 | 0.04518 |
| 59 | 0.02949 | 0.05324 | 0.05306 | 0.05290 | 0.05263 | 0.05241 |
| 60 | 0.07676 | 0.06115 | 0.06069 | 0.06033 | 0.05977 | 0.05936 |
| 61 | 0.06915 | 0.06816 | 0.06750 | 0.06699 | 0.06624 | 0.06568 |
| 62 | 0.08164 | 0.07389 | 0.07315 | 0.07258 | 0.07175 | 0.07114 |
| 63 | 0.08041 | 0.07817 | 0.07747 | 0.07695 | 0.07618 | 0.07561 |

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.08104 | 0.08052 | 0.08012 | 0.07954 | 0.07910 |
| 65 | 0.08862 | 0.08272 | 0.08245 | 0.08226 | 0.08195 | 0.08171 |
| 66 | 0.12521 | 0.08344 | 0.08351 | 0.08357 | 0.08362 | 0.08361 |
| 67 | 0.06149 | 0.08360 | 0.08408 | 0.08440 | 0.08482 | 0.08507 |
| 68 | 0.07967 | 0.08393 | 0.08472 | 0.08526 | 0.08596 | 0.08641 |
| 69 | 0.06078 | 0.08502 | 0.08592 | 0.08655 | 0.08740 | 0.08795 |
| 70 | 0.09321 | 0.08721 | 0.08800 | 0.08858 | 0.08938 | 0.08993 |
| 71 | 0.06573 | 0.09050 | 0.09100 | 0.09140 | 0.09199 | 0.09244 |
| 72 | 0.13084 | 0.09468 | 0.09480 | 0.09494 | 0.09524 | 0.09551 |
| 73 | 0.15214 | 0.09940 | 0.09915 | 0.09905 | 0.09903 | 0.09912 |
| 74 | 0.04316 | 0.10447 | 0.10393 | 0.10362 | 0.10331 | 0.10322 |
| 75 | 0.09818 | 0.10985 | 0.10910 | 0.10862 | 0.10808 | 0.10783 |
| 76 | 0.09658 | 0.11544 | 0.11456 | 0.11397 | 0.11328 | 0.11292 |
| 77 | 0.18268 | 0.12108 | 0.12019 | 0.11960 | 0.11887 | 0.11847 |
| 78 | 0.15416 | 0.12663 | 0.12592 | 0.12543 | 0.12480 | 0.12444 |
| 79 | 0.10955 | 0.13220 | 0.13183 | 0.13153 | 0.13112 | 0.13087 |
| 80 | 0.14568 | 0.13808 | 0.13811 | 0.13807 | 0.13795 | 0.13787 |
| 81 | 0.06808 | 0.14458 | 0.14501 | 0.14522 | 0.14543 | 0.14552 |
| 82 | 0.22578 | 0.15199 | 0.15274 | 0.15317 | 0.15366 | 0.15391 |
| 83 | 0.10421 | 0.16047 | 0.16143 | 0.16203 | 0.16272 | 0.16311 |
| 84 | 0.11099 | 0.17013 | 0.17117 | 0.17185 | 0.17267 | 0.17313 |
| 85 | 0.30733 | 0.18097 | 0.18196 | 0.18264 | 0.18349 | 0.18397 |
| 86 | 0.16512 | 0.19288 | 0.19374 | 0.19436 | 0.19514 | 0.19559 |
| 87 | 0.17431 | 0.20579 | 0.20644 | 0.20693 | 0.20757 | 0.20793 |
| 88 | 0.20576 | 0.21962 | 0.21998 | 0.22029 | 0.22070 | 0.22092 |
| 89 | 0.23325 | 0.23420 | 0.23422 | 0.23430 | 0.23443 | 0.23446 |
| 90 | 0.26033 | 0.24933 | 0.24900 | 0.24883 | 0.24863 | 0.24846 |
| 91 | 0.27787 | 0.26481 | 0.26412 | 0.26371 | 0.26317 | 0.26277 |
| 92 | 0.30544 | 0.28040 | 0.27941 | 0.27877 | 0.27790 | 0.27728 |
