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A MODAL APPROACH TO PRICE AN OPTION IN CONTINUOUS TIME

BY

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THIS THESIS IS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF DOCTOR OF PHILOSOPHY (PhD) IN APPLIED MATHEMATICS.

DECLARATION

I hereby declare that this submission is my own work towards the award of the Ph.D. and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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DEDICATION

Dedicated to my family.

ABSTRACT

Average value options or Asian options have been priced using geometric and arithmetic averages of the underlying asset. However, these methods do not give accurate results especially in very low volatility regimes. In this study, we develop a new option pricing model based on the modal average of the underlying asset to price options. Using data from the NASDAQ in the United States of America we use the proposed model to price options sold on some stocks listed on the exchanges using $\mathcal{R}$ software. The results consistently showed that for volatilities less than 3% of the underlying asset, the modal average option pricing model gives a better option price when compared to existing average option pricing models. Moreover, the modal average consistently does better at all levels of volatility when compared to the Black-Scholes model. We further proved analytically that the modal average model indeed does better than the geometric or arithmetic average models especially for low volatility stocks.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Background to the Study

Derivative pricing has become an important aspect in financial engineering in recent years. Option pricing in particular gained prominence in the early 1970s when the mathematical foundations of the theory was finally pinned down by Samuelson, Merton, Fisher Black and Myron Scholes. More often, derivatives are priced based on an underlying asset and therefore involves the determination of the distribution function of the underlying asset. The assets may be stocks or shares listed on a stock market, commodities such as crude oil, coffee, cocoa, a currency such as the United States dollar, Great British Pound, or possibly the Ghanaian Cedi. Trading in derivative instruments is predominantly popular in European and American markets but recently, as the economies of developing countries begins to grow, interest in derivative instruments have increased as governments and corporates realise that they can no longer rely on traditional exports for growth. In the light of this it is of vital importance that financial instruments with the capacity to generate growth and create wealth are thoroughly explored. One such instrument is options.

Options have been traded or engaged in directly or indirectly for more than millennia. Thales of Miletus is the first person recorded to have traded in options. As a philosopher he was able to read the stars and predict the amount of rainfall for the coming year. He would then go round his village and place a deposit for the use of olive presses. When heavy rains was realised and the harvest was good the demand for the olive presses will sour and Thales would then hire the presses out charging higher fees and making substantial profits. It must be noted that the deposit only gave Thales the right but he was in no way obliged to hire the presses and this is what actually distinguishes options from other derivatives such as forwards and contracts. If the harvest fails, his losses were limited only to the initial deposit he has paid; Thales has purchased the first option with olive presses as underlying assets. Today, the underlying assets or securities are no longer olive presses but financial instruments such as stocks, equities, bonds, indexes, currencies, futures and other commodities. One significant advantage options trading has over other derivative instruments is that it has the possibility of making large profits with small sums of capital outlays. Such a trading instrument should be particularly more attractive to African countries where income levels are relatively low and people have less to invest.

1.1.2 Option Classification

Options can be classified according to the time the option is exercised or how the underlying asset is computed. The most commonly traded options are European and American options. A European option is an option that can be exercised only on the date that the option expires. An American option on the otherhand is exercised at any time before the option expires. These two options are usually referred to as Vanilla options. The other options types are called Exotic options. These are path_dependent options whose values or prices depend on the averages or the functional behaviour of the underlying asset at some point in the option's life. Exotic options include:

- Asian option _ this is an option whose value is determined by the average price of the underlying asset over the period of the option's life. This average could be arithmetic or geometric or a form of some other average. If the strike price in the Asian option is replaced by the average, it is called Asian strike option.
- Barrier option – in a barrier option, the underlying asset price must exceed a certain barrier before it can be exercised.

- Lookback options – in the lookback option, the price of the option depends on the maximum or minimum value of the price of the underlying asset.
- Binary option – this is an option whose value depends on the underlying asset satisfying a defined condition on expiration.
- Compound options – this is an option whose underlying instrument is also an option.
- Bermudan option – this is an option that can be exercised only on a pre-specified dates on or before expiration.

1.1.3 Analytical Valuation of an Option Price

The theory of option pricing had its roots in the 1900's when the French Mathematician, Louis Bachelier provided an analytical valuation for stock options based on the assumption that stock prices follow a Brownian motion process. However, it was not until 1973 that the option pricing theory gained international acclaim when Fisher Black and Myron Scholes published a seminal paper in which they outlined a procedure to obtain a closed–form formula to compute the prices of European calls and puts. The underlying premise of the theory was that by constantly adjusting the proportions of stocks and options in a portfolio, an investor can create a riskless portfolio where all market risks are eliminated. In the same year, the "Theory of Rational Option Pricing" was published by Merton in which he examined the option pricing methodology introduced by Black and Scholes. Merton (1973) provided an alternative derivation of the Black–Scholes formula by relaxing some of the assumptions in the Black–Scholes model and hence his model provided a more useable formula than the original model.

The model eventually became known as the Black–Scholes–Merton (BSM) model. Merton's paper provided several extensions of the Black–Scholes model including introducing dividend payments on the stock. The basic idea of Black, Scholes, and Merton models was to construct a portfolio from risky stock and a riskless bond or cash that yields the same return as a portfolio consisting only of an option. The stock which is risky because of uncertainty in its price process is assumed to follow the lognormal distribution and modeled as a stochastic process but the bond or cash, assumed to carry no risk is modeled as a deterministic process.

Cox et al. [1979], outlined an approach in which the possible stock price paths were represented in a binomial tree diagram. The hallmark of the model is that it is possible to present in a discrete form the stock price process of an option. By discretizing the Brownian motion process, the option price is presented as a simple random walk in a binomial tree. The price of the option is then computed at each node of the trees, starting from the price at maturity and working backwards to the present value of the option. The tree method is good for derivative pricing especially for path dependent options where there are possibilities of early exercise since it is now possible to condition the option at each node of the tree. In addition, the probability measure is discrete and hence the binomial tree formula provides a more accurate approximation of the movement of the underlying asset. Several other quantitative techniques have since appeared but the BSM and the CRR models remain the fundamental models of option pricing– the BSM model suited for pricing options in continuous time, the CRR suited to discrete time modeling.

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1.1.4 Averaged Value Options

Average Value options or as it is known elsewhere, Asian options, are options whose price is determined based on the average price of the underlying stock or asset. Asian options were introduced into the oil market in the late 1970's in Asia whence it earned its name. Compared to European options, Asian options are often cheaper and better suited to hedging purposes. In addition, they can also reduce the risk of price manipulations especially near the option's maturity date. The only downside to Asian options is that they are hard to price. In the discrete case for instance, if the binomial model approach is used to price the options, it is necessary to keep track of 2^n possible paths or the sample space has 2^n elements, where n is the number of periods. This makes it very difficult to examine if n becomes large. On the otherhand, in the continuous case, if the underlying assets are assumed to have a lognormal distribution as in Black Scholes model then the arithmetic average does not have a known distribution since the average is a sum of correlated lognormal random variables. This complicates the quest for a closed form pricing formula. In this study, we shall explore the possibility of developing a model in the Asian option style. We are motivated to investigate further into options priced on averages since we believe that their robust features makes them better suited for African markets than their European and American counterparts which are more susceptible to price manipulations.

1.2 Statement of the Problem

The use of averages of the underlying asset to price options has seen considerable investigation by researchers and different methods have been suggested to analyze the average. Averaging includes discrete and continuous averages and the strike price of an Average Value options is formed by the aggregation of the underlying asset price during the lifetime of the option. Discrete averages are computed at finite time intervals and the stock price is taken at a set of regularly spaced time points. In contrast, pricing continuous Average Value options is obtained by computing the average via the integral of the underlying price path over an interval of time. To the best of our knowledge no general analytical solution to price arithmetic average options is known and as such several numerical methods have been proposed to solve the arithmetic average problem. In addition, arithmetic and geometric averages are not a satisfactory description of the stock price path especially for stocks where prices remain stagnant or assume a certain price for long periods. This situation results in very low volatility for the stock and using the arithmetic or geometric average is unsuitable to provide a true representation of the stock price path. Thus, we establish the following:

- Arithmetic and Geometric averages are currently used to price Average Value or Asian options.
- Arithmetic and Geometric averages are unsuitable to offer true statistical information of the price path of the underlying asset especially for very low volatility stocks.
- Markets in developing countries are associated with extremely low volatilities. The use of a modal average for such cases will perhaps provide a true price path representation for low volatility stocks.

1.3 Objectives of the Study

The general objectives of the study is to develop an alternative approach to price path dependent options and to establish a model to numerically evaluate an underlying asset whose average cannot be accurately captured or completely defined by existing methods of using arithmetic and geometric averages. The specific objectives of this study are:

- To develop a new option pricing model based on the modal average of the underlying asset.
- To use the model to price options on stocks listed on some stock exchanges
- To compare the option prices obtained from the modal average to option prices obtained using arithmetic average, geometric average, the median average as well as the Binomial and Black−Scholes−Merton models.
- To analytically show that the modal average model is a superior model compared to the existing model for low volatility assets

1.4 Methodology

We develop and present a simple new approach to price Average Value options from the premise of probability spaces and systematically develop an algorithm that prices an option using a stock as the underlying asset. Our fundamental view is to derive the option price based on an underlying asset whose price movement is captured as a realization of the price event in a measurable space, admitting a level of uncertainty in an economy equipped with a filtration, viewed as information available at time t and on which is defined a probability measure. Throughout the text we will assume an efficient market with continuous trading, constant risk−free interest rate and no transaction cost to both buyers and sellers. In otherwords, we will assume that our model survives in the Black−Scholes world. Thus, we are basically concerned with European style option whose underlying stock is averaged over a specified time period. In particular, we will assume the stock price to be stochastic and modeled by the Geometric Brownian Motion (GBM) with both mean and volatility assumed to be constant. Through risk−neutral pricing we are able remove the mean or drift by assuming a complete market in which there are no arbitrages and subsequently the existence of an equivalent martingale measure. That leaves us to determine the volatility. Volatility can be determined using historical volatilities or using the volatilities of options being traded on the market. This method assumes the availability of options prices in the market but options are currently not sold on the Ghanaian market so we do not have these prices available. Thus, in the case where we would like to price option on Ghanaian stocks we may not have the option prices available in the market. For this reason we shall use historical volatilities which we will estimate from empirical stock price data. Using the computed volatility we will develop the model based on an underlying stock asset existing in a risk neutral world where under equivalent martingale measures the discounted stock price is a martingale. Based on the extreme value concept we will determine the maximum of a function and proceed to derive a method for the modal average of the underlying asset price using a numerical procedure. Daily stock price data is obtained from the Nasdaq. From the data we compute annual volatilities of the stocks and using $\mathcal{R}$ we simulate the stock price process for the coming year. From the simulated stock price we compute the stock price using arithmetic, geometric, median and modal averages.

1.5 Significance of the Study

There have been numerous efforts to develop alternative pricing models that are capable of capturing the unique features of some assets found in certain financial markets and subsequently use these models to develop option prices that better reflect the behaviour of the underlying asset.

The development of an option pricing model in which the underlying asset is based on its modal average will provide an accurate model for options where the mode is a better representative of the average than the other average measures. This will improve the fit to observed stock prices especially in certain African markets where prices of assets (especially stock prices) have very low volatilities. This study thus practically develops a model which provides us with a more realistic approach to value options on stocks that have low volatilities. The other principal contribution of this work is also to establish the development of option price from the perspective of abstract spaces and systematically illustrate the comprehension of an option value from the premise of a probability space. This approach thus provides new insights into the mathematical structure of option valuation and the application of the derived model allows us to solve a previously unresolved problem, the valuation of options on extremely low volatility stocks. The paper's other contribution will be to demonstrate the use of the frequencies of the stock price rather than the actual stock price in option valuation. This is significant as it will eliminate the need for assumption of a distribution function as an approximation of the underlying asset. In addition, the paper provides an analytical explanation of why the modal average is a more suitable model for pricing options when underlying asset has low volatility.

1.6 Organization of the Study

This work is structured as follows: Chapter 1 gives introduction to the study; the problem statement, the gap in literature the study aims to bridge and a brief outline of the methods we will use to achieve this. The objectives of the study and its significance are also outlined. Chapter 2 reviews the existing literature on option pricing models focusing extensively on Average Value or Asian options. Chapter 3 introduces essential mathematical concepts especially in stochastic calculus and gives a broad outline of the theories required in derivative pricing including measure theory. Chapter 4 gives a rigorous treatment of continuous option pricing models including derivations of the famous Black–Scholes–Merton model using partial differential equations. In addition, the Binomial tree model is treated as an example in discrete time.

Emphasis is however given to the various Average Value option pricing models. Chapter 5 is Main Results 1 and it outlines the theoretical development of deriving a modal average for the underlying asset and developing a numerical algorithm to evaluate it. We proceed to obtain a pricing model for the option using the modal average under equivalent martingale measures. Chapter 6 is Main Results II and comprises of numerical results using Monte Carlo simulation in $\mathcal{R}$ for stocks listed on and the Nasdaq. In addition, graphical outputs and comparative analysis of the option prices using the averages methods and closed form models are given. Discussion of results follows in Chapter 7, where we give a theorem to back our argument for a modal average.

Conclusions and recommendations are given in this chapter.

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CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In derivative valuation, the price of an option is based primary on the underlying asset. The price of an option is therefore dependent on how the underlying asset behaves. In this study, we will use a stock price as the underlying asset in pricing options and as such we will begin the literature by looking into studies on stock price behaviour. The foundations for the use of stochastic process as a means of predicting and forecasting stock price behaviour were laid down by Louis Bachelier who developed the first mathematical model of a stock's price and tested the model by using it to price options. Bachelier evaluation assumed that the stock price process distribution can be captured as a Brownian motion process. Modelling stock price as Brownian motion process means that the price increments are normally distributed and it follows that negative stock prices are possible. Despite this Bachelier's model is widely considered as a landmark in the history of stock price modeling and it has had great influence on the whole development of stochastic calculus and financial engineering. Its use has recently resurfaced; Schaefer (2002) priced and hedged European options on future spreads using the Bachelier spread option model.

There was a lull in the development of stock price modeling until the 1960's when interest in options stimulated further investigation of stocks price behaviour. Bachelier thesis was actually unearthed by Samuelson in 1965, after he personally rediscovered the virtually forgotten thesis hiding in the archives of Harvard University library. Studies in predicting the behaviour of the stock price continued however. Kendall (1953) statistically analysed the stock market behaviour of the British stock indices and found that the changes in prices were generally random and each outcome was independent of the past. Osborne (1959) studied stocks on the NYSE and found that the changes of logarithm of stock prices were nearly normally distributed. Osborne inferred that the losses and gains should be measured by changes in logarithms and redefined Bachelier model by using the logarithm of Brownian motion process to model the stock price process.

Roberts (1959) discussed what he termed "the Chance Model" and concluded that stock price patterns familiar in technical analysis could be generated by using random numbers. It was French (1965) who premised the behaviour of a stock price to the notion of a random walk. The key idea was that changes over time in stock prices could only occur in response to the arrival of new information as any previously available information would have already been reflected in the price. Thus, stock prices already contain all the information regarding the business. However, because information arrives randomly, stock prices have to fluctuate randomly leading to the idea of a random walk. These ideas burgeoned into efficient markets hypotheses in which current prices fully reflect all available information. The limiting function of the random walk is the Brownian motion process on which Bachelier had proposed in his stock price model. Using

Brownian motion to model stock prices means that the increments are normally distributed. However, in stock price modeling we are interested in the stock's return rather than the price increments. For this reason the important aspects of the price increments are the percentages or ratios rather than the arithmetic differences. By working with ratios or percentages instead of algebraic differences, we consider logarithmic or percentage changes to be subject to uniform probabilities. This means that the first differences of the logarithms of prices are distributed in the usual absolute Brownian motion way. Since the arithmetic mean of logarithms is geometric mean of actual prices, this modified random walk is called Geometric Brownian Motion.

In effect, Samuelson suggested that since we are interested in the stock return the percentage increments must be independent rather than the arithmetic differences and introduced Geometric Brownian Motion (GBM) as a model for the evolution of the stock price. Under the GBM, the logarithm of the percentage changes are normally distributed and stock prices cannot be negative. The GBM is a more representative model for stock prices because if stock prices can be negative then it follows that investors will lose more that their investments in case of corporate bankruptcy. But this is not the case, stocks possess limited liabilities and as soon as a stock's price falls to zero it is declared insolvent and taken out of the market_investors are not liable for corporates liabilities. Even then, GBM does not give a complete description of the behaviour of a stock price as it fails to exhibit jumps caused by a sudden announcements or unpredictable events. In addition, it assumes a constant volatility for the stock price although in reality volatility is stochastic. As we shall observe later stochastic volatility and jump price models have recently been comprehensively examined. Despite these uncertainties, most financial engineers and economists have accepted the GBM as a model for stock behaviour because it is everywhere positive in contrast to Brownian motion, and it's now the most widely accepted formula for modelling stock price behaviour.

Pricing options based on stocks began with Bachelier who was the first to obtain a closed form price of an option with the underlying asset modeled as arithmetic Brownian motion. He measured volatility in absolute terms and called the quantity "the coefficient of instability" or the "nervousness" of the security. However, his model ignored discounting. Samuelson [1965] applied GBM to price options and obtained a closed form model which accounted for time value of money and in which the underlying asset is non_negative.

Samuelson model was however not very popular as it required that one has to compute individual risk characteristics. The lack of certainty about a measure of an individual's risk characteristics makes Samuelson model difficult for investors and sellers to agree on a single option price. It was not until 1973 that Fisher Black and Myron Scholes published their seminal paper in which they finally developed a closed–form model in which a hedged strategy was used to replicate a call option which depended only on observable quantities. The Black–Scholes–Merton model, as it will later become known, is often regarded as the apogee of the option pricing theory and its introduction was so illuminating in structure and function that it created inflation in option trading and marked the beginning of a rapid expansion in derivatives trading in European and North American financial markets. Black (1989) gave his own riveting accounts of the events leading to the development of the formula in his memoires. Using Itô's lemma, Black and Scholes obtained a partial differential equation (also Black–Scholes differential equation) that formulates the movement of the option price over time. The key idea of the model was to set up a portfolio consisting of one risky asset (stock) and one riskless asset (bond or cash) and to buy and sell these assets in just the right way so as to completely eliminate all the risk inherent in the stock. This way of buying and selling, known as delta hedging, has given insights into so many other hedging strategies. If one solves the Black–Scholes PDE, he obtains the Black–Scholes formula. The formula was, and still popular because it is easy to apply as it requires one to compute only the volatility of the stock in other to obtain the option price. In addition, the Black–Scholes equation is independent of the risk preference of the investor and consequently the risk preference measure does not enter the equation and so cannot affect the solution as in Samuelson's model.

Cox, Ross and Rubinstein (1979) presented a discrete−time option pricing model known as the binomial model whose limiting form is the Black–Scholes equation. The binomial model assumes that the stock price at each moment can either go up with probability u or down with probability d and by setting up a portfolio to determine the amount of stock needed to make the portfolio riskless. The model then use risk neutral and arbitrage arguments to determine the price at the end of the option's life, discounting it to the present value. The model is extremely efficient if the option has only one underlying asset. However as the number of nodes of the trees gets large it becomes difficult to examine the prices at the nodes and the model becomes unattractive.

Since the introduction of Black−Scholes−Merton and Cox−Ross−Rubinstein models several other models and techniques have emerged but they are all variants of these two fundamental approaches. Recent studies in option pricing have focused primarily on novel computational applications and testing efficiency and speed of convergence of the models. Monte Carlo simulation for instance has gained prominence and has widely been employed as an effective simulation technique. Bally, et al (2005), Egloff (2005), Moreno et. al. (2003), Dagpunar (2007) all examined the effectiveness of the Monte Carlo technique in options pricing. Mehrdoust et. al. (2017) examined the Monte Carlo option pricing under the constant elasticity of variance model.

2.2 Options on Dividend Paying Stock

Merton (1973) derived a partial differential equation model of the Black Scholes for the options on stocks with dividend payments. He compared the American call to a European call and showed that the value of the American call with dividend payment is always higher than the European call. Merton asserted that American call will not be exercised early if the underlying asset does not pay dividends.. Musiela and Rutkowsky (1997) proposed a model that adds the future value at maturity of all dividends paid during the lifetime of the option to the strike price. Roll (1977), employed a duplication technique to value an option with stock dividend payment. Gaske (1979) presented models for computing American call options with dividends. Whaley

(1981) established a model with dividends in which he corrected some of the errors in Roll (1977) and Gaske (1979). Shreve (2004) also investigated and obtained a result for pricing options when the underlying stock pays proportional dividends. Barone-Adesi and Whaley
(1988) developed pricing models for stocks with discrete dividend payments for an America put. Vellekoop and Nieuwenhuis (2006) argued that market prefers to specify dividends as discrete rather than continuous and showed that existing models admit arbitrage. They assumed a piecewise lognormal model for the underlying asset and developed a non-recombining model for the option price.

2.3 Options with Transaction Cost

Leland (1985) was the first to examine the presence of transaction cost in the Black−Scholes model. Boyle and Vorst (1992) derived a discrete version of options with transaction cost in the Cox−Ross− Rubinstein model in which a replication portfolio was created between the nodes of the trees. Bensaid et al. (1992) extended Boyle and Vorst approach by introducing a super replication strategy in place of the simple replication. The method considered a hedging strategy where the value of the option at maturity dominates the final payoff. Hodges and Neuberger (1989) introduced an optimization approach in the form of a utility maximization objective function. This approach of determining the option price using optimization techniques was further developed by Davis et al. (1993), Clewlow and Hodges (1997), Constantinides and Zariphopoulou (1999) and Musiela and Zariphopoulou (2001). Monoyios (2004) developed an algorithm that numerically computes the option price with transaction cost using Markov's chain approximation to the continuous time stochastic optimization problem.

2.4 Options with Stochastic Interest Rate

Merton (1973), Amin and Jarrow (1992) introduced options with stochastic interest rates. Other short rate models include Vasicek (1977), Cox et. al. (1985), Hull and White (1990) model, also called the extended Vasicek model. Other popular models include the one-factor short_rate models of Black and Karasinski (1990) which assumes a single source of randomness as the main driver of interest rate movements. Duffie and Dan (1996) popularised the Gaussian affine interest rate model which Ostrovski (2013) employed to simulate the Hull and White model.

2.5 Options with Stochastic Volatility

The most popular stochastic volatility model is of Heston (1993), in which the randomness in stock variability is captured as the square root of the variance. Hull and White (1987) obtained analytic models in the Black_Scholes framework using stochastic volatilities. Chen (1994) obtained the first stochastic mean and stochastic volatility model in which the dynamics of the interest rate model is captured in three stochastic differential equations. Other stochastic volatility models include Wiggins (1987), who employed numerical analysis to the study of stochastic volatility and the constant elasticity variance model of Cox (1995). Tian, et. al. (2012) examined the use of hybrid stochastic volatility and Tian et. al. (2015) again examined the use of hybrid local stochastic volatility in options pricing models.

2.6 Option Styles

The Black–Scholes model was priced using European options. Subsequent advances have explored models for other option styles.

2.6.1 American Options

The valuation of American call and put options has extensively been examined. Merton (1973) showed that the value of an American put is usually higher than the value of a European put. Brennan and Schwartz (1997), Bensoussan (1984), Geske and Johnson (1984), Geske (1979), Karatzas (1988), Duffie (1988), Barone-Adesi and Whaley (1986), (1987), (1988) all presented rigorous analytical treatment of American calls and puts with or without dividends and transaction cost. Cox, et. al. (1979) derived a discretized evaluation of the American options using the binomial framework. Longstaff and Schwartz (2001) provided a numerical approximation of American options by simulation. However, difficulties have emerged in using standard numerical procedures to price American options. To navigate these difficulties Glasserman and Yu (2004) developed a technique that combines Monte Carlo simulation with dynamic programming and showed the degree of convergence in this model grows exponentially to existing models.

2.6.2 Asian Options

Asian options or Average Value options are priced based on the entire path of the underlying assets. These options are divided into two different types depending on how the average of the underlying asset is computed. They are the Geometric Asian Options and the Arithmetic Asian options. If the underlying asset is averaged as a geometric average then we realise that the option price can be derived directly by substituting the geometric average into the Black–Scholes formula. This is the procedure proposed by Kemna and Vorst (1990). This approach is possible because the asset price is assumed to follow the lognormal distribution. The product of lognormally distributed random variables are also lognormally distributed and hence we end up with and equivalent distribution. However, in the arithmetic average case, the sums of lognormal random variables are not lognormally distributed and hence it is not possible to obtain a closed form formula if the underlying asset is averaged as arithmetic.

Turnbull and Wakeman (1991) derived an approximating distribution of the arithmetic average with the lognormal distribution. They employed the generalized Edgeworth series expansion which relies on the fact that all the moments of the arithmetic average can easily be calculated even though the distribution of the average is unknown. Milevsky and Posner (1998) discussed the reciprocal gamma distribution as a preferred approximate distribution to the lognormal distribution. This is due to the fact that infinite sums of correlated lognormal random variables are reciprocally gamma distributed. The resulting formula and the method bear a close resemblance to the Black_Sholes formula. Vorst (1992) gave an exact pricing formula for Asian options based on a geometric average. He used this to approximate the Asian option price by adjusting the strike price with the difference in expectation of the arithmetic and the geometric averages. Furthermore, he obtained upper and lower bounds on the price. Vorst method however, deteriorates as volatility and maturity increases. This is expected since the arithmetic average is approximated with a lognormal distribution. Curran (1994) conditioned on the geometric average by integrating with respect to its lognormal distribution. Ingersoll (1987), Rogers and Shi (1995) outlined a partial differential equation to model floating and fixed strike Asian options. Shreve and Vecer (2000) established a method which included replication and self_financing strategies.

Vecer (2000) also developed numerical techniques for pricing Asian option contracts. Semi_analytic techniques were developed by Hoogland and Neumann (2001) to price continuous Asian options using scale invariance methods. The use of Monte Carlo simulation has become popular in pricing Asian options. Rubinstein, and Kroese, (2007), Kechejian, et. al, (2016) have examined this method in pricing Asian options.

2.7 Other Exotic Options

Other exotic options that have also emerged in pricing other complex options include the Lookback Options, Compound Options, Bermudian options, Barrier options Passport Options etc. Merton (1973) was the first to price barrier options using partial differential equations to obtain the theoretical price of a down-and-out call option. Rubinstein and Reiner (1991) gave a list of pricing formulas for different versions of barrier options using the probability method.

Other works on barrier options include Ritchken, (1995), Broadie, et. al. (1997), Heynen, et. al. (1994). Bensoussan (1984), Karatzas (1988) in a complete Itô process model and then Kramkov (1996), have all examined models to price Bermudean options and swaptions. Kodukula & Papudesu (2006) gave more examples of compound options in real option applications. Other methods have emerged. For instance Lasserre, et. al. (2006) priced Exotic options, including Asian and Barrier options using Moments Semi-Definite Programs Relaxations methods.

CHAPTER 3

PRELIMINARY CONCEPTS IN STOCHASTIC CALCULUS

3.1 Functions

3.1.1 Continuous Functions

A continuous function $f(x)$ can be increasing or decreasing on an interval $[a, b]$.

Suppose $f(x)$ is increasing then

- if $x_1 < x_2$, it follows that $f(x_1) \leq f(x_2)$.
- If $x_1 < x_2$ and $f(x_1) < f(x_2)$, then $f(x)$ is strictly increasing.

Suppose $f(x)$ is decreasing then

- if $x_1 < x_2$ it follows that $f(x_1) \geq f(x_2)$.
- If $x_1 < x_2$ and $f(x_1) > f(x_2)$, then $f(x)$ is strictly decreasing.

Right and Left-Continuous Functions

Consider the point t_0 on the function $f(t)$ as shown in the Figure 2.1.

The limit as t approaches t_0 from the left is denoted by $\lim_{t\uparrow t_0} f(t) = f(t-)$

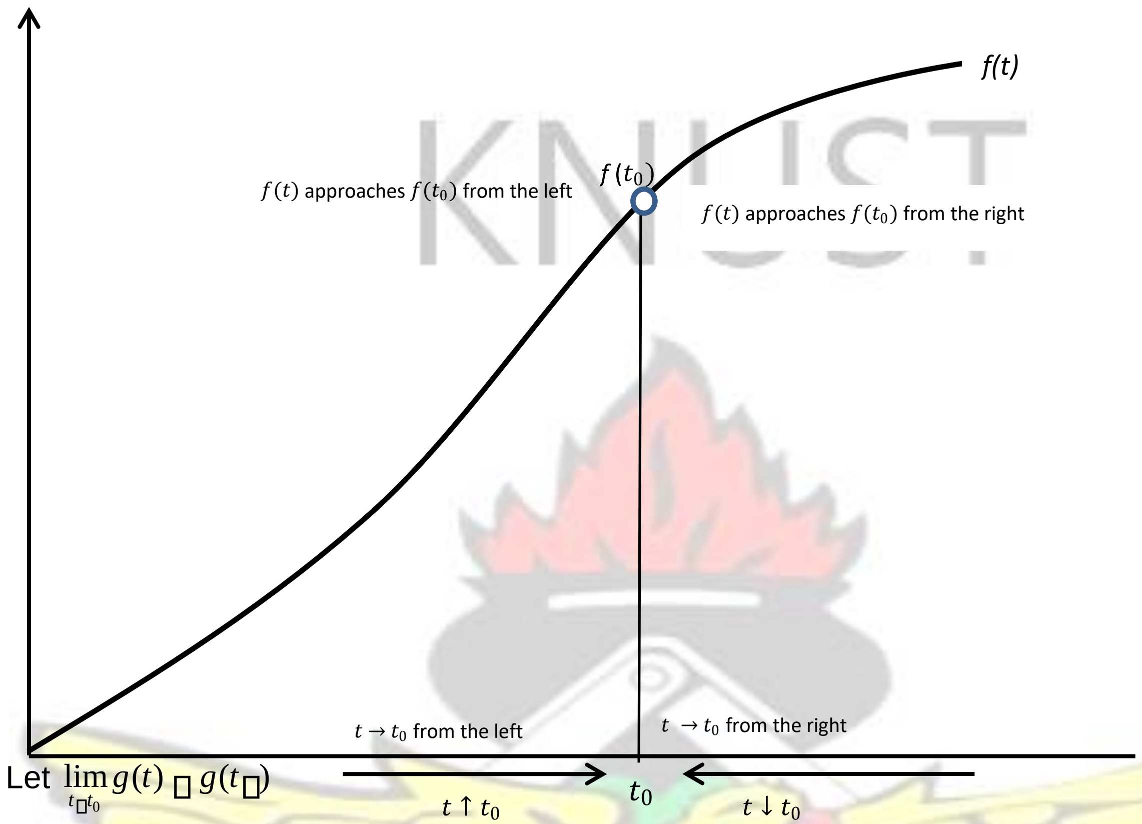
The limit as t approaches t_0 from the right is denoted by $\lim_{t\downarrow t_0} f(t) = f(t+)$

For a left continuous function the unilateral limits $\lim_{t\uparrow t_0} f(t) = f(t-)$ exist and are finite.

Similarly for a right-continuous function the unilateral limits $\lim_{t\downarrow t_0} f(t) = f(t+)$ exist and are finite. Furthermore, the limits at infinity

$$\lim_{t\downarrow -\infty} f(t) = f(t-\infty) \quad \text{and} \quad \lim_{t\uparrow +\infty} f(t) = f(t+\infty)$$

exist and may be□□and □□respectively.



Let $\lim_{t \square t_0} g(t) \square g(t_\square)$

Figure 3.1 Continuity of a function at a Point

A monotone function $f(t)$ is called continuous at the point t_0 if and only if

$$f(t-) = f(t) = f(t+) \qquad \text{..3.1}$$

That is

$$f(t-) = \sup_{-\infty<t<t_0} f(t) = f(t) = \inf_{t_0<t<+\infty} f(t) \qquad \text{..................................3.2}$$

Thus, the continuity of a monotone function at t_0 is equivalent to the assertion that

$$\lim_{t\uparrow t_0} f(t) = f(t_0) = \lim_{t\downarrow t_0} f(t) \qquad \text{..................................3.3}$$

3.1.2 Continuous on the Right with Left Limits (*ćadlàg* functions)

Stock prices are non−negative right continuous functions with limits on the left. Thus, the geometric Brownian motion required to model stock prices is a non−negative distribution function. We will be mainly concerned with the class of right−continuous functions on $[0, T]$ with left limits. These classes of functions have a special name *"continue à droite, limite à gauche"*(right continuous with left limits), *ćadlàg* functions.

3.1.3 Variation of a Function

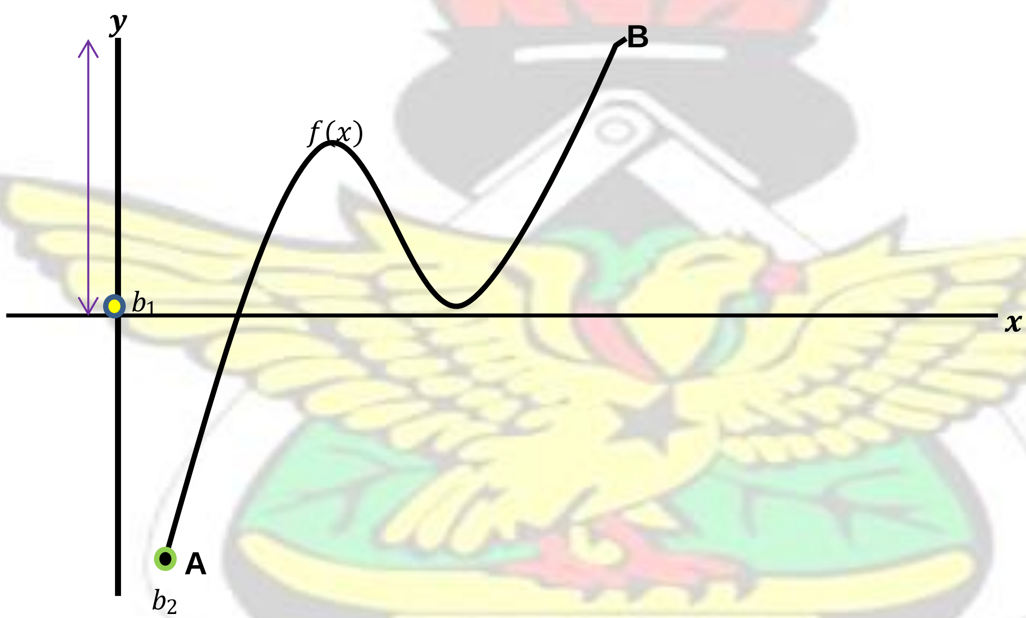


Figure 3.2 Variation of a function

Consider Figure 3.2. Suppose there is a ball (b_2) that travels along the path of the function $f(x)$. Suppose there is second ball (b_1) which also travels along the y−axis as b_2 traverse the function $f(x)$ such that the distance y travelled by b_1 is defined by the distance covered by b_2 on the graph.

The total distance (up and down) traversed by b_1 after b_2 has moved from point **A** to **B** on the graph is called the total variation of the function $f(x)$. Given $f(x)$, the variation of the function on $[a,b]$ is given as

$$V_{f[a,b]}(t) = sup\sum_{i=1}^{n}|f(t_i^n) - f(t_{i-1}^n)|$$

where the supremum is taken over partitions $a = t_0^n < t_1^n <,\ldots,< t_n^n = b.$

The addition of new points means the interval between the partitions become smaller and smaller

and $$\mathrm{V}_{f[\mathrm{a,b}]}(t) = lim_{\delta\to 0}\sum_{i=1}^{n}|f(t_i^n) - f(t_{i-1}^n)|$$

where $$\delta_n = \max_{1\le i\le n}(t_i^n - t_{i-1}^n)$$

……………………………….3.4

Thus the variation of a continuously differentiable function $f(t)$ having a derivative $f'(t)$ is given by

$$\mathrm{V}_f(t) = \int_0^t |f'(s)|ds$$

Consider the function $f(t)$ in Figure 3.3 below.

