KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

THE VALUATION OF LIFE INSURANCE LIABILITIES: NUMERICAL METHOD APPROACH

BY

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Declaration

I hereby declare that this submission is my own work towards the award of the MPhil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

This study is dedicated to my late mother, *Augustina Dwabeng Serwaa*, for her tireless effort which brought me to the level which served as a springboard for me to enrol l in this program. *May her soul rest in perfect peace.*

Abstract

Most of the life insurance contracts in Ghana contains surrender options and are also path-dependent. The presence of one or more option elements and path-dependence derivatives presents complexities in the valuation of the life insurance contract in Ghana. The study seeks to compare the multi-period binomial tree method to the Black-Scholes partial differential equation in the valuation of the life insurance contract which contains surrender options.

The paper implemented and compare the three finite difference algorithm in solving the Black-Scholes equation and concluded that, the Crank-Nicolson method gives more accurate results than the Implicit and Explicit finite difference methods and the explicit finite method. The paper also checked the stability and accuracy of the multi-period binomial tree and the implicit finite difference and Crank-Nicolson method.

The study revealed that, the multi-period binomial tree method is closer to the solution for even smaller values of N (the number of steps) than the Black-Scholes partial differential equation and recommended the use of the binomial tree method in the valuation of the life insurance contract in Ghana which contains surrender options.

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Chapter 1

INTRODUCTION

1.1 Background of the Study

If one wants to trace back the history of insurance then one need to look up to that period where most of the commerce and business deals used to happen through sea routes. During this period, people used to send the ships along with their valuable material in order to make business but if the weather turned bad then sometimes the ship may used to go out of control and sometimes they did not have any option but to remove all the material from the ship in order to save their life. So, many of the ship owners decided that they should form a common fund and the common fund should be utilized for the purpose of reimbursement of the monetary losses of the person whose ship was involved in an accident. This is how insurance started. Now it is far more than marine insurance all over the world. There are many more types of insurance which have entered the market over this period of time.

According to Financial Consumer Agency of Canada, FCAC (2011), insurance is a way of reducing your potential financial loss or hardship. It can help cover the cost of unexpected events such as theft, illness or property damage. Insurance can also provide your loved ones with a financial payment upon your death.

1.1.1 Insurance policy types

There are many different types of insurance policies available, and it is important to choose the right insurance deal for you. Examples includes auto insurance policy, life insurance, health insurance, home (property) insurance, tenants (renter's) insurance, business insurance, credit or debt insurance, buying insurance and so on.

Health insurance

Health insurance is insurance against the risk of incurring medical expenses among individuals.

Vehicle (auto) Insurance Policy and Types in Ghana

Vehicle insurance (also known as auto insurance, GAP (Guaranteed Auto Protection) insurance, car insurance, or motor insurance) is insurance purchased for cars, trucks, motorcycles, and other road vehicles. Its primary use is to provide financial protection against physical damage and/or bodily injury resulting from traffic collisions and against liability that could also arise from it.

Life insurance

Life insurance is a contract between an insurance policy holder and an insurer, where the insurer promises to pay a designated beneficiary a sum of money (the "benefits") upon the death of the insured person. Depending on the contract, other events such as terminal illness or critical illness may also trigger payment. The policy holder typically pays a premium, either regularly or as a lump sum. Other expenses (such as funeral expenses) are also sometimes included in the benefits. There are two main types of life insurance: term and permanent.

Term life insurance provides coverage if you die within a specific period of time, unless you do not pay your premium. Term life insurance premiums are generally less expensive than permanent life insurance premiums. Premiums are usually fixed for the length of the term, often at intervals of five or ten years. However, your premiums may increase when you renew the policy. For example, premiums would increase every five years on a five-year renewable policy. Most life insurance policies will only cover you up to a maximum age.

Permanent life insurance is life insurance that remains active until the policy matures, unless the owner fails to pay the premium when due. The policy cannot be cancelled by the insurer for any reason except fraudulent application, and any such cancellation must occur within a period of time defined by law (usually two years). It provides coverage throughout your lifetime, unless you fail to pay your premiums. Permanent life insurance policies generally accumulate a cash value that is either added to the face value of your policy and paid out upon your death, or returned to you if you cancel your policy and reduces the risk to which the insurance company is exposed, and thus the insurance expense over time.

Life insurance liabilities

A correct assessment of duration and convexity of insurance liabilities and equity measures is critical as they constitute the primary ingredients of any sound asset-liability management approach. In addition, the effort toward a more detailed and more accurate risk picture of life insurance operations enables one to debunk some pitfalls that are commonly encountered in the insurance industry (Eric, 1995).

According to Anders Peter (2002), the holders of life insurance contracts (LICs) have the first claim on the company's assets, whereas equity holders have limited liability; interest rate guarantees are common elements of LICs; and LICs according to the so-called contribution principle (which states that if a risk is insured by multiple carriers, and one carrier has paid out a claim, that carrier is entitled to collect proportionate coverage from other carriers) are entitled to receive a fair share of any investment surplus.

According to Eric(1995), risk-taking initially occurs on the liability side of the balance sheet. Underwriters issue insurance policies which are transformed into liabilities (read technical reserves). Because of the time lag between the premium inflow and the indemnity outflow, the reserves are invested on the financial marketplace and generate the portfolio of assets of the company.

1.1.2 Types of Life Insurance Contracts

There are several types of life insurance contracts. The most common among them are *the European-style and the American-style* contracts. These options - as well as others where the payoff is calculated similarly - are referred to as *vanilla options*. Options where the payoff is calculated differently are categorized as *exotic options*. Exotic options can pose challenging problems in valuation and hedging.

The European-style contract pays off simply the future benefit at the expiration date. The price that is paid for the asset when the contract is exercised is called the "exercise price or striking price". The last day on which the contract may be exercised is called the "expiration date" or "maturity date".

An American-style contract on the other hand may be exercised at any time before the expiration date. Since an American-style option provides an investor with a greater degree of flexibility than a European style option, the premium for an American style option is at least equal to or higher than the premium for a European-style option which otherwise has all the same features. For both, the payoff - when it occurs - is via:

$max[(K-S),0]$, for a put option.

Where K is the *Strike price* (the fixed price at which the owner of an option can sell (in the case of a put), the underlying security or commodity.) and S is the *spot price* of the underlying asset.

In finance, a spot contract, spot transaction, or simply spot, is a contract of buying or selling a commodity, security or currency for settlement (payment and delivery) on the spot date, which is normally two business days after the trade date. The settlement price (or rate) is called spot price (or spot rate).

1.1.3 Numerical Methods

The life insurance contracts are often relatively complex and consist of path-dependent derivatives and in most cases, analytical solutions to the valuation problems cannot be found. Hence the need to resort to numerical methods in finding the value of the contract. There are so many possibilities to numerically solve these valuation problems. Among these methods are the Monte Carlo Simulations which is used for the valuation of the insurance contract provided that the policyholders cannot change or (partially) surrender the contract during its term - European contracts (Gerstner et al., n.d.).

In finance, the binomial options pricing model (BOPM) provides a generalizable numerical method for the valuation of options. The binomial model was first proposed by Cox, Ross and Rubinstein (1979) cited in (Hull, 2003). Essentially, the model uses a discrete-time (lattice based) model of the varying price over time of the underlying financial instrument. In general, binomial options pricing models do not have closed-form solutions. The Binomial options pricing model approach is widely used as it is able to handle a variety of conditions for which other models cannot easily be applied. This is largely because the BOPM is based on the description of an underlying instrument over a period of time rather than a single point. As a consequence, it is used to value American options that are exercisable at any time in a given interval as well as Bermudan options that are exercisable at specific instances of time. Being relatively simple, the model is readily implementable in computer software (including a spreadsheet). Although computationally slower than the BlackScholes formula, it is more accurate, particularly for longer-dated options on securities with dividend payments. For these reasons, various versions of the binomial model are widely used by practitioners in the options markets.

1.1.4 Profile of Study Area

The services sector is one of the most important sectors of the Ghanaian economy, and obviously of all other economies. The sector has shown significant development over the past decade. Since 2008, it has contributed significantly towards Ghana's total Gross Domestic Product (GDP). One major sector that constitutes the services sector is the Financial Services Sector. Ghana's Financial Services Sector can be classified into three main categories i.e., Banking, Insurance and Capital Markets. One of these sectors which have contributed immensely towards the growth of the financial services in the Ghana is the insurance industry.

1.2 Statement of the Problem

According to Bjaerk (2001), life insurance contracts and pension plans are complex financial securities that come in many variations. The contracts which offer a guaranteed return each year until maturity are common throughout Ghana and other developing countries. The contracts in Ghana are sometimes equipped with a right to terminate the contract prior to maturity. When this applies the contract is said to contain a surrender option as well. The presence of one or more option elements in life insurance presents complexities in the valuation of the life insurance contract in Ghana.

Also, analytical solution to the valuation problems cannot be found because of the complexities and the presence of path-dependence derivatives in the life insurance contract. Hence, the need to resort to numerical methods in the valuation of the life insurance contract in Ghana.

1.3 Objective of the study

The objectives of the study are:

1. To implement multi-period binomial tree algorithm for fast and accurate numerical valuation of the life insurance contract in Ghana.

2. To determine which of the finite difference method is appropriate for the valuation of the American style life insurance contract using the Black-Scholes equation.

3. To compare the multi-period binomial tree method to the Black-Scholes partial differential equation in valuation of the life insurance contracts in Ghana which contains surrender options.

1.4 Methodology

Since path-dependence prohibits the derivation of closed-form valuation formulas, the problem can be reduced to allow for the development and implementation of a finite difference algorithm and binomial tree method for fast and accurate numerical valuation of the life insurance contracts. Agents are assumed to operate in a continuous time frictionless economy with a perfect financial market, so that tax effects, transaction costs, divisibility, liquidity, and short-sales constraints and other imperfections can be ignored. As regards the specific contracts, we also ignore the effects of expense charges, lapses and mortality. (Duffie, 1996).

1.5 Justification of the Study

The study analyses one of the most common life insurance products - the so-called participating (or with profits) policy. The study takes a finite difference algorithm approach and binomial tree method to the market valuation of equity and liabilities in life insurance companies.

1.6 Thesis Organization

This thesis is organized into five main chapters. Chapter 1 presents the introduction of the thesis. This consists of the background of the study, the research problem statement, objectives of the study, methodology, thesis justification and organization. Chapter 2 is the literature review, which looks at briefly work done by other researchers on the topic. Chapter 3 is the formulation of the mathematical model. Chapter 4 contains the data collection and analysis, Formulation of model instances, algorithms, computational procedure, results and discussion. Chapter 5 looks at summary, conclusions and recommendation of the results.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

This chapter looks into the review of related works on past insurance and insurance liabilities the some related models from other writers.

2.2 Definition and meaning of Insurance

Various writers and researchers has given diverse definitions of the term insurance. But despite the diversity in their expression, they seem to almost talk about the same thing or similar issue. Dorfman (2008), defined insurance as a financial arrangement that redistributes the cost of unexpected losses. Thus, insurance involves the transfer of loss exposures (or uncertainty of loss) to an insurance pool and the redistribution of the cost of losses among the members of the pool and loss as in insurance term is an unintentional decline in or disappearance of value arising from a contingency (Pal, Bolda, & Garg, 2007). According to Dorfman (2008), an insurance system redistributes the cost of losses by collecting premium payment from every participant (insured) in the system. In exchange for the premium payment, the insurer promises to pay the insured's claims in the event of a covered loss. Insurance companies bear risk in return for a fee called premium.

Pal et al (2007) defined insurance as a co-operative mechanism to spread the loss caused by a particular risk over a number of persons who are exposed to it and who agree to ensure themselves against that risk (Pal et al., 2007). FCAC (2011) also defined insurance as a way of reducing your potential financial loss or hardship. It can help cover the cost of unexpected events such as theft, illness or property damage.

Insurance can also provide your loved ones with a financial payment upon your death.

According to Mphasis (2009), insurance, in law and economics, is a form of risk management primarily used to hedge against the risk of a contingent loss. It can therefore be defined as the equitable transfer of the risk of a loss, from one entity to another, in exchange for a premium, and can be thought of as a guaranteed small loss to prevent a large, possibly devastating loss (Mphasis, 2009).

2.3 History of Insurance in Ghana

The British merchants in the 19th century introduced insurance in Ghana and these insurance were bounded by the British merchants and these merchants were bound by the British Merchant shipping laws. The law basically states that all goods being shipped into the British colonies should be carried by ships owned by British citizens. This means that, the goods being carried by ships owned by its citizens were insured by insurance companies in the United Kingdom. Because of this, the insurance companies in the United Kingdom sent their agents to Ghana where the goods were sent. Thus insurance transactions were done through the foreign trading companies in Ghana acting as chief agents of insurance companies in the United Kingdom and other foreign countries. The insurance industry in Ghana at that time comprises mainly insurers; sellers of insurance, insured; purchasers of insurance and intermediaries; agents of insurance companies who act between the insurers and the insured and under the Act of British Parliament, legible to accept proposal and sign and issue insurance cover on behalf of the insurance companies in the United Kingdom. There were no insurance broking, claim adjusting and reinsurance firms at the time in Ghana. Local insurance companies began to emerge towards the independence in 1957. Gold Cost Insurance Company which was formed in 1955 was among the first. In 1957 and 1958, General Insurance Company and Cooperative Insurance Company were also formed respectively. Later, Government purchased Gold Coast Insurance Company and took over Cooperative Insurance Company, merge them to formed State Insurance Company (SIC) in 1962. The period between 1962 and 1970 saw remarkable changes in Ghanaian insurance industry. A lot of rules and regulations were introduced and Acts were passed into laws in the insurance industry. This development sprung up more insurance policies into the market apart from life policies. Policies such as workmen compensation, marine insurance, aviation insurance, accident insurance such as motor, burglary, personal accident, employers' liability, goods-in-transit etc. were also introduced into the market. Most of the laws were enacted to favour and protect the local insurance companies and created opportunities for more insurance companies to spring up (Afriyie, 2006).

2.4 Economics Importance of Insurance

Researchers attention were drawn to factors and patterns of economic growth. As there were always the unexplained percentage of growth, three economic growth theories evolved, classical, neo-classical and endogenous growth theory, which is usually referred to as new growth theory. Classical economic growth theory is a theory on economic growth that argues that economic growth will end because of an increasing population and limited resources. Classical Growth Theory economists believed that temporary increases in real GDP per person would cause a population explosion that would consequently decrease real GDP. Neoclassical economic growth theory is an economic theory that outlines how a steady economic growth rate will be accomplished with the proper amounts of the three driving forces: labour, capital and technology. The theory states that by varying the amounts of labour and capital in the production function, an equilibrium state can be accomplished. When a new technology becomes available, the labour and capital need to be adjusted to maintain growth equilibrium. And endogenous economic growth theory is an economic theory which argues that economic growth is generated from within a system as a direct result of internal processes. More specifically, the theory notes that the enhancement of a nation's human capital will lead to economic growth by means of the development of new forms of technology and efficient and effective means of production. Researches within the financial literature focused on the explanation of externalities that may promote economic growth in addition to labour, capital and technology factors that were typically used by representatives of classical and neo-classical economic growth theories. However, most of the research done is focused on the impact of banking while several studies examined the impact of capital markets development. As reviewed by Levine (2005) most of the existing research on the links between the operation of the financial system and economic growth suggest that better functioning of banks and capital markets facilitates economic growth.

Researchers of the links of insurance and economic growth focused on the impact of economy on insurance development. Beenstock, Dickinson and Khajurja (1986) were among the first to do in-depth research that found a good foundation for a positive impact of income on insurance demand. They used cross-section and time-series data for ten industrialized countries for the period of 1970 to 1981 and found the life insurance demand to be directly positively dependent on income, measured as GDP per capita. A series of empirical was done on both life and nonlife insurance. Using insurance premium as dependent variable and economys income as explanatory variables it was realized that, life and nonlife insurance directly depend upon economic development.

Ward and Zurbruegg (2000) conducted the first study that examined causal relationship between insurance industry growth and economic growth. They short and long dynamic relationships between economic growth, measured by annual real GDP, and insurance industry, measured by total real premiums, for nine OECD countries for the period 1961-1996 were examined. As additional explanatory variables they used changes in private saving rates, the general government budget surplus, population size, the general government level of current expenditure and youth plus old age dependency ratios, measured as the proportion of the total population under 16 and over 65 years of age. Based on bivariate VAR methodology to test for Granger causality, they found that the causal relationship between economic growth and insurance market development vary across countries. Even though they did not found the exact causes, they suspected that possible causes are country-specific nature of cultural, regulatory and legal environment, the improvement in financial intermediation and the moral hazard effect of insurance. These conclusions were also reached by Outreville (1996) and Enz (2000). They also concluded that elasticity of the demand for insurance varies itself with the level of income that is it becomes less sensitive to income growth in more developed economies.

Webb, Grace and Skipper (2002) examine whether banks, life and nonlife insurers contribute to economic growth by facilitating the efficient allocation of capital using revised Solow-Swan model of economic growth. They use cross-country data for 55 developed and developing countries, excluding ex-communist European economies, for the period 1980- 1996. In addition to average penetration of life and non-life insurance, as explanatory variables for GDP per capita growth, they use average growth rate of capital stock per capita, average penetration of banking activity, average level of exports as a share of GDP, average government expenditure share of GDP, natural log of initial real GDP per capita and data on proportion of the population over 25 who have completed primary school. They found that the exogenous components of banking and life insurance penetration are robustly predictive of increased productivity. Synergy between banks and insurers exists, which indicates that banks and insurers collectively provide greater benefits than it would be by summing their individual contributions. Additionally, they found that there is no link between economic growth and non-life insurance. Economic growth affects life insurance penetration while it does not predict banking development.

2.5 Micro-Insurance in Ghana

National Insurance Commission (NIC) and many other agencies have come out with plans to support micorsinsurance in Ghana. NIC most especially, has taken this issue at both local and international level in an effort to gain a better understanding of the relevance of microinsurance for financial inclusion and poverty reduction. Also, the Program for Sustainable Economic Development of the German Development Cooperation (GTZ) has been supporting microsinsurance in general for many years (Wiedmaier-Pfister & Michael, 2009).

2.6 Life Insurance Contracts

Grosen and Jϕrgensen (2000) analyzes one of the most common life insurance products - the so-called participating (or with profits) policy. This type of contract stands in contrast to unit-linked (UL) products in that interest is credited to the policy periodically according to some mechanism which smooths past returns on the life insurance company's (LIC) assets. As is the case for UL products, the participating policies are typically equipped with an interest rate guarantee and possibly also an option to surrender (sell-back) the policy to the LIC before maturity. They showed that the typical participating policy can be decomposed into a risk free bond element, a bonus option, and a surrender option. A dynamic model is constructed in which these elements can be valued separately using contingent claims analysis. The impact of various bonus policies and various levels of the guaranteed interest rate was analysed numerically. They concluded that values of participating policies are highly sensitive to the bonus policy, that surrender options can be quite valuable, and that LIC solvency can be quickly jeopardized if earning opportunities deteriorate in a situation where bonus reserves are low and promised returns are high. The life insurance contract suggested by Grosen and Jϕrgensen (2000) features some annual surplus participation. In this type of contract, the greater of the guaranteed interest rate or a fraction of the asset return is annually credited to the policy and in turn becomes part of the guarantee, which is why this type is called a cliquet-style guarantee. The insurance contract's market value, as well as the insurance company's risk, depends on the guaranteed interest rate as well as on the amount of ongoing surplus.

Grosen and Jϕrgensen (2002) takes a contingent claim approach to the market valuation of equity and liabilities in life insurance companies. Their model explicitly takes into account the facts that the holders of life insurance contracts (LICs) have the first claim on the company's assets whereas equity holders have limited liability, that interest rate guarantees are common elements of LICs, and that LICs according to the so-called contribution principle are entitled to receive a fair share of any investment surplus. Furthermore, a regulatory mechanism in the form of an intervention rule was built into their model. This mechanism is shown to significantly reduce the insolvency risk of the issued contracts and it implies that the various claims on the company's assets become more exotic and obtain barrier option properties. In Grosen and Jϕrgensen (2002), the authors use a model with a point-to-point guarantee, that is, the company guarantees only a maturity payment and an optional participation in the terminal surplus at expiration of the contract. The contract's market value in this model is basically a function of the guaranteed interest rate and the terminal surplus participation and thus only the guaranteed interest rate influences shortfall risk at maturity.

Hansen and Miltersen (2002) analyze minimum rate of return guarantees for life insurance (investment) contracts and pension plans with a smooth surplus distribution mechanism as in the model suggested by Grosen and Jϕrgensen (2000). They specifically model the smoothing mechanism used by most Danish life insurance companies and pension funds and they based the annual distribution of bonus on the smoothing mechanism after the minimum rate of return guarantee into account has been taken. They considered two different methods that the company can use to collect payment for issuing these minimum rates of return guarantee contracts: The direct method where the company gets a fixed percentage fee of the customer's savings each year, e.g., 0.5% in Denmark, and the indirect method where the company gets a share of the distributed surplus.

Ballotta et al. (2003). developed suitable valuation techniques for the broad category of participating life insurance policies. The nature of the liability implied by these contracts allows treating them as options written on the reference portfolio backing the policy. In these contracts, the liabilities annually earn the greater of some guaranteed interest rate and a predetermined fraction of the arithmetic average of the last period returns of some reference portfolio. Their valuation approach is based on the classical contingent claim theory; in particular, Monte Carlo techniques were used to compute the values of the so called *"policy reserve"*, that is the guaranteed payoff and the reversionary bonus, and the terminal bonus.

2.7 Valuation of Life Insurance

Insurance Regulatory and Development Authority (IRDA) in India define Life Insurance as a financial cover for a contingency linked with human life, like death, disability, accident, retirement etc. Human life is subject to risks of death and disability due to natural and accidental causes. When human life is lost or a person is disabled permanently or temporarily, there is loss of income to the household.

Anders and Peter (2002) presented a model which explicitly takes into account the fact that holders of life insurance contracts (LICs) have the first claim on the company's assets whereas equity holders have limited liability, that is, interest rate guarantees are common elements of LICs, and that LICs according to the so-called contribution principle are entitled to receive a fair share of any investment surplus. He further built a regulatory mechanism in the form of an intervention rule into the model. The mechanism was shown to significantly reduce the insolvency risk of the issued contracts and it implies that the various claims on the company's assets become more exotic and obtain barrier option properties. He derived closed valuation formulas. Numerical were also used to illustrate how the model can be used to establish the set of initially fair contracts and to determine the market values of contracts after their inception.

Daniel et al (2010) present such a generic model for the valuation of life insurance contracts and embedded options. They describe various numerical valuation approaches within their generic setup. They particularly focus on contracts containing early exercise features since these present (numerically) challenging valuation problems. Based on an example of participating life insurance contracts, they illustrate the different approaches and compare their efficiency in a simple and a generalized Black-Scholes setup, respectively. Moreover, they study the impact of the considered early exercise feature on their example contract and analyse the influence of model risk by additionally introducing an exponential L'evy model. In their study, they realized that, the Monte Carlo approach yields fast results for European contracts, i.e.

contracts without any early exercise features, but it was inefficient for the valuation of long-term non-European contracts: In this case, the number of necessary simulation steps to obtain accurate results may be extremely high. Secondly, they presented a discretization approach based on the consecutive solution of certain partial (integro-) differential equations (PDE approach). They realized the approach was more apt for the valuation of long-term non-European contracts and allows for the calculation of the Greeks, but depending on the model specifications solving the P(I)DEs can be very complex and can slow down the algorithm considerably. Lastly, they discuss the so-called least-squares Monte Carlo approach. It combines the advantages of the Monte Carlo and the PDE approach: On one hand, it is a backward iterative scheme such that early exercise features can be readily considered and, on the other hand, it remains efficient even if the dimension of the state space becomes larger as the valuation is carried out by Monte Carlo simulations rather than the numerical solution of partial (integro) differential equations (P(I)DEs).

