

A mathematical model of corruption dynamics endowed with fractal–fractional derivative

Ugochukwu Kizito Nwajeri ^a, Joshua Kiddy K. Asamoah ^{b,c}, Ndubuisi Rich Ugochukwu ^a, Andrew Omame ^{a,d}, Zhen Jin ^{e,*}

^a Department of Mathematics, Federal University of Technology Owerri, Nigeria

^b Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

^c Department of Mathematics, Saveetha School of Engineering SIMATS, Chennai, India

^d Abdus Salam School of Mathematical Sciences, Government College University, Katchery Road, Lahore 54000, Pakistan

^e Complex Systems Research Center, Shanxi University, Taiyuan 030006, China

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ABSTRACT

Numerous organisations across the globe have significant challenges about corruption, characterised by a systematic, endemic, and pervasive nature that permeates various societal establishments. Hence, we propose the fractional order model of corruption, which encompasses the involvement of corrupt individuals across various stages of education and employment. Specifically, we examine the presence of corruption among children in elementary schools, youths in tertiary institutions, adults in civil services, adults in government and public offices, and individuals who have renounced their involvement in corrupt practices. The basic reproduction number of the system was determined by utilising the next-generation matrix. The strength number was obtained by calculating the second derivative of the corruption-related compartments. The examined model solution's existence, uniqueness, and stability were established using the Krasnoselski fixed point theorem, the Banach contraction principle, and the Ulam–Hyers theorem, respectively. Based on the numerous figures presented, our simulations indicate a positive correlation between the decline in fractal–fractional order and the increase in the number of individuals susceptible to corruption. This phenomenon results in an increase in the prevalence of corruption among designated sectors of the general population. The persistence of corruption in society is a significant challenge to its eradication, as individuals who see personal gains from engaging in corrupt practices tend to exhibit a recurring inclination towards such behaviour. Nevertheless, it is recommended that to mitigate corruption within various corruption-prone subcategories, there is a need to enhance the level of consciousness and promotion of anti-corruption measures throughout all societal establishments.

Introduction

The phenomenon of corruption is a complex social, political and economic term that affects public (government) or private (company) officeholders and the entire nation. This phenomenon is so addictive and can be termed as an endemic disease within society because it is very tempting, attractive, infectious and preposterously very indulging with a gruesome pattern. Corruption is a fundamental threat to peace and security, and this has shown that the main cause of corruption is insecurity, greed, low media freedom, low-income and instability in government. It has been evident in [1] that corruption is not just a consequence but a source of conflict, fuelling it in several ways. Corruption generates new grievances in society or drives existing ones by undermining defence and security institutions and, at the same

time eroding state legitimacy [2]. Corruption undermines democratic institutions, slows economic development, contributes to governmental instability and can lead to many tragedies, including death. Furthermore, it attacks the foundation of democratic institutions by distorting electoral processes, perverting the rule of law, democracy and human rights, and stunting economic development because foreign direct investment is discouraged, and small businesses within the confinement of corruption often find it impossible to overcome the “start-up costs” required. It is a well-known fact that corruption is very rife and a serious problem in many developing countries worldwide. Corruption has deprived many citizens of privileges such as basic amenities, security, basic education, and socio-economic empowerment. As described by the then United States of America secretary, John Kerry, at the

* Corresponding author.

E-mail addresses: jkkasamoah@knust.edu.gh (J.K.K. Asamoah), jinzhn@263.net (Z. Jin).

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world economic forum in 2016, corruption is a social danger and an endemic disease because it feeds organised crime, destroys nation-states, and paralyses opportunities, particularly for women and girls. It also facilitates environmental degradation, contributes to human trafficking and money laundering, undermines whole communities and destroys the future. Corruption is also radicalised because it destroys the fate of legitimate authority. It opens up a vacuum which allows the predators to move in, and no one knows this better than the violent extremist group who regularly uses corruption as a recruitment tool. Corruption is an opportunity destroyer because it discourages honesty and accountable investment. It renders businesses more expensive to operate and drives up the cost of public services. Corruption has no boundary since it operates in different institutions, including the education sector of a given state. Many developing and developed countries have experienced several adverse effects of corruption, which has become a constant phenomenon. Corruption has become a daily cycle, an audacious activity and a disastrous political emblem which has soiled and epitomised great nations. Massive contract fraud at the top, through petty bribery, money laundering schemes, embezzlement and withholding of salaries, sex for grades, lack of promotion when due, impersonation and cheating during examination etc., are the hallmark of corruption in so many countries. Politicians in a strong position of power and wealth use their connections to engage in corruption. They even undermine the laws they established to satisfy their selfish ambitions and interests. Law enforcement agents harass any citizens they wish, and students cut any possible corners to succeed in school examinations. Furthermore, some teachers engage in sex-for-marks and bribery to award unreasonable marks to students in our colleges and universities.

According to Corruption Perception Index (CPI) [3] which employed the following data sources: bribery, diversion of public funds, officials using their public office for private gain without facing any consequences, the ability of governments to contain corruption in the public sector, nepotistic appointments in the civil service, state capture by narrow vested interests, excessive red tape in the public sector, legal protection of whistleblowers of corruption, etc measured corrupt state on the scale of 0 to 100 where the number 0 is highly corrupt while 100 means very clean from corruption. The data shows that Sub-Sahara Africa, with 32 scores, is the most corrupt region in the world, while Western Europe and European Union, with 66 scores, are very clean from corruption. With the unending war and instability in government, Somalia has been rated the most corrupt nation in the world with a 12 out of 100 score, while Denmark, with a 90 out of 100 score, is rated the highly clean state from corruption. In 2012, developing nations were estimated to have lost over 4 trillion dollars to corruption collectively [4,5]. Nowadays, this menace runs almost on every level of government institutions, such as law enforcement agencies (immigration, police, army, etc.), education sector, judiciaries etc. Corruption seen in primary, secondary and tertiary institutions is particularly very dangerous to social trust and depreciates the development of sound, educated, competent and public-determined individuals with ethical standards and a common sense of common citizenship while normalising the acceptability of corruption among future adults [6]. Embezzlement of funds intended for teaching materials and school building, bribes paid to teachers by parents to award unreasonable grades to wards, bribes paid to auditors to obscure misuse of funds, and substandard recruitment of sound teachers because of nepotism and favouritism are some examples of corruption seen in primary schools. The effect of this level of this unreasonable practices at this stage erodes the value of integrity of the pupil and produces sub-standard and ineffective individuals in secondary schools, universities, and the workplace who will end up engaging in more sophisticated corruption.

How long could we allow this atrocious phenomenon to keep undermining the fundamentals of society without reasonable measures to control this menace? This menace needs public awareness and possible cessation of corruption, which was employed by Corruption Watch in

2018 and stated as follows: Developing an online public survey on the National Anti-Corruption Strategy. Public consultations are needed. Hosting round-table discussions with civil society organisations and the private sector. Soliciting input from anti-corruption experts at a local and international level. Making a written submission to the policymakers on key findings from the consultative process and recommendations that should be considered when finalising the strategy. If the mentioned strategies and powerful legislation against corruption are enacted, there is every chance corruption can be eradicated from the state. The level of corruption is normally affected by the monetary policy. Goel and Nelson [7] in their research found less corruption in the countries with higher economic and political freedom. Also, states with a well-controlled financial sector are less corrupt than those with the opposite. Dimant [8] claimed in his research that the level of efficiency in administration determines the level of corruption. On another note, corruption is mostly influenced by the low salaries of public officials who are therefore trying to improve their financial position by taking bribes, and consequently, the socio-economic situation of the public servant also affects the phenomenon of corruption. Corruption is a serious dent to any economy, especially from the low-level economy state. Tanzi and Davoodi [9] in their research noted that corruption increases the volume of public investment at the expense of private investments as a result of room for public expenditure manipulation. Corruption reduces the effectiveness of public investments and the infrastructure of a country. It reduces tax revenue by compromising the ability of the state to collect taxes and fees. Corruption appears to redirect the composition of public expenditure from the required fixed assets for health and education, as there is less chance of getting commissions than from other unnecessary projects. With all these economic woes, there are many underlying effects on the state. Unending wars, high levels of poverty, emigration, loss of highly skilled personnel to other developed states, poor health facilities, and constant uprising from the masses are mostly the experiences of rising corruption risk in any state. Implementing anti-corruption is necessary to avoid unintended repercussions of corruption practices in a state. Public servants in private and public sectors must be scrutinised without prejudice or discrimination. This will help to hire the best individual for the required position. Administrators, especially in education and government organisation, should consider increasing the remuneration of their employees to a substantial standard to meet their required needs. The government must tighten regulations to ensure due process is followed and be prepared to enact the law against defaulters. However, as much as the aforementioned measures may be effective, developing rigorous intervention strategies to mitigate corruption and corrupt practices requires a thorough understanding of corrupt processes, prevention and disengagement programs which involves complex mathematical modelling.

Mathematical models can provide a first step in this direction. Using the approach of differential equations to describe social science and physical problems dates back, at least to the work of Richardson [10], who pioneered the application of mathematical techniques in studying the causes of war, the relationship between the arms race and the eruption of war. Over the years, mathematical models have been used to study fanatic and chaos behaviours [11–16]. Some scholars have studied deterministic models for the dynamics of corruption. For instance, Haileyesus [17] proposed and analysed a non-linear deterministic model for the corruption dynamics using the stability theory of differential equations and optimal control. Nathan et al. formulated a new mathematical model for the dynamics of moral corruption with comprehensive age-appropriate sexual information and provision of guidance and counselling [18]. Hathroubi and Trabelsi [19] studied corruption as an epidemic phenomenon using the epidemic diffusion model of Kermack and Mc-Kendrick (1927). Most of the work done on corruption models employed classical integer-order derivative and integral, yielding interesting results. To the best of our knowledge, none of these dynamical systems of corruption have used non-integer order

derivatives and integrals, also known as fractional order derivatives and integrals. Thus, this has become an interest of this manuscript to improve the existing corruption models with the aid of classical integer order derivatives and modern fractional order derivatives and integrals. As a result, corruption is referred to as an endemic disease within society and diabolically very attractive; there is a need to represent the corruption model with fractional operators since recent studies, especially epidemiological models, are best and more accurately described by fractional order derivatives and integrals compared to classical integer-order derivative and integral. For instance, the authors [20] investigated the spread of COVID-19 with the fractal–fractional operator. In the literature, the effect of lockdown and other government policies to curb the virus before the impending vaccination was properly analysed. Akbulut Arik et al. [21] considered an epidemiological model that takes into account pathogens in the environment and social distancing with both integer and non-integer derivative. Nwajeri et al. [22] proposed the fractional order model for HPV and Syphilis using a non-singular kernel of Caputo–Fabrizio derivatives. Qureshi and Atangana investigated and analysed a mathematical model of fever outbreak by novel fractional order with field data [23]. The use of real data and fractal–fractional differentiation was employed by Qureshi et al. [24] to investigate the mathematical model of diarrhea transmission dynamics. In their analysis, they substantiated that the disease can be controlled or eradicated if the transmission rate of the epidemic is reduced and the rate of treatment is increased. Several epidemiological systems have been developed with the help of fractional operators of Riemann–Liouville, Caputo, Caputo–Fabrizio, the Atangana–Baleanu and fractal–fractional [25–32]. Other areas of real-life problems apart from epidemiological systems analysed by researchers which have followed a power law, exponential and Mittag-Leffler kernels include physical and engineering systems [33–37] and the reference therein. Most of the above-mentioned and referenced literature demonstrates the importance and effectiveness of fractional derivatives and integrals, especially fractal–fractional derivative and integrals, over classical derivatives and integrals for mathematical modelling. This is because fractal–fractional derivative and integral use fractional order and fractal dimension, and it is also a non-local operator compared to classical order, which is a local operator and operates against time only. The complexity and rigour of developing a proper mathematical model of corruption are more suitable for fractal–fractional derivatives and integrals. This is because it captures the non-localities and the memory effects for the movement of the corruption cases and will produce more realistic results for the interaction of corruption in society. Corruption is truly an endemic disease within any society. Many authorities have tried and failed to control this menace. However, we believe getting rid of this disease within any society would be great, yet minimising it would be fantastic progress in the right direction for any society. Therefore, this manuscript aims to develop and comprehensively analyse the mathematical model of this ignominious and deplorable phenomenon, corruption endowed with fractal–fractional operators, and propose how to fight and eradicate this mayhem in society. We start by formulating the model with classical derivative and integral and then elevating the model using fractal–fractional derivative and integral in Section “Model Formulation”. The rest of the manuscript is organised as follows. In Section “Preliminaries”, we present the basic properties of the fractal fractional operator. Section “Qualitative analysis of the model” presents the qualitative analysis of the model solution, including the positivity, invariant region, the two equilibrium states of the corruption system, basic reproduction number, strength number, existence and uniqueness and Ulam–Hyers stability. Section “Numerical scheme” presents a numerical algorithm of the corruption model and figures with vivid observations and conclusions.

Model formulation

Let the parameter \mathcal{N} represents the number of population in Nigeria, \mathcal{S}_p represents the susceptible population of corruption, \mathcal{K}_{ce} represents the corrupt children in elementary schools, \mathcal{K}_{yt} represents the corrupt youths in tertiary institutions, \mathcal{K}_{ac} represents the corrupt adults in civil service, \mathcal{K}_{gp} represents the corrupt adults in government and public offices and \mathcal{L}_{oc} represents the individuals who have quit corruption. We shall assume the population are susceptible at the rate of Φ_p , and they are shamefully attracted or manipulated into corruption at the rate of Ψ_p , they progress from one compartment to another at the rates of $\Omega_{ce}, \Omega_{yt}, \Omega_{ac}$, the corrupted individuals die a natural death at the rate of μ_p and they quit corruption through rehabilitation, self-determination or any relevant positive means at the rates of Y_{ce}, Y_{yt}, Y_{oc} . A fraction $(1 - \Lambda_{ce}), (1 - \Lambda_{yt}), (1 - \Lambda_{ac})$ and $(1 - \Lambda_{gp})$ with $(\Lambda_{ce} + \Lambda_{yt} + \Lambda_{ac} + \Lambda_{gp} = 1)$ who gets corrupted from the susceptible population moves into the different stages accordingly. Thus, the description of the corruption can be underpinned by the following algebraic classical differential equations:

$$\begin{aligned}
 \frac{d\mathcal{S}_p}{dt} &= \Phi_p - \Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p - \mu_p\mathcal{S}_p, \\
 \frac{d\mathcal{K}_{ce}}{dt} &= (1 - \Lambda_{ce})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 &\quad - (\Omega_{ce} + Y_{ce} + \mu_p)\mathcal{K}_{ce}, \\
 \frac{d\mathcal{K}_{yt}}{dt} &= (1 - \Lambda_{yt})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 &\quad + \Omega_{ce}\mathcal{K}_{ce} - (\Omega_{yt} + Y_{yt} + \mu_p)\mathcal{K}_{yt}, \\
 \frac{d\mathcal{K}_{ac}}{dt} &= (1 - \Lambda_{ac})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 &\quad + \Omega_{yt}\mathcal{K}_{yt} - (\Omega_{ac} + Y_{ac} + \mu_p)\mathcal{K}_{ac}, \\
 \frac{d\mathcal{K}_{gp}}{dt} &= (1 - \Lambda_{gp})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 &\quad + \Omega_{ac}\mathcal{K}_{ac} - (\Omega_{gp} + Y_{gp} + \mu_p)\mathcal{K}_{gp}, \\
 \frac{d\mathcal{L}_{oc}}{dt} &= Y_{ce}\mathcal{K}_{ce} + Y_{yt}\mathcal{K}_{yt} + Y_{ac}\mathcal{K}_{ac} + Y_{gp}\mathcal{K}_{gp} - \mu_p\mathcal{L}_{oc}.
 \end{aligned}
 \tag{1}$$

As a result, that corruption can be affected by memory, non-locality and hereditary property behaviour and can be efficiently and appropriately captured through fractal–fractional derivative and integral. We present the corruption model termed the fractal–fractional corruption model as follows.

$$\begin{cases}
 {}^{\mathcal{F}\mathcal{F}\text{-}A\mathcal{B}\mathcal{E}}\mathcal{D}_t^{\eta,\rho}\mathcal{S}_p(t) = \mathcal{H}_1(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{yt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{oc}(t)), \\
 {}^{\mathcal{F}\mathcal{F}\text{-}A\mathcal{B}\mathcal{E}}\mathcal{D}_t^{\eta,\rho}\mathcal{K}_{ce}(t) = \mathcal{H}_2(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{yt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{oc}(t)), \\
 {}^{\mathcal{F}\mathcal{F}\text{-}A\mathcal{B}\mathcal{E}}\mathcal{D}_t^{\eta,\rho}\mathcal{K}_{yt}(t) = \mathcal{H}_3(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{yt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{oc}(t)), \\
 {}^{\mathcal{F}\mathcal{F}\text{-}A\mathcal{B}\mathcal{E}}\mathcal{D}_t^{\eta,\rho}\mathcal{K}_{ac}(t) = \mathcal{H}_4(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{yt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{oc}(t)), \\
 {}^{\mathcal{F}\mathcal{F}\text{-}A\mathcal{B}\mathcal{E}}\mathcal{D}_t^{\eta,\rho}\mathcal{K}_{gp}(t) = \mathcal{H}_5(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{yt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{oc}(t)), \\
 {}^{\mathcal{F}\mathcal{F}\text{-}A\mathcal{B}\mathcal{E}}\mathcal{D}_t^{\eta,\rho}\mathcal{L}_{oc}(t) = \mathcal{H}_6(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{yt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{oc}(t)),
 \end{cases}
 \tag{2}$$

where $\mathcal{H}_{\mathcal{K}} = \mathcal{H}_{\mathcal{K}}(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{yt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{oc}(t))$, for $\mathcal{K} = 1, 2, \dots, 6$ is given by

$$\begin{cases}
 \mathcal{H}_1 = \Phi_p - \Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p - \mu_p\mathcal{S}_p, \\
 \mathcal{H}_2 = (1 - \Lambda_{ce})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 \quad - (\Omega_{ce} + Y_{ce} + \mu_p)\mathcal{K}_{ce}, \\
 \mathcal{H}_3 = (1 - \Lambda_{yt})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 \quad + \Omega_{ce}\mathcal{K}_{ce} - (\Omega_{yt} + Y_{yt} + \mu_p)\mathcal{K}_{yt}, \\
 \mathcal{H}_4 = (1 - \Lambda_{ac})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 \quad + \Omega_{yt}\mathcal{K}_{yt} - (\Omega_{ac} + Y_{ac} + \mu_p)\mathcal{K}_{ac}, \\
 \mathcal{H}_5 = (1 - \Lambda_{gp})\Psi_p(\mathcal{K}_{ce} + \varphi_{yt}\mathcal{K}_{yt} + \varphi_{ac}\mathcal{K}_{ac} + \varphi_{gp}\mathcal{K}_{gp})\mathcal{S}_p \\
 \quad + \Omega_{ac}\mathcal{K}_{ac} - (\Omega_{gp} + Y_{gp} + \mu_p)\mathcal{K}_{gp}, \\
 \mathcal{H}_6 = Y_{ce}\mathcal{K}_{ce} + Y_{yt}\mathcal{K}_{yt} + Y_{ac}\mathcal{K}_{ac} + Y_{gp}\mathcal{K}_{gp} - \mu_p\mathcal{L}_{oc},
 \end{cases}
 \tag{3}$$

and the corresponding initial conditions of the above model are presented as

$$\begin{cases} \mathcal{S}_p(0) = \mathcal{S}_p^0 \geq 0, & \mathcal{K}_{CE}(0) = \mathcal{K}_{CE}^0 \geq 0, & \mathcal{K}_{VT}(0) = \mathcal{K}_{VT}^0 \geq 0, \\ \mathcal{K}_{AC}(0) = \mathcal{K}_{AC}^0 \geq 0, & \mathcal{K}_{GP}(0) = \mathcal{K}_{GP}^0 \geq 0, & \mathcal{L}_{QC}(0) = \mathcal{L}_{QC}^0 \geq 0. \end{cases} \quad (4)$$

In this case, the total population at any time, t , is given by

$$\mathcal{N}(t) = \mathcal{S}_p + \mathcal{K}_{CE}(t) + \mathcal{K}_{VT}(t) + \mathcal{K}_{AC}(t) + \mathcal{K}_{GP}(t) + \mathcal{L}_{QC}(t).$$

Preliminaries

In this section, we present some of the definitions and lemmas necessary for investigating the subject matter in the manuscript.