| 93 | 0.27072 | 0.29593 | 0.29469 | 0.29385 | 0.29269 | 0.29187 |
| 94 | 0.30918 | 0.31120 | 0.30980 | 0.30880 | 0.30742 | 0.30643 |
| 95 | 0.30627 | 0.32607 | 0.32460 | 0.32351 | 0.32196 | 0.32085 |
| 96 | 0.39349 | 0.34042 | 0.33897 | 0.33786 | 0.33623 | 0.33505 |
| 97 | 0.39879 | 0.35415 | 0.35282 | 0.35176 | 0.35014 | 0.34896 |
| 98 | 0.37378 | 0.36722 | 0.36611 | 0.36515 | 0.36364 | 0.36251 |
| 99 | 0.33804 | 0.37965 | 0.37882 | 0.37802 | 0.37670 | 0.37569 |
| 100 | 0.48429 | 0.39146 | 0.39095 | 0.39035 | 0.38929 | 0.38846 |
| 101 | 0.30538 | 0.40271 | 0.40253 | 0.40216 | 0.40142 | 0.40081 |
| 102 | 0.46266 | 0.41347 | 0.41359 | 0.41346 | 0.41309 | 0.41276 |
| 103 | 0.37844 | 0.42377 | 0.42415 | 0.42428 | 0.42431 | 0.42429 |
| 104 | 0.44348 | 0.43365 | 0.43424 | 0.43460 | 0.43507 | 0.43541 |
| 105 | 0.43904 | 0.44311 | 0.44385 | 0.44444 | 0.44538 | 0.44611 |
| 106 | 0.47714 | 0.45216 | 0.45298 | 0.45380 | 0.45522 | 0.45639 |
| 107 | 0.47350 | 0.46077 | 0.46162 | 0.46266 | 0.46460 | 0.46625 |
| 108 | 0.45281 | 0.46895 | 0.46977 | 0.47102 | 0.47351 | 0.47568 |
| 109 | 0.40650 | 0.47668 | 0.47743 | 0.47888 | 0.48195 | 0.48467 |
| 110 | 0.43044 | 0.48398 | 0.48458 | 0.48623 | 0.48992 | 0.49323 |

Table 5.7: Graduated mortality rates using MW-H at z=4

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00263 | 0.00257 | 0.00251 | 0.00241 | 0.00233 |
| 19 | 0.00255 | 0.00191 | 0.00189 | 0.00188 | 0.00185 | 0.00184 |
| 20 | 0.00283 | 0.00158 | 0.00159 | 0.00160 | 0.00162 | 0.00165 |
| 21 | 0.00050 | 0.00156 | 0.00159 | 0.00161 | 0.00166 | 0.00170 |
| 22 | 0.00081 | 0.00179 | 0.00181 | 0.00184 | 0.00188 | 0.00192 |
| 23 | 0.00254 | 0.00217 | 0.00219 | 0.00220 | 0.00223 | 0.00225 |
| 24 | 0.00303 | 0.00261 | 0.00262 | 0.00263 | 0.00263 | 0.00264 |
| 25 | 0.00328 | 0.00306 | 0.00305 | 0.00304 | 0.00303 | 0.00301 |
| 26 | 0.00351 | 0.00343 | 0.00342 | 0.00340 | 0.00337 | 0.00334 |
| 27 | 0.00351 | 0.00371 | 0.00369 | 0.00367 | 0.00362 | 0.00359 |
| 28 | 0.00369 | 0.00388 | 0.00385 | 0.00383 | 0.00379 | 0.00376 |
| 29 | 0.00386 | 0.00395 | 0.00392 | 0.00390 | 0.00387 | 0.00385 |
| 30 | 0.00405 | 0.00393 | 0.00391 | 0.00390 | 0.00389 | 0.00389 |
| 31 | 0.00392 | 0.00387 | 0.00386 | 0.00386 | 0.00388 | 0.00390 |
| 32 | 0.00402 | 0.00380 | 0.00381 | 0.00383 | 0.00387 | 0.00391 |
| 33 | 0.00394 | 0.00377 | 0.00381 | 0.00385 | 0.00391 | 0.00395 |
