OPTIMAL LOAN PORTFOLIO

[USING KARMARKAR'S ALGORITHM]

[CASE STUDY: CAPITAL RURAL BANK LIMITED, DOMINASE-SUNYANI]

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DECLARATION

I, Anthony Donkor, hereby declare that except for reference to other people's work, which have duly been cited, this submission is my own work towards the Master of Science degree and that it contains no material previously published by another person nor presented elsewhere.

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DEDICATION

I dedicate this thesis with all my love and respect to my dear wife Mary Donkor and my dearest daughter Nhyira Yaa Agyeiwaa Donkor for their love, support and understanding throughout the study of this course.

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ABSTRACT

In Ghana, banking industry is now characterized by increasing competition and innovation. This has made most of the banks to adopt scientific approach to improve the quality of their loan structure. The decline of relevant portfolio planning models especially in Ghana is attributed mainly to the evolving dynamics of the Ghanaian banking industry where the regulatory controls have changed with a high frequency. Due to the model used in allocating funds to various loan types, a lot of banks had suffered substantial losses from a number of bad loans in their portfolio. As a result, most banks are not able to maximize their profit on loans due to poor allocation of funds. The purpose of this study is to develop a linear programming model using the Karmarkar's projective scaling algorithm to help Capital Rural Bank Limited, Dominase-Agency (Sunyani) to maximize their profit on loans. The results from the model showed that, Capital Rural Bank Limited, Dominase-Agency (Sunyani) would be making annual profit of **GH¢5,961,300.00** if they are to stick to the model. From the study, it was realized that the scientific method used to come out with this model can have a dramatic increase in the bank's profit on loans if put into practice.

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CHAPTER ONE

INTRODUCTION

Lending is the principal business activity for most commercial banks. The loan portfolio is typically the largest asset and the predominate source of revenue. As such, it is one of the greatest sources of risk to a bank's safety and sound. Effective management of the loan portfolio and the credit function is fundamental to a bank's safety and soundness. Portfolio problems have historically been the major cause of bank losses and failures. The lifeblood of each lending institution is its credit portfolio, and the success of the institutions depends on how well that portfolio is managed (Harold, 1998).

Banking in the modern sense can be traced to medieval and early Renaissance Italy, to the rich cities in the north like Florence, Venice and Genoa. The Bardi and Peruzzi families dominated banking in 14th century Florence, establishing branches in many other parts of Europe. Perhaps the most famous Italian bank was the Medici bank, set up by Giovanni Medici in 1397. The earliest known state deposit bank, Bancodi San Giorgio (Bank of St. George), was founded in 1407 at Genoa, Italy. (Wikimedia Foundation Inc, 2012).

1.0 BACKGROUND TO THE STUDY

Assessing loan portfolio management involves evaluating the steps bank management takes to identify and control risk throughout the credit process. The assessment focuses on what management does to identify issues before they become problems. For decades, good credit portfolio managers have concentrated most of their effort on prudently approving loans and carefully monitoring loan performance. Although these activities continue to be mainstays of loan portfolio management, analysis of past credit problems, such as those associated with oil and gas lending, agricultural lending, and commercial real estate lending in the 1980s, has made it clear that portfolio managers should do more. (Acerbi,2001). Traditional practices rely too much on trailing indicators of credit quality such as delinquency, nonaccrual, and risk rating trends. Banks have found that these indicators do not provide sufficient lead time for corrective action when there is a systemic increase in risk. Effective loan portfolio management begins with oversight of the risk in individual loans.

The banking sector of Ghana in the past could be divided into two groups- the elite foreign banks which concentrated on the rich of the society and the local banks mainly owned by the state. The latter served the interest of most working class people. The elite banks were Barclays Bank (formerly called the Colonial Bank) and Standard Chartered Bank (formerly, Bank of British West Africa). The second group of banks with state ownership include Ghana Commercial Bank (GCB), Social Security Bank (now SG-SSB), Agricultural Development Bank (ADB), and the National Investment Bank (NIB). The clients of the locally owned banks found business transactions very frustration especially during salary payments, for example, it was not uncommon observing long winding queues extending several meters outside the banking hall. The few foreign banks on the other hand, apply high charges and the initial deposit to open accounts was very high. The average Ghanaian could therefore not open accounts with these banks. Choices were very few and competition was virtually absent in the sector.

Bank of Ghana (BoG) with the support of government undertook a process of financial sector restructuring which transformed the financial sector. Some of the initiatives that led this transformation are the movement to universal banking, the adoption of an open licensing system and the modernization of the payments systems. According to Dr. Paul Acquah, who was the governor of the BoG, ´universal banking involves the removal of restrictions on banking activities which allow banks to choose the type of banking services that they would like to offer in line with their capital, risk appetite and their business orientation'(2006). Ghana's financial system is based on a number of banks and non-banking financial institutions, including the Bank of Ghana, which, as the Central Bank, has the responsibility of advising the government on the implementation and control of monetary policies. Other institutions include commercial and merchant banks, discount houses, insurance companies, leasing companies, venture capital, a mortgage finance institution, and a stock exchange. Direct financing of projects in the country is provided by the commercial and other banking institutions. In an effort to ensure systematic development of the banking system, the Central Bank (Bank of Ghana), in addition to its traditional functions (for example formulation of monetary policies), also has the responsibility to ensure that banking is responsive to the needs of the public. In attempt to encourage the establishment of new types of financial institutions, the Bank of Ghana pursues a liberal policy with regard to entry into the banking system, and is actively involved in the promotion of development and rural banking as well as in the establishment of discount houses.

Lending is one of the main activities of banks in Ghana and other parts of the world. This is evidenced by the volume of loans that constitute banks assets and the annual substantial increase in the amount of credit granted to borrowers in the private and public sectors of the economy.

Lending is the principal business for most commercial banks. Loan portfolio is therefore typically the largest asset and the largest source of revenue for banks in Ghana. In view of the significant contribution of loans to the financial health of banks through interest income earnings, these assets are considered the most valuable assets of banks.

The Ghanaian financial services market offers significant opportunities with the recent oil production, trends in global commodity prices, influx of Foreign Direct Investments and growth in Private Sector activities within the local market. These trends offer good prospects amidst the perennial challenges like high cost of borrowing and increases in bad loans. Interest income continues to constitute the most significant component of income derived from operations. In recent years, interest margins are gradually shrinking as competition becomes intense. A greater focus on a transaction based revenue portfolio, which is often unfunded and less prone to default, has to be considered. Going forward, players in the industry needs to transform their Governance procedures, Operating models, Information Security and Risk management activities to align them with the global trends. These changes are necessary to sustain the growing opportunities within the local market. Banks should deepen current strategies aimed at diversifying current portfolio funding from Large and Local Corporate to expanding their Retail and Small and Medium Enterprise (SME) market segment which is still very much underserved. Banks must also continue to develop products that focus on these unbanked population with the aim of converting them into regular bank account holders. In a bid to develop a long term funding portfolio and reduce the high priced purchased funds, individuals should be encouraged to develop a savings culture (Ghana Banking Survey-2011).

A survey in 2011 on the Ghanaian banking sector revealed that the rebound in profitability was propelled by improvements in the credit markets which allowed banks to improve quality of loan portfolio and release provisions set aside in the previous years for credit losses. Growth in operating assets, non-interest bearing deposit and wider interest margin also contributed to the favorable profitability.

Credit is the provision of resources (such as granting a loan) by one party to another party where that second party does not reimburse the first party immediately, thereby generating a debt, and instead arranges either to repay or return those resources (or material(s) of equal value) at a later date. It is any form of deferred payment. The first party is called a creditor, also known as a lender, while the second party is called a debtor, also known as a borrower. Movements of financial capital are normally dependent on either credit or equity transfers. Credit is in turn dependent on the reputation or creditworthiness of the entity which takes responsibility for the funds. Credit is denominated by a unit of account. Unlike money (by a strict definition), credit itself cannot act as a unit of account. However, many forms of credit can readily act as a medium of exchange. As such, various forms of credit are frequently referred to as *money* and are included in estimates of the money supply (Finlay, 2009).

1.1 TYPES OF BANKS

There are different types of banks which operate in the country to meet the financial requirements of various categories of people engaged in business, agriculture, profession, etc. On the basis of functions, the banking institutions in Ghana may be divided into the following types; (Wikimedia Foundation Inc, 2012).

1.1.1 Central Bank

A **central bank**, **reserve bank**, or **monetary authority** is a public institution that manages a state's currency, money supply, and interest rates. Central banks also usually oversee the commercial banking system of their respective countries. In contrast to a commercial bank, a central bank possesses a monopoly on increasing the nation's monetary base, and usually also prints the national currency, which usually serves as the nation's legal tender. Examples include Bank of Ghana (BoG), the European Central Bank (ECB), the Federal Reserve of the United States, and the People's Bank of China.

The primary function of a central bank is to manage the nation's money supply (monetary policy), through active duties such as managing interest rates, setting the reserve requirement, and acting as a lender of last resort to the banking sector during times of bank insolvency or financial crisis. Central banks usually also have supervisory powers, intended to prevent commercial banks and other financial institutions from reckless or fraudulent behavior. Central banks in most developed nations are institutionally designed to be independent from political interference. (Wikimedia Foundation Inc, 2012).

1.1.2 Commercial Banks

A **commercial bank** (or **business bank**) is a type of financial institution and intermediary. It is a bank that lends money and provides transactional, savings, and money market accounts and that accepts time deposits. A commercial bank is owned by a group of individuals or an individual or the state, for-profit entity that is licensed by the Central Bank (Bank of Ghana) to conduct overall banking activities.

Commercial banks engage in the following activities:

- processing of payments by way of telegraphic transfer, internet banking, or other means.
- issuing bank drafts and bank cheques
- accepting money on term deposit
- lending money by overdraft, installment loan, or other means
- providing documentary and standby letter of credit, guarantees, performance bonds, securities underwriting commitments and other forms of off balance sheet exposures
- safekeeping of documents and other items in safe deposit boxes
- cash management and treasury

Examples are: Ghana Commercial Bank, Standard Chartered Bank (Gh) Ltd., Barclays Bank (Gh) Ltd., SG-SSB Limited, Metropolitan Allied Commercial Bank, the Trust Bank, Zenith Bank, Intercontinental Bank, Standard Trust Bank, Fidelity Bank, Guaranty Trust Bank (Ghana) Limited, (Wikimedia Foundation Inc, 2012).

1.1.3 Development Banks

The Development Bank fosters, empowers and finances up coming and already exiting ventures. The Bank provides finance for private sector start-ups and expansions, equity deals, bridging finance, enterprise development finance, trade finance, small and medium enterprises, public private partnerships, public sector infrastructure, local authorities, and bulk finance to responsible micro-finance providers. Examples are National Investment Bank (NIB), Agricultural Development Bank (ADB), International Commercial Bank, the Trust Bank, Prudential Bank, Amalgamated Bank, Ghana Commercial Bank, ARB Apex Bank,.

1.1.4 Merchant Banks

Merchant bank is a financial institution primarily engaged in offering financial services and give advice to corporations and to individuals. The term can also be used to describe the private equity activities of banking. The Merchant bank is that a merchant bank invests its own capital in client companies. Merchant banks provide fee-based corporate advisory services, including those in mergers and acquisitions. Examples are; Merchant Bank of Ghana Ltd., Ecobank Ghana Ltd., Continental Acceptances Ltd. and First Atlantic Merchant Bank, CAL Bank, HFC Bank, (Wikimedia Foundation Inc, 2012).

1.2 TRADE CREDIT

The word credit is used in commercial trade in the term "trade credit" to refer to the approval for delayed payments for purchased goods. Credit is sometimes not granted to a person who has financial instability or difficulty. Companies frequently offer credit to their customers as part of the terms of a purchase agreement. Organizations that offer credit to their customers frequently employ a credit manager.

1.3 CONSUMER CREDIT

Consumer debt can be defined as money, goods or services provided to an individual in lieu of payment 'Common forms of consumer credit include credit cards, store cards, motor (auto) finance, personal loans (installment loans), retail loans (retail installment loans) and mortgages. This is a broad definition of consumer credit and corresponds with the Bank of England's definition of "Lending to individuals". Given the size and nature of the mortgage market, many observers classify mortgage lending as a separate category of personal borrowing, and consequently residential mortgages are excluded from some definitions of consumer credit - such as the one adopted by the Federal Reserve in the US. The cost of credit is the additional amount, over and above the amount borrowed, that the borrower has to pay. It includes interest, arrangement fees and any other charges. Some costs are mandatory, required by the lender as an integral part of the credit agreement. Other costs, such as those for credit insurance, may be optional. The borrower chooses whether or not they are included as part of the agreement. Interest and other charges are presented in a variety of different ways, but under many legislative regimes lenders are required to quote all mandatory charges in the form of an annual percentage rate (APR). The goal of the APR calculation is to promote truth in lending', to give potential borrowers a clear measure of the true cost of borrowing and to allow a comparison to be made between competing products. The APR is derived from the pattern of advances and repayments made during the agreement. Optional charges are not included in the APR calculation. So if there is a tick box on an application form asking if the consumer would like to take out payment insurance, then insurance costs will not be included in the APR calculation (Finlay, 2009).