Definition 1. Suppose that the function $\mathcal{W}(t)$ is continuous and fractionally differentiable on the interval $a < t < b$ with order σ , then the fractal fractional derivative of $\mathcal{W}(t)$ of order λ in Atangana–Baleanu–Caputo sense is given by [38]

$$\begin{aligned} {}^{\mathcal{F}\mathcal{F}}\text{-}\mathcal{ABC}_{\mathcal{D}_t^{\lambda,\sigma}} \mathcal{W}(t) &= \frac{\mathcal{ABC}(\lambda)}{1-\lambda} \int_a^t \frac{d}{du^\sigma} \mathcal{W}(u) E_\lambda \\ &\times \left(\frac{-\lambda(t-u)^\lambda}{1-\lambda} \right) du, \quad t \geq 0, \end{aligned} \quad (5)$$

with $0 < \lambda \leq 1$, and $0 < \sigma \leq 1$. From the definition $\mathcal{ABC}(\lambda) = 1 - \lambda + \frac{\lambda}{\Gamma(\lambda)}$ represents the normalisation function and in particular $\mathcal{ABC}(\lambda) = 1$ if $\lambda = 0, 1$. Also, the quantity

$$\frac{d}{du^\sigma} \mathcal{W}(u) = \lim_{t \rightarrow 0} \frac{\mathcal{W}(t) - \mathcal{W}(u)}{t^\sigma - u^\sigma},$$

while the quantity $E_\lambda(\cdot)$ is the famous one parameter Mittag-Leffler function given by

$$E_\lambda(-t^\lambda) = \sum_{m=0}^{\infty} \frac{(-t)^m}{\Gamma(\lambda m + 1)}.$$

The two-parameter Mittag-Leffler type function is given by the series expansion [39]

$$E_{\lambda_1, \lambda_2}(-t^\lambda) = \sum_{m=0}^{\infty} \frac{(-t)^m}{\Gamma(\lambda_1 m + \lambda_2)}, \quad \lambda_1, \lambda_2 \in \mathbb{R}_+.$$

Definition 2. Suppose that the function $\mathcal{W}(t)$ is continuous and fractionally differentiable on the interval $a < t < b$ with order σ , then the fractal fractional integral of $\mathcal{W}(t)$ of order λ in Atangana–Baleanu–Caputo sense is defined as [38,40]

$$\begin{aligned} {}^{\mathcal{F}\mathcal{F}}\text{-}\mathcal{ABC}_{\mathcal{I}_t^{\lambda,\sigma}} \mathcal{W}(t) &= \frac{\sigma(1-\lambda)t^{\sigma-1}}{\mathcal{ABC}(\lambda)} \mathcal{W}(t) \\ &+ \frac{\sigma\lambda}{\mathcal{ABC}(\lambda)\Gamma(\lambda)} \int_a^t (t-u)^{\lambda-1} u^{\sigma-1} \mathcal{W}(u) du, \quad t \geq 0, \end{aligned} \quad (6)$$

with $0 < \lambda \leq 1$, and $0 < \sigma \leq 1$.

Lemma 1 ([40]). Given the following initial value problem (IVP),

$$\begin{cases} {}^{\mathcal{F}\mathcal{F}}\text{-}\mathcal{ABC}_{\mathcal{D}_t^{\lambda,\sigma}} \mathcal{W}(t) = \mathcal{P}(t), & t \in \mathcal{J} = [0, \mathcal{T}], \quad 0 < \lambda, \sigma \leq 1, \\ \mathcal{W}(0) = \mathcal{W}_0. \end{cases} \quad (7)$$

The above initial value problem (7) can be written as [34],

$$\begin{cases} {}^{\mathcal{F}\mathcal{F}}\text{-}\mathcal{ABC}_{\mathcal{D}_t^{\lambda,\sigma}} \mathcal{W}(t) = \sigma t^{\sigma-1} \mathcal{P}(t), & t \in \mathcal{J} = [0, \mathcal{T}], \quad 0 < \lambda, \sigma \leq 1, \\ \mathcal{W}(0) = \mathcal{W}_0. \end{cases} \quad (8)$$

Thus the solution of the initial value problem (8) is equivalent to the following Volterra fractional integral equation

$$\mathcal{W}(t) = \frac{\sigma(1-\lambda)t^{\sigma-1}}{\mathcal{ABC}(\lambda)} \mathcal{P}(t) + \frac{\sigma\lambda}{\mathcal{ABC}(\lambda)\Gamma(\lambda)} \int_0^t (t-u)^{\lambda-1} u^{\sigma-1} \mathcal{P}(u) du, \quad (9)$$

with $0 < \lambda, \sigma \leq 1$.

Qualitative analysis of the model

In this section, we investigate the positivity of the model solution, the invariant region, the corruption-free equilibrium state and the basic reproduction number. More so, we present the existence and uniqueness of the solution of the corruption system using Krasnoselskii fixed point theorem and Banach contraction principle. The stability of the model solution is investigated using the Ulam–Hyers technique.

Positivity of the model solution

The classical system (1) deals with the human population; thus, every state variable and its corresponding parameters are non-negative. So, by using the following theorem, we prove the positivity of the model solution.

Theorem 2. Given the system solution set $\mathcal{Y}_{CP} = \{ (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) \in \mathbb{R}_+^6 : \mathcal{S}_p(0) > 0, \mathcal{K}_{CE}(0) > 0, \mathcal{K}_{VT}(0) > 0, \mathcal{K}_{AC}(0) > 0, \mathcal{K}_{GP}(0) > 0, \mathcal{L}_{QC}(0) > 0 \}$, then the solution set $\{ \mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC} \}$ of the classical corruption system (1) is non-negative for all $t \geq 0$.

Proof. Using similar approach in [41], we let $\mathcal{L} = \sup\{t > 0 : \mathcal{S}_p(t_*) \geq 0, \mathcal{K}_{CE}(t_*) \geq 0, \mathcal{K}_{VT}(t_*) \geq 0, \mathcal{K}_{AC}(t_*) \geq 0, \mathcal{K}_{GP}(t_*) \geq 0, \mathcal{L}_{QC}(t_*) \geq 0\}$, for all $0 \leq t \leq t_*$.

Since we have that $\mathcal{S}_p^0(t) \geq 0, \mathcal{K}_{CE}^0(t) \geq 0, \mathcal{K}_{VT}^0(t) \geq 0, \mathcal{K}_{AC}^0(t) \geq 0, \mathcal{K}_{GP}^0(t) \geq 0$ and $\mathcal{L}_{QC}^0(t) \geq 0$, it follows that $\mathcal{L} > 0$. In another note, suppose $\mathcal{L} < \infty$, it follows that $\mathcal{S}_p^0(t)$ or $\mathcal{K}_{CE}^0(t)$ or $\mathcal{K}_{VT}^0(t)$ or $\mathcal{K}_{AC}^0(t)$ or $\mathcal{K}_{GP}^0(t)$ or $\mathcal{L}_{QC}^0(t)$ is equal to zero at \mathcal{L} .

Thus, integrating the first equation of the classical corruption system (1) using the method of constant variation, we obtain the solution at \mathcal{L} as,

$$\begin{aligned} \mathcal{S}_p(\mathcal{L}) &= \mathcal{S}_p(0) \exp \left(- \int_0^{\mathcal{J}} (\Psi_p \mathcal{K}_{CE} + \Psi_p \varphi_{VT} \mathcal{K}_{VT} \right. \\ &\quad \left. + \Psi_p \varphi_{AC} \mathcal{K}_{AC} + \Psi_p \varphi_{GP} \mathcal{K}_{GP} + \mu_p) \right) (\xi) d\xi \\ &+ \int_0^{\mathcal{L}} \Phi_p \exp \left(- \int_\xi^{\mathcal{L}} (\Psi_p \mathcal{K}_{CE} + \Psi_p \varphi_{VT} \mathcal{K}_{VT} \right. \\ &\quad \left. + \Psi_p \varphi_{AC} \mathcal{K}_{AC} + \Psi_p \varphi_{GP} \mathcal{K}_{GP} + \mu_p) (\tau) d\tau \right) d\xi > 0. \end{aligned} \quad (10)$$

Similar process yields

$$\begin{aligned} \mathcal{K}_{CE}(\mathcal{L}) &= \mathcal{K}_{CE}(0) \exp \left(- \int_0^{\mathcal{J}} ((1 - \Lambda_{CE}) \Psi_p \mathcal{S}_p - (\Omega_{CE} + Y_{CE} + \mu_p)) \right) (\xi) d\xi \\ &+ \int_0^{\mathcal{L}} (1 - \Lambda_{CE}) \Psi_p (\varphi_{VT} \mathcal{K}_{VT} + \varphi_{AC} \mathcal{K}_{AC} + \varphi_{GP} \mathcal{K}_{GP}) \mathcal{S}_p \\ &\times \exp \left(- \int_\xi^{\mathcal{L}} ((1 - \Lambda_{CE}) \Psi_p \mathcal{S}_p - (\Omega_{CE} + Y_{CE} + \mu_p)) (\tau) d\tau \right) d\xi > 0. \end{aligned} \quad (11)$$

$$\begin{aligned} \mathcal{K}_{VT}(\mathcal{L}) &= \mathcal{K}_{VT}(0) \exp \left(- \int_0^{\mathcal{J}} ((1 - \Lambda_{VT}) \Psi_p \varphi_{CE} \mathcal{S}_p - (\Omega_{VT} + Y_{VT} + \mu_p)) \right) (\xi) d\xi \\ &+ \int_0^{\mathcal{L}} (1 - \Lambda_{VT}) \Psi_p (\mathcal{K}_{CE} + \varphi_{AC} \mathcal{K}_{AC} + \varphi_{GP} \mathcal{K}_{GP}) \mathcal{S}_p \\ &\times \exp \left(- \int_\xi^{\mathcal{L}} ((1 - \Lambda_{VT}) \Psi_p \varphi_{CE} \mathcal{S}_p - (\Omega_{VT} + Y_{VT} + \mu_p)) (\tau) d\tau \right) d\xi > 0. \end{aligned} \quad (12)$$

$$\begin{aligned} \mathcal{K}_{AC}(\mathcal{E}) &= \mathcal{K}_{AC}(0) \exp\left(-\int_0^T ((1 - \Lambda_{AC}) \Psi_p \varphi_{Y_{VT}} \mathcal{S}_p - (\Omega_{AC} + Y_{AC} + \mu_p))\right)(\xi) d\xi \\ &+ \int_0^t (1 - \Lambda_{AC}) \Psi_p (\mathcal{K}_{CE} + \varphi_{Y_{VT}} \mathcal{K}_{YT} + \varphi_{GP} \mathcal{K}_{GP}) \mathcal{S}_p \\ &\times \exp\left(-\int_\xi^t ((1 - \Lambda_{AC}) \Psi_p \varphi_{Y_{VT}} \mathcal{S}_p - (\Omega_{AC} + Y_{AC} + \mu_p))(\tau) d\tau\right) d\xi > 0. \end{aligned} \tag{13}$$

$$\begin{aligned} \mathcal{K}_{GP}(\mathcal{E}) &= \mathcal{K}_{GP}(0) \exp\left(-\int_0^T ((1 - \Lambda_{GP}) \Psi_p \varphi_{AC} \mathcal{S}_p - (\Omega_{YT} + Y_{YT} + \mu_p))\right)(\xi) d\xi \\ &+ \int_0^t (1 - \Lambda_{GP}) \Psi_p (\mathcal{K}_{CE} + \varphi_{Y_{VT}} \mathcal{K}_{YT} + \varphi_{AC} \mathcal{K}_{AC}) \mathcal{S}_p \\ &\times \exp\left(-\int_\xi^t ((1 - \Lambda_{GP}) \Psi_p \varphi_{AC} \mathcal{S}_p - (\Omega_{YT} + Y_{YT} + \mu_p))(\tau) d\tau\right) d\xi > 0. \end{aligned} \tag{14}$$

$$\begin{aligned} \mathcal{L}_{QC}(\mathcal{E}) &= \mathcal{L}_{QC}(0) \exp\left(-\int_0^T \mu_p\right) d\xi \\ &+ \int_0^t (Y_{CE} \mathcal{K}_{CE} + Y_{YT} \mathcal{K}_{YT} + Y_{AC} \mathcal{K}_{AC} + Y_{GP} \mathcal{K}_{GP}) \\ &\times \exp\left(-\int_\xi^t \mu_p(\tau) d\tau\right) d\xi > 0. \end{aligned} \tag{15}$$

However, since every variables are non-negative in $[0, \mathcal{E}]$, it follows that $\mathcal{S}_p(\mathcal{E}) > 0$, which is a contradiction and hence for $\mathcal{E} = \infty$. Thus, the entire model solution set \mathcal{Y}_{CP} , are non-negative for $t \geq 0$

Invariant region

To obtain the invariant region from which the solution of the classical corruption model is bounded, we consider the total human population

$$N(t) = \mathcal{S}_p + \mathcal{K}_{CE} + \mathcal{K}_{YT} + \mathcal{K}_{AC} + \mathcal{K}_{GP} + \mathcal{L}_{QC}.$$

Thus taking the first derivative of both sides concerning time t yields

$$\frac{dN}{dt} = \frac{d\mathcal{S}_p}{dt} + \frac{d\mathcal{K}_{CE}}{dt} + \frac{d\mathcal{K}_{YT}}{dt} + \frac{d\mathcal{K}_{AC}}{dt} + \frac{d\mathcal{K}_{GP}}{dt} + \frac{d\mathcal{L}_{QC}}{dt}.$$

Substituting the equivalent state equation from the classical corruption model (1) into the above yields

$$\frac{dN}{dt} = \Phi_p - \mu_p N, \tag{16}$$

so that by integrating both sides of the equation and letting $t \rightarrow \infty$, we obtain the following set

$$\hat{\mathcal{X}} = \left\{ (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) \in \mathbb{R}_+^6 : 0 \leq N \leq \frac{\Phi_p}{\mu_p} \right\},$$

which is bounded and positively invariant. Also, the set $\hat{\mathcal{X}}$ shows the corruption system solution is epidemiologically and mathematically well-posed.

The equilibrium state of the corruption system

This subsection presents the equilibrium state of the corruption system. We begin with the equilibrium state with zero contamination or desirability of corruption, referred to as the Corruption free equilibrium state, \mathcal{CFES} denoted by Q^0 . This is obtained by setting the fractal fractional derivative, ${}^{\mathcal{FF}}\mathcal{D}_t^{\eta, \rho} \mathcal{S}_p$, ${}^{\mathcal{FF}}\mathcal{D}_t^{\eta, \rho} \mathcal{K}_{CE}$, ${}^{\mathcal{FF}}\mathcal{D}_t^{\eta, \rho} \mathcal{K}_{YT}$, ${}^{\mathcal{FF}}\mathcal{D}_t^{\eta, \rho} \mathcal{K}_{AC}$, ${}^{\mathcal{FF}}\mathcal{D}_t^{\eta, \rho} \mathcal{K}_{GP}$ and ${}^{\mathcal{FF}}\mathcal{D}_t^{\eta, \rho} \mathcal{L}_{QC}$ from the corruption model (2)–(3) to zero and also setting $\mathcal{K}_{CE} = \mathcal{K}_{YT} = \mathcal{K}_{AC} = \mathcal{K}_{GP} = 0$, so that

$$\begin{aligned} Q^0 &= (\mathcal{S}_p^0, \mathcal{K}_{CE}^0, \mathcal{K}_{YT}^0, \mathcal{K}_{AC}^0, \mathcal{K}_{GP}^0, \mathcal{L}_{QC}^0), \\ &= \left(\frac{\Phi_p}{\mu_p}, 0, 0, 0, 0, 0 \right). \end{aligned}$$

On another note, the equilibrium state where corruption persists or continues within the population is known as the corruption endemic equilibrium state, \mathcal{CEES} denoted by $Q^* = (\mathcal{S}_p^*, \mathcal{K}_{CE}^*, \mathcal{K}_{YT}^*, \mathcal{K}_{AC}^*, \mathcal{K}_{GP}^*, \mathcal{L}_{QC}^*)$. This is obtainable by the same process as the free equilibrium state but in this case, $\mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP} \neq 0$. Therefore, solving simultaneously, we get

$$\begin{aligned} \mathcal{S}_p^* &= \frac{\Phi_p}{\Xi^* + \mu_p}, \\ \mathcal{K}_{CE}^* &= \frac{(1 - \Lambda_{CE}) \Phi_p \Xi^*}{(\Omega_{CE} + Y_{CE} + \mu_p)(\Xi^* + \mu_p)}, \\ \mathcal{K}_{YT}^* &= \frac{\Phi_p \Xi^*}{(\Omega_{YT} + Y_{YT} + \mu_p)(\Xi^* + \mu_p)} \left[(1 - \Lambda_{YT}) + \frac{\Omega_{CE}(1 - \Lambda_{CE})}{(\Omega_{CE} + Y_{CE} + \mu_p)} \right], \\ \mathcal{K}_{AC}^* &= \frac{\Phi_p \Xi^*}{(\Omega_{AC} + Y_{AC} + \mu_p)(\Xi^* + \mu_p)} \\ &\times \left[(1 - \Lambda_{AC}) + \frac{\Omega_{YT}}{(\Omega_{YT} + Y_{YT} + \mu_p)} \left((1 - \Lambda_{YT}) + \frac{\Omega_{CE}(1 - \Lambda_{CE})}{(\Omega_{CE} + Y_{CE} + \mu_p)} \right) \right], \\ \mathcal{K}_{GP}^* &= \frac{\Phi_p \Xi^*}{(\Omega_{GP} + Y_{GP} + \mu_p)(\Xi^* + \mu_p)} \left[(1 - \Lambda_{GP}) \right. \\ &+ \frac{\Omega_{AC}}{(\Omega_{AC} + Y_{AC} + \mu_p)} \left((1 - \Lambda_{AC}) + \frac{\Omega_{YT}}{(\Omega_{YT} + Y_{YT} + \mu_p)} \right) \\ &\times \left. \left((1 - \Lambda_{YT}) + \frac{\Omega_{CE}(1 - \Lambda_{CE})}{(\Omega_{CE} + Y_{CE} + \mu_p)} \right) \right], \\ \mathcal{L}_{QC}^* &= \frac{\Phi_p \Xi^*}{\mu_p(\Xi^* + \mu_p)} \left[\frac{Y_{CE}(1 - \Lambda_{CE})}{(\Omega_{CE} + Y_{CE} + \mu_p)} \right. \\ &+ \frac{Y_{YT}}{(\Omega_{YT} + Y_{YT} + \mu_p)} \left((1 - \Lambda_{YT}) + \frac{\Omega_{CE}(1 - \Lambda_{CE})}{(\Omega_{CE} + Y_{CE} + \mu_p)} \right) \\ &+ \frac{Y_{AC}}{(\Omega_{AC} + Y_{AC} + \mu_p)} \left((1 - \Lambda_{AC}) \right. \\ &+ \frac{\Omega_{YT}}{(\Omega_{YT} + Y_{YT} + \mu_p)} \left((1 - \Lambda_{YT}) + \frac{\Omega_{CE}(1 - \Lambda_{CE})}{(\Omega_{CE} + Y_{CE} + \mu_p)} \right) \left. \right) \\ &+ \frac{Y_{GP}}{(\Omega_{GP} + Y_{GP} + \mu_p)} \left((1 - \Lambda_{GP}) \right. \\ &+ \left. \left. \frac{\Omega_{AC}}{(\Omega_{AC} + Y_{AC} + \mu_p)} \left((1 - \Lambda_{AC}) + \frac{\Omega_{YT}}{(\Omega_{YT} + Y_{YT} + \mu_p)} \right) \right) \right], \end{aligned}$$

where

$$\Xi^* = \Psi_p (\mathcal{K}_{CE}^* + \varphi_{Y_{VT}} \mathcal{K}_{YT}^* + \varphi_{AC} \mathcal{K}_{AC}^* + \varphi_{GP} \mathcal{K}_{GP}^*).$$

Basic reproduction number

The basic reproduction number \mathcal{R}_0 computes the conventional number of the secondary contamination of corruption due to unique newly contaminated individuals inaugurated into the susceptible population. The reproduction number can be obtained using the next generation method proposed in [42].