| 34 | 0.00394 | 0.00383 | 0.00389 | 0.00394 | 0.00400 | 0.00404 |
| 35 | 0.00390 | 0.00399 | 0.00407 | 0.00411 | 0.00417 | 0.00419 |
| 36 | 0.00370 | 0.00427 | 0.00433 | 0.00436 | 0.00439 | 0.00439 |
| 37 | 0.00423 | 0.00463 | 0.00465 | 0.00465 | 0.00463 | 0.00460 |
| 38 | 0.00483 | 0.00499 | 0.00496 | 0.00492 | 0.00484 | 0.00479 |
| 39 | 0.00523 | 0.00526 | 0.00517 | 0.00510 | 0.00499 | 0.00492 |
| 40 | 0.00541 | 0.00536 | 0.00524 | 0.00515 | 0.00505 | 0.00499 |
| 41 | 0.00529 | 0.00526 | 0.00514 | 0.00508 | 0.00501 | 0.00498 |
| 42 | 0.00878 | 0.00499 | 0.00493 | 0.00491 | 0.00491 | 0.00494 |
| 43 | 0.00185 | 0.00468 | 0.00471 | 0.00475 | 0.00483 | 0.00491 |
| 44 | 0.00372 | 0.00449 | 0.00460 | 0.00470 | 0.00486 | 0.00497 |
| 45 | 0.00755 | 0.00458 | 0.00475 | 0.00488 | 0.00505 | 0.00517 |
| 46 | 0.00188 | 0.00503 | 0.00520 | 0.00531 | 0.00545 | 0.00552 |
| 47 | 0.00582 | 0.00586 | 0.00595 | 0.00600 | 0.00604 | 0.00603 |
| 48 | 0.00403 | 0.00694 | 0.00691 | 0.00687 | 0.00678 | 0.00668 |
| 49 | 0.01058 | 0.00813 | 0.00797 | 0.00784 | 0.00763 | 0.00746 |
| 50 | 0.01327 | 0.00929 | 0.00906 | 0.00889 | 0.00862 | 0.00843 |
| 51 | 0.00952 | 0.01049 | 0.01027 | 0.01011 | 0.00987 | 0.00972 |
| 52 | 0.00774 | 0.01196 | 0.01183 | 0.01174 | 0.01162 | 0.01156 |
| 53 | 0.01952 | 0.01410 | 0.01410 | 0.01412 | 0.01417 | 0.01423 |
| 54 | 0.01720 | 0.01731 | 0.01746 | 0.01760 | 0.01782 | 0.01801 |
| 55 | 0.02129 | 0.02195 | 0.02221 | 0.02243 | 0.02278 | 0.02306 |
| 56 | 0.01596 | 0.02818 | 0.02847 | 0.02871 | 0.02910 | 0.02940 |
| 57 | 0.04338 | 0.03585 | 0.03609 | 0.03629 | 0.03661 | 0.03685 |
| 58 | 0.05745 | 0.04450 | 0.04463 | 0.04474 | 0.04491 | 0.04503 |
| 59 | 0.02949 | 0.05350 | 0.05350 | 0.05349 | 0.05345 | 0.05341 |
| 60 | 0.07676 | 0.06218 | 0.06203 | 0.06190 | 0.06165 | 0.06144 |
| 61 | 0.06915 | 0.06984 | 0.06958 | 0.06933 | 0.06892 | 0.06857 |
| 62 | 0.08164 | 0.07596 | 0.07562 | 0.07531 | 0.07480 | 0.07439 |
| 63 | 0.08041 | 0.08024 | 0.07988 | 0.07957 | 0.07906 | 0.07867 |

| Age | qx | | | Graduated | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.08267 | 0.08236 | 0.08210 | 0.08171 | 0.08141 |
| 65 | 0.08862 | 0.08348 | 0.08328 | 0.08314 | 0.08295 | 0.08282 |
| 66 | 0.12521 | 0.08308 | 0.08309 | 0.08313 | 0.08322 | 0.08331 |
| 67 | 0.06149 | 0.08216 | 0.08244 | 0.08268 | 0.08308 | 0.08338 |