Christopher (2009) explain the new valuation approach based on market-consistent values and its rationale; set out the issues faced by life insurers in implementing the new regime; and explain how these issues were addressed. He did this by analyzing the valuation reports of the 38 life insurers who used the new approach and, in particular, the information about the modelling they used . He realized that, the market consistent basis offers a number of advantages over the traditional regime for valuing liabilities in the United Kingdom. However, he found out that there are further challenges ahead. He elaborated some of the challenges as:

- What economic scenario generator an insurer uses can make a big difference to the reported value of its guarantees and options; more work is needed to understand (and, perhaps, reduce) these differences
- Incorporating, in the modelling, the insurers' planned management actions more fully is important and
- Further controls are needed so that he did not see as a continuation of the errors

that arose when the new regime was introduced.

Eric (1995) addressed the issues of the duration and convexity of insurance liabilities and equity is addressed since these issues have affected the insurance landscape. He added that a correct assessment of these risk measures is critical as they constitute the primary ingredients of any sound asset-liability management approach. In addition, the effort toward a more detailed and more accurate risk picture of life insurance operations enables one to debunk some pitfalls that are commonly encountered in the insurance industry.

Fabio et al (2006) analysed both the term structure of interest and mortality rates role for evaluating a fair value of a life insurance business. They discussed a fair value accounting impact on reserve evaluations and compare it to the traditional deterministic model based on local rules for an Italian balance sheet calculation and a stochastic one based on a diffusion process for both mortality and financial risks. They separated the embedded derivatives from their host contracts so the fair value of a traditional life insurance contract would be expressed as the value of four components: the basic contract, the participation option, the option to annuities and the surrender option.

2.8 Mathematical Model

Many writers have come with different models of valuation of insurance liabilities. Differences among the various models arise from the development of liabilities due to different types of guarantees and different surplus distribution mechanisms among countries. Among these models are Black-Scholes model for asset prices, Levy model for asset prices, the asset-liability management (ALM) model and asset dynamics and interest rate modelling.

In the standard Black-Scholes framework, the total market value of assets A evolves according to a geometric Brownian motion. A Levy process is a process with independent and stationary increments that is continuous in probability (Nadine & Stefan, 2007). Levy models are viable alternative to the geometric Brownian motion for modelling price processes of financial assets. In contract to the Brownian motion, the Levy models allows for jumps in the price path, and skewness and excess kurtosis in asset return distributions, and takes into account features often observed in real-world asset prices. According to Nadine and Stefan (2007), replacing Black-Scholes with the Levy model may seem to be a serious drawback since the Levy model lead to incomplete markets with an infinite number of martingale measures and Black-Scholes model relies on perfect hedging arguments. But the insurance industries cannot and do not follow perfect hedging strategies for participating contracts. Note that, in incomplete market situations, it is not just one arbitrage-free price but a whole range of arbitrage-free prices.

The asset-liability management (ALM) is responsible for the administration of the assets and liabilities of insurance contracts. The ALM model is used for the simulation of the future development of a life insurance company. The ALM model includes the Capital Market Model (for the specification of the dynamics of the short interest rate at a time), the Management model which is used for the capital allocation, bonus declaration mechanism and the shareholder participation and the Liability model for the decrement of policies due to mortality and surrender and the development of the policyholder's accounts. It also include the Balance sheet model which is used to derive the recursive development of all items in the simplified balance sheet (Gerstner et al., n.d.).

2.9 Models Framework

There are financial and actuarial approaches to assess financial guarantees within life insurance contracts. Some of the financial approach is concerned with risk-neutral valuation and has been researched by various authors, e.g. Briys and de Varenne (1997), Grosen and Jϕrgensen (2000), Grosen and Joergensen (2002), or Bauer et al. (2006). Note that the concept of risk-neutral valuation is based on the assumption of a perfect hedging strategy or replicating portfolio. Such a perfect hedge, however, is usually not possible for insurers for several reasons (cf. e.g. Bauer et al. (2006)).

Barbarin and Devolder (2005) propose a model that combines the financial and actuarial approach. They use a simple liability model similar to Briys and de Varenne (1997) and Grosen and Jϕrgensen (2002), with a point to point guarantee and terminal surplus participation. To integrate both approaches, they use a two-step method: First, they determine a guaranteed interest rate such that certain real world risk-measures (e.g. value at risk or expected shortfall risk) are satisfied. Second, to obtain fair contracts, they use risk-neutral valuation and adjust the surplus-participation rate accordingly. This two-step approach can be applied within their model because the surplus participation is only applied at maturity. Therefore whilst it has an influence on the contract's value it has no impact on the considered risk measures. However, they do not consider cliquet-style guarantees that are predominant in many insurance markets.

In Brennan and Schwartz (1976, 1979), cited in Nielsen and Sandmann (1995), the rational insurance premium on an equity - linked insurance contract was obtained through the application of the theory of contingent claims pricing. The premium was determined in an economy with the equity following a geometric Brownian motion, whereas the interest rate was assumed to be constant. Nielsen and Sandmann (1995) realized that, further consideration with deterministic interest rate allow for interest rate risk by assuming an Ornstein - Uhlenbeck process implying a closed form solution of the single premium endowment policy. They presented a model for the multi premium case in the context of a stochastic interest rate process. It was shown that the insurance contract includes an Asian - like option contract. No closed form solution will be obtained. They discuss different numerical approaches and apply Monte Carlo simulations with a variance reduction technique.

Nielsen and Sandmann (1995) concluded that, in an economy with stochastic development of the term structure of interest rates a model for the determination of the fair premium on an equity linked life insurance contract has been established. An essential part of the premium equation consists of a contingent claim with a character as an Asian option. However it was shown that the stochastic interest rate and the long time to maturity of the insurance contract prohibited the application of the "usual" solution methods: Edgeworth expansion or Fast Fourier transform. The approximation formula developed by Vorst (1992) cited in Nielsen and Sandmann (1995) exhibited a better performance than the two just mentioned for medium term contracts. Nielsen and Sandmann (1995) applied and advocated for Monte Carlo simulations to overcome the difficulties. The result obtained was compared to the Edgeworth and Vorst approximation and found to be preferable to these. They realized that, although the Monte Carlo simulations are more time consuming than the other methods they did not take it as a serious critical point against simulation as the fair premium only has to be calculated once when the contract is entered.

According to Mike et al (2010), the use of advanced data mining techniques to improve decision making has already taken root in property and casualty insurance as well as in many other industries. However since in their opinion, the application of such techniques for more objectives, consistent and optimal decision making in the life insurance industry is still in a nascent stage, they described the ways data mining and multivariate analytic techniques can be used to improve decision making processes in such functions as life insurance underwriting and marketing, resulting in more profitable and efficient operations. They implemented predictive modelling in life insurance underwriting and marketing and demonstrated the segmentation power of predictive modelling and resulting business benefits.

The liability structure of the insurance company is implied by participating life insurance contracts and based on a model suggested by Ballota et al (2006) cited in Nadin and Stefan (2007). According to Ballota et al (2006) cited in Nadin and Stefan (2007), for policyholders to initiate contracts, they must pay a single premium P_0 and if the company's initial capital is E_0, then the sum of the initial contribution $A_0 = E_0 + P_0$. This sum of initial contribution A_0 is invested in the reference portfolio. Hence for $0 < k \leq 1$, it holds that $P_0 = k \cdot A_0$ and $E_0 = (1-k) \cdot A_0$, where k represents the leverage of the company.

If P denote the policyholders' account, that is, the book value of the policy reserves. The policy reserve P is a year-to-year, or cliquet-style, guarantee, which means it annually earns the maximum guaranteed interest rate or a fraction α of the annual surplus generated by the insurer's investment portfolio. Hence for $t = 1, 2, ..., T$, the development of the policy reserve is given by

$$P(t) = P(t-1) \cdot (1 + max[g, \alpha(\frac{A(t)}{A(t-1)} - 1)])$$

Bollotta et al (2006) summarizes the value of liabilities $L(T)$ as

$$L(T) = P(T) + \delta[k \cdot A(T)]^{+} = P(T) + \delta \cdot B(T) - D(T)$$

Where $D(T)$ denotes the default put option, $E(T)$, the residual claim of the equity holders and is determined as the difference between the market value of the reference portfolio $A(T)$ and the policyholder's claim $L(T)$, i. e.

$$E(T) = A(T) - L(T) = \max(A(T) - P(T), 0) - \delta \cdot B(T) \geq 0$$

In the standard Black-Scholes framework, the total market value of assets A evolves according to a geometric Brownian motion as stated earlier. In Black-Scholes model for asset prices, the standard Brownian motion $(W^P(t), 0 \leq t \leq T)$ on a probability space (Ω, F, P) and $(F_t), 0 \leq t \leq T$, be the filtration generated by the Brownian motion. The total market value of the assets A in standard Brownian motion evolves according to a geometric Brownian motion under the objective measure P is given by

$$dA(t) = mA(t)dt + \sigma A(t)dW^P(t),$$

with constant asset drift m, volatility σ, and P-Brownian motion W^P, assuming a complete, perfect, and frictionless market (Nadine & Stefan, 2007). The solution of the stochastic differential equation is

$$A(t) = A(0) \cdot e^{((m - \frac{\sigma^2}{2})t + \sigma \cdot W^P(t))} = A(t-1) \cdot e^{(m - \frac{\sigma^2}{2} + \sigma \cdot (W^P(t) - W^P(t-1)))}$$

Bjarke et al (2001) described the financial characteristics of the contract which is arranged between the insurance or Pension Company and the investor. If, a contract of nominal value P_0 is issued by the company at time zero and the contract is immediately acquired by an investor for a single premium of V_0, then let's assumed that there are no further payments from or to the contract prior to expiration at time T where the contract is settled by a single payment from the company to the investor. In general P_0 shall be tread as exogenously given whereas V_0 is to be determined by our model. V_0 will also be referred to as the fair value of the contract. The contract is a contingent claim and we will determine its value process using methods from the well-developed theory of contingent claims valuation as also illustrated by Duffie (1996). The benefit from the contract at the maturity date is denoted $P(T)$ and shall generally refer to $\{P(t)\}_{0\leq t\leq T}$ as the account balance process of the contract. The evolution of $P(\cdot)$ between successive time points in the set $\Upsilon \equiv \{1,2,...,3\}$ is determined by the discretely compounded policy interest rate process, $\{rP(t)\}_{t\in\Upsilon}$. Specifically, we have

$$P(t) = (1 + r_P(t)) \cdot P(t-1), t \in \Upsilon$$

which implies

$$P(t) = P_0 \cdot \prod_{i=1}^{t} (1 + r_P(i)), t \in \Upsilon$$

Time is measured in years, $P(\cdot)$ is updated annually, and the $r_P(\cdot)s$ are annualized rates as in real life contracts. Now, the way in which $rP(\cdot)$ is determined is of course of vital importance.

According to Nadine and Alexander (unpublished), fair pricing of embedded options in life insurance contracts is usually conducted by using the appropriate concept of risk-neutral valuation. This concept assumes a perfect hedging strategy, which insurance companies can hardly pursue in practice. They extended the risk-neutral valuation concept with a risk measurement approach and accomplish this by first calibrating contract parameters that lead to the same market value using risk-neutral valuation. They then measure the resulting risk assuming that insurers do not follow perfect hedging strategies. They use lower partial moments as the relevant risk measure, comparing shortfall probability, expected shortfall, and shortfall variance. Their research showed that even when contracts have the same market value, the insurance company's risk can vary widely, a finding that allows us to identify key risk drivers for participating life insurance contracts.

2.10 Numerical Methods Approach

Gerstner et al (unpublished) propose a discrete time asset-liability management (ALM) model for the simulation of simplified balance sheets of life insurance products. The model incorporates the most important life insurance product characteristics, the surrender of contracts, a reserve-dependent bonus declaration, a dynamic asset allocation and a two-factor stochastic capital market. All terms arising in the model can be calculated recursively which allows an easy implementation and efficient evaluation of the model equations. The modular design of the model permits straightforward modifications and extensions to handle specific requirements. In practise, the simulation of stochastic ALM models is usually performed by Monte Carlo methods which suffer from relatively low convergence rates and often very long run times, though. As alternatives to Monte Carlo simulation, they proposed deterministic integration schemes, such as quasi-Monte Carlo and sparse grid methods for the numerical simulation of such models. Their efficiency is demonstrated by numerical examples which show that the deterministic methods often perform much better than Monte Carlo simulation as well as by theoretical considerations which show that ALM problems are often of low effective dimension.

Russel and Collins (1962) described the application of the Monte Carlo technique to a practical situation in a company to solve the problem of rate-making with real problem in the transfer of coverage from one carrier to another by a policyholder who finds himself or herself in a large deficit position with the original carrier in the field of insurance. This situation can be avoided if the policyholder is willing to pay an additional charge for a guarantee of an upper limit on the amount of deficit carried forward from one year to the following years. In order to determine such a charge, it is necessary to know the probability of, the expected value of, and the variation of claims in excess of a given amount. The basic problem to be solved, of course, is that of determining the frequency distribution of the annual claim cost of a given group of lives for a given year. It was desired that the following properties of the group be allowed to vary over rather wide ranges: (1) the size of the group, (2) the age distribution of the group, (3) the sex distribution of the group, (4) the total amount of insurance, and (5) the distribution of the insurance on individual lives. Since the analytical solution of such a problem was complex, they used the Monte Carlo technique, which is admirably suited to a problem of this nature.

Bjarke et al (2001), sets up a model for the valuation of traditional participating life insurance policies. These claims are characterized by their explicit interest rate guarantees and by various embedded option elements, such as bonus and surrender options. Owing to the structure of these contracts, the theory of contingent claims pricing is a particularly well-suited framework for the analysis of their valuation. The eventual benefits (or pay-offs) from the contracts considered crucially depend on the history of returns on the insurance company's assets during the contract period. This path-dependence prohibits the derivation of closed-form valuation formulas but they demonstrated that the dimensionality of the problem can be reduced to allow for the development and implementation of a finite difference algorithm for fast and accurate numerical evaluation of the contracts. They also demonstrate how the fundamental financial model can be extended to allow for mortality risk and we provide a wide range of numerical pricing results. So they use finite difference approach to value the life insurance liabilities.

The finite difference approaches used include the implicit finite difference, the explicit finite difference and the Crank Nicolson scheme. Among them, it was realized was more accurate than the implicit and the explicit finite difference because the error associated with the final solution with Crank-Nicolson is smaller the other two methods.

2.11 Path-Dependent Option

A path-dependent option is an option whose payoff depends on the path followed by the price of the underlying asset. Path dependence explains how the set of decisions one faces for any given circumstance is limited by the decisions one has made in the past, even though past circumstances may no longer be relevant. Their payoffs do not merely depend on the final value of the underlying asset, but also on the way that the price was reached. American-style contract for example, is path-dependent since there is usually a probability of the option being exercised before expiry and thus ceasing to exist. There are many kinds of path-dependent options, such as *lookback* and *Asian options*. Others includes Russian, Game or Israeli, Cumulative Parisian and barrier options.

Financial derivatives (eg. options and futures) derive their value from an underlying traded financial security, whose price is modelled by some stochastic process. In their most general form, the option payoff is path dependent since it depends on the entire future path traversed by the underlying security (Andrew, 1999). According to Chance (1995) cited in Andrew (1999), path dependent options are defined using either discrete or continuous price sampling. Closed form solutions are often available for continuous sample, but in practice most traded path dependent options are discretely sampled. It is known that the application of these closed form solutions leads to substantial pricing errors for discretely sampled options (Andrew, 1999) and (Chance, 1995).

According to David (1985) cited in Scott (2006), the concept of path dependence originated as an idea that a small initial advantage or a few minor random shocks along the way could alter the course of history. But the scope of this idea has grown so wide that path dependence has dulled it value and in becoming a trendy way to say that history matters, path dependence no longer provides any analytic leverage (Pierson, 2000).

As indicated by David (1985), the concept of path dependence seems almost metaphorical. Path dependence simply means that the current and future states, actions or decisions depend on the path of previous states, actions or decisions (Scott, 2006).

Scott (2006) described a dynamic process that produces outcomes at discrete time intervals indexed by the integers, $t = \{1, 2, \ldots\}$ He denotes the outcome at time t as x_t. In addition to the outcome there are other information, opportunities, or events that may arise in a given time period which he described as the environment at time t. This contains exogenous factors that influence outcomes. A history at time T, h_T is the combination of all outcomes x_t up through time $(T-1)$ and all other factors, the y_t, up through time T.

Chapter 3

METHODOLOGY

3.1 Introduction

This chapter consist of the methodology used to implement the numerical methods used in the valuation of traditional participation of life insurance policies using the finite difference approach and the binomial tree method. There are three formulas of finite difference method, the backward difference scheme, the forward difference scheme and the central difference scheme.

3.2 Preliminaries

In this study, the algorithm was derived from the application of the finite difference scheme to solve boundary-value partial differential equations proposed by Bjarke et al (2001) and the multi-period binomial tree method. This is because in this scheme, the path-dependence variable is conveniently treated as a parameter in both the European-style contract and the American-style contract. Also, the simple Monte Carlo approach used by Grosen and Jϕrgensen (2000) requires that the exact solution to the Geometric Brownian Motion (GBM)~~, the~~ finite difference approach is not specific to this choice of the process in the evolution in the asset base (Bjarke, Peter, & Anders, 2001). The study is based on one time premium payment.

Definition 3.1: Differential Equation

A differential equation is an equation involving the unknown function $y = f(t)$, together with its derivatives $y', y'', ..., y^{(n)}$

Mathematically, a differential equation may be express implicitly as

$$F(t, y, y', y'', ... y^{(n)}) = 0 \tag{3.1}$$

Explicitly, the general form of a differential equation can be written as

$$y^{(n)} = f(t, y', y'', ..., y^{(n-1)}) \tag{3.2}$$

Definition 3.2: (Ordinary Differential Equations)

An ordinary differential equation (ODE) is an equation involving an unknown function of a single variable together with one or more of its derivatives.

Definition 3.3: (Order of Differential Equations)

A first order differential equation is of the form

$$y' = f(t, y) \tag{3.3}$$

and the equation is said to be in **normal form**.

A differential equation of **order n** is of the form

$$f(t, y, y', y'', ... y^{(n)}) = 0 \tag{3.4}$$

and ~~is~~ also said to be in **normal form**.

A typical n^{th} order linear differential equation is given by

$$y^{(n)} + a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + ... + a_{(n-1)}(t)y' + a_n(t)y = f(t) \tag{3.5}$$

Definition 3.4: Partial Differential Equations (PDE)

A partial differential equation (PDE) is an equation that involves an unknown function (the dependent variable) and some of its partial derivatives with respect to two or more independent variables. Mathematically, PDE is of the form

$$F(t_1,\dots,t_n,u,\frac{\partial u}{\partial t_1},\dots,\frac{\partial u}{\partial t_n},\frac{\partial^2 u}{\partial t_1 \partial t_1},\dots,\frac{\partial^2 u}{\partial t_1 \partial t_n},\dots)=0 \tag{3.6}$$

If F is a linear function of u and its derivatives, then the PDE is called *linear*. An n^{th}-order PDE has the highest order derivative of order n. A simple PDE is

$$\frac{\partial u}{\partial t}(t,y)=0 \tag{3.7}$$

This relation implies that the function $u(t,y)$ is independent of t. However the equation gives no information on the function's dependence on the variable y. Hence the general solution of this equation is

$$u(t,y)=f(y) \tag{3.8}$$

where f is an arbitrary function of y.

General linear second order PDE is of the form

$$a(t,y)u_{tt}+2b(t,y)u_{ty}+c(t,y)u_{yy}+d(t,y)u_t+e(t,y)u_y+g(t,y)u=f(t,y) \tag{3.9}$$

where $(t,y)\in\Omega$ is a domain in $t-y$ coordinates.

Definition 3.5: Stochastic Differential Equation (SDE)

A stochastic differential equation (**SDE**) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process (Davis, 2005). In probability theory, a stochastic process or sometimes random process (widely used) is a collection of random variables; this is often used to represent the evolution of some random value, or system, over time. This is the probabilistic counterpart to a deterministic process (or deterministic system).

SDE's can be used to model the randomness of the underlying asset in valuing insurance liabilities. For example, The dynamic behaviour of the asset price in a time interval dt can then be represented by the SDE given by

$$ds_t = \alpha(A_t, t)dt + \sigma(A_t, t)dW_t \; for \; t \in [0, \infty) \tag{3.10}$$

3.3 Finite Difference Approximation

A finite difference method typically involves the following steps:

1. Generate a grid, for example $(x_i, t^{(k)})$, where we want to find an approximate solution.

2. Substitute the derivatives in an ODE/PDE or an ODE/PDE system of equations with finite difference schemes. The ODE/PDE then become a linear/non-linear system of algebraic equations.

3. Solve the system of algebraic equations.

4. Implement and debug the computer code.

5. Do the error analysis, both analytically and numerically.

3.3.1 Types of finite difference methods

Depending on how we approximate the partial derivative with respect to time, we have three different finite difference schemes:

1. Explicit finite difference scheme, when we use the forward difference formula

2. Implicit finite difference scheme, when we use the backward difference formula

3. Crank-Nicholson finite difference scheme, when we use the centred difference formula.

3.3.2 Finite difference formulas of Ordinary Differential Equations (ODE)

There are three commonly used finite difference formulas to approximate first order derivative of a function $f(x)$. They are forward finite difference, backward finite difference and central finite difference.

Let's consider **Taylor's series** expansion of a function $f(x)$ in the neighbourhood of $x = x_i$:

$$f_{i+1} = f_i + \Delta x f_i' + \frac{(\Delta x)^2}{2!} f_i'' + \frac{(\Delta x)^3}{3!} f_i''' + \frac{(\Delta x)^4}{4!} f_i'''' + \dots \tag{3.11}$$

where $\Delta x = x_{i+1} - x_i$.

Solving equation 3.11 for f_i, we have

$$f_i' = \frac{f_{i+1} - f_i}{\Delta x} - \frac{(\Delta x)}{2!} f_i'' - \frac{(\Delta x)}{3!} f_i''' - \dots \tag{3.12}$$

Using the mean-value theorem, equation 3.12 becomes

$$f_i' = \frac{f_{i+1} - f_i}{\Delta x} - \frac{\Delta x}{2} f''(\xi);\ x_i < \xi < x_{i+1} \tag{3.13}$$

where $O(\Delta x) = -\frac{\Delta x}{2} f''(\xi)$, the order of Δx , indicates the error is proportional to the step length (Δx) and also a second derivative of f. Hence

$$f_i' \approx \frac{f_{i+1} - f_i}{\Delta x} \tag{3.14}$$

Equation 3.14 is called the **Forward Difference Formula**.