Thus the transfer Jacobian matrices are respectively given by

$$\begin{aligned} \mathcal{F} &= \begin{pmatrix} (1 - \Lambda_{CE}) \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{CE}) \varphi_{Y_{VT}} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{CE}) \varphi_{AC} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{CE}) \varphi_{GP} \Psi_p \mathcal{S}_p^0 \\ (1 - \Lambda_{YT}) \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{YT}) \varphi_{Y_{VT}} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{YT}) \varphi_{AC} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{YT}) \varphi_{GP} \Psi_p \mathcal{S}_p^0 \\ (1 - \Lambda_{AC}) \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{AC}) \varphi_{Y_{VT}} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{AC}) \varphi_{AC} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{AC}) \varphi_{GP} \Psi_p \mathcal{S}_p^0 \\ (1 - \Lambda_{GP}) \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{GP}) \varphi_{Y_{VT}} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{GP}) \varphi_{AC} \Psi_p \mathcal{S}_p^0 (1 - \Lambda_{GP}) \varphi_{GP} \Psi_p \mathcal{S}_p^0 \end{pmatrix} \\ \mathcal{V} &= \begin{pmatrix} \mathcal{K}_1 & 0 & 0 & 0 \\ -\Omega_{CE} & \mathcal{K}_2 & 0 & 0 \\ 0 & -\Omega_{YT} & \mathcal{K}_3 & 0 \\ 0 & 0 & -\Omega_{AC} & \mathcal{K}_4 \end{pmatrix}, \\ \mathcal{V}^{-1} &= \begin{pmatrix} \frac{1}{\mathcal{K}_1} & 0 & 0 & 0 \\ \frac{\Omega_{CE}}{\mathcal{K}_1 \mathcal{K}_2} & \frac{1}{\mathcal{K}_2} & 0 & 0 \\ \frac{\Omega_{CE} \Omega_{YT}}{\mathcal{K}_1 \mathcal{K}_2 \mathcal{K}_3} & \frac{\Omega_{YT}}{\mathcal{K}_2 \mathcal{K}_3} & \frac{1}{\mathcal{K}_3} & 0 \\ \frac{\Omega_{AC} \Omega_{CE} \Omega_{YT}}{\mathcal{K}_1 \mathcal{K}_2 \mathcal{K}_3 \mathcal{K}_4} & \frac{\Omega_{AC} \Omega_{YT}}{\mathcal{K}_2 \mathcal{K}_3 \mathcal{K}_4} & \frac{\Omega_{AC}}{\mathcal{K}_3 \mathcal{K}_4} & \frac{1}{\mathcal{K}_4} \end{pmatrix}, \end{aligned}$$

$$\mathcal{R}_0 = \frac{\left(\mathcal{K}_2[\mathcal{K}_3 (\mathcal{K}_4 (1 - A_{CE}) + \mathcal{K}_1 (1 - A_{GP}) \varphi_{GP}) + \mathcal{K}_1 (1 - A_{VT}) (\mathcal{K}_4 \varphi_{AC} + \varphi_{GP} \Omega_{AC})] + \Phi \right) \Psi_p \mathcal{S}_p^0}{\mathcal{K}_1 \mathcal{K}_2 \mathcal{K}_3 \mathcal{K}_4}, \tag{17}$$

Box I.

where,

$$\mathcal{K}_1 = \Omega_{CE} + Y_{CE} + \mu_p, \quad \mathcal{K}_2 = \Omega_{VT} + Y_{VT} + \mu_p, \\ \mathcal{K}_3 = \Omega_{AC} + Y_{AC} + \mu_p, \quad \mathcal{K}_4 = \Omega_{GP} + Y_{GP} + \mu_p.$$

Hence, the basic reproduction number of the corruption model (1), which is the spectral radius of the next generation matrix $\mathcal{F}\mathcal{V}^{-1}$ is given by Eq. (17) in Box I, with,

$$\Phi = [\mathcal{K}_1 (1 - A_{VT}) + (1 - A_{CE}) \Omega_{CE}] [\mathcal{K}_3 \mathcal{K}_4 \varphi_{VT} + (\mathcal{K}_4 \varphi_{AC} + \varphi_{GP} \Omega_{AC}) \Omega_{VT}].$$

The following Figs. 1 and 2 are the numerical solution varying influence of the reproduction number \mathcal{R}_0 on some of the parameters in the considered corruption system. The numerical simulation of the varying influence of \mathcal{R}_0 as a function of the contact rate Ψ_p and the progression rates are shown in Fig. 1. More so, Fig. 2 depicts the scattered simulation of the varying influence of \mathcal{R}_0 as a function of the contact rate Ψ_p and the corruption quitting rates. The figures show that a reproduction number is an effective tool for measuring corruption behaviour since \mathcal{R}_0 is less than one, as shown in the figures.

Strength number

Decades of research in epidemiological modelling have witnessed the tremendous application of reproduction numbers as a mathematical threshold for investigating reproduction in some existing diseases. The reproduction number, which is obtainable by considering the component \mathcal{F} and \mathcal{V} such that the vector infectious classes can be written as

$$\frac{d\mathcal{G}}{dt} = \mathcal{F} - \mathcal{V},$$

which leads to

$$\det(\mathcal{F}\mathcal{V}^{-1} - I\lambda) = 0.$$

The spectral radius of the above next generational matrix yields the dominant eigenvalue known as the basic reproduction number.

Taking into account the considered corruption system, we obtain the strength number by taking the second derivative of the classes involved with corruption,

$$\frac{\partial}{\partial \mathcal{K}} \left(\frac{\mathcal{K}}{\mathcal{N}} \right) = \frac{[\mathcal{N} - \mathcal{K}]}{\mathcal{N}^2}, \tag{18}$$

and

$$\frac{\partial^2}{\partial \mathcal{K}^2} \left(\frac{[\mathcal{N} - \mathcal{K}]}{\mathcal{N}^2} \right) = \frac{-2[\mathcal{N} - \mathcal{K}]}{\mathcal{N}^3}, \\ = \frac{-2(\mathcal{S}_p + \mathcal{L}_{QC})}{(\mathcal{S}_p + \mathcal{K}_{CE} + \mathcal{K}_{VT} + \mathcal{K}_{AC} + \mathcal{K}_{GP} + \mathcal{L}_{QC})^3}, \tag{19}$$

where $\mathcal{K} = \mathcal{K}_{CE} + \mathcal{K}_{VT} + \mathcal{K}_{AC} + \mathcal{K}_{GP}$. At the corruption-free equilibrium state, we get

$$\frac{-2(\mathcal{S}_p^0 + \mathcal{L}_{QC}^0)}{(\mathcal{S}_p^0 + \mathcal{K}_{CE}^0 + \mathcal{K}_{VT}^0 + \mathcal{K}_{AC}^0 + \mathcal{K}_{GP}^0 + \mathcal{L}_{QC}^0)^3} = \frac{-2}{(\mathcal{S}_p^0)^2}.$$

From this case, we obtain the following

$$\mathcal{F}_{\mathcal{A}} = \begin{pmatrix} \frac{-2(1-A_{CE})\Psi_p}{\mathcal{S}_p^0} \\ \frac{-2(1-A_{VT})\Psi_p}{\mathcal{S}_p^0} \\ \frac{-2(1-A_{AC})\Psi_p}{\mathcal{S}_p^0} \\ \frac{-2(1-A_{GP})\Psi_p}{\mathcal{S}_p^0} \end{pmatrix} \text{ and } \mathcal{V} = \begin{pmatrix} \mathcal{K}_1 & 0 & 0 & 0 \\ -\Omega_{CE} & \mathcal{K}_2 & 0 & 0 \\ 0 & -\Omega_{VT} & \mathcal{K}_3 & 0 \\ 0 & 0 & -\Omega_{AC} & \mathcal{K}_4 \end{pmatrix}.$$

Then the determinant of $(\mathcal{F}_{\mathcal{A}}\mathcal{V}^{-1} - \lambda I) = 0$ yields

$$\mathcal{A}_0 = \frac{-2(1 - A_{CE})\Psi_p \mu_p}{\Phi_p (\Omega_{CE} + Y_{CE} + \mu_p)} < 0, \text{ for } A_{CE} < 1. \tag{20}$$

Here, the spread of corruption will not have a renewal process and will possess a sole magnitude and disappear when $\mathcal{A}_0 = 0$. Also, when $\mathcal{A}_0 > 0$, there is a strength that will encourage the removal process so that the spread will possess several waves [43].

Existence and uniqueness

In this case, Krasnoselskii fixed point theorem will be employed to investigate the existence of the model solution while the Banach contraction principle will be used to show that the model has a unique solution. For the subsequent analysis, we present the following lemma.

Lemma 3. Assume that the continuous functions $\mathbf{w} \in \mathcal{E}[0, \mathcal{J}]$ and $\mathbf{u} \in \mathcal{H} \subset [0, \mathcal{J}]$, then the fractal fractional dynamic model (2)–(4) which its right hand side vanishes at $t = 0$ can be written as

$$\begin{cases} {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta, \rho} \mathbf{u}(t) = \mathbf{w}(t, \mathbf{u}(t)), & t \in \mathcal{J} = [0, \mathcal{J}], \quad 0 < \eta, \rho \leq 1, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{21}$$

where $\mathbf{w}(t, \mathbf{u}(t))$ is given in (3) and

$$\begin{cases} \mathbf{u}(t) = (\mathcal{S}_p(t), \mathcal{K}_{CE}(t), \mathcal{K}_{VT}(t), \mathcal{K}_{AC}(t), \mathcal{K}_{GP}(t), \mathcal{L}_{QC}(t))^{\mathcal{T}}, \\ \mathbf{u}(0) = (\mathcal{S}_p(0), \mathcal{K}_{CE}(0), \mathcal{K}_{VT}(0), \mathcal{K}_{AC}(0), \mathcal{K}_{GP}(0), \mathcal{L}_{QC}(0))^{\mathcal{T}}. \end{cases} \tag{22}$$

where $(\cdot)^{\mathcal{T}}$ is the vector transpose. By using Eq. (8), the solution of the initial value problem (22) is equivalent to the following Volterra fractional integral equation

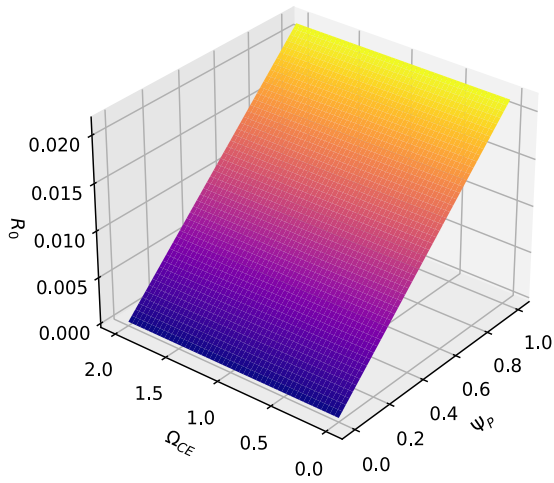
$$\mathbf{u}(t) = \frac{\rho(1 - \eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathbf{w}(t, \mathbf{u}(t)) \\ + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t - v)^{\eta-1} v^{\rho-1} \mathbf{w}(v, \mathbf{u}(v)) dv, \tag{23}$$

with $0 < \eta, \rho \leq 1$.

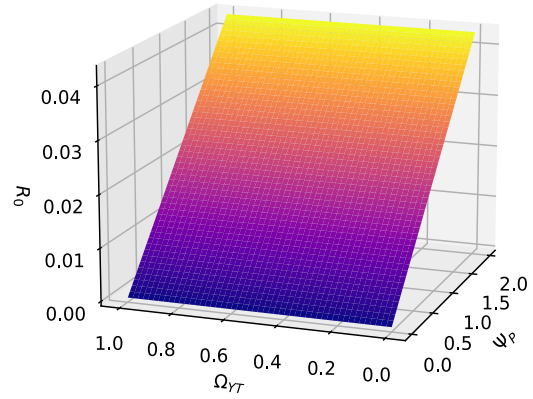
Remark 4. The proof of the above Lemma 3 can be obtained using Eq. (8) and applying the fractal fractional integral on both sides.

Thus with the aid of Lemma 3, the fractal fractional corruption model (2)–(4) is equivalent to the following Volterra fractional integral equations;

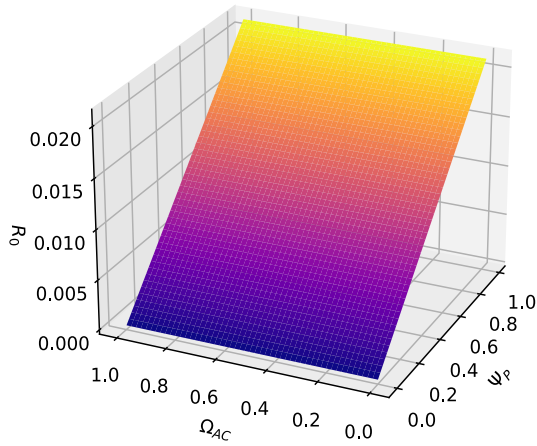
$$\mathcal{S}_p(t) = \mathcal{S}_p(0) + \frac{\rho(1 - \eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_1(t, \mathcal{S}_p(t)) \\ + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t - v)^{\eta-1} v^{\rho-1} \mathcal{H}_1(v, \mathcal{S}_p(v)) dv,$$



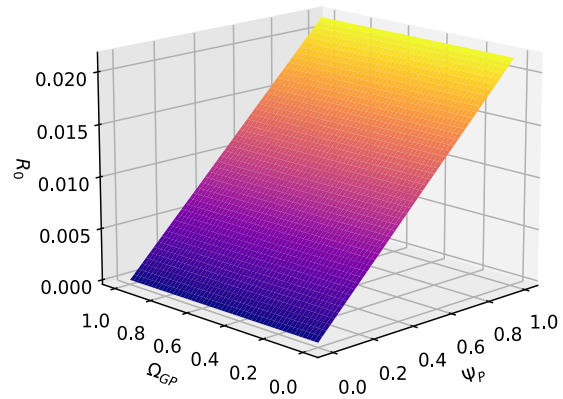
(a) \mathcal{R}_0 as function of Ψ_p and Ω_{CE}



(b) \mathcal{R}_0 as function of Ψ_p and Ω_{YT}



(c) \mathcal{R}_0 as function of Ψ_p and Ω_{AC}



(d) \mathcal{R}_0 as function of Ψ_p and Ω_{GP}

Fig. 1. Numerical simulation of varying the influence of \mathcal{R}_0 as a function of contact rates and progression rates.

$$\begin{aligned}
 \mathcal{K}_{CE}(t) &= \mathcal{K}_{CE}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_2(t, \mathcal{K}_{CE}(t)) \\
 &\quad + \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_2(v, \mathcal{K}_{CE}(v)) dv, \\
 \mathcal{K}_{YT}(t) &= \mathcal{K}_{YT}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_3(t, \mathcal{K}_{YT}(t)) \\
 &\quad + \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_3(v, \mathcal{K}_{YT}(v)) dv, \\
 \mathcal{K}_{AC}(t) &= \mathcal{K}_{AC}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_4(t, \mathcal{K}_{AC}(t)) \\
 &\quad + \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_4(v, \mathcal{K}_{AC}(v)) dv, \\
 \mathcal{K}_{GP}(t) &= \mathcal{K}_{GP}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_5(t, \mathcal{K}_{GP}(t)) \\
 &\quad + \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_5(v, \mathcal{K}_{GP}(v)) dv, \\
 \mathcal{L}_{QC}(t) &= \mathcal{L}_{QC}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_6(t, \mathcal{L}_{QC}(t))
 \end{aligned}
 \tag{24}$$

$$+ \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_6(v, \mathcal{L}_{QC}(v)) dv.$$

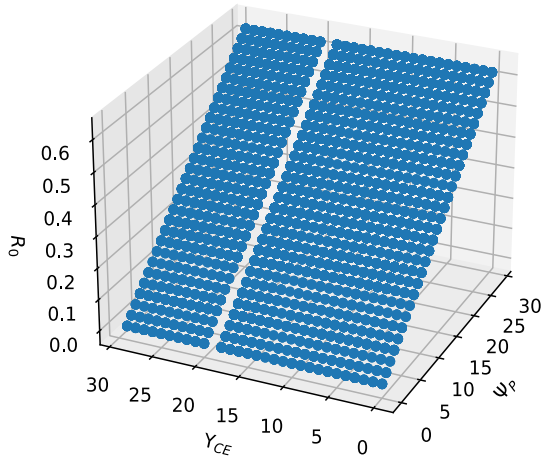
For the sake of results purposes, we let $\hat{\mathcal{U}} = \hat{\mathcal{B}} \in \mathcal{J} = [0, \mathcal{T}]$ be any space which possess the norm $\mathbf{u} \in \hat{\mathcal{U}}$ as

$$\begin{aligned}
 \|\mathbf{u}\| &:= \max_{t \in \mathcal{J}} |\mathbf{u}(t)|, \quad \mathcal{J} = [0, \mathcal{T}], \\
 &:= \max_{t \in \mathcal{J}} \left[|\mathcal{S}_p(t)| + |\mathcal{K}_{CE}(t)| + |\mathcal{K}_{YT}(t)| + |\mathcal{K}_{AC}(t)| + |\mathcal{K}_{GP}(t)| + |\mathcal{L}_{QC}(t)| \right].
 \end{aligned}$$

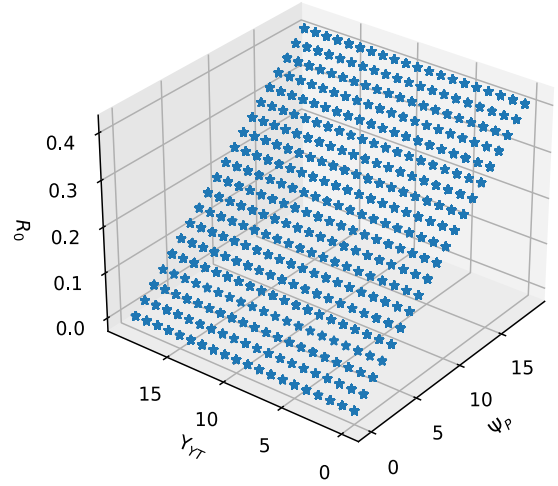
The above can equivalently be written as

$$\begin{aligned}
 \|\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}\| &:= \max_{t \in \mathcal{J}} \left[|\mathcal{S}_p(t)| + |\mathcal{K}_{CE}(t)| + |\mathcal{K}_{YT}(t)| \right. \\
 &\quad \left. + |\mathcal{K}_{AC}(t)| + |\mathcal{K}_{GP}(t)| + |\mathcal{L}_{QC}(t)| \right].
 \end{aligned}$$

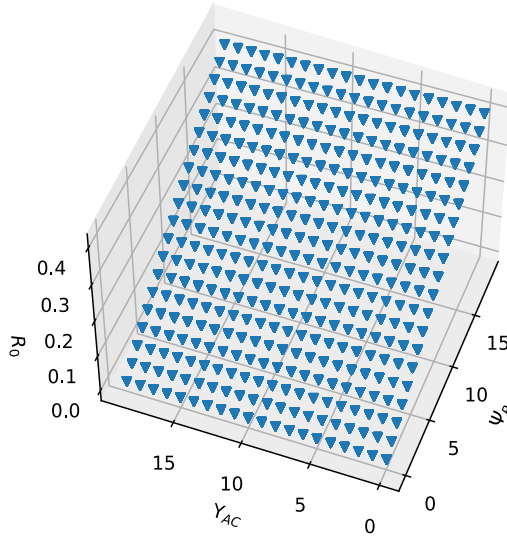
Also, for the subsequent theorems, we state the following assumptions;



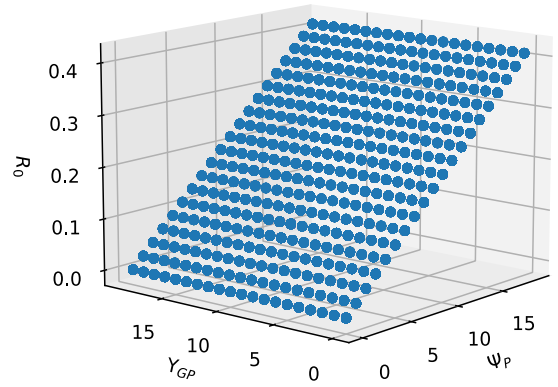
(a) \mathcal{R}_0 as function of Ψ_p and Υ_{CE}



(b) \mathcal{R}_0 as function of Ψ_p and Υ_{YT}



(c) \mathcal{R}_0 as function of Ψ_p and Υ_{AC}



(d) \mathcal{R}_0 as function of Ψ_p and Υ_{GP}

Fig. 2. Scattered simulation of varying the influence of \mathcal{R}_0 as a function of contact rates and corruption quitting rates.