| 68 | 0.07967 | 0.08164 | 0.08216 | 0.08256 | 0.08318 | 0.08365 |
| 69 | 0.06078 | 0.08236 | 0.08296 | 0.08341 | 0.08411 | 0.08464 |
| 70 | 0.09321 | 0.08478 | 0.08526 | 0.08564 | 0.08623 | 0.08668 |
| 71 | 0.06573 | 0.08888 | 0.08909 | 0.08928 | 0.08961 | 0.08988 |
| 72 | 0.13084 | 0.09424 | 0.09415 | 0.09411 | 0.09410 | 0.09412 |
| 73 | 0.15214 | 0.10032 | 0.09997 | 0.09972 | 0.09938 | 0.09916 |
| 74 | 0.04316 | 0.10665 | 0.10614 | 0.10574 | 0.10514 | 0.10470 |
| 75 | 0.09818 | 0.11293 | 0.11233 | 0.11184 | 0.11108 | 0.11050 |
| 76 | 0.09658 | 0.11886 | 0.11826 | 0.11776 | 0.11695 | 0.11632 |
| 77 | 0.18268 | 0.12424 | 0.12373 | 0.12329 | 0.12258 | 0.12200 |
| 78 | 0.15416 | 0.12896 | 0.12867 | 0.12840 | 0.12791 | 0.12750 |
| 79 | 0.10955 | 0.13325 | 0.13326 | 0.13322 | 0.13308 | 0.13291 |
| 80 | 0.14568 | 0.13756 | 0.13791 | 0.13813 | 0.13837 | 0.13848 |
| 81 | 0.06808 | 0.14251 | 0.14312 | 0.14355 | 0.14414 | 0.14452 |
| 82 | 0.22578 | 0.14862 | 0.14937 | 0.14993 | 0.15076 | 0.15135 |
| 83 | 0.10421 | 0.15628 | 0.15701 | 0.15761 | 0.15855 | 0.15926 |
| 84 | 0.11099 | 0.16568 | 0.16628 | 0.16681 | 0.16772 | 0.16845 |
| 85 | 0.30733 | 0.17682 | 0.17720 | 0.17760 | 0.17836 | 0.17902 |
| 86 | 0.16512 | 0.18954 | 0.18969 | 0.18993 | 0.19046 | 0.19096 |
| 87 | 0.17431 | 0.20367 | 0.20361 | 0.20367 | 0.20391 | 0.20420 |
| 88 | 0.20576 | 0.21899 | 0.21875 | 0.21863 | 0.21856 | 0.21860 |
| 89 | 0.23325 | 0.23521 | 0.23483 | 0.23454 | 0.23416 | 0.23394 |
| 90 | 0.26033 | 0.25201 | 0.25152 | 0.25109 | 0.25043 | 0.24996 |
| 91 | 0.27787 | 0.26905 | 0.26847 | 0.26793 | 0.26704 | 0.26635 |
| 92 | 0.30544 | 0.28595 | 0.28532 | 0.28471 | 0.28365 | 0.28281 |
| 93 | 0.27072 | 0.30238 | 0.30173 | 0.30110 | 0.29997 | 0.29904 |
| 94 | 0.30918 | 0.31804 | 0.31742 | 0.31682 | 0.31571 | 0.31478 |
| 95 | 0.30627 | 0.33269 | 0.33216 | 0.33164 | 0.33067 | 0.32982 |
| 96 | 0.39349 | 0.34618 | 0.34579 | 0.34541 | 0.34468 | 0.34402 |
| 97 | 0.39879 | 0.35844 | 0.35828 | 0.35809 | 0.35768 | 0.35728 |
| 98 | 0.37378 | 0.36957 | 0.36966 | 0.36971 | 0.36969 | 0.36959 |
| 99 | 0.33804 | 0.37974 | 0.38012 | 0.38041 | 0.38080 | 0.38104 |
| 100 | 0.48429 | 0.38926 | 0.38989 | 0.39040 | 0.39118 | 0.39173 |
| 101 | 0.30538 | 0.39849 | 0.39930 | 0.39997 | 0.40104 | 0.40185 |
| 102 | 0.46266 | 0.40781 | 0.40867 | 0.40941 | 0.41064 | 0.41160 |