1.4 DIFFERENCE BETWEEN LOANS AND CREDITS

While the terms Credit and Loans are often used interchangeably, there is an important distinction between the two. **Credit** in general refers to promissory notes backed by a promise/contingency. Merchant and commercial banks issue trade credit in amounts that exceed their net worth. These notes are typically backed by future output. The money supply in most Western industrialized nations consists almost entirely of these promissory notes. The important point to make is that they are not backed by a real asset (gold, bank assets). **Loan** on the other hand is the flip side to savings. Consumers and firms deposit a portion of their income (promissory notes) at banks which then loan it out to borrowers.

The key distinction is the very nature of the instrument that the banker uses. In the case of credit, he issues new promissory notes. In the case of loans, he lends out promissory notes deposited by other customers. The current crisis in the U.S. banking sector is credit, not loan related. Banks are more reticent to issue new promissory notes to small and medium size firms, thus hampering the recovery.

1.5 PROFILE OF CAPITAL RURAL BANK

Capital Rural Bank has its headquarters at Abesim near Sunyani (Brong – Ahafo) and has four (4) other agencies at Dominase-Sunyani, Fiapre, Magazine and New Dormaa in Sunyani Metropolis.

Capital Rural Bank was ranked first in terms of share capital in Brong – Ahafo region and fourteenth (14^{th}) nationwide according to 4^{th} quarter (2011) report on the performance of Rural and Community Banks (RCBs) carried out by ARB Apex Bank's Monitoring Unit. Capital Rural

Bank has a mission of providing quality services to customers since good customer service is the lifeblood of any business.

1.6 STATEMENT OF THE PROBLEM

Capital Rural Bank has the welfare of its customers at heart, hence it provides a flexible loan payment term across all the types of loans at their door steps. As time went on, it was identified by Management Board of the bank, that a section of loan types always end up in bad debt, both the principal and the interest which can never being retrieved. Management of the bank decided to give loan to its customers on just a few number of loan types which can be retrieved. Furthermore, some customers bank in a particular bank because of its favourable and reliable loan policy in their favour, so when this favour does not exist anymore he or she (customer) finds its way to other banks. This reduces the number of account holders as against the amount of money the bank (Capital Rural Bank) generate as profit which affect the development of the bank.

The main aim of this project is to propose a linear model subject to some constraints for Capital Rural Bank to enable them disburse their funds allocated for loans optimally leading to maximization of profits.

1.7 OBJECTIVES

The main objective of this study is to determine the optimal allocation of funds for loans for Capital Rural Bank Limited leading to maximization of profit.

1.8 METHODOLOGY

In order for Rural Banks to maximize their profits, the proposed model was based on their loan policy and its previous history on loan disbursement.

For this thesis to be successful and achieve its goals a secondary data on loan formulation portfolio was collected from Capital Rural Bank and a linear programming model (interior point method) is used to model the problem. The potential reduction algorithm (karmarkar's algorithm) was used to solve the model. Resource materials were obtained from required books, Matlab (the programming language) and more importantly the internet. The linear programming model has three (3) basic components that is the objective function which is to maximized, the constraints or limitation and the non-negativity constraints.

1.9 JUSTIFICATION

The institution of Banks is one of the fastest growing institutions in the Ghana which has a tremendous impact on the economy and the society. The bank might run at a lost or even collapse if they are not able to retrieve all the loans they give out. Due to this, a more scientific approach must be employed by banks to ensure adequate, effective and efficient distribution of funds they have available for loans to ensure constant growth of these banks. When banks run efficiently they are able to allocate a larger amount of its funds for social services in the community in which they operate.

The proposed model is going to help banks to efficiently distribute the funds they have available for loan in order to maximize their profit. The proposed model will also help decision makers at the Bank to formulate prudent and effective loan policies. This makes this study justifiable and worthwhile.

1.10 ORGANISATION OF THE THESIS

The thesis consists of five (5) chapters where Chapter 1 throws light on the introduction, problem statement, and objectives of the thesis, justification, methodology and structure of the thesis. Chapter 2 reviews work done by other people on the topic (literature review). Chapter 3 contains the method used to carry out this research. Chapter 4 talks about the analysis and results. Finally chapter five (5) contains conclusion and recommendation.

CHAPTER TWO

LITERATURE REVIEW

2.0 INTRODUCTION

In this section of the work, other people's works, journals of various fields of research using linear programming programs would be considered.

Stewart et al., (2008) examined the numerical implementation of a linear programming (LP) formulation of stochastic control problems involving singular stochastic processes. The decision maker has the ability to influence a diffusion process through the selection of its drift rate (a control that acts absolutely continuously in time) and may also decide to instantaneously move the process to some other level (a singular control). The first goal of the paper is to show that linear programming provides a viable approach to solving singular control problems. A second goal is the determination of the absolutely continuous control from the LP results and is intimately tied to the particular numerical implementation. The original stochastic control problem is equivalent to an infinite-dimensional linear program in which the variables are measures on appropriate bounded regions. The implementation method replaces the LP formulation involving measures by one involving the moments of the measures. This moment approach does not directly provide the optimal control in feedback form of the current state. The second goal of the paper is to show that the feedback form of the optimal control can be obtained using sensitivity analysis.

Consider a linear-programming problem in which the "right-hand side" is a random vector whose expected value is known and where the expected value of the objective function is to be minimized. An approximate solution is often found by replacing the "right-hand side" by its expected value and solving the resulting linear programming problem. Mandansky (1960) gave conditions for the equality of the expected value of the objective function for the optimal solution and the value of the objective function for the approximate solution; bounds on these values were also given. In addition, the relation between this problem and a related problem, where one makes an observation on the "right-hand side" and solves the (non stochastic) linear programming problem based on this observation, was discussed.

Sinha et al., (2003) proposed a modified fuzzy programming method to handle higher level multi-level decentralized programming problems (ML (D) PPs). They presented a simple and practical method to solve the same. This method overcomes the subjectivity inherent in choosing the tolerance values and the membership functions. They considered a linear ML (D) PP and applied linear programming (LP) for the optimization of the system in a supervised search procedure, supervised by the higher level decision maker (DM). The higher level DM provides the preferred values of the decision variables under his control to enable the lower level DM to search for his optimum in a narrower feasible space. The basic idea is to reduce the feasible space of a decision variable at each level until a satisfactory point is sought at the last level.

Wu et al., (2000) proposed a neural network model for linear programming that is designed to optimize radiotherapy treatment planning (RTP). This kind of neural network can be easily implemented by using a kind of `neural' electronic system in order to obtain an optimization solution in real time.

Jianq et al., (2004) proposed a novel linear programming based method to estimate arbitrary motion from two images. The proposed method always finds the global optimal solution of the linearized motion estimation energy function and thus is much more robust than traditional motion estimation schemes. As well, the method estimates the occlusion map and motion field at the same time. To further reduce the complexity of even a complexity-reduced pure linear programming method they presented a two-phase scheme for estimating the dense motion field. In the first step, they estimated a relatively sparse motion field for the edge pixels using a non-regular sampling scheme, based on the proposed linear programming method. In the second step, they set out a detail-preserving variational method to upgrade the result into a dense motion field. The proposed scheme is much faster than a purely linear programming based dense motion estimation scheme.

Banks that are considering offering funding to potential borrowers are faced with an information problem, to the extent that borrowers are usually more informed about their own firms than lenders. As a result, banks often wind up approving some loans that are ex-post unprofitable. These informational asymmetries lead to credit rationing equilibria (e.g., Stiglitz and Weiss, 1981). In addition, they may invalidate other standard competitive market results and affect the bank's loan and credit allocation. In this paper, we build an integrated theoretical model that describes the bank's loan allocation between different economic sectors. In this model, we attempt to integrate various aspects of the bank's loan allocation decision that have been discussed separately, but which have not been previously integrated into an equilibrium model. Our model discusses the following issues: uncertainty, relationship banking, industrial organization of the banking sector, bank and borrower characteristics, monitoring and switching costs.

Although in this paper our primary interest is a bank's loan allocation between economic sectors, it is useful to begin with a brief description of the theoretical and empirical literature on bank credit allocation between private and public sectors and bank credit allocation between large and small enterprisers. The importance of these issues is clear: Bank lending constitutes a major source of funding - both for individuals as well as for smaller businesses. In a business firm the objective is to maximize its shareholders' wealth, from the point of view of Finance. In order for this objective to be accomplished all the resources and all the assets of the bank both short-term and long-term should be used as efficiently as possible, with the least cost and the maximum return.

In working capital management, which concerns the short term decision of a firm according to Almgren and Chriss (2001) we distinguish four areas of importance from the point of view of practitioners and academicians, that play a very important role in the value of the firm and the owners' wealth. These are cash management, accounts receivable and accounts payable management and inventory management.

Liquidity is a major factor that determines the health of a company. Profitability only is not enough. A firm can have successful sales and show profits at the end of the year, but might suffer from lack of cash in the short run, whenever, it is needed. This situation, which is observed very often among Ghanaian banks, could lead the management of a company in seeking external sources of funds, which eventually will increase their cost of capital and financial risk, hence lowering their profitability. The financial managers and academicians have realized the importance of liquidity and efficiency in liquid assets management during the last 20 years in the USA and Europe. In Ghana, the emphasis in liquidity has been relatively new, since in general, the emphasis in working capital management is recent. At present, though, it has become obvious, that not only the working capital management strategies which a firm follows affect its shareholders' wealth and the firm's value significantly, but also, that they can be used as a competitive advantage against the other companies of their industry.

In every field of study, it is possible to look back and identify a person or event that caused a major change in the direction or development of the field. In the field of "investments" in general and "portfolio management" in particular, it is an indisputable fact that the work by Markowitz on Portfolio Theory changed the field more than any other single event. The doctoral thesis written by Markowitz (1952) at the University of Chicago dealt with portfolio selection and in it he developed the basic portfolio model. Because of this work, Markowitz is often referred to as the "father of modern portfolio theory", and much subsequent research had been based on this effort (Melnik, 1970).

The basic model, developed by Markowitz, derived the expected rate of return for a portfolio of assets and an expected risk measure. Markowitz showed that the variance of the rate of return was a meaningful measure of risk under a reasonable set of assumptions and derived the formula for computing the variance of the portfolio. This portfolio variance formulation indicated the importance of diversification for reducing risk, and showed how to properly diversify. The Markowitz model is based on certain assumptions. Under these assumptions, a single asset or portfolio of assets is considered to be efficient if no other asset or portfolio of assets offers higher expected return with the same (or lower) risk, or lower risk with the same (or higher) expected return (Markowitz, 1952 and 1959)

2.1 THE EFFICIENT FRONTIER

The Full Variance Model developed by Markowitz is based on the assumption that the purpose of portfolio management is to minimize variance for every possible combination of the expected yield (Best and Grauer, 1991). It has been argued by Blume (1970) and Hodges and Brealey (1972) that the concept of the efficient frontier is basic to the understanding of portfolio theory. Assume that in the market place, there are a fixed number of common stocks in which a businessman can invest. Each of the securities has its own expected yield and standard deviation; others have the same standard deviation but vary in expected yield. The investor will select the security that offers the highest yield for a chosen level of risk exposure as presented by the standard deviation. It is assumed that investors try to minimize risk by minimizing the deviation from the expected yield, and this is done by means of portfolio diversification.

2.2 LINEAR PROGRAMMING IN PORTFOLIO MANAGEMENT

Linear programming is now being applied even in portfolio management. In fact, more and more computer programs are being developed to help investors maximize their expected rates of return on their stock and bond investments subject to risk, dividend and interest, and other constraints. For example, one linear programming model can be used to determine when a bond dealer or other investor should buy, sell, or simply hold a bond. The model can also be used to determine the optimal strategy for an investor to follow in order to maximize portfolio returns for each level of risk exposure. Another use of linear programming is to determine the highest return that an investor can receive from holding portfolios with various proportions of different securities. Still another use of the model is in determining which of the projects that satisfy some minimum acceptance standard should be undertaken in the face of capital rationing (i.e., when all such projects cannot be accepted because of capital limitations). The most complex portfolio management problems involving thousands of variables that leading financial management firms deal with, and that previously required hours of computer time with the largest computers to solve with the simplex method, can now be solved in a matter of minutes with the algorithm developed in 1984 by Karmarkar at Bell Labs. More important for the individual investor and small firms is that more and more user-friendly computer programs are becoming available to help solve an ever-widening range of financial management decisions on personal computers. While in the final analysis these computer programs can never replace financial acumen, they can certainly help all investors improve their planning. (*Journal of Financial and Quantitative Analysis,* December 1987, pp. 439–466).