Similarly, the **Backward Difference Formula** from the Taylor series

$$f_{i-1} = f_i - \Delta x f_i' + \frac{(\Delta x)^2}{2!} f_i'' - \frac{(\Delta x)^3}{3!} f_i''' + \dots \tag{3.15}$$

is given by

$$f_i' \approx \frac{f_i - f_{i-1}}{\Delta x} \tag{3.16}$$

with the error $O(\Delta x) = \frac{\Delta x}{2} f''(\xi)$. Finally, subtracting equation 3.15 from equation 3.11 we have the central difference formula

$$f_i' \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x} \tag{3.17}$$

with the error $O(\Delta x) = -\frac{(\Delta x)^2}{2} f'''(\xi)$

3.3.3 Finite Difference Approximation for Partial Differential Equations (PDE)

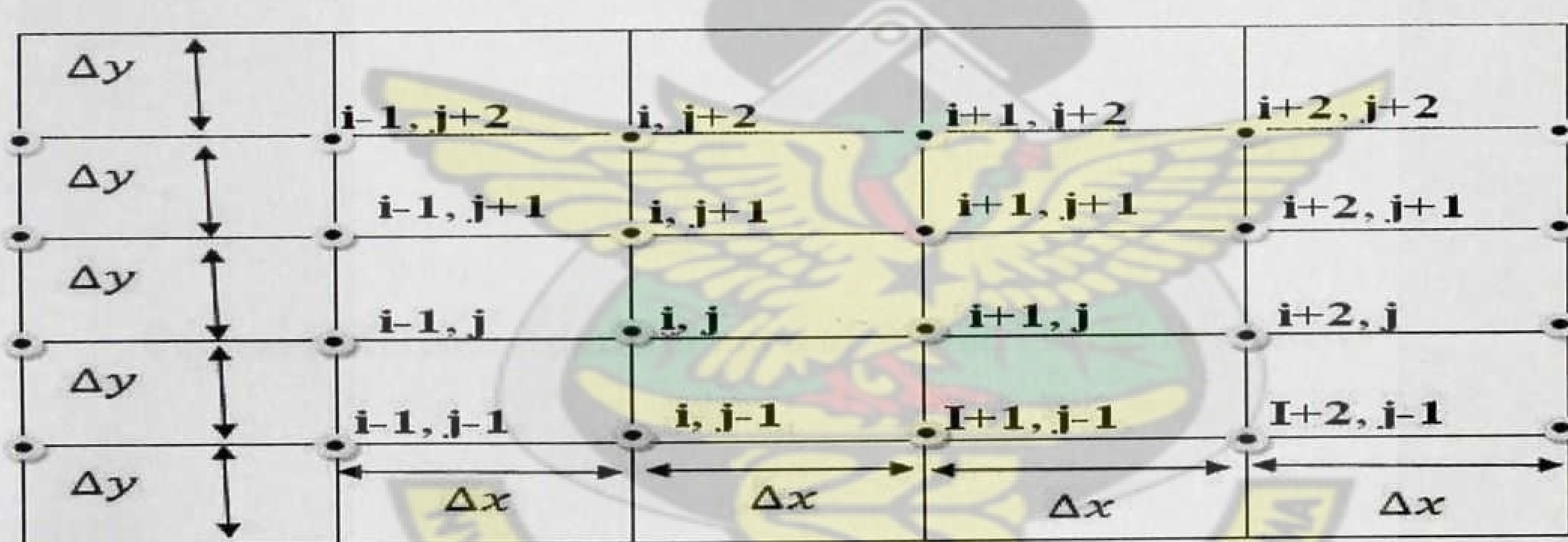

Figure 3.1: Two-dimensional grid

In many financial and engineering problems , the function f depends on two or more independent variables, hence the need for finite-difference approximation of partial derivatives. Since partial derivatives denotes the local variation of a function with respect to a particular independent variable while all other independent variables are held constant, finite difference approximation of ordinary derivatives can be adapted for the partial derivatives. If there are two independent variables, we use the notation (i, j) to designate the pivot point, and if there are three independent variables, (i, j, k) are used where i, j and k are the counters in the x, y and z directions.

Figure 3.1 above is a two-dimensional finite-difference grid. If we consider the function $f(x,y)$, then the finite-difference approximation for the partial derivative $\frac{\partial f(x,y)}{\partial x}$ at $x = x_i, y = y_i$ can be found by fixing the value of y at y_i and treating $f(x, y_i)$ as a one-variable function. The forward, backward and central difference of $\frac{\partial f}{\partial x}$ can be express as

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \approx \frac{f(x_i+\Delta x, y_j) - f(x_i, y_j)}{\Delta x} \tag{3.18}$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \approx \frac{f(x_i, y_j) - f(x_i-\Delta x, y_j)}{\Delta x} \tag{3.19}$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \approx \frac{f(x_i+\Delta x, y_j) - f(x_i-\Delta x, y_j)}{2\Delta x} \tag{3.20}$$

Central-Difference Approximation of Second Partial Derivatives

The central-difference approximation of second partial derivatives at (x_i, y_i) can be derived as

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \approx \frac{f(x_i+\Delta x, y_j) - 2f(x_i, y_j) + f(x_i-\Delta x, y_j)}{(\Delta x)^2} \tag{3.21}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j} \approx \frac{f(x_i, y_j+\Delta y) - 2f(x_i, y_j) + f(x_i, y_j-\Delta y)}{(\Delta y)^2} \tag{3.22}$$

and

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{i,j} \approx \frac{f(x_i+\Delta x, y_j+\Delta y) - f(x_i+\Delta x, y_j-\Delta y) - f(x_i-\Delta x, y_j+\Delta y) + f(x_i-\Delta x, y_j-\Delta y)}{(4\Delta x\Delta y)} \tag{3.23}$$

Error of finite-difference approximation of partial derivatives

To find the error associated with finite-difference approximation of partial derivatives, we use Taylor series expansion of $f(x,y)$ around the point (x_i, y_j). That is,

$$f_{i\pm1,j} = f_{i,j} \pm \Delta x \frac{\partial f}{\partial x}\bigg|_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2}\bigg|_{i,j} \pm \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3}\bigg|_{i,j} + \dots \tag{3.24}$$

$$f_{i,j\pm1} = f_{i,j} \pm \Delta y \frac{\partial f}{\partial y}\bigg|_{i,j} + \frac{(\Delta y)^2}{2!}\frac{\partial^2 f}{\partial y^2}\bigg|_{i,j} \pm \frac{(\Delta y)^3}{3!}\frac{\partial^3 f}{\partial y^3}\bigg|_{i,j} + \dots \tag{3.25}$$

Truncating equation 3.24 after the nth order, we have the error

$$R_{x,n} \simeq (-1)^{n+1}\frac{(\Delta x)^{n+1}}{(n+1)!}\frac{\partial^{n+1} f(x,y)}{\partial x^{n+1}}\bigg|_{i,j} \tag{3.26}$$

and truncating equation 3.25 after the nth order gives the error

$$R_{y,n} \simeq (-1)^{n+1}\frac{(\Delta y)^{n+1}}{(n+1)!}\frac{\partial^{n+1} f(x,y)}{\partial y^{n+1}}\bigg|_{i,j} \tag{3.27}$$

3.3.4 Finite difference approximation for two dimensional PDEs

Let's consider a two-dimensional PDE

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = g(x,y) \tag{3.28}$$

such that $a \le x \le b$ and $c \le y \le d$. If we let $U(a,y) = U_a$, $U(b,y) = U_b$, $U(x,c) = U_c$ and $U(x,d) = U_d$, where U_a, U_b, U_c and U_d are the boundary conditions at y and x respectively. Note that, Δx is not necessarily equal to Δy, but for this case we let $\Delta x = \Delta y = h$. Let's consider the grid below.

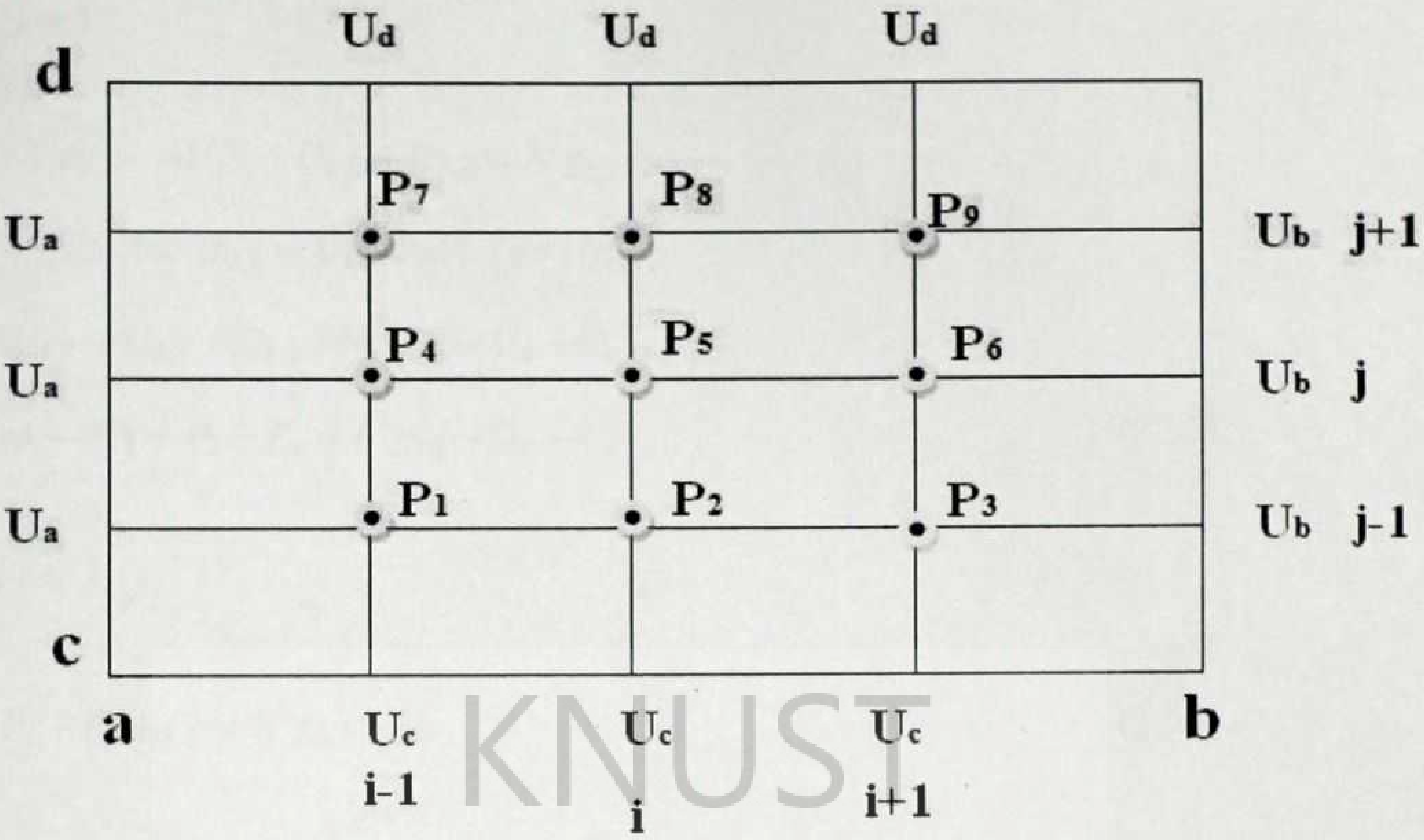

Figure 3.2: Simplify two-dimensional grid

At the generic points

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{i,j} + \left.\frac{\partial^2 U}{\partial y^2}\right|_{i,j} = g_{i,j}$$

Using the central difference scheme we have

$$\frac{\partial^2 U_{i,j}}{\partial x^2} \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} \tag{3.29}$$

and

$$\frac{\partial^2 U_{i,j}}{\partial y^2} \approx \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2} \tag{3.30}$$

Adding equations 3.29 and 3.30, we have

$$\frac{\partial^2 U_{i,j}}{\partial x^2} + \frac{\partial^2 U_{i,j}}{\partial y^2} \approx \frac{U_{i-1,j} + U_{i+1,j} - 2U_{i,j} + U_{i,j-1} + U_{i,j+1}}{h^2} = g_{i,j} \tag{3.31}$$

$$\Rightarrow U_{i-1,j} + U_{i+1,j} - 4U_{i,j} + U_{i,j-1} + U_{i,j+1} = h^2 g_{i,j} \tag{3.32}$$

At P_1: $i = 1, j = 1$

$$\Rightarrow U_{0,1} + u_{2,1} - 4U_{1,1} + U_{1,0} + U_{1,2} = h^2 g_{1,1}$$

$$but\ U_{0,1} = U_a,\ and\ U_{1,0} = U_c$$

$$\Rightarrow -4U_{1,1} + U_{2,1} + U_{1,2} = h^2 g_{1,1} - U_a - U_c$$

$$\Rightarrow -4P_1 + P_2 + P_4 = h^2 g_{1,1} - U_a - U_c \tag{3.33}$$

At P_2: $i = 2,\ j = 1$

$$P_1 - 4P_2 + P_3 + P_5 = h^2 g_{2,1} - U_c \tag{3.34}$$

Using the computational model below,

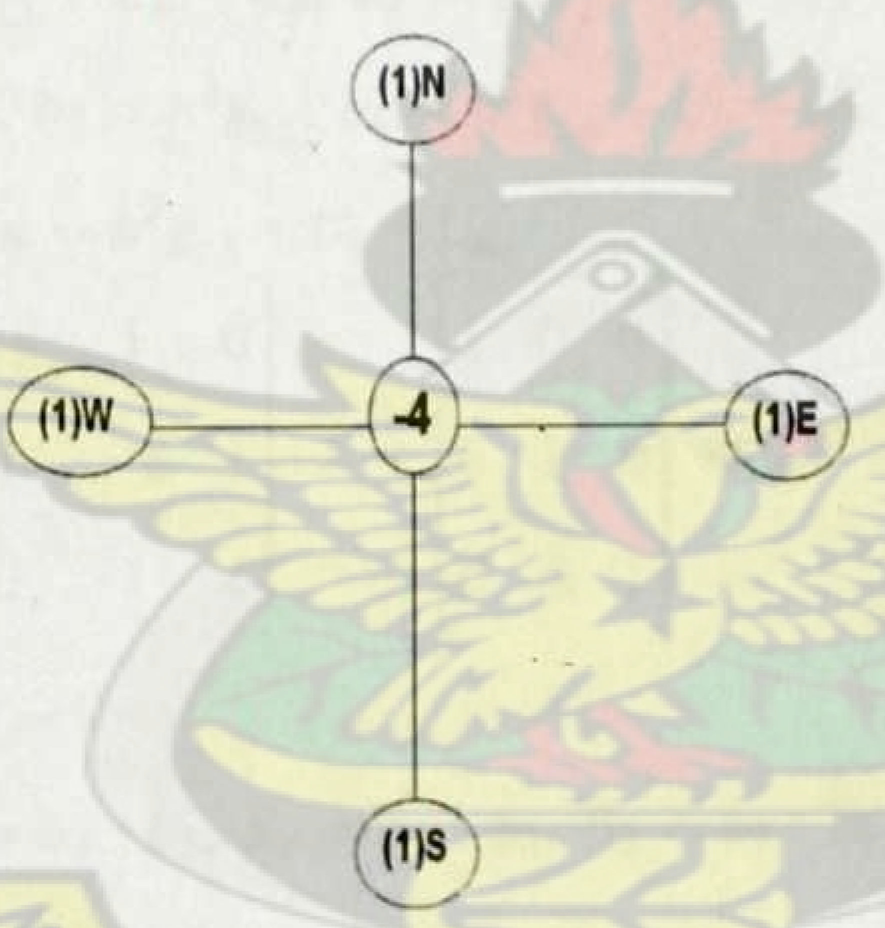

Figure 3.3: Computational model

we have $U_W + U_E - 4U_P + U_N + U_S = h^2 g_P$

Summarizing the generic points, we have

$$\begin{array}{llllllllllll}
P_1: & -4P_1 & +P_2 & & +P_4 & & & & & & = & h^2 g_{1,1} - U_a - U_c \\
P_2: & P_1 & -4P_2 & +P_3 & & +P_5 & & & & & = & h^2 g_{2,1} - U_c \\
P_3: & & P_2 & -4P_3 & & & +P_6 & & & & = & h^2 g_{3,1} - U_b \\
P_4: & P_1 & & & -P_4 & +P_5 & & +P_7 & & & = & h^2 g_{1,2} - U_a \\
P_5: & & P_2 & & +P_4 & -4P_5 & & & +P_8 & & = & h^2 g_{2,2} \\
P_6: & & & P_3 & & +P_5 & -4P_6 & & & +P_9 & = & h^2 g_{3,2} - U_b \\
P_7: & & & & P_4 & & & -4P_7 & +P_8 & & = & h^2 g_{1,3} - U_a - U_d \\
P_8: & & & & & P_5 & & +P_7 & -4P_8 & +P_9 & = & h^2 g_{2,3} - U_d \\
P_9: & & & & & & P_6 & & +P_8 & -4P_9 & = & h^2 g_{3,3} - U_b - U_d
\end{array}$$

Writing the above systems in matrix, we obtain

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \end{bmatrix}$$

where $b_1 = h^2 g_{1,1} - U_a - U_b$, $b_2 = h^2 g_{2,1} - U_c$, $b_3 = h^2 g_{2,1} - U_c$, $b_4 = h^2 g_{3,1} - U_b$, $b_4 = h^2 g_{1,2} - U_a$, $b_5 = h^2 g_{2,2}$, $b_6 = h^2 g_{3,2} - U_a - U_b$, $b_7 = h^2 g_{1,3} - U_a - U - d$, $b_8 = h^2 g_{2,3} - U_d$ and $b_9 = h^2 g_{3,3} - U_b - U_d$

If we let $A = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Also $B_1 = b_1 : b_3$, $B_2 = b_4 : b_6$, $B_3 = b_7 : b_9$, $X_1 = P_1 : P_3$, $X_2 = P_4 : P_6$ and $X_3 = P_7 : P_9$, we obtain the matrix

$$\begin{bmatrix} A & I & O \\ I & A & I \\ O & I & A \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{3.35}$$

which is simplify in the form

$$HX = B \tag{3.36}$$

When systems are express in the form $HX = B$, we have several solution techniques in solving it.

3.3.5 Solution Techniques

There exist different types of solution techniques. Notable among them are the Gaussian Elimination, Gauss-Jordan Elimination, LU and QR decomposition and iterative methods. The iterative methods includes Jacobi, Gauss-Seidel and the relation methods (Successive Under Relaxation and Successive Over Relaxation - SOR).

Iterative Methods

As stated earlier, the common iterative techniques for solving linear systems are Jacobi, Gauss-Seidel and SOR method. The basic idea is to solve the i^{th} equation in the system for the i^{th} variable ((Laurene, 2008)). Let's consider the four-by-four system below:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \tag{3.37}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \tag{3.38}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \tag{3.39}$$

$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \tag{3.40}$$

Solving for x_1, x_2, x_3, x_4 in equations 3.37 to 3.40, we have

$$x_1 = -\frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \frac{a_{14}}{a_{11}}x_4 + \frac{b_1}{a_{11}} \tag{3.41}$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \frac{a_{24}}{a_{22}}x_4 + \frac{b_2}{a_{22}} \tag{3.42}$$

$$x_3 = -\frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \frac{a_{34}}{a_{33}}x_4 + \frac{b_3}{a_{33}} \tag{3.43}$$

$$x_4 = -\frac{a_{41}}{a_{44}}x_1 - \frac{a_{42}}{a_{44}}x_2 - \frac{a_{43}}{a_{44}}x_3 + \frac{b_4}{a_{44}} \tag{3.44}$$

Iterative methods are stopped at certain conditions. Below are two possibilities:

1. Iterations are stopped when the norm of the change in the solution vector x from iteration to the next is sufficiently small or
2. when the norm of the residual vector, $\|Ax - b\|$, is below a specified tolerance.

3.3.6 Jacobi Iterative Technique

In the Jacobi method, the system $Ax = b$ is transformed into the system $X = Hx + d$, where H has the zeros on the diagonal and X is a vector which is updated from previous vector x.

The systems 3.41 to 3.44 in a matrix form is

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & -\frac{a_{14}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & -\frac{a_{24}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{30}}{a_{33}} & 0 & -\frac{a_{34}}{a_{33}} \\ -\frac{a_{41}}{a_{44}} & -\frac{a_{42}}{a_{44}} & -\frac{a_{43}}{a_{44}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \frac{b_3}{a_{33}} \\ \frac{b_4}{a_{44}} \end{bmatrix} \tag{3.45}$$

which is in the form $X = Hx + d$. Iteratively

$$X_{i+1} = Hx_i + d \tag{3.46}$$

Equation 3.46 is the Jacobi Iterative technician. In this iteration, **matlab** will be used to run the iteration.

3.3.7 Convergence, Consistency, and Stability

Norm of a sequence

For any sequence of numbers a_n, let

$$||a|| = \max_n |a_n| \tag{3.47}$$

This norm is called the "L_∞ -norm" of the sequence a_n. There are other kinds of norms, which are equally useful in their own circumstances but the researcher is interest in the L_∞ -norm because we want to know the *maximum error* of our numerical solution which is bounded by some tolerance ε_{tol}:

$$\max_n |\varepsilon| \leq \varepsilon_{tol} \ or \ ||\varepsilon|| \leq \varepsilon_{tol} \tag{3.48}$$

Convergence

Definition 3.5: A one-step finite difference scheme approximating a partial differential equation is a convergent scheme if for any solution to the partial differential equation, $u(t,x)$, and solutions to the finite difference scheme, u_i^n, such that v_i^0 converges to $u_0(x)$ as $i\Delta x$ converges to x, then v_i^n converges to $u(t,x)$ as $(n\Delta t, i\Delta x)$ converges to (t,x) as Δt, Δx converges to 0 (Singiresu, 2002).

Simply put, a numerical method is convergent if its global error computed up to a given x satisfies:

$$\lim_{h\to 0} \|\varepsilon\| \equiv \lim_{h\to 0} \|y - Y\| = \lim_{h\to 0} \max_n |y_n - Y_n| = 0. \tag{3.49}$$

This implies that, the numerical solution Y_n is computed with no round-off (i.e machine) error.

Consistency

Definition 3.6 : A numerical method $F[Y_n, h] = 0$ is consistent if

$$\lim_{h\to 0} \|\tau\| = 0 \tag{3.50}$$

where $\tau_n = F[y_n, h]$.

Stability

Definition 3.6: We say the method is *numerically stable* if the actual error e_j^n is bounded as $n \to \infty$. Conversely, a method is *numerically unstable* if the actual error grows without bound (this phenomenon is known as *numerical instability*)

This is to say that, for a stable method, the deviation between two numerical solutions arising for example, due to the round-off error, does not grow with the number of steps.

3.4 Life Insurance Contract Models

3.4.1 The Contract and Dynamic Model

Let's consider a simple balance sheet of the insurance company.

Table 3.1: Simplified balance sheet

Assets	Liabilities
$A(t) = A_t$	$L(t) = L_t$
	$B(t) = B_t$
A_t	A_t

Note that, the balance sheet is not the companies balance sheet but rather a simplified form of the asset and liability situation in relation to a given contract. In the above balance sheet, A_t denotes the market value of the insurer's asset portfolio, L_t denotes the policyholder's account balance and $B_t = A_t - L_t$ is the bonus reserve at time t.

If charges are disregarded, the policyholder's account balance at time zero, L_0, equals the single up-front premium P, that is, $L_0 = P$. Note that the policyholder may surrender his or her contract during the term of the contract. If the contract is lapsed at time $v_0 \in \{1,...,T\}$, the policyholder receives the current account balance L_{v_0}. It is assumed that shareholders are paid dividends during the anniversaries as compensations for the adopted risk.