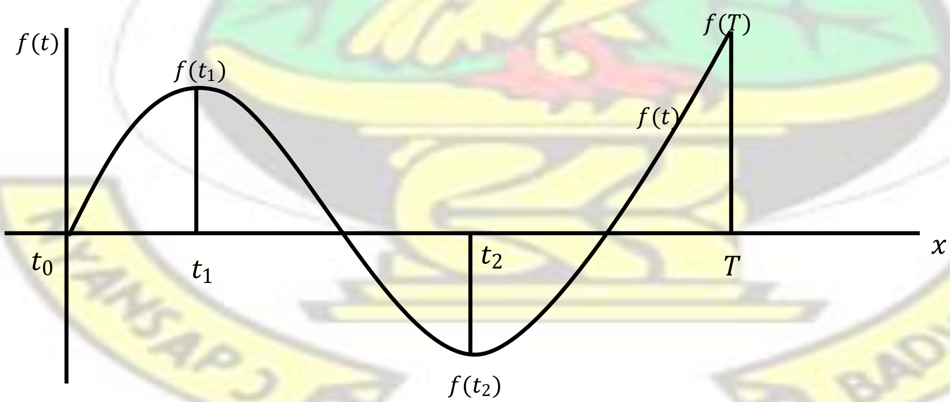


Figure 3.3 Variation of $f(x)$ over an interval $[0,\ T]$

The finite variation of $f(x)$ on $[0,T]$ is given by

$V_{f([0,T])} = \{f(t_1) - f(t_0)\} - \{f(t_2) - f(t_1)\} + \{f(T) - f(t_2)\}$

$$= \int_0^{t_1} f'(t)dt + \int_{t_1}^{t_2} -f'(t)dt + \int_{t_2}^{T} f'(t)dt = \int_{t_0}^{T} |f'(t)|dt$$

3.1.4 Quadratic Variation of a Function

If we look at the path by path behaviour of a stochastic process we can observe how much the random variables vary between the ends of the paths. This is roughly the variation of the process. Summing the variation usually gives zero since the expectation of the increments of a normal distribution is zero and we end up with no information. However, if we take the square of the variations we can analyse that. The square of the variations is called the quadratic variation.

Definition

Let $f(t)$ be a function on $[0, t]$. The quadratic variation of $f(t)$ is given as

$$[f, f](T) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} [f(t_i^n) - f(t_{i-1}^n)]^2$$

…………………………………………..3.5

where $0 = t_0{}^n < t_1{}^n < \ldots < t_n{}^n = t,$ and $\delta_n = \max_{1 \le i \le n}(t_i^n - t_{i-1}^n)$

3.2 Probability Theory

Algebra

A collection of subsets of Ω is called an algebra if

i. $\Phi \in \mathcal{F}$

ii. If $A \in \mathcal{F}$, then $A \in \mathcal{F}$

iii. If $A_1, A_2, \ldots, A_n$ is a finite collection of sunsets in Ω then

$$\bigcup_{i=1}^{n} A_i \in \mathcal{F}$$

and by De−Morgan's rule

$$\bigcap_{i=1}^{n} A_i \in \mathcal{F}$$

Sigma (σ) −Algebra

A collection $\mathcal{F}$ of subsets of Ω is called a sigma (σ) −algebra if

i. $\Phi \in \mathcal{F}$

ii. If $A \in \mathcal{F}$, then $A \in \mathcal{F}$

iii. If If $A_1, A_2, \ldots$ is a countably infinite collection of sunsets in Ω then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

and by De−Morgan's rule

$$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$$

Borel σ –Algebra $\mathcal{B}$

The Borel σ –algebra denoted by $\mathcal{B}$ is the smallest algebra that contains all open sets or all open intervals. $\mathcal{B}$ contains all open sets, all closed sets, all countable unions of closed sets, all countable intersections of such countable unions, etc.

Measurable Space

The sample space Ω endowed with a σ –algebra $\mathcal{F}$ is called a measurable space and is denoted by $(\Omega, \mathcal{F})$. Ω contains all the outcomes in the sample space and the set $\mathcal{F}$ represents the set of possible events or collection of subsets of Ω.

$\mathcal{F}$ –Measurable Function

A function $X: \Omega \to \mathbb{R}$ is called $\mathcal{F}$ –measurable if for every $B \in \mathcal{B}$ the pre-image $X^{-1}(B) \in \mathcal{F}$. where $X^{-1}(B) = \{\omega \in \Omega | X(\omega) \in \mathcal{B}\}$.

Random Variable

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X is an $\mathcal{F}$ –measurable function $X : \Omega \to \mathbb{R}$ such that

$$X^{-1}(B) \in, \forall B \in \mathcal{B}$$

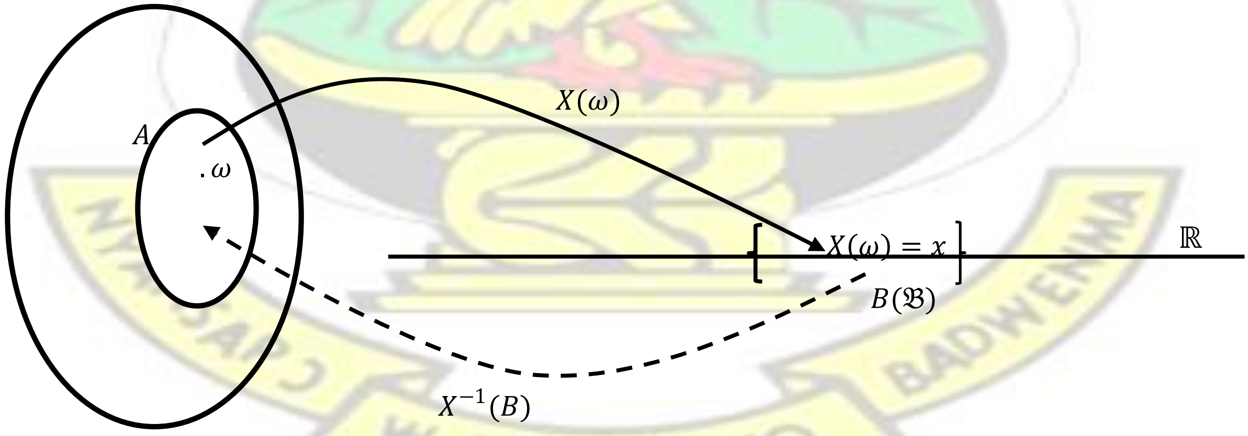


Figure 3.4 The Random Variable X

Sigma $(\boldsymbol{\sigma})$ −Algebra Generated by $\boldsymbol{X}$

A (σ) −Algebra generated by the random variable X is defined as

$$\sigma(X) \triangleq \{A \in \Omega \colon \mathrm{A} = X^{-1}(B) \in, \vee\ B \in \mathcal{B}\ \}$$

Probability of an Event

The probability of an event ω is defined as

$$\mathbb{P}(X^{-1}(B)) \triangleq \mathbb{P}\{\omega \in \Omega \colon \mathrm{X}(\omega) \in \mathcal{B}\}$$

Probability Measure

A probability measure on $(\Omega, \mathcal{F})$ is a function that maps $\mathcal{F}$ −measurable function onto the [0,1] interval. A probability measure is thus a special case of measure theory which acts on the sample space and transform or assign numeric values or probabilities to the outcomes of the sample space in the [0,1] interval on the real line.

A probability measure is a function $\mathbb{P}\colon \mathcal{F} \to [0,1]$ such that

i. $\mathbb{P}(\emptyset) = 0$. ii.

$\mathbb{P}(\Omega) = 1$.

iii. (Countable Additivity)

If $\{A_i;\, i \geq 1\}$ is a sequence of disjoint sets in $\mathcal{F}$, then the measure of the union (of countably infinite disjoint sets) is equal to the sum of measures of individual sets, i.e.,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

……………………………….3.6

Measure Space

A measurable space endowed with a probability measure is called a measure space and denoted by the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ represents the probability measure on $(\Omega, \mathcal{F})$. We use $\omega \in \Omega$ to represent an outcome which may be a path of a process.

Integration and Expectation

Riemann Integral

Define

$$U_n(f) \triangleq \sum_{i=1}^{n} \{sup f(x)\} \Delta x_i$$

$$L_n(f) \triangleq \sum_{i=1}^{n} \{sup f(x)\} \Delta x_i$$

A function f is Riemann integrable over $[a, b]$ if

$lim_{n\to\infty} U_n(f)$ and $lim_{n\to\infty} L_n(f)$ exist and equal.

That is

$$\lim_{n\to\infty} U_n(f) = \lim_{n\to\infty} L_n(f) = \int_a^b f(x)dx$$

Abstract Integration

Let $(\Omega, \mathcal{F}, w)$ be a measure space and $f : \Omega \to [0, \pm\infty]$ be a measurable function. We define

$$\int_A f d\omega, \quad A \in \mathcal{F}$$

as the integral of the measurable function f with respect to a measure over the measurable set A. This definition of integral of f generalizes the Riemann integral to include all function that cannot be defined under Riemann integral.

Lebesgue Integral

Let $(\Omega, \mathcal{F}, w)$ be a measure space transformed onto the real line $\mathbb{R}$ on which is defined a Borel set $\mathfrak{B}(\mathbb{R})$ so that $(\Omega, \mathcal{F}, w)$ is transformed to $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda)$. The Lebesgue integral is defined on the Borel set $A \in \mathfrak{B}(\mathbb{R})$ as

$$\int_A f d\lambda \qquad \text{.................................3.7}$$

λ is the measure on the $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ measurable space. The Lebesgue integral generalizes the Riemann integral as the domain of integral is shifted from $[a, b]$ to the general Borel set $A \in \mathfrak{B}(\mathbb{R})$. Since the Lebesgue integral is defined over the Borel set it exists everywhere. In this realm more complicated sets and functions can be evaluated and the Lebesgue integral exist everywhere.

Expectation of a Random Variable X

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable function $X : \Omega \to \mathbb{R}$. The integral of X over the measurable space $(\Omega, \mathcal{F})$ with respect to the probability measure $\mathbb{P}$ is denoted by

$$\int_\Omega X d\mathbb{P}$$

The integral $\int_\Omega X d\mathbb{P}$ is called the expectation of the random variable and we write

$$\mathbb{E}[X] = \int_{\Omega} X d\mathbb{P}$$

……………………………….3.8

Expectation of a Discrete Random Variable

Suppose X is a discrete random variable taking values $a_1, a_2, \ldots$, then the expected value of X is given by

$$\mathbb{E}[\mathrm{X}] = \sum_{i=1}^{\infty} x_i \mathbb{P}(X = x)$$

$$\mathbb{E}[\mathbb{X}] = \sum_{i=1}^{\infty} x_i \mathbb{P}_{\mathrm{X}}(x_i)$$

Example 3.1

The Geometric distribution has Probability Mass Function (PMF)

$$\mathbb{P}(X = x) = (1-p)^{x-1}p$$

$$\mathbb{E}[\mathrm{X}] = \sum_{i=1}^{\infty} x_i \mathbb{P}(X = x)$$

$$\mathbb{E}[\mathrm{X}] = \sum_{i=1}^{\infty} x(1-p)^{x-1}p$$

$$= \frac{p}{[1-(1-p)]^2} = \frac{1}{p}$$

Expectation of a Continuous Random Variable

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let a measurable function $X : \Omega \to \mathbb{R}$ be a continuous random variable. Let g be a measurable function which is non_negative and finite. Then

$$\mathbb{E}[X] = \int g f_{\mathrm{X}} d\lambda$$

Where λ is the Lebesgue measure defined on the Borel set.

Example 3.2

The density function of the normal random variable $X \sim N(\mu, \sigma^2)$. Thus

$$\mathbb{E}[X] = \int x f_{\mathrm{X}} d\lambda$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

Hence

$$\mathbb{E}[X] = \int_0^\infty x\left[\frac{1}{\sigma\sqrt{2\pi}} exp\left\{\frac{(x-\mu)^2}{2\sigma^2}\right\}\right] dx$$

Riemann_Stieltjes Integral

Consider $(\Omega, \mathcal{F}, w)$ to be a possible measurable space. In addition suppose $f: \Omega \to \mathbb{R}$ is a continuous function on the interval $[a, b]$ in $\mathbb{R}$. Let $g: [a, b] \to \mathbb{R}$ be a function of bounded variation then the Riemann-Stieltjes integral is given by

$$I = \int_a^b f(t) dg(t) = \sum_{k=1}^{n} f(s_k)\, g(t_k) - g(t_{k-1})$$

That is

$$\int_a^b f(t) dg(t) = \int_a^b f(t) g'(t)) dt$$

3.3 Stochastic Processes

Given a measurable space $(\Omega, \mathcal{F})$ a stochastic process $\{X(t)\}$ is represented as a filtration $\mathcal{F}_t = \sigma(\{X(s); 0 \leq s \leq t\})$ generated by random variables $X(s)$. In discrete time stochastic process t

takes only discrete positive numbers, $t = 0, 1, 2, ...$ An example of discrete stochastic process is the Poisson process. A continuous stochastic process is a collection of random variables indexed by time over the real line, that is, $t \in \mathbb{R}^{+}$. A continuous stochastic process does not necessary imply that the process generating the randomness is continuous. Rather, it is the examination of the process over continuous or discrete times that defines it nature. Thus the stochastic process can have jumps in its path but will still remain continuous. An example of continuous stochastic processes is the Brownian motion process.

3.3.1 Brownian Motion

A process on some measure space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Brownian motion denoted by $B(t)$ if it satisfies the following theorem.

Theorem 3.1

There exist a probability distribution over the set of continuous function $B(t): \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that

i. $\mathbb{P}(B(0) = 0) = 1$ a.s.

ii. The increment of Brownian motion $B(t) - B(s)$ is independent of $\mathcal{F}_s$ for all $0 < s < t$

iii. The increment $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$. $B(t)$ has continuous paths but is nowhere differentiable for $t \geq 0$.

For a Brownian motion process it is not possible to describe what happens at each point in time so we examine the process over continuous time hence it is a continuous stochastic process.

Properties of Brownian Paths

A Brownian motion process $B(t)$ is a function of t has the following properties. For almost every sample path $B(t)$, $0 \leq t \leq T$

- is a continuous function of t

- is not monotone in any interval, no matter how small the interval is
- is not differentiable at any point
- has infinite variation on any interval, no matter how small it is has quadratic variation on $[0, t]$ equal to t, for any t

3.3.2 Filtration

The collection of σ −algebras $\mathcal{F}_1, \mathcal{F}_2, \dots . \mathcal{F}_t, \dots, \mathcal{F}_T$ such that $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ is called a filtration on the measure space $(\Omega, \mathcal{F}, \mathbb{P})$. A filtration is denoted by set $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$. The sequence of $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots . \mathcal{F}_t \subset \cdots \subset \mathcal{F}_T$ forms a chain of sigma algebras such that each $\mathcal{F}_t$ contains the sigma algebra preceding it so that as time evolves more and more information is revealed about the structure of the stochastic process. Thus, the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ gives the passage of information about the process through time and contains all the information that has been observed from the process up to time t. The set $\{\mathcal{F}_t\}_{t \geq 0}$ is called a filtration associated with a stochastic process $\{X(t): t \geq 0\}$.

Adapted Process

A stochastic process $\{X(t), t \geq 0\}$ is adapted to the filtration $\mathcal{F}_t$ if all the σ −algebras generated by $X(t)$ up to t is contained in $\mathcal{F}_t$. It follows that by the time t we know all the information about $X(t)$, that is $X(t)$ is $\mathcal{F}_t-$ measurable.

3.3.3 Quadratic Variation of a Stochastic Process

Let $X(t)$ be a stochastic process. The quadratic variation of the process $X(t)$ is given by

$$[X, X](t) = lim \sum_{i=1}^{n} |X(t_i^n) - X(t_{i-1}^n)|^2$$

where, $\{t_i^n, 0 \leq i \leq n\}$ is a partition of $[0, t]$. If the process is the Brownian motion process $B(t)$ then

$$[B,B](t) = lim \sum_{i=1}^{n} |B(t_i^n) - B(t_{i-1}^n)|^2$$

……………………………….3.9

3.3.4 Martingale of a Stochastic Process

Martingales are a collections of stochastic processes which models a fair game.

Definition

A stochastic process $X(t)$ adapted to the filtration $\{\mathcal{F}_t\}$, $t \geq 0$ for continuous or $t = 0,1, \ldots T$ for discrete, is a martingale if for any t

$$\mathbb{E}(X(t+1)|\mathcal{F}_t) = X(t)$$

$$\mathbb{E}(X(t+1)|X_0, X_1, \ldots, X_t) = X(t)$$

for $0 < s \leq t \leq T$ and such that $\mathbb{E}(X(t) < \infty$.

If $\mathbb{E}(X(t)|\mathcal{F}_s) \leq X(s)$ then $X(t)$ is a supermartingale.

If $\mathbb{E}(X(t)|\mathcal{F}_s) \geq X(s)$ then $X(t)$ is a submartingale

Martingales are called fair game because if you play any game of martingales your expected value cannot be positive or negative no matter the strategy you adopt.

3.3.5 Martingales of Brownian Motion

Theorem 3.2

The Brownian motion $B(t)$ is a martingale.

Proof:

If $B(t)$ is a martingale then it follows that

$$\mathbb{E}(B(t+1)|\mathcal{F}_t) = B(t)$$

Now $$\mathbb{E}(B(t+1)|\mathcal{F}_t) = \mathbb{E}\{[B(t+1) - B(t) + B(t)]|\mathcal{F}_t\}$$

$$= \mathbb{E}[B(t+1) - B(t)|\mathcal{F}_t] + \mathbb{E}[B(t)|\mathcal{F}_t]$$

Now by the independence of increments of Brownian motion it follows that

$$\mathbb{E}[B(t+1) - B(t)|\mathcal{F}_t] = 0 \text{ Hence}$$

$\mathbb{E}(B(t+1)|\mathcal{F}_t) = \mathbb{E}[B(t)|\mathcal{F}_t]$

$$\mathbb{E}(B(t+1)|\mathcal{F}_t) = B(t)$$

■

3.4 Stochastic Calculus

Let $B(t)$, $t \geq 0$, be Brownian motion process. An equation of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t) \quad \text{……………………………….3.10}$$

where functions $\mu(x, t)$ is the mean and $\sigma(x, t)$ represents the volatility are given. $X(t)$ is the unknown process called a stochastic differential equation (SDE) driven by Brownian motion. The functions $\mu(x, t)$ and $\sigma(x, t)$ are called the coefficients. They can either be constants or stochastic processes.

3.4.1 Examples of Stochastic Differential Equations

1. Geometric Brownian Motion

A stochastic process $X(t)$ is said to follow a Geometric Brownian Motion if it satisfies the stochastic differential equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dBt) \quad \text{…………..…………………3.11}$$

where μ is the drift and σ the volatility are constants. The Geometric Brownian Motion has the solution $$X(t) = X(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)}$$

Proof

If $X(t)$ is a stochastic differential then $dX(t) = \mu X(t)dt + \sigma X(t)dBt$ then by Itô formula for Itô process $$d(\text{In}X(t)) = f'dX(t) + \frac{1}{2}f''\sigma^2(t)dt$$

If $f(x)$ is lognormally distributed then $f(x) = Inx$ and $f'(x) = \frac{1}{X(t)}$, $f''(x) = \frac{1}{X^2(t)}$.

Hence $$d(InX(t)) = \frac{1}{X(t)}(\mu X(t)dt + \sigma X(t)dB(t)) - \frac{1}{2}\sigma^2 dt$$

$$d(InX(t)) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB(t)$$

Integrating both sides we have

$$InX(t) = InX(0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)$$

$$In\left(\frac{X(t)}{X(0)}\right) = \mu - \frac{\sigma^2}{2} + \sigma B(t)$$

$$\frac{X(t)}{X(0)} = e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}$$

$$X(t) = X(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)}$$

…………………..……………..3.12

2 Ornstein_Uhlenbeck Process

A stochastic process $X(t)$ is said to follow a Ornstein_Uhlenbeck Process (U_LP) if it satisfies the stochastic differential equation

$$dX(t) = -\alpha X(t)dt + \sigma dB(t)$$

Theorem 3.3

The Ornstein_Uhlenbeck Process satisfies the stochastic differential

$$d\,X(t) = -\alpha X(t)dt + \sigma dB(t)$$

and has the solution

$$X(t) = e^{\alpha t}\left(X(0) + \int_0^t \sigma e^{-\alpha s} dB(s)\right)$$

Proof

$$dX(t) = -\alpha X(t)dt + \sigma dB(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.13$$

The integrating factor $(I.F) = e_{-\alpha t}$. Multiply through equation (3.13) by $I.F$

$$e^{-\alpha t}dX(t) = \alpha e^{-\alpha t}X(t)\,dt + \sigma e^{-\alpha t}dB(t)$$

Let $f(x,t) = xe^{-\alpha t}$, then

$$\frac{\partial f}{\partial t} = -\alpha e^{-\alpha t}X(t)dt, \quad \frac{\partial f}{\partial x} = e^{-\alpha t} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = 0$$

By Itô's lemma $df(X(t),t) = \frac{\partial f}{\partial t}(X(t),t)dt + \frac{\partial f}{\partial x}(X(t),t)dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),t)dX(t)^2$ But $f(x,t) = xe^{-\alpha t}$ and so $d(e^{-\alpha t}X(t)) = -\mu e^{-\alpha t}X(t)dt + e^{-\alpha t}\,dX(t)$

or

$$e^{-\alpha t}dX(t) = d\left(e^{-\alpha t}X(t)\right) + \mu e^{-\alpha t}X(t)dt$$

Now $d\left(e^{-\alpha t}X(t)\right) + \mu e^{-\alpha t}X(t)dt = \alpha e^{-\alpha t}X(t)\,dt + \sigma e^{-\alpha t}dB(t)$ integrate both sides from 0 to t

$$d\left(e^{-\alpha t}X(t)\right) = \sigma e^{-\alpha t}dB(t)$$

$$\int_0^t d(e^{-\alpha s}X(s)) = \int_0^t \sigma e^{-\alpha s}dB(s)$$

$$e^{-\alpha t}X(t) - X(0) = \int_0^t \sigma e^{-\alpha s}dB(s)$$

$$X(t) = X(0)e^{\alpha t} + e^{\alpha t}\int_0^t \sigma e^{-\alpha s}dB(s)$$

$$X(t) = e^{\alpha t}\left(X(0) + \int_0^t \sigma e^{-\alpha s}dB(s)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.14$$

■

3.4.2 Itô Calculus

The Brownian motion process unlike deterministic functions has quadratic variation. The quadratic variation of the Brownian motions is given by $(dB)^2$. For this reason the Brownian motion process cannot be differentiated as in Riemann sums. The differential of the Brownian motion is given by the Itô lemma.

Itô's Formula for Brownian Motion

Theorem 3.4

If $B(t)$ is a Brownian motion on $[0,T]$ and $f(x)$ is a twice continuously differentiable function on $\mathbb{R}$, then for any $t \leq T$, then the differential of a function of $B(t)$ is given by

$$df(B(t)) = f'(B(t)dB(t) + \frac{1}{2}f''\big(B(t)\big)dt \qquad \text{…………………………..3.15}$$

or in integral form

$$f\big(B(t)\big) = f(0) + \int_0^t f'(B(s)dB(s) + \frac{1}{2}\int_0^t f''\big(B(s)\big)ds \qquad \text{…………………………..3.16}$$

Proof

By Taylor's formula $f(B(t)) = f(t) + f'(B(t))dB(t) + \frac{1}{2}f''\big(B(t)\big)(dB(t))^2 + O(t,x)$

But $\quad (dB(t))^2 = dt$

$$df(B(t)) - f(t) = f'\big(B(t)\big)dB(t) + \frac{1}{2}f''\big(B(s)\big)dt$$

$$df = f'(B(t))dB(t) + \frac{1}{2}f''\big(B(t)\big)dt$$

$$f\big(B(t)\big) = f(0) + \int_0^t f'(B(s)dB(s) + \frac{1}{2}\int_0^t f''\big(B(s)\big)ds$$

■

Itô's Lemma for Functions of Two Variables Theorem 3.5

Let $X(t)$ be an Itô process with the stochastic differential

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

where μ and σ are adapted processes. Let $f(X(t), t)$ be a twice continuously differentiable function on$\big((0, \infty) \times \mathbb{R}\big)$, then $f(X(t), t)$ is also and Itô process and has the stochastic differential

$$df(X(t), t) = \frac{\partial f}{\partial t}(X(t), t)dt + \frac{\partial f}{\partial X}(X(t), t)dX(t) + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial X^2}\ (X(t), t)dX(t)^2$$

……………………………………3.17

In integral form

$$f(X(t), t) = f\big((0), 0\big) + \int_0^t f'(X(s), s)ds + \int_0^t f'(X(s), t)dX(s) + \frac{1}{2}\int_0^t \sigma^2 \frac{\partial^2 f}{\partial X^2}\ (X(s), s)dX(s)^2$$

……………………………………3.18

$$df(X(t), t) = \left(\frac{\partial f}{\partial t}(X(t), t) + \ \mu(t)\frac{\partial f}{\partial X}(X(t), t) + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial X^2}(X(t), t)\right) dt + \sigma(t)\frac{\partial f}{\partial X}(X(t), t)\ dB(t)$$

……………………………………3.19

Proof

Consider small changes in t and $X(t)$. By Taylor expansion for two variables

$f(t + \Delta t, x + \Delta x)$

$$= f(x,t) + \frac{\partial f}{\partial t}(t,x)\Delta t + \frac{\partial f}{\partial x}(t,x)\Delta x$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(t,x)(\Delta t)^2 + 2\frac{\partial^2 f}{\partial x \partial t}(t,x)\Delta x \Delta t + 2\frac{\partial^2 f}{\partial x^2}(t,x)(\Delta x)^2\right) + \cdots$$

$$= f(x,t) + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(dt)^2 + 2\frac{\partial^2 f}{\partial x \partial t}dxdt + 2\frac{\partial^2 f}{\partial x^2}(dx)^2\right)$$

$$f(t+\Delta t, x+\Delta x) - f(x,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(dt)^2 + 2\frac{\partial^2 f}{\partial x \partial t}dxdt + 2\frac{\partial^2 f}{\partial x^2}(dx)^2\right)$$

$$df(x,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(dt)^2 + 2\frac{\partial^2 f}{\partial x \partial t}dxdt + 2\frac{\partial^2 f}{\partial x^2}(dx)^2\right)$$

Now if $X(t)$ is a Brownian motion process $B(t)$ then

$$df(t,B(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB(t) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial f}{\partial x}dB(t)dt + \frac{\partial^2 f}{\partial x^2}(dB(t))^2\right)$$

But $dB(t)dt = 0$, $(dt)^2 = 0$ and $\left(dB(t)\right)^2 = dt$. Hence

$$df(t,B(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt$$

$$df(t,B(t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB(t)$$

or simply

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB(t)$$

……………………………………3.20

■

We can extend this proof to rewrite Itô lemma in other forms. From Equation 3.20 if we replace $B(t)$ by a diffusion process $X(t)$ then we can write

$$df\big(t,X(t)\big) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX(t))^2$$

But $dX(t) = \mu(t)dt + \sigma(t)dB(t)$ and so we have

$$df = \frac{\partial}{\partial t}dt + \frac{\partial f}{\partial X}[\mu(t)dt + \sigma(t)dB(t)] + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}[\mu(t)dt + \sigma(t)dB(t)]^2$$

$$df = \frac{\partial}{\partial t}dt + \frac{\partial f}{\partial X}[\mu(t)dt + \sigma(t)dB(t)]$$
$$+\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}\{\mu(t)^2dt^2 + 2\mu(t)\sigma(t)dtdB(t) + \sigma(t)^2dB(t)^2\}$$

Now by convention $dt^2 = 0$, $dtdB(t) = 0$, $dB(t)^2 = dt$, hence

$$df = \frac{\partial f}{\partial t}dt + \mu(t)\frac{\partial f}{\partial X}dt + \sigma(t)\frac{\partial f}{\partial X}dB(t) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}dt$$

$$df = \left(\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma(t)\frac{\partial f}{\partial X}dB(t)$$

…………………………………….3.21

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Equation 3.21 is the most used form of Itô's lemma. In some cases f, μ and σ are functions of $X(t)$ and t so we can write

$$df(X(t),t) = \left(\frac{\partial f}{\partial t}(X(t),t) + \mu(X(t),t)\frac{\partial f}{\partial X}(X(t),t) + \frac{1}{2}\sigma^2(X(t),t)\frac{\partial^2 f}{\partial X^2}(X(t),t)\right)dt$$
$$+\sigma(X(t),t)\frac{\partial f}{\partial X}(X(t),t)dB(t)$$

…………………………………….3.22

In integral form

$$f(X(t),t) = f(X(0),0) + \int_0^t \left(\frac{\partial f}{\partial t}(X(s),s) + \mu(X(s),s)\frac{\partial f}{\partial X}(X(s),s) + \frac{1}{2}\sigma^2(X(s),s)\frac{\partial^2 f}{\partial X^2}(X(s),s) \right) ds + \int_0^t \left(\sigma(X(s),s)\frac{\partial f}{\partial X}(X(s),s) \right) dB(s)$$

$$f(X(t),t) = f(X(0),0) + \int_0^t \left(f'(X(s),s) + \mu(s)f'(X(s),s) + \frac{1}{2}\sigma^2(s)f''(X(s),s) \right) ds + \int_0^t \left(\sigma(s)f'(X(s),s) \right) dB(s)$$

……………………………………3.23

The simplified version known as the Itô integral used more frequently is given as

$$f(t) = f(0) + \int_0^t \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2} \right) ds + \sigma\int_0^t \left(\frac{\partial f}{\partial X} \right) dB(s)$$

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3.4.3 The Itô Process

The stochastic differential equation $dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dB(t)$ with solution $X(t)$ satisfying

$$X(t) = X(0) + \int_0^t \mu(X(s),s)ds + \int_0^t \sigma(X(s),s)dB(s), \quad 0 \le t \le T$$

…………………………3.25

with initial condition $X(0) = x(0)$ has a unique solution as long as μ and σ are finite and satisfy the Lipschitz conditions such that for every T and N, there is a constant K depending only on T and N

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq K|x - y|$$

$|\mu(x,t)| + |\sigma(x,t)| \leq K(1 + |x|)$ for all $|x|, |y| \leq N$ and all $0 \leq t \leq T$

$X(0)$ is independent of $(B(t), 0 \leq t \leq T)$, and $EX^2(0) < \infty$.

Equation 3.25 is defined as the Itô integral. The Itô integral accumulates quadratic variation. This ensures that we do not have the Riemann sums of the partitions of the area under the curve. However, it is important to note that the Itô integral is the limit of the Riemann sum when we always take the leftmost point of each interval. This property ensures that all decisions taken are made based on the leftmost time so that in the time interval $[t, t_{i+1}]$, we are only allowed to use information up to time t.

Itô integral as a martingale

If $f(t, B(t))$ is adapted to $B(t)$ then

$$\int_0^t f(t, B(t)) dB(t)$$

is a martingale.

3.4.4 Stochastic Exponential

If $X(t)$ is a stochastic process then its stochastic exponential is denoted by $\mathcal{E}(X)$ and defined as

$$\mathcal{E}(B)(t) = e^{X(t)-X(0)-\frac{1}{2}[X,X](t)} \quad \text{..3.26}$$

The process $[X,X](t)$ is the quadratic variation of the process. If the given process is a Brownian motion then

$$\mathcal{E}(B)(t) = e^{B(t)-B(0)-\frac{1}{2}[B,B](t)} \quad \text{..3.27}$$

If $B(0) = 0$ and $[B,B](t) = t$ then

$$\mathcal{E}(B)(t) = \mathcal{E}(B)(t) = e^{B(t)-\frac{1}{2}t} \quad \text{..3.28}$$

3.4.5 The Stock Price and its Return Process

Let $S(t)$ denote a stock price and assume that it is an Itô process, i.e. it has a stochastic differential. The process of the return on stock $R(t)$ is defined by the relation

$$dR(t) = \frac{dS(t)}{S(t)}$$

$$dS(t) = S(t)dR(t) \quad \text{..3.29}$$

It follows that $S(t)$ is a stochastic exponential of $R(t)$. If the return process is a constant then the $R(t) = r$ is deterministic and can be modelled by the ordinary differential equation

$$dS(t) = S(t)rdt$$

$$\frac{dS(t)}{S(t)} = rdt$$

$$InS(t) = \int_t^T rds$$

$$S(t) = \exp\left(\int_t^T rds\right)$$

$$S(t) = S(0)e^{rt}$$

On the other hand if the return r is uncertain then

$$R(t) = r + \text{White noise } (\xi)$$

$$\xi(t) = \frac{dB(t)}{dt}$$

$$\xi(t) = dB'(t)$$

The stochastic differential of the return process is given by

$$dR(t) = rdt + \sigma dB(t)$$

Substituting into Equation 3.33 into we have

$$dS(t) = S(t)(rdt + \sigma dB(t))$$

$$dS(t) = rS(t)dt + \sigma S(t)dB(t) \quad \text{...3.30}$$

This is the Geometric Brownian motion with solution

$$S(t) = exp\{B(t) + \left(r - \tfrac{1}{2}\sigma^2\right)t\} \quad \text{................................3.31}$$

3.5 Stochastic Interest Rate Models

Return processes are called interest rates and are their models are called interest rate models.

Typical examples are:

3.5.1 The Vasicek Model

In the Vasicek model the stochastic differential of the return process is

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dB(t) \quad \text{..3.32}$$

Such that α, β and σ are positive constants. To obtain the solution to the Vasicek model set $\alpha = \sigma = 0$. Thus

$$dR(t) = -\beta R(t)dt$$

$$R(t) = R(0)\exp\left(-\int_0^T \beta ds\right) = R(0)e^{-\beta t}$$

Let $V(t) = R(0)$ then $R(t) = V(t)e^{-\beta t}$

Using integrating by parts $dR(t) = -\beta e^{-\beta t}V(t)dt + e^{-\beta t}dV(t)$

The return process is given by the discounted value process and so we have

$$R(t) = V(t)e^{-\beta t}$$

$$dR(t) = -\beta R(t)dt + e^{-\beta t}dV(t)$$

$$-\beta R(t)dt + e^{-\beta t}dV(t) = \alpha dt - \beta R(t)dt + \sigma dB(t)$$

$$e^{-\beta t}dV(t) = \alpha dt + \sigma dB(t)$$

$$dV(t) = \alpha e^{\beta t}dt + \sigma e^{\beta t}dB(t)$$

Thus we have

$$V(t) = V(0) + \alpha\int_0^t e^{\beta s}ds + \sigma\int_0^t e^{\beta s}dB(s)$$

$$V(t) = R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma\int_0^t e^{\beta s}dB(s)$$

$R(t) = V(t)e^{-\beta t}$ and so $V(t) = R(t)e^{\beta t}$

$$R(t)e^{\beta t} = R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma\int_0^t e^{\beta s}dB(s)$$

$$R(t) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)e^{-\beta t} + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB(s)$$

$$R(t) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB(s)$$

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Hence

$$\int_0^t e^{2\beta s} dB(s) = \frac{\alpha}{2\beta}\left(e^{2\beta t} - 1\right)$$

We realise that $R(t)$ has a mean value given by

$$mean = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

and variance $$var = \frac{\alpha^2}{2\beta}\left(e^{2\beta t} - 1\right)$$

The Vasicek model is characterized by $\alpha - \beta R(t)$. The parameters $R(t)$, α and β thus determines the behaviour of the process model.

i. If $(t) = \beta$, then $\alpha - \beta R(t)$ is zero.

ii. If $(t) > \beta$, then $\alpha - \beta R(t)$ isnegative, which pushes $R(t)$ back toward β . iii. If $R(t) < \beta$, then $\alpha - \beta R(t)$ is positive, which again pushes $R(t)$ back toward β.

