KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

TIME DEPENDENT VARIABLE IN THE BLACK-SCHOLES : AN APPLICATION IN LIFE INSURANCE CONTRACTS

BY

OTOO EZRA MICHAEL

(PG1079313)

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY IN PARTIAL FUFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF M.PHIL ACTUARIAL SCIENCE

NOVEMBER 2015

DECLARATION

I hereby declare that this submission is my own work towards the award of the M.phil Actuarial Science degree and that, to the best of my knowledge, it contains no material previously published by another person nor that which had been accepted for the award of any other degree of the university, except where due acknowledgment had been made in the text.

MICHAEL EZRA OTOO	....	....
(Student)	(Signature)	(Date)

Certified by:

Dr. G. A. OKYERE	....	....
(Supervisor)	(Signature)	(Date)

Certified by:

Prof. S. K. AMPONSAH	....	....
(Head of Department)	(Signature)	(Date)

DEDICATION

I dedicate this work to the Almighty God and to Dr. William Rockson and Mrs. Bella Rockson for making this dream a reality. God bless you for making this possible is all i can say.

ABSTRACT

Valuations of actuarial liabilities (benefits of insurance contracts) have been of major concern to managers and policy holders in the financial sector. These valuations are done through the standard BS-equation which assumes a constant rate of return. However literature has it that rate of return is time dependent and valuation under constant rate of return may impair the conclusions made on life insurance contracts. This study seeks to solve the modified Black-Scholes partial differential equation with the incorporation of time dependent rate of return in the valuation of life insurance contracts. Further, solutions to both the standard and modified model under two iterative techniques, Hopscotch and Crank-Nicolson, were investigated and compared. In line with these objectives, data was simulated and estimates for speed and level were computed for using Vasicek model. These estimates were now adopted in modeling the interest rate as a Cox-Ingersoll-Ross process. Solutions to both the standard and modified BS indicates that the Hopscotch converges faster than the Crank-Nicolson but the Crank-Nicolson gives consistent values than the Hopscotch. Further it was observed that value of life insurance contract from the modified model are much lower than that of the existing model.

ACKNOWLEDGMENT

My sincere gratitude goes to the Lord God Almighty for His unfavored grace and mercy, wisdom to complete this book. Special appreciation goes my supervisor Dr. G. A. Okyere for his endless time and support in terms of corrections and suggestions in writing this thesis. I say I am really grateful. I want to use this platform to show my gratitude to the head of Mathematics,Prof. S.K. Amponsah for his encouragement and support. To the entire staff of Mathematics department of KNUST, I take this opportunity to express my gratitude to you all for help and support. I say may God bless and sustain you in His Divine grace and mercy.

I also want to take this opportunity to say thank you to my parents and family especially Mr and Mrs Rockson for making this dream a reality and also for their unceasing encouragement, support and attention.

Lastly, I want to show gratitude to all friends, all KNUST students, my course mates especially Mac-Billa Issaka, Akwasi Boateng and Atikatu Akilu for the advice and support in all forms through this venture.

CONTENTS

LIST OF ABBREVIATION

BS .... Black-Scholes

PDE .... Partial Differential Equation

LIC's .... Life Insurance Contracts

IFRS .... International Financial Reporting Standard

IAIS .... International Association of Insurance Supervisors

FCAC .... Financial Consumer Agency of Canada

EU .... European Union

OECD .... Organisation for Economic co-operation and Development

CEE .... Central and East European

NMS .... New Member State

GDP .... Gross Domestic Products

VAR .... Value at Risk

ELEPAVG ... Equity-Linked Endownment Policy with Asset Value Guarantee

IRDA .... Insurance Regulatory and Development Authority

P(I)DEs .... Partial(Integro)Differential Equations

ALM .... Asset-Liability Management

SDE .... Stochastic Differential Equation

ODE .... Ordinary Differential Equation

CIR .... Cox-Ingersoll-Ross

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Valuation of actuarial liabilities thus benefits paid to policyholders have been of great concern to managers and policyholders in the financial sector. Since the implementation of certain regulatory requirements such as solvency II and the International Financial Reporting Standard(IFRS), valuation of life insurance contracts have become very vital in the insurance sectors, Anders and Peter (2002) In a statement by the International Association of Insurance Supervisors (I.A.I.S.) at the 2006 solvency and Actuarial Issue Subcommittee addressed the issue of embedded options in insurance contracts.It said even though the uncertainty of embedded options can not be distinguished, its essential that insurers regulate their assets and liabilities in a way that will minimize their potential impact. Pricing of life insurance contracts with embedded options were issues ruled in the international Financial Reporting Standards for insurance contracts (I.F.R.S.) and solvency II.

A life insurance contract with an embedded surrender option can be considered as an American contingent claim. The surrender option is a type of option that gives the owner the right to sell back the contract to the insurer for an amount (surrender value)before maturity after which the contracts expires. As suggested by Smith (1985) and Walden (1985), life insurance contract usually offer policyholder a variety of options and can therefore be regarded as option packages. The policyholder is the holder of the contract who makes payments to the insurer (usually insurance company) and receives a promised sum of money upon occurrence of a specific event.

Many researches have dealt with the valuation of insurance liabilities, using the Black-Scholes model developed by Black, F. and Scholes, M. in 1973.

Bacinello(2001), Giraldi, C., Susinno, G., Berti, G., Brunello, J., Buttarazzi, S., Cenciarelli, G., Daroda, C. and Stamega, G. in 2003 ; Grosen and Jorgensen, 2000 ; Hansen and Miltersen, 2002 all considered the pricing of life insurance contracts with embedded options such as insurance guarantees, bonus provisions under the Black-Scholes framework. Life insurance contracts containing options such as surrender options can be considered as option packages and pricing of options has being extensively covered by the Black-Scholes model.

Black, Scholes and Merton obtained a closed form expression for assess the price of an European call and put option written on a non dividend paying stock and also assumed the rates of return and volatility are constant. One of the main out standings is that they proposed a closed form formula that is easily evaluable without numerical methods. Also some set backs is that volatility and rates of return are not constant in real financial markets. Studies by Mandelbrot (1963) and Fama (1965) have suggested that short run returns in the financial market especially commodity and stock markets do not follow the normal distribution but have fat tails and are peaked i.e. they have leptokurtic distributions.

Valuations of life insurance contracts under the Black-Scholes(BS) equation assumes a constant rate of return and volatility but literature suggests rates of return and volatility vary with respect to time. Further recent research also suggests that these assumptions are often violated in the financial sectors. Hence the need to look at it in the valuation of life insurance contracts using the BS-equation.

Life insurance contracts are widely considered as a put option and valuations are done under the standard Black-Scholes equation. This research seeks to value a Life insurance contracts by applying the Black-Scholes equation with time dependent rates of return (interest rates).

1.1 Background

Insurance can be considered as a promise of compensation for a specific future losses in exchange for a periodic payment. It is set up to give protection to individuals , companies or other entities financially in an event of unexpected loss.The contract between the insured and insurer is the insurance policy. Agreeing to the terms of an insurance policy creates a contract between the insured (the person or entity buying the insurance) and the insurer (the insurance company selling the contract)

1.1.1 life Insurance

Life insurance is an insurance coverage that pays out a certain amount of money (benefits) to the insured or their specified beneficiaries upon a certain event such as death of the individual who is insured. Depending on the contract, other events such as terminal illness or critical illness can also trigger payments. The policyholder typically pays a premium either regularly or as a lump sum. Other expenses (such as funeral expenses) can also be included in the benefits.

1.1.2 Types Of Life Insurance

The types of life insurance is classified according to how and when the benefits are paid.

Term Insurance

Term life insurance is a basic coverage that generally does not build life insurance policy as cash value. It provides death benefit protections for a specific period of time. Because it provides coverage for a defined period, it's premium is usually lower than the other types of life insurance. However, premiums will probably increase if one renews or converts his or her coverage.

Whole Life Insurance

Whole life insurance is a type of life insurance policy in which a level premium guarantees a life time death benefit. In comparison to the term insurance, the whole life insurance premiums is much expensive. This is due to the increasing premiums of the term insurance during it's life span as the age of the insured increases. But in effect, if we are able to maintain the average life expectancy, the cumulative premiums of whole life insurance and that of the term insurance are roughly equal.

Endowment Life Insurance

This type of life insurance policy guarantees benefits at a given age rather than a death benefit amount. Its premiums are much more expensive than the other policies because premiums are paid in short periods and maturity dates are earlier.

Guarantee Universal Life Insurance

This provides a death benefit as well as to build policy cash value. This coverage is different from term and whole life insurance because, within policy limits, they can vary the amount and timing of premiums.

1.1.3 Options

According to research, life insurance contracts can be treated as options and hence the need to look at options and it's types.

An option gives the holder the right but not obliged to either buy or sell an underlying asset by a certain date (maturity) for a certain price (strike price). There are two types of options :

- The call option, which gives the holder the right to buy an underlying asset by a certain date for a specified price and

- The put option, which gives the holder the right to sell the underlying asset by a certain date for a certain price.

The price in the contract is known as the exercise price or strike price ; the date in the contract is known as the expiration date or maturity.
American options can be exercised at any time up to maturity date. European options can only be exercised on the expiration date itself.
In life insurance contracts, the policyholder has the right to sell back the contract to the insurer (typically the insurance company) for a specific price at maturity and is therefore considered as a put option. Since the American option provides the investor a greater degree of flexibility than the European style option, the premiums is at least equal or higher than the premiums for an European option otherwise they have the same features. The payoff for both the American style option and European style option for a put option (life insurance contract) is given by $max\ [(K - S), 0]$. where K is the strike price and S is the spot price of the underlying asset.
In finance, a spot contract, spot transaction or simply spot is a contract of buying or selling a commodity, security or currency for settlement (payment and delivery) on the spot date, which is normally two business days after the trade date. The settlement price or rate is called spot price or spot rate.

1.2 Problem Statement

Valuation of actuarial liabilities is a major concern in this modern era. Life insurance contracts contain other options which makes it difficult to valuate analytically because of its complexity. Also the limitations of the Black-Scholes model is been taken into account in this research. Analytical solution to the valuation problem cannot be found because of the complexities in the life insurance contract. Hence the need to consider numerical methods in the valuation of insurance contract in Ghana.

1.3 Objective

The objectives of this study are :

- To solve the Black-Scholes PDE with time dependent coefficients.
- To find solution to the Hopscotch and Crank-Nicholson method.
- To compare the value of a life insurance contract of the existing model to the modified model
- To compare the Hopscotch and Crank-Nicholson method.

1.4 Methodology

Addressing the limitations of the Black Scholes model by closed form valuation is extremely difficult and we resort to numerical methods. The study implements the finite difference algorithm for fast, and accurate numerical valuation of life insurance contracts. We compare the Crank Nicolson method to the Hopscotch method. The implementation of the finite difference algorithm will be achieved by the use of Matlab software for fast and accurate numerical valuation of life insurance contracts.

1.5 Justification

The study considers one of the vital areas in life insurance contracts that is the valuations of liabilities (benefits paid to policyholder). The study implements the finite difference method in the valuation of actuarial liabilities of life insurance companies in Ghana.

1.6 Thesis Organization

The thesis is categorized in five main chapters.The first chapter introduces the thesis and gives a brief introduction into life insurance. This section considers the background of the research, the research problem, objective of the study, the method to be used in the thesis,thesis justification and organization. Chapter 2 is the literature review, which considers briefly what other researchers have done on the topic under consideration. Chapter 3 is the formulation of the mathematical model to be analysed. Chapter 4 contains the data and analysis, formulation of model instances, algorithm, computational procedure, results and discussion. Chapter 5 summarizes, concludes and gives recommendations of the results.

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

This chapters reviews other related works on past insurance and insurance liabilities and some related models from other writers.

2.2 Definition And Meaning Of Insurance

Though by adverse explanation to the term insurance by various researches and writers, they all point to the fact that insurance is a way of hedging against adverse loss.

Pal, Bolda, and Garg, in 2007 explained insurance as a system of diversifying a loss to a number of people that are vulnerable to that particular risk and have accepted to protect themselves against that risk. In 2008, Dorfman defined insurance as a financial understanding between the policyholder and the insurer that the cost of an unanticipated loss will be re dispersed. The study further explains that the redistribution of these costs of loss are done by the collection of certain amounts (premiums) from every member that falls within this uncertainty. In exchange, for the monies collected from these people, the insurer assures the policyholder a certain amount in the event that the unanticipated loss occurs. Mostly, insurance companies absorbs these losses or risk in exchange for a fixed sum which is known as premiums.

Pal et al. (2007) described loss in insurance language as an accidental decrease in or the fading off of value as a result of occurrence of a certain event . The analogy of both definition is that insurance involves the shifting of loss vulnerability to an insurance system and diversifying the cost of losses among members of the system.

Mphasis in 2009 explained insurance in economics and law as a way of managing uncertainty with the aim of hedging against the risk of contingency loss. Furthermore it can also be defined as a way of equitably transferring the chance of a loss from one equity to the other in return for a premium. In a nut shell it can be thought as replacing a devastating loss with a small guaranteed loss. Financial Consumer Agency of Canada FCAC in 2011 defined insurance as a way of reducing your potential financial loss or hardship. Thus insurance is a way of helping reduce a devastating loss or damage. Insurance can also be thought of as means of providing loved ones financial support upon the occurrence of death.

2.3 Relations Of Insurance To The Growth Of Economy

In the last few decades, there has been an increased in the growth of insurance markets. in considering the insurance sector as both a financial mediator and as a source of risk movement and a provider of compensation for a loss encourages the growth of an economy. Thus, the premiums collected are invested and this bring revenue. Further various risk brought under control encourages the accumulation of new capital.

Literature also have suggested that, countries with advanced financial mechanisms appreciate rapid and more steady long term economic growth. There is an adverse positive impact on the total factor productivity of economies with advanced financial systems which in turn interprets into the long term economic growth. According to Haiss and Kjell in 2008 the economic financial impact on growth differs with the development for the insurance section. The significance of the insurance-growth nexus is increasing because of the growing share of the insurance sector in the aggregate financial section in almost every prominent and developed market economy. Insurance contributes to economic growth through the following ways;

- The operations expenses of firms are decreased by insurance
- Insurance promotes investment and transformation by generating an environment of greater assurance.
- Insurance companies are a strong back bone for the evolution of a feasible additional system of social security especially in the area of retirement and provision of health.
- As an established investors, insurers add to the contemporaneous nature of the financial sector and enhances the speed of access of a firm's capital.
- Insurance simulates concrete uncertainty-administration measures through the pricing systems and other techniques and grants accountable and assists economic development.
- Insurance promotes steady utilization throughout the customer's life.

Furthermore, in a worldwide economy, identified by the fast social and demographic change and by the disclosure of new uncertainties (e.g. either climate change or technological enlargements) and new necessities (health care, pensions), partnership between the independent and public organizations is vital. This understanding can generate satisfactions in so many manners, for example averting of accidents, indemnification of agricultural uncertainties , well-being of the working population, global merchandising (transmit financial standings of insurance), etc CEA (2006). The impact of insurance on the economy's transformation has been dealt with by many researches. Beenstock, Dickinson and Khanjuria in 1986 were among the first to do in-depth research on the impact of insurance on the economy. They found a positive impact of income on insurance demand. They used a cross-section and time series data for ten industrialized countries for the period of 1970 and 1981 and found the life insurance demand to be directly positive dependent on income, measured as GDP per capital. A series of empirical study was done on both life and non -life insurance. Using insurance premium as dependent variable and economy's income as explanatory variables, it was realized that, life and non-life insurance directly depends upon economic development. Haiss and Kjell (2008) also researched in the area of the relationship between insurance and economic growth. They conducted a survey on 29 organization for economic cooperation and development (OECD) countries. They used a mix-country yearly group data from 1992 to 2004. The outcome of the results indicated an affirmative effect of life insurance on the Gross Domestic Product(GDP) growth in European Union countries, Switzerland,Norway and Iceland. Further, there was a short-run effect for the Central and Eastern European Countries (CEE) / New Member States (NMS) countries for other insurance consumption other than that of life insurance consumptions. The results of the study concluded that the investigation of the finance growth nexus previously restricted to the banking and capital market sector should be expanded to take in to accounts the insurance sector. Further more, it indicated that similar to banking, the influence of insurance is also based on the level of economic evolution.

Ward and Zurbruegg (2000) conducted the first study to examine the causal relations between insurance industry growth and economic growth. The study looked at the short and long aggressive relationships between economic growths, guarded by the yearly real GDP and the insurance industry also guarded by the real sum of premiums for 9 OECD countries between the periods of 1961 to 1996. As additional explanatory variables, the study used changes in private saving rates, the general government budget surplus, population size, the general government level of current expenditure and youth plus old dependency ratios, measured as the proportion of the total population under 16 and over 65 years of age. Based on bivariate VAR methodology, the study tested for Granger causality, and found out a causal relationship between economic growth and insurance market development vary across countries. Even though they did not find the exact causes, they suspected that possible causes were based on the country's specific nature of culture, regulatory and legal environment, the improvement in financial intermediation and the moral hazard effect of insurance. Other researches also confirm to this conclusions, Outreville (1996) and Enz (2000). They also concluded that elasticity of the demand for insurance varies itself with the level of income. That is to say it becomes less sensitive to income growth in more developed economies. Webb, Grace and Skipper in 2002 examine the causal relationship of banks, life, and non-life insurance activity on economic growth in the context of a revised Solow-Swan neoclassical of economic growth model, where the authors include financial activities (bank, life and non-life insurance) as an additional input in the production function assumed to be a Cobb-Douglas type. The empirical specification falls in the context of the cross-country economic growth regressions literature Robert and Martin (1995), where the growth rate of real GDP per capita is regressed against the change in capital intensity (gross domestic investment), human capital (education enrollment), the ratio of government consumption to GDP, the degree of openness (the ratio of exports to GDP), the initial level of GDP per capita, and measures of financial intermediary activity (the ratio of bank credit to GDP, the ratio of non-life insurance premiums to GDP, and the ratio of life insurance premiums to GDP). For the estimation, the authors use the three-stage-least-squares instrumental variable approach (3SLS-IV), where the instruments used are the legal origin of the country (English, French, German, Scandinavian, or Socialist) for the banking measure, a measure of corruption and quality of the bureaucracy for the non-life insurance measure, and the religious composition of the country (Catholic, Muslim, or Protestant) for the life insurance measure. In order to assess a causal relationship with economic growth, the authors use average levels the explanatory variables. Among the main findings, the authors show that the exogenous components of the banking and life insurance measures are found to be robustly predictive of increased economic growth. However, these measures are not significant in the presence of interaction terms between banking and life insurance, and between banking and non-life insurance; when these interaction terms are included, the individual variables loss explanatory power. Adams, Andersson,J., Andersson, L. and Lindmark in 2005 evaluate both a long-run relationship and Granger-causality between insurance market size and economic growth for the United Kingdom using net written premium for each insurance market (general and long-term insurance) in the UK for the period 1966-2003. Using Johansen's co integration test, the authors find a long-run relationship between development in insurance market size and economic growth for all insurance components. They examine empirically the dynamic historical relation between banking, insurance and economic growth in Sweden using time-series data from 1830 to 1998 using co integration and Granger causality tests. Their results indicate that the development of banking, but not insurance (measure by total insurance premiums), preceded economic growth in Sweden during the nineteenth century, while Granger causality was reverse in the twentieth century. The insurance market appears to be driven more by the pace of growth in the economy rather than leading economic development.