The use of linear and other types of mathematical programming techniques has received extensive coverage in the banking literature. Chambers and Chames (1961), as well as Cohen and Hammer (1967, 1972), developed a series of sophisticated linear programming models for managing the balance sheet of larger banks, while Waterman and Gee (1963) and Fortson and Dince (1977) proposed less elegant formulations which were better suited for the small-to medium-sized bank. Several programming models have also been proposed for managing a bank's investment security portfolio, including those by Booth (1972).

2.3 RISK MANAGEMENT AND PROBABILITY OF LOSS ON LOAN PORTFOLIO

Sound credit management is a prerequisite for a financial institution's stability and continuing profitability, while deteriorating credit quality is the most frequent cause of poor financial performance and condition. The prudent management of credit risk can minimize operational risk while securing reasonable returns. Ensuring lending staffs comply with the credit union's lending license and by-laws is the first step in managing risk. The second step is to ensure board approved policies exist to limit or manage other areas of credit risk, such as syndicated and brokered loans, and the concentration of lending to individuals and their connected parties (companies, partnerships or relatives).

The board and management should also set goals or targets for their loan portfolio mix, as part of their annual planning process. The loan portfolio should be monitored on an ongoing basis, to determine if performance meets the board's expectations, and the level of risk remains within acceptable limits.

Standardized lending procedures should be adopted to reduce risk of transactional error, and ensure compliance with regulatory requirements and board policy. Approval and disbursements, documentation, lending staff and loan security are just some of the procedures recommended.
A credit union can meet standards of sound business and financial practices by ensuring it has developed and implemented credit policies, risk and performance measurement techniques, and risk management procedures, (Spring, 2005).

The traditional credit risk with the default event, the key point is data mining the characteristics of both default and non-default companies to establish the identification equations and categorize the samples. The representative model of this stage is 4C – character, capacity, collateral and condition. People try to make a full qualitative analysis about the obligators willingness and capability of payback from five aspects.

After Fisher‘s research on heuristics, there developed quickly and enormously credit risk evaluation models based on statistics, of which most represented is Edward Ahman‘s Z-score. Ahman observed manufacturing companies near or far from bankruptcy in 1968 and took 22 financial ratios to establish the most famous five Z-score based on the mathematical statistical screening.

With the fast development of information technology, recent years have seen large artificial intelligence models that have been incorporated in the credit risk analysis. For instance, neural networks as a self-organizing, self-adapting and self-learning non-parameter method are very robust and accurate in predicting especially does not rigidly follow normal.

According to Klaus Rheinberger and Martin Summer in their credit risk portfolio models, three parameters drives loan losses: The probability of default by individual obligors (PD), the loss given default (LGD) and the exposure at default (EAD). While the standard credit risk models focus on modelling the PD for a given LGD, a growing recent literature has looked closer into the issue of explaining LGD and of exploring the consequences of dependencies between PD and LGD. Most of the papers on the issue of dependency between PD and LGD have been written for US data and usually find strong correlations between these two variables. The first papers investigating the consequences of these dependencies for credit portfolio risk analysis using a credit risk model suggested by Finger (1999) and Gordy (2000). The authors used a different credit risk model in the tradition of actuarial portfolio loss models and focus directly on two risk factors: an aggregate PD and an aggregate API as well as their dependence. The authors used this approach because their interest was to investigate the implications of some stylized facts on asset prices and credit risk that have frequently been found in the macro-economic literature for the risk of collateralized loan portfolios. The authors also believe that the credit risk model we use gives us maximal flexibility with assumptions about the distribution of systematic risk factors.

There are a variety of models that try to capture the dependence between PD and LGD. These models are developed in the papers of Jarrow (2001), Jokivuolle and Peura (2003), Carey and Gordy (2003), Hu and Perraudin (2002), Bakshi et al. (2001), Gurtler and Heithecker (2005) and Altman et al., (2003). Most of these papers look at bond data but some also cover loans.

Acharya et al., (2003) investigated defaulted bonds and look into recoveries of US corporate credit exposures, Grunert and Weber (2005) investigated recoveries of German bank loans and summarizes existing knowledge about recoveries. While these papers show a nuanced picture of the determinants of recoveries that consists of many microeconomic and legal features such as the industry sector in which exposures are held or the seniority of a claim all papers find that macroeconomic conditions play a key role.

CHAPTER THREE

METHODOLOGY

3.0 INTRODUCTION

Linear programming (**LP**, or **linear optimization**) is a mathematical method for determining a way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model for some list of requirements represented as linear relationships. Linear programming is a specific case of mathematical programming (mathematical optimization).

More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. Its feasible region is a convex polyhedron, which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality. Its objective function is a real-valued affine function defined on this polyhedron. A linear programming algorithm finds a point in the polyhedron where this function has the smallest (or largest) value if such point exists.

Linear programs are problems that can be expressed in canonical form:

$$\begin{aligned} &\text{maximize} && c^T x \\ &\text{subject to} && Ax \le b \\ & && x > 0 \end{aligned}$$

where **x** represents the vector of variables (to be determined), **c** and **b** are vectors of (known) coefficients, A is a (known) matrix of coefficients, and $(\cdot)^{\mathbf{T}}$ is the matrix transpose. The expression to be maximized or minimized is called the *objective function* ($\mathbf{c}^T\mathbf{x}$ in this case). The inequalities $A\mathbf{x} \le \mathbf{b}$ are the constraints which specify a convex polytope over which the objective function is to be optimized. In this context, two vectors are comparable when they have the same dimensions. If every entry in the first is less-than or equal-to the corresponding entry in the second then we can say the first vector is less-than or equal-to the second vector.

Linear programming can be applied to various fields of study. It is used in business and economics, but can also be utilized for some engineering problems. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design (Wikimedia Foundation, Inc, 2012).

The problem of solving a system of linear inequalities dates back at least as far as Fourier, after whom the method of Fourier-Motzkin elimination is named. The earliest linear programming was first developed by Leonid Kantorovich in 1939.[1] Leonid Kantorovich developed the earliest linear programming problems in 1939 for use during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. The method was kept secret until 1947 when George B. Dantzig published the simplex method and John von Neumann developed the theory of duality as a linear optimization solution, and applied it in the field of game theory. Postwar, many industries found its use in their daily planning (Wikimedia Foundation, Inc, 2012).

The linear-programming problem was first shown to be solvable in polynomial time by Leonid Khachiyan in 1979, but a larger theoretical and practical breakthrough in the field came in 1984 when Narendra Karmarkar introduced a new interior-point method for solving linear-programming problems.

Dantzig's original example was to find the best assignment of 70 people to 70 jobs. The computing power required to test all the permutations to select the best assignment is vast; the number of possible configurations exceeds the number of particles in the observable universe. However, it takes only a moment to find the optimum solution by posing the problem as a linear program and applying the Simplex algorithm. The theory behind linear programming drastically reduces the number of possible optimal solutions that must be checked (Wikimedia Foundation, Inc, 2012).

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3.1 USES OF LINEAR PROGRAMMING

Linear programming is a considerable field of optimization for several reasons. Many practical problems in operations research can be expressed as linear programming problems. Certain special cases of linear programming, such as *network flow* problems and *multicommodity flow* problems are considered important enough to have generated much research on specialized algorithms for their solution. A number of algorithms for other types of optimization problems work by solving LP problems as sub-problems. Historically, ideas from linear programming have inspired many of the central concepts of optimization theory, such as *duality, decomposition,* and the importance of *convexity* and its generalizations. Likewise, linear programming is heavily used in microeconomics and company management, such as planning, production, transportation, technology and other issues. Although the modern management issues are ever-changing, most companies would like to maximize profits or minimize costs with limited resources. Therefore, many issues can be characterized as linear programming problems (Wikimedia Foundation, Inc, 2012).

3.2 INTERIOR POINT METHODS

Interior point methods (also referred to as **barrier methods**) are a certain class of algorithms to solve linear and nonlinear convex optimization problems.

The interior point method was invented by John von Neumann. Von Neumann suggested a new method of linear programming, using the homogeneous linear system of Gordan (1873) which was later popularized by Karmarkar's algorithm, developed by Narendra Karmarkar in 1984 for linear programming. The method consists of a self-concordant barrier function used to encode the convex set. Contrary to the simplex method, it reaches an optimal solution by traversing the interior of the feasible region (Wikimedia Foundation, Inc, 2012).

An interior point method is a linear or nonlinear programming method that achieves optimization by going through the interior of the solid defined by the problem rather than around its surface (Forsgren, et al. 2002). A polynomial time linear programming (LP) algorithm using an interior point method was founded by Narendra Karmarkar (Forsgren et al, 2002). Interior point methods were known as early as the 1960s in the form of the barrier function methods. Narendra Karmarkar proposed a new polynomial algorithm for LP that held great promise and performed well in practice. The main idea of this algorithm is quite different from Simplex Method. Unlike Simplex Method, iterates are calculated not on the boundary, but in the interior of the feasible region. This algorithm is an iterative algorithm that makes use of projective transformations and a potential function (Karmarkar's potential function) (Forsgren et al, 2002). The current iterate is mapped to the center of the special set in the interior feasible region using a projective transformation. This set is an intersection of the standard simplex and a hyperplane obtained from the constraints. Then, the potential function is minimized over the ball inscribed in the set. The minimiser is mapped back to the original space and becomes a new iterate.

3.2.1 Comparison between Simplex Method and Interior Point Methods

Moving along the edges of the polytope defined by the constraints, the simplex and revised simplex algorithms solves a linear programming problem, until the minimum is reached, from vertices to vertices with successively smaller values of the objective function. Dense linear algebra is used in the implementation of these algorithms. It is possible to solve exact or extended precision problems and this is the unique feature of the implementation. Therefore for small-sized problems these methods are suitable for which non-machine-number results are needed, or a solution on the vertex is desirable.

Loosely speaking, interior point algorithms for linear programming, defined by the constraints iterate from the interior of the polytope. Very fast they get closer to the solution, but unlike the simplex/revised simplex algorithms, do not find the solution exactly. Machine precision sparse linear algebra is used in the implementation of an interior point algorithm. Therefore the interior point method is more efficient and should be used for large-scale machine-precision linear programming problems.

On comparing simplex method with Interior point method, the runtime seems to be neither faster nor slower than the others. The implementation details are very crucial on which the efficiency of both, Interior points and simplex methods is dependent.

The simplex method needs $2n-3n$ iterations, where n is the number of primal variables but there can be 200,000 or more iterations using Karmarkar. Thus we can say that generally speaking, Interior point methods are better for large-scale problems.

3.3 METHODS OF SOLVING LINEAR PROGRAMMING PROBLEMS

3.3.1 Karmarkar's projective scaling method

Karmarkar's projective scaling method, also known as *Karmarkar's interior point LP algorithm*, starts with a trial solution and shoots it towards the optimum solution.

This algorithm addresses LP problem of the form:

$$\begin{aligned} \text{Min} \quad & C^T X \\ \text{s.t:} \quad & AX = 0 \\ & \sum_{i=1}^{n} x_i = 1 \\ & X \geq 0 \end{aligned} \tag{1}$$

where A is a m x n matrix of rank m, C^T is 1 x n vector, $X = (x_1, \ldots, x_n)^T, n \geq 2$ and A and C are all real. The following assumptions hold:

- The point $(1/n, \ldots, 1/n)$ is feasible in problem (1)
- The optimal objective value of problem (1) is zero.

Steps of Karmarkar's Algorithm

<u>Step 0</u> :

- Compute $r = \sqrt{\dfrac{1}{n(n-1)}}$ and $\alpha = \dfrac{n-1}{3n}$.
- Put $k = 0$ and let $X_0 = (1/n, \ldots, 1/n)^T$

Step 1 :

- Let $Y_0 = X_0$
- $D_0 = diag(X_0)$
- $P = \begin{bmatrix} AD_0 \\ 1 \end{bmatrix}, \quad 1 = [1 \quad , \quad . \quad . \quad . \quad , \quad 1]$
- $\overline{C} = C^T D_0$
- Compute $C_p = \left[I - P^T \left(PP^T \right)^{-1} P \right] \overline{C}^T$

  If , $C_p = 0$, any feasible solution becomes an optimal solution. Stop

Otherwise compute

- $Y_{new} = Y_0 - \alpha r \dfrac{C_p}{\left\| C_p \right\|}$
- $X_1 = \dfrac{D_0 Y_{new}}{1 D_0 Y_{new}}$
- $Z = C^T X_1$

Increase *k by* one and repeat *Step 1* until the objective function (*Z*) value is less than or equal to zero (Bazaraa, 1984 and Hamdy, 1992).