(R1). There exists a real constant $\xi_{0\mathcal{H}_k}, \delta_{1k}, \delta_{2k}, \delta_{3k}, \delta_{4k}, \delta_{5k}, \delta_{6k} > 0$ for $k = 1, 2, \dots, 6$ such that

$$\begin{aligned} & \left| \mathcal{H}_k(t, \mathcal{S}_p(t), \mathcal{K}_{CE}(t), \mathcal{K}_{YT}(t), \mathcal{K}_{AC}(t), \mathcal{K}_{GP}(t), \mathcal{L}_{QC}(t)) \right| \\ & \leq \xi_{0\mathcal{H}_k} + \delta_{1k} |\mathcal{S}_p(t)| + \delta_{2k} |\mathcal{K}_{CE}(t)| + \delta_{3k} |\mathcal{K}_{YT}(t)| \\ & \quad + \delta_{4k} |\mathcal{K}_{AC}(t)| + \delta_{5k} |\mathcal{K}_{GP}(t)| + \delta_{6k} |\mathcal{L}_{QC}(t)|. \end{aligned}$$

(R2). There exists a real number $\mathcal{E}_{\mathcal{H}_k} > 0$ such that for every solutions

$$\begin{aligned} & \mathcal{S}_p(t), \mathcal{K}_{CE}(t), \mathcal{K}_{YT}(t), \mathcal{K}_{AC}(t), \mathcal{K}_{GP}(t), \mathcal{L}_{QC}(t), \hat{\mathcal{S}}_p(t), \\ & \hat{\mathcal{K}}_{CE}(t), \hat{\mathcal{K}}_{YT}(t), \hat{\mathcal{K}}_{AC}(t), \hat{\mathcal{K}}_{GP}(t), \hat{\mathcal{L}}_{QC}(t) \in \hat{\mathcal{U}}, \end{aligned}$$

then we obtain

$$\begin{aligned} & \left| \mathcal{H}_k(t, \mathcal{S}_p(t), \mathcal{K}_{CE}(t), \mathcal{K}_{YT}(t), \mathcal{K}_{AC}(t), \mathcal{K}_{GP}(t), \mathcal{L}_{QC}(t)) \right. \\ & \quad \left. - \hat{\mathcal{H}}_k(t, \hat{\mathcal{S}}_p(t), \hat{\mathcal{K}}_{CE}(t), \hat{\mathcal{K}}_{YT}(t), \hat{\mathcal{K}}_{AC}(t), \hat{\mathcal{K}}_{GP}(t), \hat{\mathcal{L}}_{QC}(t)) \right| \\ & \leq \mathcal{E}_{\mathcal{H}_k} \left[|\mathcal{S}_p(t) - \hat{\mathcal{S}}_p(t)| + |\mathcal{K}_{CE}(t) - \hat{\mathcal{K}}_{CE}(t)| + |\mathcal{K}_{YT}(t) - \hat{\mathcal{K}}_{YT}(t)| \right] \end{aligned}$$

$$+ \left| \mathcal{K}_{AC}(t) - \hat{\mathcal{K}}_{AC}(t) \right| + \left| \mathcal{K}_{GP}(t) - \hat{\mathcal{K}}_{GP}(t) \right| + \left| \mathcal{L}_{QC}(t) - \hat{\mathcal{L}}_{QC}(t) \right|,$$

where $\mathcal{E}_{\mathcal{H}_k} > 0$ represents the Lipschitz constant.

For the existence and uniqueness of the fractal fractional corruption model solution, we consider the following useful set

$$\mathcal{B} = \left\{ (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) \in \hat{\mathcal{U}} \times \hat{\mathcal{U}} : \|(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})\| \leq \mathcal{Q}_B \right\},$$

or

$$\mathcal{B} = \left\{ (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) \in \hat{\mathcal{U}} \times \hat{\mathcal{U}} : \|\mathcal{S}_p\| + \|\mathcal{K}_{CE}\| + \|\mathcal{K}_{YT}(t)\| + \|\mathcal{K}_{AC}\| + \|\mathcal{K}_{GP}\| + \|\mathcal{L}_{QC}\| \leq \mathcal{Q}_B \right\}.$$

Theorem 5 (Krasnoselskii Fixed Point). Let $\hat{\mathcal{V}}$ be a closed convex non-empty subset of a Banach space $\hat{\mathcal{U}}$. Assume that the two operators denoted as $\mathcal{G}\mathbf{u} = \mathcal{F}\mathbf{u} + \mathcal{Z}\mathbf{u}$ with $\mathcal{F}\mathbf{u} + \mathcal{Z}\mathbf{u} \in \hat{\mathcal{V}}$ for every $\mathbf{u} \in \hat{\mathcal{V}}$, then [44]

- (I). \mathcal{F} is a contraction.
- (II). \mathcal{Z} is continuous and compact (completely continuous).

It follows that \mathcal{G} possess a fixed point, and thus the operator $\mathcal{F}\mathbf{u} + \mathcal{Z}\mathbf{u} = \mathcal{G}\mathbf{u}$ must have at least one solution.

Proof. Using Eq. (24), we have the following operators;

$$\begin{cases} \mathcal{F}_1(\mathcal{S}_p)(t) = \mathcal{S}_p(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_1(t, \mathcal{K}_{CE}(t)), \\ \mathcal{F}_2(\mathcal{K}_{CE})(t) = \mathcal{K}_{CE}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_2(t, \mathcal{K}_{CE}(t)), \\ \mathcal{F}_3(\mathcal{K}_{YT})(t) = \mathcal{K}_{YT}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_3(t, \mathcal{K}_{YT}(t)), \\ \mathcal{F}_4(\mathcal{K}_{AC})(t) = \mathcal{K}_{AC}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_4(t, \mathcal{K}_{AC}(t)), \\ \mathcal{F}_5(\mathcal{K}_{GP})(t) = \mathcal{K}_{GP}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_5(t, \mathcal{K}_{GP}(t)), \\ \mathcal{F}_6(\mathcal{L}_{QC})(t) = \mathcal{L}_{QC}(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_6(t, \mathcal{L}_{QC}(t)). \end{cases} \quad (25)$$

and

$$\begin{cases} \mathcal{Z}_1(\mathcal{S}_p)(t) = \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_1(v, \mathcal{S}_p(v)) dv, \\ \mathcal{Z}_2(\mathcal{K}_{CE})(t) = \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_2(v, \mathcal{K}_{CE}(v)) dv, \\ \mathcal{Z}_3(\mathcal{K}_{YT})(t) = \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_3(v, \mathcal{K}_{YT}(v)) dv, \\ \mathcal{Z}_4(\mathcal{K}_{AC})(t) = \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_4(v, \mathcal{K}_{AC}(v)) dv, \\ \mathcal{Z}_5(\mathcal{K}_{GP})(t) = \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_5(v, \mathcal{K}_{GP}(v)) dv, \\ \mathcal{Z}_6(\mathcal{L}_{QC})(t) = \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{H}_6(v, \mathcal{L}_{QC}(v)) dv. \end{cases} \quad (26)$$

Using assumption (R2), we show that the operator $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6)(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})$ such that for all $t \in \mathcal{J} = [0, \mathcal{T}]$, we obtain

$$\begin{aligned} \|\mathcal{F}_1(\mathcal{S}_p) - \mathcal{F}_1(\hat{\mathcal{S}}_p)\| &\leq \max_{t \in \mathcal{J}} \left| \mathcal{S}_p(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_1(t, \mathcal{K}_{CE}(t)) \right. \\ &\quad \left. - \left(\mathcal{S}_p(0) + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{H}_1(t, \mathcal{K}_{CE}(t)) \right) \right|, \\ &\leq \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \max_{t \in \mathcal{J}} \left| \mathcal{H}_1(t, \mathcal{K}_{CE}(t)) - \mathcal{H}_1(t, \hat{\mathcal{K}}_{CE}(t)) \right|, \\ &\leq \mathcal{E}_{\mathcal{H}_1} \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \|\mathcal{S}_p - \hat{\mathcal{S}}_p\|, \end{aligned}$$

which implies that

$$\|\mathcal{F}_1(\mathcal{S}_p) - \mathcal{F}_1(\hat{\mathcal{S}}_p)\| \leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \mathcal{E}_{\mathcal{H}_1} \|\mathcal{S}_p - \hat{\mathcal{S}}_p\|. \quad (27)$$

Similar approach yields the following;

$$\|\mathcal{F}_2(\mathcal{K}_{CE}) - \mathcal{F}_2(\hat{\mathcal{K}}_{CE})\| \leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \mathcal{E}_{\mathcal{H}_2} \|\mathcal{K}_{CE} - \hat{\mathcal{K}}_{CE}\|. \quad (28)$$

$$\|\mathcal{F}_3(\mathcal{K}_{YT}) - \mathcal{F}_3(\hat{\mathcal{K}}_{YT})\| \leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \mathcal{E}_{\mathcal{H}_3} \|\mathcal{K}_{YT} - \hat{\mathcal{K}}_{YT}\|. \quad (29)$$

$$\|\mathcal{F}_4(\mathcal{K}_{AC}) - \mathcal{F}_4(\hat{\mathcal{K}}_{AC})\| \leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \mathcal{E}_{\mathcal{H}_4} \|\mathcal{K}_{AC} - \hat{\mathcal{K}}_{AC}\|. \quad (30)$$

$$\|\mathcal{F}_5(\mathcal{K}_{GP}) - \mathcal{F}_5(\hat{\mathcal{K}}_{GP})\| \leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \mathcal{E}_{\mathcal{H}_5} \|\mathcal{K}_{GP} - \hat{\mathcal{K}}_{GP}\|. \quad (31)$$

$$\|\mathcal{F}_6(\mathcal{L}_{QC}) - \mathcal{F}_6(\hat{\mathcal{L}}_{QC})\| \leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \mathcal{E}_{\mathcal{H}_6} \|\mathcal{L}_{QC} - \hat{\mathcal{L}}_{QC}\|. \quad (32)$$

Thus, from (27)–(32), we can state that

$$\begin{aligned} &\|\mathcal{F}(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) - \mathcal{F}(\hat{\mathcal{S}}_p, \hat{\mathcal{K}}_{CE}, \hat{\mathcal{K}}_{YT}, \hat{\mathcal{K}}_{AC}, \hat{\mathcal{K}}_{GP}, \hat{\mathcal{L}}_{QC})\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \left(\mathcal{E}_{\mathcal{H}_1} + \mathcal{E}_{\mathcal{H}_2} + \mathcal{E}_{\mathcal{H}_3} + \mathcal{E}_{\mathcal{H}_4} + \mathcal{E}_{\mathcal{H}_5} + \mathcal{E}_{\mathcal{H}_6} \right) \\ &\quad \times \left(\left\| (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) - (\hat{\mathcal{S}}_p, \hat{\mathcal{K}}_{CE}, \hat{\mathcal{K}}_{YT}, \hat{\mathcal{K}}_{AC}, \hat{\mathcal{K}}_{GP}, \hat{\mathcal{L}}_{QC}) \right\| \right). \end{aligned} \quad (33)$$

By taking $\max\{\mathcal{E}_{\mathcal{H}_1}, \mathcal{E}_{\mathcal{H}_2}, \mathcal{E}_{\mathcal{H}_3}, \mathcal{E}_{\mathcal{H}_4}, \mathcal{E}_{\mathcal{H}_5}, \mathcal{E}_{\mathcal{H}_6} = \mathcal{E}_{\mathcal{H}} > 0\}$, Eq. (33) becomes

$$\begin{aligned} &\left\| \mathcal{F}(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) - \mathcal{F}(\hat{\mathcal{S}}_p, \hat{\mathcal{K}}_{CE}, \hat{\mathcal{K}}_{YT}, \hat{\mathcal{K}}_{AC}, \hat{\mathcal{K}}_{GP}, \hat{\mathcal{L}}_{QC}) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \right) \mathcal{E}_{\mathcal{H}} \\ &\quad \times \left(\left\| (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) - (\hat{\mathcal{S}}_p, \hat{\mathcal{K}}_{CE}, \hat{\mathcal{K}}_{YT}, \hat{\mathcal{K}}_{AC}, \hat{\mathcal{K}}_{GP}, \hat{\mathcal{L}}_{QC}) \right\| \right). \end{aligned} \quad (34)$$

From the above Eq. (34), we have shown that the operator

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6)(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})$$

is a contraction.

Next, we show that \mathcal{Z} is continuous and compact, which implies that \mathcal{Z} is completely continuous. Thus, it is sufficient to show that $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5, \mathcal{Z}_6)$ is bounded and equip-continuous. With the aid of assumption (R1), it is easy to see that $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5, \mathcal{Z}_6)$ is continuous as $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6)$ is also continuous. Therefore, for every $t \in \mathcal{J} = [0, \mathcal{T}]$, we obtain

$$\begin{aligned} &\|\mathcal{Z}_1(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})\| \\ &= \max_{t \in \mathcal{J}} \left| \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \right. \\ &\quad \left. \times \mathcal{H}_1(v, \mathcal{S}(v)_p, \mathcal{K}_{CE}(v), \mathcal{K}_{YT}(v), \mathcal{K}_{AC}(v), \mathcal{K}_{GP}(v), \mathcal{L}_{QC}(v)) dv \right|, \\ &\leq \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \\ &\quad \times \max_{t \in \mathcal{J}} \left| \mathcal{H}_1(v, \mathcal{S}(v)_p, \mathcal{K}_{CE}(v), \mathcal{K}_{YT}(v), \mathcal{K}_{AC}(v), \mathcal{K}_{GP}(v), \mathcal{L}_{QC}(v)) dv \right|, \\ &\leq \frac{\eta \mathcal{J}^{\eta+\rho-1} \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_1} + \delta_{11} \|\mathcal{S}_p\| + \delta_{21} \|\mathcal{K}_{CE}\| + \delta_{31} \|\mathcal{K}_{YT}\| \right. \\ &\quad \left. + \delta_{41} \|\mathcal{K}_{AC}\| + \delta_{51} \|\mathcal{K}_{GP}\| + \delta_{61} \|\mathcal{L}_{QC}\| \right). \end{aligned} \quad (35)$$

We used the gamma function relation $\Gamma(z+1) = z\Gamma(z)$ and the Laplace transform of a convolution of two functions. Thus, Eq. (35) shows that \mathcal{Z}_1 is bounded.

Similar process yields the following expression for the operators $\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5$ and \mathcal{Z}_6 as,

$$\begin{aligned} &\|\mathcal{Z}_2(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})\| \\ &\leq \frac{\eta \mathcal{J}^{\eta+\rho-1} \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_2} + \delta_{12} \|\mathcal{S}_p\| + \delta_{22} \|\mathcal{K}_{CE}\| + \delta_{32} \|\mathcal{K}_{YT}\| \right. \\ &\quad \left. + \delta_{42} \|\mathcal{K}_{AC}\| + \delta_{52} \|\mathcal{K}_{GP}\| + \delta_{62} \|\mathcal{L}_{QC}\| \right). \end{aligned} \quad (36)$$

$$\begin{aligned} &\|\mathcal{Z}_3(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})\| \\ &\leq \frac{\eta \mathcal{J}^{\eta+\rho-1} \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_3} + \delta_{13} \|\mathcal{S}_p\| + \delta_{23} \|\mathcal{K}_{CE}\| + \delta_{33} \|\mathcal{K}_{YT}\| \right. \\ &\quad \left. + \delta_{43} \|\mathcal{K}_{AC}\| + \delta_{53} \|\mathcal{K}_{GP}\| + \delta_{63} \|\mathcal{L}_{QC}\| \right). \end{aligned} \quad (37)$$

$$\begin{aligned} &\|\mathcal{Z}_4(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})\| \\ &\leq \frac{\eta \mathcal{J}^{\eta+\rho-1} \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_4} + \delta_{14} \|\mathcal{S}_p\| + \delta_{24} \|\mathcal{K}_{CE}\| + \delta_{34} \|\mathcal{K}_{YT}\| \right. \\ &\quad \left. + \delta_{44} \|\mathcal{K}_{AC}\| + \delta_{54} \|\mathcal{K}_{GP}\| + \delta_{64} \|\mathcal{L}_{QC}\| \right). \end{aligned} \quad (38)$$

$$\begin{aligned} &\|\mathcal{Z}_5(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{YT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})\| \\ &\leq \frac{\eta \mathcal{J}^{\eta+\rho-1} \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_5} + \delta_{15} \|\mathcal{S}_p\| + \delta_{25} \|\mathcal{K}_{CE}\| + \delta_{35} \|\mathcal{K}_{YT}\| \right. \\ &\quad \left. + \delta_{45} \|\mathcal{K}_{AC}\| + \delta_{55} \|\mathcal{K}_{GP}\| + \delta_{65} \|\mathcal{L}_{QC}\| \right). \end{aligned} \quad (39)$$

$$\begin{aligned} & \left\| \mathcal{Z}_6 (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) \right\| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_6} + \delta_{16} \|\mathcal{S}_p\| + \delta_{26} \|\mathcal{K}_{CE}\| + \delta_{36} \|\mathcal{K}_{VT}\| \right. \\ & \quad \left. + \delta_{46} \|\mathcal{K}_{AC}\| + \delta_{56} \|\mathcal{K}_{GP}\| + \delta_{66} \|\mathcal{L}_{QC}\| \right). \end{aligned} \tag{40}$$

By taking $\mathcal{Z} = \max (\mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 + \mathcal{Z}_4 + \mathcal{Z}_5 + \mathcal{Z}_6)$, $\xi_{0\mathcal{H}} = \max (\xi_{0\mathcal{H}_1}, \xi_{0\mathcal{H}_2}, \xi_{0\mathcal{H}_3}, \xi_{0\mathcal{H}_4}, \xi_{0\mathcal{H}_5}, \xi_{0\mathcal{H}_6})$, $\delta_1 = \max (\delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}, \delta_{15}, \delta_{16})$, $\delta_2 = \max (\delta_{21}, \delta_{22}, \delta_{23}, \delta_{24}, \delta_{25}, \delta_{26})$, $\delta_3 = \max (\delta_{31}, \delta_{32}, \delta_{33}, \delta_{34}, \delta_{35}, \delta_{36})$, $\delta_4 = \max (\delta_{41}, \delta_{42}, \delta_{43}, \delta_{44}, \delta_{45}, \delta_{46})$, $\delta_5 = \max (\delta_{51}, \delta_{52}, \delta_{53}, \delta_{54}, \delta_{55}, \delta_{56})$, $\delta_6 = \max (\delta_{61}, \delta_{62}, \delta_{63}, \delta_{64}, \delta_{65}, \delta_{66})$, Eqs. (35)–(40) becomes

$$\begin{aligned} & \left\| \mathcal{Z} (\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) \right\| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}} + \delta_1 \|\mathcal{S}_p\| + \delta_2 \|\mathcal{K}_{CE}\| + \delta_3 \|\mathcal{K}_{VT}\| \right. \\ & \quad \left. + \delta_4 \|\mathcal{K}_{AC}\| + \delta_5 \|\mathcal{K}_{GP}\| + \delta_6 \|\mathcal{L}_{QC}\| \right), \end{aligned} \tag{41}$$

which proves that \mathcal{Z} is bounded.