2.4 Life Insurance Contracts

The most common life insurance products the so called participating (or with profits)policy has being analyzed by Anders and Peter (2002). This type of contract varies differently from the unit-linked products according to how the interest are credited to the policy holder. The unit-linked is a type of contract in which interest is credited to the policy periodically according to some mechanism which smooths past returns on the life insurance company's assets. In the participating life products the insurance company's profit is shared with the policyholder. Both the participating policies and unit-linked products are equipped with an interest rate guarantees and possibly also an option to surrender (sell back) the policy to the life Insurance company before maturity. In the participating contracts,profit sharing is gained by various means. Generally, a mathematical reserve is created in which dividends are credited to these accounts at the end of each policy year and this involve the acquisition of extra insurance.These interest are then modified in outcome of the mathematical reserves. In some cases of the periodic premium policies, the policyholder contributes to the increment of the interest by way of an increment in the periodic premiums. According to Anders and Peter (2002) the predictable participating policy can be disintegrated into a bonus option, risk- free bond element and surrender option. An aggressive model was built in which the variables were separated using contingent claims analysis. The study also did a numerical study on the effects of the of the different bonus policies and various levels of the guaranteed interest rate. In conclusion, the research identified that the values of participating policies are very responsive to the bonus policy , that surrender options can be completely costly, and that life insurance company's solvency can be rapidly jeopardize if income favorable time decline in a situation where bonus reserves are little and guaranteed outcomes are high. As suggested by Grosen and Joergensen (2000) life insurance contract characterizes some yearly supplementary participation. In this type of contract , the majority of the pledged interest is yearly credited to the policy and in turns becomes a part of the guarantee, which is why policies of this nature are called a Cliquet-style guarantee. The insurance contracts' market value, as well as the insurance company's risk, depends on the guaranteed interest rate as well as on the amount of ongoing surplus. In 2002,Grosen and Jorgensen promoted a contingent claim approach to the market of equity and liabilities in life insurance companies. The model derived explicitly takes into account the facts that the holders of life insurance contracts have the first claim on the companies assets whereas equity holders have limited liability, that interest rates guarantees are common elements of life insurance companies, and that life insurance companies according to so-called contribution principle are entitled to receive a fair share of investment surplus. Furthermore, a regulatory mechanism in the form of an intervention rule was built into their model. This mechanism is shown to significantly reduce the insolvency risk of the issued contracts and it implies that various claims on the company's assets become more exotic and obtain barrier options properties. Anders and Peter (2002) used a model with a point-to-point guarantee, that is the company guarantees only a maturity payment and an optional participation in the terminal surplus at expiration of the contract. The contracts market value in this model is basically a function of the guaranteed interest rate and the terminal surplus participation and thus only the guaranteed interest rates influences shortfall risk at maturity. In 2002, Hansen and Miltersen modified the model by Grosen and Joergensen (2000) to analyze the minimum rate of return guarantees for life insurance (investment) contracts. A suitable model for valuation for broad category of participating life insurance policies was developed by Ballotta, Haberman and Wang in 2003. . The model treated these contracts as options written on the reference portfolio backing the policy because of the implied liabilities in the contract. The liabilities earned annually, the greater part of some guaranteed interest rate and a predetermined fraction of the arithmetic average of the last period returns of some portfolio. They used the Monte Carlo techniques to compute the values of the so called "policy reserve", that is the guaranteed payoff and the reversionary bonus,and the terminal bonus. Schmeiser and Wagner (2011) suggested that life insurance contracts have embedded, in addition to an interest rate guarantee and annual surplus participation, a paid up option that is the right to stop payments during the contract term, a resumption option ; the right to resume payments and a surrender option that is the right to terminate the contract early. The study evaluated a joint option in participating life insurance contracts. The study developed a model framework and algorithm to jointly value the options in the life insurance contracts and advanced the model framework of Gatzert and Schmeiser (2008) that is paid up and resumption options with focus on the maximum risk considered a surrender option pricing and Andersen (1999) created an algorithm for valuation of options for one option. The work dealt with a fair and joint valuation of embedded option with the implementation of a robust numerical algorithm. The results showed that substantial value of embedded options are necessary in appropriate pricing and adequate risk management.

2.5 Pricing of Insurance Contracts

Kurz (2003) priced a equity-linked life insurance policy with an asset value guarantee and periodic premiums. The paper established a quasi-explicit formula for the periodic premiums under an equity-linked endowment policy with asset value guarantee in an economy with interest rate risk. The equity-linked endowment policy with asset value guarantee(ELEPAVG) is a life insurance product of the unit-linked type where the sellers provide for a minimum benefit or asset value guarantee, payable on death or maturity. According to Bacinello (2009), in equity-linked life insurance contracts the benefits are directly linked to the value of a reference portfolio composed by units of a given asset that can be, e.g., a stock, a stock index, a mutual fund or a combination of mutual funds. Then the cash value of benefits is stochastic, while premiums are usually deterministic. Premiums, net of insurance and expense loadings, are deemed to be invested in the reference asset. These contracts are generally characterized by a high level of financial risk, that can be totally charged to the policyholder, in pure equity-linked contracts, or can be shared between the policyholder and the insurer, in guaranteed equity-linked contracts. In particular, guarantees can operate only when the benefits become due (terminal or point-to-point guarantees), or contain some ratchet features, that allow to periodically consolidate the greater between actual and guaranteed returns. Some authors considered the pricing of equity-linked life insurance policies with asset value guarantee using modern financial techniques before the work by Annette Kurz. Brennan and Schwartz (1976) and Boyle and Schwartz (1977) recognized that the payoff from an individual equity-linked contract at maturity or expiration is identical to the payoff from an

European call option plus a certain amount (guarantee amount) or to the payoff from an European put option plus the value of the reference portfolio. The insurance premium on an ELEPAVG contract was obtained by the application of the theory of contingent claim pricing. The premium was determined in an economy with the equity following a geometric Brownian motion, whereas the interest rate was assumed to be constant. Further studies with a deterministic interest rate have also been discussed in Delbaen (1986) and Aase and Persson (1994). Delbaen specifically focused on the periodic premium payments but obtained no closed form solution and used Monte Carlo simulations in order to get results. Bacinello/Ortu (1993, 1994) and Neilsen and Sandmann (1995) extended these models and allowed for interest rate risk. To obtain a closed form solution for the price of a single premium endowment policy Bacinello/Ortu described the spot rate of interest by an Ornstein-Uhlenbeck process. Neilsen and Sandmann (1995) analysed the multi premium case and obtained no closed form solution but discussed different numerical procedures. Annette Kurz developed a model for the multi premium case in the context of a stochastic interest rate process which allows to derive a quasi-explicit closed form solution. The new formula simplified the calculation of the periodic premium because it avoids time consuming numerical calculations.

2.6 Valuation Of Life Insurance With Embedded Options

Insurance Regulatory and Development Authority (IRDA) in India defined life insurance as a financial cover for a contingency linked human life, like death, disability, accident, retirement etc. Human life is subject to risks of death and disability due to natural and accidental causes. When human life is lost or a person is disabled permanently or temporally, there is a loss of income in the household.

In the late seventies, a life insurance product was introduced in Italy, this encouraged Bacinello (2001) to consider this life insurance product. These life insurance products were called rivalutabile. She said a unique portfolio of investments was created in which the mathematical reserve of all the policies are contained and are separated from the company's assets.

At each year ending, the rate of return on this portfolio in the coming year is assigned to the policyholder's account with the assurance that this does not go below the technical rate.

Bacinello studied both the policies paying a single premium and multiple premiums which are adjusted according to the performance of the reference portfolio. The study obtained a very simple closed form relation that has similar features of 'fair' contracts in the Black-Merton-Scholes framework. The study however didn't take into account the element of surrender option. She later considered life insurance contracts with embedded options with the aim of bridging the gap and the uses recursive binomial formula patterned after Cox, Ross and Rubinstein (1979) model for pricing single premiums. The problem associated when using such complex contracts by insurance companies were highlighted by Bacinello (2003). The study introduced basic ways of pricing such contracts, but the main problem was that the study didn't present a specific valuation models.

The study started with a fixed term contract that pays a single premium at issuance which is an example of the equity linked life insurance. This type of contract contained the European style options and a financial risk.

Milevsky and Posner,2001 introduced the mortality risk into contracts that turns European options in to Titanic options. They also suggested that in contracts when premiums are paid periodically, the options now contain the Asian feature. The study valuated the guaranteed annuity option and the surrender option,and paid particular attention on the definition of their payoff structure. The study then illustrated some examples of participating contracts, that are usually characterized by more complicated payoffs and can involve other implicit options, such as the bonus option and the default option, and hinted at some additional sources of risk affecting the liabilities of a pension fund. The study concluded that the payoff structures are even more complicated if we consider the liabilities of a pension fund, where the following additional sources of risk are sometimes involved:

- The salary risk, since premiums (also called contributions) are usually expressed as a function of salaries, e.g. a linear homogeneous function (rate of contribution) or a spline of linear homogeneous functions.
- The inflation risk, when pension installments are adjusted according to an inflation index.
- The GDP risk (present in some National Pension Plans), when contributions are accumulated at a rate depending on the performance of the nominal Gross Domestic Product.
- The family risk, because benefits are usually reversible to survivors.

Literature by Anders and Peter (2002) developed a model which states that the holders of a life insurance contracts (LIC's) are considered first for a claim before the equity holders who have limited liabilities, this is to say policyholders have a first chance to a fair share of any investment surplus which are the interest rate guarantees. He further built a regulatory mechanism in the form of an intervention rule into the model. The mechanism shown a significantly reduction in the insolvency risk of the issued contracts and it implied that the various claims on the company's assets become more exotic and obtain barrier option properties. The study derived a closed form of valuation formulas. Numerical techniques were also used to illustrate how the model can be used to establish the set of initially fair contracts and to determine the market values of the contracts after their inception. Daniel et al. (2010) presented a more generic model for the valuation of life insurance contracts with embedded options. The study described the various numerical approaches within the work's generic setup and particularly focused on contracts containing early exercise (surrender option) features since these present (numerically) challenging valuation problems. The study based on a couple of examples of life insurance contracts to illustrate the various approaches and compared their efficiency in a simple and generalized Black-scholes setup,respectively. More in particular, the study considered the impact of the these early exercise feature and analyzed the influence of model risk by the additional introduction of an exponential L'evy model. The results of the study suggests the Monte-Carlo approach yields fast results for European contracts but was inefficient for the valuation of non-European contracts that is contracts with the surrender option. In this case, the number of necessary simulation steps to obtain accurate results may be extremely high. Secondly, the study presented a discretization approach based on the consecutive solution of certain partial (integro-) differential equations and realized that the approach was more apt for the valuation of long term non-European contracts and allows for the calculation of the Greeks, but depending on the model specifications solving the P(I)DEs can be very complex and can slow down the algorithm considerably. Lastly the work also discussed the so-called least square Monte Carlo approach. It combines the advantages of the Monte-Carlo and the PDE approach: on one hand, it is a backward iterative scheme such that early exercise features can be readily considered and, on the other hand it remains efficient even if the dimension of the state space becomes larger as the valuation is carried out by Monte Carlo simulations rather than the numerical solution of partial (integro-) differential equations(P(I)DEs). Christopher (2009) explained the new valuation approach based on market-consistent values and its rationale; set out the issues faced by life insurers in implementing the new regime and explained how these issues were addressed. This was achieved by analyzing the valuation reports of 38 life insurers who used the new approach, and in particular, the information about the modeling been used. It was realized that, the market consistent basis offers a number of advantages over the traditional regime for valuing liabilities in the United Kingdom. The following challenges were also discovered and the study elaborated on some of the challenges:

- What economic scenario generator an insurer uses can make a big difference to the reported value of it's guarantees and options; more work is needed to understand (and, perhaps, reduce) these differences.
- Incorporating, in modeling, the insurers' planned management actions are more fully important and
- Further controls are needed so that he did not see as a continuation of the errors that arose when the new regime was introduced.

Eric (1995) addressed the issues of the duration and convexity of insurance liabilities and equity since these issues usually affects the insurance landscape. He added that a correct assessment of these risk measures is critical as they constitute the primary ingredients of any sound asset-liability management approach. In addition, the effort toward a more detailed and more accurate risk picture of life insurance operations enables one to debunk some pitfalls that are commonly encountered in the insurance industry. Fabio, Paolo, and Andrea in 2006 analyzed both the term structure of interest and mortality rates for evaluating a fair value of a life insurance business. The study discussed a fair value accounting impact on the reserve evaluation and compared it to the traditional deterministic model based on local rules for an Italian balance sheet calculation and a stochastic one based on a diffusion process for mortality and financial risks. The study separated the embedded derivatives from their host contracts so the fair value of a traditional life insurance contract would be expressed as the value of four components: the basic contract, the participation option, the option to annuities and the surrender option.

2.7 Mathematical Model

Researchers have dealt with the valuations of life insurance contracts in different models. The variations among the models is arise from the development of liabilities due to the different types of guarantees and different surplus distribution mechanisms among countries. Among these models are Black- Scholes model for asset prices, Levy model for asset prices, the asset-liability management(ALM) model and asset dynamics and interest rate modelling. In the standard Black-Scholes framework, the total market value of asset A evolves according to a geometric Brownian motion. A Levy process according to Nadine and Stefan, 2007 is a process with independent and stationary increments that is continuous in probability. This process can also be used in the modelling price process of financial assets. Unlike the Brownian motion, the Levy models allow for jumps in the price path, and skewness and excess kurtosis in asset return distributions and takes in to account features often observed in real-world asset prices. According to Nadine and Stefan(2007), replacing Black-Scholes model with the Levy model may seem to be a serious drawback since the Levy model leads to incomplete markets with an infinite number of martingale measures and Black-Scholes model relies on perfect hedging arguments. But the insurance industries cannot and do not follow perfect hedging strategies for participating contracts. Note that, in incomplete market situations, it is not just one arbitrage-free price but a whole range of arbitrage-free prices. The asset-liability management (ALM) is responsible for the administration of the assets and liabilities of insurance contracts. The ALM model is used for the simulation of the future development of a life insurance company. The ALM model includes the Capital Market Model (for the specification of the dynamics of the short interest rate at a time), the management model which is used for the capital allocation, bonus declaration mechanism and the shareholder participation and the liability model for the decrement of policies due to mortality and surrender and the development of the policyholder's accounts. It also include the Balance sheet model which is used to derive the recursive development of all items in the simplified balance sheet (Gerstner, Griebel, Holtz, Goschnick, and Haep,n.d.)

2.8 Models Framework

Various researches have been conducted in the area of financial and actuarial approaches into the assess of financial guarantees within life insurance contracts. Briys and Varenne. (1997), Grosen and Jorgensen (2000), Grosen and Jorgensen (2002) all considered financial approach using the risk neutral valuation. The concept of risk neutral valuation is based on the assumption of a perfect hedging strategy or replicating portfolio. But perfect hedging , however is not possible for insurers for several reasons, (Bauer, Kiesel, Kling, and Russ, 2006). Barbarin and Devolder (2005) proposed a model that combines the financial and actuarial approach. They use a simple liability model similar to Briys and de Varenne (1997) and Grosen and Jorgensen (2002), with a point guarantee and terminal surplus participation. To integrate both approaches, they use a two-step method:first,they determine a guaranteed interest rate such that certain real world risk-measures(e.g value at risk or expected shortfall risk) are satisfied, secondly, to obtain fair contracts, they use risk-neutral valuation and adjust the surplus-participation rate accordingly. This two-step approach can be applied within their model because the surplus participation is only applied at maturity. Therefore whilst it has an influence on the contract value, it has no impact on the considered risk measures. However, they do not consider cliquet-style guarantees that are predominant in many insurance markets. In Brennan and Schwartz (1976, 1979) as cited in Nielsen and Sandmann (1995), the rational insurance premium on an equity-linked insurance contact was obtained through the application of the theory of contingent claims pricing.The premium was determined in an economy with the equity following a geometric Brownian motion,whereas the interest rate was assumed to be constant. Nielsen and

Sandmann (1995) realized that, further consideration with deterministic interest rate allow for interest rate risk by assuming an Ornstein-Uhlenbeck process implying a closed form solution of the single premium endowment policy. They presented a model for the multi premium case in the context of a stochastic interest rate process. It was shown that the insurance contracts includes an Asian-like option contracts. No closed form solution will be obtained. They discuss different numerical approaches and apply Monte Carlo simulation with a variance reduction technique. Neilsen and Sandmann (1995) concluded that, in an economy with stochastic development of the term structure of interest rates a model for the determination of the fair premium on an equity linked life insurance contacts has been established. An essential part of the premium equation consists of a contingent claim with a character as an Asian option. However it was shown that the stochastic interest rate and the long time to maturity of the insurance contract prohibited the application of the 'usual' solution methods:Edge worth expansion of Fast Fourier transform. The approximation formula developed by Vorst (1992) cited in Nielsen and Sandmann (1995) exhibited a better performance than last two just mentioned for medium term contracts. Nielson and Sandmann (1995)applied and advocated for Monte Carlo simulation to overcome difficulties. The result obtained was compared to the Edgeworth and Vorst approximation and found to be preferable to these. They realized that although the Monte Carlo simulations take more time consuming than the other methods they did not take it as a serious critical point against simulation as the fair premium only to be calculated one when the contract is entered. According to (Mike, Arun, Alice, Cheng-Sheng, David, Chris and Jim, 2010), the use of advance data mining techniques to improve decision making has already taken root in property and casualty insurance as well as in many other industries. However since in their opinion, the application of such techniques for more objectives, consistent and optimal decision making in the life insurance industry is still in a nascent stage, they described the ways data mining and multivariate analytic techniques can be used to improve decision making processes in such functions as life insurance underwriting and marketing , resulting in more profitable and efficient operations. They implemented predictive modeling in life insurance underwriting and marketing and demonstrated the segmentation power of predictive modelling and resulting business benefits. The liability structure of the insurance company is implied by participating life insurance contracts and based on a model suggested by (Ballotta, Haberman, and Wang, 2006) cited in Nadine and Stefan (2007). According to Ballotta et al. (2006) cited in Nadin and Stefan (2007), for policyholders to initiate contracts, they must pay a single premium P_0 and if the company's initial capital is E_0, then the sum of the initial contribution $A_0 = E_0 + P_0$. This sum of initial contribution A_0 is invested in the reference portfolio. Hence for $0 < k < 1$, it holds that $P_0 = k \cdot A_0$ and $E_0 = (1 - k) \cdot A_0$, where k represents the leverage of the company. If P denotes the policyholder's accounts, that is ,the book value of the policy reserves. The policy reserve P is a year -to-year, or cliquet-style, guarantee interest rate or a fractionα of the annual surplus generated by the insurer's investment portfolio. Hence for $t = 1, 2, \cdots, T$ the development of the policy reserve is given by

$$P(t) = P(t-1) \cdot \left(1 + max\left[g, \alpha\left(\frac{A(t)}{A(t-1)} - 1\right)\right]\right)$$

Ballotta et al (2006) summarizes the value of liabilities $L(t)$ as

$$L(t) = P(T) + \delta[k \cdot A(T)]^{+} = P(T) + \delta \cdot B(T) - D(T)$$

where $D(T)$ denotes the default put option,$E(T)$, the residual claim of the equity holders and is determined as the difference between the market value of the reference portfolio $\wedge(T)$ and the policyholder's claim $L(T)$, i.e.

$$E(T) = A(T) - L(T) = \max(A(T) - P(T), 0) - \delta \cdot B(T) \geq 0$$

In the standard Black-Scholes framework, the total market value of assets A evolves according to a geometric Brownian motion as stated earlier. In Black-Scholes model for asset prices, the standard Brownian motion$W^p(t), 0 \leq t \leq T$ on a probability space (ω, F, P) and $(F_t, 0 \leq t \leq T)$, be the filtration generated by the Brownian motion. The total market value of the assets A in standard Brownian motion evolves according to a geometric Brownian motion under the objective measure P is given by

$$dA(t) = mA(t)dt + \sigma A(t)dW^p(t),$$

with constant asset drift m,volatility σ, and P-Brownian motion W^p, assuming a complete, perfect, and frictionless market(Nadine and Stefan,2007). The solution of the stochastic differential equation is

$$A(t) = A(0) \cdot e^{((m-\frac{\sigma^2}{2})t+\sigma \cdot W^p(t)} = A(t-1) \cdot e^{(m-\frac{\sigma^2}{2}+\sigma \cdot (W^p(t)-W^p(t-1))}$$

Other researchers also described the financial characteristics of contract which is between the insurance or pension company and investor. e.g. (Bjarke, Peter and Anders, 2001) . If a contract of nominal value P_0 is issued by the company at time zero and the contract is immediately acquired by an investor for a single premium of V_0, then let's assume that there are no further payments from or to the contract prior to expiration at time T, where the contract is settled by a single payment from the company to the investor. In general P_0 was treated as exogenously given where as V_0 was determined by the model. V_0 was referred to as the fair value of the contract. The contract is a contingent and they determined it's value process using methods from well-developed theory of contingent claims valuation as also illustrated by Duffie (1996). The benefits from the contract at maturity date is denoted by $P(T)$ and generally referred $P(t)_{0 \leq t \leq T}$ as the account balance process of the contract. The evolution of $P(\cdot)$ between successive time points in the set $\Upsilon \equiv 1, 2, \cdots, 3$ is determined by the discretely compounded policy interest rate process,$rP(t)_{t\epsilon\Upsilon}$. Specifically, we have

$$P(t) \;=\; (1 + r_p(t)) \cdot P(t-1), t\epsilon\Upsilon$$

which implies

$$P(t) \;=\; P_0 \cdot \prod_{i-1}^{t} (1 + r_p(i)), t\epsilon\Upsilon.$$

Time is measured in years,$P(\cdot)$ is updated annually, and the $r_p(\cdot)s$ are annualized rates as in real life contracts. Now, the way in which $rP(\cdot)$ is determined of course of vital importance. According to Nadine and Alexander (2006), fair pricing of embedded options in life insurance contracts are usually conducted using the appropriate concept of risk-neutral valuation. This concept assumes a perfect hedging strategy, which insurance companies can hrdly pursue in practice. They extended the risk-neutral valuation concept with a risk measurement approach and accomplish this by first calibrating contract parameters that lead to the same market value using risk-neutral valuation. They then measured the resulting risk assuming that insurers do not follow perfect hedging strategies. They use lower partial moments as the relevant risk measure, comparing shortfall probability, expected shortfall, and shortfall variance. Their research showed that even when contracts have the same market value, the insurance company's risk can vary widely, a finding that allows us to identify key risk drivers for participating life insurance contracts.