| 103 | 0.37844 | 0.41759 | 0.41835 | 0.41904 | 0.42023 | 0.42119 |
| 104 | 0.44348 | 0.42815 | 0.42866 | 0.42916 | 0.43007 | 0.43085 |
| 105 | 0.43904 | 0.43977 | 0.43988 | 0.44004 | 0.44042 | 0.44078 |
| 106 | 0.47714 | 0.45269 | 0.45228 | 0.45197 | 0.45152 | 0.45120 |
| 107 | 0.47350 | 0.46715 | 0.46612 | 0.46519 | 0.46359 | 0.46231 |
| 108 | 0.45281 | 0.48334 | 0.48164 | 0.47995 | 0.47688 | 0.47432 |
| 109 | 0.40650 | 0.50147 | 0.49910 | 0.49652 | 0.49162 | 0.48743 |
| 110 | 0.43044 | 0.52174 | 0.51873 | 0.51514 | 0.50805 | 0.50184 |

Table 5.8: Graduated mortality rates using MW-H at z=5

| Age | qx | graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 18 | 0.00190 | 0.00270 | 0.00272 | 0.00274 | 0.00278 | 0.00280 |
| 19 | 0.00255 | 0.00194 | 0.00194 | 0.00193 | 0.00191 | 0.00190 |
| 20 | 0.00283 | 0.00157 | 0.00156 | 0.00155 | 0.00152 | 0.00150 |
| 21 | 0.00050 | 0.00153 | 0.00152 | 0.00152 | 0.00150 | 0.00149 |
| 22 | 0.00081 | 0.00174 | 0.00174 | 0.00174 | 0.00174 | 0.00175 |
| 23 | 0.00254 | 0.00213 | 0.00213 | 0.00213 | 0.00215 | 0.00216 |
| 24 | 0.00303 | 0.00259 | 0.00259 | 0.00260 | 0.00262 | 0.00264 |
| 25 | 0.00328 | 0.00305 | 0.00305 | 0.00306 | 0.00308 | 0.00310 |
| 26 | 0.00351 | 0.00344 | 0.00345 | 0.00346 | 0.00348 | 0.00348 |
| 27 | 0.00351 | 0.00374 | 0.00375 | 0.00376 | 0.00376 | 0.00375 |
| 28 | 0.00369 | 0.00392 | 0.00393 | 0.00393 | 0.00392 | 0.00390 |
| 29 | 0.00386 | 0.00401 | 0.00400 | 0.00399 | 0.00396 | 0.00393 |
| 30 | 0.00405 | 0.00400 | 0.00398 | 0.00395 | 0.00391 | 0.00388 |
| 31 | 0.00392 | 0.00392 | 0.00388 | 0.00385 | 0.00382 | 0.00380 |
| 32 | 0.00402 | 0.00381 | 0.00377 | 0.00375 | 0.00373 | 0.00373 |
| 33 | 0.00394 | 0.00371 | 0.00369 | 0.00369 | 0.00371 | 0.00374 |
| 34 | 0.00394 | 0.00370 | 0.00372 | 0.00374 | 0.00380 | 0.00385 |
| 35 | 0.00390 | 0.00383 | 0.00388 | 0.00393 | 0.00401 | 0.00407 |
| 36 | 0.00370 | 0.00412 | 0.00420 | 0.00426 | 0.00434 | 0.00439 |
| 37 | 0.00423 | 0.00456 | 0.00463 | 0.00467 | 0.00472 | 0.00474 |
| 38 | 0.00483 | 0.00504 | 0.00507 | 0.00508 | 0.00507 | 0.00505 |
| 39 | 0.00523 | 0.00544 | 0.00541 | 0.00537 | 0.00530 | 0.00523 |
| 40 | 0.00541 | 0.00562 | 0.00553 | 0.00546 | 0.00534 | 0.00524 |
| 41 | 0.00529 | 0.00550 | 0.00539 | 0.00530 | 0.00518 | 0.00509 |
| 42 | 0.00878 | 0.00513 | 0.00503 | 0.00497 | 0.00488 | 0.00483 |