The life insurance companies invest largely in highly liquid assets like bonds and stocks for which market prices are easily observed. For this reason, we can assume that A is traded (Bjarke et al., 2001). The policy account balance, $L(t)$, is a book value. Alternatively, $L(t)$ can also be considered to be the funds set aside to cover the insurance contract liability a distributed reserve. $B(t)$ is the undistributed reserve or the buffer. The reason for keeping $B(t)$ is to partly protect the policy reserve, $L(t)$, (thus company solvency) from unfavourable fluctuations in the asset base. Before considering the policy interest rate, the dynamic asset side must be modelled.

3.4.2 Asset Dynamic Models

Two different asset models -a geometric Brownian motion with deterministic interest rate (constant short rate r) and a geometric Brownian motion with stochastic interest rates given by a Vasicek model cited in cited in Daniel et al (2010).

In the case of the geometric Brownian motion with deterministic interest rate, we use classical Black-Scholes setup. The asset process under the risk-free measure Q evolves according to stochastic differential equation

$$dA_t = rA_t dt + \sigma_A A_t dW_t,\ A_0 = P(1 + x_0) \tag{3.51}$$

where r is the constant short rate, $\sigma_A > 0$ is the volatility of the asset process A, and W is a standard Brownian motion under Q (martingale).

According to Wilmott et al (1995), asset prices move randomly because of the efficient market hypothesis. But the different forms of this hypothesis basically say two things:

- The past history is fully reflected in the present price, which does not hold any further information.
- Markets respond immediately to any new information about an asset.

The modelling of asset prices is really about modelling the arrival of new information which affects the price. Base on the two assumptions, the unanticipated changes in the asset price are a Markov process.

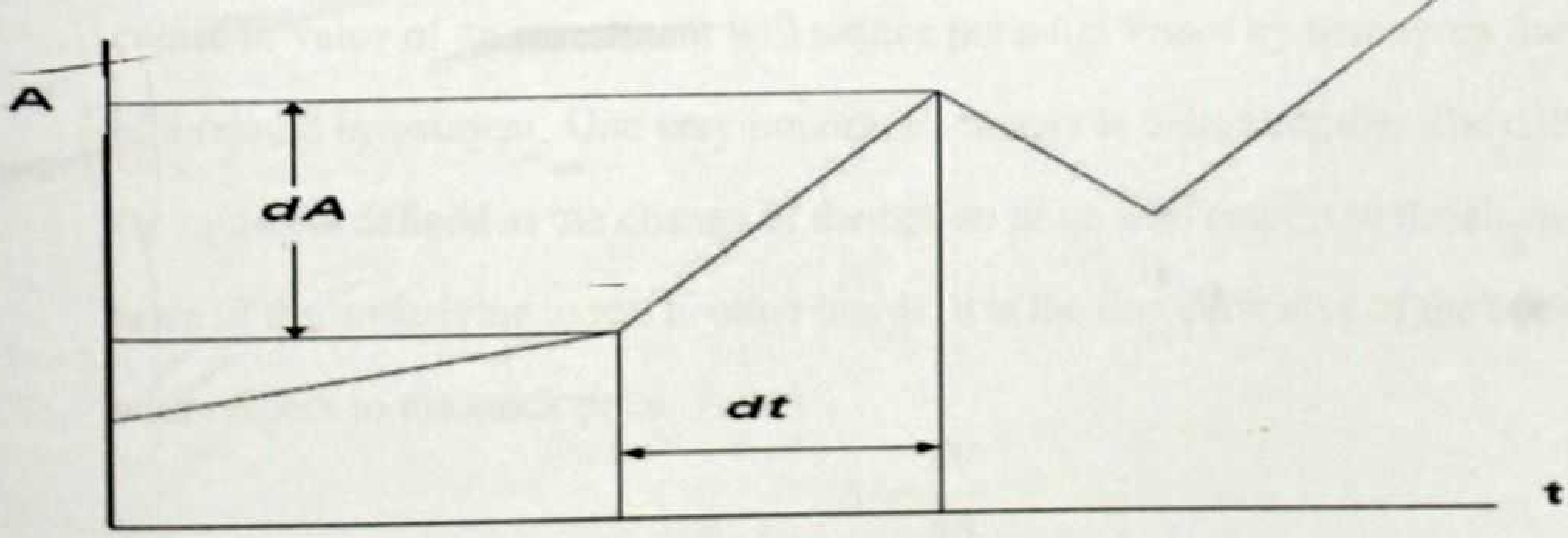

Figure 3.4: Details of a discrete random walk

3.5 Numerical Methods

Closed form solutions exist for European contracts. However, for other contracts, American or Asian contracts, a closed form solutions does not exist. And the only way a market participant will be able to obtain a price is by using an appropriate numerical method. Binomial tree method, Monte-Carlos Simulation and finite difference method (Explicit, Implicit and Crank-Nicolson method) and Risk-neutral valuation methods are some of the numerical methods. In this paper we will compare the Binomial tree method and the finite difference method in the valuation of the life insurance liabilities. In the finite difference method, we will use the Implicit and Crank-Nicolson method as the explicit method is conditionally stable.

3.6 Black-Scholes setup for the Valuation model

Before we get to the valuation function using Black-Scholes setup, let's consider the concept of arbitrage, a concept, which in certain circumstances, allows us to establish relationship between prices and hence determine them.

3.6.1 Hedging

Hedging is a financial strategy used to reduce the risk of investing in financial markets. Like insurance, hedging can avoid some losses, but it also may reduce some potential for returns on investment. An investor who believes he can make a profit on the increase in value of an investment will reduce potential losses by betting on the decline of a related investment. One very important strategy is delta hedging. The delta, Δ, of the option is defined as the change of the option price with respect to the change in the price of the underlying asset. In other words, it is the first derivative of the option price with respect to the stock price:

$$\Delta = \frac{\partial V}{\partial A}$$

.

3.6.2 Arbitrage

It is the practice of taking advantage of a price difference between two or more markets: striking a combination of matching deals that capitalize upon the imbalance, the profit being the difference between the market prices. When used by academics, an arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state; in simple terms, it is the possibility of a risk-free profit at zero cost.

One of the fundamental concepts underlying the theory of financial derivative pricing and hedging (an investment position intended to offset potential losses/gains that may be incurred by a companion investment) is that of arbitrage. Finance theory assume the existence of risk-free investments that give a guaranteed return with no chance of default (Wilmot, Howison, & Dewynne, 1995). The highest risk-free return that one can make on a portfolio of assets is the same as the return if the equivalent amount of cash were placed in a bank.

3.6.3 Options, values, pay-offs and strategies

Let V be the value of an option, where distinction is important, we use $C(A,t)$ and $P(A,t)$ to denote a *call* and a *put* respectively. A **call option**, often simply labelled a "call", is a financial contract between two parties, the buyer and the seller of this type of option. The buyer of the call option has the right, but not the obligation to buy an agreed quantity of a particular commodity or financial instrument (the underlying) from the seller of the option at a certain time (the expiration date) for a certain price (the strike price). The seller (or "writer") is obligated to sell the commodity or financial instrument should the buyer so decide. A **put or put option** is a contract between two parties to exchange an asset (the underlying), at a specified price (the strike), by a predetermined date (the expiry or maturity). One party, the buyer of the put, has the right, but not an obligation, to sell the asset at the strike price by the future date, while the other party, the seller of the put, has the obligation to buy the asset at the strike price if the buyer exercises the option.

V is the function of the current underlying asset A, and time, t. i.e. $V = V(A,t)$. The value of the option/contract depends on the following parameters.

σ - the volatility of the underlying asset.

T - the expiry or maturity time.

r - the interest rate.

3.6.4 The Black-Scholes Analysis

Before we develop a model for the price of asset, it is necessary to first develop a model for the price of the asset itself. Economic theory and historical data suggest that asset returns are composed of two components. First, the value of asset will increase with time at a rate **r** known as the drift rate. Second, at any time the value of the asset is subject to random variability. This variability is expressed by a random variable **X** with certain special properties.

The notion that asset's value will necessarily increase over time is due to the expectation that a company will, in the long run, generate a return, which is measured as a percentage of the value of the investment, for investors. This is because the expected absolute return is dependent on the price of the asset. The price change of a risk-less asset in a time interval Δt could thus be modelled as follows:

$$\Delta A = Ar\Delta t \tag{3.52}$$

It is clear, however, that no stock is risk-less. Risk is modelled by the stochastic term X which has the following two properties:

1. $\Delta X = \phi\sqrt{\Delta t}$ where ϕ is a standard normal random variable, i.e. $\phi \sim N(0,1)$.

2. The value of ΔX in time interval Δt is independent of ΔX in any other time interval.

These two properties describe X as a random variable that follows a Wiener process.

Notice that for n time steps:

$$E\left[\sum_{i=1}^{n} \Delta X_i\right] = 0 \tag{3.53}$$

$$Var\left(\sum_{i=1}^{n} \Delta A_i\right) = n\Delta t \tag{3.54}$$

Thus A is a random variable whose expected future value depends only on its present value, and whose variability increases as more time intervals are considered. The first property is consistent with an economic theory called the weak form of market efficiency that states that current asset prices reflect all information that can be obtained from the records of past asset prices. The second property shows that uncertainty about the value of A increases as one looks further into the future. This is intuitively appealing, as more events affecting the asset price are anticipated in longer time intervals. (Hull, 2003).

The first property of X implies that if Δt represents one unit of time. $Var(\Delta X) = 1$. To reflect the risk of a particular asset, the variability of ΔX needs to be adjusted. That is, it must be scaled by a quantity which is an estimate of the variability of A in Δt. This quantity is σA where σ is the standard deviation of A during Δt expressed as a percentage. Thus $\sigma A \Delta X$ is a random variable with standard deviation σA. It is now possible to model the behaviour of a asset price as follows:

$$\Delta A = rA\Delta t + \sigma A \Delta X \tag{3.55}$$

As $\Delta t \to 0$

$$dA = rAdt + \sigma AdX \tag{3.56}$$

Equation (3.56) is the **asset price model**. Note that the absolute change in the asset price is not by itself a useful quantity. With each change in asset price, we have a **return**, defined to be the change in the price divided by the original value. Suppose at time t, the asset price is A. Considering a small subsequent time interval dt, during which A changes to $A+dA$, as shown in figure 3.4 at page 42, the return on the asset, $\frac{dA}{A}$ is modelled.

If $\sigma = 0$, that is the standard deviation of the returns is zero (0)

$$\Rightarrow \frac{dA}{A} = rdt$$
$$\Rightarrow \frac{dA}{dt} = Ar$$
$$\Rightarrow A = A_0 e^{r(t-t_0)} \tag{3.57}$$

where $A(0) = A_0$ is the value of the asset at time t_0, that is $t = 0$. Thus, if $\sigma = 0$, the asset is totally deterministic and the future price of the asset can be predicted with certainty. The term dX contains the randomness that is a feature of asset price and is known as *Wiener Process* or *standard Brownian Motion*. It has the following properties

- dX is a random variable drawn from a normal distribution.
- the mean of dX is zero (0).
- the variance of dX is dt.

Equation (3.56) implies that, A follows an **Itô Process**. According to Wilmot, et al (1995), the Black-Scholes analysis assumes that the asset prices behave as just demonstrated and follows the following assumptions:

1. The asset price follows lognormal random walk: This assumption of the Black-Scholes model suggests that people cannot consistently predict the direction of the market or an individual stock. The Black-Scholes model assumes stocks move in a manner referred to as a random walk. Random walk means that at any given moment in time, the price of the underlying stock can go up or down with the same probability. The price of a stock in time $t+1$ is independent from the price in time t.

2. The risk-free interest rate r and the asset volatility σ are known functions of time over the life of the option/contract: The most significant assumption is that volatility, a measure of how much a stock can be expected to move in the

near-term, is a constant over time. While volatility can be relatively constant in very short term, it is never constant in longer term. Some advanced option valuation models substitute Black-Schole's constant volatility with stochastic-process generated estimates. The same like with the volatility, interest rates are also assumed to be constant in the Black-Scholes model. The Black-Scholes model uses the risk-free rate to represent this constant and known rate. In the real world, there is no such thing as a risk-free rate, but it is possible to use the U.S. Government Treasury Bills 30-day rate since the U. S. government is deemed to be credible enough. However, these treasury rates can change in times of increased volatility.

3. There is no transaction cost associated with hedging a portfolio: The Black-Scholes model assumes that there are no fees for buying and selling options and stocks and no barriers to trading.

4. The underlying asset has no dividends during the life of the option/contract: Another assumption is that the underlying stock does not pay dividends during the option's life. In the real world, most companies pay dividends to their share holders. The basic Black-Scholes model was later adjusted for dividends, so there is a workaround for this. This assumption relates to the basic Black-Scholes formula. A common way of adjusting the Black-Scholes model for dividends is to subtract the discounted value of a future dividend from the stock price.

5. There are no arbitrage possibilities. It is impossible to secure a risk free profit. Although there is arbitrage in certain market segments, these are not secure in the long run and relying on them violates that basic needs for Black-Scholes to work.

6. The Black-Scholes model assumes European-style options which can only be exercised on the expiration date. American-style options can be exercised at any time during the life of the option, making American options more valuable due to their greater flexibility. That is trading of the underlying asset can take place

continuously.

7. Short selling is permitted and the assets are divisible: The Black-Scholes model assumes that markets are perfectly liquid and it is possible to purchase or sell any amount of asset or options or their fractions at any given time (liquidity).

Insurers are actually selling a naked put option to the buyer of the insurance. Therefore the method of finding the value of put options can be applied in the valuation of the life insurance contract.

Consider a contract of $V(A,t)$, where V is not necessarily a call or a put but the value of the whole portfolio of different contract. We use a result from stochastic calculus known as Itô's lemma. This states that, if x follows a general Itô process

$$dx = a(x,t)dt + b(x,t)dZ \tag{3.58}$$

and $f = f(x,t)$ then

$$df = \left(\frac{\partial f}{\partial x}a(x,t) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(b(x,t))^2\right)dt + \frac{\partial f}{\partial x}b(x,t)dZ \tag{3.59}$$

Applying Itô's Lemma to the function $V(A,t)$ gives

$$dV = \left(\frac{\partial V}{\partial A}rA + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2\right)dt + \frac{\partial V}{\partial A}\sigma A dZ \tag{3.60}$$

This expression is difficult to solve for V since it contains the stochastic term dZ especially. The main idea behind Black-Scholes analysis is for one to create a portfolio which consist of shares of assets and derivatives that is instantaneously risk-less, and thus eliminates the stochastic term in equation (3.59). The instantaneously risk-less portfolio at time t consists of one long position in the derivative and a short position of exactly $\frac{\partial V}{\partial A}$ (also referred to as the delta hedging) shares of the underlying asset. The value of this portfolio is given by

$$\Pi = V - \frac{\partial V}{\partial A}A \tag{3.61}$$

The instantaneous change of Π is

$$d\Pi = dV - \frac{\partial V}{\partial A} dA \tag{3.62}$$

Combining equations (3.56), (3.60) and (3.62), we obtain

$$d\Pi = -\frac{\partial V}{\partial A} dA + \left(\frac{\partial V}{\partial A} rA + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2} \sigma^2 A^2 \right) dt + \frac{\partial V}{\partial A} \sigma A dZ$$

$$d\Pi = -\frac{dV}{dA}(rAdt + \sigma A dZ) + \left(\frac{\partial V}{\partial A} rA + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2} \sigma^2 A^2 \right) dt + \frac{\partial V}{\partial A} \sigma A dZ$$

$$\Rightarrow d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2} \sigma^2 A^2 \right) dt \tag{3.63}$$

Note that, in equation (3.63), the change in the instantaneously risk-less portfolio is not dependent on the stochastic term dZ. In order for the portfolio to maintain its risk-less property, it must be rebalanced at every point in time as $\frac{\partial V}{\partial A}$ will not remain the same for different values of t. Thus shares will need to bought and sold continuously in fractional amounts as was stated in the assumption.

Since this portfolio is risk-less, the assumption that there are no arbitrage opportunities dictates that it must earn exactly the risk free rate. If we consider the concept of arbitrage and supply and demand, with assumption that there is no transaction cost, the return on an amount Π invested in riskless assets would see a growth of $r\Pi dt$ in time dt. This implies the right-hand side of equation (3.63) is equal to $r\Pi dt$. That is

$$d\Pi = r\Pi dt.$$

Using equation (3.61) and (3.63) we have

$$d\Pi = r\Pi dt = r\left(V - \frac{\partial V}{\partial A} A \right) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2} \sigma^2 A^2 \right) dt$$

Simplifying and re-arranging results in the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} - rV = 0 \tag{3.64}$$

In analysing contracts/options on a path-dependent quantity, such as the average asset price, Black-Scholes approach become inadequate. This is because there are many realizations of the asset prices random walk leading to the current value, any two of these give a different value for the path-dependent (Wilmot et al., 1995). This led to the introduction of a third variable in addition to A and t which will measure the relevant path-dependent quantity.

Since we use continuously sampled quantities for the payoff of average strike option (the fixed price at which the owner of an option can purchase in the case of call or sell in the case of put, the underlying security or commodity), our average will depend on a time integral.

To look at the time integrals of the random walk, consider European option/contract with payoff depending on A and on

$$\int_0^T f(A(T),T)dT \tag{3.65}$$

where f is a given function of the variables A and t. The integral in eqn. (3.65) is over the path of A from $t = 0$ to $t = T$ (the expiry). The payoff at expiry for average strike call is

$$\max\left(A - \frac{1}{T}\int_0^T A(T)dT, 0\right) \tag{3.66}$$

We have $f(A,t) = A$. Let

$$P = \int_0^t f(A(T),T)dT \tag{3.67}$$

We treat P, A and t as independent variables since the history of the asset price is independent of the current price. Note that, P varies depending on the variation of the random walk. In eqn.(3.67), T is replaced by t since the payoff depends on both P and A, the value of an exotic path-dependent contract is written as $V(A,P,t)$. This means, the value of the option depends on the current asset price A, the time t and the history of the integral of the asset P.

The change in P due to small changes in t and A is given by stochastic differential equation

$$P(t+dt) = P + dP = \int_0^{t+dt} f(A(T),T)dT \tag{3.68}$$

Simplifying to $O(dt)$ - the order of dt, we have

$$P + dP = \int_0^{t} f(A(T),T)dT + f(A(t),t)dt \tag{3.69}$$

where $dP = f(A,t)dt$. Equation (3.69) is Stochastic Differential Equation (SDE) of P without random component. To value the contract that depends of A, t and P, we apply Itô's lemma to the function $V(A,P,t)$ and this gives

$$dV = \sigma A\frac{\partial V}{\partial A}dX + \left(\frac{1}{2}\sigma^2 A^2\frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} + \frac{\partial V}{\partial t} + f(A,t)\frac{\partial V}{\partial P}\right)dt \tag{3.70}$$

Since dP introduces no new source of risk, it is anticipated that the option can be hedge using the underlying asset only. Since this is European contract, we set up the usual risk-free portfolio which consist of one option and a short position with $N = \frac{\partial V}{\partial A}$ of the underling asset. Considering arbitrage leads to

$$\frac{\partial V}{\partial t} + f(A,t)\frac{\partial V}{\partial P} + \frac{1}{2}\sigma^2 A^2\frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} - rV = 0 \tag{3.71}$$

Note that, the path-dependent quantity P is updated discretely and is therefore constant between sampling dates. The PDE for the option value between sampling dates becomes just he basic Black-Scholes equation with P treated as a parameter. So in valuing the path-dependent option with discrete sampling, we start from the expiry date, when the option value is known (i.e. equal to the payoff) and work backwards.

Hence equation (3.71) becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} - rV = 0 \tag{3.72}$$

3.6.5 Dividend Paying Asset

The above Black-Scholes equation, equation 3.72 is under assumption that no dividends are paid during the life of the insurance. We will therefore cater for these dividends which are paid during the life of the contract.

Let ϕ be a constant continuous dividend yield which is known. This means that the holder receives a dividend $\phi A \Delta t$ within the time interval Δt. After the dividend, the share value is lowered making the expected rate of return r be $(r-\phi)$. So the geometric Brownian motion model in equation 3.56 becomes (Hull, 2003)

$$dA = (r-\phi)Adt + \sigma AdX \tag{3.73}$$

and the Black-Scholes equation becomes

$$\frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + (r-\phi)A\frac{\partial V}{\partial A} - rV = 0 \tag{3.74}$$

3.6.6 Finite Difference Approximation for Black-Scholes Differential Equations

The finite difference methods attempt to solve Black Scholes Partial differential equation by approximating the differential equation over the area of integration by a system of algebraic equations. They are a means of obtaining numerical solutions to Partial differential equations. The most common finite difference methods for solving the Black Scholes Partial differential equation are the Explicit method, the fully Implicit method and the Crank-Nicolson method. These are closely related but differ in stability, accuracy and execution speed. In the formulation of a partial differential equation problem, three components are considered. They are:

1. The partial differential equation.

2. The region of space-time on which the partial differential equation is required to be satisfied.

3. The auxiliary boundary and initial conditions to be met.

Discretization of Black Scholes Equation

The finite difference method consists of discretizing the partial differential pricing equation and the boundary conditions using a forward, a backward difference or central difference approximation. The Black Scholes PDE given by 3.64 can we written as

$$\frac{\partial V(A_t,t)}{\partial t}+\frac{\sigma^2 A^2 \partial^2 V(A_t,t)}{2\partial A_t^2}+\frac{rA_t \partial V(A_t,t)}{\partial A_t}=rV(A_t,t)$$

in simplify formed is written as:

$$\frac{\partial V}{\partial t}+\frac{\sigma^2 A^2 \partial^2 V}{2\partial A^2}+\frac{rA\partial V}{\partial A}=rV \tag{3.75}$$

Note: For continuous dividend paying asset, replace r with $r-\phi$

The equation is discretize with respect to time, t, and to the underlying asset price, A. The (A,t) plane is divided into grid or mesh using approximate infinitesimal steps ΔA and Δt by small fixed finite steps. An array of $N+1$ equally spaced grid points$t_0, t_1, \ldots, t_N$ is used to discretize the time derivative with $t_{i+1}-t_i=\Delta t$ and $\Delta t = T/N$.

Also, since asset price cannot go below 0 and it is assumed that $A_{max}=2A_0$. We also have $M+1$ equally spaced grid points $A_0, A_1, \ldots, A_M$ and is used to discretize the asset price derivative with $A_{j+1}-A_j=\Delta A$ and $\Delta A=\frac{A_{max}}{M}$. We then have a rectangular region on the (A,t) plane with sides $(0,S_{max})$ and $(0,T)$. Using the grid coordinates (i,j), we are able to compute the solution at discrete points with a total grid points of $(M+1)(N+1)$.

The (i,j) points on the grid corresponds to time $i\Delta t$ for $i=0,1,\ldots,N$ and the Asset price

$j\Delta A$ for $j = 0, 1, \ldots, M$. The figure below illustrates the discretized asset price and time derivatives into$(M+1)$ and $(N+1)$ grid points respectively.