CHAPTER 3

METHODOLOGY

3.1 Introduction

This section deals with the various method used to implement the numerical methods of the valuation of life insurance contracts. This chapter also explains the methods in deriving the interest rate values.

The study utilizes a model proposed by Bjarke et al. (2001). The algorithm used to solve this model is by the application of finite difference scheme (Hopscotch and Crank-Nicolson Schemes). The study is based on one time paying premiums.

3.2 Definitions

Random Variable

Let Ω be a non empty finite set, a random variable X is a function that assigns a numerical value to each state of the world. i.e;

$$X : \Omega \to \Re$$

σ - Algebra

let Ω be a non empty set. A σ-algebra is a collection of f subsets of Ω with the following three properties;

i. $\phi, \Omega \in f$

ii. If $A \in f$, then it's compliment $A^c \in f$

iii. If $A_1, A_2, ...$ is a sequence of sets in f, then $\bigcup_{k=1}^{\infty} A_k \in f$.

Probability Measure

The measure of chance of an occurance of an event,$P : F \rightarrow [0, 1]$ which satisfies the following two properties;

I. $P(\Omega) = 1$

II. For any mutually disjoint events $A_1, A_2, ... \in F$, $P(A_1, \cup A_2, ...) = P(A_1) + P(A_2) + ...$

Probability Space

A probability space is a triplet (Ω, F, P) consisting of the set of outcomes(Ω), a σ- field(algebra) with Boolean algebra properties and a probability measure P.

Expectation

A random variable $X : \Omega \rightarrow \Re$ is called integrable if $\int_{\Omega} |x(w)|dp(w) = \int_R |x|p(x)dx < \infty$.where
$p(x)$ is the probability density function of x. Then the expectation (mean)of the integrable random variable X is defined by

$$E[X] = \int_{\Omega} X(w)dp(w) = \int_R Xp(x)dx$$

.

Conditional Expectation

Let X be a random variable on the probability space (Ω, f, p) and G be a sub σ- algebra of f. Then $E\left[\frac{X}{G}\right]$ is defined to be any random variable Y that satisfies ;

I. Y is G-measurable.

II. $\int_A E\left[\frac{X}{G}\right] dp = \int_A X dp$, for all $A \in G$.

Then $E\left[\frac{X}{G}\right]$ is called the conditional expectation of X given G.

Filtration

Given a probability space (Ω, F, P) , we call a sequence of σ-algebras $\{f_t\}_{t\in T}$ a filtration if $f_s \subset f_t \subset f$ for all $s, t \in T$ such that $s \leq t$. Or simply filtration is the flow of information.

Stochastic Process

A stochastic process on the probability space (Ω, f, p) is a family of random variables X_t parameterized by $t \in T$, where $T \subset \Re$.
A stochastic process is called adapted to the filtration f_t if X_t is f_t predictable , for any $t \in T$. f_t predictable means given any two numbers $a, b \in \Re$, then all the states of the world for which X takes values between a and b forms a set that is an event.

Martingale

A martingale is a stochastic process such that ;

I. $E[X_t] < \infty \; \forall \, t$

II. $E[X_t/f_s] = X_s \;\; \forall \, s < t$

General Brownian Motion

A general Brownian Motion is a stochastic process $(W_t)_{t \geq 0}$ which satisfies the following properties ;

1. $W_0 = 0$
2. If $0 \leq s < t$, then $W_t - W_s \sim N(\mu(t-s), \delta^2(t-s))$
3. Future changes are independent of past and present ; if $0 \leq r \leq s < t$, then the random variable $W_t - W_s$ and W_r are independent.
4. The paths $t \to W_t$ are continuous with probability of 1

Where μ is the drift, δ^2 is the variance of the Brownian motion.
μ and δ^2 are constants.
In the case where $\mu = 0$ and $\delta^2 = 1$, we get the normalized Brownian motion or the Standard Brownian Motion called the Weiner Process.
i.e ; the condition III now becomes
if $0 \leq s < t$, then $W_t - W_s \sim N(0, t-s)$.

Geometric Brownian Motion

If the stochastic process X_t is a Brownian motion with drift μ and variance rate δ^2 , the process $\{Y_t = \exp X_t, t \geq 0\}$ is called a geometric Brownian motion, or the exponential Brownian motion.

Markov Process

Let (Ω, f, P) be a probability space. Let $\{f_t\}_{t=0}^{n}$ be a filtration under f. Let $\{X_t\}_{t=0}^{n}$ be a stochastic process on (Ω, f, P). This process is said to be a Markov if :

- The stochastic process X_t is adapted to the filtration $\{F_t\}$, and
- For each $t = 0, 1, 2, ..., n-1$, the distribution of X_{t+1} conditioned on F_t is the same as the distribution of X_{t+1} conditioned on X_t

That is to say :

A Markov process is a stochastic process where only the present value of the variable is relevant for predicting the future.

3.3 Differential Equations

This is an equation involving an unknown function and it's derivatives.

3.3.1 Ordinary Differential Equations

This is a differential equation in which the unknown function depends on only one independent variable.

3.3.2 Partial Differential Equations

This is a differential equation in which the unknown function depends on two or more independent variables. A general second order partial differential equation (PDE) is of the form

$$aU_{xx} + 2bU_{xy} + cU_{yy} + dU_x + eU_y + fu = g \tag{3.1}$$

If U_{xx} is formally replaced by α^2 , u_{xy} by $\alpha\beta$, U_{yy} by β^2, U_x by α, and u_y by β, then associated with (2.1) is a polynomial of degree two in α and β :

$$P(\alpha, \beta) = a\alpha^2 + 2b\alpha\beta + c\beta^2 + d\alpha + e\beta + f = g.$$

The PDE is classified according to the discriminant $b^2 - ac$

If $b^2 - ac = 0$ then the PDE is parabolic.

If $b^2 - ac > 0$ then the PDE is hyperbolic.

If $b^2 - ac < 0$ then the PDE is elliptic.

3.3.3 Stochastic Differential Equation

A stochastic differential equation (or SDE) is a differential equation in which one or more of the terms is a stochastic process, thus resulting in a solution which is itself a stochastic process.

Order of Differential Equation

The order of a differential equation is the order of the highest derivative appearing in the equation.

Example ; $\frac{dy}{dx} + x = 0.$ This of order 1.

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x = 0$ This is of order 2.

Notation

The expression $y^{'}, y^{''}, ..., y^{(n)}$ are often used to represent respectively, the first, second ,..., nth derivatives of y with respect to the independent variable under consideration.

3.3.4 Solutions to a Differential Equation

A solution to a differential equation in the unknown function y and the independent variable x on the interval l, is a function $y(x)$ that satisfies the differential equation identically for all x in L.
A particular solution of a differential equation is any one solution. The general solution of a differential equation is the set of all solutions.

Initial and Boundary- Value Problem

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable constitutes an initial value problem and the subsidiary conditions are the Initial conditions.
If the subsidiary conditions are given at more than one value of the independent variable, the problem is a Boundary-value problem and the conditions are Boundary conditions.

3.4 Finite Difference Equation

Let a region Ω in the xt-plane be covered by a grid (x_i, t_j). If all the derivatives in the PDE are replaced by difference quotients, the result is the Finite Difference Equation.

i.e. If $L[u] = g(x,t)$ in Ω is the PDE is equal to $D[u_{ij}] = g_{ij}(x_i, t_j)$ in Ω. The amount by which the solution to $L[u] = g$ fails to satisfy the difference equation is called the local truncation error. It can be represented by

$$E_{ij} = D[U_{ij}] - f_{ij} \tag{3.2}$$

In constructing a finite difference equation usually follows the following steps

1. Generate a grid for example (x_i, t_j), we want to find an approximate solution.

2. Substitute the derivatives in an ODE or PDE system of equations with finite difference scheme. The ODE or PDE now becomes a linear or non-linear system of algebraic expressions.

3. Solve the system of algebraic expressions.

4. Implement and debug the computer code.

5. Do the error analysis, both analytically and numerically.

3.4.1 Types of Difference Schemes

The types of difference methods differ from each other according to how we approximate the partial derivatives with respect to time. Based on this we have the following types of finite difference schemes:

1. Implicit Finite Difference Scheme,when we use the forward difference formula

2. Explicit Finite Difference Scheme, when we use the backward difference Scheme

3. Crank-Nicholson Finite Difference Scheme, which is regarded as the hybrid between the fully implicit and explicit schemes.

4. Hopscotch Finite Difference Scheme, this in it's implementation wee alternate between the implicit and explicit schemes at each node.

3.4.2 Finite Difference Method of Ordinary Differential Equations (ODE)

There are three main commonly used finite difference formulas to approximate first order derivative of aa function $f(x)$. These are the forward finite difference, backward finite difference and central finite difference.

Let's consider Taylor's series expansion of a function $f(x)$ about c

$$f(x) \sim f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \dots$$

Then about the origin 0, where $c = 0$

$$\implies f(x) \sim f(0) + f'(0)(x) + \frac{f''(0)x^2}{2!} + \dots$$

For small increments h we get

$$f(x+h) \sim f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \dots \tag{3.3}$$

Neglecting higher order we get

$$f(x+h) = f(x) + hf'(x) + 0(h)$$

Where $0(h)$ is the error of the approximation. Then simplifying for $f'(x)$ we get

$$f'(x) = \frac{f(x+h) - f(x)}{h} + 0(h) \tag{3.4}$$

This is the forward approximation formula

Similarly if we consider negative increment from the Taylor series

$$f(x-h) \sim f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \dots \tag{3.5}$$

If we neglect higher orders and simplify for $f'(x)$ we get

$$f'(x) = \frac{f(x) - f(x-h)}{h} + 0(h) \tag{3.6}$$

Final for central approximation we subtract equation 3.5 from equation 3.3

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + 0(h^2) \tag{3.7}$$

For higher order derivative approximation we add equation 3.3 and equation 3.5

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + 0(h^4) \tag{3.8}$$

3.4.3 Finite Difference Approximation for Partial Differential Equation (PDE)

A grid or mesh in the xt plane is a set of points (x_0+in, t_0+jk) where i and j are integers and (x_0, t_0) is a reference point.The (x_i, t_j) are the grid points, mesh points or nodes. In many financial and engineering problems, the function f depends on two or more independent variables ,hence the need for finite approximation of partial derivatives. Since partial derivatives denote the local variation of a function with respect to a particular independent variable while all other independent variables are held constant, finite difference approximation of ordinary derivatives can be adapted for the partial derivatives. The notation used to represent a two independent variables is (i, j) to designate the pivot point, and if there are three independent variables,(i, j, k) are used where i,j, and k are the counters in the x,y, and z directions. If we consider the function $f(x, y)$, then the finite-difference approximation for the partial derivative $\frac{\partial f(x,y)}{\partial x}atx = x_i, y = y_i$ can be found by fixing the value of y at y_i and treating $f(x, y_i)$ as a one variable function.

The forward, backward and central difference of $\frac{\partial f}{\partial x}$ can be expressed as

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \approx \frac{f(x_i + \Delta x, y_j) - f(x_i, y_j)}{\Delta x} \tag{3.9}$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \approx \frac{f(x_i, y_j) - f(x_i - \Delta x, y_j)}{\Delta x} \tag{3.10}$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \approx \frac{f(x_i + \Delta x, y_j) - f(x_i - \Delta x, y_j)}{2\Delta x} \tag{3.11}$$

Central Difference Approximation of Second Partial Derivatives

The central difference approximation of second partial derivatives at (x_i, y_j) can be derived as

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \approx \frac{f(x_i + \Delta x, y_j) - 2f(x_i, y_j) + f(x_i - \Delta x, y_j)}{(\Delta x)^2} \tag{3.12}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j} \approx \frac{f(x_i, y_j + \Delta y) - 2f(x_i, y_i) + f(x_i, y_i - \Delta y)}{(\Delta y)^2} \tag{3.13}$$

and

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{i,j} \approx \frac{f(x_i + \Delta x, y_i + \Delta y) - f(x_i + \Delta x, y_i - \Delta y) - f(x_i - \Delta x, y_i + \Delta y) + f(x_i - \Delta x, y_i - \Delta y)}{4\Delta x \Delta y} \tag{3.14}$$

Error of Finite Difference Approximation of Partial Derivatives

To find the error associated with finite-difference approximation of partial derivatives, we use Taylor series expansion of $f(x, y)$ around the point (x_i, y_i). That is,

$$f_{i\pm1,j} = f_{i,j} \pm \Delta x \left.\frac{\partial f}{\partial x}\right|_{i,j} + \frac{(\Delta x)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \pm \frac{(\Delta x^3)}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right| + \cdots \quad (3.15)$$

$$f_{i,j\pm1} = f_{i,j} \pm \Delta y \left.\frac{\partial f}{\partial y}\right|_{i,j} + \frac{(\Delta y)^2}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j} \pm \frac{(\Delta y^3)}{3!}\left.\frac{\partial^3 f}{\partial y^3}\right| + \cdots \quad (3.16)$$

Truncating equation 3.15 after the nth order, we have the error

$$R_{x,n} \simeq (-1)^{n+1}\frac{(\Delta x)^{n+1}}{(n+1)!}\left.\frac{\partial^{n+1} f(x,y)}{\partial x^{n+1}}\right|_{i,j} \quad (3.17)$$

and truncating equation 3.16 after the nth order gives the error

$$R_{y,n} \simeq (-1)^{n+1}\frac{(\Delta y)^{n+1}}{(n+1)!}\left.\frac{\partial^{n+1} f(x,y)}{\partial y^{n+1}}\right|_{i,j} \quad (3.18)$$

3.4.4 Stability of Numerical Schemes for PDEs

Definition Of Stability For Differential Equation

A differential equation is said to be stable if for every set of initial data (at$t = 0$) the solution of the differential equation remains bounded as t approaches infinity.

Definition For Stability For Difference Equation

A linear difference equation with constant coefficients is said to be stable if and only if all the roots of it's characteristic equation have an absolute value at must be 1 and those of absolute value of 1 are simple. The equation is strongly stable if and only if all of the roots have absolute values strictly less than 1.

Consistency

A finite difference scheme/operator is consistent if the operator reduces to the original differential equation as the increments in the independent variables vanish. For us to get a specific solution to a partial differential equation, additional conditions must be imposed on the solution function.Typically these conditions occur in the form of boundary values that are prescribed on all/ part of the perimeter of the region in which the solution is sought. The nature of the boundary and boundary values are usually the determining factors in setting up an appropriate numerical scheme for obtaining the appropriate solution.

Stability

Stability of the scheme is when the errors caused be small perturbation in the numerical solution remains bound.

Convergence

The scheme converges if the finite difference solution approaches the true solution to the partial differential equation as the increments Δx , Δt go to zero. The basic idea of convergence and stability analysis for a linear PDE consist of writing the solution in the as a complex Fourier series and analyzing the generic component of the solution. The Von Neumann stability analysis helps to identify when (and if) the numerical scheme behave properly. Stability under this method means the scheme does not amplify errors. The Artificial Viscosity is also a tool for stabilizing the numerical schemes.

Lax Equivalence Theorem

Given a well-posed initial boundary value problem and a finite-difference problem consistent with it stability is both necessary and sufficient for convergence.

Theorem

For a two level difference methods, Von Neumann stability is both necessary and sufficient for stability.

Von Neumann Stability Criterion

A difference method for an initial-boundary value problem with a bounded solution is Von-Neumann stable if extended solution to difference scheme of the form

$$U_{nj} = \xi^j e^{ikn\Delta x}$$

has the property $|\xi| \leq 1$ Where i and j are the grid points.x is the space variable.

Matrix Stability Criterion

Consider an initial- boundary value problem with N nodes in the x direction and define a column vector of errors at level j $E_j = (E_{1j}, E_{2j}, \cdots E_{Nj})^T$. For two level difference methods, the errors at levels j and $j+1$ are related by

$$E_{j+1} = CE_j,$$

where $C = N \times N$matrix Let $\rho(C)$, the spectral radius of C, denote the maximum of the magnitudes of the eigenvalues of C.

Theorem

A two level difference method for an initial boundary value problem with a bounded solution is matrix stable if $\rho(C) \leq 1$. The matrix stability criterion is a necessary condition for the stability of a two level method.

Theorem

Let C be a symmetric or similar to a symmetric matrix, whereby all eigenvalues of C are C.Then matrix stability is necessary and sufficient for stability.

3.5 The Black-Scholes Model

The Black-Scholes model is a famous model used in the pricing of complex financial instruments. It was developed by Myron Scholes, Robert Merton and Fisher Black in the early . In 1997, the importance of their was recognized world wide when Myron Scholes and Robert Merton received the Noble prize for Economics. Unfortunately, Fisher Black died in 1995 or he would have also received the award Hull (2003).

Before we derive the Black scholes model, we consider a few definitions;

ITÔ Process

The stochastic process $X = X_t, t \geq 0$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s$$

is an Itô process. The corresponding stochastic differential is given by

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

ITÔ Lemma

Suppose B_t follows a Brownian motion and $f(x)$ is twice differentiable, then by Taylor theorem of the second order

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2$$

but $(dB_t)^2 = dt$ and this implies that

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt.$$

This is the Itô lemma for the functions of Brownian motion.