3.3.1.1 Conversion Of LP Standard Form Into Karmarkar's Standard Form

In converting the LP problem :

$$\text{Maximize } Z = C^T X$$

Subject to

$$AX \le b$$

$$X \ge 0$$

to Karmarkar's Canonical form :

$$\text{Minimize } Z = C^T X$$

Subject to

$$AX = 0$$

$$\sum_{i=1}^{n} x_i = 1$$

$$X \ge 0$$

The following are the steps involved:

Step 1:

- Write the dual of the given primal problem

$$\text{Minimize } Z = bW$$

Subject to

$$A^T W \ge C^T$$

$$W \ge 0$$

Step 2:

- Introduce Slack and Surplus variables to primal and dual problems
- Combine these two(2) problems

Step 3 :

- Introduce a bounding constraint $\sum x_i + \sum w_i \le K$, where K should be sufficiently large to include all feasible solutions of original problem
- Introduce a slack variable in the bounding constraint and obtain

$$\sum x_i + \sum w_i + s = K$$

Step 4:

- Introduce a dummy variable d (subject to condition $d = 1$) to homogenize the constraints
- Replace the equations $\sum x_i + \sum w_i + s = K$ and $d = 1$ with the following equations :

$$\sum x_i + \sum w_i + s - Kd = 0 \quad \text{and} \quad \sum x_i + \sum w_i + s + d = K + 1$$

Step 5:

- Introduce the following transformations so as to obtain one on the RHS of the last equation:

$$x_i = (K+1)y_i, i = 1,2,....m+n$$

$$w_i = (K+1)y_{m+n+i}, i = 1,2,....m+n$$

$$s = (K+1)y_{2m+2n+1}$$

$$d = (K+1)y_{2m+2n+2}$$

Step 6:

- Introduce an artificial variable $y_{2m+2n+3}$ (to be minimized) in all the equations such

  that the sum of the coefficients in each homogenous equation is **zero** and

  coefficients of the artificial variable in the last equation is **one**.
  (Dhamija, A.K ,2012)

Example 1:

Convert the following LP problem to Karmarkar's canonical form:

$$\text{Max} \quad Z = 2x_1 + x_2$$
$$\text{s.t}$$
$$x_1 + x_2 \leq 5$$
$$x_1 - x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution 1:

- Write the dual of the given primal problem:

$$\text{Min } Z = 5w_1 + 3w_2$$
$$\text{s.t}$$
$$w_1 + w_2 \geq 2$$
$$w_1 - w_2 \geq 1$$
$$w_1, w_2 \geq 0$$

- Introduction of slack and surplus variables and combination of primal and dual problems:

$$x_1 + x_2 + x_3 = 5$$
$$x_1 - x_2 + x_4 = 3$$
$$w_1 + w_2 - w_3 = 2$$
$$w_1 - w_2 - w_4 = 1$$
$$2x_1 + x_2 = 5w_1 + 3w_2$$

- Addition of bounding constraint with slack variable s:

$$\sum_{i=1}^{4} x_i + \sum_{i=1}^{4} w_i + s = K$$

- Homogenized equivalent system with dummy variable d:

$$x_1 + x_2 + x_3 - 5d = 0$$
$$x_1 - x_2 + x_4 - 3d = 0$$
$$w_1 + w_2 - w_3 - 2d = 0$$
$$w_1 - w_2 - w_4 - d = 0$$

$$2x_1 + x_2 - 5w_1 - 3w_2 = 0$$
$$\sum_{i=1}^{4} x_i + \sum_{i=1}^{4} w_i + s - Kd = 0$$
$$\sum_{i=1}^{4} x_i + \sum_{i=1}^{4} w_i + s + d = K + 1$$
$$x_i \geq 0, w_i \geq 0, d > 0, s > 0, i = 1,2,...,4$$

- Introduction of transformations:

$$x_i = (K+1)y_i, i = 1,2,....4$$

$$w_i = (K+1)y_{4+i}, i = 1,2,....4$$

$$s = (K+1)y_9$$

$$d = (K+1)y_{10}$$

- The system of equations obtained are as follows:

$$y_1 + y_2 + y_3 - 5y_{10} = 0$$
$$y_1 - y_2 + y_4 - 3y_{10} = 0$$
$$y_5 + y_6 - y_7 - 2y_{10} = 0$$
$$y_5 - y_6 - y_8 - y_{10} = 0$$
$$2y_1 + y_2 - 5y_5 - 3y_6 = 0$$
$$\sum_{i=1}^{9} y_i - Ky_{10} = 0$$
$$\sum_{i=1}^{10} y_i = 1$$
$$y_i \geq 0, i = 1,2,........,10$$

- Introduction of an artificial variable y_{11}

Minimize y_{11}

Subject to :

$$y_1 + y_2 + y_3 - 5y_{10} + 2y_{11} = 0$$
$$y_1 - y_2 + y_4 - 3y_{10} + 2y_{11} = 0$$
$$y_5 + y_6 - y_7 - 2y_{10} + y_{11} = 0$$
$$y_5 - y_6 - y_8 - y_{10} + 2y_{11} = 0$$
$$2y_1 + y_2 - 5y_5 - 3y_6 + 5y_{11} = 0$$
$$\sum_{i=1}^{9} y_i - Ky_{10} + (K-9)y_{11} = 0$$
$$\sum_{i=1}^{11} y_i = 1$$
$$y_i \geq 0, i = 1,2,........,11$$

- The values of y are then determined and using the transformation

$x_i = (K+1)y_i, i = 1,2,....4$ and $K = 10$, the corresponding

values of x can be determined.

- Solving, using MATLAB codes, $y_1 = 0.3636$ and $y_2 = 0.0909$

Using $K = 10 \quad \Rightarrow x_i = 11y_i$

Hence $x_1 = 11y_1 = 11 \times 0.3636 = 4.0$ and $x_2 = 11y_2 = 11 \times 0.0909 = 1.0$

$\Rightarrow Z = 2x_1 + x_2 = 2(4) + 1 = 9.0$ (The same value if Simplex is used)

Example 2:

Carry out the first three iterations of Karmarkar's method for the following problem:

Minimize Z = $2x_2 - x_3$

Subject to

$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Solution 2:

$$n = 3 \quad , \quad C = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} , \quad A = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \quad , \quad r = \frac{1}{\sqrt{n(n-1)}} = \frac{1}{\sqrt{3(3-1)}} = \frac{1}{\sqrt{6}} \quad , \quad \alpha = \frac{n-1}{3n} = \frac{3-1}{3(3)} = \frac{2}{9}$$

Iteration 0 (k = 0)

$$D_0 = diag(X_0) = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\overline{C} = C^T D_0 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \end{bmatrix}$$

$$AD_0 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \end{bmatrix}$$

$$P=\begin{bmatrix} AD_0 \\ 1 \end{bmatrix}=\begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$PP^T=\begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix}\times\begin{bmatrix} 1/3 & 1 \\ -2/3 & 1 \\ 1/3 & 1 \end{bmatrix}=\begin{bmatrix} 2/3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\left(PP^T\right)^{-1}=\begin{bmatrix} 3/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$C_p=\left[I-P^T\left(PP^T\right)^{-1}P\right]\times\overline{C}^T=\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}\right)\times\begin{bmatrix} 0 \\ 2/3 \\ -1/3 \end{bmatrix}=\begin{bmatrix} 1/6 \\ 0 \\ -1/6 \end{bmatrix}$$

$$\left\|C_p\right\|=\sqrt{(1/6)^2+0+(-1/6)^2}=\sqrt{\frac{2}{36}}=\frac{\sqrt{2}}{6}$$

$$Y_{new}=X_0-\alpha r\frac{C_p}{\left\|C_p\right\|}=\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}-\left(\frac{\frac{2}{9}\times\frac{1}{\sqrt{6}}}{\frac{\sqrt{2}}{6}}\right)\times\begin{bmatrix} 1/6 \\ 0 \\ -1/6 \end{bmatrix}=\begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix}$$

$$X_1=Y_{new}=\begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix}$$

$$Z = C^T X_1 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix} = 0.2692$$

Since $Z > 0$, we continue

<u>Iteration 1(k = 1)</u>

$$D_1 = diag(X_1) = \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix}$$

$$\overline{C} = C^T D_1 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} = \begin{bmatrix} 0 & 0.6666 & -0.3974 \end{bmatrix}$$

$$AD_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} = \begin{bmatrix} 0.2692 & -0.6666 & 0.3974 \end{bmatrix}$$

$$P = \begin{bmatrix} AD_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2692 & -0.6666 & 0.3974 \\ 1 & 1 & 1 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 0.2692 & -0.6666 & 0.3974 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 1 \\ -0.6666 & 1 \\ 0.3974 & 1 \end{bmatrix} = \begin{bmatrix} 0.675 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\left(PP^T\right)^{-1} = \begin{bmatrix} 1.482 & 0 \\ 0 & 0.333 \end{bmatrix}$$

$$P^T\left(PP^T\right)^{-1}P = \begin{bmatrix} 0.2692 & 1 \\ -0.6666 & 1 \\ 0.3974 & 1 \end{bmatrix} \times \begin{bmatrix} 1.482 & 0 \\ 0 & 0.333 \end{bmatrix} \times \begin{bmatrix} 0.2692 & -0.6666 & 0.3974 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.441 & 0.067 & 0.492 \\ 0.067 & 0.992 & -0.059 \\ 0.492 & -0.059 & 0.567 \end{bmatrix}$$

$$C_p = \left[I - P^T\left(PP^T\right)^{-1}P\right]\overline{C}^T = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.441 & 0.067 & 0.492 \\ 0.067 & 0.992 & -0.059 \\ 0.492 & -0.059 & 0.567 \end{bmatrix}\right) \times \begin{bmatrix} 0 \\ 0.6666 \\ -0.3974 \end{bmatrix} = \begin{bmatrix} 0.151 \\ -0.018 \\ -0.132 \end{bmatrix}$$

$$\left\|C_p\right\| = \sqrt{(0.151)^2 + (-0.018)^2 + (-0.132)^2} = 0.2014$$

$$Y_{new} = X_0 - \alpha r \frac{C_p}{\left\|C_p\right\|} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} - \left(\frac{\frac{2}{9} \times \frac{1}{\sqrt{6}}}{0.2014}\right) \times \begin{bmatrix} 0.151 \\ -0.018 \\ -0.132 \end{bmatrix} = \begin{bmatrix} 0.2653 \\ 0.3414 \\ 0.3928 \end{bmatrix}$$

$$D_1 Y_{new} = \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} \times \begin{bmatrix} 0.2653 \\ 0.3414 \\ 0.3928 \end{bmatrix} = \begin{bmatrix} 0.0714 \\ 0.1138 \\ 0.1561 \end{bmatrix}$$

$$1 D_1 Y_{new} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.0714 \\ 0.1138 \\ 0.1561 \end{bmatrix} = 0.3413$$

$$X_2 = \frac{D_1 Y_{new}}{1 D_1 Y_{new}} = \frac{1}{0.3413} \times \begin{bmatrix} 0.0714 \\ 0.1138 \\ 0.1561 \end{bmatrix} = \begin{bmatrix} 0.2092 \\ 0.3334 \\ 0.4574 \end{bmatrix}$$

$$Z = C^T X_2 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2092 \\ 0.3334 \\ 0.4574 \end{bmatrix} = 0.2094$$