We realise that as the process behaviour is determined primarily be how close $R(t)$ gets to β. This is known as mean reversion. The speed of $R(t)$ to move closer to β gives us the level of the long run interest rate.

3.5.2 The Cox−Ross−Ingesoll Model

This is an interest rate model proposed by Cox, Ross and Ingesoll. In this model the stochastic differential for the return process is

$$dR(t) = \left(\alpha - \beta R(t)\right)dt + \sigma\sqrt{R(t)}dB(t) \quad \text{………..........……….3.34}$$

where $\alpha > 0$, $\beta > 0$ and $\sigma > 0$ are all constants.

3.6 Diffusion Processes

A diffusion process is a continuous time continuous state Markov process.

Definition

The stochastic process $\{X(t),\, t \geq 0\}$ is a continuous state continuous time Markov process if it satisfies the Markov property

$$\mathbb{P}(X(t) \in \mathcal{B} | X(s) = x) = \{\mathbb{P}(X(t) \in \mathcal{B} |\, X(t_1) = x_1, \dots, X(t_n) = x_n,\;\; X(s) = x\}$$

for all Borel subsets $B \subseteq \mathfrak{B}(\mathbb{R})$, and time instants $0 \leq t_1 \leq \dots \leq t_n \leq s \leq t$ and all $x_1 \dots . x_n \in \mathbb{R}$.

For a fixed s, x and t the transition probability $\mathbb{P}(X(t) \in \mathcal{B} | X(s) = x)$ is a probability measure on the σ −algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ such that

$$\mathbb{P}(X(t)|X(s) = x) = \int_B \mathbb{P}(s, x; t, y) dy$$

for all $B \in \mathfrak{B}$.

The quantity $\mathbb{P}(s, x; t, y)$ is the transition density. From the Markov property it follows that

$$\mathbb{P}(s, x; t, y) = \int_{-\infty}^{\infty} \mathbb{P}(s, x; \tau, z)\mathbb{P}(\tau, z; t, y) dz$$

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for all $0 \leq s \leq t$ and $x, y \in \mathbb{R}$. This is the Chapman−Kolmogorov equation.

For example the standard Brownian motion with non-overlapping, independent increments is a homogenous Markov process with transition probability density

$$\mathbb{P}(s, x; t, y) = \frac{1}{\sqrt{2\pi(t-s)}} exp\left\{-\frac{(y-x)^2}{2(t-s)}\right\}$$

The transition probability of the Ornstein−Uhlembeck process with parameter γ is given by

$$\mathbb{P}(s, x; t, y) = \frac{1}{\sqrt{2\pi(1-e^{-2\gamma(t-s)})}} exp\left\{-\frac{\left(y - xe^{-\gamma(t-s)}\right)^2}{2(1-e^{-2\gamma(t-s)})}\right\}$$

For all $0 \leq s \leq t$ and $x, y \in \mathbb{R}$

Diffusion processes are specified using operators or generators. An Itô process has an operator of the form $Lf = Lf(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x)$……………….……….....3.36

Where $f(x)$ is differentiable to the second order.

3.6.1 Generators of Some Basic Diffusion Processes

1. For an arithmetic Brownian motion with diffusion $dX(t) = \mu dt + \sigma dB(t)$ the generator is:

$$\mathcal{L}f(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x)$$

or

$$\mathcal{L}f(x) = \mu \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}$$

2. The Geometric Brownian Motion has the form

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$$

The generator of this diffusion process is

$$\mathcal{L}f(x) = \mu x f'(x) + \frac{1}{2}\sigma^2 x f''(x)$$

In this case the operator $\mathcal{L}_s$ satisfy

$$\mathcal{L}f(x,s) = (\mathcal{L}f)(x,s) = \mu(x,s)\frac{\partial}{\partial x}(x,s) + \frac{1}{2}\sigma^2(x,s)\frac{\partial^2}{\partial x^2}(x,s)$$

3. For an Ornstein−Uhlenbeck process the generator is

$$\mathcal{L}f(x) = -\alpha x f'(x) + \frac{1}{2}\sigma^2 x f''(x)$$

The operator $\mathcal{L}_s$ satisfy

$$\mathcal{L}f(x,s) = (\mathcal{L}f)(x,s) = -\alpha(x,s)\frac{\partial}{\partial x}(x,s)\frac{1}{2}\sigma^2(x,s)\frac{\partial^2}{\partial x^2}(x,s)$$

Example 3.3

Consider the GBM $dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$.

The generator of this diffusion is

$$Lf(x) = \frac{1}{2}\sigma^2(x)x^2 f''(x) + \mu(x)f'(x).$$

Its density is the fundamental solution of the PDE

$$\frac{\partial}{\partial t}p(y,t,x,s) = \mathcal{L}(t,x)$$

Where $$Lf(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x).$$

The solution to $dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$ is

$$X(t) = X(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}.$$

Thus, the transition probability function of $X(t)$ is given as

$$P(y,t,x,s) = P(xe^{(\mu - \frac{\sigma^2}{2})(t-s) + \sigma(B(t)-B(s))} \leq y | X(s) = x).$$

Its density is

$$P(y,t,x,s) = \frac{\partial}{\partial y}\Phi\left(\frac{\ln(y/x) - (\mu - \sigma^2/2)(t-s)\sigma\sqrt{t-s}}{\sigma\sqrt{(t-s)}}\right)$$

3.6.2 Forward and Backward Equations

The Forward equation addresses the following: If at time t the state of the system is x, essentially, $(X(t) = x)$ what can we say about the distribution of the state at a future time s where $s \geq t$. Since we are interested in the dynamics of the diffusion process looking forward in time we call this equation Kolmogorov's forward equation or Fokker−Plank equation. The backward equation addresses the following: Given that the system at a future time s has a particular behavior, what can we say about the distribution at time where $t \leq s$. This imposes a terminal condition on the PDE, which is integrated backward in time, from s to t. Since we are interested in the dynamics of the diffusion process looking backwards in time, we call this equation Kolmogorov's backward equation.

Theorem 3.6

Let the stochastic process $X(t), t \geq 0$ be a diffusion process such that

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$$

Then the forward evolution of its transition density $p(s, x; t, y)$ is given by Kolmogorov's forward equation

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial y}\{(\mu(t, y))\} - \frac{1}{2}\frac{\partial^2 p}{\partial y^2}\{\sigma^2(t, y)\} = 0$$

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For a fixed initial state (s, x) and a fixed final state (t, y) the backward evolution of the transition density $p(s, x; t, y)$ is given by Kolmogorov's backward equation

$$\frac{\partial p}{\partial s} + \frac{\partial p}{\partial x}\{(\mu(s, x))\} + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\{\sigma^2(s, x)\} = 0$$

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3.6.3 Feynman_Kac Theorem

Let $X(t)$ satisfies the stochastic differential equation $dX(t) = \mu dt + \sigma dB(t)$where $B(t)$ is Brownian motion under the measure $\mathbb{Q}$. Let $V(X(t), t)$ be the value of a contingent claim on $X(t)$, then by Itô formula

$$\frac{\partial V(X(t), t)}{\partial t} + \mu(X(t), t)\frac{\partial V(X(t), t)}{\partial x} + \frac{1}{2}\sigma(X(t), t)^2\frac{\partial^2 V(X(t), t)}{\partial x^2} - r(X(t), t)V(X(t), t) = 0$$

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and $V(X(t),t)$ has the solution

$$V(X(t),t) = \mathbb{E}\left(e^{-\int_t^T r(X(u),u)du}\right) V(X(T),T)|\mathcal{F}_t$$ …………………….3.40 The generator of the process $dX(t) = \mu dt + \sigma\, dB(t)$ is the operator

$$\mathcal{L} = \mu(X(t),t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(X(t),t)^2\frac{\partial^2}{\partial x^2}$$

so we can write

$$\frac{\partial V(X(t),t)}{\partial t} + \mathcal{L}V(X(t),t) - r(X(t),t)V(X(t),t) = 0$$

The Fayman_Kac's theorem is restatement or proof of the Black_Scholes Model. It asserts that if $X(t)$ follows the Itô process $dX(t) = \mu dt + \sigma dB(t)$ then a contingent claim $V(X(t),t)$ on $X(t)$ satisfies the Black Scholes formula

$$\frac{\partial V(X(t),t)}{\partial t} + \mu(X(t),t)\frac{\partial V(X(t),t)}{\partial x} + \frac{1}{2}\sigma(X(t),t)^2\frac{\partial^2 V(X(t),t)}{\partial x^2} - r(X(t),t)V(X(t),t) = 0$$

and $V(X(t),t)$ is given by the expectation

$$V(X(t),t) = \mathbb{E}\left[e^{-\int_t^T r(X(u),u)du}\, V(X(T),T)|\mathcal{F}_t\right]$$

Example 3.4

Consider the stock price process $S(t)$ following the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t).$$

A contingent claim on the stock with value $V(X(t),t)$ satisfies the Black–Scholes PDE so that

$$\frac{\partial V(X(t),t)}{\partial t} + rS\frac{\partial V(X(t),t)}{\partial x} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V(X(t),t)}{\partial x^2} - rV(X(t),t) = 0$$

The generator of the process given by

$$\mathcal{L} = rS\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial x^2}$$

By the Feynman−Kac theorem, the payoff $C(X(T),T) = V(X(T),T)$ is given by

$$C(S(t),t) = e^{-r(T-t)}\mathbb{E}[V(S(T),T)|\mathcal{F}_t]$$

Example 3.5

Solve this PDE

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \mu x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} = rf$$

$$0 \le t \le \mathrm{T} \quad f(x,t) = x^2$$

using the solution of the corresponding stochastic differential equation.

And give a probabilistic representation of the solution $f(x,t)$ of the PDE

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \mu x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} = rf$$

$0 \le t \le \mathrm{T}$ $f(x,t) = x^2$ where σ, μ and r are positive constants,

Solution

The SDE corresponding to the PDE is $dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$. It has a solution

$$X(t) = X(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}$$

By the Feynman−Kac formula $f(x,t) = \mathbb{E}(e^{-r(\mathrm{T}-t)}X^2(T)|X(t) = x)$

$$= e^{-r(\mathrm{T}-t)}\mathbb{E}(X^2(T)|X(t) = x)$$

But

$$X(T) = X(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma B(t)}$$ with $t = T - t$ we can write

$$X(T) = X(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)t+\sigma\left(B(T)-B(t)\right)}$$

We can obtain $\mathbb{E}(X^2(T)|X(t) = x) = x^2e^{-\left(2\mu-\frac{\sigma^2}{2}\right)(\mathrm{T}-t)}$

$$f(x,t) = e^{-r(\mathrm{T}-t)}\mathbb{E}(X^2(T)|X(t) = x) = e^{-r(\mathrm{T}-t)}\left(x^2e^{-\left(2\mu-\frac{\sigma^2}{2}\right)(\mathrm{T}-t)}\right)$$

Hence $f(x,t) = x^2e^{-\left(2\mu-\frac{\sigma^2}{2}\right)(\mathrm{T}-t)}$

3.6.4 Martingale Property of Diffusion Processes

Consider the stochastic process

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dB(t), \quad t \geq 0$$

By Itô formula

$$df(X(t),t) = \frac{\partial f}{\partial x}(X(t),t)dX(t) + \frac{\partial f}{\partial t}(X(t),t)dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),t)\left(dX(t)\right)^2$$

$$= \frac{\partial f}{\partial x}(X(t),t)[\mu(X(t),t)dt + \sigma(X(t),t)dB(t)] + \frac{\partial f}{\partial t}(X(t),t)dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),t)\left(dX(t)\right)^2$$

$$= \frac{\partial f}{\partial x}(X(t),t)[\mu(X(t),t)dt + \sigma(X(t),t)dB(t)] + \frac{\partial f}{\partial t}(X(t),t)dt$$

$$+ \frac{1}{2}\sigma^2(X(t),t)\frac{\partial^2 f}{\partial x^2}(X(t),t)dt$$

$$= \mu(X(t),t)\frac{\partial f}{\partial x}(X(t),t)dt + \sigma(X(t),t)\frac{\partial f}{\partial x}(X(t),t)dB(t) + \frac{\partial f}{\partial t}(X(t),t)dt$$

$$+ \frac{1}{2}\sigma^2(X(t),t)\frac{\partial^2 f}{\partial x^2}(X(t),t)dt$$

$$df(X(t),t) = \left(\frac{1}{2}\sigma^2(X(t),t)\frac{\partial^2 f}{\partial x^2}(X(t),t) + \mu(X(t),t)\frac{\partial f}{\partial x}\ (X(t),t) + \frac{\partial f}{\partial t}(X(t),t)\right)dt$$

$$+\left(\sigma(X(t),t)\frac{\partial f}{\partial x}\ (X(t),t)\right)dB(t)$$

Let the operator $\mathcal{L}_t f(x,t) = +\frac{1}{2}\sigma(X(t),t)^2\frac{\partial^2}{\partial x^2} + \mu(X(t),t)\frac{\partial}{\partial x}$ then

$$df(X(t),t) = \left(\mathcal{L}_t f\ (X(t),t) + \frac{\partial f}{\partial t}(X(t),t)\right)dt + \left(\sigma(X(t),t)\frac{\partial f}{\partial x}\ (X(t),t)\right)dB(t)$$

$$df(X(t),t) = \left(\mathcal{L}_t f\ + \frac{\partial f}{\partial t}\right)(X(t),t)dt + \left(\sigma(X(t),t)\frac{\partial f}{\partial x}\right)dB(t)$$

Integrating we have

$$f(X(t),t) = f(X(0),0) + \int_0^t \left(\mathcal{L}_u f\ + \frac{\partial f}{\partial s}\right)(X(u),u)du + \int_0^t \left(\sigma(X(u),u)\frac{\partial f}{\partial x}\right)(X(u),u)dB(u)$$

Let

$$M_f(t) = f(X(t),t) - \int_0^t \left(\mathcal{L}_u f\ + \frac{\partial f}{\partial s}\right)(X(u),u)du$$

……………………….……….3.41

Then the compensated process of $f(X(t),t)$ is a martingale.

Theorem 3.7

Let $X(t)$ be a solution to $dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dB(t)$ with coefficients $\mu(x,t)$ and $\sigma(x,t)$. Then the process

$$M_f(t) = f(X(t),t) - \int_0^t \left(\mathcal{L}_u f\ + \frac{\partial f}{\partial s}\right)(X(u),u)du$$

where $f(x,t)$ is a differentiable in x and t is a martingale.

Proof

We know that by Itô formula

$$M_f(t) = f(X(t),t) - \int_0^t \left(\mathcal{L}_u f + \frac{\partial f}{\partial s}\right)(X(u),u)du$$

Let $A > 0$, then $\left(\frac{\partial f}{\partial s}(x,u)\right)^2 < A_1$ and so $\int_0^T \mathbb{E}\left(\frac{\partial f}{\partial x}(X(u),u)\sigma(X(u),u)\right)^2 du$

is bounded and we can write

$$\int_0^T \mathbb{E}\left(\frac{\partial f}{\partial x}(X(u),u)\sigma(X(u),u)\right)^2 du < K_1 \int_0^T \mathbb{E}\big(\sigma^2(X(u),u)\big)\, du$$

By the linear growth condition

$$\int_0^T \mathbb{E}\left(\frac{\partial f}{\partial x}(X(u),u)\sigma(X(u),u)\right)^2 du < 2K_1 K^2 T\left(1 + \mathbb{E}\left(\sup_{u\le T} X^2(u)\right)\right)$$

And by the existence and uniqueness result

$$\int_0^T \mathbb{E}\left(\frac{\partial f}{\partial x}(X(u),u)\sigma(X(u),u)\right)^2 du < \infty$$

and so

$$M_f(t) = f(X(t),t) - \int_0^t \left(\mathcal{L}_u f + \frac{\partial f}{\partial s}\right)(X(u),u)du$$

……………………………..……….3.41

is a martingale.

■

3.7 Change of Probability Measure

Change of probability measure describes what happens when the measure assigning probabilities to random variables are changed to another the probability measure on the same measurable space. Change of measure allows us to switch from one distribution (measure) with drift to anther distribution without drift. The transformation of a density with drift to another density without drift under measure theory ensures that we can transform a non-martingale process into a martingale process.

3.7.1 Radon−Nikodym Derivative

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures on the measurable space $(\Omega, \mathcal{F})$. The change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is governed by the Radon−Nikodym derivative defined by

$$d\mathbb{Q} = \Lambda d\mathbb{P}$$

Λ is known as the Likelihood Ratio, Radon−Nikodym derivative or Radon−Nikodym density. The Radon−Nikodym derivative is given by Girsanov's theorem.

3.7.2 Change of Measure for Brownian Motion

Let $\mathbb{P}$ be a probability measure on $[0, T]$ defined by the Brownian motion with drift μ and let $\mathbb{Q}$ be a probability measure on $[0, T]$ defined by the Brownian motion without drift. Then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent and The Radon−Nikodym derivative is

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\mu B(T) - \frac{1}{2}\mu^2 T}$$

……………………………………….…3.42

3.7.3 Change of Measure for Normal Random Variables

As an example, we consider the change of measure of two normally distributed functions. Let $f_{\mu}(x)$ be normally distributed as $N(\mu, 1)$ and let $f_0(x)$, also be normally distributed as $N(0, 1)$. If the probability measure associated with the distribution $N(\mu, 1)$ is $\mathbb{P}$ and the probability measure associated with the distribution $N(0, 1)$ is $\mathbb{Q}$ then the Nikodym derivative establishing the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is given by $e^{\mu X - \frac{\mu^2}{2}}$. That is

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\mu X - \frac{\mu^2}{2}}$$

To see this we proceed as follows. Since $f_{\mu}(x)$ is normally distributed its density is given by

$$f_{\mu}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$$

$$f_{\mu}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2\mu x + \mu^2)}$$

Now $$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2)}$$

Hence $$f_{\mu}(x) = f_0(x) e^{\mu x - \frac{\mu^2}{2}}$$

Then $$\frac{f_{\mu}(x)}{f_0(x)} = e^{\mu x - \frac{\mu^2}{2}}$$

But $$\frac{f_{\mu}(x)}{f_0(x)} = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

Hence $$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\mu X - \frac{\mu^2}{2}}$$

Thus $\Lambda(X) = e^{\mu X - \mu 2/2}$ is the Radon−Nikodym derivative from the original measure $\mathbb{P}$ under the normal distribution with mean 0 and variance 1 to the distribution of the process under a new measure $\mathbb{Q}$ with the normal distribution with mean μ and variance 1. Generally the Radon−Nikodym derivative is obtained via Girsanov's theorem.

Theorem 3.8 (Girsanov's Theorem)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space on which is defined a Brownian motion process $B(t)$. Define $\psi(t)(t)$ as satisfying $X(t) = -\int_0^t \psi(t)(s)dB(s)$, then under the equivalent martingale measure $\mathbb{Q}$, the Radon-Nikodym derivative is given by

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}(B) = exp\left(-\int_0^T \psi(t)(s)dB(s) - \frac{1}{2}\int_0^T \psi(t)^2(s)ds\right) = \mathcal{E}(X)(T\,).$$

…………………..…………….3.43

and the process

$$B^{\mathbb{Q}}(t) = B^{\mathbb{P}}(t) + \int_0^t \psi(t)(s)ds$$

……….……………………..3.44

is a $\mathbb{Q}$ −Brownian motion. For proof of Girsanov's theorem see Bernt Øksendal (2000).

Expectation under Equivalent Probability Measures

Equivalent probability measures assign different probabilities to the outcomes on a sample space and thus have different random variables and consequently different expectations.

Theorem 3.9

Let $X(t)$ be a diffusion process. The expectation of $X(t)$ under the probability measure $\mathbb{P}$ is given by $\mathbb{E}\mathbb{P}(X(t))$ and the expectation of $X(t)$ under the measure $\mathbb{Q}$ is given by $\widetilde{\mathbb{E}}\mathbb{P}(X(t))$. If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent then

$$\widetilde{\mathbb{E}}\,[X(t] = \mathbb{E}[\Lambda(\mathrm{t})\ (X(t)]$$ ……………...……………………3.45

CHAPTER 4

OPTION PRICING MODELS

4.1 Introduction

The mathematical theories underpinning option pricing are well rooted in stochastic calculus, measure theory, martingales and partial differential equations. These fields of mathematical study have their own concepts, and are far advanced and well developed in depth. The two most basic models of option pricing are the Black–Scholes Merton model and the discrete binomial tree model introduced by Cox, Rox and Rubinstien. We shall trace the development of the option pricing theories beginning from Bachelier through Samuelson to the Black–Scholes model. Although we are mostly concerned with continuous models, we will also examine the discrete model proposed by Cox, Rox and Rubinstien and show that its limiting function is the Black Scholes. The other approach to pricing options involves the use of expectations. In this approach the price of the option is valued as a replicating portfolio whose value equals the price of its discounted expected payoff.

4.2 Background to Continuous Time Option Pricing Theory

In a differential form, the Bachelier model can be written

$$dS(t) = \mu dt + \sigma dB(t) \qquad \text{...4.1}$$

where $S(t)$ is the stock price, $B(t)$ is the Brownian motion process, μ is the return on the stock price and σ is the volatility of the stock price. Bachelier proposed that an option sold on this stock has a price $C(S(t), t)$ given by

$$C(S(T),T) = S(T)N\left(\frac{S(T)-K}{\sigma\sqrt{T}}\right) - KN\left(\frac{S(T)-K}{\sigma\sqrt{T}}\right) \qquad \text{...4.2}$$

where S is the initial price of the stock, K is the exercise or strike price and σ is the volatility of the stock price, T is the time to the option's maturity and $N(x)$ has standard normal density

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} exp\left(\frac{u}{2}\right) du$$

The only weakness of that model is that stock prices can be negative which is contrary to reality. Sprenkle provided a new formula that rules out negative option prices.

$$C(S(T),T) = e^{\rho T}S(T)N(d_1) - (1-A)KN(d_2) \qquad \ldots\ldots 4.3$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[In\left(\frac{S}{K}\right) + \left(\rho + \tfrac{1}{2}\sigma^2\right)T\right]$$

where $$d_2 = d_1 - \sigma\sqrt{T}$$

where the parameters have their usual meanings. Boness (1964) accounted for the time value of money through the discounting of the terminal stock price. In Boness model the Bachelier's formula was modified as

$$C(S(T),T) = S(T)N(d_1) - Ke^{-\rho T}N(d_2) \qquad \ldots\ldots 4.4$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[In\left(\frac{S}{K}\right) + \left(\rho + \tfrac{1}{2}\sigma^2\right)T\right]$$

where $$d_2 = d_1 - \sigma\sqrt{T}$$

Samuelson (1965) allowed for different levels of risk for the stock and obtained

$$C(S(T),T) = S(T)e^{(\rho-\alpha)T}N(d_1) - Ke^{-\alpha T}N(d_2) \qquad \ldots\ldots 4.5$$

where $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[In\left(\frac{S}{K}\right) + \left(\rho + \tfrac{1}{2}\sigma^2\right)T\right]$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The weakness in Samuelson's model was the assignment of different returns to individual investors. This was addressed in the Black–Scholes model.

4.3 The Black_Scholes Model

Fisher Black and Myron Scholes (1973) provided the Black Scholes partial differential equation. The model's main underlying mathematical theory is the Itô lemma

$$df(X(t),t) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma\frac{\partial f}{\partial X}\,dB(t)$$

……………………………………4.6

Derivation of the Black–Scholes partial differential equation

The derivation of the Black Scholes model can be summarized in three key arguments:

i. Itô formula application to the value of a replicating portfolio of the option ii. The hedging argument to create a riskless portfolio iii. The no arbitrage arguments of a risk free return of the portfolio

i. Itô formula application to the value of a replicating portfolio

Consider a stock whose price process $S(t)$ follows the Geometric Brownian Motion such that

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \quad \text{…………..………………………..4.7}$$

Equation 4.7 is an Itô process with mean $(\mu S(t))$ and $(Var(S(t)) = [\sigma S(t)]^2)$.

Consider a contingent claim on $S(t)$ whose value $V(S(t), t)$ depends on the $S(t)$ and t. By Itô's lemma the change in $V(S(t), t)$ is given by

$$dV(S(t).t) = \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t}\right)dt + \left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)dB(t)$$

…………………………………………….….4.8

Thus, the stochastic process followed by $V(S(t), t)$ is also an Itô process with mean

$$\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t}\right)$$

and variance

$$\left(\sigma S \frac{\partial V(S(t),t)}{\partial S(t)}\right)^2$$

ii. The hedging argument to create a riskless portfolio

Now let's construct a portfolio in which we buy 1 option with value $V(S(t), t)$ and an unknown amount of stocks. The question here is how much of the stocks must be purchased in order to create a riskless or hedged portfolio. Let this amount be Δ stocks. The portfolio now consist of an option and Δ amount of stocks and has a value given by $\pi = V(S(t), t) - \Delta S(t)$. In an infinitesimal time step the change in the portfolio's value is given by

$$d\pi = dV(S(t),t) - \Delta dS(t) \quad \text{……………………………….4.9}$$

Substitute Equations 4.7 and Equation 4.8 into Equation 4.9 gives

$$d\pi = \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t}\right)dt$$
$$+ \left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)dB(t) + \Delta[\mu S(t)dt + \sigma S(t)dB(t)]$$

Grouping terms with dt and $dB(t)$ we have

$$d\pi = \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t} - \mu\Delta S(t)\right)dt$$
$$+ \left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)} - \sigma\Delta S(t)\right) dB(t)$$

……………………………………..……4.10

We again realise in Equation 4.10 that the stochastic process for the hedged portfolio is an Itô process with drift parameter

$$\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t} - \mu\Delta S(t)\right)$$

$$\left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)} - \sigma\Delta S(t)\right)^2$$

and variance

Equation 4.10 consist of two parts: a deterministic part given by

$$\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu\Delta S(t)\right)dt$$

and a stochastic part given by

$$\left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}-\sigma\Delta S(t)\right)dB(t)$$

……………………………………………………………4.11

To make the portfolio completely riskless the stochastic part in Equation 4.10 must vanish. In otherwords we should have

$$\left(\sigma S(t)\frac{\partial V(S(t),t)}{\partial S(t)}-\sigma\Delta S(t)\right)\,dB(t)=0$$

……………………………………………………………4.12

Solving for Δ in Equation 4.12 yields

$$\Delta=\left(\frac{\partial V(S(t),t)}{\partial S(t)}\right)$$

If Equation 4.12 holds, then Equation 4.10 becomes

$$d\pi=\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu\Delta S(t)\right)dt$$

…………………………………..4.13

Replace Δ in Equation 4.13 by $\frac{\partial V(S(t),t)}{\partial S(t)}$, then we have

$$d\pi=\left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2}+\frac{\partial V(S(t),t)}{\partial t}-\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)dt$$

…………………………………..4.14

It follows that to completely hedge the portfolio we must purchase $\frac{\partial V(S(t),t)}{\partial S(t)}$ of the underlying asset. It means that to have a riskless portfolio we must purchase an amount of stocks that is equal to the ratio of how much the option value changes relative to the change in value of the stock. However, this situation is only valid in a small time interval and so we must continuously change the amount of stocks purchased to rebalance $\frac{\partial V(S(t),t)}{\partial S(t)}$.

iii. The no arbitrage arguments and a risk free return of the portfolio

We claim that the infinitesimal change in the portfolio's value is $d\pi$. If this is true then what is the return of this riskless portfolio in a small time step dt? Black and Scholes suggested that the return must be the risk free rate r otherwise there will be arbitrage opportunities. If this is the case then owning π amount of the portfolio would provide a return of $r\pi dt$ in a small time interval dt. Consequently,

$$d\pi = r\pi dt$$

Replacing $d\pi$ by $r\pi dt$ in Equation 4.14 we have

$$r\pi dt = \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t} - \mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)dt$$

$$r\pi = \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t} - \mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)$$

But $\pi = (V(S(t),t) - \Delta S(t)) = \left(V(S(t),t) - S(t)\frac{\partial V}{\partial S(t)}\right)$ and so

$$r\left(V(S(t),t) - S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)$$

$$= \left(\mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t} - \mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)}\right)$$

$$rV(S(t),t) - rS(t)\frac{\partial V(S(t),t)}{\partial S(t)} = \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \frac{\partial V(S(t),t)}{\partial t}$$

$$\frac{\partial V(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + rS(t)\frac{\partial V(S(t),t)}{\partial S(t)} - rV(S(t),t) = 0$$

……………………………4.15

At maturity, the price of the option $C(S(t),t)$ is equal to the value of the hedged portfolio and so $C(S(t),t) = V(S(t),t)$. Hence Equation 4.15 is rewritten as

$$\frac{\partial C(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 C(S(t),t)}{\partial S(t)^2} + rS(t)\frac{\partial C(S(t),t)}{\partial S(t)} - rC(S(t),t) = 0$$

……………………………4.16

For a European call option the boundary conditions are

$$C(0,t) = 0 \qquad C(S(T),T) = max\,(S(T) - K, 0),\ \ t \geq 0$$

Equation 4.16 is known as the Black Scholes partial differential equation. In arriving at the equation, Black and Scholes made the following assumptions: The stock does not pay any dividends

- The option is European-styled.
- The market is efficient
- The option trading does not attract any transaction costs
- The rate of interest on the stock is equal to the risk-free rate and assumed constant
- The underlying stock is assumed to be lognormally distributed
- Investors can own a fractional share of the stock

An important outcome of the Black–Scholes derivation is that it does not contain the drift parameter μ of the underlying asset. As a consequence, although two investors may differ on their estimates on the return of the stock asset, they will still agree on the price of the option.

This is the tenet of risk neutral pricing.

Solution to the Black−Scholes PDE

The relation between the option price and the heat equation in Physics has been observed as far back as Bachelier who considered the distribution of the option price as analogous to the dissipation of heat along the horizontal path of an infinite rod. In effect, the Black–Scholes equation is a partial differential equation which by suitable transformation of variables is equivalent to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

…………………………….4.17

$$t \geq 0, \quad -\infty < x < \infty$$

with initial value $u(0, x) = V(x)$

The solution to Equation 4.17 is given by

$$u(t, x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} exp\left\{\frac{-(x-s)^2}{4t}\right\} V(s)ds$$

…………………………….4.18

It follows that once we change the Black−Scholes equation into the heat equation, we will have a closed form solution. In that case the initial condition is the final payout function $V(s)$ in Equation 4.18. The complete transformation of Black−Scholes equation to the heat equation and the solution are given in the Appendix.

4.4 The Binomial Tree or Cox−Rox−Rubenstein Model

The other option pricing model is the discrete time model proposed by Cox, Rox, and Rubenstein and so it is sometimes known as the Cox−Rox−Rubenstein model or simply the binomial tree model. The binomial tree model follows a simple stationary random walk binomial process. At each moment in time, considered as a node of the binomial tree, the price of the stock can either go up or down by a given probability. The option price is then computed at each node of the trees with the assumption that the rate of return of the stock is the risk−free rate. In the limit as the number of nodes gets large the binomial tree converges to the lognormal price process and the formula converges to the Black−Scholes formula.

4.4.1 One Step Binomial Model

Consider a stock whose current price is $S_0 = ¢20$.

We want to price an option sold on this stock.

Let the option has the following characteristics

Initial Stock Price $S_0 = ¢20$

Strike Price K_T = $¢21$

Time to maturity = 3 months

Return on stock price $r = risk\ free\ rate = r_f$

At the end of the option's life the stock price would either go up or down. Let's assume it either goes up to ¢22 or down to ¢18. If the price turns at to be ¢22 then the value of the option or the profit made on the option is given by

$$max(S_T - K_T, 0) = (S_T - K_T, 0)^+ = (22 - 21, 0)^+ = ¢1$$

The option value situation is illustrated in Figure 4.1.

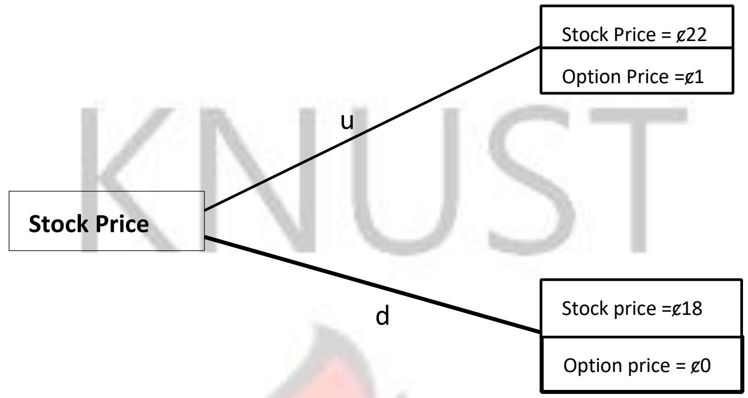

Figure 4.1 Two-Step Binomial Lattice

To price the option we proceed as follows:

1. Set up a portfolio that consists of buying Δ shares of the stock and 1 share of the option (call option).

We are confronted with the following problem: how much shares of the stock (Δ) we need to buy in order to make the portfolio riskless. For the portfolio to be riskless the interest earned on the portfolio must be the risk-free interest rate (r_f).

Suppose the stock price moves up to ¢22, then the value of the shares is 22Δ and the value of the option is

$$(22-21,\ 0)^{+} = ȼ1.00$$

Hence total value of the portfolio $= 22\Delta - 1$

On the other hand, if the stock price moves down to ȼ18, then the total value of the share is 18Δ and the value of the option is

$$(18-22, 0)^{+} = ȼ0.00$$

Total value of the portfolio $= 18\Delta + 0 = 18\Delta$

The key argument here is that the portfolio is riskless if the value of the shares we buy (Δ) is such that the final value of the portfolio is same in both scenarios i.e.

$$22\Delta - 1 = 18\Delta$$

$$\Delta = ¼ = 0.25$$

This means that to make the portfolio riskless we must purchase 0.25 or $^1/_4$ of stocks. This is what led Black and Scholes to allow for fractional ownership of shares in their assumptions. It follows that to make the portfolio riskless we must buy 0.25 of shares and 1.0 of options.

If the stock price moves up to ¢22 the value of the portfolio is $22 \times 0.25 - 1 =$ ¢4.5

If the stock price moves down to ¢18, the value of the portfolio is $18 \times 0.25 =$ ¢4.5

So for a riskless portfolio, the value of the portfolio is always the same (¢4.5) at the end option's life regardless of whether the stock price moves up or down. Riskless portfolio must in the sense of arbitrage opportunities earn a risk-free interest and so we can say that the present value of the portfolio is given by

$$PV_p = e^{rt}\, 4.5$$

If $t = 3\ months = \frac{3}{12} = 0.25, r_f = 12\%$ per annum, then

$$PV_p = e^{0.12 \times 0.25}\, 4.5$$
$$= 4.367$$

This present value of the portfolio must equal to the value of the portfolio when the stock price was at ¢20.

If the price of the option price is ¢ $C.00$, then the value of the portfolio minus the price of the option must be equal to the present value of the portfolio at the end of the period.

Portfolio Value at $t = 0$ _Option Price =Present Value of portfolio at expiration T.

$$20 \times (0.25) - C = PV_P$$
$$5 - C = 4.367$$

$$C = 5 - 4.367$$
$$= 0.633$$

Thus, in absence of any arbitrage opportunities the price of the option must be

$$C = 63p$$

We conclude that in the absence of arbitrage opportunities the price of the option must be 63p. If the price of the option is greater than 0.63 then $5 - C_1$ world be less than ¢4.367. This will make the portfolio cost less than ¢4.367 to set up which means you can set up a riskless portfolio which can earn more interest than the risk free rate and that is not possible. On the other hand, if the price of the option is less than 0.63, then it follows that the value of the portfolio would be more than ¢4.367 and so you can have an asset whose interest is less than the risk free rate i.e. you can borrow money from the asset at less than the risk free rate.