Diffusion Process

Diffusion process is a stochastic process X_t that satisfy the stochastic differential equation of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t.$$

This is process is classified as an itô process.

3.5.1 The Stochastic Model

A model given by Vasicek as cited in Daniel et al(2010) suggested two different asset models; a geometric Brownian motion with deterministic interest rate (constant short rate,r) and a geometric Brownian motion with stochastic interest rate. This research considers a geometric Brownian motion with deterministic interest rate The asset price gives the price of the underlying asset of an option]. Let A_t be the price of an asset, then the price change /difference in the asset is assumed to be a Markov process. The rate of return on the asset, is the change in the price of the asset divided by its original value ; $\frac{dA_t}{A_t}$.

By the Samuelson model, there are two contributions to the returns of the asset : the deterministic part which contributes to the mean growth rate of the asset and the stochastic part which contributes to the random change in the asset price due to external effects, such as unanticipated news. This stochastic part measures the risk faced by the insurance company. So that the return on the asset is given by

$$\frac{dA_t}{A_t} = \mu dt + \sigma dW_t, \text{for t} \in [0, \infty) \qquad (3.19)$$

where μ is the drift contributing to the mean return and σ is the volatility.

dW_t is the Weiner process under the risk neutral probability. The Weiner process $\{W_t, t \geq 0\}$ has the stochastic distribution

$W_t \sim N(0, t)$.So that

$dW_t := W(t + dt) - W(t) \sim N(0, dt)$

$dW_t := \emptyset\sqrt{dt}, \emptyset \sim N(0, 1)$. $\emptyset$ is the standard normal random variable. This Weiner process is assumed to follow a Geometric Brownian Motion and eqn.(2.1) is a stochastic differential equation. The drift and the volatility is assumed to satisfy the following conditions;

$$P\left[\int_0^t |\mu| ds < \infty\right] = 1$$

$$P\left[\int_0^t \sigma^2 ds < \infty\right] = 1.$$

3.5.2 Portfolio

A portfolio is a pair of predictable process $(\emptyset_t, \psi_t)$ and we interpret $\emptyset_t$ as the number of risky assets held at time t, ψ_t as the number of bonds. A self financing portfolio is a portfolio which requires only an initial investment and no money is withdrawn.

3.5.3 Hedging

Hedging in finance is a strategy for minimizing risk. Hedgers take a position in the derivative securities opposite those taken in the underlying asset in order to manage risk. One very important strategy in hedging is the delta hedging. The delta, $\triangle$ of an option is defined as the change in the option price with respect to the change in the price of the underlying asset. In other words, delta is the first derivative of the option price with respect to the stock price : $\triangle = \frac{\partial V}{\partial A}$.

3.5.4 Arbitrage

An arbitrage means guaranteed risk free profit with no invested capital, in other words an arbitrage is "free lunch". It is a practice of taking full advantage of the in balances in a financial market.

In the pricing of options, we assume the market is arbitrage free. Further more wee say that the no arbitrage principle is that a portfolio yielding a zero return in every possible scenario must have a zero present value, any other value will imply an arbitrage opportunity, which one can realize by shorting the portfolio if its value is positive and buying it if its value is negative.

According to (Wilmot, Howison, and Dewynne, 1995), the Black Scholes analysis assumes that the asset price behave as just demonstrated above and follows the following assumptions :

1. The asset price follows lognormal random walk. This assumption of the Black Scholes suggests that the market direction or individual stock cannot be predicted consistently. The prices of stocks moves in a manner referred to random walk. Random walk simply means given any time,t the prices of stock can either increase or decrease with the same probability. The price of stock at time $t + 1$ is independent of the price at time, t.

2. The risk free interest rate r and the asset volatility σ are known functions of time over the option/ contract. The most significant assumption is that volatility, a measure of how much a stock can be expected to move in the near-term is constant over time. While volatility can be relatively constant in very short term, it is never constant in longer term. Some advanced option valuation models substitute Black-Scholes constant volatility with stochastic process

generated estimates. The same like with the volatility, interest rates are also assumed to be constant in the Black-Scholes model. The Black-Scholes model uses the risk-free rate to represent this constant and known rate.

In the real world, there is no such thing as a risk-free rate, but it is possible to use the US Government Treasury Bills 30-day rate since the US government is deemed to be credible enough. However, these treasury rates can change in times of increased volatility.

3. There is no transaction cost. Even though there are cost involved in the buying and selling of options, the Black-Scholes model assumes non of this exist.

4. There are no dividend payments during the life of the contract or option. Dividends are amount paid to share holders for bearing the risk of the insurance company. This assumption has been excluded in an upgraded that takes care of the dividends since in the real around dividends are paid.

5. There are no riskless arbitrage opportunities. It is not possible gain profit without bearing any risk. Although there must an arbitrage in certain markets this can not be assumed to be true in the long run.

6. Short selling of securities with full use of proceeds is permitted and assets are divisible.The Black-Scholes model assumes it is possible

to purchase any amounts of assets or fractions at any point in time.

7. Security trading is continuous.

life insurance contracts can be assumed to be a put option since the policyholder has the right to sell the contract to the insurer. So there wee can use the valuation of put option can be applied in the valuations of life insurance contracts.

We now consider the value of the contract which is a function of time and the asset. Let $V(t, A)$ be the value of the portfolio. If the asset A_t follows the Geometric Brownian motion and then

$$\frac{dA_t}{A_t} = \mu dt + \sigma dW_t \tag{3.20}$$

then

$$dA_t = \mu A_t dt + \sigma A_t dW_t \tag{3.21}$$

.Then let $V(t, A)$ be twice differentiable function of t and of the random process A_t. Then by applying the Itô Lemma to the function $V(t, A)$, the resulting equation is given by

$$dV(t, A_t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial A_t}dA_t + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 V}{\partial {A_t}^2}dt \tag{3.22}$$

Then substituting equation (3.21) into equation (3.22) we get

$$dV(t, A_t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial A_t}(\mu A_t dt + \sigma A_t dW_t) + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 V}{\partial A_t^2}dt \tag{3.23}$$

Simplifying we arrive at the resulting equation

$$dV(t, A_t) = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 V}{\partial A_t^2} + \frac{\partial V}{\partial A_t}\right] dt + \sigma A_t \frac{\partial V}{\partial A_t} dW_t \quad (3.24)$$

This equation contains the stochastic term dW_t which makes it difficult to solve. So in order to reduce the volatile fluctuations and preserve the portfolio value, we dynamically re balance the portfolio by buying / selling stocks /options / Bonds.

Assumptions

From fundamental economics balance, the conservation of the total value of the portfolio is

$$sQ_s + vQ_v + B = 0 \quad (3.25)$$

and the requirement for self financing the portfolio is

$$sdQ_s + vdQ_v + \delta B = 0 \quad (3.26)$$

Where ;

Q_s is the number of underlying asset with a unit price s in the portfolio.

Q_v is the number of financial derivative (option) with a unit price v

B is the cash money in the portfolio

dQ_s is the change in the number of assets.

dQ_v is the change in the number of options.

δB is the change in the cash due to buying and selling of assets and options.

Bonds are appreciated by the fixed interest rate, $r > 0$

$$\frac{dB(s)}{B} = rds$$

,

$$\int_0^t \frac{dB(s)}{B} = r\int_0^t ds$$
$$\ln(B(t) - B(0)) = rt$$

then

$$B(t) = B(0)e^{rt}$$

The total change in value of bonds in the portfolio is given by

$$dB = rBdt + \delta B.$$

Because we sell bonds ($\delta B < 0$) or buy bonds ($\delta B > 0$) when hedging (re balancing)the portfolio in the time period $(t, t + dt)$.

Differentiating equation (3.25) we get

$$0 = d(sQ_s + vQ_v + B) = d(sQ_s + vQ_v) + dB$$

But $dB = rBdt + \delta B$. This implies that

$$0 = d(sQ_s + vQ_v) + rBdt + \delta B$$

$$\implies 0 = dsQ_s + sdQ_s + dvQ_v + vdQ_v + rBdt + \delta B.$$

But from equation (3.26) , we get

$$0 = Q_s ds + Q_v dv + rBdt \tag{3.27}$$

Making B the subject from equation (3.25) we get $B = -(sQ_s + vQ_v)$ and substituting it into equation (3.27)

$$0 = Q_s ds + Q_v dv - r(sQ_s + vQ_v)dt \tag{3.28}$$

Dividing through by Q_v

$$0 = \frac{Q_s}{Q_v} ds + dv - \left(rs\frac{Q_s}{Q_v} + rv \right) dt$$

Taking $\triangle = -\frac{Q_s}{Q_v}$

$$\implies 0 = \triangle ds - rs\triangle dt + dv - rvdt$$

$$0 = dv - rvdt - \triangle(ds - rsdt) \tag{3.29}$$

From the Itô Lemma if the asset price is assumed to follow a Geometric Brownian motion, then we obtain a smooth function $V = v(t, A)$

$$dV(t, A_t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial A_t} dA_t + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 V}{\partial {A_t}^2} dt$$

substituting equation 3.22 into equation 3.29

$$0 = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 V}{\partial A_t^2}\right) dt + \frac{\partial V}{\partial A_t} dA_t - rvdt - \triangle(dA_t - rA_t dt)$$

$$0 = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 V}{\partial A_t^2} - rV + \triangle r A_t\right) dt + \left(\frac{\partial V}{\partial A_t} - \triangle\right) dA_t \tag{3.30}$$

Holding a strategy in buying/ selling stocks and options with the goal to eliminate possible volatile fluctuation. The only volatile term in equation 3.30 is $\left(\frac{\partial V}{\partial A_t} - \triangle\right) dA_t$ due to the stochastic differential dA_t . After setting $\triangle = \frac{\partial V}{\partial A_t}$ (Delta hedging) and divide equation 3.30 by dt we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A_t^2 \frac{\partial^2 V}{\partial A_t^2} + rA_t \frac{\partial V}{\partial A_t} - rV = 0 \tag{3.31}$$

This is the Black-Scholes PDE.

3.6 Finite Difference Approximation for Black - Scholes

The finite difference method seeks to solve the Black-Scholes partial differential equation(PDE)over the area of integration by a system of algebraic equations. These are means of solving the PDE numerically.The most common finite difference mmethods for solving the Black-Scholes PDE are the Explicit method,Implicit method, Crank Nicholson method and the Hopscotch method. These methods are closely related but differ in implementation, efficiency, stability and accuracy. The study looks 3 components in formulating the PDE problem and these are :

- The partial differential equation.
- The region of space-time on which the PDE is required to be satisfied.
- The auxiliary boundary and initial conditions to be met.

3.6.1 Discretization of the Black- Scholes Equation

The finite difference approximations result from replacing the partial differential pricing equation and the boundary conditions using a forward, backward or central difference approximation. The Black-Scholes PDE derived in equation 3.31 can be written as ;

$$\frac{\partial V(t,A_t)}{\partial t}+\frac{\sigma^2 A_t^2 \partial^2 V(t,A_t)}{2\partial A_t^2}+rA_t\frac{\partial V(t,A_t)}{\partial A_t}=rV(t,A_t) \quad (3.32)$$

which in simplified form is written as :

$$\frac{\partial V}{\partial t}+\frac{1}{2}A^2\sigma^2\frac{\partial^2 V}{\partial A^2}+rA\frac{\partial V}{\partial A}=rV \quad (3.33)$$

The discretization of the equation is done by approximating with respect to time, t and to the underlying asset price,A. the t-A computational domain is divided into grid or mesh using approximate infinitesimal steps $\triangle A$ and $\triangle t$ by small fixed finite steps. The term $\triangle t$ is equally spaced and an array of $N+1$ grid points $t_0, t_1, \ldots t_N$ is used to approximate the time derivative with $\triangle t = t_{i+1} - t_i = t_i - t_{i-1}$ and $\triangle t = \frac{T}{N}$. T is the maturity of the contract.

We then construct the boundary conditions, since the asset price cannot go below 0 and it is used that $A_{max} = 2A_0$. We also have $M+1$ equally spaced grid points $A_0, A_1, \ldots, A_M$ is used to discretized the asset price derivative with $\triangle A = A_{j+1} - A_j = A_j - A_{j-1}$and $\triangle A = \frac{A_{max}}{M}$. This gives us a rectangular region on the t-A plane with sides $(0, A_{max})$ and $(0, T)$. Using the grid coordinates (x, y), we are able to compute the solution at discrete points with a total grid points of $(M+1)(N+1)$.

The (x, y) points on the grid corresponds to time $x\Delta t$ for $x = 0, 1, \ldots, N$ and the asset price $y\Delta A$ for $y = 0, 1, \ldots, M$. The figure below illustrates the discretized asset price and the time derivative into $(M+1)$ and $(N+1)$ grid points respectively. We are interested

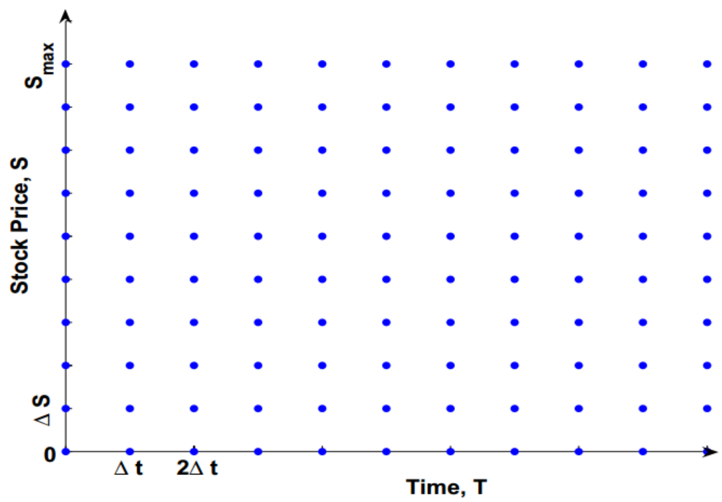

Figure 3.1: The mesh points for finite difference scheme

in the value of the contract $V(t, A)$ so we represent values of $V(t, A)$

at mesh points $x\Delta t, y\Delta A$, denoted as

$$V_x^y = V(x\Delta t, y\Delta A) = V(t_x, A_y) = V(t, A)$$

Using the Taylor's expansions for $V(t, A + \Delta A)$ and $V(t, A - \Delta A)$ around the point (t, A) we get

$$V(t, A + \Delta A) = V + \frac{\partial V}{\partial A}\Delta A + \frac{\partial^2 V}{2\partial A^2}\Delta A^2 + 0(\Delta A^2) \qquad (3.34)$$

$$V(t, A - \Delta A) = V - \frac{\partial V}{\partial A}\Delta A + \frac{\partial^2 V}{\partial A^2} - 0(\Delta A^2) \qquad (3.35)$$

Working for the forward difference using equation 3.34, we derive

$$\frac{\partial V}{\partial A} = \frac{V(t, A - \Delta A) - V(t, A)}{\Delta A} + 0(\Delta A) \qquad \text{so that}$$

$$\frac{\partial V}{\partial A} \approx \frac{V_x^{y+1} - V_x^y}{\Delta A} \qquad (3.36)$$

and equation 3.35 gives the backward difference

$$\frac{\partial V}{\partial A} = \frac{V(t, A) - V(t, A - \Delta A)}{\partial A} + 0(\Delta A)$$

$$\frac{\partial V}{\partial A} \approx \frac{V_x^y - V_x^{y-1}}{\Delta A} \qquad (3.37)$$

Subtracting equation 3.35 from equation 3.34 and taking the first derivative results in the central difference is given by

$$\frac{\partial V}{\partial A} = \frac{V(t, A - \Delta A) - V(t,\ A - \Delta A)}{2\,\Delta A} + 0(\Delta A^2)$$

$$\frac{\partial V}{\partial A} = \frac{V_x^{y+1} - V_x^{y-1}}{2\,\Delta A} \tag{3.38}$$

By adding 3.34 and 3.35, we estimate the second order partial derivative, with which is given by

$$\frac{\partial^2 V}{\partial A^2} = \frac{V(t, A + \Delta A) - 2V(t, A) + V(t, A - \Delta A)}{\Delta A^2} + 0(\Delta^2)$$

$$\frac{\partial^2 V}{\partial A^2} \approx \frac{V_x^{y+1} - 2V_x^{y} + V_x^{y-1}}{\Delta A^2} \tag{3.39}$$

Similarly, considering the expansion for the time derivative we derive the following

$$\text{Forward}: \frac{\partial V}{\partial t} = \frac{V(t + \Delta t, A) - V(t, A)}{\Delta t} + 0(\Delta t)$$

$$\frac{\partial V}{\partial t} \approx \frac{V_{x+1}^{y} - V_x^{y}}{\Delta t} \tag{3.40}$$

$$\text{Backward}: \frac{\partial V}{\partial t} = \frac{V(t, A) - V(t - \Delta t, A)}{\Delta t} + 0(\Delta t)$$

$$\frac{\partial V}{\partial t} = \frac{V_x^{y} - V_{x-1}^{y}}{\Delta t} \tag{3.41}$$

3.6.2 Boundary and Initial Conditions

We need the discretization of the boundary and initial conditions,since without the initial and boundary conditions the Black-Scholes PDE will have either infinitely many solutions or no solution.The payoff for an European put option is given by Max $(K - A_t, 0)$. When an asset is worth nothing, the put is worth it's strike price K. That is :

$$V_x^0 = K \qquad \text{for}\, x = 0, 1, \ldots, N.$$

Then

$$V_x^M = 0 \qquad \text{for}\, x = 0, 1, \ldots, N.$$

Since the value of the contract approaches zero as the price of the underlying asset increases. The value of the contract at maturity T is known, therefore we can find the final conditions(otherwise the initial condition) of the contract.

$$V_N^y = Max(k - y\Delta A\, 0) \qquad \text{for}\, y = 0, 1, \ldots, M$$

$$V_N^y = Ke^{-r(N-x)\Delta t} - Amax, \qquad \text{where} \qquad Amax = M\Delta A$$

Depending on which combination of the finite difference schemes we use in the discretization, we have the Explicit, Implicit, Crank-Nicholson or the Hopscotch methods. We consider the European contract stated in equation 3.33, with maturity date $T = N\Delta t$and $A_{max} = M\Delta A$ the maximum asset price. Let V_x^y denote the value of the asset at $(x\Delta t,\, y\Delta A)$, now we consider the approaches to the various finite difference schemes

3.6.3 Explicit Finite Difference Scheme

The discretization of the Black-Scholes partial differential equation in equation 3.33 is done by taking the Backward difference for the time derivative and the central difference for the asset price derivative. Since we know the value of the contract at maturity time, we can express the next value V_x^y explicitly in terms of V_{x+1}^{y-1}, $V_{x=1}^{y}$ and V_{x+1}^{y+1}. Discretization of equation 3.33, we derive :

$$\frac{V_{x+1}^{y} - V_x^y}{\Delta t} + \frac{rj\Delta A}{2\Delta A}\left[V_{x+1}^{y+1} - V_{x+1}^{y-1}\right] + \frac{\sigma^2 y^2 \Delta^2}{2\Delta^2}\left[V_{x+1}^{y+1} - 2V_{x+1}^{y} + V_{x+1}^{y+1}\right] = rV_x^y$$

Making V_x^y the subject we have

$$V_x^y = \frac{1}{1 + r\Delta t}\left[\alpha_y V_{x+1}^{y+1} + \beta_y V_{x+1}^{y} + \gamma_y V_{x+1}^{y-1}\right] \qquad (3.42)$$

$$\left\{\begin{array}{l} \alpha_y = \frac{ry\Delta t}{2} + \frac{\sigma^2 y^2 \Delta t}{2} = \frac{1}{1+r\Delta t}\pi_u \\ \beta_y = 1 - \sigma^2 y^2 \Delta t = \frac{1}{1+r\Delta t}\pi_o \\ \gamma_y = \frac{\sigma^2 y^2 \Delta t}{2} - \frac{ry\Delta t}{2} = \frac{1}{1+r\Delta t}\pi_d \end{array}\right\} \qquad (3.43)$$

for $x = N-1, N-2, \ldots, 1, 0$ and $y = 1, 2, \ldots M-1$. These coefficients can be interpreted as probabilities times a discount factor. If one of these probabilities < 0, instability occurs. Therefore the conditions to have non-negative probabilities is that $\sigma^2 y^2 \Delta t < 1$ and $r < \sigma^2 y$(Hull,2003)

Thus,this case is actually equivalent to the trinomial tree where the coefficients are the probabilities that can be assigned ass the likelihood of moving upwards, no movement and downward movements respectively. The system of equations is represented in the form

$$\begin{bmatrix} \beta_0 & \gamma_0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{M-1} & \beta_{M-1} & \gamma_{M-1} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_M & \beta_M \end{bmatrix} \begin{bmatrix} V_{x+1,0} \\ V_{x+1,1} \\ \vdots \\ V_{x+1,M-1} \\ V_{x+1,M} \end{bmatrix} = \begin{bmatrix} V_{x,0} - \alpha_0 \\ V_{x,1} \\ \vdots \\ V_{x,M-1} \\ V_{x,M} - \gamma_M \end{bmatrix} \tag{3.44}$$

This system of equations can be written in the form $DV_{x+1,y} = V_{x,y}$ for $y = 0, 1, \cdot, M$ and the error terms are ignored since the boundary conditions cater for them.