Since $Z > 0$, we continue

Iteration 2(k = 2)

$$D_2 = diag(X_2) = \begin{bmatrix} 0.2092 & 0 & 0 \\ 0 & 0.3334 & 0 \\ 0 & 0 & 0.4574 \end{bmatrix}$$

$$\overline{C} = C^T D_2 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2092 & 0 & 0 \\ 0 & 0.3334 & 0 \\ 0 & 0 & 0.4574 \end{bmatrix} = \begin{bmatrix} 0 & 0.6668 & -0.4574 \end{bmatrix}$$

$$AD_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2092 & 0 & 0 \\ 0 & 0.3334 & 0 \\ 0 & 0 & 0.4574 \end{bmatrix} = \begin{bmatrix} 0.2092 & -0.6668 & 0.4574 \end{bmatrix}$$

$$P = \begin{bmatrix} AD_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2092 & -0.6668 & 0.4574 \\ 1 & 1 & 1 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 0.2092 & -0.6668 & 0.4574 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2092 & 1 \\ -0.6668 & 1 \\ 0.4574 & 1 \end{bmatrix} = \begin{bmatrix} 0.6976 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\left(PP^T\right)^{-1} = \begin{bmatrix} 1.4335 & 0 \\ 0 & 0.3333 \end{bmatrix}$$

$$P^T\left(PP^T\right)^{-1}P = \begin{bmatrix} 0.2092 & 1 \\ -0.6668 & 1 \\ 0.4574 & 1 \end{bmatrix} \times \begin{bmatrix} 1.4335 & 0 \\ 0 & 0.3333 \end{bmatrix} \times \begin{bmatrix} 0.2092 & -0.6668 & 0.4574 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.3961 & 0.1333 & 0.4706 \\ 0.1333 & 0.9706 & -0.1039 \\ 0.4706 & -0.1039 & 0.6333 \end{bmatrix}$$

$$C_p = \left[I - P^T\left(PP^T\right)^{-1}P\right]\overline{C}^T = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3961 & 0.1333 & 0.4706 \\ 0.1333 & 0.9706 & -0.1039 \\ 0.4706 & -0.1039 & 0.6333 \end{bmatrix}\right) \times \begin{bmatrix} 0 \\ 0.6668 \\ -0.4574 \end{bmatrix} = \begin{bmatrix} 0.1263 \\ -0.0279 \\ -0.0984 \end{bmatrix}$$

$$\left\|C_p\right\| = \sqrt{(0.1263)^2 + (-0.0279)^2 + (-0.0984)^2} = 0.1625$$

$$Y_{new} = X_0 - \alpha r \frac{C_p}{\left\|C_p\right\|} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} - \left(\frac{\frac{2}{9} \times \frac{1}{\sqrt{6}}}{0.1625}\right) \times \begin{bmatrix} 0.1263 \\ -0.0279 \\ -0.0984 \end{bmatrix} = \begin{bmatrix} 0.2628 \\ 0.3489 \\ 0.3883 \end{bmatrix}$$

$$D_2 Y_{new} = \begin{bmatrix} 0.2092 & 0 & 0 \\ 0 & 0.3334 & 0 \\ 0 & 0 & 0.4574 \end{bmatrix} \times \begin{bmatrix} 0.2628 \\ 0.3489 \\ 0.3883 \end{bmatrix} = \begin{bmatrix} 0.0550 \\ 0.1163 \\ 0.1776 \end{bmatrix}$$

$$1D_2Y_{new} = [1 \quad 1 \quad 1] \times \begin{bmatrix} 0.0550 \\ 0.1163 \\ 0.1776 \end{bmatrix} = 0.3489$$

$$X_3 = \frac{D_2Y_{new}}{1D_2Y_{new}} = \frac{1}{0.3489} \times \begin{bmatrix} 0.0550 \\ 0.1163 \\ 0.1776 \end{bmatrix} = \begin{bmatrix} 0.1576 \\ 0.3334 \\ 0.5090 \end{bmatrix}$$

$$Z = C^T X_3 = [0 \quad 2 \quad -1] \times \begin{bmatrix} 0.1576 \\ 0.3334 \\ 0.5090 \end{bmatrix} = 0.1578$$

Since $Z > 0$, we continue

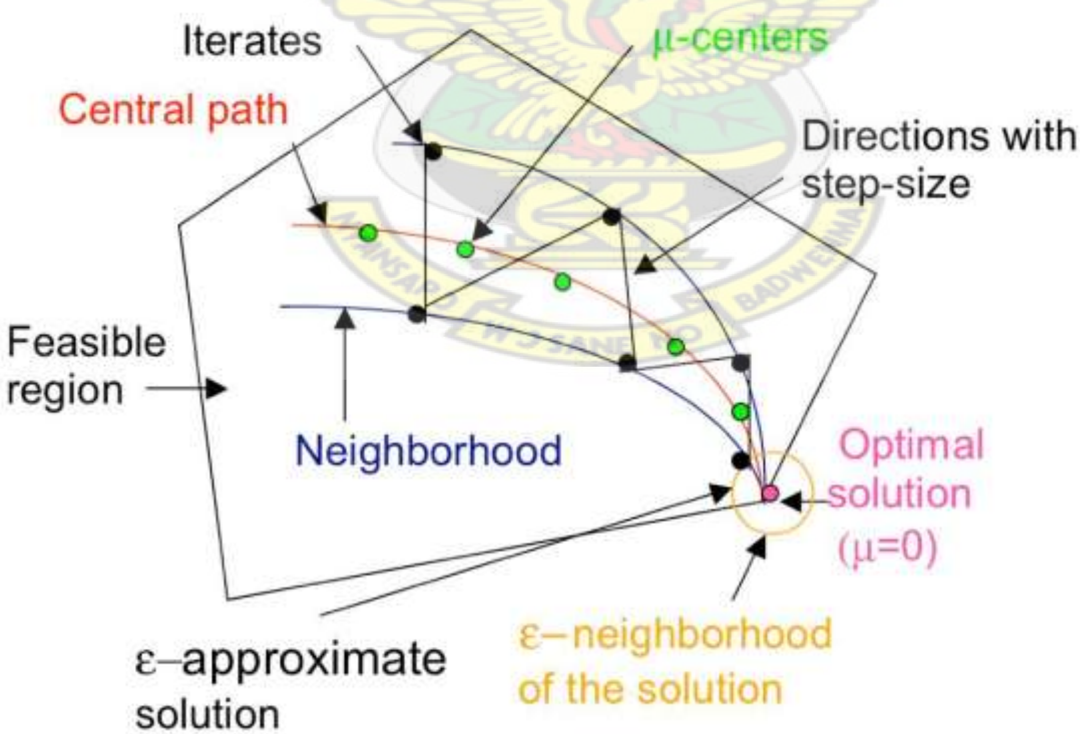

Figure 3.3 - Graphical representation of the Karmarkar algorithm: where $\varepsilon = tolerance$

The main idea of the method is to perform orthogonal projection of an iteration to the optimal set when the iteration is near the optimal set. Another interesting fact is that in the case when LP problem has infinitely many optimal solutions, Karmarkar's method tends to find an exact optimal solution that is in the center of the optimal set as opposed to the Simplex method (SM) that finds the corner (vertex) of the optimal set. However, it is possible to recover a vertex optimal solution as well. The method is also a path-following algorithm since iterates are required to stay in the horn neighborhoods of the central path. These algorithms are designed to reduce the primal-dual gap directly in each iteration. There is another group of interior-point algorithms that are designed to reduce the primal-dual gap indirectly in each iteration. This algorithm directly reduces the objective function to a constant number in each iteration. Iterates of these algorithms do not necessarily stay in the horn neighborhood of the central path, *(Lesaja, 2009).*

3.3.2 The Two – Phase Simplex Method

Given an LP problem

$$\text{Maximize } Z = C^T X$$
$$\text{Subject to}$$
$$AX \leq b$$
$$X \geq 0$$

In Linear Programming if the initial Basic Feasible Solution (BFS) exists then the normal **Simplex Method** can be used. In cases where such an obvious initial BFS does not exist, the **Two-Phase Simplex Method** can be used.

Steps:

- Convert each inequality constraint to the standard form

- If constraint i is a $\geq$ or = constraint, add an artificial variable a_i. Also add the sign restriction $a_i \geq 0$

- For now, ignore the original LP's objective function. Instead solve an LP whose objective function is **min *w*** (sum of all the artificial variables). (**Phase I LP).**

- The act of solving the Phase I LP will force the artificial variables to be zero.

- Because each $a_i \geq 0$, solving the Phase I LP will result in one of the following ***three cases***

Case 1. The optimal value of ***w*** is greater than zero. The original LP has no feasible solution.

Case 2. The optimal value of **w** is equal to zero, and no artificial variables are in the optimal Phase I basis. Drop all columns in the optimal Phase I tableau that correspond to the artificial variables, combine the original objective function with the constraints from the optimal Phase I tableau (Phase II LP). The optimal solution to the Phase II LP is the optimal solution to the original LP.

Case 3. The optimal value of w is equal to zero and at least one artificial variable is in the optimal Phase I basis. drop from the optimal Phase I tableau all nonbasic artificial variables and any variable from the original problem that has a negative coefficient in row 0 of the optimal Phase I tableau combine the original objective function with the constraints from the optimal Phase I tableau (Phase II LP). The optimal solution to the Phase II LP is the optimal solution to the original LP.

Example

Consider the problem

min z = $4x_1 + x_2 + x_3$
subject to

$$2x_1 + x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + x_3 = 3$$
$$x_1, x_2, x_3 \geq 0$$

Solution
There is no basic feasible solution so the two-phase method is used.

Phase I : The auxiliary form is written below:

min w = $y_1 + y_2$
subject to

$$2x_1 + x_2 + 2x_3 + y_1 = 4$$
$$3x_1 + 3x_2 + x_3 + y_2 = 3$$
$$x_1, x_2, x_3, y_1, y_2 \geq 0$$

The starting tableau (in non standard form) is :

BASIS	x_1	x_2	x_3	y_1	y_2	RHS
y_1	2	1	2	1	0	4
y_2	3	3	1	0	1	3
(-z)	4	1	1	0	0	0
(-w)	0	0	0	1	1	0

The tableau is converted to standard form by zeroing out the coefficients of the basic variables in the *w*-row:

BASIS	x_1	x_2	x_3	y_1	y_2	RHS	Min Ratio
y_1	2	1	2	1	0	4	2
y_2	3	3	1	0	1	3	1
(-z)	4	1	1	0	0	0	
(-w)	-5	-4	-3	0	0	-7	

The coefficient of x_1 in the w-row is the most negative so it is introduced in to the basis. The minimum ratio test is **min{4/2, 3/3}=1** so y_2 leaves the basis**.** The pivot element is circled.

Using $R_{12} = (-2/3)R_2+R_1$, $R_{22} = (1/3)R_2$, $R_{32} = (-4/3)R_2+R_3$, $R_{42} = (5/3)R_2+R_4$ the table becomes:

BASIS	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{y_1}$	$\mathbf{y_2}$	**RHS**	**Min Ratio**
$\mathbf{y_1}$	0	-1	**1.33**	1	-0.67	2	1.5
$\mathbf{x_1}$	1	1	0.33	0	0.33	1	3
(-z)	0	-3	-0.33	0	-1.33	-4	
(-w)	0	1	-1.33	0	1.67	-2	

The coefficient of x_3 in the w-row is the most negative so it is introduced in to the basis.

The minimum ratio test is **min{2/(4/3), 1/(1/3}=1.5** so y_1 leaves the basis**.** The pivot element is circled. Using $R_{13} = (3/4)R_{12}$, $R_{23} = (-1/4)R_{12}+R_{22}$, $R_{33} = (1/4)R_{12}+R_{32}$, $R_{43} = R_{12}+R_{42}$,the table becomes:

BASIS	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{y_1}$	$\mathbf{y_2}$	**RHS**
$\mathbf{x_3}$	0	-0.75	1	0.75	-0.5	1.5
$\mathbf{x_1}$	1	1.25	0	-0.25	0.5	0.5
(-z)	0	-3.25	0	0.25	-1.5	-3.5
(-w)	0	0	0	1	1	0

All the coefficients in the w-row are non negative, w =0, and there are no artificial variables in the basis, so the phase I is completed. Phase II begins with the tableau shown below:

BASIS	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	**RHS**	**Min Ratio**
$\mathbf{x_3}$	0	-0.75	1	1.5	
$\mathbf{x_1}$	1	**1.25**	0	0.5	0.4
(-z)	0	-3.25	0	-3.5	

The coefficient of x_2 in the z-row is the most negative so it is introduced in to the basis.

The minimum ratio test is **min{0.5/1.25} = 0.4** so x_1 leaves the basis**.** The pivot element is bolded. Using $R_{12} = 0.6R_2+R_1$, $R_{22} = 0.8R_2$, $R_{32} = 2.6R_2+R_3$, the table becomes:

BASIS	x_1	x_2	x_3	RHS
x_3	0.6	0	1	1.8
x_2	0.8	1	0	0.4
(-z)	2.6	0	0	-2.2

All the coefficients in the z-row are non negative so the Phase II is also completed. The optimum solution is ***x* = (0, 0.4, 1.8)** and ***z* = 2.2**

3.3.3 The Dual Simplex Method

In the tableau implementation of the primal simplex algorithm, the right-hand-side column is always nonnegative so the basic solution is feasible at every iteration. The basis for the tableau is *primal feasible* if all elements of the right-hand side are nonnegative. Alternatively, when some of the elements are negative, we say that the basis is *primal infeasible*.

For the primal simplex algorithm, some elements in row 0 will be negative until the final iteration when the optimality conditions are satisfied. In the event that all elements of row 0 are nonnegative, we say that the associated basis is *dual feasible*.

The Algorithm

Given a standard LP problem

$$\text{Maximize } Z = C^T X$$
$$\text{Subject to}$$
$$AX \le b$$
$$X \ge 0$$

The following are the steps involved in the algorithm:

***Step* 1 (*Initialization*)**

Start with a dual feasible basis and let $k = 1$. Create a tableau for this basis in the simplex form. If the right-hand side entries are all nonnegative, the solution is primal feasible, so stop with the optimal solution.