Generalization of the one step option pricing model

We generalize the argument above. Consider a stock whose price is S_0 and an option on the stock with price C. Let the option have a lifetime of T and assume that the stock price can go up or down during the life of the option.

Let S_0u be the new price level where $u > 1$ and S_0d be the new price level when the stock price gives down where $d < 1$.

The percentage increase when the price goes up is given by

$$\frac{S_0u - S_0}{S_0} = \frac{S_0(u-1)}{S_0} = u - 1$$

……………………………….4.19 The percentage decrease when the price goes down is given by

$$\frac{S_0 - S_0d}{S_0} = \frac{S_0(1-d)}{S_0} = 1 - d$$

……………………………….4.20

If the stock moves up to S_0u, we suppose that the payoff from the option is C_u if the stock moves down to S_0d we suppose the payoff is C_d. The option valuing situation is illustrated as shown in

Figure 4.2

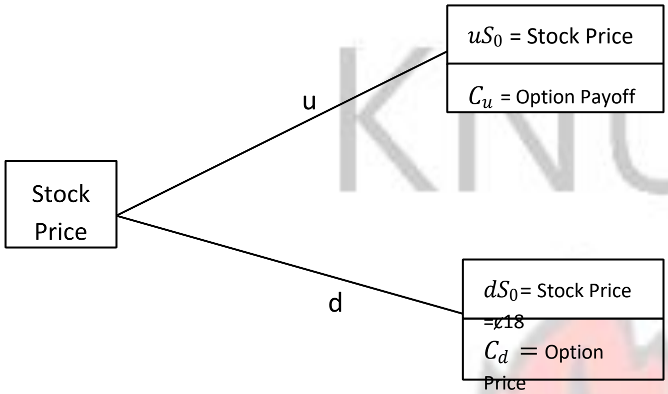

Figure 4.2 Generalised Two-Step Binomial Lattice

If we invest in Δ amount of shares and hold 1 option then if there is an upward movement in the stock price the value of the portfolio is

$$\pi = \Delta uS_0 - C_u \quad \ldots\ldots 4.21$$

If there is a downward movement in stock price the value of the portfolio is

$$\pi = \Delta dS_0 - C_d \quad \ldots\ldots 4.22$$

For no arbitrage conditions we equate Equation 4.21 and Equation 4.22 so we have

$$\Delta uS_0 - C_u = \Delta dS_0 - C_d$$

$$\Delta uS_0 - \Delta dS_0 = C_u - C_d$$

$$\Delta = \frac{C_u - C_d}{uS_0 - dS_0}$$

$$\ldots\ldots 4.23$$

For the portfolio to be riskless, the value of the option when the stock price goes up and its value when the stock price goes down must be the same and should be given by

$$\Delta uS_0 - C_u$$

Thus, the present value of the stock option at the beginning of the life of the option is thus given by $e^{-rt}\,(uS_0\Delta - C_u)$

The cost of setting up the portfolio is given by $\Delta S_0 - C_0$. But the cost of setting up the portfolio must be equal to the option value at the end of the option's life discounted to the beginning of the options life. Hence

$$\Delta S_0 - C = e^{-rt}\,(uS_0\Delta - C_u)$$

$$-C = e^{-rt}\,(uS_0\Delta - C_u) - \Delta S_0$$

$$-C = uS_0\Delta e_{-rt} - \Delta S_0 - C_u e_{-rt}$$

$$C = \Delta S_0(1 - ue_{-rt}) + C_u e_{-rt}$$

Now
$$\Delta = \frac{C_u - C_d}{uS_0 - dS_0}$$

$$C = \frac{C_u - C_d}{uS_0 - dS_0} S_0(1 - ue^{-rt}) + C_u e^{-rt}$$

$$C = \frac{C_u - C_d}{u - d}(1 - ue^{-rt}) + C_u e^{-rt}$$

....4.24

We can rewrite Equation 4.24 as

$$C = e^{-rt}(pC_u + (1 - p)C_d \qquad \text{...4.25}$$

where $p = \frac{e^{-rt} - d}{u - d}$

Equation 4.25 is the formula for pricing an option in a one−step model. It is important to realize that the price of the option does not have the probabilities of the stock price going down or up. This is because we do not value the option alone in absolute terms, rather we value the option in terms of the price of the underlying stock. The probabilities of future up or down movements are already incorporated into the stock price and so we do not need to take them into account again when valuing the option in terms of the stock price.

We shall now look at the expected return from the stock when we assume the stock price moves up with probability p and down with provability $1 - p$. The expected payoff from the option is thus given by

$$C = pC_u + (1 - p)\, C_d \qquad \text{...4.26}$$

The expected stock price at the end of the option's life T is given by

$$\mathbb{E}(S(T)) = pS_0 u + (1 - p)\, S_0 d$$

$$\mathbb{E}(S(T)) = pS_0(u - d) + S_0 d$$

$$p = \frac{e^{-rt} - d}{u - d}$$

$$\mathbb{E}(S(T)) = \frac{e^{-rt} - d}{u - d} S_0(u - d) + S_0 d$$

$$\mathbb{E}(S(T)) = S_0 e^{rt} - dS_0 + dS_0$$

$$\mathbb{E}(S(T)) = S_0 e^{rt} \text{..} 4.27$$

Equation 4.25 shows that the discount rate must be the risk_free rate. This means the only return on the stock that ensure arbitrage free price is the risk_free rate. This leads us to the idea of a risk-neutral world. In risk–neutral world all individuals are indifferent to risk and so investors require no compensation for bearing risk.

Example 4.1

Let $u = 1.1$ and $d = 0.9$ $r = 0.12$, $T = 0.25$, $C_u = 1, C_d = 0$

$$p = \frac{e^{-0.12\times0.25} - 0.9}{1.1 - 0.9}$$

$$= 0.6523$$

Price of option $$C = e^{-rt}(pC_u + (1-p)C_d$$

$$C = e^{-0.12\times0.25}(0.6523(1) + (1 - 0.6523)(0)$$

$$C = e^{-0.03} \times 0.6523$$

$$C = 0.633$$

4.4.2 Two Step Binomial Trees

We now increase the procedure to two steps. The price of the stock at the beginning is ¢20 and the steps may go up or down by 10%. Each step takes 3 months with a risk-free interest of 12% per annum. We consider an option whose underlying asset is this stock with strike price ¢21. We want to compute the price of an option which goes through these two modes in its life-time.

Figure 4.3 illustrates this. At each stage or step we would compute the price of the stock and then the corresponding option price. We would then compute the option price at the end of the option life and use it to price the option by working backwards. We proceed as follows

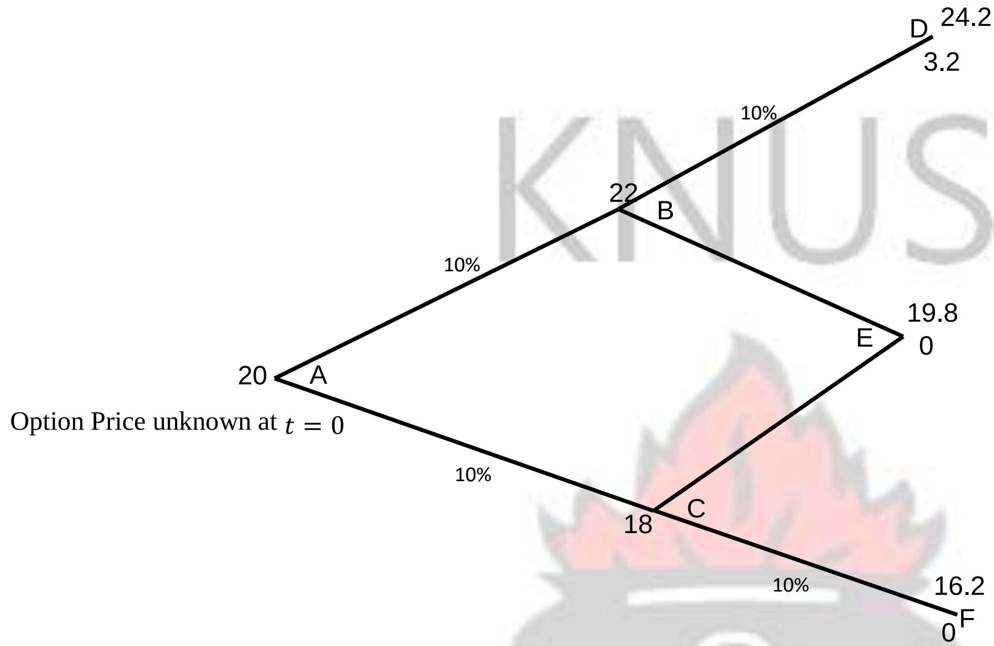

Figure 4.3 Stock and Option prices in a two_step binomial lattice

At node A

Stock price = $¢20$

Option price = unknown (this is what we want to determine)

At node B

Stock Price = $20 + 20 \times 10\%$

$= 20 + 20 \times 0.1$

$= 20 \times 1.1$

$= ¢22$

At Node D

Stock Price = $22 + 22 \times 10\%$

$= 22\,(1 + 0.1)$

$= 22 \times 1.1$

$= ¢24.2$

Now the option is exercised at node D. The strike price is ¢21 and so at the end of the option's life its value is given by

$$Option\ Value\ = Stock\ price - Strike\ price$$

$$= 24.2 - 21.0$$

$$= ¢3.2$$

We will use this to find the option price at B but before we do this let's complete the price at node C

AT NODE C

Stock price $= 20 - 20 \times 10\%$

$$= 20 - 20 \times 0.1$$

$$= 20(1 - 0.1)$$

$$= 20\ x\ 0.9$$

$$= ¢18$$

The stock price at C (¢18) is below the strike price (¢21) and so the option is not exercised at C.

AT NODE F

Stock price $= 18 \times 0.1$

$$= 18(1 - 0.1)$$

$$= 18\ x\ 0.9$$

$$= ¢16.2$$

The stock price at F (¢16.2) is below the strike price (¢21) and so the option is not exercised at F.

AT NODE E

The stock price is computed using the price at node B or C. Using the price at node B we have

Stock Price $= 22 - 22 \times 10\%$

$= 22(1 - 0.1)$

$= 22\ x\ 0.9$

$= ¢19.8$

The stock price at E (¢19.8) is below the strike price (¢21) and so the option is not exercised at E. We realise that when the stock price goes up we multiply the initial stock price by a constant u to obtain the stock price at the next step. If the probability of the stock price going up or down is 10% then $u = 1.1$. and $d = 0.9$.

Computing option prices

After computing the stock price, we can now obtain the option prices. We will do this by working backwards from node D. At the end of the trading period the stock price is ¢24.2 and at this time we exercise the option since the strike price is ¢21 (i.e. we buy the stock for ¢21 and immediately sell it at ¢24.2 making a profit of $24.2 - 21.0 = ¢3.20$. Thus, at expiration the option's value is ¢3.20 so the option price at node D is ¢3.20. Using this we can calculate the options price or value at node B using the formula for 1 step binomial tree.

$$C = e^{-rt}(pC_u + (1 - p)C_d)$$

C_u is the price of the option at node D (¢3.20) and C_d is the price of the option at node E (¢0.00).

The option price when exercised at node B is given by

$$C_B = e^{-0.12 \times 0.25}(pC_u + (1 - p)C_d$$

$$p = \frac{e^{-rt} - d}{u - d}$$

$$t = \frac{3}{12} = 0.25, \quad r = 0.12, \quad d = 0.9, \quad u = 1.1$$

$$p = \frac{e^{-0.12\times0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

$$C_{\mathrm{B}} = e^{-0.12\times0.25}(pC_u + (1-p)C_d$$

$$C_{\mathrm{B}} = e^{-0.12\times0.25}(0.6523 \times 3.2 + (1 - 0.6523) \times 0$$

$$= ¢2.0257$$

This is the option price at node B. We now calculate the option price at node A. We do this by focusing on the first step of the tree. We know that the value of the option at node B is $¢2.0257$ and at node C is 0. Hence

$$C_{\mathrm{A}} = e^{-0.12\times0.25}(0.6523 \times 2.0257 + (1 - 0.6523) \times 0$$

$$C_{\mathrm{A}} = ¢1.2823$$

This is the price of the option at node A and it is the real value of the option.

4.4.3 General Formula for Binomial Option Pricing

Given a time step Δt and r, the option price in the one step model is

$$C = e^{-r\Delta t}[pC_u + (1-p)]C_d \quad \text{...4.28}$$

For the two-step model C_u and C_d are given by

$$C_u = e{-r\Delta t}(pC_{uu} + (1-p)C_{ud}$$

$$C_d = e{-r\Delta t}(pC_{ud} + (1-p)C_{dd}$$

Substituting C_u and C_d into 4.26 we have

$$C = e^{-r\Delta t}p[p\{C_{uu} + (1-p)C_{ud}\}] + (1-p)[e^{-r\Delta t}(pC_{ud} + (1-p)]C_{dd}$$

Simplifying these we have

$$C = e{-2r\Delta t}[p2C_{uu} + 2p(1-p)C_{ud} + (1-p)2C_{dd}]$$

$$C = e^{-2r\Delta t}[p^2(0, u^2S - K)^+ + 2p(1-p)(0, uS - K)^+ + (1-p)^2(0, d^2S - K)^+]$$

............................………4.29

We now have the formula for a two period model. For 3, 4, … . , n period model we will have a recursive procedure from which the option price is given by the Binomial formula. In an n −period model there are x times in which the stock can go up and $(n - x)$ times in which the stock price can go down. The option has value only if it is exercised at the n^{th} period. The price is given by

$$\sum_{x=1}^{n} \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}\, C_{u,u,u,\dots u(x\ times)\ d,d,d,\dots,d(n-x\ times)}$$

............................……..4.30

The discounted price is thus given by

$$C = e^{-r\Delta t} \sum_{x=1}^{n} \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}\, C_{u,u,u,\dots u(x\ times)\ d,d,d,\dots,d(n-x\ times)}$$

C can be seen as the expected payoff, expressed in cedis value today, of the final payoff C_n.

For a call option

$$C_{u,u,u,\dots u(x\ times) d,d,d,\dots,d(n-x\ times)} = (u^x d^{n-x} S - K, 0)^+$$

The option price is thus given by

$$C = e^{-rn\Delta t} \sum_{x=1}^{n} \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x} (u^x d^{n-x} S - K, 0)^+$$

Let $r = e^{r\Delta t}$ then when the option is exercised we have

$$(u^x d^{n-x} S - K, 0)^+ = u^x d^{n-x} S - K \text{ and so}$$

$$C = \frac{1}{r^n} \sum_{x=1}^{n} \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x} (u^x d^{n-x} S - K)$$

............................……..4.31

By breaking up C into two terms we have

$$C = S\left[\sum_{x=1}^{n} \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x} \frac{u^x d^{n-x}}{r^n}\right] - Kr^{-n}\left[\sum_{x=1}^{n} \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}\right]$$

……………………………4.32

4.5 Method of Equivalent Martingale Measures (Risk Neutral Pricing)

Another popular method used vastly to price options is the method of equivalent martingale measures of the risk_adjusted (neutral) probability measure. The method employs the idea of absence of arbitrage opportunities in the market to imply the existence of an equivalent probability measure to determine expected discounted future payments. However, before we delve into the model derivation we need to understand some basic terminology.

4.5.1 Arbitrage Pricing Theorem terminology

Contingent Claim

A contingent claim (option payoff) X with maturity date T is an arbitrary non-negative $\mathcal{F}-$ measurable random variable representing cash flow.

Portfolio strategy

A portfolio strategy consist of setting up a portfolio consisting of the underlying assets called the risky asset and a riskless asset usually a bond, cash, or savings account.

$$Portfolio \ = \ Risky\ Asset \ + \ Riskless\ Asset$$

If $V(t)$ is the value of the portfolio and $a(t)$ is the number of risky assets and $b(t)$ is the number of riskless asset then

$$V(t) = a(t)S(t) + b(t)B(t) \qquad 4.33$$

This portfolio replicates the option price at maturity and it is sometimes called the replicating portfolio.

Self_financing strategy

A trading strategy is called self_financing if and only if the discounted value of a replicating portfolio $V(t)$ equals the sum of initial value plus the net gain from the portfolio

$$V(t)S(t) = V(t+1)S(t) \qquad 4.34$$

Attainable Claim

A contingent claim is attainable if there exist a self−financing portfolio such that

$$V(T) = X = max\,(S(T) - K, 0) \qquad 4.35$$

Complete Market

A market is complete if every contingent claim X is attainable. That is, there exist a replicating self−financing portfolio such that

$$V(T) = X$$

Arbitrage

Arbitrage is a possibility that one can start a transaction with no capital and yet has a positive probability of making some money. Given an initial capital $V(0)$ we have

$$\mathbb{P}[V(T\,) > 0] > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.36$$

where T is the time of expiry of the trade or transaction period. A

self−financing trading strategy θ is called an arbitrage if

i. $\theta(0)S(0) < 0$ and $\mathbb{P}[(\theta(t)S(t) > 0)] = 1$ ii. $\theta(0)S(0) = 0$,

$\mathbb{P}[(\theta(t)S(t) > 0)] = 1$ and $\mathbb{P}[(\theta(t)S(t) > 0)] > 0$

In (i), the strategy turns a negative initial investment into a non−negative final wealth with probability 1. In (ii), the strategy turns an initial net investment of zero (0) into a non−negative final wealth that is positive with probability 1. The arbitrage theory is made clearer with the following exposition called the arbitrage and fair price argument

The arbitrage and fair price argument

Let the price of a contract at time $t = 0$ be $C(0)$. Let the price of the stock at time $t = 0$ be $S(0)$. The argument is that arbitrage situation would not arise only if $C(0) = S(0)$. To see this we consider the following scenarios.

- If the price of the contract at $t = 0$ is greater than the stock price $t = 0$ then it follows that $C(0) > S(0)$. If this is the case, the contract can be sold for $C(0)$, and then you buy the stock at $S(0)$. Since $C(0) > S(0)$ it follows that you are left with $C(0) - S(0)$ after the transaction. Invest

$C(0) - S(0)$ in an instrument bearing no risk so that at maturity T you will receive $S(0)$ plus the return on your investment i.e. $S(0)$ plus the interest on the risk free instrument. Then $C(T)$ $= S(0) + e^{-rt}[C(0) - S(0)]$, where $e^{-rt}[C(0) - S(0)]$ is the free money you have made without taking any risk.

- On the otherhand if $S(0) > C(0)$, then you can borrow the stock and sell it and receive $S(0)$. Now buy the contract at $C(0)$. Since $C(0)$ is less than $S(0)$ you are left with a change of $S(0)$ $- C(0)$. Invest $S(0) - C(0)$ in an instrument bearing no risk so that at maturity T you will receive $S(0)$ plus the return on your investment i.e. $S(0)$ plus the interest on the risk free instrument. Then $C(T) = S(0) + e^{-rt}[C(0) - S(0)]$, where $e^{-rt}[C(0) - S(0)]$ is the free money you have made without taking any risk. We realise that the only price that guarantee no arbitrage situation is when $C(0) = S(0)$.

Discounted stock price process

The price of an asset $S(t)$ is called a discounted price process if

$$\overline{S(t)} = \frac{1}{\beta} S(t)$$

where $\overline{S(t)}$ is the discounted stock price and $\frac{1}{\beta}$ is the discount factor.

Discounted value process

Let $\mathbb{Q}$ be an equivalent martingale probability measure, then for any self_financing strategy the discounted value process is a martingale if the discounted stock price process is a martingale.

Martingale Measure

A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called a martingale measure if the discounted stock price process $[S(t)\,/\beta(t)]$ is a martingale under the measure $\mathbb{Q}$. $\beta(t)$ is called the discount factor or the numerai.

Equivalent Martingale Measures

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if they have the same null sets. That is for any set A,

$$\mathbb{P}(A) \geq 0 \Leftrightarrow \mathbb{Q}(A) \geq 0$$

The two probability measures live on the same space but assign different probabilities to the outcomes on the sample space. We can change from one probability measure to the other if and only if they are equivalent. It follows that if there exists an equivalent martingale measure $\mathbb{Q}$, equivalent to the original measure $\mathbb{P}$, then the market model does not have arbitrage opportunities and discounted stock price process is a martingale under the measure $\mathbb{Q}$. These ideas are captured in the fundamental theorem of arbitrage.

Theorem 4.1 (First Fundamental Theorem of Arbitrage)

A market model does not have arbitrage opportunities if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted stock process

$$Z(t) = \frac{S(t)}{\beta(t)}$$

....…………………..…..4.37

is a $\mathbb{Q}$ −martingale.

Proof

By the first fundamental theorem our assumption of existence of no arbitrage in the market makes the discounted stock price process a martingale under the measure $\mathbb{Q}$ equivalent to the measure $\mathbb{P}$. We now give the theorem that relates the discounted stock process and the discounted value process

Theorem 4.2

Suppose there is a probability measure $\mathbb{Q}$ such that the discounted stock process $Z(t) = \frac{S(t)}{\beta(t)}$ is a $\mathbb{Q}$ −martingale. Then for any admissible trading strategy the discounted value process $V(t)/\beta(t)$ is also a $\mathbb{Q}$ −martingale. Such $\mathbb{Q}$ is called an equivalent martingale measure (EMM) or a risk−neutral probability measure. (See Appendix for proof)

4.5.2 The Arbitrage Free Option Pricing Model

Given that a market model is complete it follows that there are no arbitrages in the market. If there are no arbitrages then by the first fundamental theorem of arbitrage there exists a unique equivalent martingale measure such that the discounted stock price process is a martingale.

It follows that for any admissible strategy the discounted value process of a replicating portfolio is a martingale. Now if the discounted value process $e^{-rt}V(T)$ is a martingale, then for any time $t \leq T$ we can write

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{V(T)}{e^{rT}}\middle|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\frac{V(t)}{e^{rt}}\middle|\mathcal{F}_t\right)$$

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{V(T)}{\beta(T)}\middle|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\frac{V(t)}{\beta(t)}\middle|\mathcal{F}_t\right)$$

For any admissible strategy $\qquad V(T) = X(T)$

$$V(t) = \mathbb{E}_{\mathbb{Q}}\left(\frac{Z(t)}{\beta(T)}X(T)\bigg|\mathcal{F}_t\right)$$

$$V(t) = \mathbb{E}_{\mathbb{Q}}\left(\frac{\beta(t)}{\beta(T)}[max\, S(T) - K, 0]\bigg|\mathcal{F}_t\right)$$

……………………………4.38

Theorem 4.3

Suppose that the market model does not admit arbitrage, and X is an attainable claim with maturity T. Then $C(t)$, the arbitrage-free price of X at time $t \leq T$, is given by $V(t)$, the value of a portfolio of any admissible strategy replicating X. Moreover

$$C(t) = V(t) = \mathbb{E}_{\mathbb{Q}}\left(\frac{\beta(t)}{\beta(T)}X\bigg|\mathcal{F}_t\right)$$

……………………………4.39

where $\mathbb{Q}$ is an equivalent martingale probability measure. (See Appendix for proof). Another approach to obtain this model is via the Fayman−Kac's theorem.

Fayman−Kac's theorem

Consider the Geometric Brownian Motion process

$$dS(t) = \mu S(t)dt + \sigma S(t)dB^{\mathbb{P}}(t)$$

where μ is the expected return and σ is the volatility of $S(t)$ and $B(t)^{\mathbb{P}}$ is the Brownian motion process under the measure $\mathbb{P}$. If $V(S(t), t)$ is the value of a contingent claim on $S(t)$ and $V(S(t), t)$ satisfies

$$\frac{\partial V(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + \mu S(t)\frac{\partial V(S(t),t)}{\partial S(t)} - rV(S(t),t) = 0$$

then $V(S(t), t)$ can be written as an expectation and

$$V(X(T), T = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(X(u),u)du}\, V(X(T), T)\,\Big|_{\mathcal{F}_t}\right]$$

$\blacksquare$

Thus, by Fayman−Kac's theorem if $V(X(T), T)$ satisfies the Black–Scholes PDE then under the probability measure $\mathbb{Q}$

$$V(X(T), T) = \mathbb{E}_{\mathbb{Q}}\,[e^{r(T-t)}(max(S(T) - K, 0))|_{\mathcal{F}}\;] \qquad \text{……………………………4.40}$$

4.6 Option Pricing by Discounting

Consider an investor who purchases a European call option written on the stock $S(t)$. At expiration if $S(T) > K$, the holder exercise the option and receives $C(T) = S(T) - K$. On the otherhand, if $S(T) < K$ then he receives $C(T) = 0$. The investor's payoff function can be written as

$$C(T) = max(S(T) - K) = (S(T) - K)^{+} \qquad \text{…………………………………4.41}$$

Since at $t = 0$ we do not know $C(T)$, we would rather want to discount the option to time $t = 0$. In otherwords we want the Present Value of the option.

Suppose we invest an amount $M(t)$ at time t in a savings account. In the subsequent small time $(t, t + \Delta t)$, the balance $M(t)$ in the account becomes $M(t + \Delta t)$. The return $R(t)$ on the investment μ can be expressed as the ratio

$$\frac{M(t + \Delta t)}{M(t)}$$

……………………………………4.42

Now in the limit as Δt approaches 0, we can write Equation 5.42 as

$$\frac{dM(t)}{M(t)}$$

The interest rate or the return on the investment now satisfies

$$R(t) = \frac{dM(t)}{M(t)dt}$$

$$\int_t^T \frac{dM(s)}{M(s)} = \int_t^T R(s)ds$$

$$ln\frac{M(T)}{M(t)} = \int_t^T R(s)ds$$

$$M(T) = M(t)exp\left(\int_t^T R(s)ds\right)$$

$$M(t) = M(T)exp\left(-\int_t^T R(s)ds\right)$$

…………………………………..4.43

The expression $exp\left(-\int_t^T R(s)ds\right)$ is called the discount factor of the amount $M(t)$. Thus, in valuing the option we calculate the expected present value (at time $t = 0$) by multiplying the option payoff $(S(T) - K,)^+$ at maturity ($t = T$) by the discount factor $exp\left(-\int_t^T R(s)ds\right) =$

e^{-Rt}. Thus, valuing the option reduces to computing the present value of the option as the expectation of the final value $[max(S(T) - K, 0)]$ of the option discounted at the $R(t)$ and under the filtration $\mathcal{F}_t$ so that

$$V(S(t), T) = \mathbb{E}\left[exp\left(-\int_t^T R(s)ds\right)(S(T) - K, 0)^+\bigg|\mathcal{F}_t\right]$$

It happens that under risk_neutral pricing $R(t)$ is the risk_free interest rate r so we write

$$V(S(t), T) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_t^T r(s)ds\right)(S(T) - K, 0)^+\bigg|\mathcal{F}_t\right]$$

4.7 Average Value or Asian Options

An Average Value or Asian option is the case where the option's payoff depends on the average of underlying stock asset. Asian option was introduced on the Asian market in the early 1970s to combat the common problem in European options where speculators could drive up the prices before maturity. The averaging procedure in Asian options reduces the significance of the closing price and reduces the effects of abnormal price changes at the maturity of the option. Most Asian options are traded in a discretely sample data, but the discrete sample case can be approximated by the continuous model. Generally, Asian options are less expensive than their European counterparts and are therefore more attractive to many different investors. Asian options come in two basic forms as regards its terminal payoff. Given an underlying stock $S(t)$ with average A and strike price K the payoffs of the option is

$$C(S(t), t) = max(A - K, 0)$$

The average can be arithmetic or geometric and can be structured as discrete or continuous.

4.7.1 Discrete Asian Options

In discrete averages options the stock price average at times $0 \leq t_1 < t_2 \dots . < t_n \leq T$ is taken over the period of the option's life. These times may be daily, weekly or monthly. Arithmetic discrete average

$$S(t)_{avg} = \frac{1}{n} \sum_{i=1}^{n} S(t_i)$$

………………………………….…………..4.44

 Geometric discrete average

$$S(t)_{avg} = \left[\prod_{i=1}^{n} S(t_i) \right]^{\frac{1}{n}}$$

…………………….……………………..4.45

4.7.2 Continuous Asian Options

In the continuous Asian option the average is integrating the price path over the time period.

Arithmetic continuous average

$$S(t)_{avg} = \frac{1}{T-t}\int_t^T S(u)du$$

…………………………………..4.46

 Geometric continuous average

$$S(t)_{avg} = exp\left[\frac{1}{T-t}\int_t^T InS(u)du\right]$$

…………………….………..4.47

4.7.3 Pricing Asian Options

Asian options are priced by representing the option value as an expectation or integral. Representing options as expectations allows the use of numerical methods such as Monte Carlo simulations. The idea is to simulate large number of the option prices and expect that as n gets large the price will converge to the expected price as dictated by the law of large numbers. In this approach lies that fact that we must describe the dynamics of the asset price not as we observe them but as they would be under a risk-adjusted/neutral probability measure. In pricing European options under the Black-Scholes, in simulating the stock's path, it is assumed that the path depends only on the terminal stock price $S(T)$ and initial stock prices $S(0)$. This sort of modeling ignores the fact that the payoff of the option may depend on the intermediate values of the underlying asset. Asian option accounts for the option by considering all values at all points of the stock price in the interval $[0, T]$. We thus price the Asian option by replacing the terminal stock price by the average stock price in the time interval $[0, T]$.

Consider an underlying stock $S(t)$ following the diffusion process and having the stochastic differential equation

$$dS(t) = rS(t)dt + \sigma S(t)dB(t) \qquad \text{……………….……………..4.48}$$

with average stock price A and strike price K. The arbitrage free price of an Asian option sold on the stock $S(t)$ under the risk neutral valuation is given by

$$C(S(t), t) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_t^T r(s)ds\right)\left[max(A - K, 0)\Big|_{\mathcal{F}_t}\right]\right]$$

……………….……………..4.49

where $t \geq 0$ and $\mathcal{F}_t$ is the filtration for the Brownian motion $B(t)$ and the expectation is taken with respect to the equivalent martingale measure $\mathbb{Q}$.

The option price thus depends on the nature of the average of the underlying asset. As has been explained earlier, currently the average is determined either by arithmetic average or geometric average. In the case of arithmetic average, no analytical tractability has been found and so a closed−form solution does not exist. For this reason, several numerical methods have been developed to obtain the option price. These methods include Monte Carlo simulation, binomial methods, numerical solutions to PDEs, Fourier transform techniques, inversion of non−trivial Laplace transforms, etc. On the other hand, options based on geometric average have a closed form solution of the price can obtained analytically.

If the stock price process is discrete then to obtain the average stock price we simulate the paths of $S(t_1), S(t_2), \ldots S(t_n)$ and compute the average. For continuous Asian option the average is the continuous average in the interval $[0, T]$ and is given by

$$\bar{S}(t) = \frac{1}{T}\int_0^T S(\tau)d\tau$$

4.7.4 Some Closed−Form Models of Geometric Average Asian Options The option payoff of an Asian option is given by

$$C(S(t),t) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_t^T r(s)ds\right)\left[max(A-K,0)\Big|_{\mathcal{F}_t}\right]\right]$$

……………………..4.50

It follows that the option price can be represented as an integral of the form

$$C(S(t),t) = e^{-rT}\int_{-\infty}^{\infty} max(x-K)\,f(x)dx$$

$$C(S(t),t) = e^{-rT}\int_K^{\infty}(x-K)f(x)dx \quad \text{…………………..4.51}$$

where $f(x)$ is the true but unknown density function of the average stock price. The general idea is to approximate $f(x)$ by some known probability distribution whose parameters and moments are well known.

- Turnbull and Wakeman approximated $f(x)$ by the lognormal distribution of the generalized Edgworth series

$$f(x) = g(x) + \frac{\mathcal{K}_2(F)-\mathcal{K}_2(G)}{2!}\frac{d^2g}{dx^2}(x) - \frac{\mathcal{K}_3(F)-\mathcal{K}_3(G)}{3!}\frac{d^3g}{dx^3}(x) + \frac{\mathcal{K}_4(F)-\mathcal{K}_4(G)+3\left(\mathcal{K}_2(F)-\mathcal{K}_2(G)\right)^2}{4!}\frac{d^4g}{dx^4}(x) + \mathcal{E}(x)$$

………………………………..4.52

where $\mathcal{E}(x)$ is the residual term and $\mathcal{K}_1(G)$ has been set equal to $\mathcal{K}_1(F)$ and $g(x)$ is the lognormal distribution. The option price is

$$C(S(t),t) = e^{-rT}\int_K^\infty (x-K)g(x)dx + \frac{\mathcal{K}_2(F)-\mathcal{K}_2(G)}{2!}\frac{d^2g}{dx^2}(k)$$

$$-\frac{\mathcal{K}_3(F)-\mathcal{K}_3(G)}{3!}\frac{d^3g}{dx^3}(k)$$

$$+\frac{\mathcal{K}_4(F)-\mathcal{K}_4(G)+3\left(\mathcal{K}_2(F)-\mathcal{K}_2(G)\right)^2}{4!}\frac{d^4g}{dx^4}(k)+\mathcal{E}(x)$$

……………………………..4.53

where G is the approximating lognormal distribution. The option price is thus given by

$$C(S(t),t) = e^{-rT}\frac{S_0}{n}\left\{\nu + \frac{1}{2}\lambda^2\Phi(d_1) - K\Phi(d_2) + \frac{\mathcal{K}_2(F)-\mathcal{K}_2(G)}{2!}g(k) - \frac{\mathcal{K}_3(F)-\mathcal{K}_3(G)}{3!}\frac{dg}{dx}(k)\right.$$

$$\left.+\frac{\mathcal{K}_4(F)-\mathcal{K}_4(G)+3\left(\mathcal{K}_2(F)-\mathcal{K}_2(G)\right)^2}{4!}\frac{d^2g}{dx^2}(k)\right\}$$

…………………..……………..4.54

where ϕ is the standard normal distribution with mean 0 and variance 1.

$$\phi(d_1) = \frac{\nu + \lambda^2 - InK}{\lambda}$$

$$\phi(d_2) = d_1 - \lambda$$

- Milevsky and Posner (1998) approximated $f(x)$ with the reciprocal gamma distribution given

$$C(S(t),t) = e^{-rT}\int_{-\infty}^{\infty} max(x-K)\, g_R(y)dy$$

……………..……………..…..4.55

where $g_R(y)$ is the reciprocal gamma distribution. The solution to Milevsky and Posner equation is

$$C(S(t),t) = exp(-rT)\mathbb{E}[A]G\left(\frac{1}{K}\bigg|_{\alpha-1,\beta}\right) - exp(-rT)KG\left(\frac{1}{K}\bigg|_{\alpha,\beta}\right)$$

$$C(S(t),t) = exp(-rT)\mathbb{E}[A]\Phi(d_1) - exp(-rT)K\Phi(d_2)$$

…………………………….4.56

where

$$\Phi(d_1) = G\left(\frac{1}{K}\bigg|_{\alpha-1,\beta}\right)$$

$$\Phi(d_2) = G\left(\frac{1}{K}\bigg|_{\alpha,\beta}\right)$$

- Vorst [1992] gave an exact pricing formula by approximating $f(x)$ with an adjusted strike price which is given as the difference in expectation of the arithmetic and geometric averages. The option payoff function is given as

$$C(S(t),t) = e^{-rT}\mathbb{E}[(G-K)^+]$$

$$C(S(t),t) = e^{-rT}\int_K^\infty (x-K)\,g(x)dx$$

………………………….………..4.57

where $g(x)$ is the lognormal density function of the geometric average G. $G \sim N(\mu_G, \sigma_G{}^2)$.