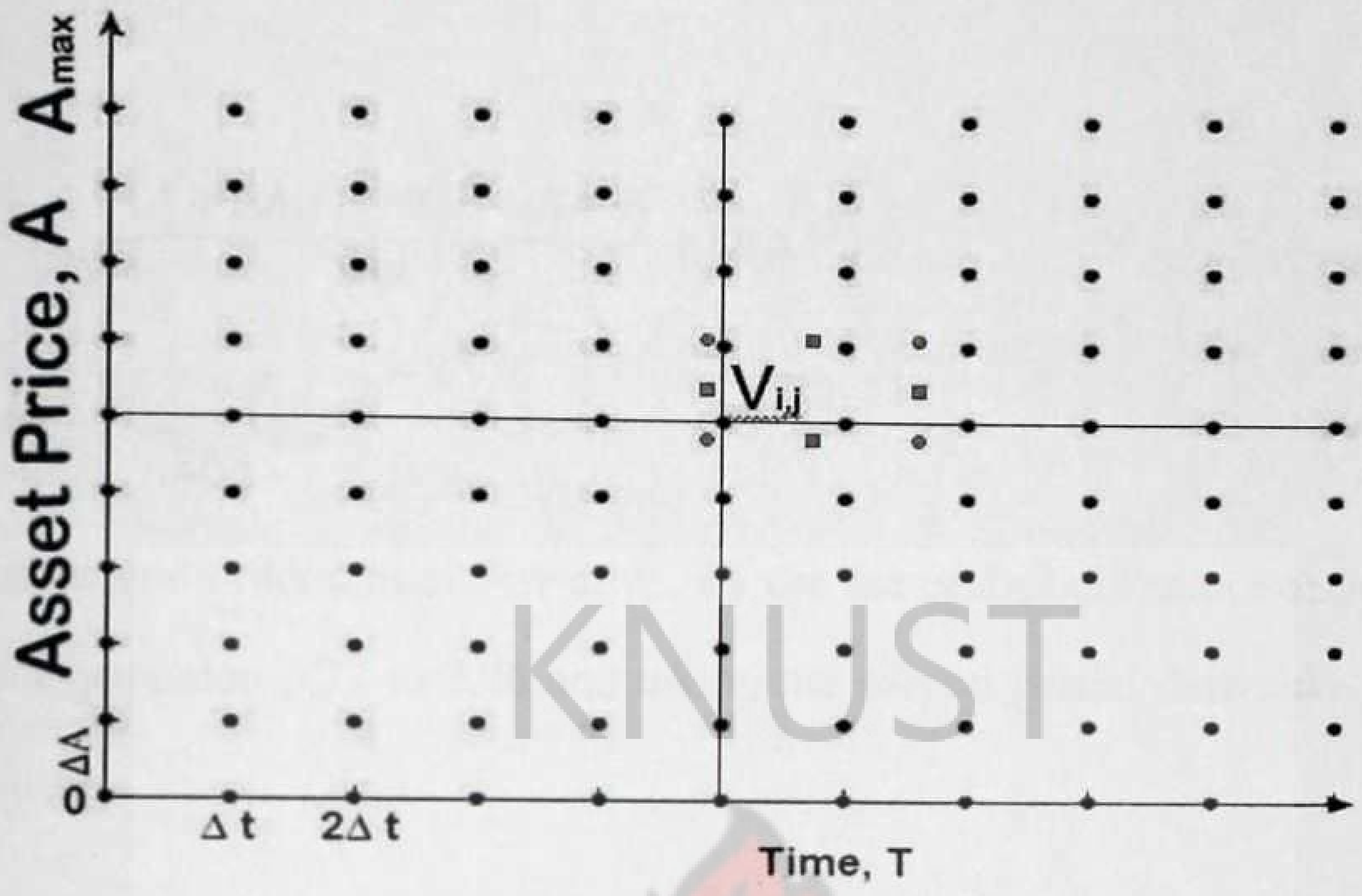

Figure 3.5: The mesh points for the finite difference approximation

Representing $V(A,t)$ in the grid by $V_{i,j}$, their respective expansions of $V(A+\Delta A,t)$ and $V(A-\Delta A,t)$ in Taylor series are:

$$V(A+\Delta A,t) = V + \frac{\partial V}{\partial A}\Delta A + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\Delta A^2 + \frac{1}{6}\frac{\partial^3 V}{\partial A^3}\Delta A^3 + O(\Delta A^4) \tag{3.76}$$

and

$$V(A-\Delta A,t) = V - \frac{\partial V}{\partial A}\Delta A + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\Delta A^2 - \frac{1}{6}\frac{\partial^3 V}{\partial A^3}\Delta A^3 + O(\Delta A^4) \tag{3.77}$$

Using equation 3.76, the forward difference is given by

$$\frac{\partial V}{\partial A} = \frac{V(A+\Delta A,t) - V(A,t)}{\Delta A} + O(\Delta A)$$

$$\frac{\partial V}{\partial A} \approx \frac{V_{i,j+1} - V_{i,j}}{\Delta A} \tag{3.78}$$

and 3.77 gives the corresponding backward difference

$$\frac{\partial V}{\partial A} = \frac{V(A,t) - V(A-\Delta A,t)}{\partial A} + O(\Delta A)$$

$$\frac{\partial V}{\partial A} \approx \frac{V_{i,j} - V_{i,j-1}}{\partial A} \tag{3.79}$$

Eqn (3.76)-eqn.(3.77), taking the first partial derivative results in the central difference which is given by

$$\frac{\partial V(A,t)}{\partial A} = \frac{V(A+\Delta A,t) - V(A-\Delta A,t)}{2\Delta A} + O(\Delta A^2)$$

$$\frac{\partial V(A,t)}{\partial A} \approx \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta A} \tag{3.80}$$

To estimate the second order partial derivative, we use the central difference approximation. Adding equation 3.77 to 3.76 and taking the second partial derivative, we get

$$\frac{\partial^2 V}{\partial A^2} = \frac{V(A+\Delta A,t) - 2V(A,t) + V(A-\Delta A,t)}{\Delta A^2} + O(\Delta A^2)$$

$$\frac{\partial^2 V}{\partial A^2} \approx \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta A^2} \tag{3.81}$$

Expanding $V(A,t+\Delta t)$ in Taylor series, we obtain

$$\frac{\partial V}{\partial t} = \frac{V(A,t+\Delta t) - V(A,t)}{\Delta t} + O(\Delta t)$$

$$\frac{\partial V}{\partial t} \approx \frac{V_{i+1,j} - V_{i,j}}{\Delta t} \tag{3.82}$$

Boundary and Initial Conditions

Without boundary or initial conditions, the solution of the Black-Scholes PDE will either have an infinity of solutions or no solution. We therefore need to specify the boundary and initial conditions for the European style contract whose payoff is given by $max(K - A_T, 0)$. When an asset is worth nothing, a put is worth it strike price K. That is

$$V_{i,0} = K \; for \; i = 0,1,...,N \tag{3.83}$$

The value of the contract approaches zero (0) as the price of the underlying asset price increases. Hence $A_{max} = A_M$ and this means

$$V_{i,M} = 0 \; for \; i = 0, 1, \ldots, N \tag{3.84}$$

Since the value of the contract is known at time T, we can find the initial condition

$$V_{N,j} = max(K - j\Delta A, 0) \; for \; j = 0, 1, \ldots, M \tag{3.85}$$

The initial condition results in the value of the contract V at the end of the period of the contract and not the beginning, implying a backward move from maturity to time zero. The American style are also handled almost the same way,

$$V_{N,j} = max(j\Delta A - K, 0) \; for \; j = 0, 1, \ldots, M \tag{3.86}$$

3.6.7 Approaches of Finite Difference Scheme

Let's consider the European contract stated in equation 3.75, suppose that T is the maturity of the asset and A_{max} is the maximum asset price. Let $M\Delta A = A_{max}$ and $N\Delta t = TV_{i,j}$ denotes the asset value at $(i\Delta t, j\Delta t)$. We will take a look at three approaches of the finite difference scheme: Implicit finite difference method, explicit finite difference method and Crank Nicolson method.

Explicit Finite Difference Method

Since we know the value of the contract at maturity time, we can give the expression that gives the next value $V_{i,j}$ explicitly in terms of $V_{i+1,j-1}$, $V_{i+1,j}$ and $V_{i+1,j+1}$. We therefore discretize Black-Scholes partial differential equation (PDE) in equation 3.75 by taking forward difference for time and central difference for the asset price discretization. This gives

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A}\left[V_{i+1,j+1} - V_{i+1,j-1}\right] + \frac{\sigma^2 j^2 \Delta A^2}{2\Delta A^2}\left[V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}\right] = rV_{i,j}$$

(3.87)

Making $V_{i,j}$ the subject, we have

$$V_{i,j} = \frac{1}{1+r\Delta t}\left[a_j V_{i+1,j-1} + b_j V_{i+1,j} + c_j V_{i+1,j+1}\right] \; for\, i = 0,1,\ldots,N \; and \; j = 1,2,\ldots,M \tag{3.88}$$

where the weights a_j, b_j and c_j are given by

$$\left\{\begin{matrix} a_j = \frac{1}{2}\sigma^2 j^2 \Delta t - \frac{1}{2} r j \Delta t \\ b_j = 1 - \sigma^2 j^2 \Delta t \\ c_j = \frac{1}{2} r j \Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t \end{matrix}\right\} \tag{3.89}$$

Since the finite difference for the discretization of the time is accurate to $O(\Delta t)$ and that of the central difference of the asset discretization is $O(\Delta t, \Delta A^2)$.

The weights, which are the risk neutral probabilities of the 3 assets prices $A - \Delta A$, A and $A + \Delta A$ at $t + \Delta t$ adds up to one (1) and $\frac{1}{1+r\Delta t}$ is the discounted factor. But we can get negative probabilities unless further restrictions are imposed on Δt and ΔA. This produces results that do not converge to the solution of the PDE and this shows the explicit method is unstable unless those restrictions are imposed on Δt and ΔA. The conditions to have non-negative probabilities is that $\sigma^2 j^2 \Delta t < 1$ and $r < \sigma^2 j$ (Hull, 2003).

The system is represented in the matrix form as

$$\begin{bmatrix} b_0 & c_0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & \cdot & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & \cdots & 0 & a_M & b_M \end{bmatrix} \begin{bmatrix} V_{i+1,0} \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M} \end{bmatrix} = \begin{bmatrix} V_{i,0} - a_0 \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} - c_M \end{bmatrix} \tag{3.90}$$

The system of equations can be written in the form $AV_{i+1,j} = V_{i,j}$ for $j = 0, 1, ..., M$ and the error terms are ignored since the boundary conditions cater for them.

The vector of the asset price $V_{i+1,j}$ is known at time T from the initial condition . We solve for $V_{i,j}$ by working backward using the matrix A which comprises of the probabilities a_j, b_j and c_j which are known probabilities and the backward iteration leads to the value of the contract obtained at time zero.

Stability of the Finite Difference Scheme

Truncation error in the asset price discretization and in the time discretization are the two fundamental sources of error. Consistency, stability and convergence are the three fundamental factors that characterized a numerical scheme. They are linked together by **Lax Equivalence theorem** which states that *given a property posed linear initial value problem and a consistent finite difference scheme, stability is the necessary and sufficient condition for convergence (Smith, 1985).*

The eigenvalues λ_i of $N \times N$ matrix

$$\begin{bmatrix} y & z & & & \\ x & y & z & & \\ \ddots & \ddots & \ddots & & \\ & x & y & z & \\ & & x & y & \end{bmatrix}$$

is given by $\lambda_i = y + 2(\sqrt{xz}) \cos \frac{i\pi}{N+1}$ *for* $i = 1, 2, ..., N$ where x, y *and* z may be real or complex no. The system is stable if $|\lambda_i| \leq 1$. (Smith, 1985).

Stability of the Explicit Finite Difference Scheme

To analyze the stability of the Explicit Difference Method, we use the matrix **A**. Matrix **A** is real and symmetric. If λ_i is the ith eigenvalue of **A**, then

$$||A||_2 = \rho(A) = \max|\lambda_i| \tag{3.91}$$

The eigenvalues λ_i are given by

$$\lambda_i = b_j + 2(\sqrt{a_j c_j})\cos\frac{i\pi}{N} \; for \; i = 1,2,...,N-1$$

. Substituting a, b and c and re-arranging the results in

$$\lambda_i \approx 1 - 2\sigma^2 j^2 \Delta t \sin^2\frac{i\pi}{2N} \tag{3.92}$$

The scheme stable when

$$||A||_2 = \max|1 - 2\sigma^2 j^2 \Delta t \sin\frac{i\pi}{2N}| \leq 1$$

$$\Rightarrow -1 \leq 1 - 2\sigma^2 j^2 \Delta t \sin^2\frac{i\pi}{2N} \leq 1 for \; i = 1,2,...,N-1$$

$$as \; \Delta t \to 0, N \to \infty \; and \; \sin^2\frac{(N-1)\pi}{2N} \to 1$$

Hence $0 \leq \sigma^2 j^2 \Delta t \leq 1$.

Therefore the scheme is stable, convergent and consistent for $0 \leq \sigma^2 j^2 \Delta t \leq 1$.

Hence the explicit finite difference method is conditionally stable.

The Implicit Finite Difference Method

We substitute equations 3.80, 3.81 and 3.82 in equation 3.75 and express $V_{i+1,j}$ explicitly in terms of the unknowns $V_{i,j-1}$, $V_{i,j}$ and $V_{i,j+1}$. That is, we discretize Black-Scholes PDE in 3.75 using forward difference for time and central difference for the asset price we have

$$\frac{V_{i+1,j}-V_{i,j}}{\Delta t}+\frac{rj\Delta A}{2\Delta A}\left[V_{i,j+1}-V_{i,j-1}\right]+\frac{\sigma^2 j^2\Delta A^2}{2\Delta A^2}\left[V_{i,j+1}-2V_{i,j}+V_{i,j-1}\right]=rV_{i+1,j} \tag{3.93}$$

Making $V_{i+1,j}$ the subject in eqn. 3.93 gives

$$V_{i+1,j}=\frac{1}{1-r\Delta t}\left[x_j V_{,j-1}+y_j V_{i,j}+z_j V_{i,j+1}\right] \tag{3.94}$$

for $i=0,1,...,N$ *and* $j=1,2,...,M-1$.

Similarly to the explicit method, the implicit method is accurate to $O(\Delta t,\Delta A^2)$. The weights x, y *and* z are given by

$$\left\{\begin{array}{c} x_j=\frac{1}{2}r\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t \\ y_j=1+\sigma^2 j^2\Delta t \\ z_j=-\frac{1}{2}rj\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t \end{array}\right\} \tag{3.95}$$

The system of equations in tridiagonal matrix form is

$$\begin{bmatrix} V_{i+1,0}-x_0 \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M}-z_M \end{bmatrix}=\frac{1}{1-r\Delta t}\begin{bmatrix} y_0 & z_0 & 0 & \cdots & 0 & 0 & 0 \\ x_1 & y_1 & z_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{M-1} & y_{M-1} & z_{M-1} \\ 0 & 0 & 0 & \cdots & 0 & x_M & y_M \end{bmatrix}\begin{bmatrix} V_{i,0} \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} \end{bmatrix} \tag{3.96}$$

The system is written as $AV_{i,j}=V_{i+1,j}$ for $j=0,1,...,M$. The matrix A has $y_j=1+\sigma^2 j^2\Delta t$ in the diagonal which is positive. The product of the diagonal elements are non-zero and therefore the matrix is non-singular . We can therefore solve by finding the inverse of the matrix, A^{-1}. Applying the boundary conditions with 3.94 changes the element $y_0, y_M=1$ and $z_0, x_M=0$ in the matrix A.

Stability of the Implicit Finite Difference Method

The eigenvalues are given by

$$\lambda_i = y_j + 2\sqrt{(x_j z_j)}\cos\frac{i\pi}{N} \; for\; i = 1,2,...,N-1 \tag{3.97}$$

Substituting for x, y and z in (3.97) and simplifying, we obtain

$$\begin{aligned}\lambda_i &= 1+\sigma^2 j^2\Delta t + \sigma^2 j^2 \Delta t\left[1-\frac{r^2}{\sigma^4 j^2}\right]^{\frac{1}{2}}\left[1-2\sin^2\frac{i\pi}{2N}\right]\\ &\Rightarrow \lambda_i \approx 1+2\sigma^2 j^2\Delta t - 2\sigma^2 j^2\Delta t\sin^2\frac{i\pi}{2N}\end{aligned} \tag{3.98}$$

The change of sigh is due to the truncation of the binomial expansion. The scheme is stable when

$$\|A\|_2 = \max\left|1+2\sigma^2 j^2\Delta t - 2\sigma^2 j^2\Delta t\sin^2\frac{i\pi}{2N}\right| \leq 1$$

$$\Rightarrow -1 \leq 1+2\sigma^2 j^2\Delta t - 2\sigma^2 j^2\Delta t\sin^2\frac{i\pi}{2N} \leq 1 \tag{3.99}$$

As $\Delta t \to 0$, $N \to \infty$ and $\sin^2\frac{(N-1)\pi}{2N} \to 1$, $|\lambda_i| \leq 1$. Therefore the scheme is unconditionally stable, convergent and consistent.

The Crank Nicolson Method

The Crank scheme is the average of explicit finite difference methods and the implicit finite difference. Adding equations 3.87 and 3.93 and finding their average gives

$$\begin{aligned}&\frac{V_{i+1,j}-V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A}\left[V_{i+1,j+1}-V_{i+1,j-1}+V_{i,j+1}-V_{i,j-1}\right] +\\ &\frac{\sigma^2 j^2 \Delta A^2}{4\Delta A^2}\left[V_{i,j-1}-2V_{i,j}+V_{i,j+1}+V_{i+1,j-1}-2V_{i+1,j}+V_{i+1,j+1}\right] =\\ &\qquad\qquad\frac{1}{2}\left[rV_{i,j}+rV_{i+1,k}\right]\end{aligned} \tag{3.100}$$

and re-arranging

$$a_j V_{i,j-1} + b_j V_{i,j} + c_i V_{i,j+1} = x_j V_{i+1,j-1} + y_j V_{i+1,j} + z_j V_{i+1,j+1} \tag{3.101}$$

for $i = 0, 1, ..., N$ and $j = 1, 2, ..., M-1$. Where the parameters a_j, b_j, c_j, x_j, y_j *and* z_j are given by

$$\begin{cases} a_j = \frac{1}{4} r j \Delta t - \frac{1}{4} \sigma^2 j^2 \Delta t \\ b_j = 1 + \frac{1}{2} r \Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \\ c_j = -\frac{1}{4} \sigma^2 j^2 \Delta t - \frac{1}{4} r j \Delta t \\ x_j = \frac{1}{4} \sigma^2 j^2 \Delta t - \frac{1}{4} r j \Delta t \\ y_j = 1 - \frac{1}{2} r \Delta t - \frac{1}{2} \sigma^2 j^2 \Delta t \\ z_j = \frac{1}{4} r j \Delta t + \frac{1}{4} \sigma^2 j^2 \Delta t \end{cases} \tag{3.102}$$

The system of equations in eqn. (3.101) is express as $C\vec{V}_i = D\vec{V}_{i+1}$ and resulting in tridiagonal gives

$$\begin{bmatrix} b_0 & c_0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & \cdots & 0 & a_M & b_M \end{bmatrix} \begin{bmatrix} V_{i,0} \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} \end{bmatrix} =$$

$$\begin{bmatrix} y_0 & z_0 & 0 & \cdots & 0 & 0 & 0 \\ x_1 & y_1 & z_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{M-1} & y_{M-1} & z_{M-1} \\ 0 & 0 & 0 & \cdots & 0 & x_M & y_M \end{bmatrix} \begin{bmatrix} V_{i+1,0} \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M} \end{bmatrix} \tag{3.103}$$

Solving the system

The elements of vector $\mathbf{V_{i+1}}$ are known at maturity time T, and so we express the system (3.103) as $V_i = C^{-1}DV_{i+1}$. By repeatedly iterating from time T to time zero, we obtain the value of V as the value of the life insurance contract. The diagonal entries of the matrix C is $b_j = 1 + \frac{1}{2}r\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t$ are always positive and thus the diagonal elements are non-zero. Therefore the matrix is non-singular as the diagonal entries are non-zero.

Accuracy of Crank-Nicolson Method

The Crank-Nicolson method is more accurate than the explicit and implicit method because it is with an accuracy of $O(\Delta t^2, \Delta A^2)$. Equating the central difference and symmetric central difference at $V_{i+\frac{1}{2},j} \equiv V(t + \frac{\Delta t}{2}, A)$.

Expanding $V_{i+1,j}$ in Taylor series at $V_{i+\frac{1}{2},j}$ to yield

$$V_{i+1,j} = V_{i+\frac{1}{2},j} + \frac{\partial V}{2\partial t}\Delta t + O(\Delta t^2) \tag{3.104}$$

and expanding $V_{i,j}$ *at* $V_{i+\frac{1}{2},j}$ gives

$$V_{i,j} = V_{i+\frac{1}{2},j} - \frac{\partial V}{2\partial t}\Delta t + O(\Delta t^2) \tag{3.105}$$

Taking the average of the equations (3.104) and (3.105) gives

$$\frac{V_{i,j} + V_{i+1,j}}{2} = V_{i+\frac{1}{2},j} + O(\Delta t^2) \tag{3.106}$$

This implies

$$V_{i+\frac{1}{2},j-1} - 2V_{i+\frac{1}{2},j} + V_{i+\frac{1}{2},j+1} = \frac{1}{2}[V_{i,j-1} - 2V_{i,j} + V_{i,j+1}] + \frac{1}{2}[V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}] + O(\Delta t^2) \tag{3.107}$$

The right-hand side of equation (3.107) is the average of the two symmetric central difference centred at i and $i+1$. Dividing by ΔA^2 we obtain the equality

$$\frac{\partial^2 V(t+\frac{\Delta t}{2},\Delta T)}{\partial A^2} = \frac{1}{2}\left[\frac{\partial^2 V(t,A)}{\partial A^2} + \frac{\partial^2 V(t+\Delta t,A)}{\partial A^2}\right] + O(\Delta t^2,\Delta A^2) \qquad (3.108)$$

Equation (3.108) is the symmetric central difference approximation. The subscript j is arbitrary and we deduce the central difference approximation as follows:

$$V_{i+\frac{1}{2},j+1} - V_{i+\frac{1}{2},j-1} = \frac{1}{2}\left[V_{i,j+1} - V_{i,j-1}\right] + \frac{1}{2}\left[V_{i+1,j+1} - V_{i+1,j-1}\right] + O(\Delta t^2) \quad (3.109)$$

Dividing by $2\Delta A$, we get

$$\frac{\partial V(t+\frac{\Delta t}{2},\Delta T)}{\partial A} = \frac{1}{2}\left[\frac{\partial V(t,A)}{\partial A} + \frac{\partial V(t+\Delta t,A)}{\partial A}\right] + O(\Delta t^2,\Delta A^2) \qquad (3.110)$$

and is the first order central difference approximation. Subtracting equation (3.105) from (3.104) we obtain the approximation of $\frac{\partial V}{\partial t}$ at $(t+\frac{1}{2}\Delta t, A)$. That is,

$$\frac{\partial V(t+\frac{1}{2}\Delta t,A)}{\partial t} = \frac{V_{i+1,j} - V_{i,j}}{\Delta t} + O(\Delta t^2) \qquad (3.111)$$

Hence the Black-Scholes PDE centred at $(t+\frac{1}{2}\Delta t, A)$ has a finite difference approximation

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{(r-\lambda)j\Delta A}{4\Delta A}\left[V_{i,j+1} - V_{i,j-1} + V_{i+1,j+1} - V_{i+1,j-1}\right] +$$

$$\frac{\sigma^2 j^2 \Delta A^2}{4\Delta A^2}\left[V_{i,j-1} - 2V_{i,j} + V_{i,j+1} + V_{i+1,j-1} + 2V_{i+1,j} + V_{i+1,j+1}\right] = rV_{i,j} \quad (3.112)$$

Re-arranging (3.112), we get an equation of the form (3.101) at page 63 which is the exact Crank-Nicolson scheme. Therefore the scheme has a leading error of order $O(\Delta t^2,\Delta A^2)$ (Kerman, 2002).