The vector of the asset price $V_{x+1,y}$ is known at time T from the initial condition. We solve for $V_{x,y}$ by working backward using matrix D which comprises of the probabilities α_y,β_y and γ_y which are known probabilities and the backward iteration leads to the value of the contract obtained at time zero.

3.6.4 Stability of Explicit Finite Scheme

We use matrix D to analysis the stability of the explicit difference method. By theorem 1 is the matrix is stable then the finite scheme is also stable. Matrix D is real and symmetric, if λ_i is the ith eigenvalue of D then

$$\| D \| = \rho(D) = \max | \lambda_i |$$

The eigenvalues of λ_i are given by

$$\lambda_i = \beta_y + 2(\sqrt{\alpha_y \gamma_y}) cos\frac{i\pi}{N} \quad \text{for} \quad i = 1, 2, \cdots, N-1.$$

If we substitute α,β and γ and rearrange we get

$$\lambda_i = 1 - 2\sigma^2 y^2 \Delta t sin^2\left(\frac{i\pi}{2N}\right)$$

Then the scheme is stable when

$$\parallel D \parallel_2 = \max \mid 1 - 2\sigma^2 y^2 \Delta t sin^2\left(\frac{i\pi}{2N}\right) \mid < 1$$

$$\Rightarrow -1 \leq 1 - 2\sigma^2 y^2 \Delta t sin^2\left(\frac{i\pi}{2N}\right) \leq 1 \text{ for } i = 1, 2, \cdots, N-1$$

as $\Delta t \to 0$, $N \to \infty$ $sin^2\left(\frac{(N-1)\pi}{2N}\right) \to 1$ so that $0 \leq \sigma^2 y^2 \Delta t \leq 1$. Therefore the scheme is stable, convergent and consistent for $0 \leq \sigma^2 y^2 \Delta t \leq 1$. Therefore the explicit finite difference method is conditionally stable.

3.6.5 Implicit Finite Difference Scheme

The discretization of the Black-scholes PDE in equation 3.33 is done by taking the forward difference with respect to the time derivative and central difference with respect to the asset price derivative.The difference equation in this case has the following form :

$$\frac{V_{x+1}^y - V_x^y}{\Delta t} + \frac{ry\Delta A}{2\Delta A}[V_x^{y+1} - V_x^{y-1}] + \frac{\sigma^2 y^2 \Delta A^2}{2\Delta A^2}[V_x^{y+1} - 2V_x^y + V_x^{y-1}] = rV_{x+1}^y \tag{3.45}$$

We then express V_{x+1}^{y} explicitly in terms of the unknowns V_{x}^{y+1},V_{x}^{y} and V_{x}^{y+1}. In making V_{x+1}^{y} the subject gives

$$V_{x+1}^{y} = \frac{1}{1 - r\Delta t}\left[a_y V_x^{y-1} + b_y V_x^y + c_y V_x^{y+1}\right] \tag{3.46}$$

$$\left\{\begin{array}{lcl} a_y &=& \frac{ry\Delta t}{2} - \frac{\sigma^2 y^2 \Delta t}{2} = \frac{1}{1-r\Delta t}\pi_d \\ b_y &=& 1 + \sigma^2 y^2 \Delta t = \frac{1}{1-r\Delta t}\pi_0 \\ c_y &=& -\frac{ry\Delta t}{2} - \frac{\sigma^2 y^2 \Delta t}{2} = \frac{1}{1-r\Delta t}\pi_u \end{array}\right\} \tag{3.47}$$

for $x = N-1, N-2, \cdots, 1, 0$ and $y = 1, 2, \cdots, M-1$. Similarly to the explicit method, the implicit method is accurate to $0(\Delta t, \Delta A^2)$. The system of equations in tridiagonal matrix form is given by

$$\begin{bmatrix} V_{x+1,0} - a_0 \\ V_{x+1,1} \\ \vdots \\ V_{x+1,M-1} \\ V_{x+1,M} - c_M \end{bmatrix} = \frac{1}{1-r\Delta t}\begin{bmatrix} b_0 & c_0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0\cdots & 0 & a_M & b_M & c_M \end{bmatrix}\begin{bmatrix} V_{x,0} \\ V_{x,1} \\ \vdots \\ V_{x,M-1} \\ V_{x,M} \end{bmatrix} \tag{3.48}$$

The system of equations derived can be written in the form $DV_{x,y} = V_{x+1,y}$ for $y = 0, 1, \cdots, M$. The matrix has $b_y = 1 + \sigma^2 y^2 \Delta t$ in the diagonal which is positive.The matrix is non-singular since the product of the diagonal elements are non-zero. We then solve using the inverse of matrix D,D^{-1}. Now applying the boundary conditions to equation 3.46 changes elements b_0,$b_1 = 1$ and c_0,$a_M = 0$ in the matrix D.

3.6.6 Stability of the Implicit Scheme

The eigenvalues of the matrix is given by

$$\lambda_i = b_y + 2(\sqrt{a_y c_y})cos\frac{i\pi}{N} \text{ for } i = 1, 2, \cdots, N-1 \tag{3.49}$$

We substitute a, b, and y into 3.49 and simplifying, the following result is obtain:

$$\lambda_i = 1 + \sigma^2 y^2 \Delta t + \sigma^2 y^2 \Delta t \left[1 - \frac{r^2}{\sigma^4 y^2}\right]^{\frac{1}{2}} \left[1 - 2sin^2\frac{x\pi}{2N}\right]$$

$$\Rightarrow \lambda_i \approx 1 + 2\sigma^2 y^2 \Delta t - 2\sigma^2 y^2 \Delta t sin^2\frac{x\pi}{2N} \tag{3.50}$$

The change of sign is due to the truncation of the binomial expansion. The scheme is stable when

$$\| D \|_2 = \max \left| 1 + 2\sigma^2 y^2 \Delta t - 2\sigma^2 y^2 \Delta t sin^2\frac{x\pi}{2N} \right| \leq 1$$

$$\Rightarrow -1 \leq 1 + 2\sigma^2 y^2 \Delta t - 2\sigma^2 y^2 \Delta t sin^2\frac{x\pi}{2N} \leq 1 \tag{3.51}$$

as $\Delta t \to 0$, $N \to \infty$ and $sin^2\frac{(N-1)\pi}{2N} \to 1$, $|\lambda_i| \leq 1$.Therefore the scheme is unconditionally stable, convergent and consistent.

3.6.7 Crank-Nicolson Method

The crank-Nicolson may be regarded as a hybrid between the explicit and fully implicit approach. It takes the average of the explicit and fully implicit scheme. In applying the Crank-Nicolson method to the

Black-Scholes PDE gives the following grid equations:

$$\frac{V_{x+1}^{y} - V_{x}^{y}}{\Delta t} + \frac{ry\Delta A}{2\Delta A}\left[\frac{V_{x+1}^{y+1} - V_{x+1}^{y-1}}{2}\right] + \frac{ry\Delta A}{2\Delta A}\left[\frac{V_{x}^{y+1} - V_{x}^{y-1}}{2}\right] +$$
$$\frac{\sigma^2 y^2 \Delta A^2}{2\Delta A^2}\left[\frac{V_{x+1}^{y+1} - 2V_{x+1}^{y} + V_{x+1}^{y-1}}{2}\right] + \frac{\sigma^2 y^2 \Delta A^2}{2\Delta A^2}\left[\frac{V_{x}^{y+1} - 2V_{x}^{y} + V_{x}^{y-1}}{2}\right]$$
$$= \frac{rV_{x}^{y}}{2} + \frac{rV_{x+1}^{y}}{2} \tag{3.52}$$

$$\Rightarrow V_{x+1}^{y} - V_{x}^{y} + \frac{ry\Delta t}{4}[V_{x+1}^{y+1} - V_{x+1}^{y-1} + V_{x}^{y+1} + V_{x}^{y-1}]$$
$$\frac{\sigma^2 y^2 \Delta t}{4}[V_{x+1}^{y+1} - 2V_{x+1}^{y} + V_{x+1}^{y-1} + V_{x}^{y+1} - V_{x}^{y} + V_{x^{y-1}}]$$
$$= \frac{r\Delta t}{2}[V_{x+1}^{y} + V_{x}^{y}]$$

simplifying and rearranging we get :

$$V_{x}^{y-1}\alpha_y + V_{x}^{y}\beta_y + V_{x}^{y+1}\gamma_y = V_{x+1}^{y-1}a_y + V_{x+1}^{y}b_y + V_{x+1}^{y+1}c_y \tag{3.53}$$

where

$$\left\{\begin{array}{c} \alpha_y = \frac{ry\Delta t}{4} - \frac{\sigma^2 y^2 \Delta t}{4} \\ \beta_y = 1 + \frac{r\Delta t}{2} + \frac{\sigma^2 y^2 \Delta t}{2} \\ \gamma_y = -\frac{ry\Delta t}{4} - \frac{\sigma^2 y^2 \Delta t}{4} \\ a_y = \frac{\sigma^2 y^2 \Delta t}{4} - \frac{ry\Delta t}{4} \\ b_y = 1 - \frac{ry\Delta t}{2} - \frac{\sigma^2 y^2 \Delta t}{2} \\ c_y = \frac{ry\Delta t}{4} + \frac{\sigma^2 y^2 \Delta t}{4} \end{array}\right\} \tag{3.54}$$

for $x = 0, 1, \cdots, N$ and $y = 1, 2, \cdots, M-1$. We express the system of equations in 3.53 as $AV_x = BV_{x+1}$ and this results into a tridiagonal system :

$$\begin{bmatrix} \beta_0 & \gamma_0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{M-1} & \beta_{M-1} & \gamma_{M-1} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_M & \beta_M \end{bmatrix} \begin{bmatrix} V_{x,0} \\ V_{x,1} \\ \vdots \\ V_{x,M-1} \\ V_{x,M} \end{bmatrix} =$$

$$\begin{bmatrix} b_0 & c_0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & \cdots & 0 & a_M & b_M \end{bmatrix} \begin{bmatrix} V_{x+1,0} \\ V_{x+1,1} \\ \vdots \\ V_{x+1,M-1} \\ V_{x+1,M} \end{bmatrix} \tag{3.55}$$

In solving this system, the elements of vector V_{x+1} are known at maturity time T, and so we express the system 3.55 as $V_x = A^{-1}BV_{x+1}$. By repeatedly iterating from time T to zero, we obtain the value of V as the value of the life insurance contract. The diagonal entries of the matrix A is $\beta_y = 1 + \frac{1}{2}r\Delta t + \frac{1}{2}\sigma^2 y^2 \Delta t$ are always positive and thus the diagonal elements are non-zero. Therefore the matrix is non-singular as the diagonal entries are non-zero.

Accuracy Of Crank-Nicolson

The accuracy of the crank-Nicolson is better than the Implicit and Explicit Scheme because it of the accuracy of $0(\Delta t^2\,,\, \Delta A^2)$. This accuracy can be demonstrated by equating the central difference and symmetric central difference at $V^y_{x+\frac{1}{2}} \;\equiv\; V(t\,+\,\frac{\Delta t}{2}\,,\,A)$. Expanding $V_{x+1,y}$ in Taylor series at $V_{x+\frac{1}{2},y}$ to yield

$$V_{x+1,y} \;=\; V_{x+\frac{1}{2}} \;+\; \frac{\partial V}{2\partial t}\Delta t \;+\; 0(\Delta t^2) \tag{3.56}$$

and expanding V^y_x at $V_{x+\frac{1}{2},\,y}$ yields the following

$$V_{x,y} \;=\; V_{x+\frac{1}{2},y} \;-\; \frac{\partial V}{2\partial t}\Delta t \;+\; 0(\Delta t^2) \tag{3.57}$$

and taking the average of the equations 3.56 and 3.57 gives

$$\frac{V_{x,y} \;+\; V_{x+1,y}}{2} \;=\; V_{x+\frac{1}{2},y} \;+\; 0(\Delta t^2) \tag{3.58}$$

This implies

$$V^{y-1}_{x+\frac{1}{2}} - 2V^y_{x+\frac{1}{2}} + V^{y+1}_{x+\frac{1}{2}} \;=\; \frac{1}{2}\left[V^{y-1}_x \;-\; 2V^y_x \;+\; V^{y+1}_x\right] + \frac{1}{2}\left[V^{y-1}_{x+1} \;-\; 2V^y_{x+1} \;+\; V^{y+1}_{x+1}\right] \tag{3.59}$$

The right-hand side of equation 3.59 is the average of the two symmetric central difference centered at x and $x + 1$. Dividing by

ΔA^2 we obtain the equality

$$\frac{\partial^2 V(t\,+\,\frac{\Delta t}{2}\,,\,\Delta T)}{\partial A^2} = \frac{1}{2}\left[\frac{\partial^2 V(t\,,\,A)}{\partial A^2} + \frac{\partial^2 V(t\,+\,\Delta t,\,A)}{\partial A^2}\right] + 0(\Delta t^2,\,\Delta A^2) \tag{3.60}$$

Equation 3.60 is the symmetric central difference approximation. The superscript y is arbitrary and we deduce the central difference approximation as follows :

$$V^{y+1}_{x+\frac{1}{2}} - V^{y-1}_{x+\frac{1}{2}} = \frac{1}{2}\left[V^{y+1}_{x} - V^{y-1}_{x}\right] + \frac{1}{2}\left[V^{y+1}_{x+1} - V^{y-1}_{x+1}\right] + 0(\Delta t^2) \tag{3.61}$$

Dividing through by $2\Delta A$, we get

$$\frac{\partial V(t+\frac{\Delta t}{2},\Delta T)}{\partial A} = \frac{1}{2}\left[\frac{\partial V(t,A)}{\partial A} + \frac{\partial V(t+\Delta t,A)}{\partial A}\right] + 0(\Delta t^2,\Delta A^2) \tag{3.62}$$

and is the first central difference approximation. Subtracting equation 3.57 from 3.56 we obtain the approximation of $\frac{\partial V}{\partial t}$ at $(t+\frac{1}{2}\Delta t, A)$. That is,

$$\frac{\partial V(t+\frac{1}{2}\Delta t,A)}{\partial t} = \frac{V^{y}_{x+1} - V^{y}_{x}}{\Delta t} + 0(\Delta t^2) \tag{3.63}$$

Hence the Black-Scholes PDE centered at $(t+\frac{1}{2}\Delta t, A)$ has a finite difference approximation

$$\frac{V^{y}_{x+1} - V^{y}_{x}}{\Delta t} + \frac{ry\Delta A}{4\Delta A}\left[V^{y+1}_{x} - V^{y-1}_{x} + V^{y+1}_{x+1} - V^{y-1}_{x+1}\right] +$$
$$\frac{\sigma^2 y^2 \Delta A^2}{4\Delta A^2}\left[V^{y-1}_{x} - 2V^{y}_{x} + V^{y+1}_{x} + V^{y-1}_{x+1} - 2V^{y}_{x+1} + V^{y+1}_{x+1}\right] = rV^{y}_{x} \tag{3.64}$$

Rearranging equation 3.64, we obtain the equation of the form 3.53

which is the exact Crank-Nicloson Scheme. Therefore the scheme has a leading error of order $0(\Delta t^2, \Delta A^2)$, Kerman (2002).

3.6.8 Hopscotch Method

The Hopscotch method is fashioned in such a way that the unknown grid points are obtained at two stages. The first stage involves solving the Black-Scholes PDE for all points on time level x such that $(x+y)$ are even values using an explicit scheme. At odd values $(x+y)$ the points are calculated at time y using an 'effectively' implicit scheme. The basic idea of the Hopscotch scheme is to divide the mesh points into two-dimensional x-y mesh (xh, yh)as follows;

1. $x+y$ odd - effective implicit scheme.

2. $x+y$ even explicit scheme.

We predominantly structure the Hopscotch scheme in two steps

- An Implicit scheme

  $$\frac{V_{x+1}^{y} - V_x^y}{\Delta t} + \frac{ry\Delta A}{2\Delta A}\left[V_x^{y+1} - V_x^{y-1}\right] + \frac{\sigma^2 y^2 \Delta A^2}{2\Delta A^2}\left[V_x^{y-1} - 2V_x^y + V_x^{y+1}\right] = rV_{x+1}^{y}$$

  for odd values of $(x+y)$ that is x is even and y must be odd and vice verse which provides the difference scheme

$$V_{x+1}^{y} = \frac{1}{1-r\Delta t}[\alpha_y V_x^{y-1} + \beta_y V_x^y + \gamma_y V_x^{y+1}]$$

for $x = 0, 1, \cdots, N$ and $y = 1, 2, \cdots, M-1$

- An Explicit scheme

$$\frac{V_{x+1}^{y} - V_{x}^{y}}{\Delta t} + \frac{ry\Delta A}{\Delta A}\left[V_{x+1}^{y+1} - V_{x+1}^{y-1}\right] + \frac{\sigma^2 y^2 \Delta A^2}{2\Delta A^2}\left[V_{x+1}^{y-1} - 2V_{x+1}^{y} + V_{x+1}^{y+1}\right] = rV_x^y$$

for even values of $(x + y)$ that is either x and y either both be even or both be odd, this leads to the difference scheme

$$V_x^y = \frac{1}{1 + r\Delta t}[a_y V_{x+1}^{y-1} + b_y V_{x+1}^{y} + c_y V_{x+1}^{y+1}]$$

for $x = 0, 1, 2, \cdots, N$ and $y = 1, 2, \cdots M$

3.7 Interest Rate Models

In the modeling of interest rates, various interest rate models can be applied. Some of these models are

1. Vasicek Model
2. Black, Derman and Toy (BDT)model
3. Cox-Ingersoll-Ross (CIR) model
4. Hull and White (HW) model
5. Ho-Lee model

This study utilizes the Vasicek model and the Cox-Ingersoll-Ross (CIR) model.