The price at time $t = 0$ of the geometric Asian option with strike price K and maturity T is

$$C(S(t),t) = exp(-rT)\left\{exp(\mu_G + \tfrac{1}{2}\sigma_G^2\right\}\Phi(d_1) - K\Phi(d_2)$$

……………………………….4.58

where

$$d_1 = \frac{\mu_G - InK + \sigma_G^2}{\sigma_G}$$

$$d_2 = d_1 - \sigma_G$$

- Curran (1994) conditioned on the geometric average and integrated with respect to its lognormal distribution. In Curran's method

$$C(S(t),t) = e^{-rT}\mathbb{E}[(A-K)^+]$$

$$C(S(t),t) = e^{-rT}\mathbb{E}\left[\mathbb{E}(A-K)^+ \mid_{G}\right]$$

$$C(S(t),t) = \int_0^{\infty} e^{-rT}\mathbb{E}\left[\mathbb{E}(A-K)^+ \mid_{G=x}\right] g(x)dx$$

……………….……………..4.59

where $g(x)$ is the lognormal density function of G. The lower bound is defined as

$$C_1(S(t),t) = \int_0^{K} e^{-rT}\mathbb{E}\left[\mathbb{E}(A-K)^+ \Big|_{G=x}\right] g(x)dx$$

and the upper bound is

$$C_2(S(t),t) = \int_K^{\infty} e^{-rT}\mathbb{E}\left[\mathbb{E}(A-K)^+ \Big|_{G=x}\right] g(x)dx$$

With solution $$C(S(t),t) = e^{-rT}[C_1(S(t),t) + C_2(S(t),t)]$$

The price of the Asian option with strike price K and maturity T is thus given by

$$C(S(t),t) = e^{-rT}\left\{\frac{1}{n}\sum exp\left(\mu_1 + \tfrac{1}{2}\sigma_1^2\right)\left(\frac{\mu_G - InL + \gamma_i}{\sigma_G}\right) - K\Phi\left(\frac{\mu_G - InK}{\sigma_G}\right)\right\}$$

…………………………..4.60

where

$$\mu_i = InS(0) + \left(r - q - \tfrac{1}{2}\sigma^2\right)t_i$$

$$\mu_G = InS(0) + \left(r - q - \tfrac{1}{2}\sigma^2\right)\frac{T+h}{2}$$

$$t_i = ih, \qquad h = \frac{T}{n}$$

q is the continuous dividend paid on $S(t)$. If $q = 0$ then

$$\mu_i = InS(0) + \left(r - q - \tfrac{1}{2}\sigma^2\right) t_i$$

$$\mu_i = InS(0) + \left(r - \tfrac{1}{2}\sigma^2\right) t_i$$

$$\mu_G = InS(0) + \left(r - \tfrac{1}{2}\sigma^2\right)\frac{T+h}{2}$$

$$t_i = ih, \qquad h = \frac{T}{n}.$$

Although the geometric and more widely the arithmetic average models have been used to price options they fail to address the problem when the variability in the underlying stock price is very low giving rise to low volatility stocks. In such a case the variability in the stock price process is not completely explained by the geometric and the arithmetic averages which are more suited to cases where the spread is uniformly distributed. In cases where the distribution is more crowded around the central value the use of the arithmetic or geometric average is unsuitable. We are convinced that for such cases the modal average is a better estimate of the average than the geometric or arithmetic average. We will proceed to develop a new model in the nest chapter and analytically show that the modal average is indeed a better model in the case of low volatility options.

CHAPTER 5

MAIN RESULTS 1−THEORETICAL RESULTS

5.1 The Modal Average of a Stock Price

In this chapter we would develop a model to price an average value option whose underlying stock price is based on the modal average. We begin by considering the distribution of the behaviour of prices of a typical stock listed on a stock exchange. We would observe the price behaviour through time and determine the average price over a given time interval use it as an underlying asset to price an option. We are guided in this derivation from the basic premise of a sample space on which the realization of a price event is captured a filtration of information onto a sample space and on which is defined a sequence of sigma algebras such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Such filtration process ensures that as time passes, more and more detailed information is revealed about the stock price as finer and finer partitions of omega is realised and eventually we know the true state of the stock price.

Consider the historical price behaviour of a stock as shown in Figure 5.1.

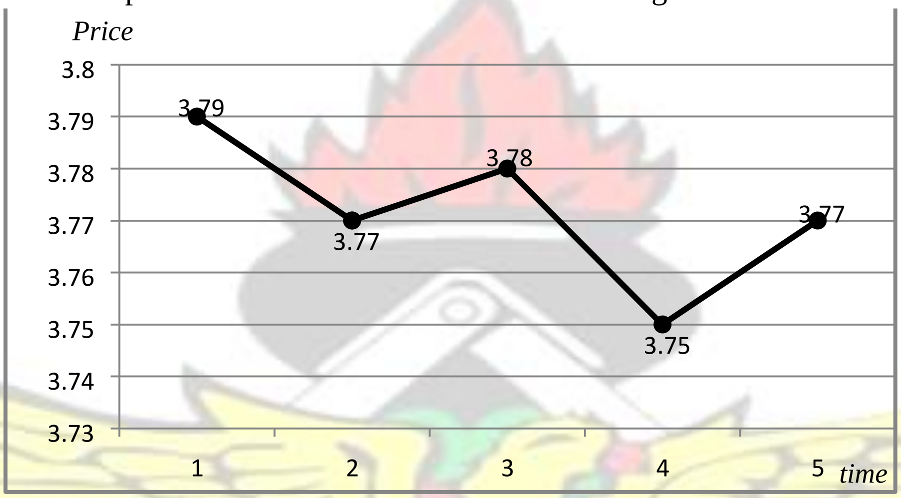

Figure 5.1 Movements of a stock price

Figure 5.1 shows the up and down movement of the stock price.

Let Ω denote the sample space for the stock price between time periods $[0, T]$, then $\Omega = 3.79$, $3.77, 3.78, 3.75, 3.77, \ldots \ldots$.

Let ω_i denote the outcome of a stock price at times $t = 1, 2, 3, \ldots \ldots$, then it follows that $\omega_i \in \Omega$.

Let $\omega_1 = 3.79$, $\omega_2 = 3.77$, $\omega_3 = 3.78$, $\omega_4 = 3.75$ $\omega_5 = 3.77$ then ω_i is an elementary outcome in Ω. Since ω can take any positive real value we can say that $\Omega = \mathbb{R}^+$. Let A denote the weekly prices of stocks, then A is a subset of Ω and the elementary outcome $\omega \in A$. We realise that the outcome of ω is generally not certain or predictable. We will thus always consider the next outcome of ω as the outcome of some random experiment whose underlying randomness is dictated solely by some act whose behaviour is based purely on chance. The occurrence of ω is seen as the realization of one particular outcome in a set of several possible outcomes. We will realise the occurrence of ω through some measurable space.

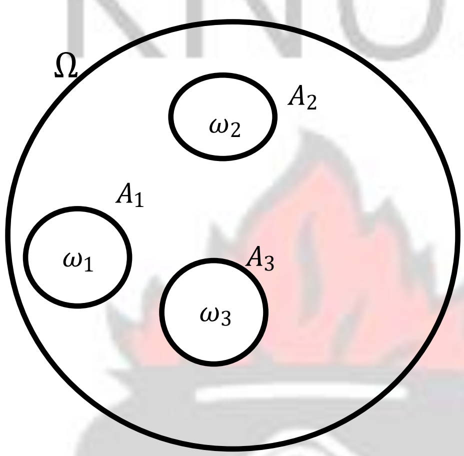

Figure 5.2 Measurable Space of a stocks

$A_1 = 3.79,$	$3.79,$	$3.77,$	$3.77,$	3.78
$A_2 = 3.78,$	$3.75,$	$3.75,$	$3.75,$	3.75
$A_3 = 3.75,$	$3.75,$	$3.77,$	$3.75,$	3.75
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$A_n = \omega_{n1},$	$\omega_{n2},$	$\omega_{n3},$	$\omega_{n4},$	ω_{n5}

Let $\mathcal{F}$ denote the collection of such subsets of Ω of the stock prices above. Clearly

i. $\phi \in \Omega$

ii. Given $A_1 = 3.79, 3.79, 3.77, 3.77, 3.78$ then $(A_1)^C = A_2, A_3 \ldots . . . , A_n$ also belongs to

$\mathcal{F}$. That is $A_1 \in \mathcal{F}_0 \Longrightarrow {A_1}^C \in \mathcal{F}$

iii. If $A_1 = 3.79, 3.79, 3.77, 3.77, 3.78 \in \mathcal{F}$ and $A_2 = 3.78, 3.75, 3.75, 3.75, 3.75 \in \mathcal{F}$, The union $A_1 \cup A_2 = 3.79, 3.79, 3.77, 3.77, 3.78, 3.78, 3.75, 3.75, 3.75, 3.75$ also belongs to $\mathcal{F}$. That is, $A_1 \in \mathcal{F}$ and $A_2 \in \mathcal{F} \Longrightarrow A_1 \cup A_2 \in \mathcal{F}$.

For now we would develop a measure to assign values to the $\mathcal{F}$ σ $-$algebra sets on Ω. The sample space Ω and the $\mathcal{F}$ $-$measurable sets form a measurable space for the stock prices. Let denote this space as $(\Omega, \mathcal{F})$. Let define a measure $\mu_s : \mathcal{F} \longrightarrow [0, \infty]$ such that

i. $\mu_s(\emptyset) = 0.$

ii. If $\{A_i, i \geq 1\}$ is a sequence of disjoint sets in $\mathcal{F}$, then the measure of the union (of countably infinite disjoint sets) is equal to the sum of measures of individual sets, i.e

$$\mu_s\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_s(A_i)$$

…………………………………………..5.1

5. 2 Measure Space of a Stock Price

The measure space of a stocks for the σ $-$algebra is defined as the triple $(\Omega, \mathcal{F}, \mu_s)$. μ_s is a measure or rule that assigns numerical values to the subset of Ω onto the semi_infinite interval $[0, \infty)$. Since $\mu_s(\Omega) < \infty$, it follows that μ_s is a finite measure. A random variable defined on $(\Omega, \mathcal{F}, \mu_s)$ is an f measurable function such that $X^{-1}(B) \in \mathfrak{B}(\mathbb{R})$. The probability of an event is defined as the Probability of a Random Variable X such that

$$\overline{\mathbb{P}}_{X_s} \triangleq \mathbb{P}\big(X^{-1}(B)\big) = \mathbb{P}(\omega \in \Omega | X(\omega) \in B)$$ …………………………5.2

The probability law thus defined for X gives the probability that the random variable X takes a value on the Borel set $\mathfrak{B}$. We can illustrate this in Figure 5.3.

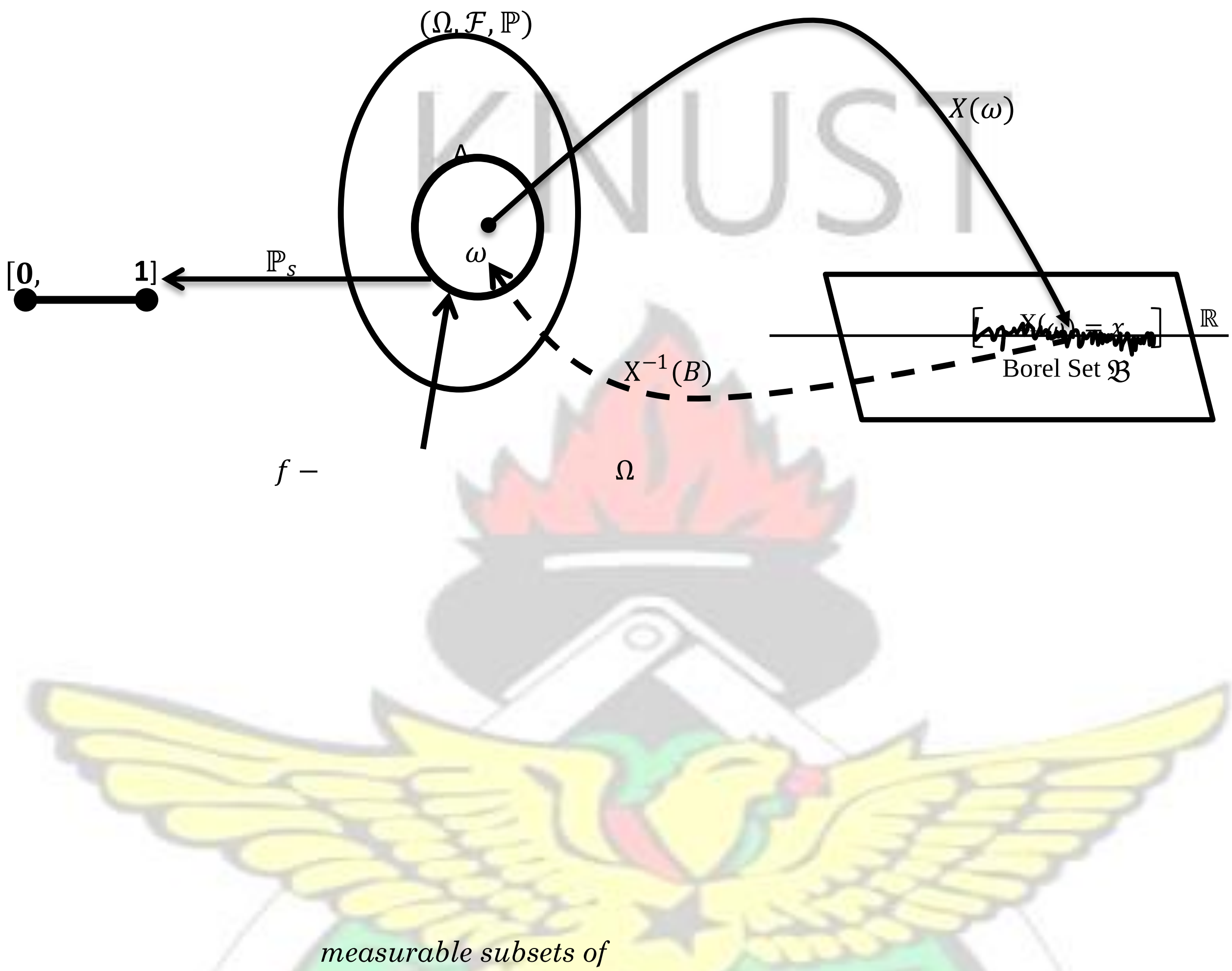

Figure 5.3. The Probability law of a Random Variable on Borel sets

In Figure 5.3, $\mathbb{P}$ is the probability measure on $(\Omega, \mathcal{F})$, that is from event ω to the probability space $[0,1]$ on the real line. $\overline{\mathbb{P}}_s$ is the measure from the Borel set $\mathfrak{B}$ on the real line onto the probability space $[0,1]$. $\overline{\mathbb{P}}_{s_X}$ is a composition of $\mathbb{P}$ with X^{-1}. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and X is a real-valued random variable, then, $\overline{\mathbb{P}}_X$ of X is a probability measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$.

In otherwords, the probability law for a random variable X is the probability measure induced by the random variable on the Borel $\sigma -$algebra on the real line $\mathbb{R}^+$.

5.3 Random Walk Process of a Stock Price

Suppose we observe the stock price movement starting from $t = 0$. In the next time step we realise that the stock price can either go up or go down and that such movements are based purely on chance. Let's denote the outcome of the stock price of the i^{th} stage by ω_i so that the outcome of the n^{th} successive price is $\omega_i = \omega_1, \omega_2, \dots, \omega_n$ and the infinite successive sequence of outcomes of ω is given by $\omega_i = \omega_1, \omega_2, \dots$..

Define a random variable ξ such that when the price goes up $\xi = 1$ and when the price goes down $\xi = -1$ and each case has a probability $= \frac{1}{2}$.

$$\xi_i = \begin{cases} 1 & if\ price\ goes\ up \\ -1 & if\ price\ goes\ down \end{cases}$$

……………………………………..…………..5.3

Since the outcome of $\omega_i{}'s$ is based on pure chance, it follows that ξ_i's are independently distributed. Let X_k be defined as

$$X_k = X(0) + \sum_{k=i}^{n} \xi_k$$

…………………………..………………..5.4

The process for the stock price is thus given by X_n, $n = 0, 1, 2, \dots$. and is said to follow the random walk. The path for the first five steps of the stock price movement in Figure 5.1 is the random walk process is shown in Figure 5.4.

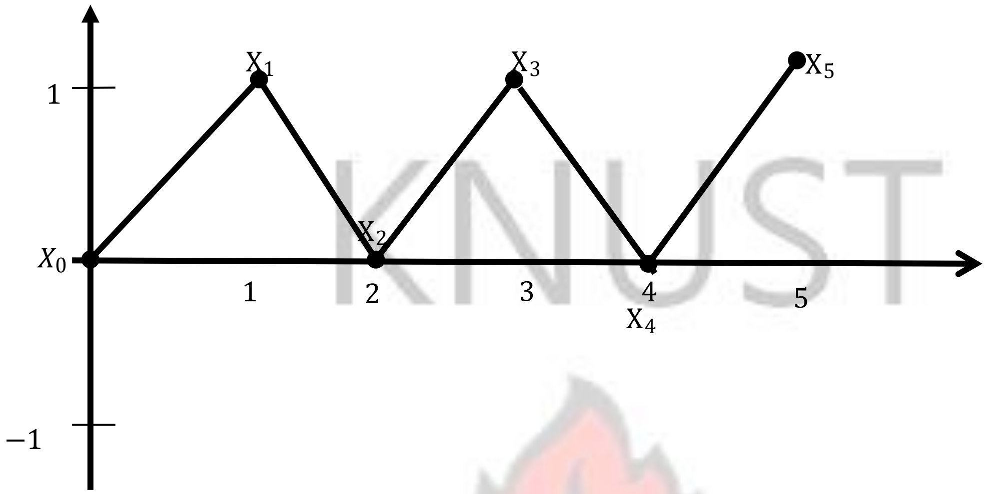

Figure 5.4 The first five steps of the random walk of the stock price

The random walk is thus characterized by the following: Each stage is independent and thus they have independent increments. This means that if we choose a non−negative integers $0 = k_0 < k_1 < k_2 < \cdots < k_n$, the random variables $X_{k_i} = (X_{k_1} - X_{k_0}), (X_{k_2} - X_{k_1}), \ldots, (X_{k_n} - X_{k_{n-1}})$ are independent. The random variable for the distribution of the increments is defined as

$$X_{k_{i+1}} - X_{k_i} = \sum_{i=k_j+1}^{k_j+1} X_k$$

……………………....…………………..5.5

The increments are normally distributed and independent as the underlying process governing their behaviour is purely random. The expectation $\mathbb{E}(X_{k_{n+1}} - X_{k_n}) = 0$ and $Var(\xi) = 1$ and so

5.4 Brownian Motion Realization of a Stock Price

Suppose in Figure 5.1 that we now speed up time i.e. reduce the time interval between changes in stock price so that the time between two price changes $t_i - t_{i-1}$ approaches zero and we increase the number of steps n of the random walk. This is possible in the event of increase in trading activity and automation of stock trading on an Exchange. It must be emphasized here that we are developing these models based on the assumption that the level of trading activity is high. The limit of the random walk is the Brownian motion process $B(t)$ and so limiting distribution function of the random walk is now inherited by the Brownian motion process which then assumes all the properties of the random walk with increments normally distributed with mean zero and variance t. The properties of Brownian motion are well documented in literature. The

Brownian motion distribution of a typical stock price on a stock exchange is shown in Figure 5.5

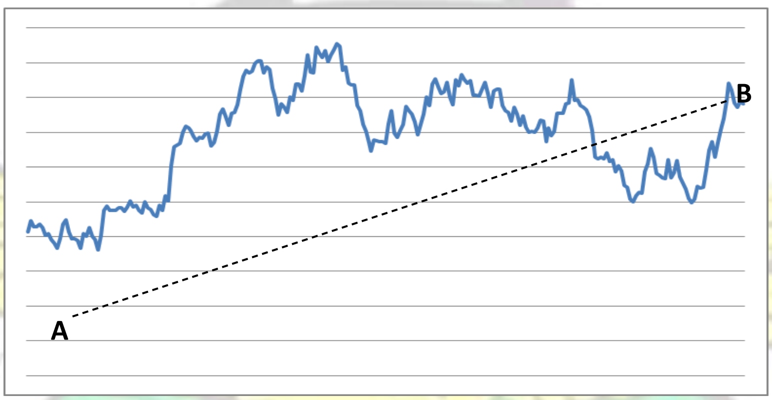

Figure 5.5 Brownian motion realization of a stock price

5.5 The Frequency Distribution of a Stock Price

Consider a stock with prices $\omega_0, \omega_1, \dots, \omega_n$ at times $0 = t_0 < t_1 < \cdots < t_n = T$.

Let $f_0, f_1, \dots, f_n$ denote the frequencies of $\omega_0, \omega_1, \dots, \omega_n$. Define Ω such that $\Omega = \{f_0, f_1, \dots, f_n\}$. Let A be a subset of Ω and let $\mathcal{F}$ denote a collection of subsets of $\omega_i's$ on Ω. Then surely, by earlier proposition $\mathcal{F}$ forms a σ −algebra of frequencies $f_i's$ on Ω with measurable space given by

$(\Omega, \mathcal{F})$. Let $\mathbb{P}$ denote a probability measure defined on this space, then $(\Omega, \mathcal{F}, \mathbb{P})$ is the measure space for the frequency of stock prices.

To develop the continuous process let ω be the realization of a stock price on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X(\omega)$ be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X: \Omega \to \mathbb{R}$, then $X(\omega)$ is a measurable function on $(\Omega, \mathcal{F}, \mathbb{P})$. By considering the frequencies of $X(\omega)$ we are now interested in some function of $X(\omega)$. The distribution of the frequency of $X(\omega)$ is thus a composite of some function f and $X(\omega)$, where $f: \mathbb{R} \to \mathbb{R}$. Let's denote this as $f(X(\omega))$. Since the pre-images $f^{-1}(X(\omega)) \in (\Omega, \mathcal{F}, \mathbb{P})$, it follows that $f\big(X(\omega)\big)$ are measurable in $(\Omega, \mathcal{F}, \mathbb{P})$ and consequently $f\big(X(\omega)\big)$ are random variables. If there are n random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, it follows that $f(X_1(\omega), X_2(\omega), \dots, X_n(\omega))$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Thus we have some random variables from some measure space being mapped onto some vector in $\mathbb{R}^n$, such that for any Borel sets in $\mathbb{R}^n$ the preimages $f^{-1}(X_1, X_2, \dots, X_n)$ are in $(\Omega, \mathcal{F}, \mathbb{P})$. The frequency distribution of X is now a mapping of X from $\mathbb{R}^n$ onto some vector space $\mathbb{R}^m$. Since f is Borel measurable it follows that $f(X_1, X_2, \dots, X_n)$ is a random vector.

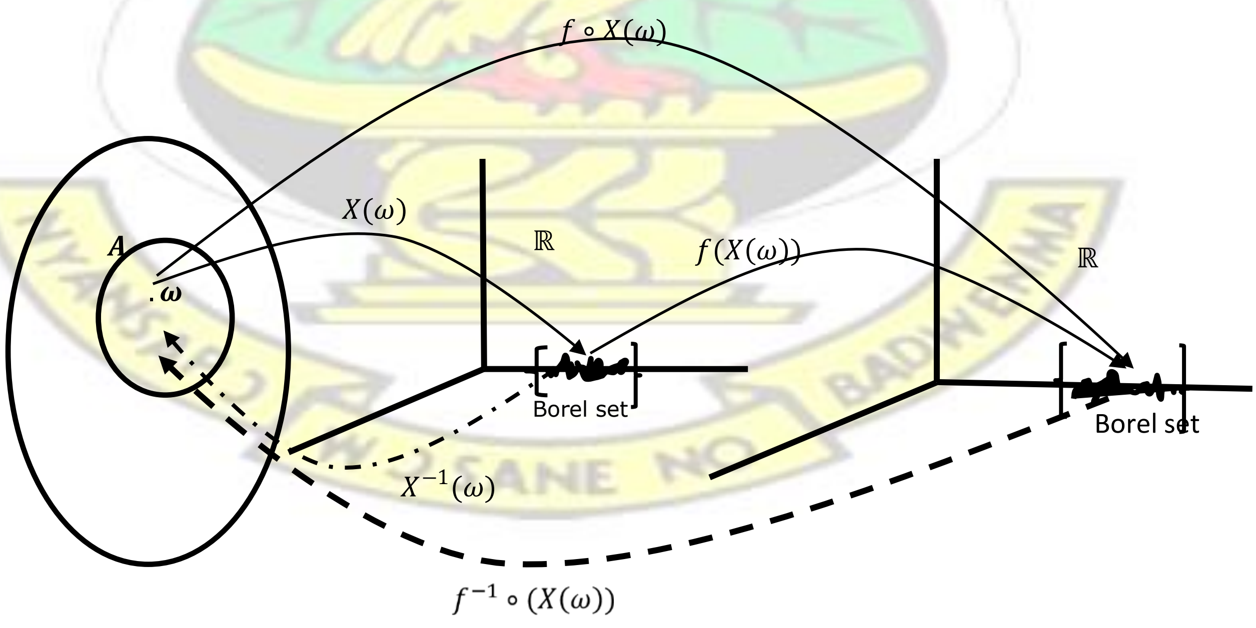

Figure 5.6 The frequency distribution of stock price on measure spaces

For each realization ω the random variable $X(\omega)$ from some underlying measure space $(\Omega, \mathcal{F}, \mathbb{P})$ is mapped onto $\mathbb{R}$. The pre$-$image of $X(\omega)$ of some Borel sets in $\mathbb{R}$ are measurable in $(\Omega, \mathcal{F}, \mathbb{P})$. The function f maps $X(\omega)$ onto some other measure space of $\mathbb{R}$. The pre$-i$mages of $f(X(\omega))$ are also measurable in $(\Omega, \mathcal{F}, \mathbb{P})$. For this study we are interested in the value of $x \in \mathbb{R}$ corresponding to the maximum frequency.

5.6 Mode of a Random Variable

Let $X_i \geq 0$ be a random variable. Let $\mathbb{P}(X_i) \geq 0$ be the frequency distribution of X_i then function $f: \mathbb{R} \to \mathbb{R}$ represents the probability distribution of X_i. The mode of this distribution is the value of X on the real line $\mathbb{R}$ corresponding to the maximum frequency of the distribution. If X represents the stock price and $\mathbb{P}(X_i)$ represent the frequency that the stock will assume the price X_i then we seek for the value of X that corresponds to the maximum frequency.

5.6.1 Mode of a Discrete Probability Distribution Function

Let $X_1, X_2, \dots, X_n$ be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{P}(x_1), \mathbb{P}(x_2), \dots, \mathbb{P}(x_n)$ be the probability distribution of $X_1, X_2, \dots, X_n$ then the function $f: \mathbb{R} \to \mathbb{R}$ represents the probability distribution of $X_1, X_2, \dots, X_n$. The mode of this distribution is the value of X corresponding to the maximum of the probabilities in the distribution. If X represent the stock price and $\mathbb{P}(x_1)$ frequency of occurrence of x or the probability that x occurs, then we seek for the numerical value of X corresponding to maximum $\mathbb{P}$.

5.6.2 Mode of Some Simple Probabilistic Experiments

Example 5.1

As an example we consider the simple toss of a single coin. Let the random variable X be the event that a head occurs. The possible outcome is a head H or a tail T. Let $X(\mathrm{H}) = 1$ and $X(\mathrm{T}) = 0$, then each outcome has a probability ½. We summarize the results in Table 5.1

$X = x$	0	1
$\mathbb{P}(X = x)$	1/2	1/2

Table 5.1 Probability distribution in a single toss of a coin

The graphical representation is given in Figure 5.7

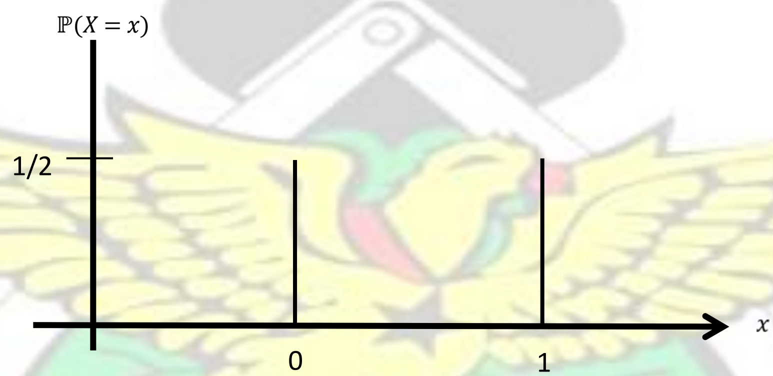

Figure 5.7 Mode of a single coin toss

The mode of the experiment is 0 or 1.

Example 5.2

A die is tossed once. Let the random variable X be the event that 4 occurs. The random variable X transforms the outcomes onto the real line as

$$X(4) = 1 \quad X(not\ 4) = 0, \quad \mathbb{P}(4) = 1/6 \quad \mathbb{P}(not\ 4) = 5/6$$

$X = x$	0	1

$P(X = x)$	5/6	1/6

Table 5.2 Probability distribution in a single toss of a die The graphical representation is given in Figure 5.8

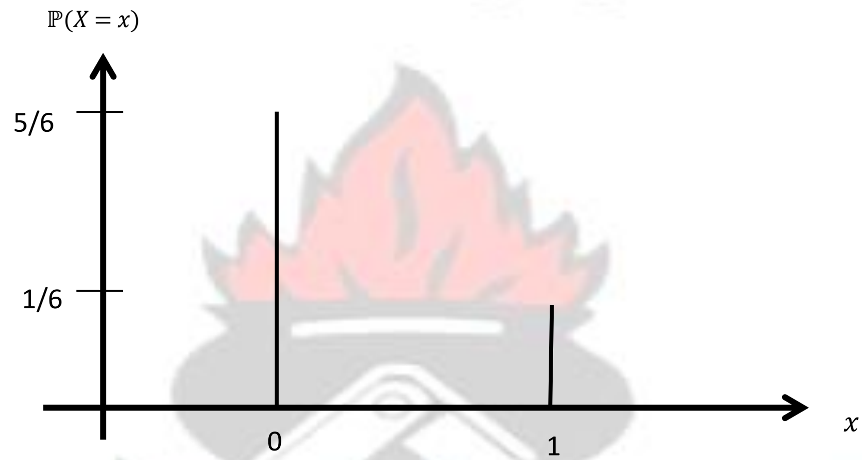

Figure 5.8 Mode of an event in a single die toss

The value of X corresponding to the maximum probability is 0 and hence the mode of the experiment is 0.

Example 5.3

Consider an experiment in which a pair of dice is tossed once. Suppose we define the random variable X as the sum of the outcomes. We wish to determine the mode of the distribution of the random variable X. The outcomes and the probability of the outcome is given in Table 5.3

$X = x$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Table 5.3. Table of probability distribution in a single toss of a pair of dice

The graphical representation is given in Figure 5.9

$P(X = x)$

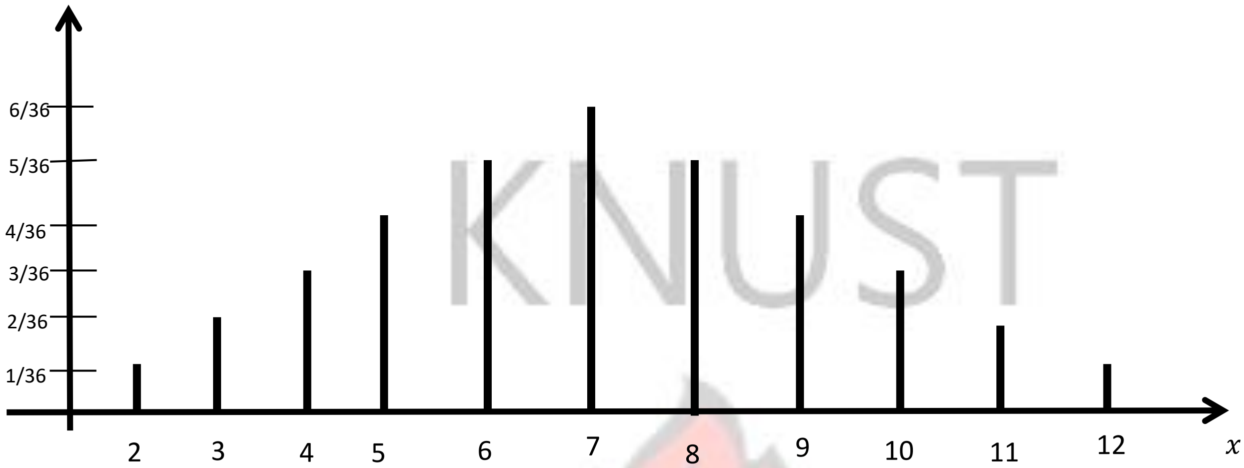

Figure 5.9 Mode of a single toss of a pair of dice

The mode of the distribution is the value of the random variable X that corresponds to the highest probability i.e. $X = 7$. Therefore the mode =7. This occurs when the sum of the two faces equals 7. The events leading to this outcome are $\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

5.6.3 Mode of Some Standard Discrete Probability Distribution Functions

The Binomial distribution

Consider the Binomial distribution

$$B(n,p;x) = \binom{n}{x} p^x (1-p)^{n-x}$$

We wish to determine the mode when $n = 5$ and $p = 0.4$

$$P(X = x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$P(X = 0) = \frac{5!}{(5-0)0!} 0.4^0 (1-0.4)^{5-0} = 0.0775$$

$$P(X = 1) = \frac{5!}{(5-1)1!} 0.4^1 (1-0.4)^{5-1} = 0.2592$$

$$P(X = 2) = \frac{5!}{(5-2)2!} 0.4^2 (1-0.4)^{5-2} = 0.3456$$

$$P(X = 3) = \frac{5!}{(5-3)3!} 0.4^3 (1-0.4)^{5-3} = 0.2304$$

$$P(X = 4) = \frac{5!}{(5-4)4!} 0.4^4 (1-0.4)^{5-4} = 0.0768$$

$$P(X = 5) = \frac{5!}{(5-5)5!} 0.4^5 (1-0.4)^{5-5} = 0.0102$$

$X = x$	0	1	2	3	4	5
$\mathbb{P}(X = x)$	0.0775	0.2592	0.3456	0.2304	0.0768	0.0102

Table 5.4 Probabilities generated by the Binomial distribution The graphical illustration is given in Figure 5.10

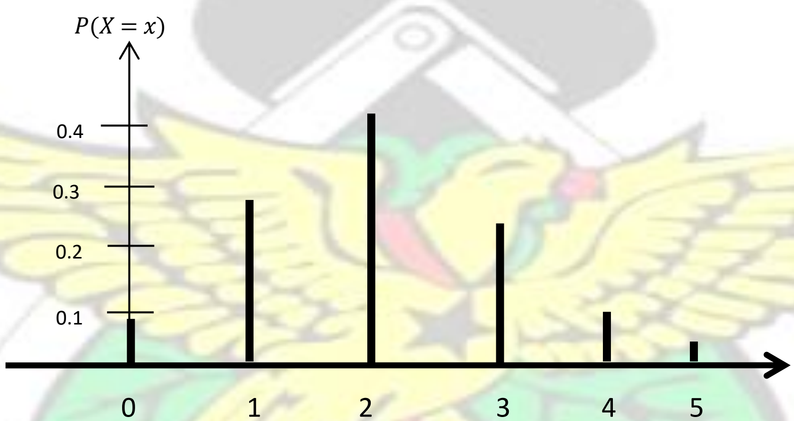

Figure 5.10 Mode of the Binomial Distribution

Clearly the numerical value of X that corresponds to the highest probability is 2, that is, $X = 2$.