3.7 Binomial tree method for the valuation of life insurance contract

In finance, the binomial options pricing method (BOPM) provides a generalized numerical method for the valuation of options. It was first proposed by Cox, Ross and Rubinstein in 1979 cited in (Hull, 2003). In contrast to the Black-Scholes and other complex option-pricing model that requires solution to stochastic differential equations, the binomial option-pricing model (two-state option-pricing model for example) is mathematically simple. It is based on the assumption of no arbitrage.

The assumption of no arbitrage implies that all risk-free investments earns the risk-free rate of return and no investment opportunities exist that requires zero amount of money of investment but yields positive returns. It is the activity of many individuals operating within the context financial markets that, in fact, upholds these conditions. The activities of arbitragers or speculators are often maligned in the media, but their activities insure that our financial market works.

There are two cases of binomial model: a one-step and two-step binomial model.

3.7.1 One-Step Binomial Model

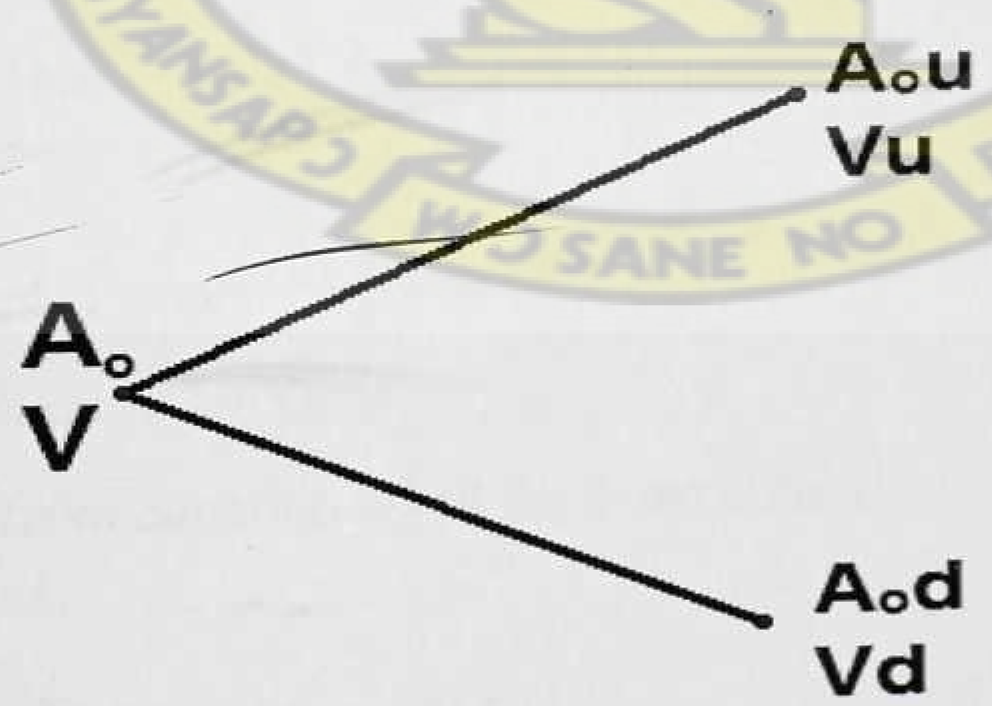

Figure 3.6: Asset and contract value in one-step tree

Let's consider an asset whose price is A_0 and the current contract value V. Let's assume the contract lasts for time T and that during the life of the contract, the asset price can either move up from A_0 to A_0u or down to A_0d where $d < 1 < u$. The proportional increase in the asset price when there is an up movement is $u-1$ and that for the down movement is $1-d$. Let the payoff from the contract when the asset move up to A_0u be Vu and that when move down to A_0d be Vd as shown in figure 3.6 above.

Let's imagine a portfolio consisting of a long position in Δ shares and a short position in one contract. If there is an up movement in the asset price, the value of the portfolio at the end of the life of the contract is $A_0u\Delta - Vu$ and that if there is a down movement is $A_0d\Delta - Vd$. Since the two are equal, it implies that

$$A_0u\Delta - Vu = A_0d\Delta - Vd \tag{3.113}$$

$$\Rightarrow \Delta = \frac{Vu - Vd}{A_0u - A_0d} \tag{3.114}$$

In this case, the contract is riskless and must earn a risk-free interest rate. Equation 3.114 shows the Δ is the change in value of the contract to the change in the asset price as we moves between nodes at time T.

If r is the risk-free interest rate, the present value of the contract is $(A_0u\Delta - V_u)e^{-rT}$. The cost of setting up contract is $A_0\Delta - V$. It follows that $A_0\Delta - V = (A_0u\Delta - V_u)e^{-rT}$.

$$V = A_0\Delta(1 - ue^{-rT}) + Vue^{-rT} \tag{3.115}$$

Substituting from equation 3.114 for Δ we obtain

$$V = e^{-rT}[pVu + (1-p)Vd] \tag{3.116}$$

where

$$p = \frac{e^{rT} - d}{u-d} \tag{3.117}$$

Equations 3.116 and 3.117 enables contracts to be valued when the asset price movements are given by a one-step model.

3.7.2 Two-Step Binomial Model

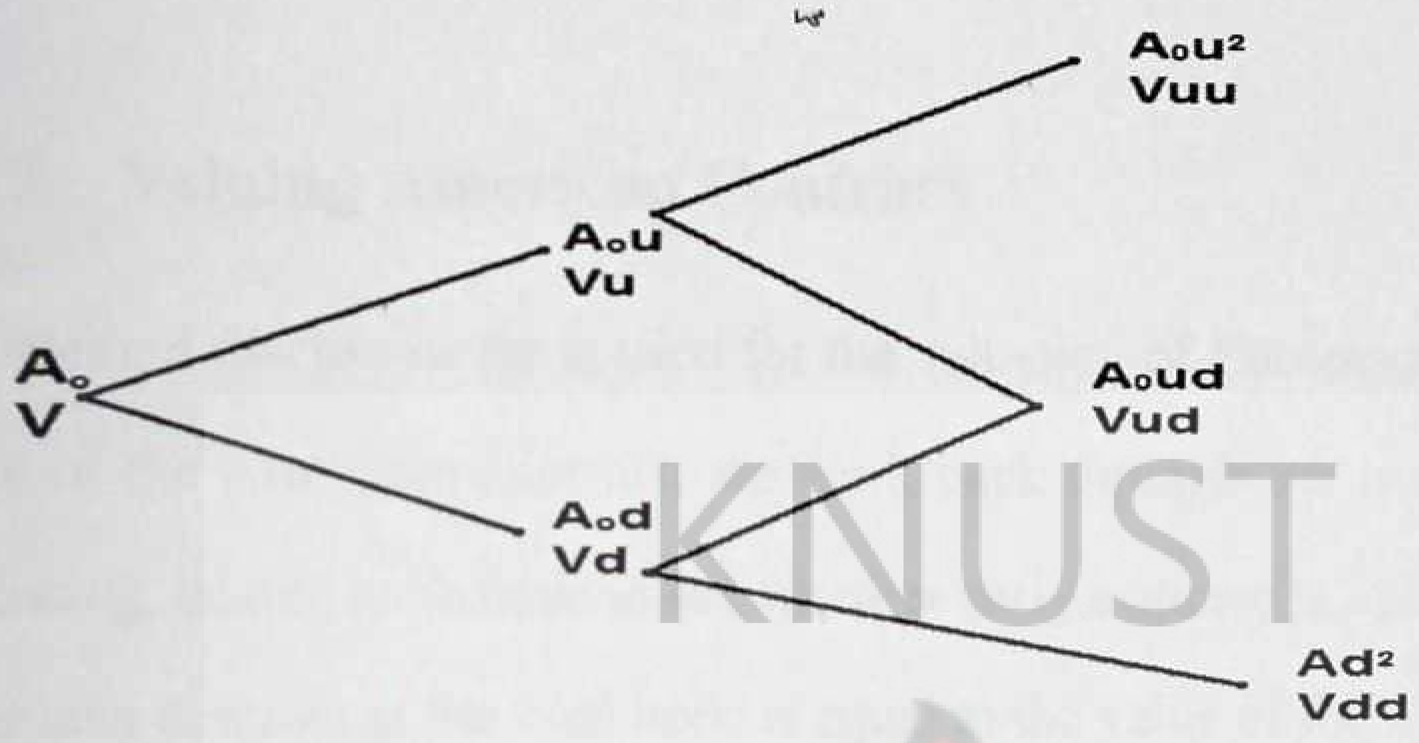

Figure 3.7: Asset and contract value in two-step tree

Let's consider figure 3.7, similar to the one-step, the asset price moves up to A_0u of down to A_0d. Given that the length of the time-step is Δt, equations 3.116 and 3.117 becomes

$$V = e^{-r\Delta t}[pVu + (1-p)Vd] \tag{3.118}$$

$$p = \frac{e^{r\Delta t} - d}{u-d} \tag{3.119}$$

Repeating equation 3.118,

$$Vu = e^{-r\Delta t}[pVuu + (1-p)Vud] \tag{3.120}$$

$$Vd = e^{-r\Delta t}[pVud + (1-p)Vdd] \tag{3.121}$$

Substituting equations 3.120 and 3.121 in 3.118 gives

$$V = e^{-2r\Delta t}[p^2Vuu + 2p(1-p)Vud + (1-p)^2Vdd] \tag{3.122}$$

The variables p^2, $2p(1-p)$ and $(1-p)^2$ are the probabilities that the upper, middle and lower nodes will be reached. The contrast value is equal to the expected payoff in a risk-neutral world discounted a risk-free interest rate.

Note that, if the asset price move up followed by a down move, it will be the same as a down move followed by and upward move. The parameters u, d and p satisfy the conditions for risk-neutral valuation and the lognormal distribution of the asset price.

3.7.3 Valuing American Contract

The method discuss so far is used for the valuation of European contract. To find the value of the American contract, we work back through the tree from the end to the beginning, testing each node to see whether early exercise is optimal. The value of the American contract at the final node is equal to the value of the European contract. The value of the contract at earlier node is max(V, payoff from early exercise) where

$$V = e^{-r\Delta t}[pVu + (1-p)Vd]$$

In practice, u and d are determine from the asset price volatility, σ , and the length of the step time interval Δt. Crox, Ross and Rubinstein (CRR) choose the up and down ratios to be

$$u = e^{\sigma\sqrt{\Delta t}} \tag{3.123}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{3.124}$$

$$\Rightarrow d = \frac{1}{u}$$

$$\Rightarrow d * u = 1$$

From equation 3.119, $p = \frac{e^{r\Delta t} - d}{u-d}$.

In practice, the life of a contract is divided into 30 or more time step. In each time step, there is a binomial asset price movements of 30 time steps, 31 terminal asset prices and 2^{30} or about 1 billion possible asset price paths are considered. The equations which define the trees are equations 3.122 and $e^{r\Delta t}$.

3.7.4 Numerical Implementation of Binomial tree

The CRR model in equation 3.122 can only be used in the valuation of European contract as mentioned earlier. To find the value of American style contract, we use different multi-period binomial model. The no arbitrage arguments are used and no assumptions are required for the probabilities of up and down movements in the asset price at each node.

At time zero, the asset price A is known. At time Δt, there are two possible asset prices Au and Ad. At time $2\Delta t$, there are three possible asset prices, Au^2, Aud and Ad^2, and so on. In general, at time $i\Delta t$, where $0 \leq i \leq N, (i+1)$ asset prices are considered, given by

$$Au^j d^{N-j} \; for \; j = 0, 1, ..., N \tag{3.125}$$

where N is the total number of movements and j is the total number of up movements.

Note that, we are adopting the European contract for an American contract. In this case we use backwards induction to fill in the nodes on the contract value tree and compare the value we get by using the formula from before to the value of early exercise at that respective node. The actual value of the node is the greater of the two. That is,

Value of contract at a node=max(Binomial value, Exercise Value)

If the life of an European contract on a non-dividend paying asset is divided into N sub-interval of length $i\Delta t$ and the j^{th} node at time $i\Delta t$ at the (i, j) node where $0 \leq i \leq N$ and $0 \leq j \leq i$. Then the value of the contract is $V_{i,j} at (i, j)$ node. Given that the asset price at node (i, j) is $Au^i d^{i-j}$, then the value of the European contract is given by

$$V_{i,j} = \max(Au^j d^{N-j} - K, 0) \; for \; j = 0, 1, ..., N \tag{3.126}$$

The probability p of moving from (i, j) node at time $i\Delta t$ to $(i+1, j+1)$ node at time

$(i+1)\Delta t$, and probability $(1-p)$ of moving from the (i,j) node at time $i\Delta t$ to the $(i+1,j)$ node at time $(i+1)\Delta t$ give the risk neutral valuation $V_{i,j}$ as

$$V_{i,j} = e^{-r\Delta t}[pV_{i+1,j+1} + (i-p)V_{i+1,j}] \text{ and } 0 \leq i \leq N-1, 0 \leq j \leq i \tag{3.127}$$

For American contract, we find the value of the contract at any node (i,j) earlier than the maturity time T. When early exercise is taken into account, the value $V_{i,j}$ of the contract must be compared with the contracts intrinsic value (Davis, 2005), (Hull, 2003) and is given by

$$V_{i,j} = \max[K - Au^j d^{i-j}, e^{-r\Delta t}(pV_{i+1,j+1} + (1-p)V_{i+1,j})] \tag{3.128}$$

Dividend Paying Asset

The Merton's model adjust the Black-Scholes model to cater for European options on stocks that pay continuous dividend. In so doing, the risk-free rate is modified from r to $(r-\lambda)$, where λ is the continuous dividend yield. Applying the same principle for the valuation of the contract, the risk-neutral probability in equation 3.119 is modified but the other parameters remains the same. That is,

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \\ \text{and } p &= \frac{e^{(r-\lambda)\Delta t} - d}{u-d} \end{aligned} \tag{3.129}$$

When generating the binomial tree of the asset for the American and the European contract as well on assets paying a continuous dividend, these parameters are applied and the tree will be identical in both cases. The probability of an asset price increase varies inversely with level of the continuous dividend rate λ (Davis, 2005).

On a particular date τ, it is assumed that there is a single known dividend and the dividend yield is a percentage of the asset price which is known. If $i\Delta$ t is prior to the asset going ex-dividend, the nodes on the tree correspond to asset prices

$$Au^{j}d^{N-j} \ for \ j = 0, 1, ..., N \tag{3.130}$$

If the time $i\Delta t$ is after the asset goes ex-dividend the nodes correspond to asset prices (Davis, 2005).

$$A(1-\lambda)u^{j}d^{N-j} \ for \ j = 0, 1, ..., N \tag{3.131}$$

3.7.5 Stability of Binomial tree method

For the Binomial tree method to converge, $0 \leq p \leq 1$. And for this condition to be met, $d < e^{r\Delta t} < u$. The choice of u and d is constrained to ensure that the limiting tree is the BlackScholes model. Since p ensures that the mean is correct, we have one essential condition left: the variances must converge correctly. Since we have two sequences and only one condition, there is still quite a lot of flexibility.

Chapter 4

ANALYSIS AND RESULTS

4.1 Introduction

In this chapter, we look at the application Black-Scholes partial differential equation and Binomial tree method in the valuation of life insurance liabilities. The chapter compare and contrast the convergence of the Black-Scholes PDE and the Binomial tree method in the valuation of life insurance liabilities. The assumptions underlying the application of Black-Scholes equation are fully elaborated in page 47.

4.2 Matlab Implementation

The matrices obtained using the finite difference method are generally a very large tri-diagonal matrices and requires more computational time. For this reason, Matlab was used find the solutions to the systems. See appendix I and II for Matlab codes for Implicit Method and Crank-Nicolson method respectively. Matlab code was also implemented for the multi-period binomial tree method. See appendix III and IV for matlab code.

4.3 Stability analysis of Implicit Finite Difference Method

The eigenvalues of the resulting matrix are given in table 4.1 below for volatility of 0.2231 and surrender time of 2 years for maturity of 30 years.

Table 4.1: Eigenvalues of the Implicit Finite Difference Method as $N \to \infty$

N=2000		N=4000		N=5000		N=6000		N=7000	
j	λ	j	λ	j	λ	j	λ	j	λ
1991	1.0197	3991	1.0099	4991	1.0079	5991	1.0066	6991	1.0057
1992	1.0156	3992	1.0078	4992	1.0063	5992	1.0052	6992	1.0045
1993	1.0120	3993	1.0060	4993	1.0048	5993	1.0040	6993	1.0034
1994	1.0088	3994	1.0044	4994	1.0035	5994	1.0029	6994	1.0025
1995	1.0061	3995	1.0031	4995	1.0025	5995	1.0021	6995	1.0018
1996	1.0039	3996	1.0020	4996	1.0016	5996	1.0013	6996	1.0011
1997	1.0022	3997	1.0011	4997	1.0009	5997	1.0007	6997	1.0006
1998	1.0010	3998	1.0005	4998	1.0004	5998	1.0003	6998	1.0003
1999	1.0003	3999	1.0001	4999	1.0001	5999	1.0001	6999	1.0001

N=10000		N=11000	
j	λ	j	λ
9991	1.0040	10091	1.0036
9992	1.0031	10092	1.0029
9993	1.0024	10093	1.0022
9994	1.0018	10094	1.0016
9995	1.0012	10095	1.0011
9996	1.0008	10096	1.0007
9997	1.0004	10097	1.0004
9998	1.0002	10098	1.0002
9999	1.0001	10099	1.0000

It could be verify from table 4.1 that, as the number of steps N increases, the eigenvalue approaches 1 and this shows that the implicit finite difference scheme is unconditionally stable.

4.4 Stability of Crank-Nicolson Method

The table below shows the eigenvalues of the Matrix of the scheme as $N \to \infty$.

Table 4.2: The eigenvalues of the Crank-Nicolson method as $N \to \infty$

N=100		N=500		N=1000		N=2000		N=4000	
j	λ_j	j	λ_j	j	λ_j	j	λ_j	j	λ_j
93	1.0572	493	1.0119	993	1.0060	1993	1.0030	3993	1.0015
94	1.0398	494	1.0088	994	1.0044	1994	1.0022	3994	1.0011
95	1.0284	495	1.0062	995	1.0031	1995	1.0016	3995	1.0008
96	1.0189	496	1.0040	996	1.0020	1996	1.0010	3996	1.0005
97	1.0111	497	1.0023	997	1.0012	1997	1.0006	3997	1.0003
98	1.0055	498	1.0011	998	1.0006	1998	1.0003	3998	1.0001
99	1.0020	499	1.0004	999	1.0002	1999	1.0001	3999	1.0000

Table 4.2 indicates that as $N \to \infty$, the eigenvalues approaches one (1) showing the stability of the Crank-Nicolson's method. Also the Crank-Nicolson method is with an accuracy of $O(\Delta t^2, \Delta A^2)$ and that also indicates that how accurate the results is to the actual value.

4.5 Stability of the Binomial tree method

Table below shows how the convergence of the Binomial tree for $\sigma = 0.2231$, $r = 0.05$ and $T = 2$ years. The multi-period binomial tree method is stable if $d < e^{r\Delta t} < u$.

Table 4.3: Stability of Binomial tree method

N	d	$e^{r\Delta t}$	u
100	0.9689	1.0010	1.0321
150	0.9746	1.0007	1.0261
200	0.9779	1.0005	1.0226
250	0.9802	1.0004	1.0202
300	0.9819	1.0003	1.0184
350	0.9833	1.0003	1.0170
400	0.9843	1.0003	1.0159
450	0.9852	1.0002	1.0150
700	0.9881	1.0001	1.0120
900	0.9895	1.0001	1.0106
2050	0.9931	1.0000	1.0070
3000	0.9943	1.0000	1.0058
10000	0.9968	1.0000	1.0032

Table 4.3 above indicates that, $e^{r\Delta t}$ lies between d and u and this shows that the Binomial tree is stable as $N \to \infty$ and hence the multi-period binomial tree method is stable.

4.6 Comparing the convergence of the Binomial tree, Implicit and Crank-Nicolson's Method Life Insurance Valuation with no-dividend

Earlier in chapter 3, we considered the convergence of the fully implicit, the Crank-Nicolson method and the multi period model with relation to the Black Scholes value of the life insurance contract. Table 4.4 shows the value of life insurance contract containing surrender option with data from Life Insurance Company A and table 4.5 shows that of the data from Life Insurance Company B. The data from company A are as follows:

Asset price, $A = 50$, strike price, $K = 52$, risk-free interest rate, $r = 0.05$, surrender period, $t = 2$ years, maturity period, $T = 30$ years, volatility, $\sigma = 0.2231$ and the dividend payment rate, $\phi = 0.03$. The surrender value of the life insurance contract is **5.4650** with the value at maturity being **8.22** for non-dividend paying asset.

The data from Insurance Company B are as follows:

$A = 250$, $K = 260$, $r = 0.06$, $t = 7$ years, $T = 30$ years, and $\sigma = 0.24$. The value of the life insurance contract at maturity is **40.15** and the surrender value is **36.04**.

Note: *In the tables below, the figures in bracket are the difference between the actual values and values obtained from the various numerical methods.*

Table 4.4: The comparison of the three methods in the valuation of Life Insurance Liabilities with no-dividend payment for company A. Surrender value at t= 2 years. - Expected value = 5.4650

No. of steps	Multi-Period Binomial	Fully Implicit	Crank-Nicolson
30	5.4834(-.0184)	5.3770(.0880)	5.4204(.0446)
90	5.4842(-.0192)	5.4311(.0339)	5.4465(.0185)
150	5.4582(.0068)	5.4413(.0237)	5.4503(.0147)
210	5.4554(.0096)	5.4462(.0188)	5.4531(.0119)
270	5.4551(.0099)	5.4486(.0164)	5.4541(.0109)
330	5.4572(.0078)	5.4501(.0149)	5.4546(.0104)
390	5.4581(.0069)	5.4511(.0139)	5.4550(.0100)
450	5.4584(.0066)	5.4518(.0132)	5.4552(.0098)
510	5.4584(.0066)	5.4525(.0125)	5.4554(.0096)
570	5.4583(.0067)	5.4529(.0121)	5.4556(.0094)
630	5.4508(.0142)	5.4533(.0117)	5.4557(.0093)
660	5.4558(.0070)	5.4534(.0116)	5.4558(.0092)
690	5.4578(.0072)	5.4536(.0114)	5.4558(.0092)
720	5.4577(.0073)	5.4537(.0113)	5.4559(.0091)
750	5.4575(.0075)	5.4538(.0112)	5.4559(.0091)
780	5.4574(.0076)	5.4554(.0110)	5.4559(.0091)
810	5.4572(.0078)	5.4541(.0109)	5.4560(.0090)
840	5.4571(.0079)	5.4542(.0108)	5.4560(.0090)
870	5.4569(.0081)	5.4543(.0107)	5.4560(.0090)

Table 4.5: The valuation of Life Insurance Liabilities with no-dividend payment at Maturity (T=30 years) for company A. Expected Value=8.220

No. of steps	Multi-Period Binomial	Fully Implicit	Crank-Nicolson
100	8.1763(.0437)	7.4244(.7956)	7.5107(.7093)
250	8.1831(.0369)	7.5275(.6925)	7.5622(.6578)
400	8.2045(.0155)	7.5535(.6665)	7.5757(.6443)
550	8.2076(.0124)	7.5656(.6544)	7.5817(.6383)
700	8.2074(.0126)	7.5725(.6475)	7.5851(.6349)
850	8.2040(.0160)	7.5770(.6430)	7.5874(.6326)
1000	8.2075(.0125)	7.5801(.6399)	7.5890(.6310)
1150	8.2116(.0084)	7.5824(.6376)	7.5901(.6299)
1300	8.2085(.0115)	7.5842(.6358)	7.5910(.6290)
1450	8.2085(.0115)	7.5846(.6354)	7.5917(.6283)
1500	8.2085(.0115)	7.5860(.6340)	7.5919(.6281)

Table 4.6: The comparison of the three methods in the valuation of Life Insurance Liabilities with no-dividend payment for company B. Surrender value at t= 7 years. - Expected value = 36.04

No. of steps	Multi-Period Binomial	Fully Implicit	Crank-Nicolson
30	35.8113(.2287)	34.4780(1.5620)	35.0667(.9733)
90	35.9807(.0593)	35.2300(.8100)	35.4022(.6378)
330	36.0343(.0057)	35.4678(.5722)	35.5187(.5213)
390	36.0315(.0085)	35.4817(.5583)	35.5249(.5151)
450	36.0263(.0137)	35.4919(.5481)	35.5294(.5106)
510	36.0319(.0081)	35.4998(.5402)	35.5329(.5071)
570	36.0334(.0066)	35.5060(.5340)	35.5356(.5044)
630	36.0287(.0113)	35.5110(.5290)	35.5378(.5022)
660	36.0324(.0076)	35.5131(.5269)	35.5387(.5013)
690	36.0339(.0061)	35.5151(.5249)	35.5396(.5004)
720	36.0300(.0100)	35.5169(.5231)	35.5404(.4996)
750	36.0304(.0096)	35.5185(.5215)	35.5412(.4988)
780	36.0336(.0064)	35.5201(.5199)	35.5419(.4981)
810	36.0331(.0069)	35.5215(.5185)	35.5425(.4975)
840	36.0302(.0098)	35.5228(.5172)	35.5431(.4969)
870	36.0319(.0081)	35.5241(.5159)	35.5436(.4964)

Tables 4.4, 4.5 and 4.6 shows that the Crank-Nicolson finite scheme in converges faster than the fully implicit finite scheme in as $N \to \infty$. The multi-period binomial model is closer to the value of the life insurance contract for small values of N than the two finite difference methods and it also give more accurate results than the Black-Scholes partial differential equation (see also figures 4.1, 4.2, 4.3 and 4.4).