3.7.1 Vasicek Model

The Vasicek model is mainly used in modeling a mean reverting process. The process for the rate of return is modeled by

$$dr(t) = \lambda(\mu - r(t))dt + \sigma dW(t) \tag{3.65}$$

where

λ measures the speed of mean reversion. (Speed)

μ is the long run mean which reverts the process. (Level)

σ measures the volatility of the process.

$W(t)$ is the Brownian Motion.

3.7.2 Cox-Ingersoll-Ross Model

Cox-Ingersoll-Ross model was developed in 1980 and published in 1985 and it is used in modeling the instantaneous interest rate. The modeling process for the rate of return is given by

$$dr(t) = K(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t). \tag{3.66}$$

where $r(0), K, \theta, \sigma > 0$

λ measures the speed of mean reversion. (Speed)

μ is the long run mean which reverts the process. (Level)

σ measures the volatility of the process.

$W(t)$ is the Brownian Motion.

CHAPTER 4

ANALYSIS AND RESULTS

4.1 Introduction

In this chapter, we analysis the application of time dependent interest rate in the Black-Scholes partial differential equation in the valuation of life insurance contract. The chapter further looks at the comparison of the Hopscotch and the Crank-Nicloson method in the valuation of life insurance contracts. The study considers a data from an insurance company as cited in Ali (2013)

4.2 Matlab Implementation

Matlab is used in the computational analysis since the matrices developed using the finite difference have generally very large tridiagonal matrices and requires more computational time.

4.3 Stability of Crank-Nicolson Method

In this section, the researcher looks at the stability of the scheme i.e. the convergence of the scheme. This section describes how the eigen values approaches 1 as we move along the grids. The table below displays the eigenvalues of the matrix of the scheme as $N \longrightarrow \infty$

Table 4.1: The eigenvalues of the Crank-Nicolson method as $N \longrightarrow \infty$

N=100		N=500		N=1000		N=2000		N=4000	
j	λ_j	j	λ_j	j	λ_j	j	λ_j	j	λ_j
93	1.0572	493	1.0119	993	1.0060	1993	1.0030	3993	1.0015
94	1.0398	494	1.0088	994	1.0044	1994	1.0022	3994	1.0011
95	1.0284	495	1.0062	995	1.0031	1995	1.0016	3995	1.0008
96	1.0189	496	1.0040	996	1.0020	1996	1.0010	3996	1.0005
97	1.0111	497	1.0023	997	1.0012	1997	1.0006	3997	1.0003
98	1.0055	498	1.0011	998	1.0006	1998	1.0003	3998	1.0001
99	1.0020	499	1.0004	999	1.0002	1999	1.0001	3999	1.0000

Table 4.1 displays the stability of the Crank-Nicolson method as the eigenvalues approaches one(1) when $N \longrightarrow \infty$. Also the Crank-Nicolson method is with an accuracy of $0(\Delta t^2, \Delta A^2)$ and also indicates how accurate the results is to the actual value.

4.4 Derivation of Interest Rate

In this section, the study looks at how to generate the interest rate (rates of return).

The study utilizes the Vasicek model and the Cox-Ingersoll-Ross model. The Vasicek model for rate of return is given by :

$$dr(t) = S(L - r(t))dt + \sigma dW(t) \tag{4.1}$$

In order to calibrate the simple short rate of return we reduce equation 4.1 into a simple regression model. In reducing 4.1 into a regression model , we obtain

$$Y(t) = \alpha + \beta r(t) + \epsilon(t)$$

where $Y(t) = dr(t)$

$\alpha = SLdt$ and $\beta = -Sdt$.

The model's volatility is proportional to the standard error of the residuals $\sigma = \sqrt{var(\epsilon(t))dt}$

Solving the data in appendix C for speed and level, we get the speed $=$ 0.0618686 and the level $=$ 0.165532. The study then uses these values in estimating the rates of return using the Cox-Ingersoll-Ross model (CIR).

The figure above illustrates the movement of the rate of return from time zero to maturity using the CIR model with an initial rate of 0.05

4.5 Interest Rates Valuations

This section looks at the valuations of interest rate at various grid points. Table 4.2 to table 4.7 show the values of the interest rate at a grid level of 100 and 1000 and also the valuation of the life insurance

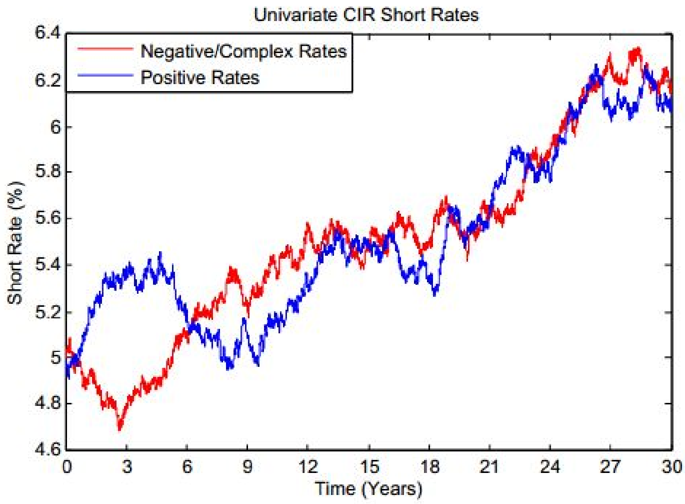

Figure 4.1: The short growth rate of return from time zero to maturity

contracts at these points.

Table 4.2: Table of Rate of Return with a step level of 100

step size (Δt)	rate of return	step size (Δt)	rate of return
0.3	0.0500	3.3	0.0593
0.6	0.0754	3.6	0.0568
0.9	0.0486	3.9	0.0549
1.2	0.0702	4.2	0.0541
1.5	0.0510	4.5	0.0523
1.8	0.0589	4.8	0.0576
2.1	0.0565	5.1	0.0598
2.4	0.0498	5.4	0.0605
2.7	0.0424	5.7	0.0630
3.0	0.0495	6.0	0.0633

Table 4.2 illustrates the rate of return with a discretized step level of 100 with maturity of the life insurance contract at 30 years. This is with an equal step size of 0.3.

Table 4.3: Table of price of contract with a step level of 100 under the Crank-Nicolson Scheme

step size (Δt)	rate of return	value of contract
0.3	0.0500	7.5107
0.6	0.0754	5.7542
0.9	0.0486	7.6401
1.2	0.0702	6.0381
1.5	0.0510	7.4113
1.8	0.0589	6.7818
2.1	0.0565	6.9615
2.4	0.0498	7.5291
2.7	0.0424	8.2731
3.0	0.0495	7.5568

Table 4.3 discusses some of the prices of a life insurance contract at a grid level of 100 with an equal step size of 0.3. This valuation is done under the Crank-Nicolson scheme.

Table 4.4: Valuation of life Insurance Contract under the Hopscotch with a step level of 100

step size (Δt)	rate of return	value of contract
0.3	0.0500	7.5688
0.6	0.0754	5.4154
0.9	0.0486	7.7342
1.2	0.0702	6.3296
1.5	0.0510	7.4539
1.8	0.0589	6.8665
2.1	0.0565	6.9160
2.4	0.0498	7.6433
2.7	0.0424	8.3481
3.0	0.0495	7.6311

Table 4.4discusses some valuation of life insurance contracts under the Hopscotch scheme with a step level of 100. This valuation is done with an equal step size of 0.3.

Table 4.5: Table of rate of return with a step level of 1000

step size ((Δt))	rate of return
0.03	0.0569
0.06	0.0573
0.09	0.0592
0.12	0.0551
0.15	0.0596
0.18	0.0587
0.21	0.0556
0.24	0.0525
0.27	0.0446
0.30	0.0500

Table 4.5 displays a table value for the rate of return at a discretized level of 1000 and an equal step size of 0.03.

Table 4.6: Valuation of life insurance contract at a 1000 level under the Crank-Nicolson scheme

step size (Δt)	rate of return	value of contract
0.03	0.0569	7.0217
0.06	0.0573	6.9916
0.09	0.0592	6.8523
0.12	0.0551	7.1607
0.15	0.0596	6.8238
0.18	0.0587	6.8884
0.21	0.0556	7.1215
0.24	0.0525	7.3723
0.27	0.0446	8.1075
0.30	0.0500	7.5890

Table 4.6 discusses the valuation of life insurance contract under the Crank-Nicolson Scheme at a step level of 1000. This valuations is done with an equal step size of 0.03

Table 4.7: valuation of life insurance contract under the Hopscotch with a step level of 1000

step size (Δt)	rate of return	Value of contract
0.03	0.0569	7.0295
0.06	0.0573	7.0020
0.09	0.0592	6.8625
0.12	0.0551	7.2180
0.15	0.0596	6.8304
0.18	0.0587	6.8981
0.21	0.0556	7.1910
0.24	0.0525	7.3021
0.27	0.0446	8.2132
0.30	0.0500	7.5909

Table 4.7 above illustrates the value of a life insurance contract under the Hopscotch scheme at astep level of 1000 with an equal step size of 0.03.

4.6 Comparing the convergence of the Crank-Nicolson and the Hopscotch in the valuation of life insurance contract with no dividend.

We already considered the convergence of the Crank-Nicloson and Hopscotch method in relation to the Black-Scholes model in the valuation of life insurance contracts in chapter 3.

The data from the company A is as follows : Asset price,$A = 50$,$K = 52$, risk-free interest rate,$r = 0.05$,maturity period, $T = 30 years$ with the value of the contract at maturity being 8.22 for non-dividend paying asset. The data from company B is as follows : the asset price,$A = 250$,$K = 260$, $r = 0.06$,$T = 30 years$ with the value of the contract at maturity being 40.15. Table 4.8 shows the valuation

Table 4.8: The valuation of life insurance liabilities with no dividend payment with a maturity of 30 years for company A. Expected value = 8.220

No. of steps	Hopscotch	Crank-Nicolson
100	7.5236 (0.6964)	7.5107 (0.7093)
200	7.5404 (0.6796)	7.5529 (0.6671)
300	7.5570 (0.663)	7.5683 (0.6517)
500	7.5810 (0.639)	7.5801 (0.6399)
600	7.6081 (0.6119)	7.5830 (0.637)
700	7.6251 (0.5949)	7.5851 (0.6349)
800	7.6423 (0.5777)	7.5867 (0.6333)
850	7.6509 (0.5691)	7.5874 (0.6326)
1000	7.6766 (0.5434)	7.6509 (0.5691)

of life insurance contract of company A under the standard Black-Scholes model. This valuations are done under the two finite schemes, that is the Hopscotch and the Crank-Nicolson schemes.

Table 4.9: The valuation of life insurance contract liabilities for company B with a maturity of 30 years with an expected value of 40.15

No of steps	Hopscotch	Crank Nicolson
50	38.9995 (1.1505)	36.1788 (3.9712)
90	39.1050 (1.0450)	36.6105 (3.5395)
120	39.1105 (1.0395)	36.7513 (3.3987)
150	39.1279 (1.0221)	36.8373 (3.3127)
210	39.1244 (1.0256)	36.9352 (3.2148)
320	39.1260 (1.0240)	37.0196 (3.1304)
400	39.1267 (1.0233)	37.0522 (3.0978)
450	39.1280 (1.0220)	37.0678 (3.0822)
490	39.1273 (1.0227)	37.0773 (3.0727)
550	39.1275 (1.0225)	37.0893 (3.0607)
750	39.1281 (1.0219)	37.1149 (3.0351)
890	39.1283 (1.0217)	37.1258 (3.0242)
900	39.1282(1.0218)	37.1267 (3.0233)

Table 4.9 displays the value of contract for company B under the two finite schemes (Hopscotch and the Crank-Nicolson method).

As the number of steps increases, the difference method solution approaches the true solution with the Hopscotch converging faster than the Crank-Nicolson method.

NB : The values in the brackets are the difference between the observed value and the estimated value of the contract at each step size.

Table 4.10: Comparison of the value of the contract between the modified Black-Scholes and the standard Black-Scholes model using the Crank-Nicolson method

No of steps	Modified B-S model	Standard B-S model
100	6.7023	7.5107
120	6.7179	7.5232
150	6.7363	7.5387
180	6.7469	7.5489
200	6.7533	7.5529
250	6.7638	7.5622
300	6.7708	7.5683
330	6.7738	7.5707
390	6.7788	7.5752
420	6.7808	7.5768
470	6.7834	7.5790
510	6.7852	7.5804
580	6.7878	7.5825

Table 4.10 illustrates the value of a life insurance contract using the standard B-S model and the modified B-S model under the Crank-Nicolson scheme. The value of the contract under the modified B-S model is much lower than that of the standard B-S model.

Table 4.11: Comparison of the value of contract between the standard B-S model and the modified B-S model under the Hopscotch

No. of steps	Modified B-S model	Standard B-S model
100	6.7734	7.5813
120	6.7755	7.5839
150	6.7780	7.5852
180	6.7800	7.5862
200	6.7814	7.5867
250	6.7826	7.5873
300	6.7836	7.5883
330	6.7841	7.5886
390	6.7848	7.5891
420	6.7850	7.5893
470	6.7853	7.5896
510	6.7855	7.5898
580	6.7859	7.5900

Table 4.11 illustrates the comparison of the valuation of life insurance contracts between the standard B-S model and the modified B-S model under the Hopscotch scheme. The price of the life insurance contract under the modified is much lower than that of the standard B-S model.

4.7 Charts showing the valuation of life Insurance Contracts

In this section, the study presents the pictorial view of the convergence of the two finite schemes on both the standard and modified Black - Scholes model.

Figure 4.2 illustrates the convergence of the Crank-Nicolson scheme in estimating the price of a life insurance contract.

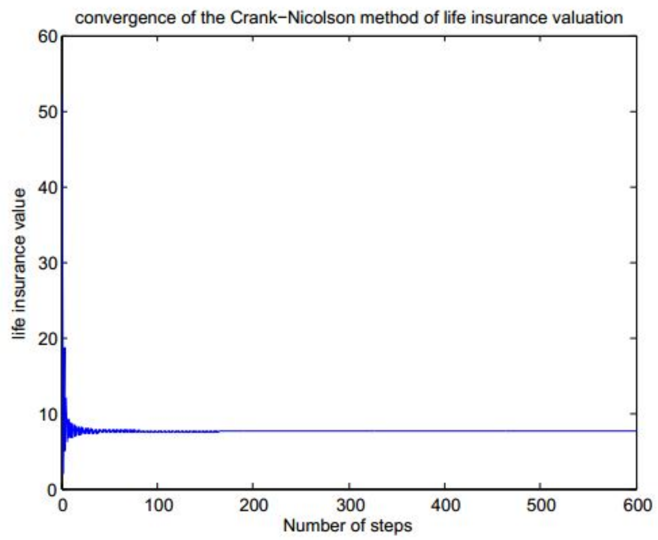

Figure 4.2: Chart on the convergence of the Crank-Nicolson

Figure 4.3 demonstrates the fluctuations of the Hopscotch scheme as we move along the grids in the valuation of life insurance contracts. This convergence is in with respect to the old rate of return.

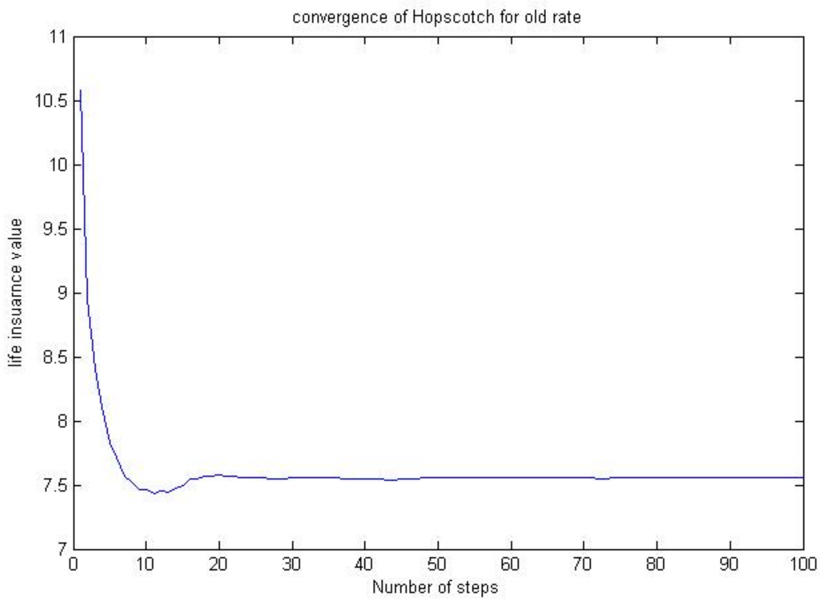

Figure 4.3: Convergence of the Hopscotch for the standard B-S model

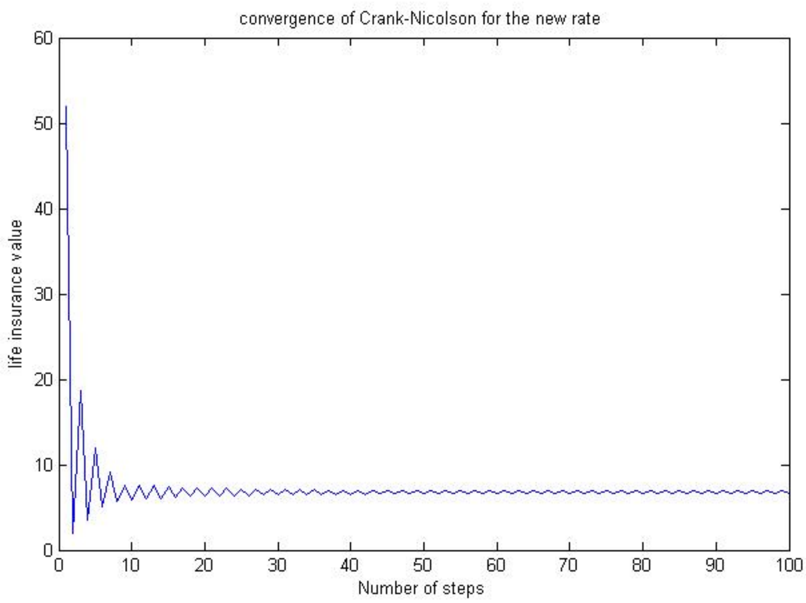

Figure 4.4: Displays the convergence Crank-Nicolson for the modified B-S model

Figure 4.4 displays the convergence of Crank-Nicolson in the valuation of life insurance contracts using a new rate of return

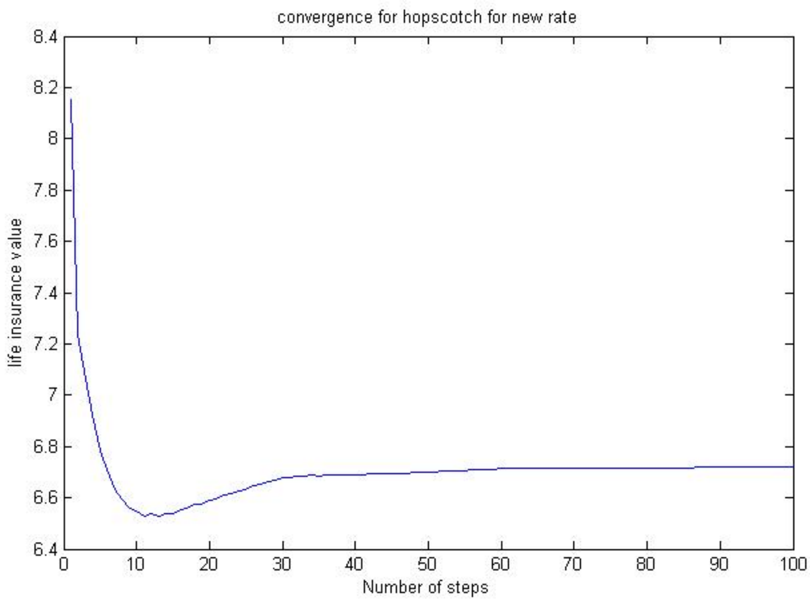

Figure 4.5: Chart on the convergence of the Hopscotch for the modified B-S model

Figure 4.5 demonstrates the convergence of the Hopscotch using the modified Black-Scholes model.