For equi-continuity, we let $0 \leq t_1 < t_2 \leq \mathcal{J}$, $\mathcal{Z}_1(\cdot)(t_2) = \mathcal{Z}_1(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})(t_2)$ and $\mathcal{Z}_1(\cdot)(t_1) = \mathcal{Z}_1(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC})(t_1)$, so that

$$\begin{aligned} & \left| \mathcal{Z}_1(\cdot)(t_2) - \mathcal{Z}_1(\cdot)(t_1) \right| \\ & = \frac{\eta \rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \left| \left(\int_0^{t_2} (t_2 - v)^{\eta-1} v^{\rho-1} - \int_0^{t_1} (t_1 - v)^{\eta-1} v^{\rho-1} \right) \right. \\ & \quad \left. \times \mathcal{H}_1 \left(v, \mathcal{S}_p(v), \mathcal{K}_{CE}(v), \mathcal{K}_{VT}(v), \mathcal{K}_{AC}(v), \mathcal{K}_{GP}(v), \mathcal{L}_{QC}(v) \right) dv \right| \\ & \leq \frac{\eta \rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \left[\int_0^{t_1} \left((t_1 - v)^{\eta-1} - (t_2 - v)^{\eta-1} \right) v^{\rho-1} \right. \\ & \quad \left. \times \left| \mathcal{H}_1 \left(v, \mathcal{S}_p(v), \mathcal{K}_{CE}(v), \mathcal{K}_{VT}(v), \mathcal{K}_{AC}(v), \mathcal{K}_{GP}(v), \mathcal{L}_{QC}(v) \right) dv \right| \right. \\ & \quad \left. + \int_{t_1}^{t_2} (t_2 - v)^{\eta-1} v^{\rho-1} \left| \mathcal{H}_1 \left(v, \mathcal{S}_p(v), \mathcal{K}_{CE}(v), \mathcal{K}_{VT}(v), \mathcal{K}_{AC}(v), \mathcal{K}_{GP}(v), \mathcal{L}_{QC}(v) \right) dv \right| \right] \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_1} + \delta_{11} |\mathcal{S}_p(v)| + \delta_{21} |\mathcal{K}_{CE}(v)| + \delta_{31} |\mathcal{K}_{VT}(v)| \right. \\ & \quad \left. + \delta_{41} |\mathcal{K}_{AC}(v)| + \delta_{51} |\mathcal{K}_{GP}(v)| + \delta_{61} |\mathcal{L}_{QC}(v)| \right) \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) \right. \\ & \quad \left. + 2 (t_2 - t_1)^{\eta+\rho-1} \right), \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_1} + (\delta_{11} + \delta_{21} + \delta_{31} + \delta_{41} + \delta_{51} + \delta_{61}) \mathcal{Q}_B \right) \\ & \quad \times \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) + 2 (t_2 - t_1)^{\eta+\rho-1} \right). \end{aligned} \tag{42}$$

We observe that the right hand side of (42), $\left| \mathcal{Z}_1(\cdot)(t_2) - \mathcal{Z}_1(\cdot)(t_1) \right|$ tends to zero as t_2 tends to t_1 . Thus by the Arzela–Ascoli theorem, the operator \mathcal{Z}_1 is bounded, uniformly continuous and compact and hence \mathcal{Z}_1 is completely continuous.

Similar approach yields

$$\begin{aligned} & \left| \mathcal{Z}_2(\cdot)(t_2) - \mathcal{Z}_2(\cdot)(t_1) \right| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_2} + (\delta_{12} + \delta_{22} + \delta_{32} + \delta_{42} + \delta_{52} + \delta_{62}) \mathcal{Q}_B \right) \\ & \quad \times \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) + 2 (t_2 - t_1)^{\eta+\rho-1} \right). \end{aligned} \tag{43}$$

$$\begin{aligned} & \left| \mathcal{Z}_3(\cdot)(t_2) - \mathcal{Z}_3(\cdot)(t_1) \right| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_3} + (\delta_{13} + \delta_{23} + \delta_{33} + \delta_{43} + \delta_{53} + \delta_{63}) \mathcal{Q}_B \right) \\ & \quad \times \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) + 2 (t_2 - t_1)^{\eta+\rho-1} \right). \end{aligned} \tag{44}$$

$$\begin{aligned} & \left| \mathcal{Z}_4(\cdot)(t_2) - \mathcal{Z}_4(\cdot)(t_1) \right| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_4} + (\delta_{14} + \delta_{24} + \delta_{34} + \delta_{44} + \delta_{54} + \delta_{64}) \mathcal{Q}_B \right) \\ & \quad \times \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) + 2 (t_2 - t_1)^{\eta+\rho-1} \right). \end{aligned} \tag{45}$$

$$\begin{aligned} & \left| \mathcal{Z}_5(\cdot)(t_2) - \mathcal{Z}_5(\cdot)(t_1) \right| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_5} + (\delta_{15} + \delta_{25} + \delta_{35} + \delta_{45} + \delta_{55} + \delta_{65}) \mathcal{Q}_B \right) \\ & \quad \times \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) + 2 (t_2 - t_1)^{\eta+\rho-1} \right). \end{aligned} \tag{46}$$

$$\begin{aligned} & \left| \mathcal{Z}_6(\cdot)(t_2) - \mathcal{Z}_6(\cdot)(t_1) \right| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}_6} + (\delta_{16} + \delta_{26} + \delta_{36} + \delta_{46} + \delta_{56} + \delta_{66}) \mathcal{Q}_B \right) \\ & \quad \times \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) + 2 (t_2 - t_1)^{\eta+\rho-1} \right). \end{aligned} \tag{47}$$

Using a similar argument from above, the right-hand sides of the inequalities (43)–(47) tend to zero as t_2 tends to t_1 . Thus by Arzela–Ascoli theorem, the operators $\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5$ and \mathcal{Z}_6 are bounded, uniformly continuous and compact and hence $\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5$ and \mathcal{Z}_6 are also completely continuous.

By taking $\mathcal{Z} = \max (\mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 + \mathcal{Z}_4 + \mathcal{Z}_5 + \mathcal{Z}_6)$, $\xi_{0\mathcal{H}} = \max (\xi_{0\mathcal{H}_1}, \xi_{0\mathcal{H}_2}, \xi_{0\mathcal{H}_3}, \xi_{0\mathcal{H}_4}, \xi_{0\mathcal{H}_5}, \xi_{0\mathcal{H}_6})$, $\delta_1 = \max (\delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}, \delta_{15}, \delta_{16})$, $\delta_2 = \max (\delta_{21}, \delta_{22}, \delta_{23}, \delta_{24}, \delta_{25}, \delta_{26})$, $\delta_3 = \max (\delta_{31}, \delta_{32}, \delta_{33}, \delta_{34}, \delta_{35}, \delta_{36})$, $\delta_4 = \max (\delta_{41}, \delta_{42}, \delta_{43}, \delta_{44}, \delta_{45}, \delta_{46})$, $\delta_5 = \max (\delta_{51}, \delta_{52}, \delta_{53}, \delta_{54}, \delta_{55}, \delta_{56})$, $\delta_6 = \max (\delta_{61}, \delta_{62}, \delta_{63}, \delta_{64}, \delta_{65}, \delta_{66})$, Eqs. (42)–(47) becomes

$$\begin{aligned} & \left| \mathcal{Z}(\cdot)(t_2) - \mathcal{Z}(\cdot)(t_1) \right| \\ & \leq \frac{\eta \Gamma(\rho+1)}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \left(\xi_{0\mathcal{H}} + (\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6) \mathcal{Q}_B \right) \\ & \quad \times \left(\left(t_1^{\eta+\rho-1} - t_2^{\eta+\rho-1} \right) + 2 (t_2 - t_1)^{\eta+\rho-1} \right). \end{aligned} \tag{48}$$

Thus the operator $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5, \mathcal{Z}_6)$ is bounded, uniformly continuous and compact which implies that the operator \mathcal{Z} is completely continuous. This translates that $\mathcal{G}\mathbf{u} = \mathcal{F}\mathbf{u} + \mathcal{Z}\mathbf{u}$ has a fixed point and hence our fractal fractional corruption model (2)–(4) possess at least one solution.

On the other hand, we investigate the **uniqueness** of the fractal fractional model solution. By Banach’s contraction principle, let the operator

$$\mathcal{G} : \hat{\mathbf{u}} \rightarrow \hat{\mathbf{u}}$$

be defined as

$$\begin{aligned} \mathcal{G}(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) &= (\mathcal{G}_1(\cdot), \mathcal{G}_2(\cdot), \mathcal{G}_3(\cdot), \mathcal{G}_4(\cdot), \mathcal{G}_5(\cdot), \mathcal{G}_6(\cdot)) (t) \\ &= (\cdot)(t). \end{aligned}$$

Theorem 6. *With the aid of assumption (R2) and the operator*

$$\begin{aligned} \mathcal{G}(\mathcal{S}_p, \mathcal{K}_{CE}, \mathcal{K}_{VT}, \mathcal{K}_{AC}, \mathcal{K}_{GP}, \mathcal{L}_{QC}) &= (\mathcal{G}_1(\cdot), \mathcal{G}_2(\cdot), \mathcal{G}_3(\cdot), \mathcal{G}_4(\cdot), \mathcal{G}_5(\cdot), \mathcal{G}_6(\cdot)) (t) \\ &= (\cdot)(t). \end{aligned}$$

which is a contraction, then the operator \mathcal{G} possess a unique fixed point which translates that the fractal fractional corruption system (2)–(4) has a

unique solution if and only if

$$\left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}} < 1. \tag{49}$$

Proof. Let the solutions $(\mathcal{S}_p, \mathcal{K}_{ce}, \mathcal{K}_{yt}, \mathcal{K}_{ac}, \mathcal{K}_{gp}, \mathcal{L}_{qc})$, $(\hat{\mathcal{S}}_p, \hat{\mathcal{K}}_{ce}, \hat{\mathcal{K}}_{yt}, \hat{\mathcal{K}}_{ac}, \hat{\mathcal{K}}_{gp}, \hat{\mathcal{L}}_{qc}) \in \hat{\mathcal{U}}$, $\mathcal{G}(\cdot) = \mathcal{G}(\mathcal{S}_p, \mathcal{K}_{ce}, \mathcal{K}_{yt}, \mathcal{K}_{ac}, \mathcal{K}_{gp}, \mathcal{L}_{qc})$ and $\mathcal{G}(\cdot) = \mathcal{G}(\hat{\mathcal{S}}_p, \hat{\mathcal{K}}_{ce}, \hat{\mathcal{K}}_{yt}, \hat{\mathcal{K}}_{ac}, \hat{\mathcal{K}}_{gp}, \hat{\mathcal{L}}_{qc})$ so that $t \in \mathcal{J} = [0, \mathcal{T}]$, we obtain

$$\begin{aligned} & \left\| \mathcal{G}_1(\cdot) - \mathcal{G}_1(\cdot) \right\| \\ &= \max_{t \in \mathcal{J}} \left\| \mathcal{G}_1(\mathcal{S}_p, \mathcal{K}_{ce}, \mathcal{K}_{yt}, \mathcal{K}_{ac}, \mathcal{K}_{gp}, \mathcal{L}_{qc}) - \mathcal{G}_1(\hat{\mathcal{S}}_p, \hat{\mathcal{K}}_{ce}, \hat{\mathcal{K}}_{yt}, \hat{\mathcal{K}}_{ac}, \hat{\mathcal{K}}_{gp}, \hat{\mathcal{L}}_{qc}) \right\|, \\ &\leq \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \max_{t \in \mathcal{J}} \left| \mathcal{H}_1(v, \mathcal{S}_p(v), \mathcal{K}_{ce}(v), \mathcal{K}_{yt}(v), \mathcal{K}_{ac}(v), \mathcal{K}_{gp}(v), \mathcal{L}_{qc}(v)) \right. \\ &\quad \left. - \mathcal{H}_1(v, \hat{\mathcal{S}}_p(v), \hat{\mathcal{K}}_{ce}(v), \hat{\mathcal{K}}_{yt}(v), \hat{\mathcal{K}}_{ac}(v), \hat{\mathcal{K}}_{gp}(v), \hat{\mathcal{L}}_{qc}(v)) \right| \\ &\quad + \frac{\eta\rho}{\mathcal{ABC}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \\ &\quad \times \left(\max_{t \in \mathcal{J}} \left| \mathcal{H}_1(v, \mathcal{S}_p(v), \mathcal{K}_{ce}(v), \mathcal{K}_{yt}(v), \mathcal{K}_{ac}(v), \mathcal{K}_{gp}(v), \mathcal{L}_{qc}(v)) \right. \right. \\ &\quad \left. \left. - \mathcal{H}_1(v, \hat{\mathcal{S}}_p(v), \hat{\mathcal{K}}_{ce}(v), \hat{\mathcal{K}}_{yt}(v), \hat{\mathcal{K}}_{ac}(v), \hat{\mathcal{K}}_{gp}(v), \hat{\mathcal{L}}_{qc}(v)) \right| dv \right), \\ &\leq \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} \mathcal{E}_{\mathcal{J}\mathcal{C}_1} \left(\left| \mathcal{S}_p - \hat{\mathcal{S}}_p \right| + \left| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right| + \left| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right| \right. \\ &\quad \left. + \left| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right| + \left| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right| + \left| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right| \right) \\ &\quad + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \mathcal{E}_{\mathcal{J}\mathcal{C}_1} \left(\left| \mathcal{S}_p - \hat{\mathcal{S}}_p \right| + \left| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right| + \left| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right| \right. \\ &\quad \left. + \left| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right| + \left| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right| + \left| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right| \right), \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}_1} \left(\left| \mathcal{S}_p - \hat{\mathcal{S}}_p \right| + \left| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right| \right. \\ &\quad \left. + \left| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right| + \left| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right| + \left| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right| + \left| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right| \right). \end{aligned}$$

This then implies that

$$\begin{aligned} & \left\| \mathcal{G}_1(\cdot) - \mathcal{G}_1(\cdot) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}_1} \left(\left\| \mathcal{S}_p - \hat{\mathcal{S}}_p \right\| + \left\| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right\| \right. \\ &\quad \left. + \left\| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right\| + \left\| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right\| + \left\| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right\| + \left\| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right\| \right). \end{aligned} \tag{50}$$

Similar approach gives

$$\begin{aligned} & \left\| \mathcal{G}_2(\cdot) - \mathcal{G}_2(\cdot) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}_2} \left(\left\| \mathcal{S}_p - \hat{\mathcal{S}}_p \right\| + \left\| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right\| \right. \\ &\quad \left. + \left\| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right\| + \left\| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right\| + \left\| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right\| + \left\| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right\| \right). \end{aligned} \tag{51}$$

$$\begin{aligned} & \left\| \mathcal{G}_3(\cdot) - \mathcal{G}_3(\cdot) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}_3} \left(\left\| \mathcal{S}_p - \hat{\mathcal{S}}_p \right\| + \left\| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right\| \right. \\ &\quad \left. + \left\| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right\| + \left\| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right\| + \left\| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right\| + \left\| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right\| \right). \end{aligned} \tag{52}$$

$$\begin{aligned} & \left\| \mathcal{G}_4(\cdot) - \mathcal{G}_4(\cdot) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}_4} \left(\left\| \mathcal{S}_p - \hat{\mathcal{S}}_p \right\| + \left\| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right\| \right. \\ &\quad \left. + \left\| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right\| + \left\| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right\| + \left\| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right\| + \left\| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right\| \right). \end{aligned} \tag{53}$$

$$\begin{aligned} & \left\| \mathcal{G}_5(\cdot) - \mathcal{G}_5(\cdot) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}_5} \left(\left\| \mathcal{S}_p - \hat{\mathcal{S}}_p \right\| + \left\| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right\| \right. \\ &\quad \left. + \left\| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right\| + \left\| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right\| + \left\| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right\| + \left\| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right\| \right). \end{aligned} \tag{54}$$

$$\begin{aligned} & \left\| \mathcal{G}_6(\cdot) - \mathcal{G}_6(\cdot) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}_6} \left(\left\| \mathcal{S}_p - \hat{\mathcal{S}}_p \right\| + \left\| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right\| \right. \\ &\quad \left. + \left\| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right\| + \left\| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right\| + \left\| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right\| + \left\| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right\| \right). \end{aligned} \tag{55}$$

Thus since $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6)$ and $\max \{ \mathcal{E}_{\mathcal{J}\mathcal{C}_1}, \mathcal{E}_{\mathcal{J}\mathcal{C}_2}, \mathcal{E}_{\mathcal{J}\mathcal{C}_3}, \mathcal{E}_{\mathcal{J}\mathcal{C}_4}, \mathcal{E}_{\mathcal{J}\mathcal{C}_5}, \mathcal{E}_{\mathcal{J}\mathcal{C}_6} = \mathcal{E}_{\mathcal{J}\mathcal{C}} > 0 \}$, it follows that (50)–(55) can be represented as

$$\begin{aligned} & \left\| \mathcal{G}(\cdot) - \mathcal{G}(\cdot) \right\| \\ &\leq \left(\frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{ABC}(\eta)} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{ABC}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{J}\mathcal{C}} \left(\left\| \mathcal{S}_p - \hat{\mathcal{S}}_p \right\| + \left\| \mathcal{K}_{ce} - \hat{\mathcal{K}}_{ce} \right\| \right. \\ &\quad \left. + \left\| \mathcal{K}_{yt} - \hat{\mathcal{K}}_{yt} \right\| + \left\| \mathcal{K}_{ac} - \hat{\mathcal{K}}_{ac} \right\| + \left\| \mathcal{K}_{gp} - \hat{\mathcal{K}}_{gp} \right\| + \left\| \mathcal{L}_{qc} - \hat{\mathcal{L}}_{qc} \right\| \right). \end{aligned} \tag{56}$$

From Eq. (56), operator \mathcal{G} is a contraction and thus possesses at least a unique solution. Hence the fractal fractional corruption model (2)–(4) possess a unique solution.

Ulam–Hyers stability

Definition 3. The fractal fractional system (2)–(4) is Ulam–Hyers stable if there exists a real number $\mathcal{d}_{cp} > 0$ such that, for a real number $\mathfrak{h}_{cp} > 0$ and for every solution $\mathbf{u} \in \mathcal{E}([0, \mathcal{T}], \mathbb{R}^+)$ of the inequality

$$\left| {}_0^{\mathcal{FF}-\mathcal{ABC}} \mathcal{D}_t^{\eta, \rho} \mathbf{u}(t) - \mathbf{w}(t, \mathbf{u}(t)) \right| \leq \mathfrak{h}_{cp}, \quad t \in \mathcal{J} = [0, \mathcal{T}], \quad 0 < \eta, \rho \leq 1, \tag{57}$$

there exists a second solution $\hat{\mathbf{u}} \in \mathcal{E}([0, \mathcal{T}], \mathbb{R}^+)$ of the fractal fractional system (2)–(4) such that

$$\|\mathbf{u}(t) - \hat{\mathbf{u}}(t)\| \leq \mathcal{d}_{cp} \mathfrak{h}_{cp} \quad t \in \mathcal{J} = [0, \mathcal{T}].$$

We note that a function $\mathbf{u} \in \mathcal{E}([0, \mathcal{T}], \mathbb{R}^+)$ is a solution of inequality (57) if there exists a function $\mathcal{A}_{cp} \in \mathcal{E}([0, \mathcal{T}], \mathbb{R}^+)$ which depends on the solution of fractal fractional model, \mathbf{u} such that

- $|\mathcal{A}_{cp}(t)| < \mathfrak{h}_{cp}, \quad t \in \mathcal{J} = [0, \mathcal{T}].$
- ${}_0^{\mathcal{FF}-\mathcal{ABC}} \mathcal{D}_t^{\eta, \rho} \mathbf{u}(t) = \mathbf{w}(t, \mathbf{u}(t)) + \mathcal{A}_{cp}(t), \quad t \in \mathcal{J} = [0, \mathcal{T}].$

Lemma 7. The solution of the initial value problem below with the perturbed term $\mathcal{A}_{cp}(t)$,

$$\begin{cases} {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta,\rho}\mathbf{u}(t) = \rho t^{\rho-1}\mathbf{w}(t, \mathbf{u}(t)) + \mathcal{A}_{cp}(t), & t \in \mathcal{J} = [0, \mathcal{T}], \quad 0 < \eta, \rho \leq 1, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{58}$$

satisfies

$$\begin{aligned} & \left| \mathbf{u}(t) - \left(\mathbf{u}_0 + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathbf{w}(t, \mathbf{u}(t) + \mathcal{A}_{cp}(t)) \right. \right. \\ & \quad \left. \left. + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \right. \right. \\ & \quad \left. \left. \times (\mathbf{w}(v, \mathbf{u}(v) + \mathcal{A}_{cp}(v))) dv \right) \right| \\ & \leq \frac{\rho(1-\eta)\Gamma(\eta+\rho)t^{\rho-1} + \eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+\rho)} \mathbf{h}_{cp} = \mathbf{h}_{cp}\mathcal{Q}_{cp}, \end{aligned} \tag{59}$$

where

$$\mathcal{Q}_{cp} = \frac{\rho(1-\eta)\Gamma(\eta+\rho)t^{\rho-1} + \eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+\rho)} > 0.$$

Proof. The solution of (58) is equivalent to the following Volterra fractional integral equation

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{u}_0 + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} (\mathbf{w}(t, \mathbf{u}(t) + \mathcal{A}_{cp}(t))) \\ & \quad + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \\ & \quad \times (\mathbf{w}(v, \mathbf{u}(v) + \mathcal{A}_{cp}(v))) dv, \\ &= \mathbf{u}_0 + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} (\mathbf{w}(t, \mathbf{u}(t) + \mathcal{A}_{cp}(t))) \\ & \quad + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathbf{w}(v, \mathbf{u}(v)) dv \\ & \quad + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathcal{A}_{cp}(v) dv, \end{aligned}$$

so that

$$\begin{aligned} & \left| \mathbf{u}(t) - \mathbf{u}_0 - \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathbf{w}(t, \mathbf{u}(t)) \right. \\ & \quad \left. - \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathbf{w}(v, \mathbf{u}(v)) dv \right| \\ & \leq \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} |\mathcal{A}_{cp}(t)| + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} |\mathcal{A}_{cp}(v)| dv, \\ & \leq \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathbf{h}_{cp} + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+\rho)} \mathbf{h}_{cp}, \\ & \leq \left(\frac{\rho(1-\eta)\Gamma(\eta+\rho)t^{\rho-1} + \eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+\rho)} \right) \mathbf{h}_{cp} = \mathbf{h}_{cp}\mathcal{Q}_{cp}, \end{aligned}$$

which proves the above Lemma.