***Step* 2 *(Iteration k*)**

- *Select the leaving variable*. Find a row, call it $\boldsymbol{r}$, with a negative right-hand-side constant; i.e., $b_r < 0$ .Let row $\boldsymbol{r}$ be the pivot row and let the leaving variable be $x_{B(r)}$.

  A common rule for choosing r is to select the most negative RHS value; i.e., $b_r = \min\{b_i : i = 1, \dots, m\}$.

- *Determine the entering variable*. For each negative coefficient in the pivot row, compute the negative of the ratio between the reduced cost in row 0 and the structural coefficient in row $\boldsymbol{r}$. If there is no negative coefficient, $a_{rj} < 0$, stop; there is no feasible solution. Let the column with the minimum ratio, designated by the index $\boldsymbol{s}$, be the pivot column; let x_s is the entering variable. The pivot column is determined by the following ratio test.

$$\frac{-c_s}{a_{rs}} = \min\left\{\frac{-c_j}{a_{rj}} : a_{rj} < 0, j = 1, \dots, n\right\}$$

- *Change the basis*. Replace $x_{B(r)}$ by x_s in the basis. Create a new tableau by performing the following operations (these are the same as for the primal simplex algorithm). Let $\mathbf{a}_i$ be the vector of the i^{th} row of the current tableau, and let $\mathbf{a}_{i\,new}$ be the i^{th} row in the new tableau. Let b_i be the RHS for row i in the current tableau, and let $b_{i\,new}$ be the RHS of the new tableau. Let a_{is} be the element in the i^{th} row of the pivot column ***s***.

The pivot row in the new tableau is:

$$a_{rnew} = \frac{a_r}{a_{rs}} \quad \text{and} \quad b_{rnew} = \frac{b_r}{a_{rs}}$$

The other rows in the new tableau are:

$$a_{inew} = (-a_{is} \times a_{rnew}) + a_i \quad \text{and} \quad b_{inew} = (-a_{is} \times b_{rnew}) + b_i \quad \text{for } i = 0,1,\ldots,m,\ i \neq r$$

***Step* 3 (*Feasibility test*)**

If all entries on the right-hand side are nonnegative the solution is primal feasible, so stop with the optimal solution. Otherwise, put $k \leftarrow k+1$ and return to Step 2.

<u>**Example**</u>

Consider the standard LP problem below to which slack variables z_1 and z_2 are already added:

minimize $w = 2x_1 + 3x_2 + 4x_3$
subject to

$$x_1 + 2x_2 + x_3 - z_1 = 3$$
$$2x_1 - x_2 - 3x_3 - z_2 = 4$$
$$x_1, x_2, x_3, z_1, z_2 \geq 0$$

Solution

The primal simplex algorithm would have to use two phases, since the initial solution where $z_1 = -3$ and $z_2 = -4$ is not feasible. On the other hand, $c \geq 0$, so the dual solution with $\lambda_1 = \lambda_2 = 0$ satisfies $c^T - \lambda^T A \geq 0$ and is therefore feasible. The following tableau is obtained:

z_1	-1	-2	-1	1	0	-3
z_2	**-2**	1	3	0	1	-4
w	2	3	4	0	0	0

In the dual simplex algorithm the pivot is selected by picking a row *i* such that $a_{i0} < 0$ and a column $j \in \{j' : a_{ij'} < 0\}$ that minimizes $-a_{0j} / a_{ij}$. Pivoting then works just like in the primal algorithm. Pivoting on a_{21}, the tableau becomes:

z_1	0	$-\frac{5}{2}$	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-1
x_1	1	$-\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	2
w	0	4	7	0	1	-4

Pivoting on a_{12} changes the tableau to reach optimality:

x_2	0	1	1	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
x_1	1	0	-1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$
w	0	0	3	$\frac{8}{5}$	$\frac{1}{5}$	$\frac{28}{5}$

Hence the optimal solution is $\{x_1 = \frac{11}{5}, x_2 = \frac{2}{5}, x_3 = 0, w = \frac{28}{5}\}$. Also $\lambda_1 = \frac{8}{5}$ and $\lambda_2 = \frac{1}{5}$

3.3.4 Gomory's Cutting - Plane Method

In mathematical optimization, the **cutting-plane method** is an umbrella term for optimization methods which iteratively refine a feasible set or objective function by means of linear inequalities, termed *cuts*. Such procedures are popularly used to find **integer solutions to mixed integer linear programming** (MILP) problems, as well as to solve general, not necessarily differentiable convex optimization problems.

Let an integer programming problem be formulated as

$$\text{Maximize } c^T x$$
$$\text{Subject to } Ax = b,$$
$$x \geq 0,\ x_i \text{ all integers.}$$

The method proceeds by first dropping the requirement that the x_i be integers and solving the associated linear programming problem to obtain a basic feasible solution. Geometrically, this solution will be a vertex of the convex polytope consisting of all feasible points. If this vertex is not an integer point then the method finds a hyperplane with the vertex on one side and all feasible integer points on the other. This is then added as an additional linear constraint to exclude the vertex found, creating a modified linear programming program. The new program is then solved and the process is repeated until an integer solution is found.

Using the simplex method to solve a linear program produces a set of equations of the form

$$x_i + \sum \bar{a}_{i,j} x_j = \bar{b}_i$$

Where x_i is a basic variable and the x_j's are the non basic variables. Rewrite this equation so that the integer parts are on the left side and the fractional parts are on the right side:

$$x_i + \sum \lfloor \bar{a}_{i,j} \rfloor x_j - \lfloor \bar{b}_i \rfloor = \bar{b}_i - \lfloor \bar{b}_i \rfloor - \sum (\bar{a}_{i,j} - \lfloor \bar{a}_{i,j} \rfloor) x_j.$$

For any integer point in the feasible region the right side of this equation is less than 1 and the left side is an integer, therefore the common value must be less than or equal to 0. So the inequality

$$\bar{b}_i - \lfloor \bar{b}_i \rfloor - \sum (\bar{a}_{i,j} - \lfloor \bar{a}_{i,j} \rfloor) x_j \leq 0$$

must hold for any integer point in the feasible region. Furthermore, if x_i is not an integer for the basic solution *x*,

$$\bar{b}_i - \lfloor \bar{b}_i \rfloor - \sum (\bar{a}_{i,j} - \lfloor \bar{a}_{i,j} \rfloor) x_j = \bar{b}_i - \lfloor \bar{b}_i \rfloor > 0.$$

So the inequality above excludes the basic feasible solution and thus is a cut with the desired properties. Introducing a new slack variable x_k for this inequality, a new constraint is added to the linear program, namely

$$x_k + \sum (\lfloor \bar{a}_{i,j} \rfloor - \bar{a}_{i,j}) x_j = \lfloor \bar{b}_i \rfloor - \bar{b}_i,\ x_k \geq 0,\ x_k \text{ an integer.}$$

Example

Consider the integer optimization problem

Maximize

$$x_1 + x_2 + x_3$$

Subject to

$$x_1 + x_2 \leq 1$$

$$x_1 + x_3 \leq 1$$

$$x_2 + x_3 \leq 1$$

$$x_i \geq 0, x_i \text{ an integer for } i = 1, 2, 3$$

Introduce slack variables x_i, i=4, 5, 6 to produce the standard form

Maximize

$$x_1 + x_2 + x_3$$

Subject to

$$x_1 + x_2 + x_4 = 1$$

$$x_1 + x_3 + x_5 = 1$$

$$x_2 + x_3 + x_6 = 1$$

$$x_i \geq 0, x_i \text{ an integer for } i = 1, 2, \ldots, 6$$

Solving this with the simplex method gives the solution $x_1=x_2=x_3=1/2$ and the equations

$$x_1 + \frac{1}{2}x_4 + \frac{1}{2}x_5 - \frac{1}{2}x_6 = \frac{1}{2}$$

$$x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 + \frac{1}{2}x_6 = \frac{1}{2}$$

$$x_3 - \frac{1}{2}x_4 + \frac{1}{2}x_5 + \frac{1}{2}x_6 = \frac{1}{2}$$

Each of these equations produces the same cutting plane, and with the introduction of a new slack variable x_7 it can be written as a new constraint

$$-\frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2}x_6 + x_7 = -\frac{1}{2}.$$

An analysis of the updated linear program quickly shows that the original three constraints are now redundant and the corresponding slack variables can be eliminated, leaving the simplified problem

Maximize

$$x_1 + x_2 + x_3$$

Subject to

$$x_1 + x_2 + x_3 + x_7 = 1$$

$x_i \geq 0$, x_i an integer for i=1, 2, 3, 7.

This is easily solved giving three solutions (x_1, x_2, x_3) = (1, 0, 0), (0, 1, 0), or (0, 0, 1).

3.4 SUMMARY

This chapter discussed the Karmarkar's Interior point method and other Linear Programming methods. In the next chapter, we shall put forward the data collected, its modeling and the analysis.

CHAPTER 4

DATA COLLECTION, MODELING AND ANALYSIS

4.0 INTRODUCTION

In this chapter we shall analyze the data taken from Capital Rural Bank Limited. A model is proposed and solved to help the bank maximize its net profit.

4.1 DATA

Capital Rural Bank Limited (Dominase-Agency) is in the process of formulating a loan policy involving GH¢20,000,000.00 for the year 2013. Being a full-service facility, the bank is obligated to grant loans to different clientele.

Table 4.1 provides the type of loans, the interest rate charged by the bank and the probability of bad debt as estimated from past experience.

Table 4.1: Loans available to Capital Rural Bank Limited.

S/N	TYPE OF LOAN	INTEREST RATE	PROBABILITY OF BAD DEBT
1	COMMERCIAL	0.39	0.020
2	FUNERAL	0.36	0.030
3	SALARY	0.36	0.010
4	SUSU	0.36	0.055
5	AGRICULTURE	0.36	0.150
6	HOUSING	0.30	0.075

Source: Capital Rural Bank Ltd. - Dominase , Sunyani (Supervisor, 2012)

Bad debts are assumed irretrievable and hence produce no principal or interest revenue.
For policy reasons, there are limits on how the bank allocates its funds. Competition with other financial institutions in the Sunyani Metropolis requires that the bank disburses the funds using the conditions below:

- Salary, Funeral and Commercial loans should be at most 60% of the total funds.
- To assist people in the area (Sunyani Metropolis) to undertake more housing projects, the Housing loan should be at most 50% of Salary and Funeral loans.
- The sum of Susu and Agriculture loans must not be more than 40% of Commercial and Housing loans.
- The sum of the Agriculture and Funeral loans should be at most 15% of the total funds.
- The overall ratio for bad debts in all loans should not exceed 4.5%.

4.2 FORMULATION OF THE STANDARD LP MODEL

The variables (in millions of Ghana cedis) of the model can be defined as follows:

$x_1 =$ Commercial Loan

$x_2 =$ Funeral Loan

$x_3 =$ Salary Loan

$x_4 =$ Susu Loan

$x_5 =$ Agriculture Loan

$x_6 =$ Housing Loan

The objective of Capital Rural Bank Ltd. is to maximize its profit comprising of the difference between the revenue from interest and lost funds due to bad debts.