Therefore Mode $= 2$

The Poisson distribution

Consider the Poisson distribution

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

Given $\lambda = 2.5$, we compute the Poisson probabilities as

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

$$P(X = 0) = e^{-2.5}\frac{2.5^0}{0!} = 0.0820$$

$$P(X = 1) = e^{-2.5}\frac{2.5^1}{1!} = 0.2050$$

$$P(X = 2) = e^{-2.5}\frac{2.5^2}{2!} = 0.2570$$

$$P(X = 3) = e^{-2.5}\frac{2.5^3}{3!} = 0.2130$$

$$P(X = 4) = e^{-2.5}\frac{2.5^4}{4!} = 0.1330$$

$$P(X = 5) = e^{-2.5}\frac{2.5^5}{5!} = 0.0670$$

$$P(X = 6) = e^{-2.5}\frac{2.5^5}{5!} = 0.0280$$

$$P(X = 7) = e^{-2.5}\frac{2.5^5}{5!} = 0.0100$$

$X = x$	0	1	2	3	4	5	6	7
$P(X = x)$	0.0820	0.2050	0.2570	0.2130	0.1330	0.0102	0.0280	0.0100

Table 5.5 Probabilities generated by a Poisson distribution.

The graphical representation is given in Figure 5.11

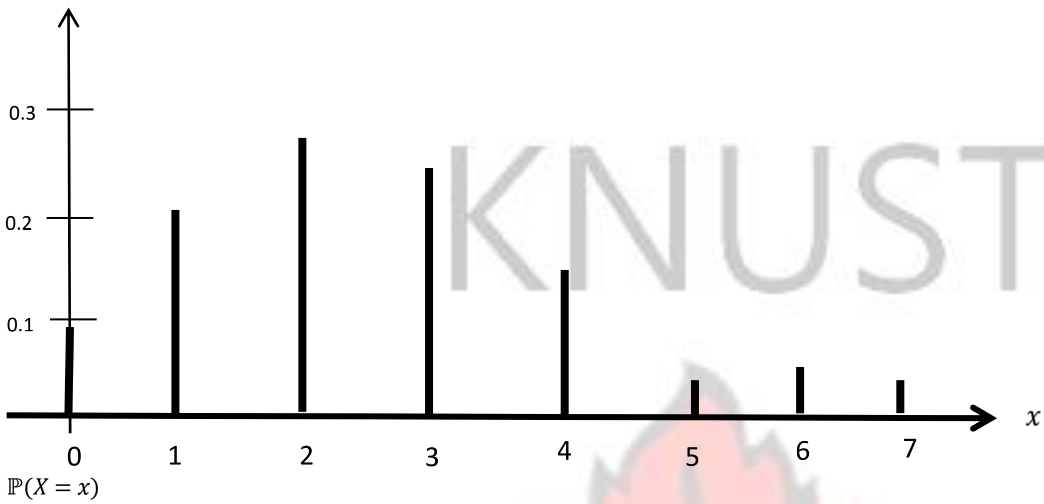

Figure 5.11 Mode of the Poisson Distribution

The mode of the distribution is $X = 2$

5.6.4 Mode of a Continuous Probability Distribution Function

Let $\mathbb{P}(x)$ be the function that assigns values to the distribution of the unique number X_i such that $\mathbb{P}(X_i) = \mathbb{P}^{max}$ is the maximum of the probability distribution of the random variable X, then X_i is the mode of $\mathbb{P}(x)$. That is $\{X_{mode} = X$ such that $\mathbb{P}$ is the maximum probability$\}$. We illustrate this in Figure 5.12

$\mathbb{P}(X)$

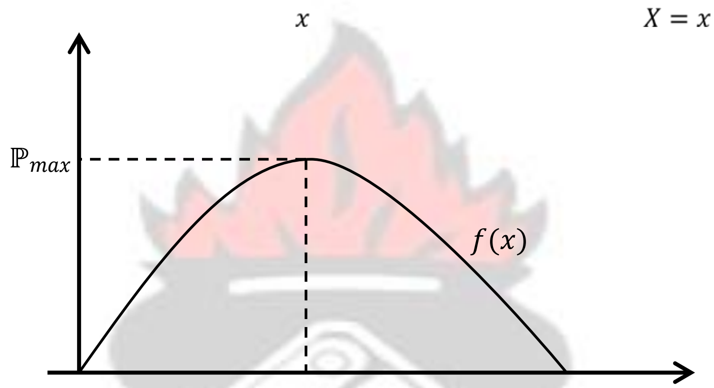

Figure 5.12 Mode of a Continuous Probability Distribution Function

In Figure 5.12 the mode of the random variable X is the value of x corresponding to the highest point on the probability density function or the point where the random variable is most dense. If $f(x)$ is a probability distribution function then the maximum of $f(x)$ is given by finding the stationary points of $f(x)$, conditioned by the fact that $\frac{d^2f}{dx^2} < 0$.

Example 5.4

Consider the $pdf \quad f(x) = \frac{1}{2} + 3x$ of the continuous random variable $X \geq 0$ as shown in Figure 5.13

$\mathbb{P}(x)$

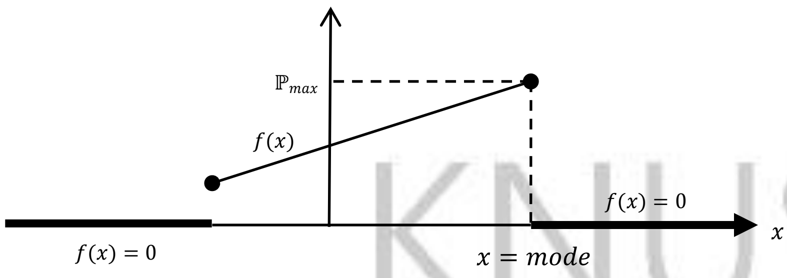

Figure 5.13 Mode of a linear function

In Figure 5.13 the mode of the straight line is the value of x corresponding to maximum of

$f(x) = \mathbb{P}_{max}$

Example 5.5

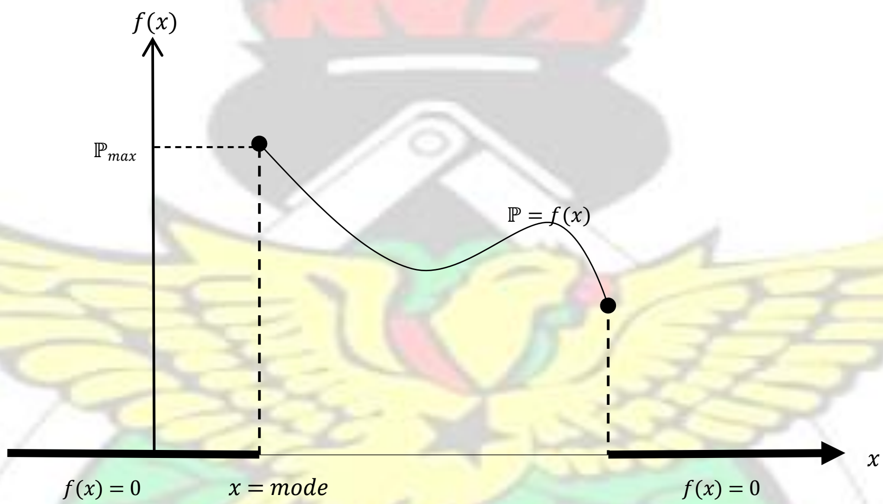

Figure 5.14 Mode of an arbitrary Curve

In Figure 5.14 the mode of the curve is the value of x corresponding to maximum of $f(x) = \mathbb{P}_{max}$

Example 5.6

Consider the curve as shown in Figure 5.15

$f(x)$

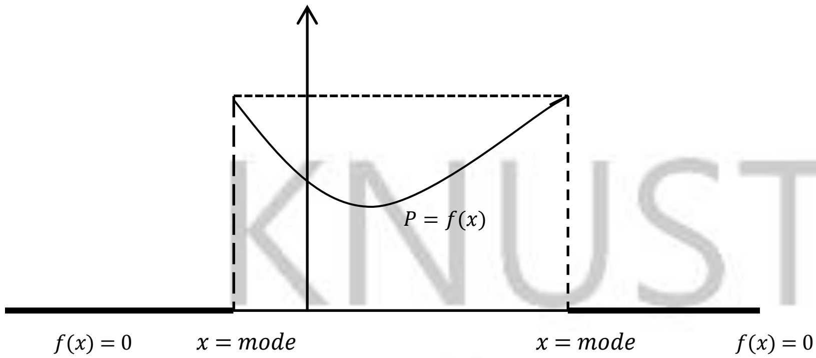

Figure 5.15   Mode of a bimodal curve

The curve has two values of x corresponding to two maximum points of $f(x)$. Hence $f(x)$ is bimodal

Example 5.7

Consider the function $f(x)$ with probability density function

$$f(x) = \begin{cases} \frac{3}{4}x^2(2-x) & 0 \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

The graph of $f(x)$ is shown in Figure 5.16

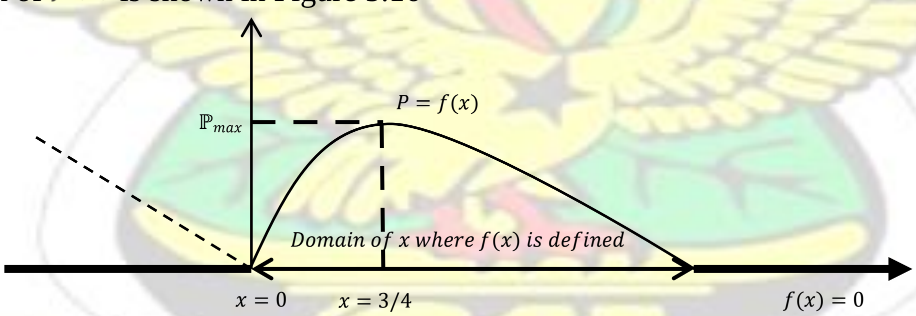

Figure 5.16   Mode of a probability density function

To determine the mode of $f(x)$ we first have to determine the stationary points of $f(x)$

Now $f(x) = \frac{3}{4}x^2(2-x)$

$$\frac{dy}{dx} = \frac{3}{4}(4x - 3x^2)$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{3}{4}(4x - 3x^2) = 0$$

The stationary points are $x = 0$ or $x = \frac{4}{3}$

Now to determine nature of the stationary point we have

$$\frac{d^2y}{dx^2} = 4 - 6x$$

Consider the stationary points

When $= 0$, $\frac{d^2y}{dx^2} = 4 - 6x = 4 - 6 \times 0 = 4 > 0$

When $x = \frac{4}{3}$, $\frac{d^2y}{dx^2} = 4 - 6x = 4 - 6 \times \frac{4}{3} = 4 - 8 = -4 < 0$

Since $\frac{d^2y}{dx^2} < 0$ when $x = \frac{4}{3}$ it follows that the maximum of $f(x)$ occurs when $x = \frac{4}{3}$

Hence the mode of $f(x)$ is $x = \frac{4}{3}$

Theorem 5.1

Let X be a continuous random variable such that $f(x)$ is the probability density function of X. Suppose $f(x)$ is smooth enough such that the first and second derivative exists. Let x be the value of X that maximizes $f(x)$ i.e. the value of x at $\frac{df}{dx} = 0$. If $\frac{d^2f}{dx^2} < 0$ for this value of x then x is the mode of the distribution function $f(x)$ and we write

$$\bar{X}_M = \frac{df}{dx}\Big|_x = 0 : \frac{d^2f}{dx^2}\Big|_x < 0$$

…………………………………..5.6

or

$$\bar{X}_M = \frac{df}{dx}\Big|_x = 0 \left| \frac{d^2f}{dx^2}\right|_x < 0$$

…………………………………..5.7

where $\bar{X}_M$ is the mode of $f(x)$.

5.7 Numerical Algorithm to Determine the Mode

We will employ numerical methods to determine the mode of a real valued function on a bounded real interval $[a, b]$. In effect, the mode is the value of the random variable X corresponding to the maximum of the function provided the function satisfies the axioms of a probability density function and smooth enough such that the first and second derivatives exist.

The problem thus reduces to solving the optimization problem of $maximize\ f(x)$ on $[a, b]$. i.e. we solve

$$max\ f(x)$$

subject to $$a \leq x \leq b$$

We note that if the first and second derivatives of $f(x)$ exists and is continuous on $[a, b]$ then it can be to solved by first computing all distinct zeros of $f'(x)$ in the interior of the interval $[a, b]$, and then evaluate $f(x)$ at these zeros and at the endpoints a and b and testing if $f''(x) = 0$. We should state here that the numerical approximation methods can be used to determine the maximum for both continuous as well as the discrete case.

5.7.1 The Golden Section Search Optimization Method

The Golden section search method is a technique for finding the extremum (minimum of maximum) of a strictly unimodal function. The algorithm is the limit of the Fibonacci search.

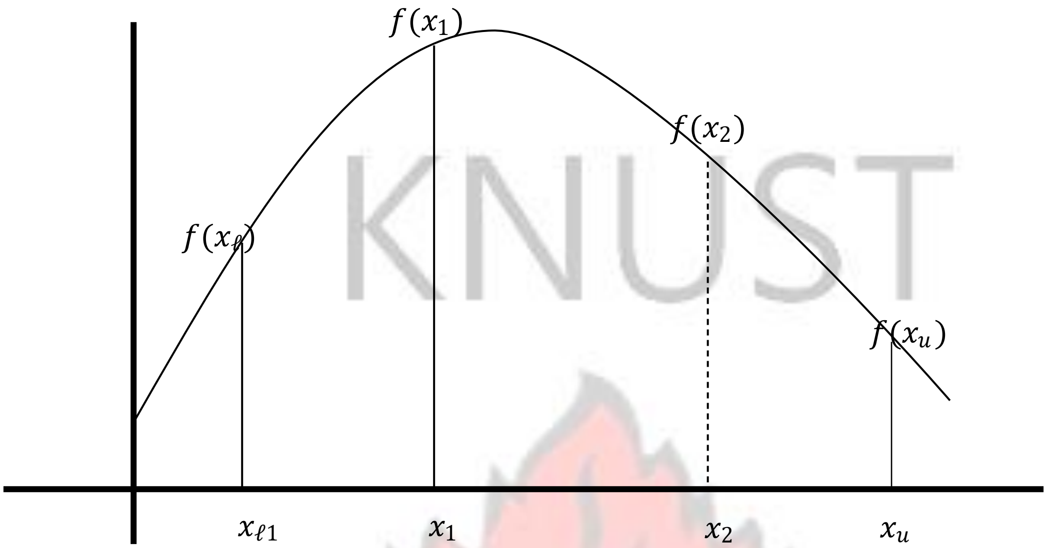

Figure 5.17 The Golden section search algorithm

Consider the values x_ℓ, x_1, x_u of the random variable X on the real line. Suppose$x\ \ell$ and x_u are the lower and upper limits respectively of X and x_ℓ, x_1, x_u have corresponding functional values $f(x_\ell)$, $f(x_1)$, $f(x_u)$ respectively as shown in figure 5.17. We realise that $f(x_1) > f(x_\ell)$, and $f(x_1) > f(x_u)$ and it follows that $max(f(x))$ lies inside the interval x_ℓ and x_u. For

If this was not the case then $f(x_\ell) < f(x_1) < f(x_u)$ and there is no turning point in the interval $[a, b]$ and $f(x)$ is a monotone increasing or decreasing function. Now we examine the function at a new point x_2 such that x_2 lies somewhere in the larger intervals of $[x_\ell, x_1]$ and $[x_1, x_u]$. Now the criterion for the examination of the function is as follows:

- If $f(x_1) > f(x_2)$, then we know that the lower bound on the function is $f(x_2)$ and it follows that the function maximum must be greater than x_2. Therefore the maximum of the function must lie in the range $[x_\ell, x_2]$. The three points to be examined are $[x_\ell, x_1, x_2]$
- On the otherhand if $f(x_1) < f(x_2)$, then we know that lower bound on the function is $f(x_1)$ and it follows that the function maximum must be greater than x_1. Therefore the maximum of the function must lie in the range $[x_1, x_u]$. The three points to be examined are $[x_1, x_2, x_u]$

To locate the two interior points x_1 and x_2 we employ Euclid's definition based on dividing a line into two segments so that the ratio of the whole line to the larger segment is equal to the ratio of the larger segment to the smaller segment. This ratio is called the Golden ratio. Thus, the values of x_1 and x_2 are not picked at random but based on the golden search. By the Golden ratio search algorithm we choose x_1 and x_2 such that each point sub-divides the interval of uncertainty into two parts where:

$$\frac{Length\ of\ whole\ line}{Length\ of\ larger\ segment} = \frac{Length\ of\ larger\ segment}{Length\ of\ smaller\ segment}$$

We subdivide the line segment as shown in Figure 5.18

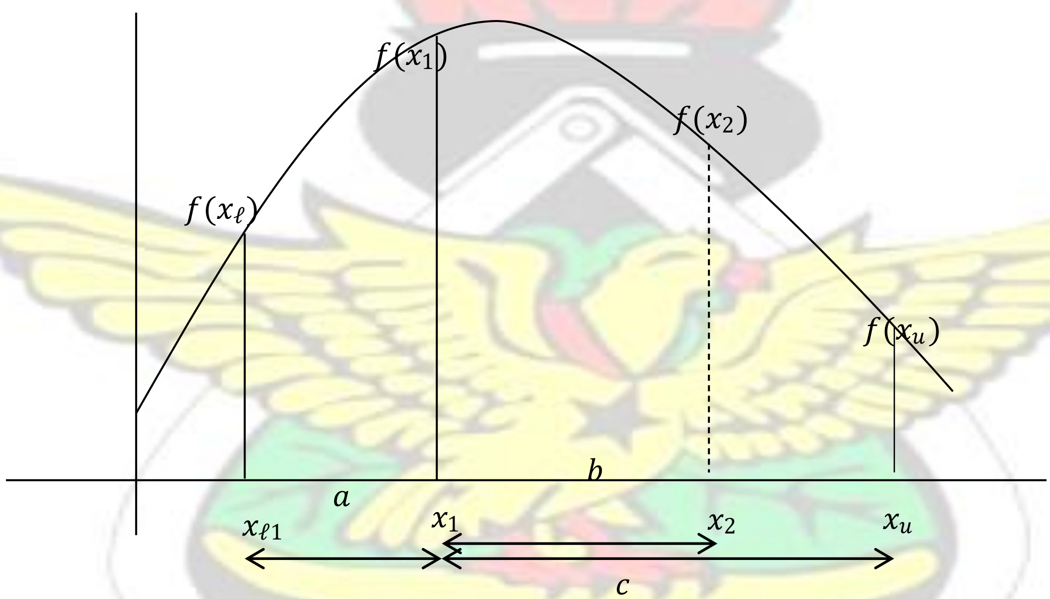

Figure 5.18 The Golden Ratio

From Figure 5.18 let a be the smaller segment and b be the larger segment of the line on which $f(x)$ is defined. Then by the golden ratio

$$\frac{a+b}{b} = \frac{b}{a}$$

$$a(a+b) = b^2$$

$$a^2 + ab = b^2$$

Let $= \frac{b}{a}$, then $1 + \frac{b}{a} = \left(\frac{b}{a}\right)^2$

$\varphi^2 - \varphi - 1 = 0$ $\quad$ $1 + \varphi = \varphi^2$

Solving for the positive roots of φ we have

$$\varphi = \frac{1+\sqrt{5}}{2}$$

The ratio $\varphi = \frac{b}{a} = \frac{1+\sqrt{5}}{2} = 1.618034$ is known as the Golden Ratio. If we select x_1 and x_2 such that they meet the Golden Ratio criteria then we can evaluate x_1 and x_2 as

$$x_1 = x_\ell + d$$

$$x_2 = x_u - d$$

where $d = (\varphi - 1)(x_u - x_\ell) = \left(\frac{1+\sqrt{5}}{2} - 1\right)(x_u - x_\ell) = 0.618034(x_u - x_\ell)$

$$x_1 = x_\ell + 0.618034(x_u - x_\ell)$$

$$x_2 = x_u - 0.618034(x_u - x_\ell)$$

We again evaluate $f(x_2)$ and $f(x_1)$. if $f(x_1) < f(x_2)$, then the new region of interest will be$[x_1, x_2, x_u]$. We then recalibrate the intervals as follows $x_1$becomes new x_ℓ

x_2 becomes new x_1

x_u remains x_u

And the new $x_2 = x_u - 0.618034(x_u - x_\ell)$

We then calculate $x_u - x_\ell$. If $x_u - x_\ell <$ then we stop the iteration and determine

$$x_{max} = \frac{x_u + x_\ell}{2}$$

Otherwise we re-evaluate the function at the new x_1 and x_2 and repeat the process

x_{max} is the value of x corresponding to the maximum $f(x)$ and therefore x_{max} is the mode of the distribution. The algorithm can be summarized as follows

Initialization:

Consider the function $f(x)$. Determine x_ℓ and x_u which is known to contain the maximum of the function $f(x)$.

Step 1

Determine two intermediate points x_1 and x_2 such that they satisfy the Golden Ratio criteria

$$x_1 = x_\ell + d$$

$$x_2 = x_u - d$$

where $d = (\varphi - 1)(x_u - x_\ell) = 0.618034\ (x_u - x_\ell)$,

Step 2

Evaluate $f(x_1)$ and $f(x_2)$.

If $f(x_1) < f(x_2)$, then the new region of interest will be $[x_1, x_1, x_u]$,

x_1 becomes new x_ℓ

x_2 becomes new x_1

x_u remains x_u

and the new $x_2 = x_u - 0.618034(x_u - x_\ell)$

Else if $f(x_2) < f(x_1)$, then the new region of interest will be $[x_\ell, x_1, x_2]$.

x_2 becomes new x_u

x_1 becomes new x_2

x_ℓ remains x_ℓ

and the new $x_1 = x_\ell + 0.618034(x_u - x_\ell)$

Step 3

If $x_u - x_\ell <$ (a sufficiently small number), then the maximum occurs at

$$\frac{x_u + x_\ell}{2}$$

and we stop the iteration, else go to Step 2.

Example 5.8

Consider probability density function defined by

$$f(x) = \begin{cases} \dfrac{3}{4}x^2(2-x) & 0 \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

We wish to find the value of x corresponding to the maximum of $f(x)$ in the interval $[0, 2]$ given that $= 0.05$

Solution

Given the interval $[0, 2]$ we choose $x_\ell = 0$ and $x_u = 2$

We determine the two intermediate points x_1 and x_2 such that

$$x_1 = x_\ell + d$$

$$x_2 = x_u - d$$

where $d = (\varphi - 1)(x_u - x_\ell) = \left(\frac{1+\sqrt{5}}{2} - 1\right)(x_u - x_\ell) = 0.618034(x_u - x_\ell)$

$$x_1 = x_\ell + 0.618034(x_u - x_\ell)$$

$$x_1 = 0 + 0.618034(2.0 - 0) = 1.236068$$

$$x_2 = x_u - 0.618034(x_u - x_\ell)$$

$$x_2 = -3.0 - 0.618034(2 - 0) = 0.763932$$

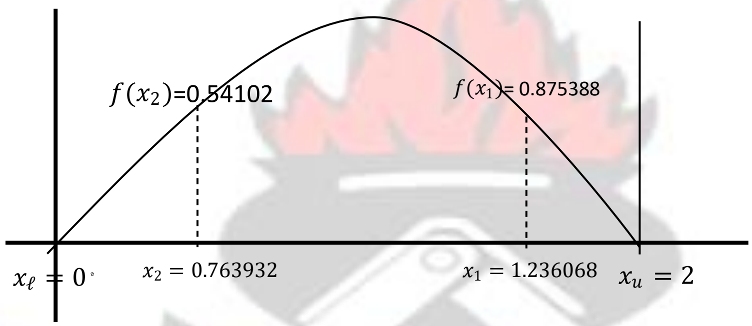

Figure 5.19 The Golden search algorithm of a pdf

$$x_u - x_\ell = 2 - 0 = 2 > 0.05$$

Hence we go back to Step 2.

Iteration 2

$$f(x_1) = f(1.236068) = \frac{3}{4}(1.23607)^2(2 - 1.23607) = 0.875388$$

$$f(x_2) = f(0.763932) = \frac{3}{4}(0.763932)^2(2 - 0.763932) = 0.54102$$

$f(x_2) < f(x_1)$ and so the lower boundary of $f(x)$ is x_2. Hence the maximum of $f(x)$ lies in the interval $[x_2, x_u]$. The new region of interest is $[x_2, x_1, x_u]$, and we recalibrate the x –axis as follows:

$x_2$becomes new $x_\ell = 0.763932$ x_1

becomes new $x_2 = 1.23607$ x_u remains

$x_u = 2.0$

Compute new x_1 as

$$x_1 = x_\ell + 0.618034(x_u - x_\ell)$$

$$x_1 = 0.763932 + 0.618034(2 - 0.763932)$$

$$x_1 = 1.527864$$

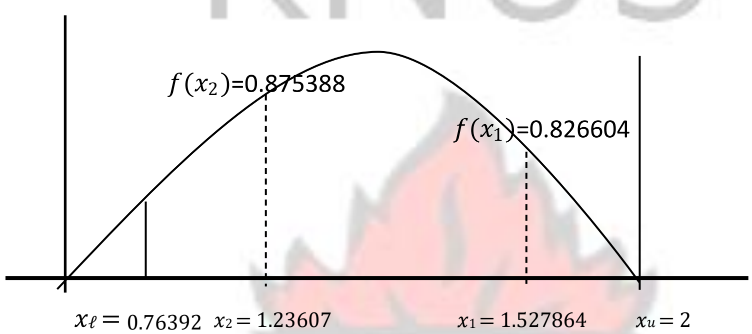

Figure 5.20 Iteration of the Golden section search algorithm

$$x_u - x_\ell = 2 - 0.763932 = 1.236068 > 0.05$$

we proceed to the next iteration using Step 2.

Iteration 3

$$f(x_1) = f(1.527864) = \frac{3}{4}(1.23607)^2(2 - 1.23607) = 0.826604$$

$$f(x_2) = f(1.236068) = \frac{3}{4}(0.763932)^2(1.527864 - 0.763932) = 0.875388$$

$f(x_1) < f(x_2)$ and so the lower boundary of $f(x)$ is x_1. Hence the maximum of $f(x)$ lies in the interval $[x_\ell, x_1]$. The new region of interest is $[x_\ell, x_2, x_1]$ and we recalibrate the x –axis as follows:

x_ℓ remains $x_\ell = 0.763932$ x_2

is the new $x_1 = 1.23607$ x_1

becomes new $x_u = 1.527864$

A new x_1 is recalculated as $x_2 = x_u - 0.618034(x_u - x_\ell)$

$$x_2 = 1.527864 - 0.618034(1.527864 - 1.23607) = 0.8291$$

$$x_u - x_\ell = 1.527864 - 0.763932 = 0.763932 > 0.05$$

We proceed to the next iteration using step 2.

Iteration 4

$$f(x_1) = f(1.527864) = \frac{3}{4}(1.527864)^2(2 - 1.23607) = 5.165437695$$

$$f(x_2) = f(0.829174775) = \frac{3}{4}(0.829174775)^2(2 - 0.763932) = 4.9418361$$

$$f(x_2) < f(x_1)$$

Hence x_2 is the minimum and $x_2 = x_\ell = 0.829174775$. The summary of the entire iteration results is given in Table 5.6

Iteration	X_ℓ	X_u	X_1	X_2	$f(X_1)$	$f(X_2)$	$X_u - X_\ell$
1	0	2.0	1.236068	0.763932	0.8753882	0.5410196	2
2	0.763932	2	1.5278641	1.2360679	0.8266045	0.8753882	1.236068
3	0.763932	1.527864	1.2360679	1.0557281	0.8753882	0.789337	0.763932
4	0.829174	1.527864	1.2609882	1.0960498	0.8813221	0.8144536	0.69869
5	0.829174	1.2609882	1.096050	0.9941123	0.8144536	0.7455584	0.4318142
6	0.9941123	1.2609882	1.159051	1.0960498	0.8472975	0.8144536	0.2668758
7	1.0960498	1.2609882	1.197987	1.1590507	0.8632706	0.8472975	0.1649383
8	1.1590507	1.2609882	1.222052	1.1979873	0.871347	0.8632706	0.1019375

Table 5.6 Table of values of The Golden section search algorithm

On the 9th iteration $= x_u - x_\ell = 0.03343729 < 0.05$. Hence we stop and compute

$$x_{max} = \frac{x_u - x_\ell}{2} = \frac{1.05835737 - 1.024920074}{2} = \frac{0.03343729}{2} = 0.0167186$$

Thus the value of x corresponding to the maximum density function is

$$x_{max} = 0.0167186$$

Hence the mode of the function is $\bar{X}_M = x_{max} = 0.0167186$

We have obtained an algorithm to determine the maximum value of a probability density function. we will now proceed to develop the modal average method to obtain an option price.

5.8 The Modal Average Method in Pricing Asian Options

Suppose the stock price process follow the SDE

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dB(t) \quad \text{...5.8}$$

Where $\mu(t)$ is the mean and $\sigma(t)$ is the volatility of stock. If $\bar{X}_M$ is the modal average of the stock in the interval $[0, T]$ then under the physical measure $\mathbb{P}$ the price of an option on the stock is given by

$$C(S(t), t) = \mathbb{E}_{\mathbb{P}}\left[exp\left(-\int_t^T \mu(s)ds\right)\left[max(\bar{X}_M - K, 0)\Big|_{\mathcal{F}_t}\right]\right]$$

...5.9

where

$$C(S(t), t) = \mathbb{E}_{\mathbb{P}}\left[exp\left(-\int_t^T \mu(s)ds\right)\left[max\left(\bar{X}_M : \frac{df}{dx}\Big|_{\bar{X}_M} = 0 \left|\frac{d^2f}{dx^2}\right|_{\bar{X}_M} < 0 - K, 0\right)\Big|_{\mathcal{F}_t}\right]\right]$$

...5.10

Now by the fundamental theorem of arbitrage there exists an equivalent martingale measure $\mathbb{Q}$ such that the discounted stock price process is a martingale. It follows that for any replicating portfolio $V(S(t), t)$ the discounted value process is a martingale. Thus, for any admissible strategy the arbitrage free price of the option $C(S(t), t)$ is given by

$$C(S(t), t) = V(S(t), t) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_t^T r(s)ds\right)\left[max\left(\left(\bar{X}_M : \frac{df}{dx}\Big|_{\bar{X}_M} = 0\left|\frac{d^2f}{dx^2}\right|_{\bar{X}_M} < 0\right) - K, 0\right)\Big|_{\mathcal{F}_t}\right]\right]$$

……………………………...5.11

Where r is the risk free rate and the density transformation from $\mathbb{P}$ to the $\mathbb{Q}$ is obtained via Girsanov theorem. This $\mathbb{Q}$ is the risk neutral measure which was earlier derived via the Radon–Nikodym derivative. By Girsanov's theorem we write

$$dB^{\mathbb{Q}}(t) = dB^{\mathbb{P}}(t) + \theta(t)dt$$

$$dB^{\mathbb{P}}(t) = dB^{\mathbb{Q}}(t) - \theta(t)dt \quad \text{………………………………..5.12}$$

But $$dS(t) = \mu S(t)dt + \sigma S(t)dW^{\mathbb{P}}(t) \quad \text{…………………………..5.13}$$

Putting Equation 5.12 into 5.13

$$dS(t) = \mu S(t)dt + \sigma S(t)[dB^{\mathbb{Q}}(t) - \theta(t)dt]$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dB^{\mathbb{Q}}(t) - \sigma\theta(t)S(t)dt$$

$$dS(t) = (\mu(t) - \sigma(t)\theta(t))S(t)dt + \sigma(t)S(t)dB^{\mathbb{Q}}(t)$$ Let

$r(t) = \mu - \sigma\theta(t)$, so that

$$\theta(t) = \frac{r - \mu}{\sigma(t)}$$

Then $$dS(t) = rS(t)dt + \sigma S(t)dB^{\mathbb{Q}}(t) \quad \text{…………………………....5.14}$$ where

$B^{\mathbb{Q}}(t)$ is the Wiener process under the measure $\mathbb{Q}$ and $r(t)$ is the risk_free rate which we would assume equal to the 90_day sovereign treasury bill rate. The stock price process under the measure $\mathbb{Q}$ is

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB^{\mathbb{Q}}(t)$$

Now the discounted option price process under the equivalent martingale measure $\mathbb{Q}$ is

$$C(S(t),t) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_t^T r(s)ds\right)\left[max\left(\bar{X}_M : \frac{df}{dx}\Big|_x = 0\left|\frac{d^2f}{dx^2}\right|_x < 0 - K\right)\Big|_{\mathcal{F}_t}\right]\right]$$

……………..…………………..5.15

Define an indicator random variable $\mathbb{I}_{\mathbb{A}}$ by

$$\mathbb{I}_{\mathbb{A}} = \begin{cases} 1, \quad if \;\; \bar{X}_M : \frac{df}{dx}\Big|_x = 0\left|\frac{d^2f}{dx^2}\right|_x < 0 > K \\ \\ 0, \quad if \bar{X}_M : \frac{df}{dx}\Big|_x = 0\left|\frac{d^2f}{dx^2}\right|_x < 0 \leq K \end{cases}$$

Then under the risk_neutral measure $\mathbb{Q}$ the condition to exercise the option is given by

$$C(S(t),t) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_t^T r(s)ds\right)\left[max\left(\left(\bar{X}_M : \frac{df}{dx}\Big|_x = 0\left|\frac{d^2f}{dx^2}\right|_x < 0\right) - K, 0\right)\Big|_{\mathcal{F}_t}\right]\right] \times \mathbb{I}_{\mathbb{A}}$$

…..………..…………………..5.16

$$C(S(t),t) = \mathbb{E}_{\mathbb{Q}}\left[exp\left(-\int_t^T r(s)ds\right)\left[max\left(\bar{X}_M : \frac{df}{dx}\Big|_x = 0\left|\frac{d^2f}{dx^2}\right|_x < 0 - K, 0\right)\Big|_{\mathcal{F}_t}\right]\right] \times \left\{\left(\bar{X}_M : \frac{df}{dx}\Big|_x = 0\left|\frac{d^2f}{dx^2}\right|_x < 0\right) > K\right\}$$

……………..…………………..5.17

CHAPTER 6

MAIN RESULTS 2− OPTION PRICING USING MODAL AVERAGES

6.1 Monte Carlo Simulation of a Stock Price Process

We employ Monte Carlo simulations to obtain the price of an Asian option. By the law of large numbers as the sample size gets large the mean of the identically independently distributed random samples converges to the expected value (the population mean) which in effect, is the true option price. It is upon this premise that the Monte Carlo simulation works. We will first obtain results using the modal average and then simulate for options where the average is computed as arithmetic, geometric or median averages. For our model option whose payoff depends on the complete path $S(t_i), \ldots, S(t_n)$ at fixed times $t_i, \ldots, t_n$ it is important that we begin by simulating the path of the stochastic process describing the evolution of the stock price. Given a call option with payoffs $max(\bar{X}_M - K)$ we generate samples of $S(t)$ and determine the modal value of the stock. For the modal average we will simulate the path of $S(t_i), \ldots, S(t_n)$ and then compute the value of x corresponding to $max\,\{f_0, f_1, \ldots, f_n\}$, where f_i denote the frequency of the stock prices at times $t_i, \ldots, t_n$. The Arithmetic and geometric averages are similarly obtained using Equation 4.39 and Equation 4.40 respectively. It must be noted that although we use discrete models the large volume of simulation ensures that the process is approximately continuous. We begin by first generating sample price paths of the stock. To do this we assume the stock price to follow the GBM process in the risk−neutral world so that

$$dS(t) = rS(t)dt + \sigma S(t)dB(t) \quad \ldots\ldots 6.1$$

The Brownian motion (Wiener process) has the following properties:

The change $\Delta B(t)$ in a short time period Δt is

$$\Delta B(t) = \xi(t)\sqrt{\Delta t}$$

129 ……………………….…………..6.2

where $\xi(t)$ has a standard normal distribution $N(0,1)$. In this time notation the stock price process becomes

$$\Delta S(t) = rS(t)\Delta t + \sigma S(t)\xi(t)\sqrt{\Delta t}$$ ……………….…………..6.3

$$\frac{\Delta S(t)}{S(t)} = r\Delta t + \sigma\xi(t)\sqrt{\Delta t}$$

……………….…………..6.4

Thus the percentage rate of return of the stock price is normally distributed with mean $r\Delta t$ and variance $\sigma^2\Delta t$. That is,

$$\frac{\Delta S(t)}{S(t)} \sim N(r\Delta t, \sigma^2\Delta t)$$

……………….…………..6.5

Now given a function $f(S(t), t)$, we know by Ito process that

$$df(S(t).t) = \left(rS(t)\frac{\partial f(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 f(S(t),t)}{\partial S(t)^2} + \frac{\partial f(S(t),t)}{\partial t}\right)dt$$

$$+ \left(\sigma S(t)\frac{\partial f(S(t),t)}{\partial S(t)}\right)dB(t)$$

Applying the process to a stock price with $f = InS(t)$ gives

$$dInS(t) = \left(\frac{rS(t)}{S(t)} + 0 + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right)\sigma^2 S(t)^2\right)dt + \left(\frac{\sigma S(t)}{S(t)}\right)dB(t)$$

……………….…………..6.6

$$dInS(t) = \left(r - \tfrac{1}{2}\sigma^2\right)dt + \sigma dB(t)$$ ……….……………………..……..6.7

In discrete time this becomes

$$\Delta InS(t) = \left(r - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma\xi(t)\sqrt{\Delta t}$$ ……………….…………..6.8

This gives $In(S(t) + \Delta t) - InS(t) = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\xi(t)\sqrt{\Delta t}$ …………….…..6.9

$S(t + \Delta t) = S(t)\left\{exp[r\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\xi(t)\sqrt{\Delta t}\right\}$…………………..6.10

Equation 6.10 is the path constructing formula for the Monte Carlo simulation of the stock price To determine $S(t + \Delta t)$ we sample from the standard normal distribution to produce a sequence of independent standard normal variables. We estimate the return and the volatility of each stock empirically by computing the daily stock price (closing price) of the individual stocks for one year. We use a one year data (2015) from the NASDAQ in the United States of America from which we compute the daily returns of stock. From this we determine the annual return (μ) for each stock in the given year (2015) by multiplying the daily returns by the number of trading days. The annual standard deviation or volatility (σ) of each stock is similarly computed. We now simulate the daily price paths of each stock for the coming year (2016) using a trading time interval of ($\frac{1}{trading\ days}$). We assume that there are no price changes from the last trading day of

2015 and the first trading day of 2016. Thus, the initial stock price $S(0)$ for the simulation is the closing price for the last trading day of 2015. The procedure can be summarized as follows: 1. Compute the return μ of a stock

$$\mu = \frac{1}{n}\sum_{i=1}^{n} ln\left(\frac{S(t)_i}{S(t)_{i-1}}\right)$$

……………….…………...6.11

2. Find standard deviation (volatility) of each stock

$$Standard\ deviation = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\mu - \bar{\mu})^2}$$

……………….…………..6.12

3. Determine the time interval Δt, where

$$\Delta t = (\frac{1}{trading\ days})$$

4. Simulate the stock price path using

$$S(t + \Delta t) = S(t)\left\{exp[r\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\xi(t)\sqrt{\Delta t}\right\}$$

………………….…………..6.13

For each stock we simulate for several price paths by generating different sets of random numbers. Random numbers are uniformly distributed so we need to transform them into normal (lognormal) distribution by using the inverse distribution method. Realized Brownian motion paths for a stock with parameters $\mu = 0.00864$, volatility $\sigma = 4.23$, $S(0) = ¢5.30$ $\Delta t = \frac{1}{trading\ days} = \frac{1}{248}$ are shown in Figure 6.1−6.4. We generate several price paths but we show only a few here.