Theorem 8. Suppose that $\mathbf{w} : \mathcal{J} \times \mathbb{R}^6 \rightarrow \mathbb{R}$ is a continuous function and with the help of (R2), the fractal fractional corruption system (2)–(4) is Ulam–Hyers stable on $t \in \mathcal{J} = [0, \mathcal{T}]$ if there exists $1 > \mathcal{E}_{\mathcal{H}\mathcal{Q}_{cp}}$.

Proof. Let the function $\hat{\mathbf{u}} \in \mathcal{E}([0, \mathcal{T}], \mathbb{R}^+)$ denote the unique solution of (21) and let $\mathbf{u} \in \mathcal{E}([0, \mathcal{T}], \mathbb{R}^+)$ denotes the solution of (58) and thus,

by employing Lemma 7 and triangle inequality we get

$$\begin{aligned} |\mathbf{u}(t) - \hat{\mathbf{u}}(t)| &= \left| \mathbf{u}(t) - \left(\mathbf{u}_0 + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathbf{w}(t, \hat{\mathbf{u}}(t)) \right. \right. \\ & \quad \left. \left. + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathbf{w}(v, \hat{\mathbf{u}}(v)) dv \right) \right| \\ & \leq \left| \mathbf{u}(t) - \left(\mathbf{u}_0 + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathbf{w}(t, \mathbf{u}(t)) \right. \right. \\ & \quad \left. \left. + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \mathbf{w}(v, \mathbf{u}(v)) dv \right) \right| \\ & \quad + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} |\mathbf{w}(t, \hat{\mathbf{u}}(t)) - \mathbf{w}(t, \mathbf{u}(t))| + \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \int_0^t (t-v)^{\eta-1} v^{\rho-1} \\ & \quad \times |\mathbf{w}(v, \hat{\mathbf{u}}(v)) - \mathbf{w}(v, \mathbf{u}(v))| dv, \\ & \leq \mathbf{h}_{cp}\mathcal{Q}_{cp} + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{E}_{\mathcal{H}\mathcal{C}} |t - \hat{\mathbf{u}}(t)| \\ & \quad + \frac{\eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+\rho)} \mathcal{E}_{\mathcal{H}\mathcal{C}} |t - \hat{\mathbf{u}}(t)| \\ & \leq \mathbf{h}_{cp}\mathcal{Q}_{cp} + \left(\frac{\rho(1-\eta)\Gamma(\eta+\rho)t^{\rho-1} + \eta\Gamma(\rho+1)\mathcal{J}_{\max}^{\eta+\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+\rho)} \right) \mathcal{E}_{\mathcal{H}\mathcal{C}} |t - \hat{\mathbf{u}}(t)|, \\ & \leq \mathbf{h}_{cp}\mathcal{Q}_{cp} + \mathcal{Q}_{cp} |t - \hat{\mathbf{u}}(t)| \mathcal{E}_{\mathcal{H}\mathcal{C}}. \end{aligned}$$

The above inequality implies that,

$$\|\mathbf{u}(t) - \hat{\mathbf{u}}(t)\| \leq \mathbf{h}_{cp}\mathcal{Q}_{cp} + \mathcal{Q}_{cp} \|t - \hat{\mathbf{u}}(t)\| \mathcal{E}_{\mathcal{H}\mathcal{C}},$$

so that,

$$\|\mathbf{u}(t) - \hat{\mathbf{u}}(t)\| \leq \mathcal{d}_{cp} \mathbf{h}_{cp}, \quad \mathcal{d}_{cp} = \frac{\mathcal{Q}_{cp}}{1 - \mathcal{E}_{\mathcal{H}\mathcal{C}}\mathcal{Q}_{cp}}. \tag{60}$$

Similar process from (57) to (60) yields the following

$$\begin{cases} \|\mathcal{S}_p(t) - \hat{\mathcal{S}}_p(t)\| \leq \mathcal{d}_{cp} \mathbf{h}_{cp}, & \|\mathcal{S}_c(t) - \hat{\mathcal{S}}_c(t)\| \leq \mathcal{d}_{cp} \mathbf{h}_{cp}, & \|\mathcal{S}_v(t) - \hat{\mathcal{S}}_v(t)\| \leq \mathcal{d}_{cp} \mathbf{h}_{cp}, \\ \|\mathcal{S}_p(t) - \hat{\mathcal{S}}_p(t)\| \leq \mathcal{d}_{cp} \mathbf{h}_{cp}, & \|\mathcal{S}_c(t) - \hat{\mathcal{S}}_c(t)\| \leq \mathcal{d}_{cp} \mathbf{h}_{cp}, & \|\mathcal{S}_p(t) - \hat{\mathcal{S}}_p(t)\| \leq \mathcal{d}_{cp} \mathbf{h}_{cp}, \end{cases} \tag{61}$$

for every $t \in \mathcal{J} = [0, \mathcal{T}]$. Thus we say that the fractal fractional corruption model (2)–(4) is Ulam–Hyers stable in conjunction with $\mathbf{h}_{cp} > 0$.

Numerical scheme

In this section, we present the numerical solution of the fractal-fractional corruption model (2)–(4) by employing the well-known fractional Adams–Bashforth iterative technique in Atangana–Baleanu–Caputo derivative sense, proposed by Toufik and Atangana [45]. The numerical scheme is shown to be consistent and stable and thus converges to the exact solution.

For this purpose, we state the fractal fractional corruption system (2) in a simplified form in conjunction with Eq. (8) as follows;

$$\begin{cases} {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta,\rho} \mathcal{S}_p(t) = \rho t^{\rho-1} \mathcal{H}_1(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{vt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{qc}(t)), \\ {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta,\rho} \mathcal{K}_{ce}(t) = \rho t^{\rho-1} \mathcal{H}_2(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{vt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{qc}(t)), \\ {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta,\rho} \mathcal{K}_{vt}(t) = \rho t^{\rho-1} \mathcal{H}_3(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{vt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{qc}(t)), \\ {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta,\rho} \mathcal{K}_{ac}(t) = \rho t^{\rho-1} \mathcal{H}_4(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{vt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{qc}(t)), \\ {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta,\rho} \mathcal{K}_{gp}(t) = \rho t^{\rho-1} \mathcal{H}_5(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{vt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{qc}(t)), \\ {}^{\mathcal{F}\mathcal{F}-\mathcal{A}\mathcal{B}\mathcal{C}}\mathcal{D}_t^{\eta,\rho} \mathcal{L}_{qc}(t) = \rho t^{\rho-1} \mathcal{H}_6(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{vt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{qc}(t)), \end{cases} \tag{62}$$

where $0 < \eta, \rho \leq 1$, $\mathcal{H}_{\mathcal{K}_\ell} = \mathcal{H}_{\mathcal{K}_\ell}(t, \mathcal{S}_p(t), \mathcal{K}_{ce}(t), \mathcal{K}_{vt}(t), \mathcal{K}_{ac}(t), \mathcal{K}_{gp}(t), \mathcal{L}_{qc}(t))$, for $\mathcal{K}_\ell = 1, 2, \dots, 6$ is given by Eq. (3) and the corresponding initial condition is giving by (4). Also $\mathcal{H}_{\mathcal{K}_\ell} : \mathcal{J} \times \mathbb{R}^6 \rightarrow \mathbb{R}$ are continuous functions for $\mathcal{K}_\ell = 1, 2, \dots, 6$ and $\mathcal{J} = [0, \mathcal{T}]$.

Plugging $t = t_{\mathcal{V}+1}$ for $\mathcal{V} = 0, 1, 2, \dots$ into (24) (the Volterra fractional integral equation equivalent to (62) and the initial condition (4)) leads

to the following system,

$$\begin{aligned}
 S_p(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_1(t_{\mathbf{r}}, S_p(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \int_{\mathbf{m}}^{t_{\mathbf{m}+1}} (t_{\mathbf{r}+1}-v)^{\eta-1} v^{\rho-1} \mathcal{H}_1(v, S_p(v)) dv, \\
 \mathcal{K}_{ce}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_2(t_{\mathbf{r}}, \mathcal{K}_{ce}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \int_{\mathbf{m}}^{t_{\mathbf{m}+1}} (t_{\mathbf{m}+1}-v)^{\eta-1} v^{\rho-1} \mathcal{H}_2(v, \mathcal{K}_{ce}(v)) dv, \\
 \mathcal{K}_{vi}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_3(t_{\mathbf{r}}, \mathcal{K}_{vi}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \int_{\mathbf{m}}^{t_{\mathbf{m}+1}} (t_{\mathbf{r}+1}-v)^{\eta-1} v^{\rho-1} \mathcal{H}_3(v, \mathcal{K}_{vi}(v)) dv, \\
 \mathcal{K}_{ac}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_4(t_{\mathbf{r}}, \mathcal{K}_{ac}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \int_{\mathbf{m}}^{t_{\mathbf{m}+1}} (t_{\mathbf{r}+1}-v)^{\eta-1} v^{\rho-1} \mathcal{H}_4(v, \mathcal{K}_{ac}(v)) dv, \\
 \mathcal{K}_{gp}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_5(t_{\mathbf{r}}, \mathcal{K}_{gp}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \int_{\mathbf{m}}^{t_{\mathbf{m}+1}} (t_{\mathbf{r}+1}-v)^{\eta-1} v^{\rho-1} \mathcal{H}_5(v, \mathcal{K}_{gp}(v)) dv, \\
 \mathcal{L}_{oc}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_6(t_{\mathbf{r}}, \mathcal{L}_{oc}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \int_{\mathbf{m}}^{t_{\mathbf{m}+1}} (t_{\mathbf{r}+1}-v)^{\eta-1} v^{\rho-1} \mathcal{H}_6(v, \mathcal{L}_{oc}(v)) dv.
 \end{aligned} \tag{63}$$

Equipped with two points Lagrange interpolation polynomial (see the numerical scheme section in [38]), we evaluate the discretised integral (63) at $t_{\mathbf{m}} = \mathbf{m}\mathbf{h}$, for $\mathbf{m} = 0, 1, 2, \dots, \mathbf{r}$ and where \mathbf{h} is the step-size, then the numerical scheme for the fractal fractional corruption system (2)–(4) are stated as follows:

$$\begin{aligned}
 S_p(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_1(t_{\mathbf{r}}, S_p(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+2)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \left[\frac{\mathcal{H}_1(t_{\mathbf{m}}, S_p(t_{\mathbf{m}}))}{\mathbf{h}} \frac{t_{\mathbf{r}}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}+1)} \right. \\
 &\times \left((\mathbf{r}-\mathbf{m}+1)^\eta (\mathbf{r}-\mathbf{m}+2+\eta) - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+2+2\eta) \right) \\
 &- \frac{\mathcal{H}_1(t_{\mathbf{m}-1}, S_p(t_{\mathbf{m}-1}))}{\mathbf{h}} \\
 &\times \left. \frac{t_{\mathbf{r}-1}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}-1)} \left((\mathbf{r}-\mathbf{m}+1)^{\eta+1} - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+\eta+1) \right) \right].
 \end{aligned} \tag{64}$$

$$\begin{aligned}
 \mathcal{K}_{ce}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_2(t_{\mathbf{r}}, \mathcal{K}_{ce}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+2)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \left[\frac{\mathcal{H}_2(t_{\mathbf{m}}, \mathcal{K}_{ce}(t_{\mathbf{m}}))}{\mathbf{h}} \frac{t_{\mathbf{r}}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}+1)} \right. \\
 &\times \left((\mathbf{r}-\mathbf{m}+1)^\eta (\mathbf{r}-\mathbf{m}+2+\eta) - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+2+2\eta) \right) \\
 &- \frac{\mathcal{H}_2(t_{\mathbf{m}-1}, \mathcal{K}_{ce}(t_{\mathbf{m}-1}))}{\mathbf{h}} \\
 &\times \left. \frac{t_{\mathbf{r}-1}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}-1)} \left((\mathbf{r}-\mathbf{m}+1)^{\eta+1} - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+\eta+1) \right) \right].
 \end{aligned} \tag{65}$$

$$\begin{aligned}
 \mathcal{K}_{vi}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_3(t_{\mathbf{r}}, \mathcal{K}_{vi}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+2)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \left[\frac{\mathcal{H}_3(t_{\mathbf{m}}, \mathcal{K}_{vi}(t_{\mathbf{m}}))}{\mathbf{h}} \frac{t_{\mathbf{r}}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}+1)} \right. \\
 &\times \left((\mathbf{r}-\mathbf{m}+1)^\eta (\mathbf{r}-\mathbf{m}+2+\eta) - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+2+2\eta) \right) \\
 &- \frac{\mathcal{H}_3(t_{\mathbf{m}-1}, \mathcal{K}_{vi}(t_{\mathbf{m}-1}))}{\mathbf{h}} \\
 &\times \left. \frac{t_{\mathbf{r}-1}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}-1)} \left((\mathbf{r}-\mathbf{m}+1)^{\eta+1} - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+\eta+1) \right) \right].
 \end{aligned} \tag{66}$$

$$\begin{aligned}
 \mathcal{K}_{ac}(t_{\mathbf{r}+1}) &= \mathcal{K}_{ac}^0 + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_4(t_{\mathbf{r}}, \mathcal{K}_{ac}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+2)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \left[\frac{\mathcal{H}_4(t_{\mathbf{m}}, \mathcal{K}_{ac}(t_{\mathbf{m}}))}{\mathbf{h}} \frac{t_{\mathbf{r}}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}+1)} \right. \\
 &\times \left((\mathbf{r}-\mathbf{m}+1)^\eta (\mathbf{r}-\mathbf{m}+2+\eta) - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+2+2\eta) \right) \\
 &- \frac{\mathcal{H}_4(t_{\mathbf{m}-1}, \mathcal{K}_{ac}(t_{\mathbf{m}-1}))}{\mathbf{h}} \\
 &\times \left. \frac{t_{\mathbf{r}-1}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}-1)} \left((\mathbf{r}-\mathbf{m}+1)^{\eta+1} - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+\eta+1) \right) \right].
 \end{aligned} \tag{67}$$

$$\begin{aligned}
 \mathcal{K}_{gp}(t_{\mathbf{r}+1}) &= \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_5(t_{\mathbf{r}}, \mathcal{K}_{gp}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+2)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \left[\frac{\mathcal{H}_5(t_{\mathbf{m}}, \mathcal{K}_{gp}(t_{\mathbf{m}}))}{\mathbf{h}} \frac{t_{\mathbf{r}}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}+1)} \right. \\
 &\times \left((\mathbf{r}-\mathbf{m}+1)^\eta (\mathbf{r}-\mathbf{m}+2+\eta) - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+2+2\eta) \right) \\
 &- \frac{\mathcal{H}_5(t_{\mathbf{m}-1}, \mathcal{K}_{gp}(t_{\mathbf{m}-1}))}{\mathbf{h}} \\
 &\times \left. \frac{t_{\mathbf{r}-1}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}-1)} \left((\mathbf{r}-\mathbf{m}+1)^{\eta+1} - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+\eta+1) \right) \right].
 \end{aligned} \tag{68}$$

$$\begin{aligned}
 \mathcal{L}_{oc}(t_{\mathbf{r}+1}) &= \mathcal{L}_{oc}^0 + \frac{\rho(1-\eta)t^{\rho-1}}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)} \mathcal{H}_6(t_{\mathbf{r}}, \mathcal{L}_{oc}(t_{\mathbf{r}})) \\
 &+ \frac{\eta\rho}{\mathcal{A}\mathcal{B}\mathcal{C}(\eta)\Gamma(\eta+2)} \sum_{\mathbf{m}=0}^{\mathbf{r}} \left[\frac{\mathcal{H}_6(t_{\mathbf{m}}, \mathcal{L}_{oc}(t_{\mathbf{m}}))}{\mathbf{h}} \frac{t_{\mathbf{r}}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}+1)} \right. \\
 &\times \left((\mathbf{r}-\mathbf{m}+1)^\eta (\mathbf{r}-\mathbf{m}+2+\eta) - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+2+2\eta) \right) \\
 &- \frac{\mathcal{H}_6(t_{\mathbf{m}-1}, \mathcal{L}_{oc}(t_{\mathbf{m}-1}))}{\mathbf{h}} \\
 &\times \left. \frac{t_{\mathbf{r}-1}^{\rho-1} \mathbf{h}^{\eta+1}}{\mathbf{r}(\mathbf{r}-1)} \left((\mathbf{r}-\mathbf{m}+1)^{\eta+1} - (\mathbf{r}-\mathbf{m})^\eta (\mathbf{r}-\mathbf{m}+\eta+1) \right) \right].
 \end{aligned} \tag{69}$$

Simulations and observation

In this section, the numerical outcomes of the various compartments in the corruption model are observed using literature values and some assumptions. All parameter values and initial state conditions are in Table 1.

Fig. 3 shows the trajectory effect of changing the fractal–fractional order from the classical integer order to memory dynamics with repeated patterns at every scale. It is noticed from Figs. 3(a)–3(f) that as the fractal–fractional order reduces simultaneously, the number of

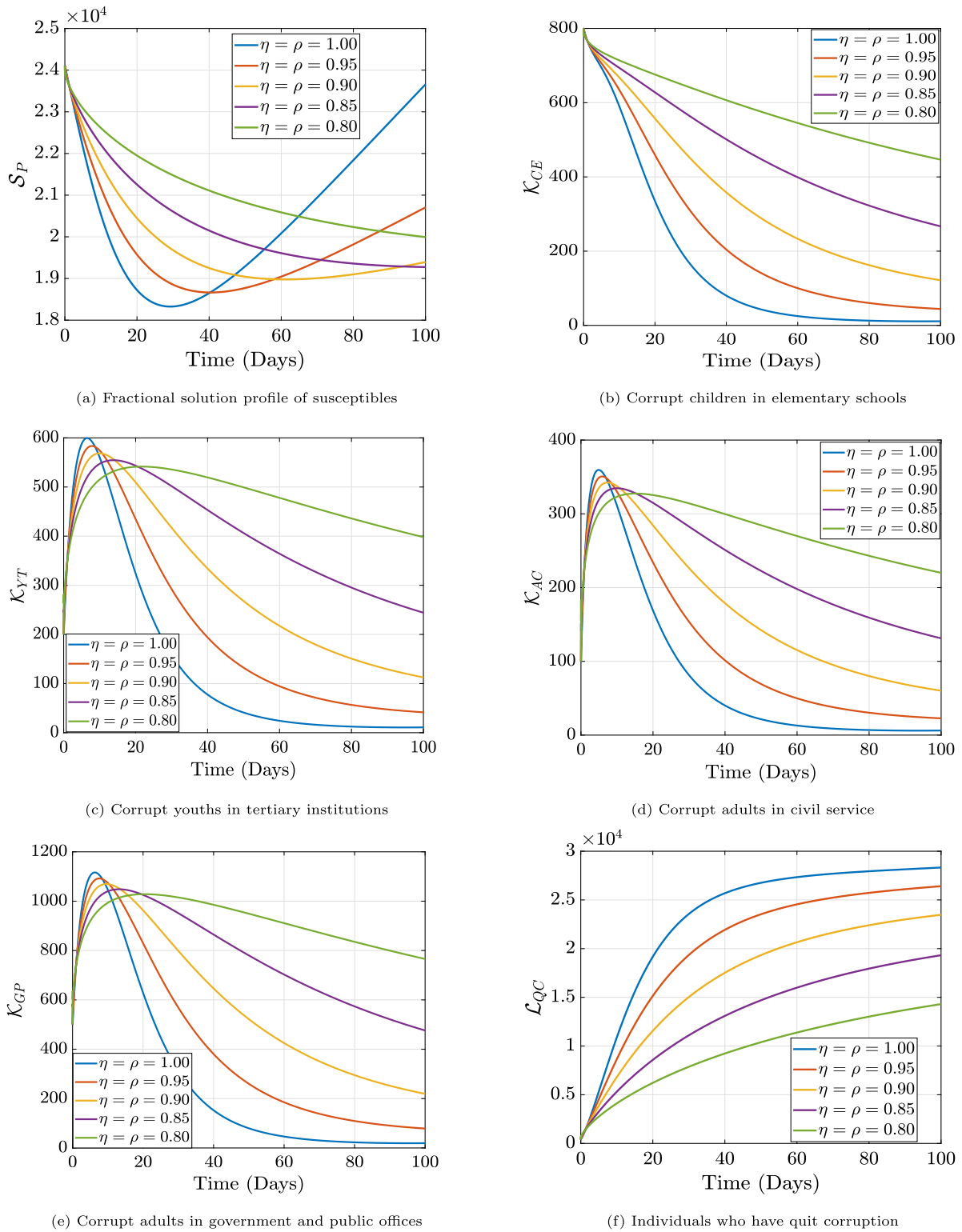


Fig. 3. Fractal fractional numerical simulation of the corruption model with 0.05 deference in Fractal–fractional order.