| 43 | 0.00185 | 0.00464 | 0.00460 | 0.00459 | 0.00458 | 0.00460 |
| 44 | 0.00372 | 0.00427 | 0.00432 | 0.00436 | 0.00444 | 0.00452 |
| 45 | 0.00755 | 0.00424 | 0.00436 | 0.00445 | 0.00460 | 0.00471 |
| 46 | 0.00188 | 0.00471 | 0.00485 | 0.00496 | 0.00511 | 0.00523 |
| 47 | 0.00582 | 0.00568 | 0.00578 | 0.00585 | 0.00596 | 0.00603 |
| 48 | 0.00403 | 0.00698 | 0.00700 | 0.00701 | 0.00702 | 0.00701 |
| 49 | 0.01058 | 0.00839 | 0.00831 | 0.00825 | 0.00815 | 0.00807 |
| 50 | 0.01327 | 0.00970 | 0.00956 | 0.00945 | 0.00929 | 0.00915 |
| 51 | 0.00952 | 0.01090 | 0.01076 | 0.01065 | 0.01048 | 0.01034 |
| 52 | 0.00774 | 0.01223 | 0.01215 | 0.01208 | 0.01196 | 0.01186 |
| 53 | 0.01952 | 0.01411 | 0.01412 | 0.01411 | 0.01409 | 0.01407 |
| 54 | 0.01720 | 0.01706 | 0.01715 | 0.01721 | 0.01729 | 0.01735 |
| 55 | 0.02129 | 0.02151 | 0.02164 | 0.02173 | 0.02190 | 0.02203 |
| 56 | 0.01596 | 0.02766 | 0.02777 | 0.02788 | 0.02807 | 0.02823 |
| 57 | 0.04338 | 0.03537 | 0.03544 | 0.03552 | 0.03568 | 0.03583 |
| 58 | 0.05745 | 0.04418 | 0.04420 | 0.04425 | 0.04435 | 0.04445 |
| 59 | 0.02949 | 0.05343 | 0.05343 | 0.05343 | 0.05345 | 0.05347 |
| 60 | 0.07676 | 0.06238 | 0.06236 | 0.06233 | 0.06226 | 0.06220 |
| 61 | 0.06915 | 0.07032 | 0.07028 | 0.07022 | 0.07007 | 0.06992 |
| 62 | 0.08164 | 0.07665 | 0.07659 | 0.07649 | 0.07629 | 0.07609 |
| 63 | 0.08041 | 0.08104 | 0.08093 | 0.08081 | 0.08057 | 0.08036 |

| Age | qx | Graduated | | | | |
|---|---|---|---|---|---|---|
| | | h=100 | h=150 | h=200 | h=300 | h=400 |
| 64 | 0.05937 | 0.08340 | 0.08324 | 0.08310 | 0.08289 | 0.08271 |
| 65 | 0.08862 | 0.08389 | 0.08373 | 0.08363 | 0.08349 | 0.08339 |
| 66 | 0.12521 | 0.08301 | 0.08294 | 0.08292 | 0.08293 | 0.08296 |
| 67 | 0.06149 | 0.08158 | 0.08168 | 0.08178 | 0.08197 | 0.08213 |
| 68 | 0.07967 | 0.08066 | 0.08093 | 0.08113 | 0.08147 | 0.08174 |
| 69 | 0.06078 | 0.08121 | 0.08154 | 0.08178 | 0.08216 | 0.08246 |
| 70 | 0.09321 | 0.08377 | 0.08401 | 0.08420 | 0.08449 | 0.08473 |
| 71 | 0.06573 | 0.08829 | 0.08835 | 0.08841 | 0.08851 | 0.08861 |
| 72 | 0.13084 | 0.09427 | 0.09414 | 0.09405 | 0.09394 | 0.09388 |
| 73 | 0.15214 | 0.10102 | 0.10076 | 0.10057 | 0.10028 | 0.10007 |
| 74 | 0.04316 | 0.10790 | 0.10760 | 0.10736 | 0.10698 | 0.10667 |
| 75 | 0.09818 | 0.11444 | 0.11417 | 0.11394 | 0.11353 | 0.11318 |