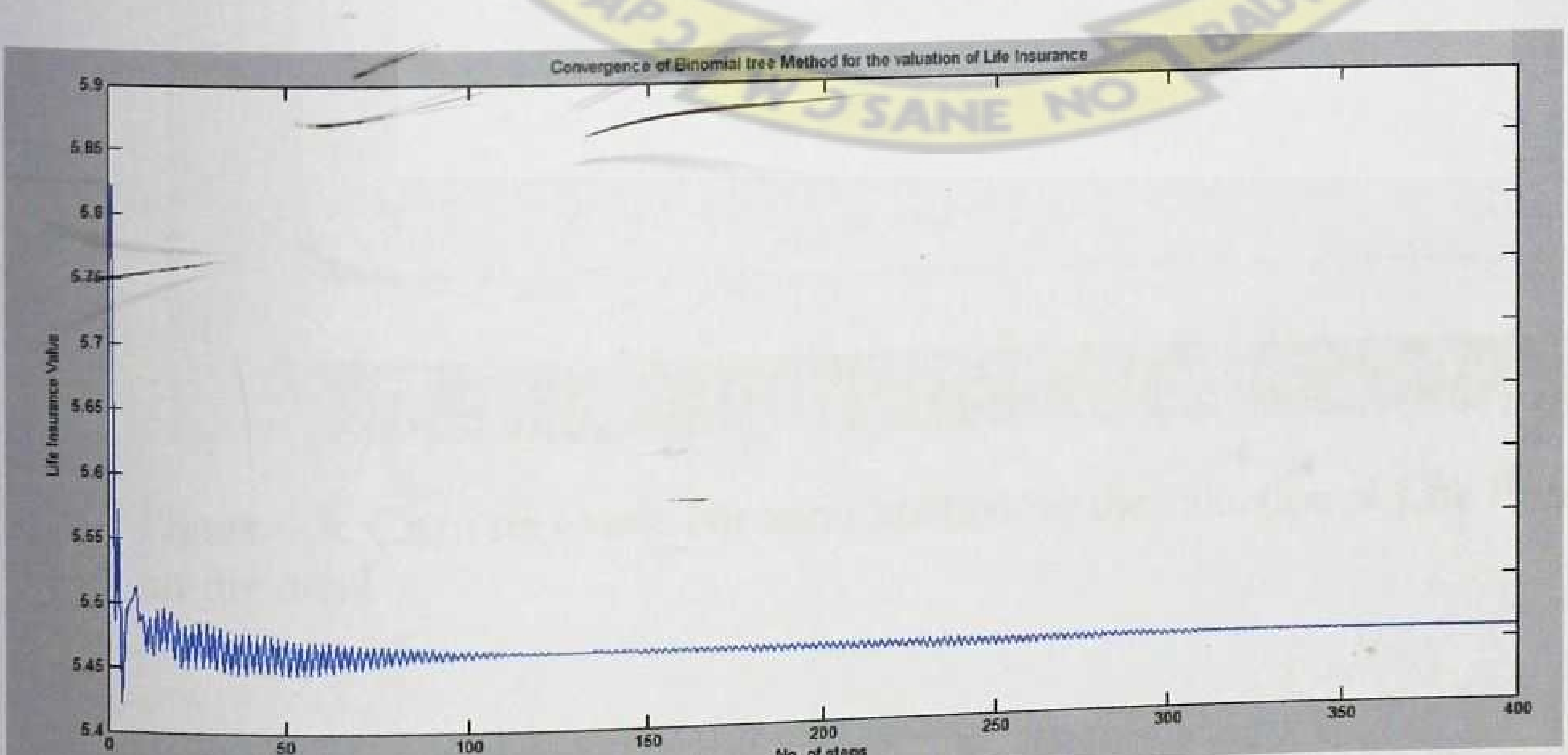

Figure 4.1: Chart on Binomial tree for the value of Life Insurance with no-dividend

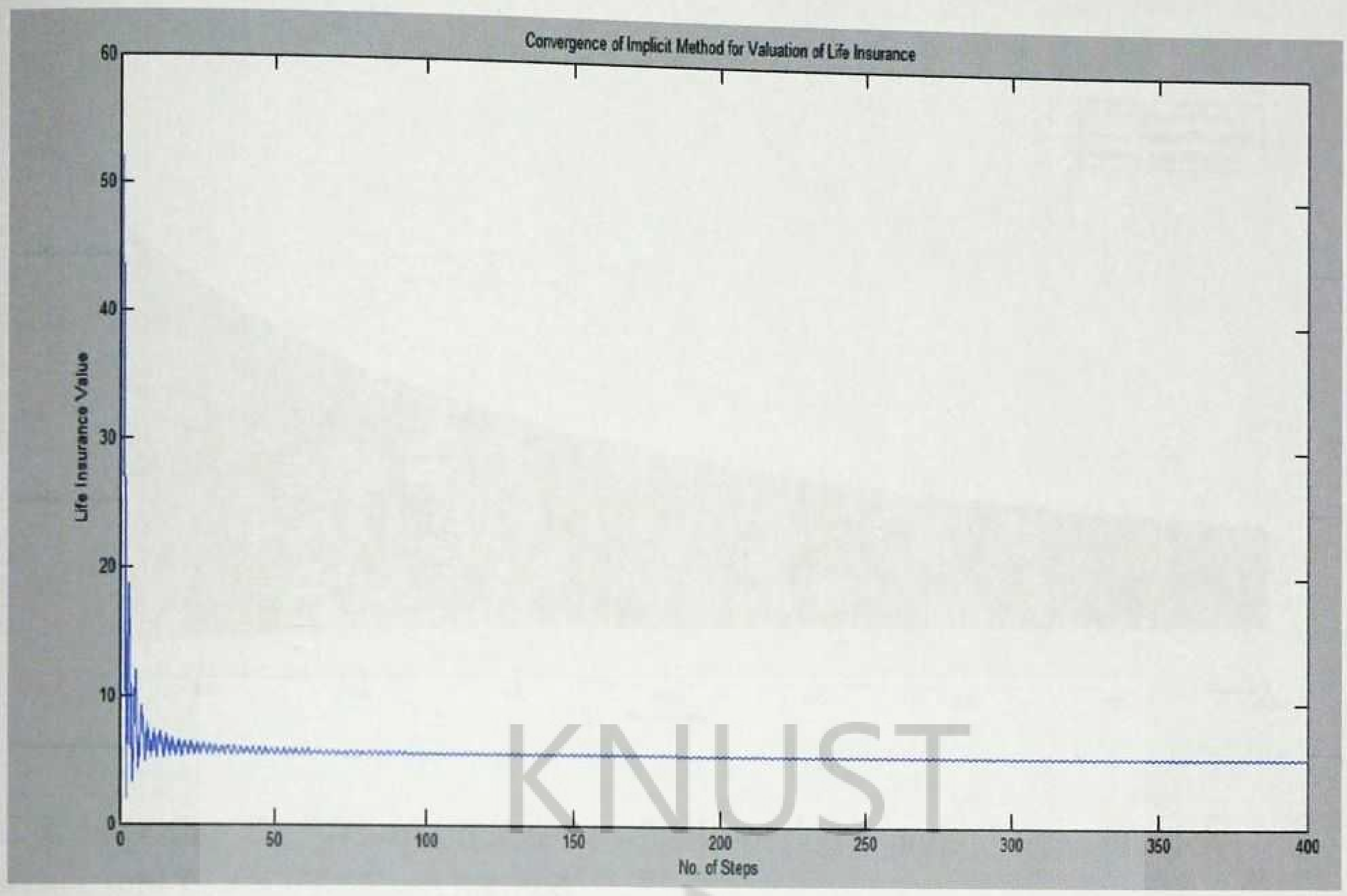

Figure 4.2: Chart on Fully Implicit Method for the valuation of Life Insurance with no-dividend

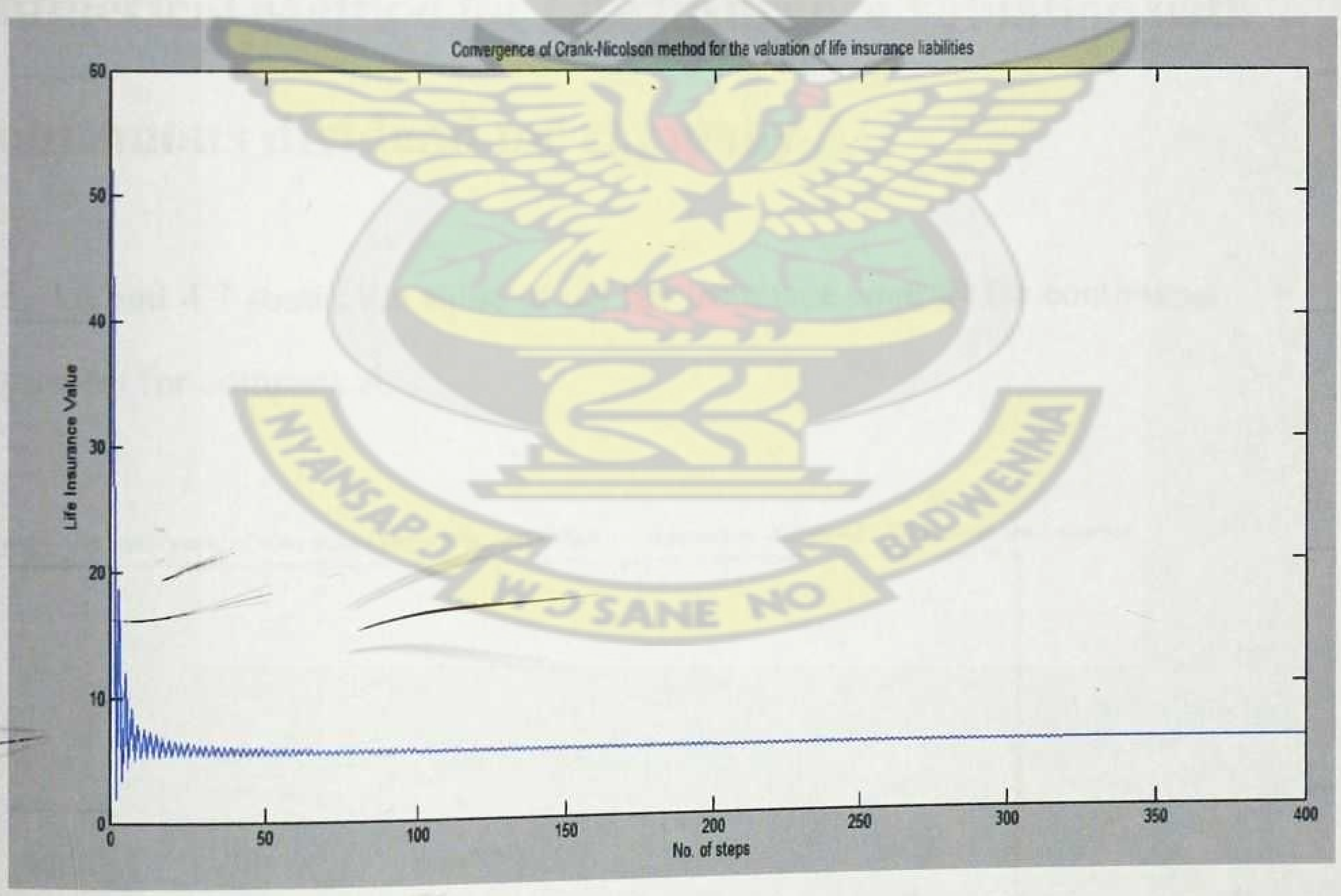

Figure 4.3: Chart on Crank-Nicolson Method for the valuation of Life Insurance with no-dividend

Figure 4.4 which is the graph drawn from table 4.4 shows the comparison of the three numerical methods.

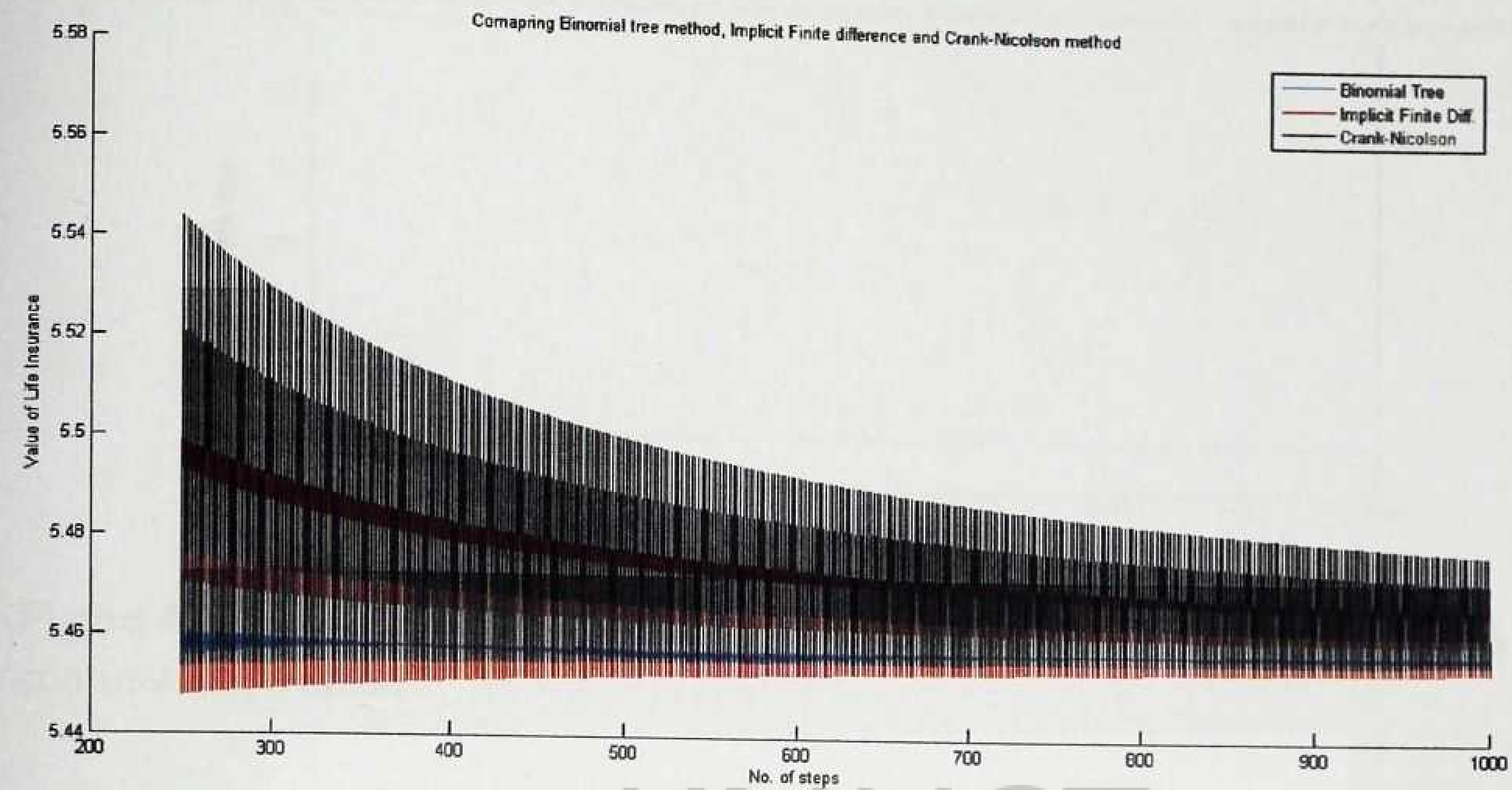

Figure 4.4: Comparing the Values obtained by Multi-period binomial tree, Implicit finite difference and Crank-Nicolson method

4.7 Numerical Method for Life Insurance Valuation with continuous dividend for company A

Figures 4.5, 4.6 and 4.7 shows the value of the life insurance contract for continuous dividend payment for company A.

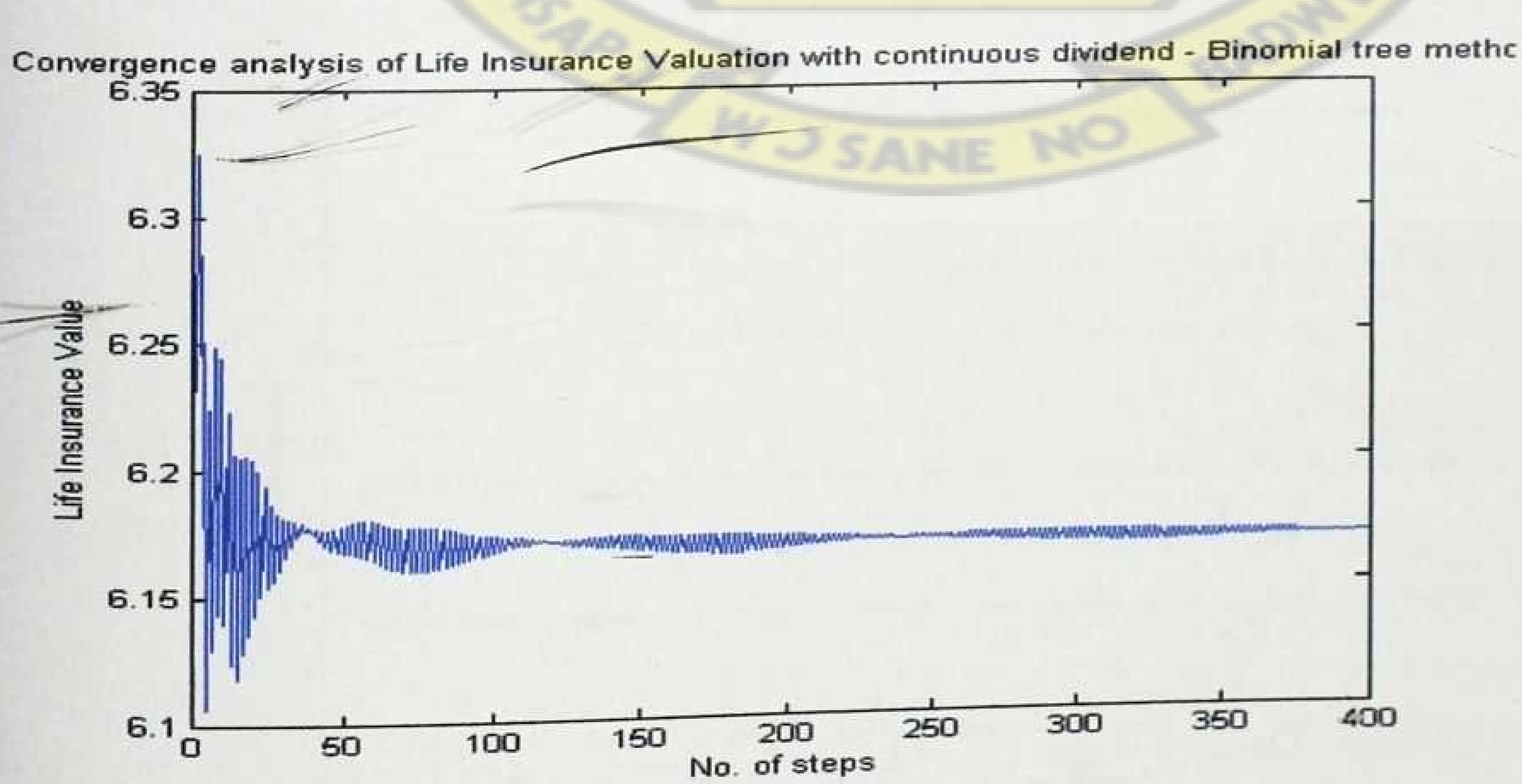

Figure 4.5: Chart on Multi-Period Binomial tree Method for the valuation of Life Insurance with continuous dividend