LOAN TYPE	LOAN AMOUNT (x)	INTEREST RATE (I)	PROBABILITY OF BAD DEBT (B)	BAD DEBT AMOUNT (B x)	PROFIT AMOUNT $(1-B)x$
Commercial	x_1	0.39	0.020	0.020 x_1	0.980 x_1
Funeral	x_2	0.36	0.030	0.030 x_2	0.970 x_2
Salary	x_3	0.36	0.010	0.010 x_3	0.990 x_3
Susu	x_4	0.36	0.055	0.055 x_4	0.945 x_4
Agriculture	x_5	0.36	0.150	0.150 x_5	0.850 x_5
Housing	x_6	0.30	0.075	0.075 x_6	0.925 x_6

The objective function is given by the formula

$$Z = \sum_{i=1}^{6} I_i (1 - B_i) x_i - \sum_{i=1}^{6} B_i x_i$$

$$Z = 0.39(0.980x_1) + 0.36(0.970x_2) + 0.36(0.990x_3) + 0.36(0.945x_4) + 0.36(0.850x_5) + 0.30(0.925x_6)$$
$$-(0.020x_1 + 0.030x_2 + 0.010x_3 + 0.055x_4 + 0.150x_5 + 0.075x_6)$$

$$\Rightarrow Z = 0.3622x_1 + 0.3192x_2 + 0.3464x_3 + 0.2852x_4 + 0.1560x_5 + 0.2025x_6$$

Subject to the following constraints:

- Total funds available for disbursement is GH¢20,000,000.00

  $$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 20$$

- Salary, Funeral and Commercial loans should be at most 60% of the total funds

$$x_1 + x_2 + x_3 \le 0.6(20)$$

$$\Rightarrow x_1 + x_2 + x_3 \le 12$$

- Housing loan should be at most 50% of Salary and Funeral loans.

$$\Rightarrow x_6 \le 0.5(x_2 + x_3)$$

$$\Rightarrow -0.5x_2 - 0.5x_3 + x_6 \le 0$$

- The sum of Susu and Agriculture loans must not be more than 40% of Commercial and Housing loans.

$$\Rightarrow x_4 + x_5 \le 0.4(x_1 + x_6)$$

$$\Rightarrow -0.4x_1 + x_4 + x_5 - 0.4x_6 \le 0$$

- The sum of the Agriculture and Funeral loans should be at most 15% of the total funds.

$$\Rightarrow x_2 + x_5 \le 0.15(20)$$

$$\Rightarrow x_2 + x_5 \le 3$$

- The overall ratio for bad debts in all loans should not exceed 0.045

$$\Rightarrow \frac{0.020x_1 + 0.030x_2 + 0.010x_3 + 0.055x_4 + 0.150x_5 + 0.075x_6}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} \le 0.045$$

$$\Rightarrow -0.025x_1 - 0.015x_2 - 0.035x_3 + 0.010x_4 + 0.105x_5 + 0.030x_6 \le 0$$

- Non-negativity

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \text{ and } x_6 \geq 0$$

Hence the Standard LP form is as shown below:

Maximize $Z = 0.3622x_1 + 0.3192x_2 + 0.3464x_3 + 0.2852x_4 + 0.1560x_5 + 0.2025x_6$

Subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 20$$

$$x_1 + x_2 + x_3 \leq 12$$

$$-0.5x_2 - 0.5x_3 + x_6 \leq 0$$

$$-0.4x_1 + x_4 + x_5 - 0.4x_6 \leq 0$$

$$x_2 + x_5 \leq 3$$

$$-0.025x_1 - 0.015x_2 - 0.035x_3 + 0.010x_4 + 0.105x_5 + 0.030x_6 \leq 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

4.3 CONVERTING THE STANDARD LP FORM INTO KARMARKAR'S FORM

The LP standard form is shown below:

Maximize $0.3622x_1 + 0.3192x_2 + 0.3464x_3 + 0.2852x_4 + 0.1560x_5 + 0.2025x_6$

Subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 20$$

$$x_1 + x_2 + x_3 \leq 12$$

$-0.5x_2 - 0.5x_3 + x_6 \leq 0$

$-0.4x_1 + x_4 + x_5 - 0.4x_6 \leq 0$

$x_2 + x_5 \leq 3$

$-0.025x_1 - 0.015x_2 - 0.035x_3 + 0.010x_4 + 0.105x_5 + 0.030x_6 \leq 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

- Writing the dual of the given primal problem:

  minimize $20w_1 + 12w_2 + 3w_5$

  subject to :

$$w_1 + w_2 - 0.4w_4 - 0.025w_6 \geq 0.3622$$

$$w_1 + w_2 - 0.5w_3 + w_5 - 0.015w_6 \geq 0.3192$$

$$w_1 + w_2 - 0.5w_3 - 0.035w_6 \geq 0.3464$$

$$w_1 + w_4 + 0.010w_6 \geq 0.2852$$

$$w_1 + w_4 + w_5 + 0.105w_6 \geq 0.1560$$

$$w_1 + w_3 - 0.4w_4 + 0.030w_6 \geq 0.2025$$

$$w_1, w_2, w_3, w_4, w_5, w_6 \geq 0$$

- Introduction of slack and surplus variables to the constraints of the primal and the dual respectively and then combine them

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 20$

$x_1 + x_2 + x_3 + x_8 = 12$

$$-0.5x_2 - 0.5x_3 + x_6 + x_9 = 0$$

$$-0.4x_1 + x_4 + x_5 - 0.4x_6 + x_{10} = 0$$

$$x_2 + x_5 + x_{11} = 3$$

$$-0.025x_1 - 0.015x_2 - 0.035x_3 + 0.010x_4 + 0.105x_5 + 0.030x_6 + x_{12} = 0$$

$$w_1 + w_2 - 0.4w_4 - 0.025w_6 - w_7 = 0.3622$$

$$w_1 + w_2 - 0.5w_3 + w_5 - 0.015w_6 - w_8 = 0.3192$$

$$w_1 + w_2 - 0.5w_3 - 0.035w_6 - w_9 = 0.3464$$

$$w_1 + w_4 + 0.010w_6 - w_{10} = 0.2852$$

$$w_1 + w_4 + w_5 + 0.105w_6 - w_{11} = 0.1560$$

$$w_1 + w_3 - 0.4w_4 + 0.030w_6 - w_{12} = 0.2025$$

$$0.3622x_1 + 0.3192x_2 + 0.3464x_3 + 0.2852x_4 + 0.1560x_5 + 0.2025x_6 = 20w_1 + 12w_2 + 3w_5$$

$$w_i \ge 0, x_i \ge 0 \quad i = 1,2,3,\ldots,12$$

- Addition of bounding constraint with slack variable s :

$$\sum_{i=1}^{12} x_i + \sum_{i=1}^{12} w_i + s = k$$

$$k = 100$$

$$\Rightarrow \sum_{i=1}^{12} x_i + \sum_{i=1}^{12} w_i + s = 100$$

- Homogenized equivalent system with dummy variable $\boldsymbol{d}$:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - 20d = 0$$

$$x_1 + x_2 + x_3 + x_8 - 12d = 0$$

$$-0.5x_2 - 0.5x_3 + x_6 + x_9 = 0$$

$$-0.4x_1 + x_4 + x_5 - 0.4x_6 + x_{10} = 0$$

$$x_2 + x_5 + x_{11} - 3d = 0$$

$$-0.025x_1 - 0.015x_2 - 0.035x_3 + 0.010x_4 + 0.105x_5 + 0.030x_6 + x_{12} = 0$$

$$w_1 + w_2 - 0.4w_4 - 0.025w_6 - w_7 - 0.3622d = 0$$

$$w_1 + w_2 - 0.5w_3 + w_5 - 0.015w_6 - w_8 - 0.3192d = 0$$

$$w_1 + w_2 - 0.5w_3 - 0.035w_6 - w_9 - 0.3464d = 0$$

$$w_1 + w_4 + 0.010w_6 - w_{10} - 0.2852d = 0$$

$$w_1 + w_4 + w_5 + 0.105w_6 - w_{11} - 0.1560d = 0$$

$$w_1 + w_3 - 0.4w_4 + 0.030w_6 - w_{12} - 0.2025d = 0$$

$$0.3622x_1 + 0.3192x_2 + 0.3464x_3 + 0.2852x_4 + 0.1560x_5 + 0.2025x_6 - 20w_1 - 12w_2 - 3w_5 = 0$$

$$\sum_{i=1}^{12} x_i + \sum_{i=1}^{12} w_i + s - 100d = 0$$

$$\sum_{i=1}^{12} x_i + \sum_{i=1}^{12} w_i + s + d = 101$$

$$w_i \geq 0, x_i \geq 0 \quad i = 1,2,3,\ldots,12$$

- Introduction of transformations:

$$x_i = (k+1)y_i \quad \Rightarrow x_i = 101y_i \quad i = 1,2,3,\ldots,12$$

$$w_i = (k+1)y_{12+i} \quad \Rightarrow w_i = 101y_{12+i} \quad i = 1,2,3,\ldots,12$$

$$s = (k+1)y_{25} \qquad \Rightarrow s = 101y_{25}$$

$$d = (k+1)y_{26} \qquad \Rightarrow d = 101y_{26}$$

- Using the above transformations, the system becomes:

$$101(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 - 20y_{26}) = 0$$

$$101(y_1 + y_2 + y_3 + y_8 - 12y_{26}) = 0$$

$$101(-0.5y_2 - 0.5y_3 + y_6 + y_9) = 0$$

$$101(-0.4y_1 + y_4 + y_5 - 0.4y_6 + y_{10}) = 0$$

$$101(y_2 + y_5 + y_{11} - 3y_{26}) = 0$$

$$101(-0.025y_1 - 0.015y_2 - 0.035y_3 + 0.010y_4 + 0.105y_5 + 0.030y_6 + y_{12}) = 0$$

$$101(y_{13}+y_{14}-0.4y_{16}-0.025y_{18}-y_{19}-0.3622y_{26})=0$$

$$101(y_{13}+y_{14}-0.5y_{15}+y_{17}-0.015y_{18}-y_{20}-0.3192y_{26})=0$$

$$101(y_{13}+y_{14}-0.5y_{15}-0.035y_{18}-y_{21}-0.3464y_{26})=0$$

$$101(y_{13}+y_{16}+0.010y_{18}-y_{22}-0.2852y_{26})=0$$

$$101(y_{13}+y_{16}+y_{17}+0.105y_{18}-y_{23}-0.1560y_{26})=0$$

$$101(y_{13}+y_{15}-0.4y_{16}+0.030y_{18}-y_{24}-0.2025y_{26})=0$$

$$101(0.3622y_1+0.3192y_2+0.3464y_3+0.2852y_4+0.1560y_5+0.2025y_6-20y_{13}-12y_{14}-3y_{17})=0$$

$$101(\sum_{i=1}^{12}y_i+\sum_{i=13}^{24}y_i+y_{25}-100y_{26})=0$$

$$101(\sum_{i=1}^{12}y_i+\sum_{i=13}^{24}y_i+y_{25}+y_{26})=101$$

- The system is further simplified to become:

$$y_1+y_2+y_3+y_4+y_5+y_6+y_7-20y_{26}=0$$

$$y_1+y_2+y_3+y_8-12y_{26}=0$$

$$-0.5y_2-0.5y_3+y_6+y_9=0$$

$$-0.4y_1+y_4+y_5-0.4y_6+y_{10}=0$$

$$y_2+y_5+y_{11}-3y_{26}=0$$

$$-0.025y_1 - 0.015y_2 - 0.035y_3 + 0.010y_4 + 0.105y_5 + 0.030y_6 + y_{12} = 0$$

$$y_{13} + y_{14} - 0.4y_{16} - 0.025y_{18} - y_{19} - 0.3622y_{26} = 0$$

$$y_{13} + y_{14} - 0.5y_{15} + y_{17} - 0.015y_{18} - y_{20} - 0.3192y_{26} = 0$$

$$y_{13} + y_{14} - 0.5y_{15} - 0.035y_{18} - y_{21} - 0.3464y_{26} = 0$$

$$y_{13} + y_{16} + 0.010y_{18} - y_{22} - 0.2852y_{26} = 0$$

$$y_{13} + y_{16} + y_{17} + 0.105y_{18} - y_{23} - 0.1560y_{26} = 0$$

$$y_{13} + y_{15} - 0.4y_{16} + 0.030y_{18} - y_{24} - 0.2025y_{26} = 0$$

$$0.3622y_1 + 0.3192y_2 + 0.3464y_3 + 0.2852y_4 + 0.1560y_5 + 0.2025y_6 - 20y_{13} - 12y_{14} - 3y_{17} = 0$$

$$\sum_{i=1}^{25} y_i - 100y_{26} = 0$$

$$\sum_{i=1}^{26} y_i = 1$$

$$y_i \geq 0\,, \quad i = 1,2,3,\ldots,26$$

- An artificial variable y_{27} is introduced and the Karmarkar's form is :

Minimize $z = y_{27}$

Subject to:

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 - 20y_{26} + 13y_{27} = 0$$

$$y_1 + y_2 + y_3 + y_8 - 12y_{26} + 8y_{27} = 0$$

$$-0.5y_2 - 0.5y_3 + y_6 + y_9 - y_{27} = 0$$

$$-0.4y_1 + y_4 + y_5 - 0.4y_6 + y_{10} - 2.2y_{27} = 0$$

$$y_2 + y_5 + y_{11} - 3y_{26} = 0$$

$$-0.025y_1 - 0.015y_2 - 0.035y_3 + 0.010y_4 + 0.105y_5 + 0.030y_6 + y_{12} - 1.07y_{27} = 0$$

$$y_{13} + y_{14} - 0.4y_{16} - 0.025y_{18} - y_{19} - 0.3622y_{26} - 0.2128y_{27} = 0$$

$$y_{13} + y_{14} - 0.5y_{15} + y_{17} - 0.015y_{18} - y_{20} - 0.3192y_{26} - 1.1658y_{27} = 0$$

$$y_{13} + y_{14} - 0.5y_{15} - 0.035y_{18} - y_{21} - 0.3464y_{26} - 0.1186y_{27} = 0$$

$$y_{13} + y_{16} + 0.010y_{18} - y_{22} - 0.2852y_{26} - 0.7248y_{27} = 0$$

$$y_{13} + y_{16} + y_{17} + 0.105y_{18} - y_{23} - 0.1560y_{26} - 1.949y_{27} = 0$$

$$y_{13} + y_{15} - 0.4y_{16} + 0.030y_{18} - y_{24} - 0.2025y_{26} + 1.5725y_{27} = 0$$

$$0.3622y_1 + 0.3192y_2 + 0.3464y_3 + 0.2852y_4 + 0.1560y_5 + 0.2025y_6 - 20y_{13} - 12y_{14} - 3y_{17} + 33.3285y_{27} = 0$$

$$\sum_{i=1}^{25} y_i - 100y_{26} + 75y_{27} = 0$$

$$\sum_{i=1}^{27} y_i = 1$$

$$y_i \geq 0, \quad i = 1,2,3,\ldots,27$$

4.4 SUMMARY OF MODEL INSTANCE

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -20 & 13 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 8 \\
0 & -0.5 & -0.5 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -0.4 & 0 & 1 & 1 & -0.4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.2 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\
-0.025 & -0.015 & -0.035 & 0.01 & 0.105 & 0.03 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.07 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -0.4 & 0 & -0.025 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3622 & -0.2128 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -0.5 & 0 & 1 & -0.015 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -0.3192 & -1.1658 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -0.5 & 0 & 0 & -0.035 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -0.3464 & -0.1186 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0.010 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -0.2852 & -0.7248 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0.105 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.1560 & -1.9490 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -0.4 & 0 & 0.030 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -0.2025 & 1.5725 \\
0.3622 & 0.3192 & 0.3464 & 0.2852 & 0.156 & 0.2025 & 0 & 0 & 0 & 0 & 0 & 0 & -20 & -12 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 33.3285 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -100 & 75
\end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

4.5 COMPUTATIONAL PROCEDURE

The computer brand used in running the programming code was PACKARD BELL with 100GB capacity hard disk drive, processing speed of 1.80GHz and a random access memory (RAM) of 0.99GB. The programming code was written in matlab to run data: ***A***, ***C*** and ***E***. The programming code can be found in the appendix. It was run using a tolerance of 1.0×10^{-15} and converges at iteration 12,757 to produce the final results as shown below;

4.6 RESULTS

The results for the six (6) basic variables are:

$y_1 = 0.0476$, $y_2 = 0.0220$, $y_3 = 0.0565$, $y_4 = 0.0254$, $y_5 = 0.0003$, $y_6 = 0.0390$

Using the transformation $x_i = 101y_i$, the variables for the main LP problem are calculated as:

$$x_1 = 101y_1 = 101(0.0476) = 4.8076$$

$$x_2 = 101y_2 = 101(0.0220) = 2.2220$$

$$x_3 = 101y_3 = 101(0.0565) = 5.7065$$

$$x_4 = 101y_4 = 101(0.0254) = 2.5654$$

$$x_5 = 101y_5 = 101(0.0003) = 0.0303$$

$$x_6 = 101y_6 = 101(0.0390) = 3.9390$$

The main objective is to:

Maximize $Z = 0.3622x_1 + 0.3192x_2 + 0.3464x_3 + 0.2852x_4 + 0.1560x_5 + 0.2025x_6$

$$\Rightarrow Z = 0.3622(4.8076) + 0.3192(2.2220) + 0.3464(5.7065) + 0.2852(2.5654) + 0.1560(0.0303) + 0.2025(3.9390)$$

$$\Rightarrow Z = 5.9613$$

4.7 SUMMARY OF RESULTS

The computational results presented in this thesis illustrate the use of Karmarkar's interior point method for linear programming. The iteration converges at iteration 12,757 and using a tolerance of 1.0×10^{-15} the outcome is as shown in the table below:

TABLE 4.7 SUMMARY OF RESULTS FOR THE ALLOCATION OF FUNDS

Variable	Loan Type	Amount to be Allocated(GH¢)
x_1	Commercial Loan	4,807,600.00
x_2	Funeral Loan	2,222,000.00
x_3	Salary Loan	5,706,500.00
x_4	Susu Loan	2,565,400.00
x_5	Agricultural Loan	30,300.00
x_6	Housing Loan	3,939,000.00

Using the above allocations, Capital Rural Bank Ltd. (Dominase-Agency, Sunyani) could realize a maximum profit of GH¢5,961,300.00 on loans.

The solution meets the policy of loan preference of the bank since the result shows Salary loan takes the lead followed by commercial loan then housing loan then susu loan then funeral loan and finally agricultural loan.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

The data collected from Capital Rural Bank Ltd. was modeled into a standard linear programming problem. Karmarkar's projective scaling method which is a polynomial time algorithm was used for the computation.

Optimizing the disbursement of funds available for loans from Capital Rural Bank Limited would result in the appropriate allocation of funds to their customers. The disbursement is as follows: Commercial loan = GH¢4,807,600.00, Funeral loan = GH¢2,222,000.00, Salary loan = GH¢5,706,500.00, Susu loan = GH¢2,565,400.00, Agricultural loan = GH¢30,300.00 and Housing loan = GH¢3,939,000.00. The result shows that the bank would be able to make a maximum profit of **GH¢5,961,300.00** on loans alone as against **GH¢ 2,653,570.00** made at the end of 2011 if this optimum disbursement policy is strictly followed.

5.2 RECOMMENDATIONS

It was realised from the conclusion that the use of scientific methods to give out loans help banks to avoid giving out loans that do not yield profit there by allocating funds to areas they are sure to get maximum returns. It is therefore recommended that Capital Rural Bank Ltd. (Dominase-Agency, Sunyani) adopt this model in their allocation of funds for loans.

Training is therefore necessary for the loan officers to help them in the implementation of this model of maximization of profit on loans.

It is also recommend that Banks and other financial institutions be educated to employ
scientific methods such as the use of mathematical models to help them disburse funds of the bank more effectively and profitably.

Lastly, it is also recommended that Karmarkar's algorithm should further be researched by students.

REFERENCES

1. Abbas, T.E and Majid, S (2011). "Bank Loan Portfolio, an MOLP Based Modeling". International Journal of Finance and Economics. Vol. 25, pp. 28 - 41

2. Acerbi, C. and Tasche, D (2001), "On the coherence of expected shortfall". Working paper , http://www.gloriamundi.org. Accessed September, 2012

3. Acharya, U., (2003). Does *Industry-Wide Distress Affect Defaulted Firms – Evidence from Creditor Recoveries*. London School of Business Working Paper.

4. Almgren, R. and Chriss, N. (2001). "Optimal execution of portfolio transactions". Journal of Risk, 3, 5-39.

5. Bakshi G. D., D. Madan, F. Zhang (2001). Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates, *Finance and Economics Discussion Series* 2001-37, Federal Reserve Board of Governors.

6. Bazara, 1984 and Hamdy, 1992. "Karmarkar's Projective Algorithm"

7. Blume M.E**.** (1970). Portfolio Theory - A Step Towards its Practical Application. Journal of Business. Vol. 43, pp. 152-173.

8. Booth, G. G., (1972). Programming Bank Portfolios under Uncertainty, "*Journal of Bank Research* Vol. 3, No. 4, pp. 28-40.

9. Brealey, R. A. and Hodges, S. D (1972). Playing with Portfolios. The Journal of Finance, Vol. 30, No. 1, pp. 125-134

10. Carey M., M. Gordy (2003). Systematic risk in recoveries on defaulted debt, Unpublished.

11. Chambers and Chames. (1961). Intertemporal Analysis and optimization of bank portfolios, Management Science II, July 1961.

12. Cohen and Hamer, (1972). Linear programming and optimal Bank Asset Management Decision.Journal of Finance vol 22, No.2, pp.147-165.

13. Dhamija, A.K (2012). "Karmarkar's Algorithm for Linear Programming Problem" www.slideshare.net/akdhamija/karmarkars-algorithm-for-linear-programming. (Accessed October, 2012).

14. Dince, R. R., and Fortson, J. C, (1977). An Application of Goal Programming to the Management of a Country Bank, *Journal of Bank Research.* Vol. 7, No. 4 , pp. 311-319.

15. Finger C. (1999). *A Comparison of Stochastic Default Rate Models*, Working Paper 00-022.

16. Finlay, S. (2009). "Consumer Credit Fundamentals". Second Edition. Palgrave Macmillan, New York, U.S.A

17. Forsgren, A., Philip E., Gill, M. and Wright, H. (2002). Interior Methods for Nonlinear Optimization. Society for Industrial and Applied Mathematics. Vol. 44. No. 4. pp. 525–597.

18. Gordy, M. (2000). A comparative anatomy of credit risk models. *Journal of Banking and Finance* 24, 119-149

19. Grunert, J., Norden, L., Weber, M. (2005). The Role of Non-Financial Factors in Internal

20. Gurtler M., D. Heithecker (2005). Systematic credit cycle risk of financial collaterals:

21. Harold, H (1998). "The Human Factor and Effective Management of Capital and Technology"

22. Hodges S. and Brealey R**.** (1972). Portfolio Selection in a Dynamic and Uncertain World. *Financial Analyst Journal*. Vol 5, pp 58-69.

23. Hu Y. T., Perraudin W. (2002). *The dependence of recovery rates and defaults*, Birkbeck College.

24. Jarrow R. A., Turnbull S. (2001). The Intersection of Market and Credit Risk, *Journal of Banking and Finance* 24, 271–299.

25. Jianq H., Li Z. and Drew M.S. (2004), *Optimizing motion Estimation with Linear Programming and Detail – Preserving variational Method*, IEEE Computer Society Conference on computer vision and pattern recognition, Vol. 1, pp. 738 – 745.

26. K. Duellmann and M. Trapp (2004). "Systematic risk in recovery rates–an empirical analysis of US corporate credit exposures". Deutsche Bundesbank Discussion Paper 02/04, 2004.

27. Lesaja, G. (2009). Introducing Interior-Point Methods for Introductory Operations Research Courses and/or Linear Programming Courses. Open Operational Research Journal. Vol. 3. pp. 1-12.

28. Mandansky A. (1960), *Inequalities for stochastic Linear Programming Problems,* Management Science, Vol. 6, pp. 197 – 204.

29. Markowitz H.M. (1952). , Portfolio Selection. *Journal of Finance*.Vol 7, pp. 77-91.

30. Melnik A. (1970). Short-run Determinants of Commercial Bank Investment Portfolios - An Empirical Analysis. *Journal of Finance*. Vol. 25, No. 3**,** pp. 639 - 649.

31. Paul, A. J and Jonathan, F.B (2009). "Operation Research Models and Methods"

32. Sinha S.B. and Sinha S. (2003), *A linear Programming approach for Linear multi – level Programming Problems*, Department of Mathematics, India institute of Technology (IIT), Kharagpur, India.

33. Stewart N.E., Feng J. and Stockbridge R.H. (2008), *Determine the optimal control of singular stochastic processes using linear programming*, Beachwood, Ohio, USA.

34. Waterman, R. H. and Gee, R. E.,(1981). An Application of Linear Programming to Bank Financial. *The Institute of Management Sciences*. Vol. 11, No. 5, pp. 77-83.

35. Wikimedia Foundation Inc. Central bank. http://en.wikipedia.org/wiki/Central_bank, (Accessed August, 2012).

36. Wikimedia Foundation Inc. History of banking. http://en.wikipedia.org/wiki/History_of_ Banking. (Accessed August, 2012).

37. Wikimedia Foundation Inc. Karmarkar's algorithm. http://en.wikipedia.org/wiki/Karmarkar% 27s_algorithm. (Accessed September,2012)

38. Wikimedia Foundation Inc. Loan. http://en.wikipedia.org/wiki/Loan. (Accessed August, 2012).

39. Wikimedia Foundation, Inc. Bank of Ghana. http://en.wikipedia.org/wiki/Bank_of_Ghana. (Accessed August, 2012).

40. Wu X., Zhu Y. and Luo L. (2000), *Linear Programming base on neural networks for radiotherapy treatment planning*, Institute of Physics and IOP limited, USA

APPENDIX A: MATLAB CODES FOR KARMARKAR’S ALGORITHM


```
% A is m x n matrix
% C is  a colum vector
% E is a row vector  with 1’s as its entries
% k is the number of iterations
% tol is the tolerance
A=input(‘Enter the matrix A:’)
C=input(‘Enter  C:’)
E=input(‘Enter  E:’)
mn =size(A)
m=mn(1)
n=mn(2)
X1= ([E]/n)'
I=eye (n)
r=1/sqrt(n*(n-1))
α= (n-1)/(3*n)
X= ([E]/n)'
tol=10^(-15)
k=0
while(C'*X>tol)
D=diag(X)
T=A*D
P= [T; E]
```

```
Q=C'*D

R= (I-P'/(P*P' )*P)*Q'

RN=norm (R)

Y=X1-(r*α)*(R/RN)

X= (D*Y)/(E*D*Y)

Z=C'*X

k=k+1

end
```