susceptibles increases, leading to an increase in the number of corruption in children in elementary schools; the number of corruption in youths in tertiary institutions; the number of corruption in adults in civil service; and the number of corruption in adults in government and public offices; and a total decline in the number of individuals who quit corruption. This may be explained like this: eradicating corruption in society will be very challenging as long as people keep the benefits of

corruption in mind and have a repeated pattern of corruption. In Figs. 4 and 5, the memory aspect of the corruption dynamics is fixed at 1, and the repeated pattern of the model is varied from 0.99 to 0.90 and from 0.90 to 0.1. Thus, we notice a crossover effect in the susceptible class during day 38 and a decrease in the number of susceptible individuals afterwards. Figs. 4(b)–4(f) show very close behaviour in corruption dynamics when the repeated pattern is small. Fig. 5(a) shows that

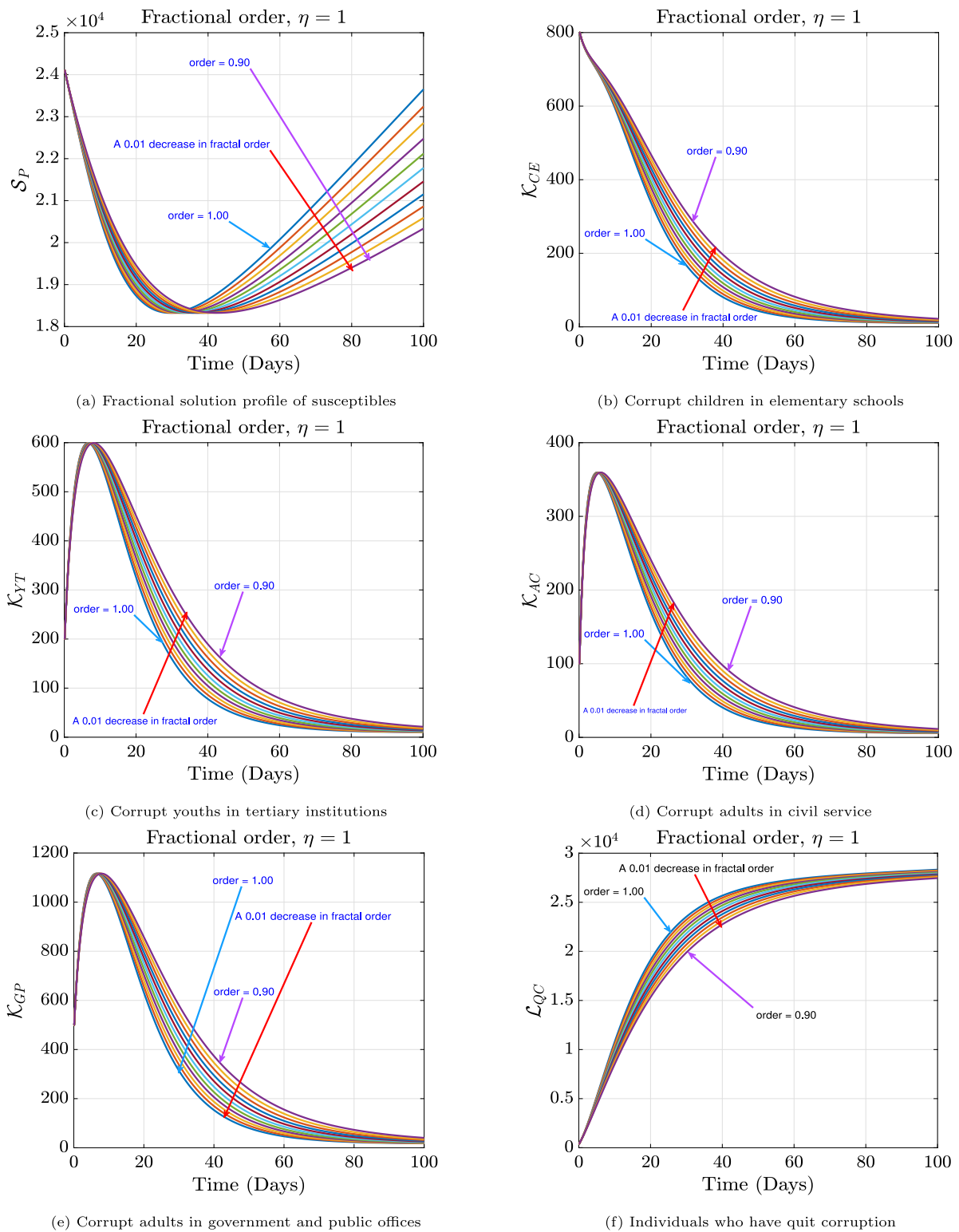


Fig. 4. Fractal numerical simulation of the corruption model with 0.01 difference in Fractal–fractional order up to 0.90, when fractional order is fixed.

there is a high increase in susceptibles when the repeated pattern of corruption is increased, whereas, as illustrated in Fig. 5(b), the pattern of corruption in elementary school children rapidly increases as the fractal order moves from 0.9 to 0.4 and then slowly decreases from 0.3 to 0.1. Figs. 5(c)–5(e) show that corruption increases rapidly when the fractal order moves from 0.9 to 0.5 and decreases when the fractal order reaches 0.4 to 0.1. Fig. 5(f) shows that a change in the fractal

order with a difference of 0.1 does not change the pattern of individuals who quit corruption as the fractal order varied from 0.90 to 0.1.

In Figs. 6 and 7, the repeated pattern aspect of the corruption dynamics is fixed at 1, and the memory aspect of the model is varied from 0.99 to 0.90 and from 0.90 to 0.1. Thus, we notice a crossover effect in the susceptible class during day 45 and a decrease in the number of susceptible individuals afterwards. Figs. 6(b)–6(f) show similar

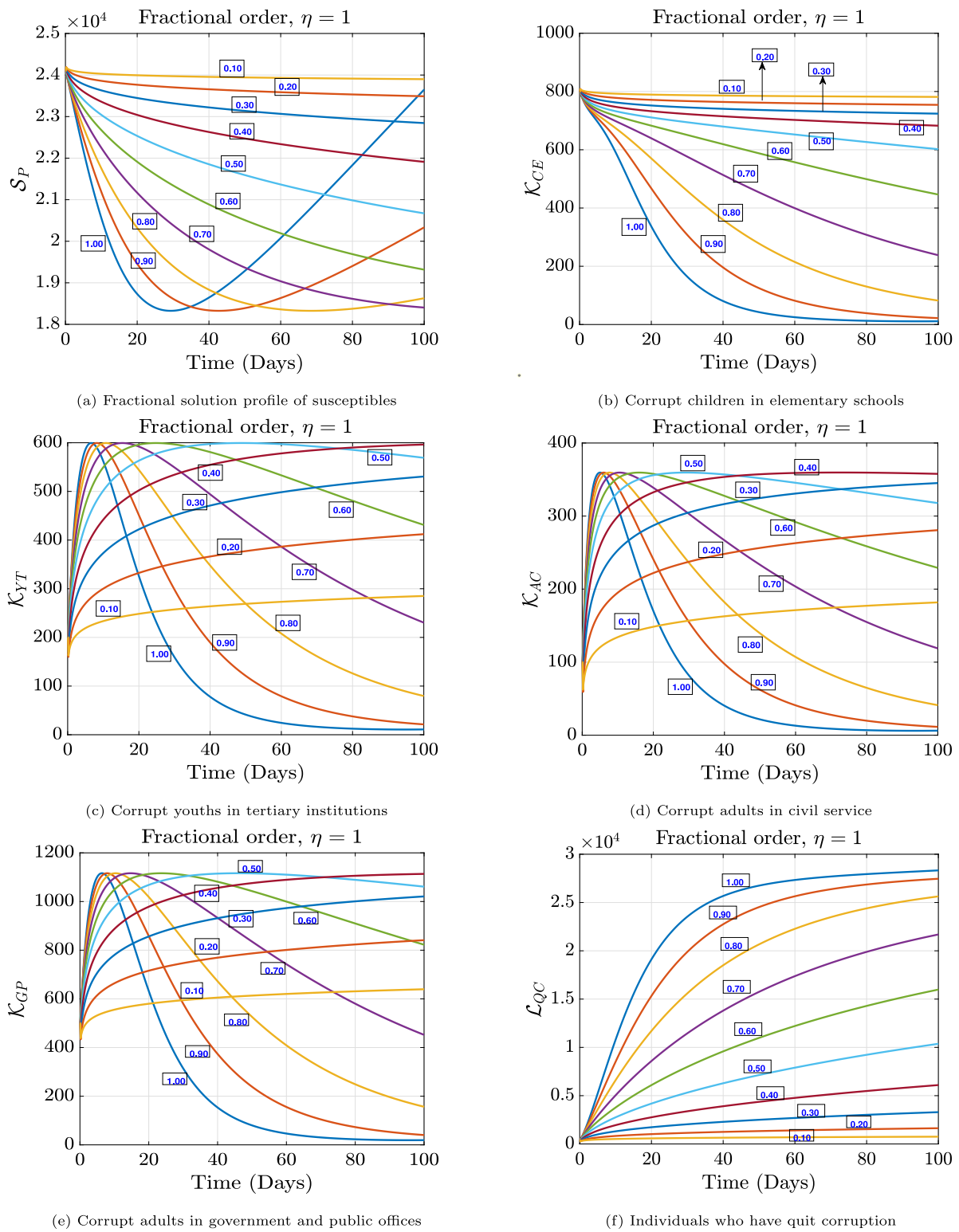


Fig. 5. Fractal numerical simulation of the corruption model with 0.10 deference in fractional order up to 1.0, when one fixed the fractional order as one.

behaviour in corruption dynamics when the model’s memory aspect is varied in small orders. Fig. 7(a) shows a high increase in susceptibles when the memory aspect of corruption is increased. Fig. 7(b) shows that the rate of corruption among elementary school kids goes up quickly as the fractional order goes from 0.9 to 0.4 and then slows down as it goes from 0.3 to 0.1. Figs. 7(c)–7(e) show that corruptions increase rapidly when the fractional order moves from 0.9 to 0.5 and decreases when

the fractional order reaches 0.4 to 0.1. Fig. 7(f) shows that a change in the fractional order with a difference of 0.1 does not change the pattern of individuals who quit corruption as the fractal order varied from 0.90 to 0.1.

Figs. 8–12 shows the model’s sensitivity analysis of key parameters. In Fig. 8, we noticed that an increase in corruption rate (contact rate) increases the number of susceptible individuals who become corrupt,

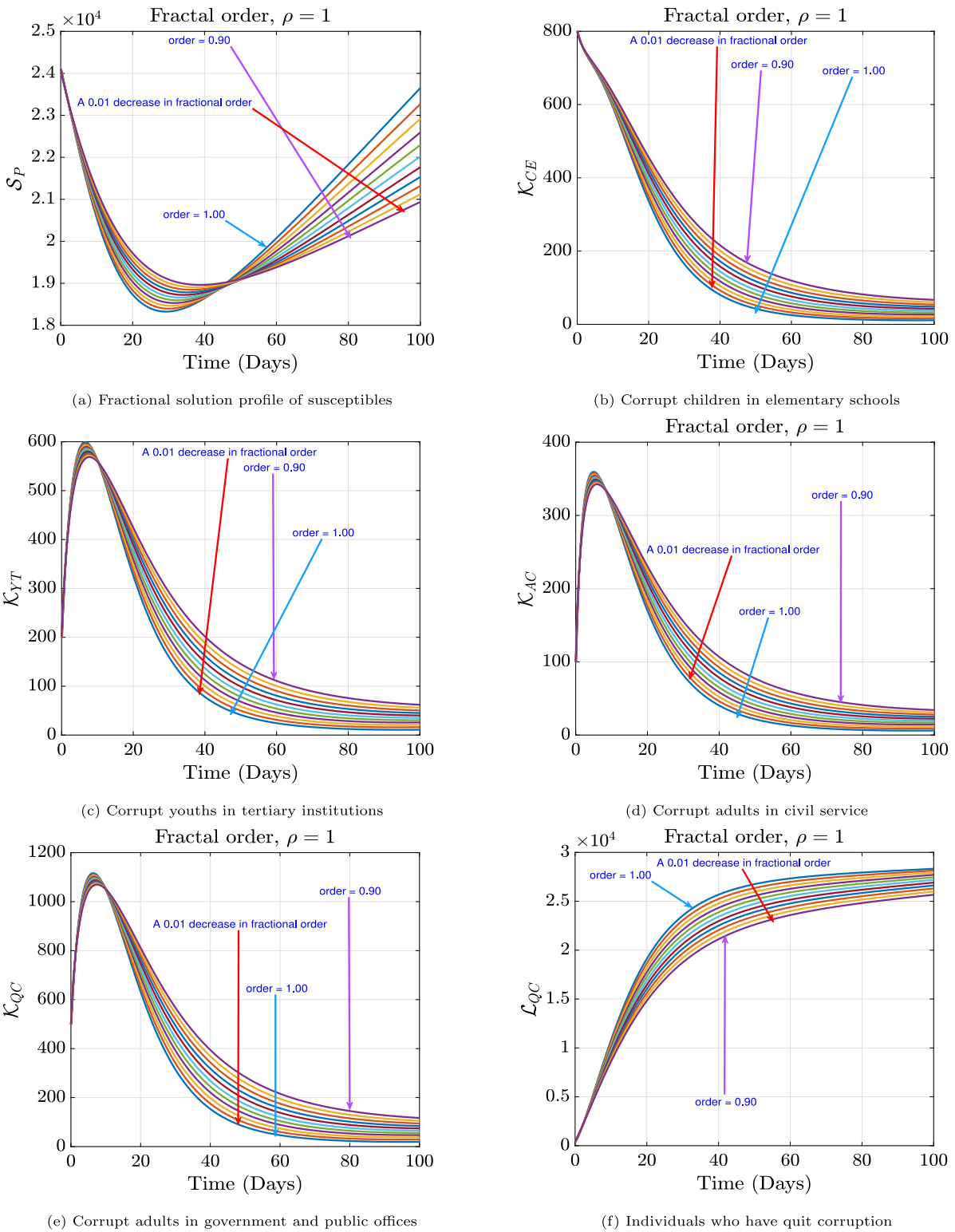


Fig. 6. Fractional numerical simulation of the corruption model with 0.01 deference in fractional order up to 0.90, when one fixed the fractional order as one.

hence leading to an increase in corruption peak in corrupt children in elementary schools, corrupt youths in tertiary institutions, corrupt adults in civil service, and corrupt adults in government and public offices. In Fig. 9, we varied the corrupt quitting rate in children in elementary schools and noticed that reducing the corruption quitting rate in children in elementary schools increases the progression rate of corruption in all the corruption-induced subclasses. In Fig. 10,

we varied the corrupt quitting rate in youths in tertiary institutions and noticed that reducing the corruption quitting rate in youths in tertiary institutions increases the progression rate of corruption in all the corruption-induced subclasses. In Fig. 11 we varied the corrupt quitting rate in adults in the civil service and noticed that reducing the corruption quitting rate in adults in the civil service increases the progression rate in adults in the civil service but has a minimal

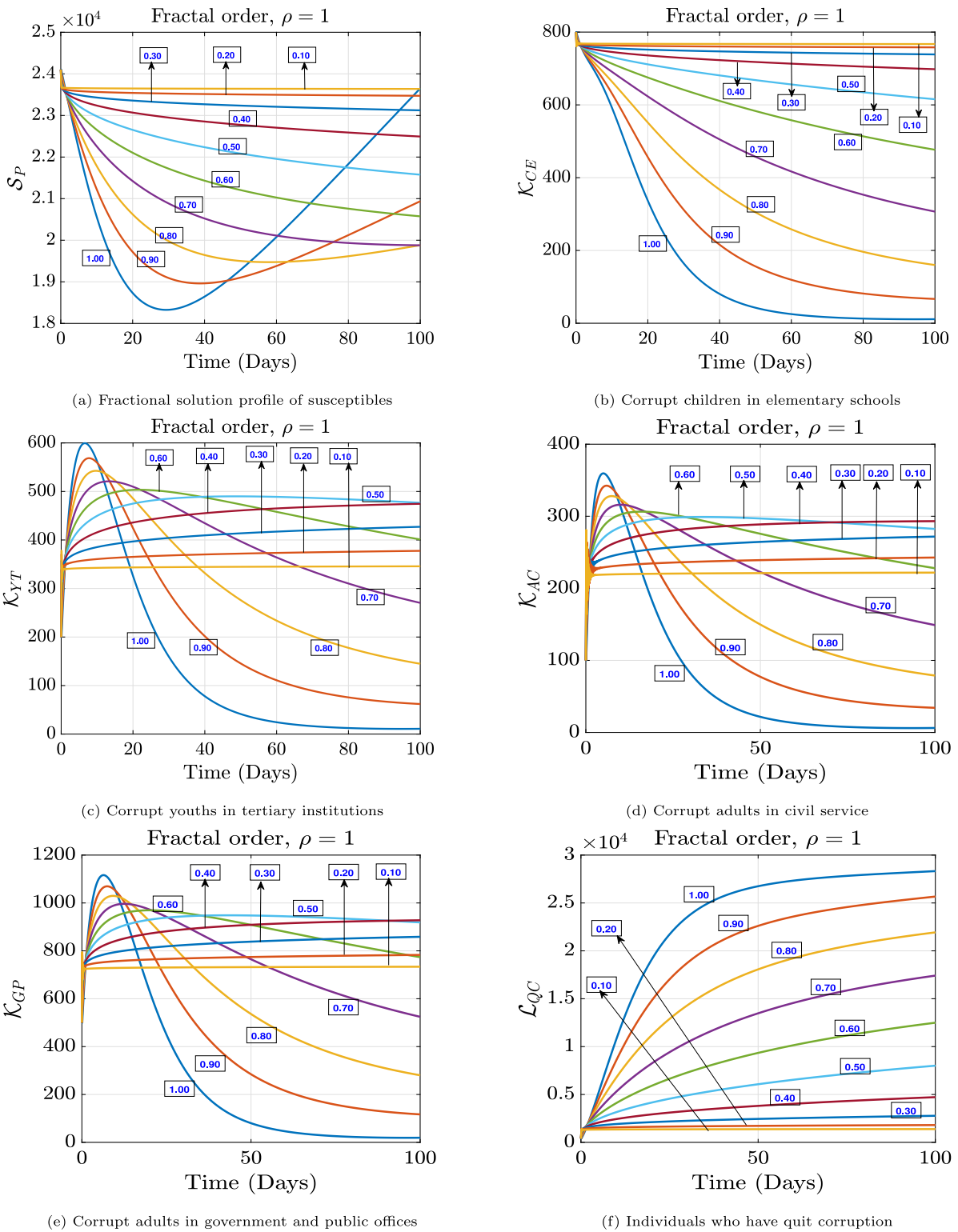


Fig. 7. Numerical simulation under hybrid derivative with fractional order $c_1 = 1.00, 0.95, 0.90, 0.85$, with constant fractal dimension.

effect on other corruption-induced subclasses. In Fig. 12, we varied the corrupt quitting rate in adults in government and public offices and noticed that reducing the corruption quitting rate in adults in government and public offices reduces the number of individuals who quit corruption but does not have any effects on all the corruption

induced subclasses. Hence, we did not show it here. Therefore, from the sensitivity analysis, it is advised that to reduce corruption in all the corruption-induced subclasses, one needs to increase the advocacy rate in quitting corruption in all population sectors.