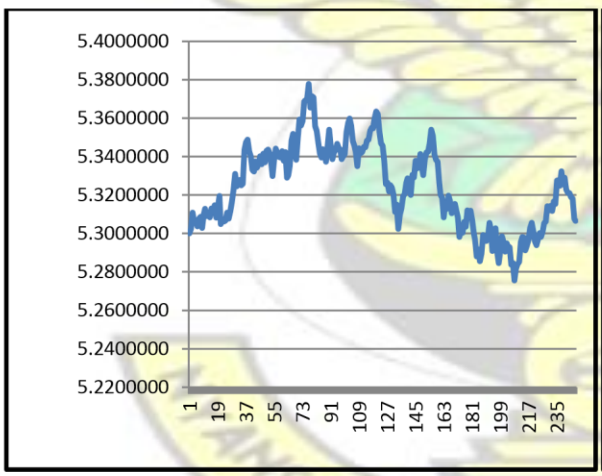

Figure 6.1 Stock price path for simulation *i*

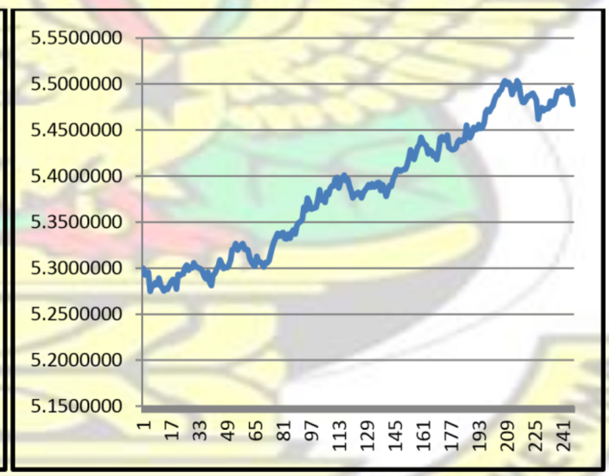

Figure 6.2 Stock price path for simulation *ii*

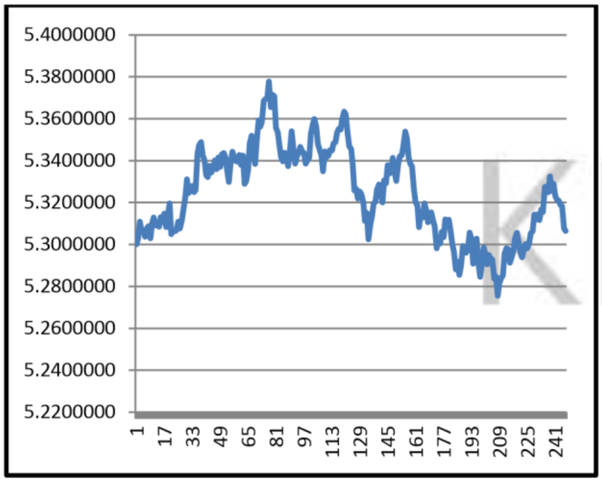

Figure 6.3 Stock price path for simulation *iii*

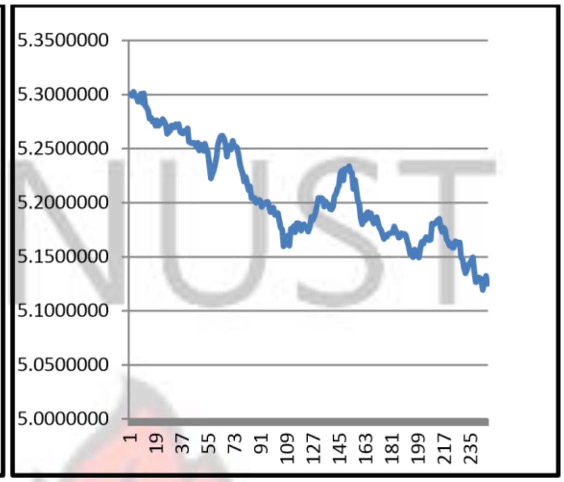

Figure 6.4 Stock price path for simulation *iv*

Simulation of the stock price process under measure $\mathbb{P}$ and under the equivalent martingale measure $\mathbb{Q}$ are also shown in Figure 6.2 (a, b).

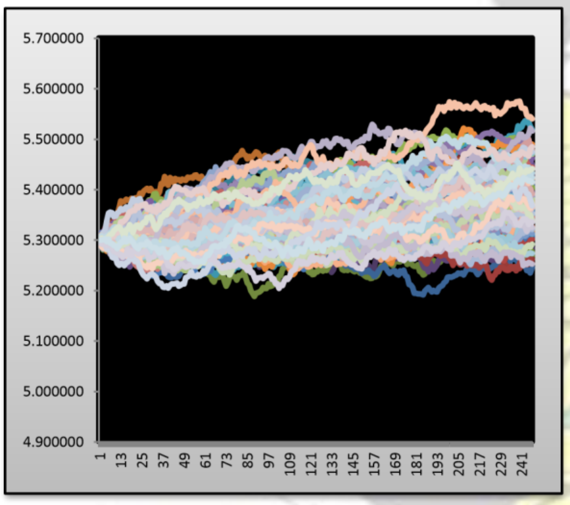

Figure 6.5 Stock price under measure $\mathbb{P}$

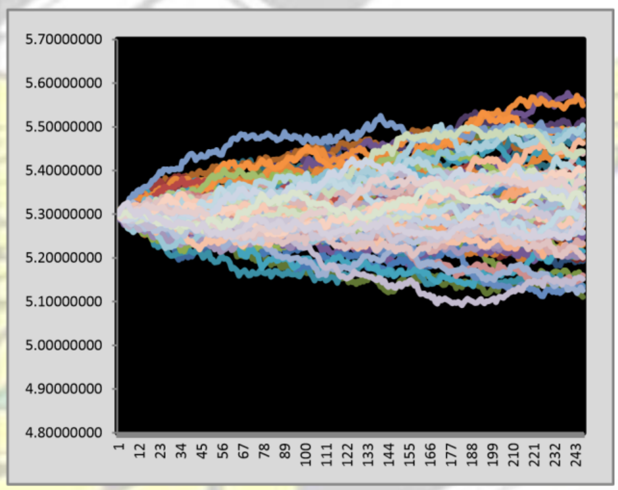

Figure 6.6 Stock price under the risk neutral measure $\mathbb{Q}$

$dS(t) = \mu S(t)dt + \sigma S(t)dB^{\mathbb{P}}(t)$ $\quad$ $dS(t) = rS(t)dt + \sigma S(t)dB^{\mathbb{Q}}(t)$

The density transformation from $\mathbb{P}$ to $\mathbb{Q}$ is given via Girsanov theorem in Equation 5.13 and Equation 5.14.

6.2 The Option Price

After simulating several paths of the stock price we obtain the average stock price as

$$S_{avg} = \frac{S_i(t_1) + S_i(t_2) + \cdots + S_i(t_m)}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 6.7$$

where S_{avg} is either arithmetic, geometric, modal or median average. We proceed to price a 3−month option on the stock using the modal, arithmetic, geometric or median average of the stock price as the underlying asset. Whichever average is used we obtain large samples of $C(S(t), t)'$s at times $t_i, \ldots\ldots, t_n$ and determine the option price as

$$C(S(t), t) = \frac{C_1(S(t), t) + C_2(S(t), t) + \cdots + C_n(S(t), t)}{n} = \frac{1}{n}\sum_{i=1}^{n} C_i(S(t), t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 6.8$$

6.3 Comparison of Modal Average Results with Existing Models

In the Monte Carlo simulations we first obtain option prices using modal average of some stocks listed on the NASDAQ as the underlying assets. We proceed to use the same data to obtain option prices using arithmetic, geometric or median averages. In addition, we also compute the option prices using the Black–Scholes formula. We proceed to compare the option price results using the modal average to all other models. The results are presented in Table 6.1 and Table 6.2.

Graphs of option prices against volatilities are also given in Figures 6.7−6.10.

6.3.1 Table of Values for Average Stock Price and Option Prices

Table 6.1 shows the average stock prices for a given volatility of selected stocks on NASDAQ. and Table 6.2 shows the corresponding option prices.

	Stock/Equity	Strike Price	Volatility	Average Stock Price			Price at Maturity
				Geometric	Arithmetic	Modal	S(T))
1	**BarclaysUS**	2.68	3.32	2.7120	2.7167	2.7136	2.7923
2	**1347PIH**	7.80	0.70	7.8418	7.8416	7.8344	7.8520
3	**Amazon**	749.87	5.33	770.5815	771.8901	768.0482	772.1159
4	**Apple**	115.82	3.44	117.7973	117.8611	118.6621	119.2983
5	**AT&T**	42.53	1.39	43.7633	43.7203	43.3087	42.7689
6	**BCOM**	20.18	1.18	20.7807	20.7763	20.5957	21.1045
7	**Facebook**	115.05	4.83	117.6367	118.0125	118.1533	122.5548
8	**Ford Motors**	12.13	4.39	12.4536	12.3851	12.3730	12.5558
9	**General Electric**	31.60	18.20	34.3291	35.2814	33.9188	38.2256
10	**General Motors**	34.84	4.12	35.5684	35.5536	35.4175	37.8018
11	**Intel Corp**	36.27	32.40	41.3124	41.3478	41.4441	42.3118
12	**Microsoft**	62.14	3.24	63.2238	63.0061	62.8056	62.7161
13	**NY Times**	13.30	2.75	13.4765	13.4862	13.4398	13.8560
14	**ODML**	85.79	4.53	87.7146	88.5126	87.6008	90.4254
15	**Starbucks**	55.52	5.00	56.7970	56.4879	56.6176	57.7570
16	**Tesla**	213.69	9.30	223.2820	220.8227	221.0606	228.1394
17	**Verizon**	51.40	16.53	54.8668	54.7580	55.3195	57.9553
18	**American Airlines**	51.40	12.00	53.7148	53.3479	53.7916	56.9553
19	**Airbus**	66.25	15.00	71.2220	69.4272	71.3456	73.1680
20	**Boeing**	155.68	13.68	165.0347	162.9145	164.1845	167.6839

Table 6.1 Average Stock Price of stocks on Nasdaq using Geometric, Arithmetic, Modal averages and the Terminal stock price

Table 6.2 shows the option prices of selected stocks listed on Nasdaq for a given strike price and volatility.

	Stock/Equity	Stock Parameter		Option Price			
		Strike Price	Volatility	Geometric	Arithmetic	Modal	Black Scholes
1	Barclays US 2	2.68	3.32	0.03202	0.03674	0.03360	0.04171

	1347PIH	7.80	0.70	0.04176	0.04155	0.03444	0.06094
3	Amazon	749.87	5.33	20.70967	18.17659	22.01807	20.82411
4	Apple 5	115.82	3.44	1.97717	2.04097	2.83951	2.99233
	AT&T	42.53	1.39	1.23223	1.18919	0.77802	1.02467
6	BCOM	20.18	1.18	0.60016	0.59578	0.41535	0.41814
7	Facebook 8	115.05	4.83	2.58647	2.96227	3.10304	3.44955
	Ford Motors 9	12.13	4.39	0.32357	0.25509	0.24300	0.33319
	General Electric 10	31.60	18.20	2.72882	2.31861	3.68109	3.73332
	General Motors	34.84	4.12	0.72830	0.57745	0.71351	0.62875
11	Intel Corp	36.27	32.40	5.04190	5.07738	5.17368	5.25889
12	Microsoft 13	62.14	3.24	1.08371	0.86606	0.66550	0.87518
	NY Times 14	13.30	2.75	0.17648	0.18617	0.13975	0.18351
	ODML	85.79	4.53	1.92443	1.80971	2.72019	1.88061
15	Starbucks 16	55.52	5.00	1.27686	0.96778	1.09748	1.22458
	Tesla	213.69	9.30	9.59110	7.13202	9.70333	7.36998
17	Verizon	51.40	16.53	4.23520	3.50626	4.51593	4.70806
18	American Airlines 19	51.40	12.00	2.69403	1.82698	2.82816	1.96990
	Airbus 20	66.25	15.00	4.97160	3.17687	5.09510	5.48788
	Boeing	155.68	13.68	9.35389	7.23382	9.88606	8.50372

Table 6.2 Option prices on Nasdaq using Geometric, Arithmetic, Modal averages and the Terminal stock prices

6.3.2 Graphs of Option Prices Using Modal Averages Against Other Option Price Models

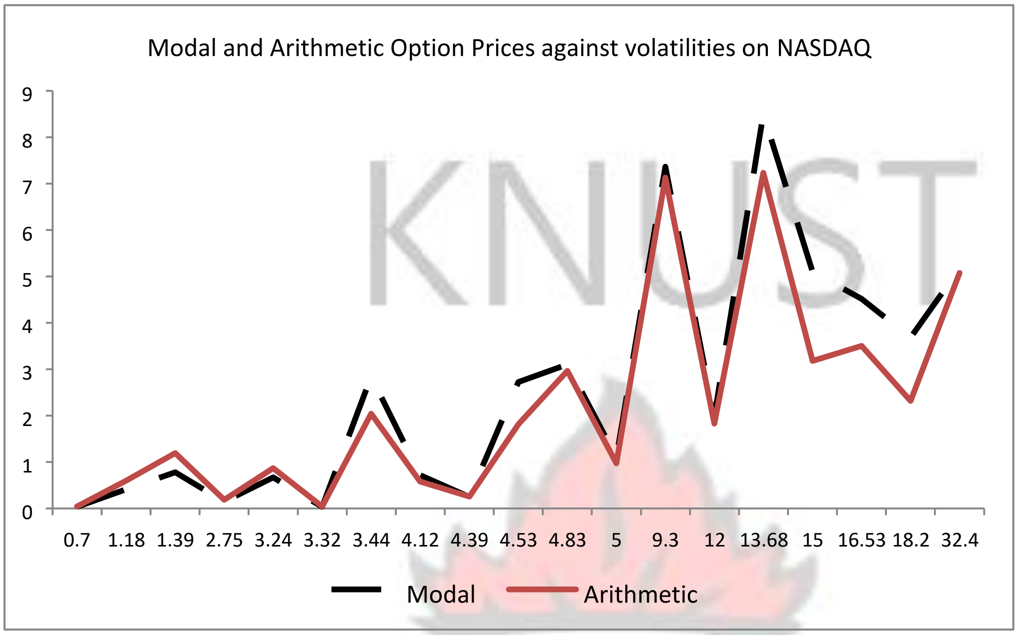

Figure 6.7 Modal and Arithmetic Option Prices against volatilities on NASDAQ

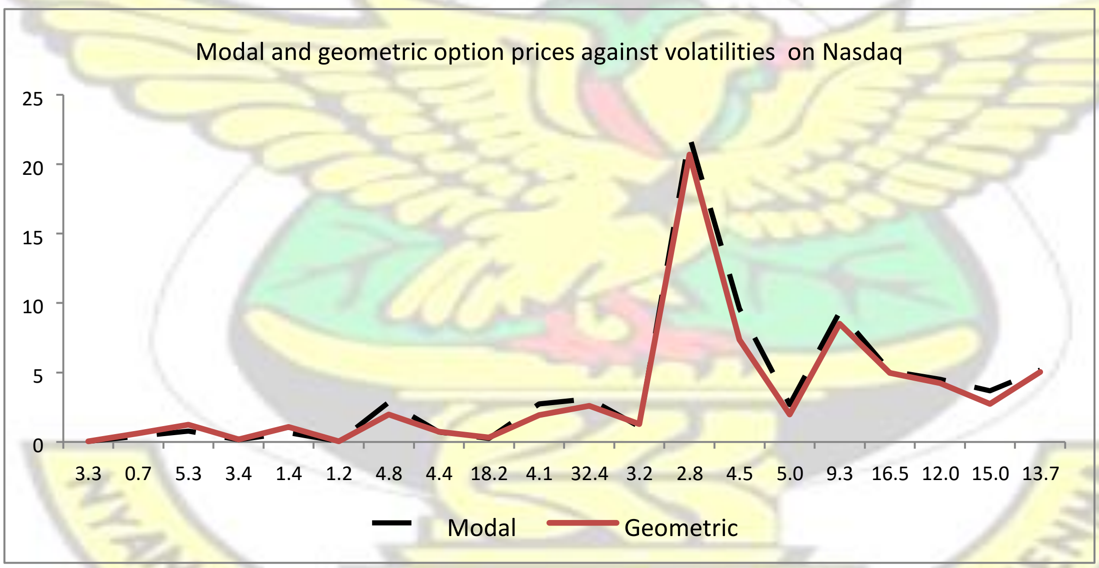

Figure 6.8 Modal and Geometric Option Prices against volatilities on NASDAQ

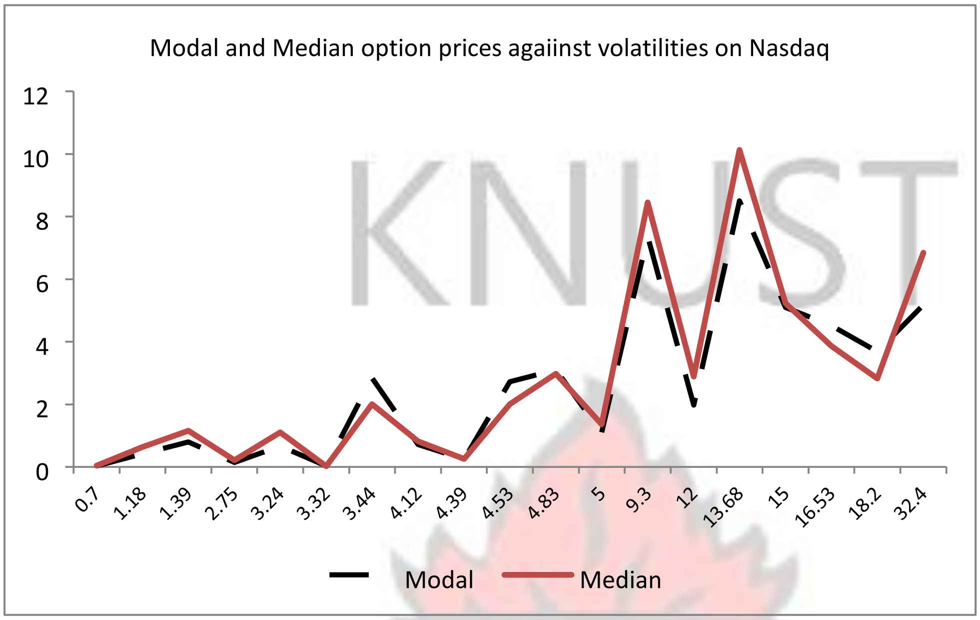

Figure 6.9 Modal and Median Option Prices against volatilities on NASDAQ

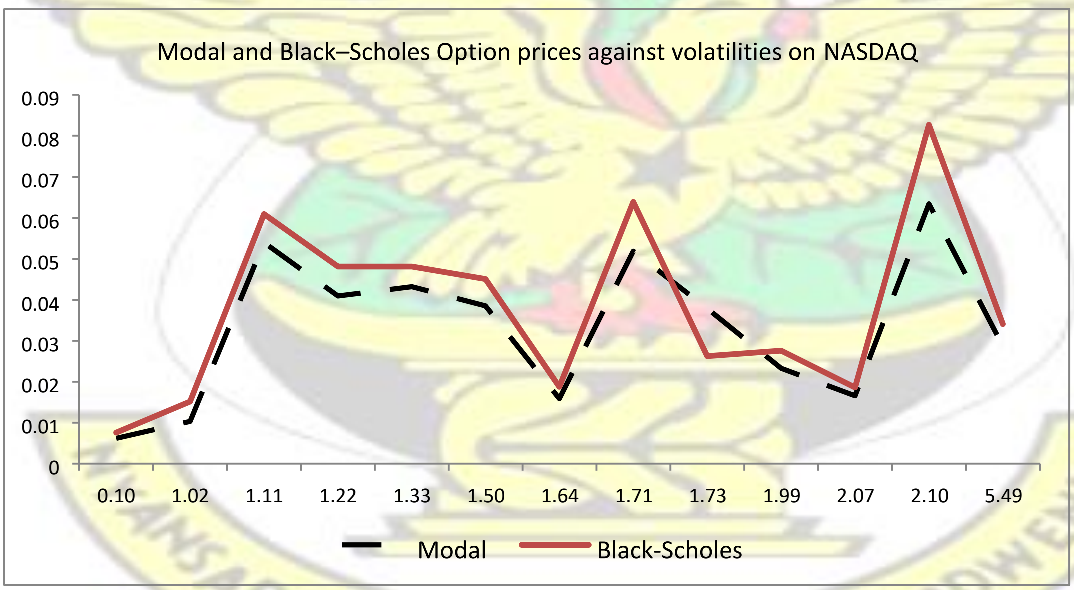

Figure 6.10 Modal and Black-Scholes Option Prices against volatilities on NASDAQ

CHAPTER 7

CONCLUSIONS

7.1 Discussion of Results

Figures 6.7_6.10 shows the graph of volatilities against option prices obtained from stocks listed on the NASDAQ. We observe the following:

- The price of an option based on a modal average is consistently lower than the price on an option based on arithmetic average for volatilities less than approximately 3.0
- The price of an option based on a modal average is consistently lower than the price on an option based on geometric average for volatilities less than approximately 3.0
- The price of an option based on a modal average is consistently lower than the price on an option based on median average for volatilities less than approximately 3.0
- The price of an option based on a modal average is consistently lower than the price on an option based on Binomial model at all levels of volatility
- The price of an option based on a modal average is consistently lower than the price on an option based on Black Scholes at all levels of volatility
- Beyond a volatility of 3% the geometric average option does almost as good as the modal average although in most cases the modal average does better.
- Beyond volatility of 5% both the geometric and the arithmetic average option consistently produce better results than the modal average
- We realised that the average option pricing models are consistently lower than the closed form Black_Scholes model.

7.2 The Modal Average as the Best Estimator of Averages for Low Volatility Options

We have proposed the modal average to estimate the average of the underlying asset over the life of the option. We will show here that the modal average is a better estimator of the average stock price especially for low volatility stocks as against other averages such as the arithmetic and geometric averages which are currently being used. Consider a probability density function $f(x)$ which describes the graph below.

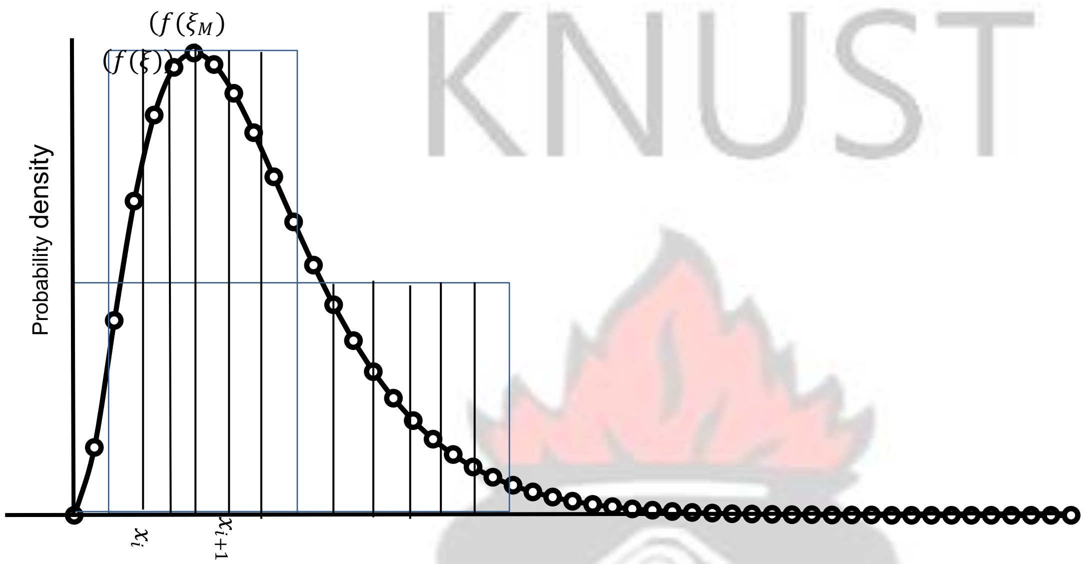

Figure 7.1 The mode as the best estimate of the average in a small interval

Let Suppose there are n partitions of the interval $[a, b]$. By construction the upper and lower Riemann sums, denoted by U_n and L_n respectively, are defined as

$$U_n(f) = \sum_{i=1}^{n} \Big(\sup_{x_{i+1,i}} f(x)\Big)\, \Delta x_i$$

$$L_n(f) = \sum_{i=1}^{n} \Big(\inf_{x_{i+1,i}} f(x)\Big)\, \Delta x_i$$

By the Riemann theorem as n increases in a manner such that each Δx_i decreases to zero, it can be seen that L_n is monotone increasing, while U_n is monotone decreasing. So, as $n \to \infty$ it follows that L_n and U_n will both converge and f is integrable if and only if

$$\lim_{n\to\infty} U_n f(x) = \lim_{n\to\infty} L_n f(x) = \int_a^b f(x)dx$$

……………..……………………..5.1

By the Mean Value Theorem there exist $\xi \in [a, b\,]$ such that

$$f(\xi) = \frac{\int_a^b f(x)dx}{\Delta x} = \frac{Area\ under\ the\ curve}{\Delta x}$$

$$\Delta x f(\xi) = \int_a^b f(x)dx$$

……………..……………………..5.2 Let $\xi_M \in [a, b\,]$ such that $f(\xi) \leq f(\xi_M)$, then ξ_M is the value of x corresponding to the maximum value of $f(x)$. Thus, the area under the curve estimated using ξ and ξ_M is such that

$$\Delta x f(\xi) \leq \Delta x f(\xi_M)$$

……………..……………………..5.3

Since $f(x)$ is a pdf it follows that the maximum area under the curve defined by $f(x)$ is 1 and by the squeeze theorem

$$0 \leq \Delta x f(\xi) \leq \Delta x f(\xi_M) \leq 1$$

……………..…………………..5.4

In estimating the mean by the arithmetic average, the area under the curve is obtained as $\Delta x f(\xi)$. Similarly, the area under the curve when we estimate by the modal average is $\Delta x f(\xi_M)$. It follows from the squeeze theorem that in the limit as $\Delta x \to 0$, the area under the curve approaches the maximum 1 if and only if $f(\xi) \to f(\xi_M)$.

Thus, we can write that

$$\lim_{\Delta x \to 0} f(\xi) = f(\xi_M)$$

……………..……………………..5.5

Hence for sufficiently small Δx the area under the curve $\Delta x f(\xi)$ is always bounded by $\Delta x f(\xi_M)$. We state the following theorem

Theorem 7.1

For sufficiently small Δx, the modal average is the best estimate of the average of a probability density function.

7.3 Conclusions

This study examined the use of modal average as an underlying asset to price options. The study introduced a new option pricing model and established a new framework for the valuation of option with the modal average stock price as the underlying asset. This study thus provides new insights into the use of the modal average in the valuation of options. In addition, we proceeded to establish the model's legitimacy by using it to price options on stocks listed on an existing stock exchange, the NASDAQ.

Moreover, we also numerically compared the option price results using modal average to the results of option price using arithmetic, geometric, median averages and the Black–Scholes formulas. The results consistently show that for low volatility options, which we considered as volatility below 3% for the purpose of this study, the other models overprice the option. This shows that the modal average model is a more superior pricing model than the other averages for low volatilities. We further showed analytically that the modal average option pricing model indeed produces better results compared to the other models for such low volatilities. These results suggest that the Modal average is more suitable for pricing options in low volatility regimes whiles the other models is more suitable in high volatility regimes.

We conclude that:

- For stocks $(0 < \sigma < 3.0)$, it is appropriate to price the option using the modal average models but for stocks with volatility $(\sigma \geq 3.0)$ it is more appropriate to price the options using the existing models.
- For stocks $(3.0 < \sigma < 5.0)$, it is appropriate to price the option using the modal average model or geometric average model.
- For stocks with volatility $(\sigma \geq 5.0)$ it is more appropriate to price the options using arithmetic or geometric averages

We therefore make the claim that the modal average model is a more suitable model for pricing options on stocks in developing countries where stock markets are associated with very low volatilities. We therefore conclude that for low volatility stocks the model of this study is an improvement over the existing option pricing models presented in the literature. In addition, a major difference between our work and all previous work is that we highlight on the need for low volatility options to be priced under a different model rather than previous models where all levels of volatility is treated same.

7.4 Recommendations

We believe this work will generate interest for further work to be conducted when some of the assumptions underlying the modal average model are relaxed. For instance, further work can look at the model in the presence of transaction cost and when the underlying stock pays dividends. In addition, the model can be examined when interest rates and volatilities are stochastic. We have obtained this result based on the premise that the option is a European option. However, the model can also be examined when it is considered as an American option.

REFERENCES

Amin, K. I. and Jarrow, R. A. (1992). Pricing options on risky assets in a stochastic interest rate economy. *Mathematical Finance*, 2(4): 217_237.

Bachelier, L. (1900). Theórie de la spéculation, Annales Scientifiques de l'E´cole Normale Supe´rieure Se´r. 3(17), 21–86.

Bally, V., Caramellino, L. and Zanette A. (2005). Pricing and hedging American Options by Monte Carlo methods using a Malliavin calculus approach. *Monte Carlo Methods and Applications*, Vol. 11, 97-133.

Barone_Adesi, G. and Whaley, R. (1986). The valuation of American call options and the expected ex_dividend stock price decline. *Journal of Financial Economics*, Vol. 17, No.1, 91–111.

Barone_Adesi, G. and Whaley, R. (1987). Efficient analytic approximation of American option values. *Journal of Finance*, Vol. 42, 302–320.

Barone-Adesi, G. and Whaley, R. (1988). On the valuation of American put options on dividend paying stocks. *Advances in Futures and Options Research*, 31-14.

Bensaid, B., Lesne, J.P., PagZes, H. and Scheinkman, J. (1992). Derivative asset pricing with transaction costs. *Mathematical Finance*. Vol. 2, No.2, 63–86.

Bensoussan A. (1984). On the theory of option pricing, *Acta Appl Math*. Vol. 2, 139–158.

Black, F. and Karasinski, P. (1991). Bond and option pricing when short rates are lognormal. *Financial Analysts Journal*, 52–59.

Boness A. J. (1964). Elements of a theory of stock option value. *Journal of Political Economy*, Vol.72, 163-75.

Black, F. (1989). How we came up with the option formula. *The Journal of Portfolio Management*, Vol.15, 4_8.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities, *Journal of Political Economics*. Vol.81, 637–659.

Brennan, M. and Schwartz, E. (1977). The valuation of American put options, *Journal of Finance*. Vol.32, 449–462.

Broadie, M., Glasserman, P. and Kou, S. G. (1997). A continuous correction for discrete barrier options. *Mathematical Finance*, Vol.7, 325_349.

Boyle, P. and Vorst T. (1992). Option replication in discrete time with transaction costs. *Journal of finance*. Vol.47, 271_293.

Clewlow, L., Hodges, S. D. (1997). Optimal delta_hedging under transactions costs. *Journal of Economic, Dynamics and Control*, Vol.21, 1353–1376.

Constantinides, G.M. and Zariphopoulou, T. (1999). Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences. *Finance and Stochastics*, Vol.3, 345–369.

Cox, J., Ross, S. and Rubinstein, M. (1979). Option Pricing: A simplified approach. *Journal of Financial Economics*, Vol.7, 229–263.

Cox, J., Ingersoll, J. and Ross, S. (1985). A Theory of the term structure of interest rates. *Econometrica*, Vol.53, 385-408.

Curran, M., (1994). Valuing Asian and portfolio options by conditioning on the geometric mean price. *The Institute of Management Sciences. Management Science*, Vol. 40, No. 12, 1705_1711

Dagpunar, J. S. (2007). Simulation and Monte Carlo.., Chichester John Wiley & Sons Ltd

Davis, M.H.A., Panas, V.G., Zariphopoulou, T. (1993). European option pricing with transaction costs. *Journal of Control and Optimization*, Vol.31, 470–493.

Duffie, D. (1998). *Securities Markets*. Boston. Academic Press.

Duffie, D. and Kan R. (1996). A yield_factor model of interest rates. *Mathematical Finance*, Vol 6, 379–406.