CHAPTER 5

CONCLUSION

5.1 INTRODUCTION

This chapter deals with the summary of the results from the analysis, conclusion and some recommendations for future studies.

5.2 Summary of Results

Under the stability analysis, the study utilizes the eigenvalue to test the stability of the two finite schemes, thus the Crank-Nicolson and Hopscotch schemes. The results also proved that the Crank-Nicolson is unconditionally stable. From the analysis of the previous chapter,the difference between the actual value of the life insurance contract and the estimated value at each step is as a result of the discretization of the asset price and time. This contributes to the sources of errors of the schemes. This is in accordance with the Lax Equivalence theorem, which states that the underlying factors that characterizes a numerical scheme are consistency, stability and convergence.

From the tables, the study realized that the level of accuracy of approximation increases as we increase the number of steps. By comparing the Hopscotch to the Crank- Nicolson, the Hopscotch converges faster than the Crank-Nicolson but the Hopscotch's increment with respect to the step size is not consistent. Thus to say the Hopscotch level of accuracy with approximation of the life insurance value is much lower than that of the Crank-Nicolson. This is the same in relation to the modified Black-Scholes partial differential equation. The values of the life insurance contract with a fixed rate of return(interest rate) is much higher than modified Black-Scholes model(thus when the rate of interest is time dependent).

5.3 Conclusion

Since both the Hopscotch and the Crank-Nicolson are a mix of the implicit scheme and the explicit scheme at different nodes, they both tend to generate similar values. Only that the Crank-Nicolson tend to be consistent in its values than the Hopscotch as we increase the number of steps. In analysis of the modified Black-Scholes partial differential equation of life insurance contract, it was realized that the value of contract at maturity with the existing model(thus constant interest rate) is higher than the value of contract of the modified Black-Scholes partial differential equation(thus time dependent interest rate). The Crank-Nicolson gives a better approximation to the modified Black-Scholes Partial differential equation then the

Hopscotch because of it's consistency in values.

5.4 Recommendation

The study suggests that the Crank-Nicolson will be appropriate in finding the solution to the modified Black-Scholes partial differential equation than the Hopscotch. It also suggests that one should look at the interest rates carefully in the valuation of life insurance contracts.

5.5 Further Studies

This study looked at one area of the constants in the Black-Scholes partial differential equation that is the rate of interest. Further research could also be study on the other constant term, thus the volatility.

REFERENCES

Aase, K. and Persson, S., A. (1994). Pricing of unit-linked insurance policies. *Scandinavian Actuarial Journal*, pages 26–52.

Adams, M., Andersson, J., Andersson, L., and Lindmark, M. (2005). The historical relation between banking, insurance and economic growth in sweden: 1830-1998.

Albizzati, M., O. and Geman, H. (1994). Interest rate risk management and valuation of the surrender option in life insurance policies. *The Journal of risk and insurance*, 61:616–637.

Ali, M. (2013). The valuation of life insurance liabilities : Numerical method approach. Master's thesis, Kwame Nkrumah University of science and technology.

Anders, G. and Peter, L., J. (2002). Life insurance liabilities at market value : An analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option frame. *The Journal of risk and insurance*, 69:63–91.

Andersen, L. (1999). A simple appoach to the pricing of bermudan swap option in the multi-factor libor market model. *Jounal of computational Finance*, 3:5–32.

Bacinello, A., R. (2001). Fair pricimg of life insurance participating policies with a minimum interest rate guaranteed. *Astin Bullentin*, 31:275–297.

Bacinello, A., R. (2009). Fair valuation of life insurance contracts with embedded options. 17.

Bacinello, A., R. and Ortu, F. (1993). Pricing guaranteed securities-linked life insurance under interest rate risk. *Acctuarial Appoach for Financial Risk, 3rd AFIR International Colloquium*, 4:35–55.

Bacinello, A., R. and Ortu, F. (1994). *Financial Modelling : Single and Periodic Premiums for Guaranteed Equity-linked life insurance under Interest Rate Risk : The Lognormal + Vasicek Case*. Heidelberg.

Ballotta, L., Haberman, S., and Wang, N. (2003). Modelling and valuation of guarantees in with profit and unitised with-profit life insurance contracts. *Actuarial research paper no. 146, Department of Actuarial Science and Statistics, Cass Business School London.*

Ballotta, L., Haberman, S., and Wang, N. (2006). Guarantees in with-profit and unitized with-profit life insurance contracts : Fair valuation problem in presence of the default option. *Journal of risk and insurance*, 73(1):97–121.

Barbarin, J. and Devolder, P. (2005). Risk measure and fair valuation of an investment guarantee in life insurance. *Insurance : Mathematics and Economics*, 37(2):297–323.

Bauer, D., Kiesel, R., Kling, A., and Russ, J. (2006). Risk - neutral valuation of participating life insurance contracts. *Insurance : Mathematics and Economics*, 39(2):171–183.

Beenstock, M., Dickinson, G., and Khanjuria, S. (1986). The determination of life premiums: An international cross-section analysis 1970-1981. *Insurance, Mathematics and Economics*, 5(4):261–270.

Bjarke, J., Peter, L., J., and Anders, G. (2001). A finite difference approach to the valuation of path dependent life insurance liabilities. *The Geneva Papers on Risk and Insurance Theory*, 26:57–84.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Econoomy*, 81:637–654.

Boyle, P., P. and Schwartz, E. (1977). Equilibrium prices of guarantees under equity-linked contracts. *The Journal of risk and insurance*, 4:639–600.

Brennan, M., J. and Schwartz, E. (1976). The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics*, 3:195–213.

Brennan, M., J. and Schwartz, E. (1979). Alternative investment strategies for equity-linked life insurance policies with an asset value guarantee. *Journal of Business*, 52:63–93.

Briys, E. and Varenne., F. (1997). On the risk of life insurance liabilities ; debunking some common pitfalls. *The Journal of risk and insurance*, 64:673–694.

CEA (2006). The contribution of the insurance sector to economic growth and employment in the eu. *Brussels : CEA*.

Christopher, O. (2009). Valuation of life insurance liabilities on a market- consistent basis : Experience from the united kingdom. *Actuarial Practice Forum*, pages 1–49.

Cox, J., C., Ross, S., A., and Rubinstein, M. (1979). Optiion pricimg : A simplified approach. *Journal of Financial Economics*, 7:229–263.

Daniel, B., Daniel, B., and Rudiger, K. (2010). On the risk-neutral valuation of life insurance contracts with numerical methods in view. *ULM University.*

Davis, M., S. (2005). Numerical methods for the valuation of financial derivatives. Master's thesis, University of Western Cape, South Africa.

Delbaen, F. (1986). Equity linked policies. *Bullentin Association Royal Actuaries Belge*, pages 33–52.

Dorfman, M. (2008). *Introduction to risk management and insurance.* Pearson Prentice Hall.

Duffie, D. (1996). *Dynamics Asset Pricing Theory.* Princeton University Press, 3rd edition.

Enz, R. (2000). The curve relations between per-capita income and insurance penetration. *Geneva Papers on risk and insurance : issues and practice*, 25(3).

Eric, B. (1995). On the risk of life insurance liabilities : Debunking some common pitfalls. *Financial Institution center*, pages 24–96.

Fabio, B., Paolo, D., A., and Andrea, F. (2006). On a fair value model for participating life insurance policies. *Investment Management and Financial Innovations*, 3(2):105–113.

Fama, E. F. (1965). The behavior of stock market returns. *Journal of Business*, 38:34–105.

FCAC (2011). Understanding insurance basics. financial consumer agency of canada. pages 1–25.

Gatzert, N. and Schmeiser, H. (2008). Assessing the risk potential of premium payment options in participating life insurance contracts. *Journal of Risk and Insurance*, 75(3):691–712.

Gerstner, T., M., Griebel, M., Holtz, M., Goschnick, R., and Haep, M. (n.d.). Numerical simulation for asset - liability management in life insurance.

Giraldi, C., Susinno, G., Berti, G., Brunello, J., Buttarrazzi, S., Cenciarelli, G., Daroda, C., and Stamega, G. (2003). *Exotic Options: the cutting edge collection : technical papers publlished in risk 1999-2003*. London : Risk books.

Grosen, A. and Joergensen, P., L. . (2000). Fair valuation of life insurance liabilities : The impact of interest rate guarantees, surrender optiions, and bonus policies. *Insurance: Mathematics and Economics*, 27:37–57.

Grosen, A. and Joergensen, P., L. . (2002). Life insurance liabilities at market value : An analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. *Journal of Risk and Insurance*, 69:63–69.

Haiss, P. and Kjell, S. (2008). The relationship between insurance and economic growth in europe: a theoretical and empirical analysis.

Hansen, M. and Miltersen, K., R. (2002). Minimum rate of return guarantees : The danish case. *Scandinavian Actuarial Journal*, 4:280–318.

Hull, J. (2003). *Options,futures and other derivatives.* Pearson Education Inc., New Jersey, 5 edition.

Kerman, J. (2002). Numerical methods for option pricingn : Binomial and finite difference approximation. *Retrieved from www.stat.columnbia.edu/ Kerman.*

Kurz, A. (2003). Pricing of equity-linked life insurance policies with an asset value guarantee and periodic premiums.

Luca, A. and Cesare, L. D. (2007). Valuation of surrender option in unit linked life insurance policies in non-rational behaviour framework. 20.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business*, 36:394–419.

Merton, R., C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management science*, 4:141–183.

Mike, B., Arun, T., Alice, K., Cheng-Sheng, p., W., Chris, S., and Jim, G. (2010). Predictive modeling for life insurance, ways life insurers can participate in the business analytics revolution. *Deloitte Consulting LLP*, pages 1–29.

Milevsky, M., A. and Posner, S., E. (2001). The titanic option : Valuation of the guaranteed minimum death benefit in variable annuities and mutual funds. *Journal of Risk and Insurance*, 68(1):93–128.

Mphasis (2009). Healthcare insurance certification program.

Nadine, G. and Alexander, K. (2006). Analysis of participating life insurance contracts : A unification approach. *Journal of Risk and Insurance*.

Nadine, G. and Stefan, K. (2007). Risk assessment of life insurance contracts : A comparative study in a levy framework. *JEL Classification*, 37.

Neilsen, J., A. and Sandmann, K. (1995). Equity-linked life insurance: A model with stochastic interest rate. *Insurance: Mathematics and Economics*, 16:225–253.

Outreville, J., F. (1996). Life insurance markets in developing countries. *Journal of risk and insurance.*, 63(2):263–278.

Pal, k., Bolda, B., and Garg, M. (2007). *Insurance management : Principles and Practices.* Deep and Deep Publication PVT Ltd.

Paul, D. and David, W., Z. (1988). *Schaum's series of Theory and Problems of Partial Differential Equations.* McGraw-Hill.

Robert, J., B. and Martin, X., S. (1995). *Economic Growth.* McGraw-Hill.

Schmeiser, H. and Wagner, J. (2011). A joint valuation of premium payment and surrender option in participating life insurance contracts. *Institute of insurance economics(University of St.Gallen,Switzerland.*

Smith, G. (1985). *Numerical solution of partial differentiaal equations : Finite difference method.* Clarendon Press, Oxford.

Steven, C., C. (2012). *Applied Numerical Methods with Matlab for Engineers and Scientists.* McGraw-Hill.

Walden, M., L. (1985). The whole life insurance policy on an option package : An empirical investigation. *Journal of Risk and Insurance*, 52:44–58.

Ward, C. and David, K. (2004). *Numerical Mathematics and Computing.* Hearthside Publishing Service.

Ward, D. and Zurbruegg, R. (2000). Does insurance promote economic growth? evidence from oecd countries. *Journal of risk and insurance*, 67(4):489–506.

Webb, I., Grace, M., and Skipper, H., D. (2002). The effects of banking and insurance on the growth of capital and output. *(Center for risk management and insurance working paper).*

Wilmot, P., Howison, S., and Dewynne, J. (1995). *The Mathematics of Financial Derivatives - a Student Introduction.* Cambridge University Press.

APPENDIX A

Appendix : Matlab Codes For Hopscotch

```
% function price = HopPut(S0,K,r,T,sigma,Smax,dS,dt)

% K is the strike price.

% Smax is the asset price.

% r is the rate of return.

% sigma is the volatility

% T is the maturity.

% M is the number of iteration in the asset direction.

% N is the number of iteration in the time direction.

% S0 is the initial price of the asset.


dS=Smax/M;

dt=T/N;

matval=zeros(M+1,N+1);

vetS=linspace(0,Smax,M+1)';

veti=0:M;

vetj=0:N;

%set up boundary conditions

matval(:,N+1) = max(K-vetS,0);

matval(1,:) = K*exp(-r*dt*(N-vetj));
```

```
matval(M+1,:) = 0;


for j=N:-1:1
    for i=2:M
        if mod(j+i,2)==1
            %Use E
            a=0.5*dt*(sigma^2*veti-r).*veti;
            b=1-dt*(sigma^2*veti.^2+r);
            c=0.5*dt*(sigma^2*veti+r).*veti;
            matval(i,j)=a(i)*matval(i-1,j+1)+b(i)*matval(i,j+1)+c(i
        end
    end
    for i=2:M
        if mod(j+i,2)==0
            %Use I


            x=0.5*(r*dt*veti-sigma^2*dt*(veti.^2));
                y=1+sigma^2*dt*(veti.^2)+r*dt;
                z=-0.5*(r*dt*veti+sigma^2*dt*(veti.^2));
                matval(i,j)=(1/y(i))*matval(i,j+1)-(z(i)/y(i))*matv
        end
    end
end
price=interp1(vetS,matval(:,1),S0)
```

APPENDIX B

Appendix B : Matlab Codes For Crank-Nicolson

```
% S is the asset price
% K is the strike price
% r is the rate of return
% sigma is the volatility
% T is the maturity
% N is the number of iteration in the time direction
% M is the number of iteration in the asset direction
y=length(N);
Table=zeros(y,3);
for j=1:y
dt=T/N ;
 ds=2*S/M ;
 A=zeros(M(j)+1,M(j)+1) ;
 f=max(K-(0:M(j))*ds,0)';
 for  m=1:M-1
     A(m+1,m)=(r*m*dt-sigma.^2*m.^2*dt)/4 ;
     A(m+1,m+1)=1+0.5*r*dt+0.5*sigma.^2*m.^2*dt ;
     A(m+1,m+2)=(-r*m*dt-sigma.^2*m.^2*dt)/4 ;
end
```

```
 A(1,1)=1 ;
 A(M+1,M+1)=1 ;
 A ;
 for m=1:M(j)-1
     B(m+1,m)=(-r*m*dt+sigma.^2*m.^2*dt)/4 ;
     B(m+1,m+1)=1-0.5*r*dt-0.5*sigma.^2*m.^2*dt ;
     B(m+1,m+2)=(r*m*dt+sigma.^2*m.^2*dt)/4 ;
end
 B(1,1)=1 ;
 B(M+1,M+1)=1 ;
 B ;
 for  i=N:-1:1
     f=A^(-1)*(B*f);
     f=max(f,(K-(0:M)*ds)') ;
end
 f ;
 P=f(round((M+1)/2)) ;
```

Appendix C

Interest Rate Data

0.0043338158 0.0022076096 0.0004898991 0.0063408421 0.0098351786

0.0110155282 0.0212510173 0.0395349175 0.0427786990 0.065764

0.0763111626 0.0789668655 0.0694371750 0.0772583147 0.075935

0.0503923637 0.0374595512 0.0565283916 0.0762821890 0.0764904

0.0820406913 0.0704653471 0.0797656785 0.0782070883 0.091867

0.0980189145 0.0922539915 0.0942222296 0.0966987893 0.097827

0.1011769414 0.0843531632 0.0869617855 0.0823050171 0.086585

0.0806914954 0.0933617110 0.0959889435 0.1121557552 0.120834

0.1052299680 0.1140807000 0.0912893772 0.1023507504 0.106260

0.1202893214 0.1133487498 0.1116767919 0.1022799862 0.102980

0.0960754859 0.0958059511 0.0927844060 0.0937178770 0.096340

0.0966387962 0.0878895332 0.0816005315 0.0729174108 0.071067

0.0449872108 0.0558011004 0.0565780610 0.0732495739 0.080180

0.0544489055 0.0601371758 0.0582722828 0.0764230994 0.0722650

0.0625553264 0.0577891733 0.0574120504 0.0688801470 0.0777109

0.0634471169 0.0719866027 0.0840755556 0.0801836847 0.069875

0.0675733714 0.0699193882 0.0861407881 0.0771477089 0.0668272

0.0724383613 0.0672849760 0.0757338655 0.0676028631 0.0853875

0.0697309616 0.0714972929 0.0664409024 0.0567129358 0.0466919

0.0584135627	0.0657349215	0.0624697530	0.0603892433	0.068899
0.0651899374	0.0458139757	0.0521920884	0.0358431416	0.055825
0.0833460877	0.0863927077	0.0767210056	0.0701952099	0.063669
0.0463973414	0.0496774965	0.0593419830	0.0632602128	0.052789
0.0707053174	0.0751351604	0.0777018245	0.0777857963	0.087088
0.1031554020	0.1022967311	0.1124775444	0.1192994394	0.09875
0.0953257536	0.1002549233	0.1316133502	0.1282469188	0.116832
0.1071400545	0.1128665836	0.1139586439	0.1204412757	0.125028
0.1252552443	0.1133696644	0.1191584237	0.1106630878	0.123795
0.1320605980	0.1209992165	0.1170971159	0.1435059295	0.160920
0.1521193552	0.1456352270	0.1626853570	0.1475078653	0.145906
0.1252435051	0.1200425441	0.1124802128	0.1293025642	0.121509
0.1296306960	0.1303001848	0.1354061880	0.1451102822	0.132938
0.1388308998	0.1248146941	0.1357140410	0.1287337678	0.126959
0.1339652028	0.1235165403	0.1316708861	0.1429294289	0.152690
0.1341468900	0.1446649773	0.1517843728	0.1419320554	0.142326
0.1531892248	0.1518750610	0.1636471531	0.1638167755	0.174631
0.1956203383	0.2111511928	0.2262630713	0.2284450324	0.225355
0.2407628702	0.2479185720	0.2390920502	0.2404621869	0.234344
0.2207612136	0.2139992104	0.2142304598	0.2232560918	0.232774
0.2107600722	0.2083664021	0.2110378978	0.2063012525	0.214783
0.2137877887	0.2131550511	0.1977812795	0.2065513025	0.211662
0.2192716546	0.2144151132	0.2222258450	0.2168244837	0.222698
0.2265455095	0.2329110149	0.2442619719	0.2526104008	0.240018