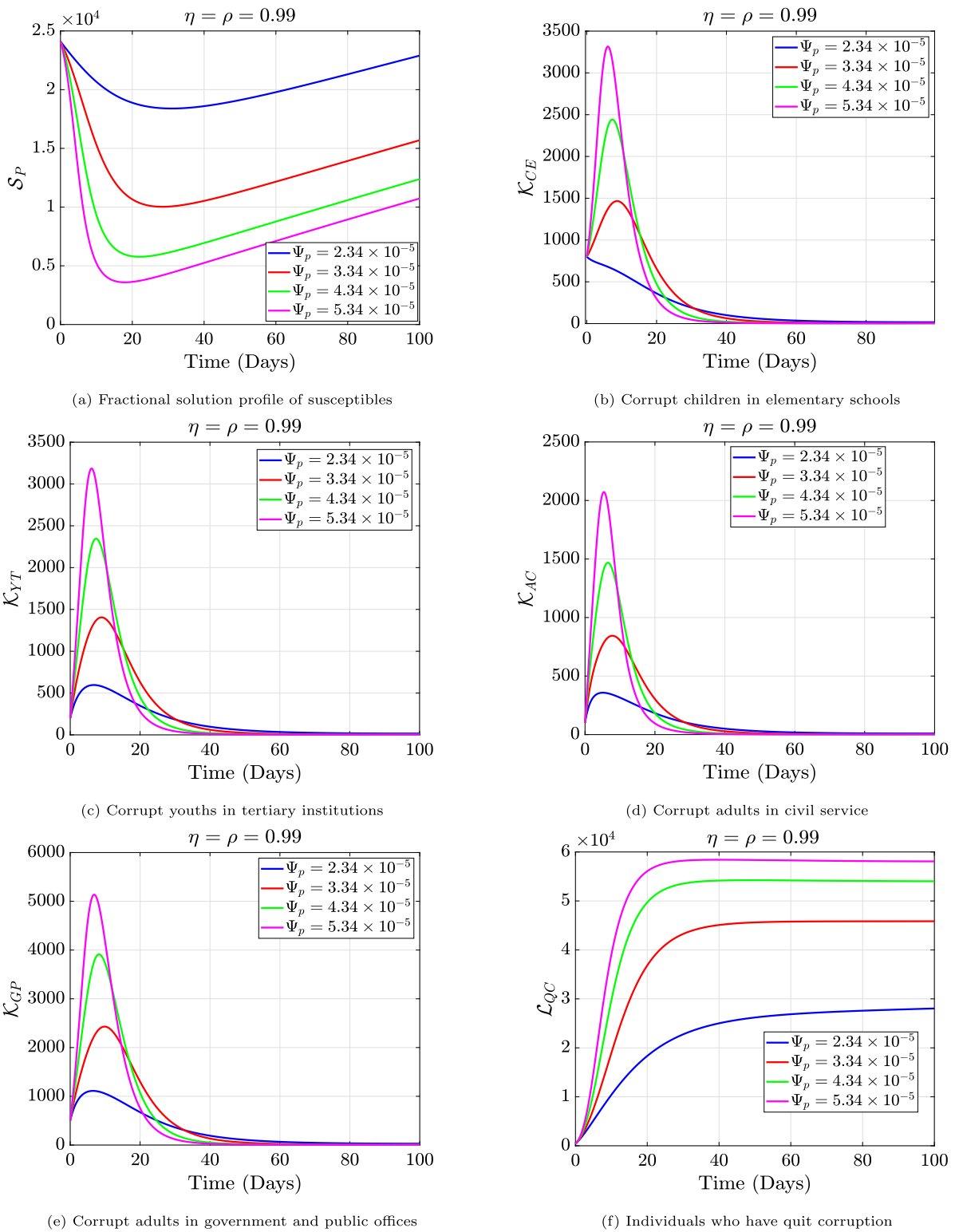


Fig. 8. Numerical simulation of varying the influence of corruption on others (contact rate) with fractal-fractional of order 0.99.

Conclusion

Though mathematical models do not present a quick fix to this phenomenon, corruption within society, however, mathematical models replicate a possible positive direction to the mayhem. Corruption is endemic and embedded in every social institution in society. The corruption witnessed in so many institutions is pariah, obnoxious, and

preposterous and has destroyed so many livelihoods without any means of remedy. While society awaits a possible way out of this menace, we have presented in this manuscript the mathematical model using classical calculus and fractional derivative and integrals to mimic the spread of corruption within the population. Banach contraction principle and Krasnoselski fixed point theorem were applied to establish the existence and uniqueness of the considered model solution, and

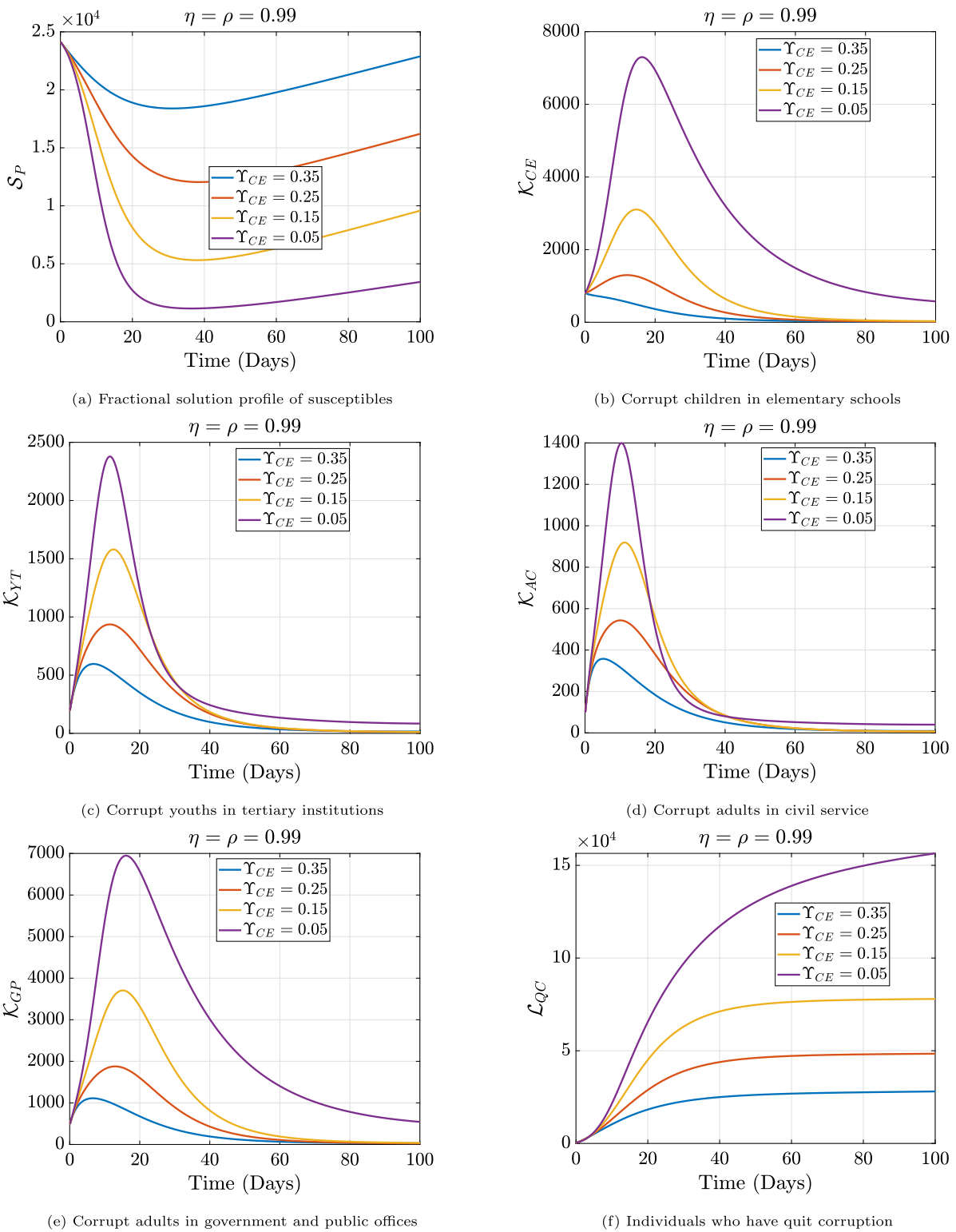


Fig. 9. Numerical simulation of varying the corruption quitting rate in elementary school children using a fractal–fractional of order 0.99.

the model solution was established to be Ulam–Hyers stable. In the simulation, the use of fractal–fractional derivative and integral indicates that as the fractal–fractional order decreases simultaneously, the number of susceptibles to corruption increases, leading to an increase in the number of corruption in the population at the prescribed sectors.

Some of our simulations suggest that eradicating corruption in society will prove challenging as long as people have in mind the benefits of involvement in corruption and have a repeated pattern. However, it is advised that to reduce corruption in all the corruption-induced subclasses, we need to increase the rate of awareness and advocacy in

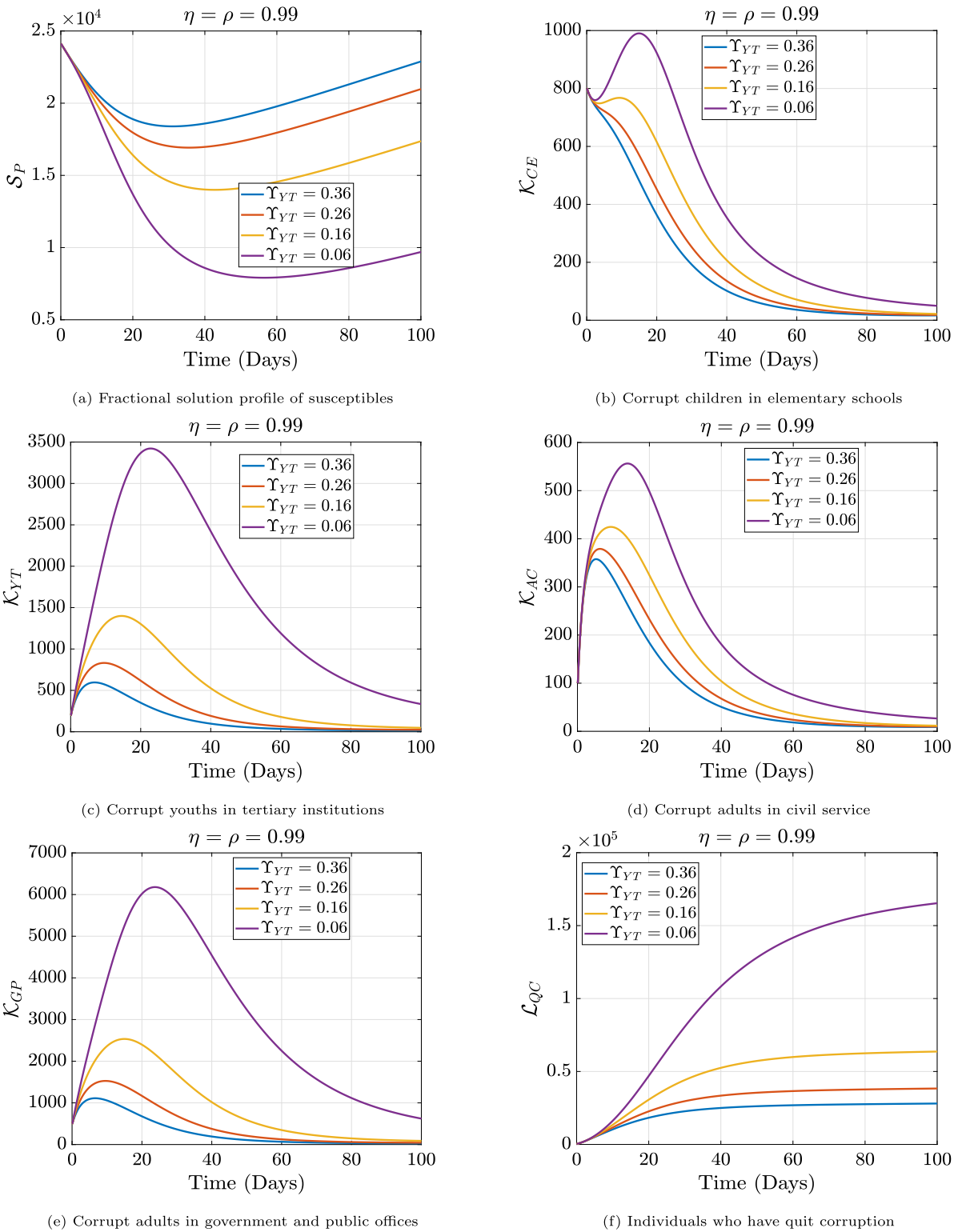
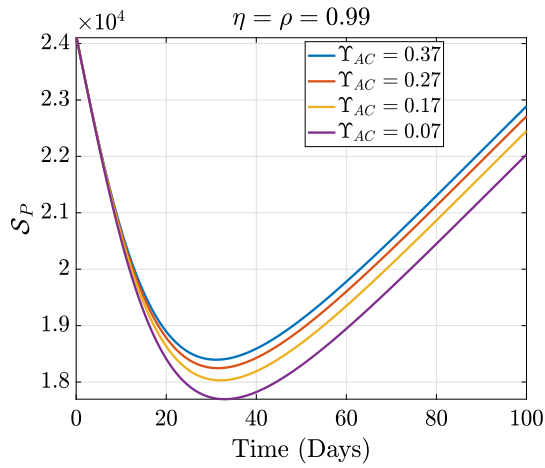
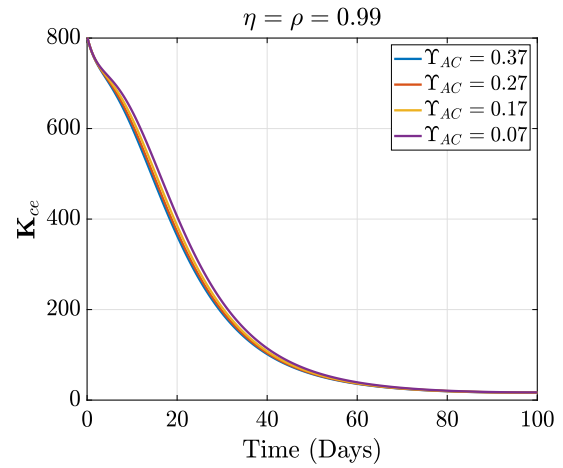


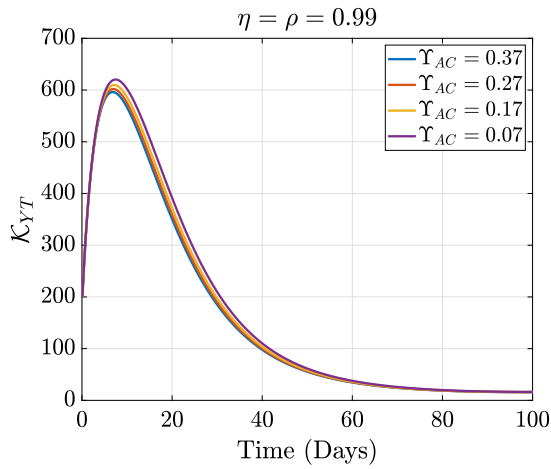
Fig. 10. Numerical simulation of varying the corruption quitting rate in youths in tertiary institutions using a fractal-fractional of order 0.99.



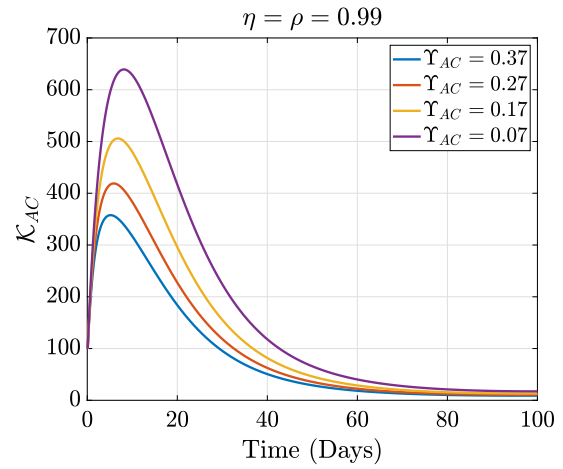
(a) Fractional solution profile of susceptibles



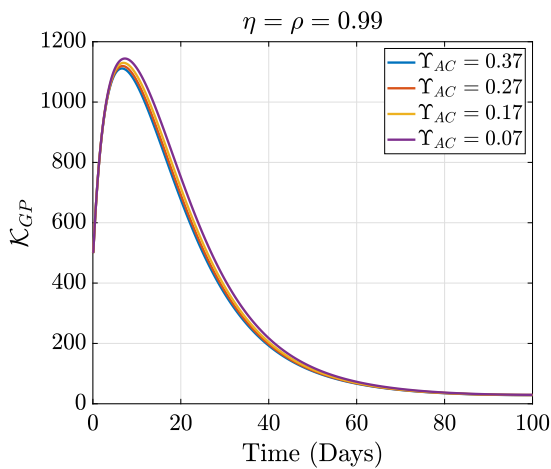
(b) Corrupt children in elementary schools



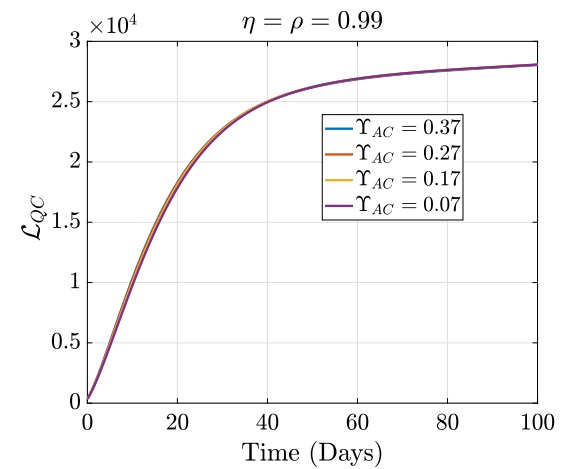
(c) Corrupt youths in tertiary institutions



(d) Corrupt adults in civil service



(e) Corrupt adults in government and public offices



(f) Individuals who have quit corruption

Fig. 11. Numerical simulation of varying the corruption quitting rate adults in civil service using a fractal-fractional of order 0.99.

Table 1
Description of parameters in the model (1).

Parameter	Description	Value	References
Ψ_p	contact rate	2.34×10^{-5}	Variable
Φ_p	Recruitment rate	100	Variable
μ_p	Natural death rate	$\frac{1}{(65 \times 365)}$	Variable
$\Omega_{CE}, \Omega_{YT}, \Omega_{AC}$	progression rates	0.1, 0.2, 0.3	Variable
$\Upsilon_{CE}, \Upsilon_{YT}, \Upsilon_{AC}$	Corruption quitting rate in $\mathcal{K}_{CE}, \mathcal{K}_{YT}$ and \mathcal{K}_{AC}	0.35, 0.36, 0.37	[17]
Υ_{GP}	Corruption quitting rate in \mathcal{K}_{GP}	0.5	Variable
S_p	Susceptibles	24103	Variable
\mathcal{K}_{CE}	Corrupt children in elementary schools	800	Variable
\mathcal{K}_{YT}	Corrupt youths in tertiary institutions	200	Variable
\mathcal{K}_{AC}	Corrupt adults in civil service	100	Variable
\mathcal{K}_{GP}	Corrupt adults in government and public offices	500	Variable
\mathcal{L}_{QC}	Individuals who have quit corruption	400	Variable

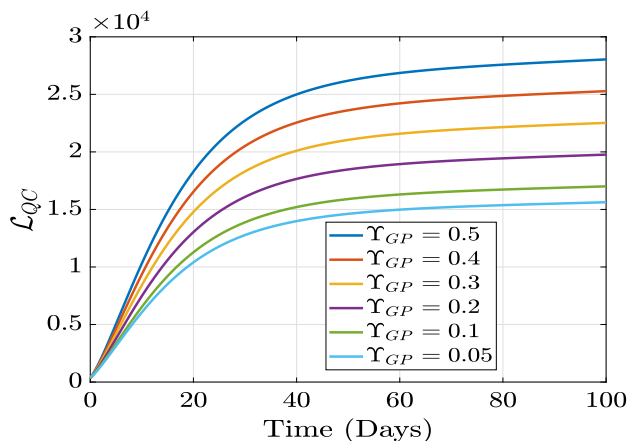


Fig. 12. Numerical simulation of varying the corruption quitting rate adults in civil service using a fractal-fractional of order 0.99.

quitting corruption in all the institutions of society. In our future work, we hope to include real data on corruption for improved analysis.

CRedit authorship contribution statement

Ugochukwu Kizito Nwajeri: Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing. **Joshua Kiddy K. Asamoah:** Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing. **Ndubuisi Rich Ugochukwu:** Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing. **Andrew Omame:** Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing. **Zhen Jin:** Supervision, Methodology, Formal analysis, Funding, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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