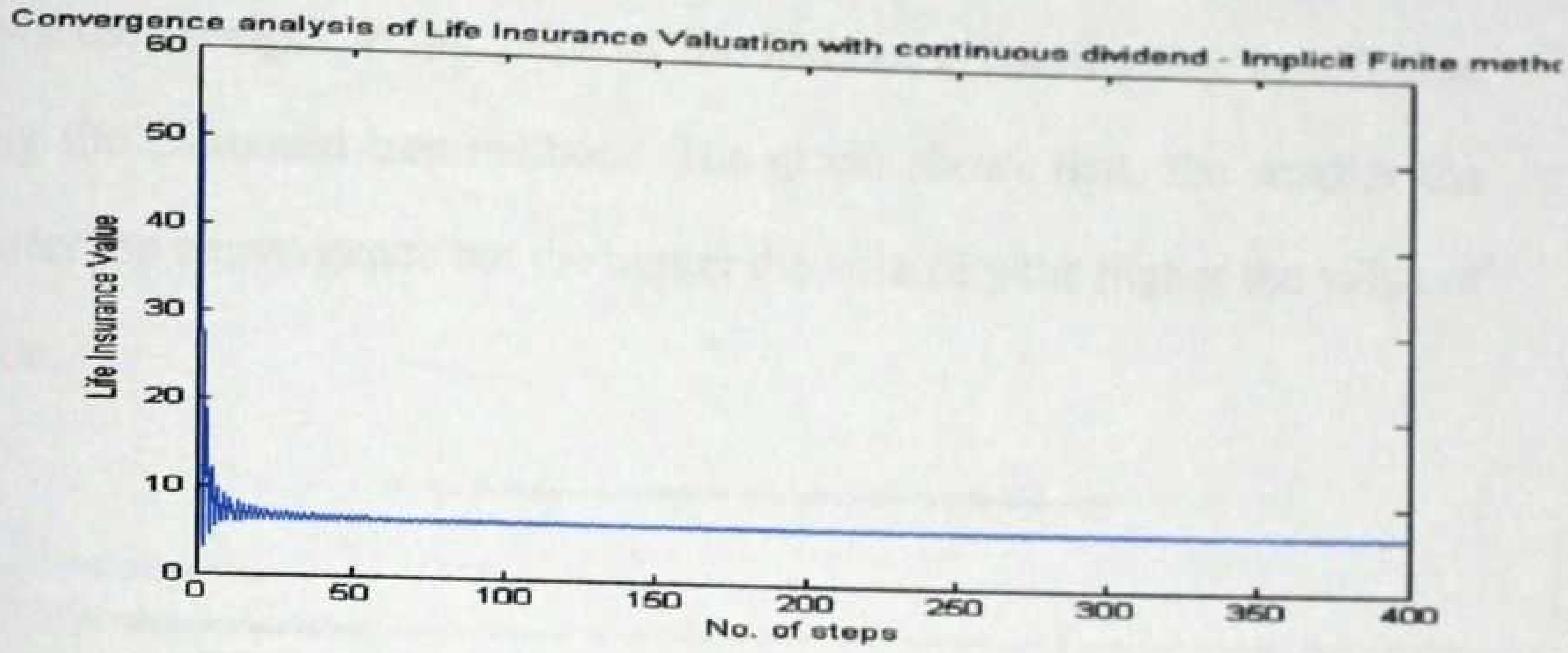

Figure 4.6: Chart on Fully Implicit Method for the valuation of Life Insurance with continuous dividend

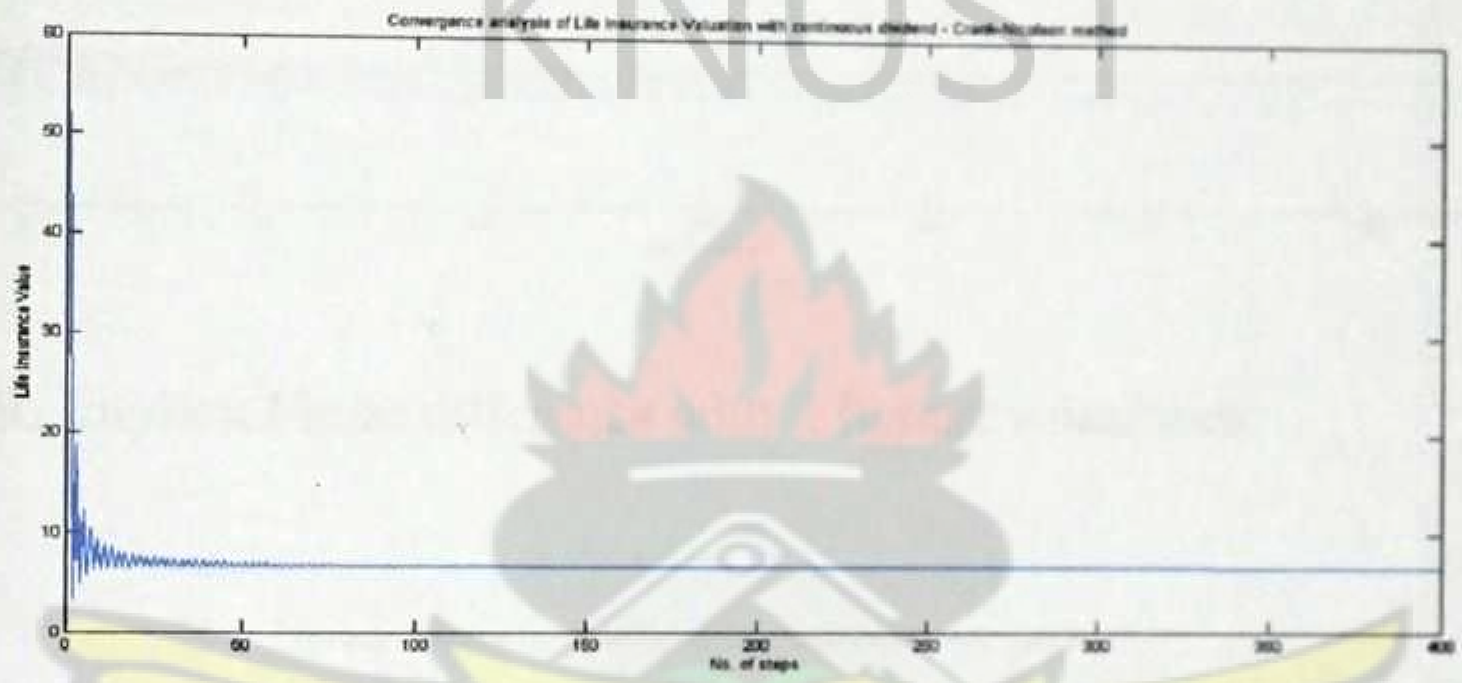

Figure 4.7: Chart on Crank-Nicolson Method for the valuation of Life Insurance with continuous dividend

4.8 Effect on volatility

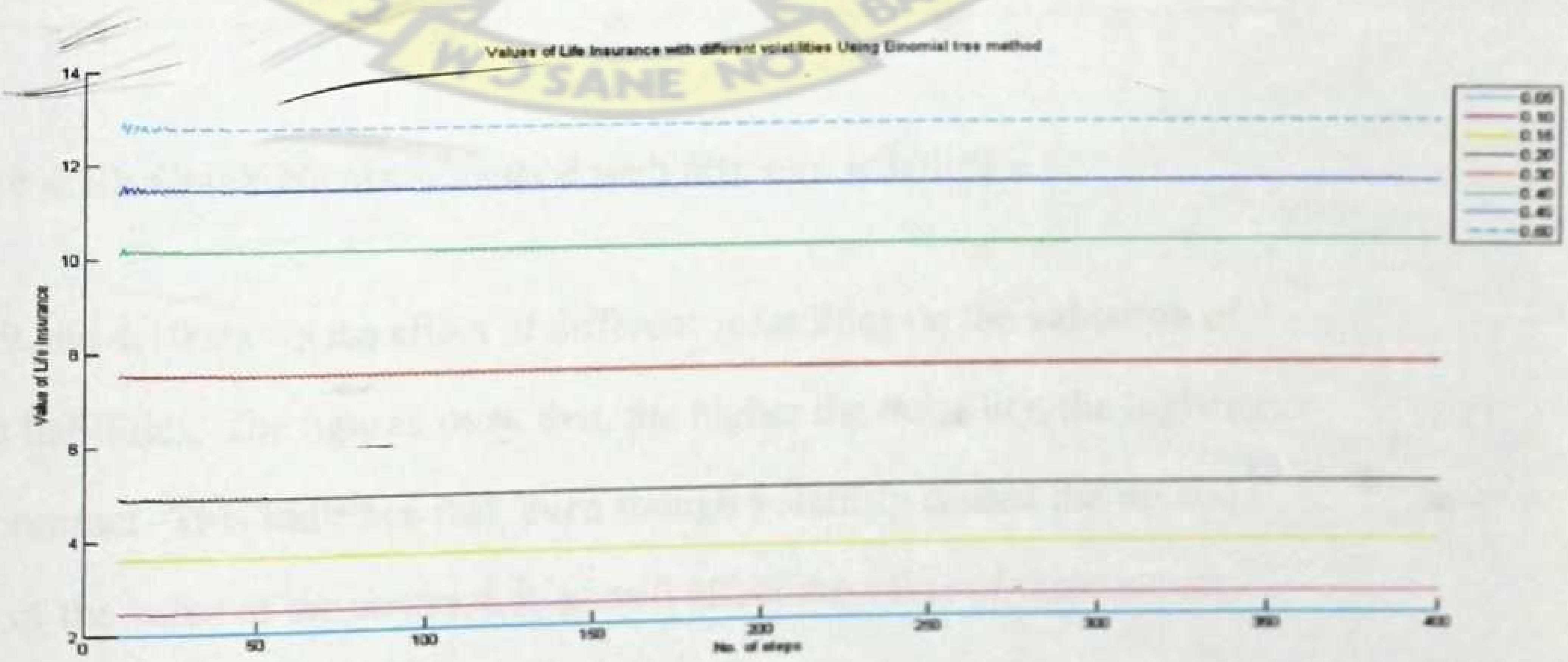

Figure 4.8: Binomial tree with different volatilities

Figure 4.8 shows the changes in the value of life insurance liability due to different volatilities using the Binomial tree method. The graph shows that, the smaller the volatility the faster the convergence but the bigger the volatility the higher the value of the life insurance.

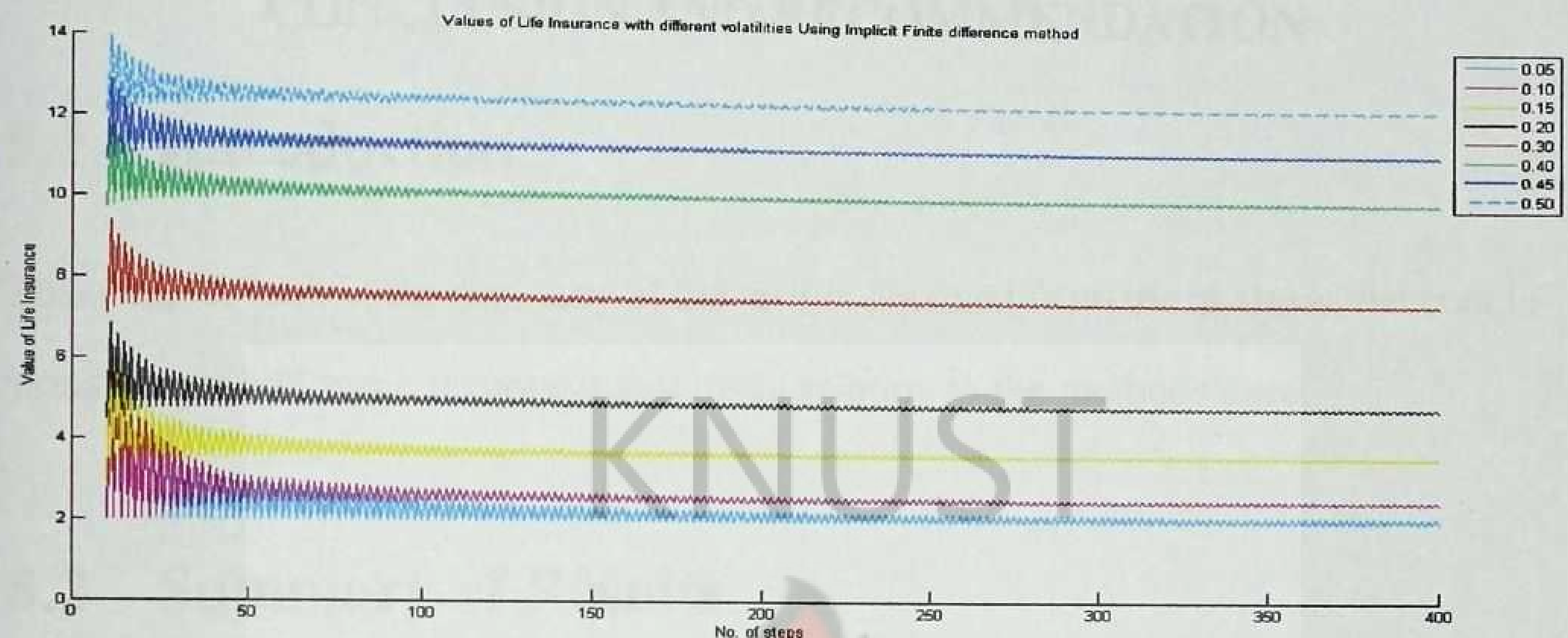

Figure 4.9: Implicit Finite difference with different volatilities

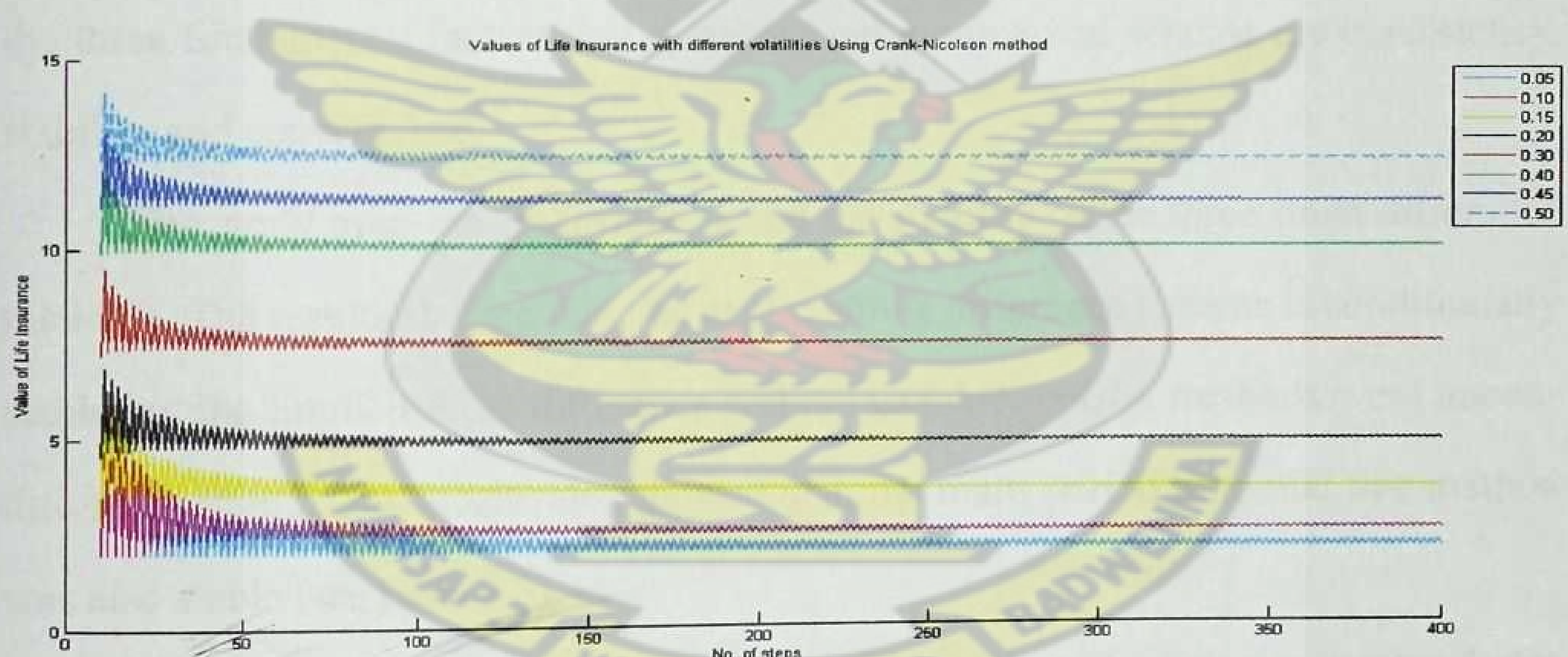

Figure 4.10: Crank-Nicolson method with different volatilities

Figures 4.9 and 4.10 shows the effect of different volatilities on the valuation of the life insurance liabilities. The figures show that, the higher the volatility, the higher the value of the contract. This indicates that, even though volatility causes the up and down movement of the value of the contract, it as well affect the value of the contract.

Chapter 5

CONCLUSION AND RECOMMENDATION

5.1 Introduction

This chapter look at the summary of the results obtained from the analysis, the conclusions drawn and some recommendations in relation to the methods used.

5.2 Summary of Results

From the analysis, it was realized that, asset price discretization and time discretization are two fundamental sources of error. Lax Equivalence theorem indicated that, the three fundamental factors that characterized a numerical scheme are consistency, stability and convergence.

The study used the eigenvalue to test the stability of the three finite difference methods. The results showed that, the explicit finite difference scheme is conditionally stable but the implicit finite difference and the Crank-Nicolson methods were unconditionally stable. The results also showed that, the multi-period binomial tree method was also stable (see table 4.3).

The results also showed that, the Crank-Nicolson method also converges faster than the implicit finite difference method when used in solving the Black-Scholes partial differential equation. But the multi-period binomial tree method was found to give more accurate results for life insurance contract containing surrender options in Ghana than the Black-Scholes partial differential equation.

5.3 Conclusion

The Crank-Nicolson method converges faster than the implicit finite difference method when these schemes are used in solving the Black-Scholes pde. That is, the Crank-Nicolson method gives more accurate results than the implicit finite difference scheme.

The multi-period binomial tree method was found to be closer to the solution for even smaller values of N (the number of steps) than the Black-Scholes pde when the valuing the life insurance contract in Ghana. The big advantage the binomial model has over the Black-Scholes model is that it can be used to accurately price American options. This is because with the binomial model it's possible to check at every point in an option's life (ie at every step of the binomial tree) for the possibility of early exercise (eg where, due to eg a dividend, or a put being deeply in the money the option price at that point is less than its intrinsic value). Hence the Binomial tree method gives more accurate results than the Black-Scholes model.

5.4 Recommendations

In finding the value of American style life insurance contract, the binomial model gives more accurate results than the Black-Scholes model. In the case where the Black-Scholes model is going to be used, then the Crank-Nicolson method converges faster and gives more accurate results than the implicit finite difference scheme. But the explicit finite difference scheme may not give accurate results because of its conditional stability.

5.5 Further Studies

This study looked at the valuation of a single premium American life insurance contract. The researcher will take a look at the valuation of continuous-instalment options which contains surrender option as well.

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APPENDIX

Appendix I - Matlab Code for Implicit Finite Scheme

```
function[Rec,V]=ImplicitFDBS_nodividend2(A,K,r,volatility,T,N,M)
% If no dividend payment was made, enter zero for the dividend_yield
% A is the asset price
% K is the strike price
% T is the maturity period
% N is the number of iterations in the time direction
% M is the number of iterations in the asset direction
sigma=volatility;
y=length(N);
Table=zeros(y,3);
for j=1:y
 dt=T/N(j);
dA=2*A/M(j);
B=zeros(M(j)+1,M(j)+1);
f=max(K-(0:M(j))*dA,0)';
f(1)=f(1)-(0.5*r*1*dt-0.5*sigma^2*1*dt);
x=1/(1-r*dt);
for m=1:M(j)-1
    B(m+1,m)=x*(r*m*dt-sigma.^2*m.^2*dt)/2;
    B(m+1,m+1)=x*(1+sigma.^2*m.^2*dt);
    B(m+1,m+2)=x*(-r*m*dt-sigma.^2*m.^2*dt)/2;
end
B(1,1)=1;
B(M(j)+1,M(j)+1)=1;
```

```
for i=N(j):-1:1
    f=B\f;
    f=max(f,(K-(0:M(j))*dA)');
end
f;
V=f(round((M(j)+1)/2));
Table(j,1)=j;
Table(j,2)=N(j);
Table(j,3)=V;
end
Rec=Table;
```


Appendix II - Matlab Code for Crank-Nicolson Method

$function[P] = CrankNicolsonFDBS(S,K,r,\sigma,T,N,M,dividend_yield)$

% If no dividend payment was made, enter zero for the dividend_yield

% S is the asset price

% K is the strike price

% T is the maturity period

% N is the number of iterations in the time direction

% M is the number of iterations in the asset direction

σ is the volatility

$\lambda = dividend\_yield;$

$dt = T/N;$

$ds = 2*S/M;$

$A = zeros(M+1,M+1);$

$f = max(K-(0:M)*ds,0)';$

$for\ m = 1:M-1$

$A(m+1,m) = ((r-\lambda)*m*dt - \sigma^2*m^2*dt)/4;$

$A(m+1,m+1) = 1+0.5*(r-\lambda)*dt+0.5*\sigma^2*m^2*dt;$

$A(m+1,m+2)=(-(r-\lambda)*m*dt-\sigma^2*m^2*dt)/4;$

end

$A(1,1)=1;$

$A(M+1,M+1)=1;$

$A;$

$form=1:M-1$

$B(m+1,m)=(-(r-\lambda)*m*dt+\sigma^2*m.^2*dt)/4;$

$B(m+1,m+1)=1-0.5*(r-\lambda)*dt-0.5*\sigma^2*m.^2*dt;$

$B(m+1,m+2)=((r-\lambda)*m*dt+\sigma^2*m^2*dt)/4;$

end

$B(1,1)=1;$

$B(M+1,M+1)=1;$

$B;$

$for\,i=N:-1:1$

$f=A^{-1}(B*f);$

$f=max(f,(K-(0:M)*ds)');$

end

$f;$

$P=f(round((M+1)/2));$

Appendix - Matlab Code for Binomial tree with no dividend

```
function[Rec,V]=american_binomial_nodividend(Asset,Strike_Price,
Maturity,rate,sigma,N)
S=Asset;K=Strike_Price;T=Maturity;r=rate;z=sigma;
x=length(N);
Table=zeros(x,3);
for j=1:x
```

```
    dt=T/N(j);
    A=zeros(N(j)+1);
    a=exp(r*dt);
    u=exp(z*sqrt(dt));
    p=(a*u-1)/(u^2-1);
    A(N(j)+1,:)=max(K-S*u.^(2*(0:N(j))-N(j)),0);
for i=N(j):-1:1
 A(i,1:i)=max((K-S*u.^(2*(1:i)-i-1)),(p*A(i+1,(1:i)+1)+(1-p)*A(i+1,1:i))/a);
end
V=A(1,1);
Table(j,1)=j;
Table(j,2)=N(j);
Table(j,3)=V;
end
Rec=Table;
```


Appendix III - Matlab Code for Binomial Tree with dividend

```
function[V]=american_binomial_dividend(Asset, Strike_Price, Maturity,
rate, volatility, N, dividend_percentage, dividend_payment_date)
% time remaining until expiration is expressed as a percent of a year
% volatility is the annual volatility of stock price (the standard deviation
of the short-term returns over one year)
% rate is the current continuously compounded risk-free interest rate
% N should be in a row matrix form and it's indicates the number of
% iterations.
A=Asset;K=Strike_Price;T=Maturity;r=rate;g=volatility;rd=dividend_percentage;
tau=dividend_payment_date;
```

```
x=length(N);

Table=zeros(x,3);

for j=1:x

dt=T/N(j);

B=zeros(N(j)+1);

a=exp(r*dt);

u=exp(g*sqrt(dt));

d=1/u;


n=floor(tau/dt);


if n==0

    n=1;

end

p=(a-d)/(u-d);


B(N(j)+1,:)=max(K-A*(1-rd)*u.^(2*(0:N(j))-N(j)),0);



for i=N(j):-1:n

        B(i,1:i)=max((K-A*(1-rd)*u.^(2*(1:i)-i-1)),

        (p*B(i+1,(1:i)+1)+(1-p)*B(i+1,1:i))/a);

end

for i=n:-1:1

  B(i,1:i)=max((K-A*(1-rd)*u.^(2*(1:i)-i-1)),(p*B(i+1,(1:i)+1)+(1-p)*B(i+1,1:

end

%B

%Iteration=N(j);
```

KNUST

```
V=B(1,1);
Table(j,1)=j;
   Table(j,2)=N(j);
   Table(j,3)=V;
end
Table
plot(Table(:,2),Table(:,3)),xlabel('No. of steps'),ylabel('Life Insurance
 Value'),title('Convergence analysis of Life Insurance Valuation with
  continuous dividend - Binomial tree method')
```