0.2423882568	0.2554812958	0.2651262479	0.2654708250	0.2588069
0.2830295606	0.2978150082	0.3059938975	0.3167463599	0.3091553
0.3248594396	0.3178380093	0.3049050889	0.2923589783	0.2904085
0.2905620106	0.2953702294	0.2954367194	0.2975168415	0.2936286
0.2952951652	0.2936358060	0.3070377083	0.2989177780	0.2735189
0.2700040462	0.2573760778	0.2573106806	0.2482418535	0.2399525
0.2365887440	0.2282948379	0.2346549893	0.2358498168	0.2287912
0.2103279826	0.2028059541	0.2007632321	0.2070158322	0.2176402
0.2333198141	0.2316364079	0.2387901582	0.2500462904	0.2495897
0.2539774760	0.2644317729	0.2492487543	0.2592612342	0.2653389
0.2685925435	0.2816146526	0.2745361775	0.2573675474	0.2737279
0.2824954578	0.2998323857	0.3050340249	0.3113402882	0.2959625
0.3038951498	0.2923734632	0.2805745178	0.3007386191	0.3075961
0.2964709178	0.2820932140	0.2602163439	0.2514751856	0.2584501
0.2605918315	0.2389999783	0.2464323892	0.2502581382	0.2323248
0.2179104418	0.2265835286	0.2250932783	0.2243532325	0.2266819
0.2110110028	0.2012676729	0.2104676116	0.2012976198	0.1899176
0.1902052175	0.1812330853	0.1837395864	0.1878486226	0.1895034
0.2205416945	0.2280419197	0.2155899609	0.2116035695	0.2274110
0.2317634202	0.2355316670	0.2375811141	0.2410308664	0.2379974
0.2320083755	0.2515622810	0.2636143746	0.2710832322	0.2584965
0.2505107001	0.2403393161	0.2362296580	0.2413954878	0.2335483
0.2301470795	0.2296735493	0.2304372700	0.2167969565	0.2139707
0.2077357911	0.2112143019	0.1932343201	0.1962061319	0.1996887

0.1709547136	0.1728132223	0.1638733521	0.1702899529	0.1699290
0.1731007940	0.1629542036	0.1749057192	0.1663529081	0.1861052
0.1828066384	0.1753525095	0.1683888296	0.1769892013	0.1580397
0.1484156221	0.1420327365	0.1184027201	0.0934460446	0.0849067
0.0692576674	0.0613992685	0.0480897117	0.0522103932	0.0463499
0.0539839485	0.0625598609	0.0640548713	0.0683848283	0.0757048
0.0910948012	0.0838228730	0.0884307538	0.0661592654	0.0547972
0.0181452618	0.0149155917	0.0204487600	0.0230857120	0.02249741
0.0246634663	0.0173456560	0.0156228085	0.0261036265	0.0295730
0.0360168543	0.0464889622	0.0510102841	0.0461873099	0.0429017
0.0298102040	0.0531683222	0.0376489120	0.0272098545	0.0303259
0.0307789824	0.0402840903	0.0262257576	0.0270293495	0.02464547
0.0030740068	0.0108998041	0.0078638602	0.0101553737	0.0057857308
0.0469987139	0.0697168950	0.0731040190	0.0694290765	0.05161320
0.0305974998	0.0379180726	0.0476399467	0.0550805292	0.0737354
0.0775582767	0.0829055320	0.0832264072	0.0849127688	0.0757630
0.0826898006	0.0719725288	0.0833468398	0.0814123381	0.079642
0.0746268177	0.0758236568	0.0741629134	0.0748052402	0.0827420
0.0922613635	0.0931614865	0.0768777558	0.1061414317	0.1264100
0.1599190610	0.1674802879	0.1841423042	0.1916843579	0.1837759
0.1652112972	0.1716562532	0.1676480517	0.1656158931	0.1771410
0.1714209323	0.1481992534	0.1674988857	0.1550351442	0.1510368
0.1579064318	0.1566169070	0.1319958286	0.1530705399	0.1453019
0.1371752583	0.1409341598	0.1485888996	0.1491567149	0.1548570

0.1718382911	0.1715551120	0.1760759145	0.1831726548	0.17932
0.1856515779	0.1956161162	0.2157396978	0.2145439257	0.213993
0.2174007279	0.2107574642	0.2057179299	0.2123046779	0.214828
0.2069693522	0.2078981003	0.2099162714	0.1820323344	0.186430
0.1940996478	0.1758804535	0.1862417500	0.1976010264	0.18892
0.1924708652	0.1830308509	0.1705211380	0.1754171306	0.188513
0.1827123562	0.1784770027	0.1925988827	0.1978268743	0.210475
0.1885585830	0.1872482381	0.2017837204	0.2069794423	0.195851
0.1855649021	0.1821347083	0.1781615840	0.1700764047	0.159370
0.1795530318	0.1816768590	0.1711773139	0.1807984474	0.18554
0.2167076001	0.2228989062	0.2283453180	0.2394263609	0.230362
0.2498375791	0.2553573325	0.2720828583	0.2709400656	0.256192
0.2440850198	0.2490054567	0.2450307033	0.2204620057	0.19572
0.1845105538	0.1743128690	0.1710812111	0.1693366319	0.176700
0.1633797431	0.1748730886	0.1709903303	0.1632186959	0.170339
0.1769495519	0.1841738389	0.1902296924	0.2022153912	0.20380
0.2091339806	0.2261804306	0.2249606492	0.2188641093	0.224026
0.2315250871	0.2449764350	0.2323035489	0.2454674032	0.252225
0.2165457549	0.2228369813	0.2224832798	0.2179120115	0.212916
0.1993879442	0.1883700364	0.1777227860	0.1726111254	0.162212
0.1465561960	0.1463062624	0.1382829541	0.1358862848	0.14698
0.1254659702	0.1168513489	0.1172603519	0.1234381960	0.116013
0.1292452039	0.1269288827	0.1260360644	0.1203073484	0.123619
0.1048212007	0.0971715583	0.0931323043	0.1054247089	0.104992

0.1074504143 0.1151336060 0.1014505320 0.0923422479 0.088430

0.0806309170 0.0851097158 0.0839506452 0.0877530428 0.095175

0.1016365131 0.1181594900 0.1193350482 0.1105011117 0.105755

0.1074977816 0.0988990166 0.0907247447 0.1130669441 0.107907

0.1018583857 0.0871801922 0.0888189727 0.1010185116 0.10239

0.1117609811 0.0816958455 0.1039275727 0.1133676472 0.116979

0.0935814644 0.0954039208 0.0786568393 0.0962433347 0.101928

0.1094808233 0.1139403951 0.1260273113 0.1112710571 0.136514

0.1344508055 0.1307916860 0.1244782595 0.1091750518 0.0981331

0.1064065796 0.0950141026 0.0970909641 0.1086629178 0.114168

0.1271163978 0.1331964416 0.1197703725 0.1276017033 0.1147877

0.1216361919 0.1289222201 0.1204246963 0.1361553634 0.12665

0.1493296162 0.1488756274 0.1711088174 0.1771040443 0.179804

0.1822103131 0.1907565028 0.1988499791 0.2009944965 0.192591

0.1833105352 0.1916461489 0.1980175924 0.2008970336 0.193886

0.1991537982 0.2081584359 0.2176678837 0.2077246819 0.210836

0.2198656112 0.2236767719 0.2214809841 0.2215694184 0.21270

0.2091048083 0.1944271278 0.2138822837 0.2275645540 0.215541

0.2333129181 0.2270958788 0.2149204733 0.2160004229 0.198904

0.1805139637 0.1755832097 0.1606726932 0.1587660886 0.150968

0.1495553163 0.1461946162 0.1574622160 0.1757666645 0.174178

0.1681728379 0.1694907740 0.1674834296 0.1646875110 0.16997

0.1565372800 0.1651708947 0.1690300674 0.1778451455 0.186526

0.1925495196 0.2100410447 0.2082924209 0.2131536289 0.2082959

0.1891410991	0.1965853856	0.1933415015	0.2028195877	0.19850
0.2153571463	0.2007182727	0.1964787474	0.1931615196	0.199916
0.2143368985	0.2016462659	0.2052628259	0.2021769831	0.202615
0.2042156298	0.1928711127	0.1843209592	0.1811328792	0.166065
0.1885516571	0.1780635108	0.1752565014	0.1640863625	0.16610
0.1620733202	0.1556101730	0.1541181303	0.1488270740	0.129642
0.1302782664	0.1455032734	0.1548238699	0.1716847837	0.17438
0.1748939968	0.1736964253	0.1747017180	0.1850563722	0.19008
0.1819637388	0.1797679684	0.1944540977	0.1833887325	0.185530
0.1730955837	0.1560337741	0.1633482812	0.1624102698	0.158180
0.1562784900	0.1415123489	0.1520828548	0.1639077102	0.180391
0.1983308952	0.2043685592	0.2245255491	0.2385702481	0.216639
0.2185716704	0.2164685844	0.2169120098	0.2239964677	0.225397
0.2255995417	0.2424658161	0.2522251407	0.2651120528	0.237733
0.2221983313	0.2160367911	0.2058268205	0.2039279822	0.202545
0.1875632904	0.1998108421	0.1896254677	0.1927290756	0.170840
0.1750103062	0.1898852756	0.2030410918	0.1936500835	0.185043
0.1725354339	0.1739734276	0.1714673626	0.1765525809	0.164121
0.1746991320	0.1634518394	0.1548368693	0.1517978303	0.137827
0.1011943388	0.1033911229	0.1087287954	0.0958763517	0.095963
0.1202137850	0.1305026560	0.1302728767	0.1180603364	0.139497
0.1480312333	0.1425742343	0.1506980766	0.1563558813	0.153444
0.1559649024	0.1462594822	0.1453787427	0.1411821828	0.147622
0.1379770156	0.1294056322	0.1459377146	0.1404571677	0.150565

0.1370842330	0.1422992524	0.1410387284	0.1366666139	0.141449
0.1464825036	0.1580005713	0.1436908385	0.1496607082	0.158333
0.1562785912	0.1633757321	0.1694335582	0.1832604837	0.1685620
0.1880116179	0.1937211139	0.2110781558	0.2262701313	0.2331447
0.2357603925	0.2354418418	0.2373695787	0.2475448997	0.24355298
0.2582627472	0.2663428245	0.2744287946	0.2833579895	0.27859188
0.2758654959	0.2744271517	0.2921122219	0.2914396248	0.30609819
0.2981317332	0.2940425170	0.3014053850	0.3018086830	0.27947800
0.2727622782	0.2637659333	0.2510254277	0.2610124973	0.2562899
0.2718238206	0.2830957440	0.2984215641	0.2838547595	0.27997665
0.2755239038	0.2794366526	0.2867623451	0.2877026485	0.2945127
0.3005728096	0.3044986483	0.2986632118	0.2986966059	0.28751410
0.3024201031	0.3018844921	0.3022131227	0.3064246030	0.31500711
0.3243995232	0.3277808373	0.3342042229	0.3324920296	0.32541683
0.3508292951	0.3600301270	0.3728893650	0.3770759790	0.37127078
0.3774359241	0.3989415416	0.4048609540	0.3804866306	0.37816888
0.3735607717	0.3570103293	0.3468003386	0.3390444327	0.33848252
0.3270183916	0.3313963673	0.3276844140	0.3222064952	0.31531525
0.2887355839	0.2795407896	0.2970876190	0.2896128210	0.29190593
0.2999153295	0.3049916061	0.2965436573	0.2774709041	0.29542797
0.2863246368	0.2891511626	0.3016603717	0.2992658394	0.30832081
0.2953546924	0.2776905282	0.2547850782	0.2594347371	0.25548905
0.2524080716	0.2614346262	0.2662965400	0.2765580026	0.2874950
0.3149936462	0.3144866349	0.3259868280	0.2983149980	0.29688387

0.2716087167	0.2816483331	0.2784030526	0.2671700637	0.2586451
0.2669970568	0.2623861545	0.2711947762	0.2615790518	0.2584878
0.2591672711	0.2618248319	0.2620822025	0.2575502225	0.2578813
0.2698444080	0.2765287472	0.2795123006	0.2925669530	0.2868623
0.2928795639	0.2956223108	0.3000220361	0.2950250155	0.2983100
0.3123967853	0.3287755417	0.3064822852	0.3027870405	0.29446013
0.3088774889	0.2949065201	0.2803629197	0.2821776447	0.2917654
0.2871128784	0.2865566793	0.2798312884	0.2906165472	0.266378
0.2574403425	0.2490098465	0.2359949144	0.2322608140	0.246925
0.2482426641	0.2519440395	0.2564033923	0.2515019578	0.248338
0.2301249254	0.2336584455	0.2350076170	0.2224231470	0.19285604
0.1936245390	0.2008796191	0.1883758728	0.1823176746	0.17917182
0.1730348174	0.1594746393	0.1786150391	0.1717941607	0.16213515
0.1720487855	0.1676816846	0.1638762594	0.1626007993	0.16168985
0.1656703751	0.1783658133	0.1856307569	0.1907011436	0.18983344
0.1746540617	0.1824588271	0.1757671868	0.1944090726	0.19792899
0.1906769364	0.1934667872	0.1977803258	0.2051035534	0.19033078
0.1858641181	0.1861067390	0.2068485627	0.2150375032	0.20188682
0.2276402164	0.2219868825	0.2085469177	0.2238070963	0.234479
0.2334750867	0.2523833437	0.2564050048	0.2663605465	0.26126341
0.2779796639	0.2837973266	0.2858533946	0.2887146080	0.29530267
0.2812236650	0.2761642262	0.2753605067	0.2522062319	0.24137894
0.2391630170	0.2366541235	0.2439076858	0.2412511746	0.22378786
0.2066937872	0.2164448960	0.2158164999	0.2153969588	0.19664278

0.1936635080 0.2131465674 0.2077401883 0.2030847211 0.20567878

0.1990373252 0.1891035938 0.1752872005 0.1729831360 0.180080

0.1643143571 0.1613498195 0.1618477315 0.1488380098 0.14236393

0.1639586811 0.1591931081 0.1687603593 0.1700528947 0.19055547

0.1749927862 0.1711815636 0.1736525253 0.1962884658 0.19309505

0.1848637270 0.1949804663 0.2137765612 0.2112156214 0.20280281

0.1837870948 0.1764588071 0.1745736298 0.1605625048 0.15777896

0.1543546410 0.1576097080 0.1621462319 0.1660609117 0.17000602

0.1578583775 0.1627704491 0.1531950185 0.1672538781 0.16562040

0.1432951028 0.1381400024 0.1407742521 0.1447537592 0.14632279

0.1424023674 0.1351128076 0.1342896521 0.1292603770 0.131538

0.1530426982 0.1345342669 0.1446321810 0.1358213590 0.13212818

0.1559504442 0.1625641490 0.1693989771 0.1698108584 0.1688284

0.1457208768 0.1417700963 0.1420141750 0.1459792207 0.14138298

0.1433890201 0.1557325521 0.1560066868 0.1591060608 0.166315

0.1839968462 0.1751938058 0.1776092017 0.1730040773 0.1586061

0.1500405602 0.1523000527 0.1511879542 0.1533690023 0.152401

0.1572564512 0.1488756292 0.1305131028 0.1320735277 0.118358

0.1291637549 0.1412019041 0.1422466475 0.1366729639 0.1348759

0.1029588961 0.1147169614 0.1159955245 0.1189320398 0.104837

0.1005215714 0.1039500687 0.1121026410 0.1270032632 0.1328059

0.1419526333 0.1296775059 0.1379001472 0.1449927210 0.136398

0.1176623274 0.1173374701 0.1223536341 0.1160954940 0.127920

0.1166660617 0.1159272872 0.1120984326 0.1148305731 0.107735

0.1020181618 0.1110475447 0.1109638040 0.0977804299 0.100471

0.0990845724 0.0890163048 0.0921078999 0.1010555682 0.0952416

0.1028419982 0.1113214439 0.1173043755 0.1085405397 0.1275630

0.1383300674 0.1444739925 0.1528544918 0.1501926329 0.151959

0.1601091771 0.1506365160 0.1689121729 0.1729623115 0.1752264

0.1622950042 0.1516398627 0.1464583184 0.1520056108 0.139660

0.1678583635 0.1622270107 0.1631263258 0.1703680063 0.197324

0.2029638760 0.2041733724 0.2188579580 0.2319425200 0.222338

0.2201187175 0.2176098681 0.2188087839 0.2316678196 0.217834

0.2033655518 0.1983815670 0.1845198053 0.1736877588 0.176961

0.1625758725 0.1491391664 0.1377495521 0.1430305513 0.144692

0.1503300528 0.1716602543 0.1898452383 0.1680453996 0.165892

0.1464389058 0.1376821016 0.1315389848 0.1248456485 0.129507

0.1352104138 0.1483849376 0.1514638375 0.1484885910 0.130019

0.1248235379 0.1234872555 0.1323329135 0.1450426770 0.147201

0.1265454204 0.1260913671 0.1303988581 0.1108939001 0.099103

0.1063219894 0.1074073956 0.0922843544 0.0906538199 0.1012796

0.1199008935 0.1036476779 0.1086912816 0.1005290847 0.101955

0.1040452494 0.1106826017 0.1135273858 0.1146772648 0.097077

0.0816788236 0.0788188398 0.0697661828 0.0657273457 0.063941

0.0513023121 0.0479555816 0.0657081457 0.0559040116 0.064970

0.0791707711 0.0802747037 0.0714082204 0.0694062201 0.049570

0.0568074734 0.0606863989 0.0417134599 0.0451888154 0.053102

0.0426577538 0.0640849246 0.0673819657 0.0759301707 0.070524

0.0473913742	0.0437843796	0.0558907943	0.0531144009	0.0515091
0.0641840558	0.0659975789	0.0585497910	0.0675005789	0.0625395
0.0609119137	0.0391549681	0.0469626221	0.0360990499	0.0271397
0.0631217246	0.0605731744	0.0462592403	0.0575712807	0.0478281
0.0483072183	0.0386676779	0.0265091277	0.0345539237	0.0498374
0.0624354186	0.0506188389	0.0520775088	0.0557624549	0.046987
0.0425435997	0.0412127192	0.0410205632	0.0472435757	0.0527941
0.0504172920	0.0590741498	0.0523500457	0.0556804684	0.0707096
0.1020750504	0.1066710701	0.1066137884	0.1221908060	0.0981483
0.1061684878	0.1015243721	0.1027644515	0.0880917898	0.0786701
0.0704134751	0.0718836884	0.0630239359	0.0453360663	0.0522034
0.0654387105	0.0740714179	0.0817293892	0.0820259039	0.0836760
0.0996464134	0.1167807682	0.1317721839	0.1265163330	0.145352
0.1237491986	0.1224036912	0.1232061007	0.1214614234	0.130778
0.1288638755	0.1363413799	0.1292851927	0.1201292565	0.122562
0.1368632260	0.1235766571	0.1129769446	0.1028461529	0.102049
0.0894115148	0.1002818641	0.1067409151	0.0990301830	0.0866179
0.1038323217	0.0962762017	0.0985142786	0.1033218694	0.117672
0.1123998584	0.1079565837	0.1173612506	0.1191090711	0.125867
0.1240964224	0.1313582606	0.1395543466	0.1414902963	0.136975
0.1278664629	0.1227015653	0.1238250347	0.1269586330	0.124719
0.1471680419	0.1320887616	0.1168422937	0.1104576757	0.1005153
0.1201164997	0.1284940464	0.1273569311	0.1160337067	0.112523
0.1479594593	0.1486957391	0.1574341016	0.1651723500	0.173448

0.1826408649 0.1925047537 0.1841956395 0.1801579812 0.1850780

0.2008021802 0.1811033454 0.1915818595 0.2064153605 0.1969048

0.1813723675 0.1642914540 0.1655705935 0.1723632938 0.1722408

0.1517772987 0.1382988204 0.1290706406 0.1238065545 0.135380444

0.1173770141 0.1181985056 0.1256000911 0.1323494406 0.1146479

0.1124029818 0.1065086633 0.0866061231 0.0848491277 0.096877

0.0909938554 0.0894241233 0.1060816474 0.1193206802 0.11468429

0.1342952255 0.1242070999 0.1307336909 0.1205738733 0.105094

0.1133940317 0.1178338662 0.1305231174 0.1209586953