Egloff, D. (2005). Monte Carlo algorithms for optimal stopping and statistical learning. *The Annals of Applied Probability*, Vol.15, No.2, 1396_1432.

Fama, E. F. (1965). The behavior of stock_market prices. *Journal of Business*, Vol.38, No.1, 34_105

Klebener, F.C. (2005). *Introduction to stochastic calculus with applications*. 2nd edition. London. Imperial College Press.

Geman, H. and Yor, M. (1993). Bessel processes, Asian options, and perpetuities. *Mathematical Finance*, Vol.3, 349_375.

Geske, R. and Johnson, H. (1984). The American put option valued analytically. *Journal of Finance*, Vol.39, 1511–1524.

Geske, R. (1979). A note on an analytical valuation formula for unprotected American call options on stocks with known dividends. *Journal of Financial Economics*, Vol.7, 375–380.

Glassserman, P. (2004). *Monte Carlo methods in financial engineering*. New York. Springer_Verlag.

Glassserman, P. and Yu, B. (2004). Number of paths versus number of basis functions in American option pricing. *Annals of Applied Probability*, Vol.14, 2090–2119.

Heynen, R.C. and Kat, H.M. (1994). Partial Barrier Options. *The Journal of Financial Engineering*, Vol.3, 253-274

Hodges, S.D. and Neuberger, A. (1989). Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, Vol.8, 222–239.

Hoogland, J. and Neumann, D. (2001). Local scale invariance and contingent claim pricing. *International Journal of Theoretical and Applied Finance*, Vol.4, No.1, 1–21.

Hull, J. C. and White, A. (1987). The pricing of options on assets with stochastic volatilities. *Journal of Finance*, Vol.42, 281_300.

Hull, J. C. (2006). *Option, futures and other derivatives*. 6th Edition. New Jersey. Pearson Education Inc. Prentice Hall. 263_312.

Ingersoll, J. (1987). *Theory of Financial Decision Making*. First Edition. Oxford. Rowman & Littlefield.

Hull J. C. and White, A. (1990). Pricing interest_rate derivative securities. *The Review of Financial Studies*, Vol.3, No.4, 573–592

Karatzas, I. (1988). On the pricing of the American option. *Applied Mathematics and Optimization*, Vol.17, 37–60

Kemma, A. and Vorst, A. (1990). A pricing method for options based on average asset values. *Journal of Banking and Finance*, Vol.14, 113–129.

Kendall, M. G. (1953). The Analysis of Economic Time_Series. Part I: Prices. *Journal of the Royal Statistical Society*, Vol.116, 11_25.

Kramkov, D.O. (1996). Optional decomposition of super_martingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields*, Vol.105, 459_479.

Lasserre, J.B., Prieto_Rumeau T. and Zervos, M. (2006). Pricing a class of exotic options via moments SDP relaxations. *Mathematical Finance*, Vol.16, No.3, 469_494.

Leland, H.E. (1985). Option pricing and replication with transactions costs. *The Journal of finance*, Vol.XL, No.5, 1283_1301.

Chen, L. (1996). Stochastic mean and stochastic volatility — A three-factor model of the term structure of interest rates and its application to the pricing of interest rate derivatives. *Financial* Markets, Institutions & Instruments, Vol.5, 1–88.

Linetsky, V. (2002). Exact pricing of Asian options: An application of spectral theory. Working Paper

Moreno, M and Navas, J. F. (2003). On the robustness of least_squares Monte Carlo (LSM) for pricing American derivatives. *Review of Derivatives Research*, Vol.6, No.2, 107–128.
Mehrdoust, F., Babaei S. and Fallah, S. (2017). Efficient Monte Carlo option pricing under CEV model. *Communications in Statistics*, Vol. 46, No. 3

Merton, R. (1973). Rational theory of option pricing. *Bell Journal of Economics and Management Science*, Vol.4, 141–183.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, Vol.3, 125−144.

Milevsky, M. A. and Posner, S. E. (1998). Asian options, the sum of lognormals, the reciprocal of gamma distribution. *Journal of Finance and Quantitative analysis*, Vol.33, No.3, 409−422.

Musiela, M. and Rutkowski, M. (1997). *Martingale methods in financial modeling*. Berlin. Springer.

Musiela, M. and Zariphopoulou, T. (2001). Pricing and risk management of derivatives written on non−traded assets. Working paper, BNP Paribas, London.

Ohanyan V., Hagop. K. and Bardakhchyan, V. G. (2016). Monte Carlo method for geometric average options on several futures. *Modeling of Artificial Intelligence*, Vol.10, No.2, 88−93

Øksendal, B. (2000). Stochastic differential equations: an introduction with applications, Fifth Edition. New York. Springer−Verlag.

Osborne, M. F. M. (1959). Brownian motion in the stock market. *Operations Research* Vol.7, No.2, 145−73.

Ostrovski, Vladimir, (2013). Efficient and exact simulation of the Hull−White model.

Ritchken, P. (1995). On pricing barrier options, *Journal of Derivatives*. Vol.3, 9−28.

Roberts, H. V. (1959). Stock−Market patterns and financial analysis: Methodological suggestions. *Journal of Finance*, Vol.14, No.1, 1−10

Roll, R. (1977). An analytical valuation formula for unprotected American call options on stocks with known dividends. *Journal of Financial Economics*, Vol.5, 251–58

Rogers, L. and Shi, Z. (1995). The value of an Asian option. *Journal of Applied Probability*, Vol. 32, 1077–1088.

Rubinstein, M. and Reiner, E. (1991). Breaking down the barriers, *RISK*, Vol.4, No.8, 28−35.

Rubinstein, R.Y. and Kroese, D. P. (2007). *Simulation and the Monte Carlo Method*. Wiley−Interscience.

Samuelson, P. A. (1965a). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, Vol.6, No.2, 41–49.

Samuelson, P. A. (1965b). Rational theory of warrant pricing. *Industrial management review*, Vol.6, No.2, 13

Schachermayer, W. and Teichmann, J. (2008). How close are the option pricing formulas of Bachelier and Black_Merton_Scholes? *Mathematical Finance*, Vol.18, No.1, 155_170.

Shreve, S. and Vecer, J. (2000). Options on a traded account: Vacation calls, vacation puts and passport options. *Finance and Stochastics*, 255_274.

Shreve, S. (2004). *Stochastic calculus for finance II, continuous–time models*. New York, Springer.

Sprenkle C. (1964). Warrant Prices as indicator of expectation. *Yale Economic Essays*, Vol.1 412_74.

Turnbull, S.M. and Wakeman, L.M. (1991). A quick algorithm for pricing European average options. *Journal of Financial and Quantitative Analysis*, Vol.26, 377_389.

Vellekoop, M. H. and Nieuwenhuis, J. W. (2006). Efficient pricing of derivatives on assets with discrete dividends. *Applied Mathematical Finance*, Vol.13, No.3, 265_284.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of financial economics*, Vol.5. 177_188.

Vorst, T. (1992). Prices and hedge ratios of average exchange rate options. *International review of financial Analysis*, Vol.1, No.3, 179_193

Vorst, A. C. F. (1996). Averaging options. *The Handbook of exotic options*. Chicago. Irwin Professional Publishing.

Whaley, R. E. (1981). On the valuation of American call options on stocks with known dividends. *Journal of Financial Economics*, Vol.9, No.2, 207_211.

Wiggins, J. B. (1987). Option values under stochastic volatility: Theory and empirical estimates. *Journal of Financial Economics*, Vol.19, No.2, 351_372

Wilmott P., Howison, S. and Dewynne J. (1995). *The Mathematics of financial derivatives*. Cambridge University Press.

APPENDIX

Differentiable Functions

A function f is called differentiable at the point $t = t_0$ if at that point $\Delta f \sim C\Delta t$ or this constant C

is denoted by $f'(t_0)$. If f is differentiable at every point of its domain, it is called differentiable.

Theorem 1

If f is continuous function and of finite variation then

$$[f, f](T) = 0$$

Proof The quadratic variation of f is

$$\sum_{i=1}^{n}[f(t_i^n) - f(t_{i-1}^n)]^2 = \sum_{i=1}^{n}|f'(t_i^*)|^2(t_i^n - t_{i-1}^n)^2 \leq \lim_{\delta_n \to 0}\sum_{i=1}^{n}|f'(t_i^*)|^2(t_i^n - t_{i-1}^n)$$

$$[f,f](T) \leq \lim_{\delta_n \to 0}\sum_{i=1}^{n}\left(f(t_i^n) - f(t_{i-1}^n)\right)^2$$

$$\leq \lim_{\delta_n \to 0}\max_{1\leq i\leq n}|f(t_{i+1}^n) - f(t_i^n)|\sum_{i=1}^{n}|f(t_{i+1}^n) - f(t_i^n)|$$

$$\leq \lim_{\delta_n \to 0}\max_{1\leq i\leq n}|f(t_{i+1}^n) - f(t_i^n)|\,\mathrm{V}_f(t)$$

Since f is continuous, it is uniformly continuous on $[0, t]$, it follows that

$$\lim_{\delta_n \to 0}\max_{1\leq i\leq n}|f(t_{i+1}^n) - f(t_i^n)| = 0$$

and hence

$$[f,f](T) = 0 \times [f,f](T) \text{ as required.}$$

Taylor's Formula

If $f(x)$ has n –power series representation at a i.e if

$$f(x) = \sum_{i=1}^{n}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

then

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

$$+\frac{1}{3!}f^{(3)}(x_0)(x - x_0)^3 + \ldots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + R_n(x, x_0)$$

where R_n is the remainder, and $f^{(n)}$ is the derivative of $f^{(n-1)}$. The remainder can be written in the form

$$R_n(x, x_0) = \frac{1}{(n+1)!}f^{(n+1)}(\theta_n)(x - x_0)^{n+1}$$

for some point $\theta_n \in (x_0,\ x)$.

Taylor's formula for functions of several variables

Let $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, x_2, \ldots, x_n)$ has continuous partial derivatives up to order two, $x = (x_1, x_{,2} \ \ldots, x_n)$, $x + \Delta x = (x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n)$ then by considering the function of one variable $g(t) = f(x + t\Delta x)\ for\ 0 \leq t \leq 1$, the following result is obtained.

$$f(x_1, x_2, \ldots, x_n) = f(x + \Delta x) - f(x)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial xi}(x_1, x_2, \ldots, x_n) dx_i$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1 + \theta \Delta x_1, \ldots, x_n + \theta \Delta x_n) dx_i dx_j)$$

where just like in the case of one variable the second derivatives are evaluated at some "middle" point, $(x_1 + \theta \Delta x_1, \ldots, x_n + \theta \Delta x_n)$ for some $\theta \in (0, 1)$, and $dx_i = \Delta x_i$.

Proof of Black–Scholes formula

We want to compute the price of the expected payoff

$$C = e^{-r(T-t)} E_{\mathbb{Q}}[max(S(T) - K, 0)] = e^{-r(T-t)} \int_0^{+\infty} f(x) max(x - K, 0)\, dx$$

Where $f(x)$ is the density of the lognormal random variable X given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{(\mu - Inx)^2}{2\sigma^2}\right)$$

Where

$\mu = E(logS(T))$ = mean stock price and $\sigma^2 = Var(logS(T))$ the variance of the return. Now If $S(T) < K$, the option will not be exercised and so $max(S(T) - K, 0)$ will be 0. We are therefore interested in the price distribution when $S(T) > K$. So we can write

$+\infty$

$$C = e^{-r(T-t)} \int_{k} f(x)(x - K)\, dx$$

$$C = e_{-r(T-t)} \int_{k}^{+\infty} xf(x)dx - Ke_{-r(T-t))} \int_{k}^{+\infty} f(x)dx)\, dx$$

Now the last integral $\int_k^{+\infty} f(x)dx)$ is the probability of the event that $S(T) > K$. So the last integral is equivalent to the statement $P(S(T) > K)$.

Now $S(t) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma B(T)}$ and so

$$P(S(T) > K) = P\left(S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma B(T)} > K\right)$$

$$P(S(T) > K) = P\left(e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma B(T)} > \frac{K}{S(0)}\right)$$

$$P(S(T) > K) = P\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B(T) > In\left(\frac{K}{S(0)}\right)\right)$$

$$= P\left(\sigma B(T) > In\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T\right)$$

$$= P\left(B(T) > \frac{In\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma}\right)$$

Dividing by $\sqrt{T}$ we have

$$P(S(T) > K) = P\left(\frac{B(T)}{\sqrt{T}} > \frac{In\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= 1 - N\left(\frac{In\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

But $1 - N(x) = N(-x)$, hence

$$1 - N\left(\frac{In\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = N\left((-)\frac{In\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= N\left(\frac{In\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Let

$$d_2 = \left(\frac{In\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

then we have $P(S(T) > K) = N(d_2)$

$P(S(T) > K)$ gives the probability that you exercise the option. It is the risk neutral probability that you exercise the option. It follows that $1 - N(d_2) = N(-d_2)$ is the probability that the option will not be exercised i.e. the risk neutral probability that the option will default. Note that if $N(d_2)$ refers to the probability that you exercise the call option, then $1 - N(d_2) = N(-d_2)$ is the probability that you exercise the put option.

We now compute the first integral $\int_k^{+\infty} xf(x)dx$.

Let $\mathrm{I} = \int_k^{+\infty} xf(x)dx$

But

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{(\mu - Inx)^2}{2\sigma^2}\right)$$

Hence

$$\mathrm{I} = \int_k^{+\infty} x\frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{(\mu - Inx)^2}{2\sigma^2}\right)dx$$

$$\mathrm{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_{k}^{+\infty} exp\left(-\frac{(\mu - Inx)^2}{2\sigma^2}\right)dx$$

Now the first natural change of variables is $Inx = s,\ x = e^s$, and $dx = e^s ds$ and this gives

$$\mathrm{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_{k}^{+\infty} exp\left(-\frac{(\mu - s)^2}{2\sigma^2}\right)e^s ds$$
$$\mathrm{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_{k}^{+\infty} exp\left(-\frac{(\mu - s)^2}{2\sigma^2} + s\right)ds$$

Now completing the square we have

$$-\frac{(\mu - s)^2}{2\sigma^2} + s = \frac{(s - (\mu + \sigma^2)^2}{2\sigma^2} + \frac{\sigma^2}{2} + \mu$$

Hence

$$\mathrm{I} = \frac{1}{\sigma\sqrt{2\pi}}\int_{Ink}^{+\infty} exp\left[-\frac{(s - (\mu + \sigma^2)^2}{2\sigma^2} + \frac{\sigma^2}{2} + \mu\right]ds$$
$$\mathrm{I} = exp\left(\frac{\sigma^2}{2} + \mu\right)\frac{1}{\sigma\sqrt{2\pi}}\int_{Ink}^{+\infty} exp\left[-\frac{(s - (\mu + \sigma^2)^2}{2\sigma^2} + \right]ds$$

The expression under the integrand is the density function of a normal variable with mean $\mu = InS_0 + \left(r - \frac{\sigma^2}{2}\right)T$ and variance $\sigma^2 T$.

Now $$\mathrm{I} = exp\left(\frac{\sigma^2}{2} + \mu\right)\cdot 1 - N(InK;\ \mu + \sigma^2, \sigma^2)$$

and $$S(0)e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma B(T)}$$

Hence we replace σ^2 in Equation 15 by $\sigma^2 T$ and μ by $InS_0 + \left(r - \frac{\sigma^2}{2}\right)T$. We obtain

$$\mathrm{I} = exp\left(\frac{\sigma^2 T}{2} + InS(0) + \left(r - \frac{\sigma^2}{2}\right)T\right)\cdot 1 - N(InK;\ InS(0) + \left(r + \frac{\sigma^2}{2}\right)T, \sigma^2 T)$$

Now

$$= 1 - N\left(\frac{InK - InS(0) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\mathrm{I} = S(0)\mathrm{e}^{rT}\left[1 - N\left(\frac{InK - InS(0) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right] S(0)\mathrm{e}^{rT} N(-)\left(\frac{InK - InS(0) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\mathrm{I} = S(0)\mathrm{e}^{rT} N\left(\frac{-InK + InS(0) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = S(0)\mathrm{e}^{rT} N\left(\frac{In\frac{S(0)}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$exp\left(\frac{\sigma^2 T}{2} + InS(0) + rT - \frac{\sigma^2 T}{2}\right) = \exp(rT + InS(0)) = S(0)\exp(rT)$$

$$\mathrm{I} = S(0)e^{rT} \cdot 1 - N(x)$$

$$1 - N(x) = 1 - N(InK;\ InS(0) + \left(r + \frac{\sigma^2}{2}\right)T\ , \sigma^2 T)$$

And so

Let

$$d_1 = \left(\frac{In\frac{S(0)}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

then $\mathrm{I} = S(0)\mathrm{e}^{rT} N(d_1)$

Now from Equation (13) we can write

$$C(S(t), t) = e_{-r(T-t)} I - K e_{-r(T-t)} P(S(T) > K)$$

Replacing T by $T - t$ and in Equation 17 and substituting Equations (14) and (17) into Equation (18) we have

$C(S(t),t) = e^{-r(T-t)}I - Ke^{-r(T-t)}P(S(T) > K) = e^{-r(T-t)}S(0)e^{r(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$

$$C(S(t),t) = S(0)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Relacing $S(0)$ by S we obtain the $Black - Scholes\ formula$ for the price of an European call option as

$$C(S(t),t) = S(0)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Where

$$d_1 = \frac{In\left(\frac{S(0)}{K}\right) + \left(r + \sigma^2/_2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{In\left(\frac{S(0)}{K}\right) + \left(r - \sigma^2/_2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$$

$N(x)$ is the cumulative distribution function of a standard normal variable.

Converting the Black-Scholes PDE to the Heat Equation

The Black-Scholes partial differential equation and boundary value problem is

$$\mathcal{L}V\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$S(t) \geq 0, \qquad 0 \leq t \leq T \qquad V(S(t),t) = f(s), \qquad V(0,t) = 0$$

If V is the price of a call option, then the boundary condition. $f(S) = max(S-K,0)$, where K denotes the strike price of the call option. The following change of variables transforms the Black-Scholes boundary value problem into a standard boundary value problem for the heat equation.

$$S = e^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V\,(S(t), t) = v\,(x, \tau\,) = v\left(ln\,(S(t)), \qquad \frac{\sigma^2}{2}(T - t)\right)$$

The partial derivatives of V with respect to S and $t$expressed in terms of partial derivatives of v in terms of x and τ are:

$$\frac{\partial V}{\partial t} = -\sigma^2 \frac{\partial v}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = -\frac{1}{S}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}$$

Placing these expressions into the Black-Scholes partial differential equation and simplifying we have

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2\tau}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2v}{\sigma^2}v$$

Setting $\kappa = \frac{2r}{\sigma^2}$ and $t = r$, the Black-Scholes boundary value problem becomes

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (\kappa - 1)\frac{\partial v}{\partial x} - \kappa v$$

$$-\infty \leq x \leq \infty, \qquad 0 \leq t \leq \frac{\sigma^2 T}{2},$$

$v(x, 0) = V(e^x, T) = f(e^x) \qquad -\infty \leq x \leq \infty$

$$v(x, t) = e^{\alpha x + \beta t}u(x, t) = \phi u$$

Where $\frac{\partial v}{\partial t} = \beta\phi u + \phi\frac{\partial u}{\partial t}$

$$\frac{\partial v}{\partial x} = \alpha\phi u + 2\alpha\phi\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}$$

$\kappa = \frac{2r}{\sigma^2}$ and $t = \tau$

Placing these expressions into the partial differential equation which v satisfies, and setting

$$\alpha = -\frac{1}{2}(\kappa - 1) = \frac{\sigma^2 - 2r}{2\sigma^2}$$

$$\beta = -\frac{1}{4}(\kappa + 1)^2 = -\left(\frac{\sigma^2 - 2r}{2\sigma^2}\right)^2$$

We have

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$-\infty \leq x \leq \infty,$ $\quad 0 \leq t \leq \frac{\sigma^2 T}{2}, \quad v(x,0) = V(e^x, T) = f(e^x), \quad -\infty \leq x \leq \infty$

Proof of Theorem 4.2

Since the market is finite, the value process $V(t)$ takes only finitely many values therefore $\mathbb{E}V(t)$ exist. The martingale property is verified as follows.

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{V(t+1)}{\beta(t+1)}\bigg|_{\mathcal{F}_t}\right) = \mathbb{E}_{\mathbb{Q}}(t+1)Z(t+1) + b(t+1)|\mathcal{F}_t$$

since $a(t)$ and $b(t)$ are predictable

$$= (t+1)\mathbb{E}_{Q}Z(t+1)|\mathcal{F}_t + b(t+1)$$

$= a(t+1)Z(t) + b(t+1)$ since $Z(t)$ is a martingale

$= a(t)Z(t) + b(t)$ since $(a(t), b(t))$ is self-financing

$$= \frac{V(t)}{\beta(t)}$$

To value options as a risk neutral investor, we discount at the risk free rate and take expectations under the risk neutral measure. To take expectations under the risk neutral measure we need to change $\mathbb{P}$ in the stochastic differential equation, which requires Girsanov's Theorem. Girsanov's Theorem tells us how a stochastic differential equation (SDE) changes as probability measure $\mathbb{P}$ changes.

Proof of Theorem 4.3

Let $V(t)$, $0 \leq t \leq T$, represent the value of the replicating portfolio. Then at maturity since X is attainable, it is replicated by an admissible strategy $X = V(T)$. Fix one such strategy. To avoid arbitrage, the price of X at any time $t < T$ must be given by the value of this portfolio $V(t)$, otherwise arbitrage profit is possible. Since the model does not admit arbitrage a martingale probability measure $\mathbb{Q}$ exists and the discounted value process is a $\mathbb{Q}$ −martingale. Hence by the martingale property

$$\frac{V(t)}{\beta(t)} = \mathbb{E}_{\mathbb{Q}}\left(\frac{V(T)}{\beta(T)}\Big|_{\mathcal{F}_t}\right)$$

But $V(T) = X$, and we have

$$\frac{V(t)}{\beta(t)} = \mathbb{E}_{\mathbb{Q}}\left(\frac{1}{\beta(T)}X\Big|_{\mathcal{F}_t}\right)$$

$$V(t) = \mathbb{E}_{\mathbb{Q}}\left(\frac{\beta(t)}{\beta(T)}X\Big|_{\mathcal{F}_t}\right)$$

At expiration $X = max(S(T) - K, 0) = (S(T) - K, 0)^{+}$

And

$$\frac{\beta(t)}{\beta(T)} = \int_t^T r(s)ds$$

Hence the

$$C(t) = \mathbb{E}_{\mathbb{Q}}\left(\frac{\beta(t)}{\beta(T)}X\,\Big|_{\mathcal{F}_t}\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_t^T r(s)ds\,\big(max(S(T)-K,0)\big)\Big|_{\mathcal{F}_t}\right)$$

$$\mathbb{E}_{\mathbb{Q}}\left(exp\left(\int_t^T r(s)ds\right)\big(max(S(T)-K,0)\big)\Big|_{\mathcal{F}_t}\right)$$

$$C(S(t),t) = \mathbb{E}_{\mathbb{Q}}\left(e^{r(T-t)}\big(max(S(T)-K,0)\big)\Big|_{\mathcal{F}_t}\right)$$

and by the property of independence of increments of Brownian motion we have

$$C(S(t),t) = \mathbb{E}_{\mathbb{Q}}\left(e^{r(T-t)}(max(S(T)-K,0))\right)$$

$C(S(t),t)$ is the discounted expected payoff on a call option if all investors were risk-neutral, and is the call option at time t. $\mathbb{E}_{\mathbb{Q}}\{e^{r(T-t)}[max(S(T)-K,0)]\}$ is the expected value of the call at time in a risk-neutral world.

$\mathcal{R}$ Codes for Simulation

Stock Price Function stock.return <- function(strike.price, n,

risk.free, stock.sigma){ delta.t <- 1/n # one period for (i in

seq(n)){ epsilon <- runif(n=1, min=0, max=1) # random

generated number # calculate stock price (using quantile function

of normal distribution) stock.price <- strike.price*(1 +

qnorm(epsilon,risk.free * delta.t,stock.sigma*

sqrt(delta.t)))

}

return(stock.price)

}

Parameters n <- 247 # trading days

strike.price<- 3.2 # initial stock price

risk.free<- 0.23 stock.sigma<-

0.017272519

Stock price simulations

stock.prices <- c() for (i

in seq(n)){ stock.prices

<-

```
c(stock.prices,stock.retur
n(strike.price=strike.pric
e,n=n,
risk.free=risk.free,stock.
sigma=stock.sigma))
}
# Simulated data of GSE trading.days<-c(1:247) stock.path.prices<-
stock.prices truncated.stock.price<-round(stock.path.prices,2)
simulateGSE=data.frame(trading.days,stock.path.prices,truncated.stock.price)
View(simulateGSE)
#Plot of simulated data of GSE library(ggplot2) ggplot(data=simulateGSE,
aes(x=trading.days,y=stock.path.prices, group=1)) + geom_line(colour="blue")
```

Option Price Function

```
## Mean Option Price Function stock.return<-function(strike.price, n, risk.free,
stock.sigma){ delta.t <- 1/n # one period   for (i in seq(n)){   epsilon <- runif(n=1,
min=0, max=1) # random generated number     # calculate stock price (using
quantile function of normal   distribution) stock.price <- strike.price * (1
+qnorm(epsilon,risk.free          * delta.t,stock.sigma* sqrt(delta.t)))
  }
return(stock.price)
}
```

Option Price Simulations simulations<-1000

number of simulations generate.options <-

function(){

Parameters n <- 247 # trading days strike.price<- 3.2 # initial stock price risk.free<- 0. 023 stock.sigma<- 0.017272519 #Stock prices per each trading day stock.prices <- c() for (i in seq(n)){ stock.prices <- c(stock.prices,stock.return(strike.price=strike.price,n=n, risk.free=risk.free,stock.sigma=stock.sigma)) stock.path.prices<- stock.prices truncated.stock.price<-round(stock.path.prices,2)

Average stock price meanstockprice<-mean(truncated.stock.price)

#Option price per each trading day future.payoff =

max((meanstockprice-strike.price),0) discounted.payoff

= future.payoff * exp(-risk.free *0.25)

mean.options=mean(discounted.payoff)

} return(mean.options)

}

 # Monte Carlo simulations for Option prices

option.price<- replicate(simulations,generate.options())

simulateoptionprice<-data.frame(option.price)

View(simulateoptionprice) #

Average option prices

mean(option.price)

Option Price Function ## Median Option Price Function

stock.return<-function(strike.price, n, risk.free, stock.sigma){ delta.t <-

1/n # one period for (i in seq(n)){ epsilon <- runif(n=1, min=0, max=1)

random generated number # calculate stock price (using quantile

function of normal distribution) stock.price <- strike.price * (1

+qnorm(epsilon,risk.free * delta.t,stock.sigma*sqrt(delta.t)))

}

return(stock.price) }

Option price simulations simulations<-1000

number of simulations generate.options <-

function(){

Parameters n <- 247 # trading days strike.price<- 3.2 # initial stock

price risk.free<- 0.23 stock.sigma<- 0.017272519 #Stock prices per

each trading day stock.prices <- c() for (i in seq(n)){ stock.prices <-

c(stock.prices,stock.return(strike.price=strike.price,n=n,

risk.free=risk.free,stock.sigma=stock.sigma)) stock.path.prices<-

stock.prices truncated.stock.price<-round(stock.path.prices,4)

Median stock price medianstockprice<-

median(truncated.stock.price)

#Option price per each trading days future.payoff =

max((medianstockprice-strike.price),0)

```
discounted.payoff = future.payoff * exp(-risk.free *0.25)
mean.options=mean(discounted.payoff)
  }
  return(mean.options)
}
```

 # Monte Carlo simulations for Option prices

```
option.price<- replicate(simulations,generate.options())
simulateoptionprice<-data.frame(option.price)
View(simulateoptionprice) #
Average of option prices
mean(option.price)
```

Option Price Function

```
## Mode Option Price Function stock.return<-function(strike.price, n, risk.free,
stock.sigma){   delta.t <- 1/n # one period   for (i in seq(n)){    epsilon <- runif(n=1, min=0,
max=1) # random generated number # calculate stock price (using quantile function of
normal   distribution) stock.price <- strike.price * (1 +qnorm(epsilon,risk.free * delta.t,
stock.sigma*sqrt(delta.t)))
  }
return(stock.price)
}
```

```
## Option price simulations simulations<-1000

# number of simulations generate.options <-

function(){

  # Parameters   n <- 247 # trading

days   strike.price<- 3.2 # initial stock

price   risk.free<- 0.23

stock.sigma<- 0.017272519


  #Stock prices per each trading day   stock.prices <- c()   for (i in

seq(n)){     stock.prices <-

c(stock.prices,stock.return(strike.price=strike.price,n=n,

risk.free=risk.free,stock.sigma=stock.sigma))


    stock.path.prices<-stock.prices                              truncated.stock.price<-

round(stock.path.prices,3)


   # Mode stock price     mode <- function(StockPricePath) {       uniqStockPricePath <-

unique(StockPricePath)

uniqStockPricePath[which.max(tabulate(match(StockPricePath, uniqStockPricePath)))]

    }
    StockPricePath <- truncated.stock.price

mode.stock.price <- mode(StockPricePath)     modestock<-

mode.stock.price
```

```
  #Option price per each trading days     future.payoff =
max((modestock-strike.price),0)     discounted.payoff =
future.payoff * exp(-risk.free *0.25)
mean.options=mean(discounted.payoff)
  }
  return(mean.options)
}
# Monte Carlo simulations for Option prices option.price<-
replicate(simulations,generate.options())
simulateoptionprice<-data.frame(option.price)
View(simulateoptionprice)

# Average of option prices mean(option.price)
```

Black–Scholes Option Pricing Formula

```
Blackscholes <- function(S, X, rf, T, sigma) {  d1 <-
(log(S/X)+(rf+sigma^2/2)*T)/(sigma*sqrt(T))   d2
<- d1 - sigma * sqrt(T)
 option.price<-      S*pnorm(d1)      -      X*exp(-rf*T)*pnorm(d2)
option.price
}
```

Golden Search Method to Find the Mode f <- function(x)

3/4*x^2*(2-x) maximum.x<-optimise(f,c(0,2), maximum =

TRUE,tol =0.05) maximum.x

library(graphics) plot(f, -2,2,

ylim = 0:1, col = 2)

R-codes for the Graphs

```
# Read Modal and Arithmetic option prices against volatility of GSE
GSEMODAL<-(read.csv(file.choose()))
View(GSEMODAL)
Modal<-GSEMODAL$Modal
Arithmetic<-GSEMODAL$Arithmetic Volatility<-GSEMODAL$Volatility
plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option prices",
col.lab=rgb(0,0.5,0)) lines(Volatility,Arithmetic, type="l", pch=22, lty=2, col="red")
title(main="Modal and Arithmetic option prices against volatility-GSE", col.main="black",
font.main=1) legend("topleft", legend=c("Modal", "Arithmetic"),
    col=c("blue", "red"), lty=1:2, cex=0.8)
# Read Modal and Median option prices against volatility-GSE
GSEMEDIAN<-(read.csv(file.choose()))
View(GSEMEDIAN)
Modal<-GSEMEDIAN$Modal
Median<-GSEMEDIAN$Median
```

Volatility<-GSEMEDIAN$Volatility plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option prices", col.lab=rgb(0,0.5,0)) lines(Volatility,Median, type="l", pch=22, lty=2, col="red") title(main="Modal and Median option prices against volatility-GSE", col.main="black", font.main=1) legend("topleft", legend=c("Modal", "Median"), col=c("blue", "red"), lty=1:2, cex=0.8)

```
# Read Modal and Geometric option prices against volatility-GSE
GSEGEO<-(read.csv(file.choose()))
View(GSEGEO)
Modal<-GSEGEO$Modal
Geometric<-GSEGEO$Geometric Volatility<-GSEGEO$Volatility
plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option
prices", col.lab=rgb(0,0.5,0)) lines(Volatility,Geometric, type="l", pch=22, lty=2,
col="red")
title(main="Modal and Geometric option prices against volatility-GSE", col.main="black",
font.main=1) legend("topleft", legend=c("Modal", "Geometric"),        col=c("blue", "red"),
lty=1:2, cex=0.8)
# Read Modal and Black–Scholes option prices against volatility-GSE
GSEBLACKSCHOLES<-(read.csv(file.choose()))
View(GSEBLACKSCHOLES)
Modal<-GSEBLACKSCHOLES$Modal
Black_Scholes<-GSEBLACKSCHOLES$Black.Scholes Volatility<-GSEBLACKSCHOLES$Volatility
```

```
plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option prices",
col.lab=rgb(0,0.5,0)) lines(Volatility,Black_Scholes, type="l", pch=22, lty=2, col="red")
title(main="Modal and Black–Scholes option prices against volatility-GSE", col.main="black",
font.main=1) legend("topleft", legend=c("Modal", "Black–Scholes"),      col=c("blue", "red"),
lty=1:2, cex=0.8)
# Read Modal and Geometric option prices against volatility-Nasdaq
NASDAQMODE<-(read.csv(file.choose()))
View(NASDAQMODE)
Modal<-NASDAQMODE$Modal
Arithmetic<-NASDAQMODE$Arithmetic
Volatility<-NASDAQMODE$Volatility
plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option prices",
col.lab=rgb(0,0.5,0)) lines(Volatility,Arithmetic, type="l", pch=22, lty=2, col="red")
title(main="Modal and Arithmetic option prices against volatility-Nasdaq", col.main="black",
font.main=1) legend("topleft", legend=c("Modal", "Arithmetic"),      col=c("blue", "red"),
lty=1:2, cex=0.8)
# Read Modal and Median option prices against volatility-Nasdaq
NASDAQMEDIAN<-(read.csv(file.choose()))
View(NASDAQMEDIAN)
Modal<-NASDAQMEDIAN$Modal
Median<-NASDAQMEDIAN$Median
Volatility<-NASDAQMEDIAN$Volatility
plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option prices",
col.lab=rgb(0,0.5,0)) lines(Volatility,Median, type="l", pch=22, lty=2, col="red")
```

```
title(main="Modal and Median option prices against volatility-Nasdaq", col.main="black",
font.main=1) legend("topleft", legend=c("Modal", "Median"),        col=c("blue", "red"),
lty=1:2, cex=0.8)
# Read Modal and Geometric option prices against volatility-Nasdaq
NASDAQGEO<-(read.csv(file.choose()))
View(NASDAQGEO)
Modal<-NASDAQGEO$Modal
Geometric<-NASDAQGEO$Geometric Volatility<-NASDAQGEO$Volatility
plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option prices",
col.lab=rgb(0,0.5,0)) lines(Volatility,Geometric, type="l", pch=22, lty=2, col="red")
title(main="Modal and Geometric option prices against volatility-Nasdaq", col.main="black",
font.main=1) legend("topright", legend=c("Modal", "Geometric"),        col=c("blue", "red"),
lty=1:2, cex=0.8)
# Read Modal and Black–Scholes option prices against volatility-Nasdaq
NASDAQBLACKSCHOLES<-(read.csv(file.choose()))
View(NASDAQBLACKSCHOLES)
Modal<-NASDAQBLACKSCHOLES$Modal
Black_Scholes<-NASDAQBLACKSCHOLES$Black.Scholes                                Volatility<-
NASDAQBLACKSCHOLES$Volatility
plot(Volatility,Modal, type="l", col="blue",xlab="Volatility",ylab="Option prices",
col.lab=rgb(0,0.5,0)) lines(Volatility,Black_Scholes, type="l", pch=22, lty=2,
col="red") title(main="Modal and Black–Scholes option prices against volatility-
Nasdaq", col.main="black", font.main=1) legend("topright", legend=c("Modal",
"Black–Scholes"),        col=c("blue", "red"), lty=1:2, cex=0.8)
```