| 76 | 0.09658 | 0.12032 | 0.12011 | 0.11991 | 0.11956 | 0.11925 |
| 77 | 0.18268 | 0.12531 | 0.12519 | 0.12507 | 0.12485 | 0.12464 |
| 78 | 0.15416 | 0.12942 | 0.12944 | 0.12944 | 0.12941 | 0.12935 |
| 79 | 0.10955 | 0.13297 | 0.13317 | 0.13331 | 0.13351 | 0.13362 |
| 80 | 0.14568 | 0.13658 | 0.13694 | 0.13721 | 0.13759 | 0.13787 |
| 81 | 0.06808 | 0.14101 | 0.14144 | 0.14176 | 0.14226 | 0.14264 |
| 82 | 0.22578 | 0.14691 | 0.14728 | 0.14758 | 0.14806 | 0.14847 |
| 83 | 0.10421 | 0.15471 | 0.15489 | 0.15508 | 0.15544 | 0.15579 |
| 84 | 0.11099 | 0.16452 | 0.16445 | 0.16448 | 0.16466 | 0.16488 |
| 85 | 0.30733 | 0.17619 | 0.17592 | 0.17580 | 0.17576 | 0.17583 |
| 86 | 0.16512 | 0.18944 | 0.18908 | 0.18888 | 0.18868 | 0.18860 |
| 87 | 0.17431 | 0.20399 | 0.20368 | 0.20346 | 0.20317 | 0.20299 |
| 88 | 0.20576 | 0.21959 | 0.21943 | 0.21927 | 0.21897 | 0.21871 |
| 89 | 0.23325 | 0.23600 | 0.23604 | 0.23597 | 0.23572 | 0.23543 |
| 90 | 0.26033 | 0.25298 | 0.25320 | 0.25321 | 0.25301 | 0.25273 |
| 91 | 0.27787 | 0.27025 | 0.27054 | 0.27060 | 0.27045 | 0.27018 |
| 92 | 0.30544 | 0.28745 | 0.28769 | 0.28773 | 0.28760 | 0.28737 |
| 93 | 0.27072 | 0.30418 | 0.30425 | 0.30423 | 0.30409 | 0.30390 |
| 94 | 0.30918 | 0.32000 | 0.31984 | 0.31973 | 0.31958 | 0.31944 |
| 95 | 0.30627 | 0.33453 | 0.33414 | 0.33396 | 0.33382 | 0.33377 |
| 96 | 0.39349 | 0.34748 | 0.34697 | 0.34677 | 0.34669 | 0.34675 |
| 97 | 0.39879 | 0.35877 | 0.35830 | 0.35816 | 0.35821 | 0.35841 |
| 98 | 0.37378 | 0.36856 | 0.36830 | 0.36830 | 0.36855 | 0.36890 |
| 99 | 0.33804 | 0.37728 | 0.37736 | 0.37755 | 0.37803 | 0.37851 |
| 100 | 0.48429 | 0.38557 | 0.38602 | 0.38641 | 0.38706 | 0.38762 |
| 101 | 0.30538 | 0.39416 | 0.39491 | 0.39541 | 0.39613 | 0.39668 |
| 102 | 0.46266 | 0.40377 | 0.40459 | 0.40509 | 0.40572 | 0.40614 |
| 103 | 0.37844 | 0.41489 | 0.41552 | 0.41585 | 0.41620 | 0.41638 |
| 104 | 0.44348 | 0.42769 | 0.42787 | 0.42790 | 0.42782 | 0.42770 |
| 105 | 0.43904 | 0.44190 | 0.44149 | 0.44117 | 0.44066 | 0.44025 |
| 106 | 0.47714 | 0.45669 | 0.45583 | 0.45528 | 0.45454 | 0.45402 |
| 107 | 0.47350 | 0.47066 | 0.46990 | 0.46949 | 0.46905 | 0.46881 |
| 108 | 0.45281 | 0.48177 | 0.48223 | 0.48268 | 0.48352 | 0.48423 |
| 109 | 0.40650 | 0.48732 | 0.49087 | 0.49336 | 0.49698 | 0.49967 |
| 110 | 0.43044 | 0.48390 | 0.49334 | 0.49959 | 0.50818 | 0.51431 |