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Research Article

Some New Variants of Hermite–Hadamard and Fejér-Type Inequalities for Godunova–Levin Preinvex Class of Interval-Valued Functions

Zareen A. Khan,[1] **Waqar Afzal**,[2,3] **Waqas Nazeer**,[2] **and Joshua Kiddy K. Asamoah**[4,5]

[1]*Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[2]*Department of Mathematics, Government College University Lahore (GCUL), Lahore 54000, Pakistan*
[3]*Department of Mathematics, University of Gujrat, Gujrat 50700, Pakistan*
[4]*Department of Mathematics, Saveetha School of Engineering SIMATS, Chennai, India*
[5]*Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana*

Correspondence should be addressed to Joshua Kiddy K. Asamoah; jkkasamoah@knust.edu.gh

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The theory of inequalities is greatly influenced by interval-valued concepts, and this contribution is explored from several perspectives and domains. The aim of this note is to develop several mathematical inequalities such as Hermite–Hadamard, Fejér, and the product version based on center radius ($\mathfrak{CR}$)-order relations. Furthermore, we develop several nontrivial examples and remarks to support the main findings.

Keywords: CR-order relation; Fejér-type inequality; Godunova-convexity; Hermite–Hadamard inequality; interval based functions

MSC2010 Classification: 05A30, 26D10, 26D15

1. Introduction

An interval enclosure can be illustrated by the Archimedes method, an old and famous example. In history, interval enclosures have been used to calculate circle circumferences. Many mathematical, logical, and computerized models of real-world problems deal with uncertainty in intervals. Interval variables are now utilized in computation outcomes rather than point variables in order to eliminate this uncertainty. This method was designed primarily to consider errors associated with finite-state machine numerical computations. The first concept related to intervals was developed by Moore [1] for handling uncertainly in finite automatic machines. Numerous scientists started investigating the hypothesis and use of arithmetic after the publication of this book.

As a matter of fact, analysis related to intervals plays a vital role in both pure and computing sciences. As one way to deal with interval uncertainty in structural systems, since the 1950s, engineering models relating to mathematics have used interval analysis. There are several real-life phenomena with uncertain outcomes that can be addressed by interval mathematics. It is possible to resolve load uncertainty in electrical systems by enacting an interval power flow model and solving Krawczyk's method for the resulting interval nonlinear system, see Reference [2]. Furthermore, the stochastic differential interval model is applicable to open birth intervals regardless of parity, see Reference [3]. Regarding interval uncertainty, some recent developments have occurred, see References [4–8].

A mapping $\mathfrak{T}\colon \Theta \subseteq R \longrightarrow R$ is called convex if one has

$$\mathfrak{T}(\mathfrak{s}\xi + (1-\mathfrak{s})\varepsilon) \leq \mathfrak{s}\mathfrak{T}(\xi) + (1-\mathfrak{s})\mathfrak{T}(\varepsilon), \tag{1}$$

which holds for all $\xi, \varepsilon \in \Theta$ and $\mathfrak{s} \in [0,1]$.

In the meantime, the generalized convexity of maps has shown to be an effective tool in mathematics and the analysis of practical and nonlinear issues. There has been extensive research into generalizations and refinements of convex mappings in recent years. Research on integral inequalities, both analytically and numerically, is interesting.

In the theory of inequality, convexity is a well-known concept. In addition to the study of integral equations, various extended convex mappings have been found to be valuable. Many areas of science have been significantly impacted by them, such as symmetry analysis, decision-making, operations research, electrical networks, finance, and numerical analysis. It is investigated how to promote the arbitrary characteristics of convexity through the use of a number of fundamental integral inequalities.

Let $\mathfrak{T}\colon \Theta \subseteq R \longrightarrow R$ be a convex function with $\xi < \varepsilon$ and $\xi, \varepsilon \in \Theta$. Then, the double inequality is defined as follows [9]:

$$\mathfrak{T}\left(\frac{\xi+\varepsilon}{2}\right) \leq \frac{1}{\varepsilon-\xi}\int_{\xi}^{\varepsilon}\mathfrak{T}(x)\,dx \leq \frac{\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)}{2}. \tag{2}$$

Many mathematicians recently expanded and generalized the well-known $\mathscr{H}.\mathscr{H}$ inequality based on some intriguing new definitions of a convex function. On the other hand, some scholars have contributed in a variety of ways to the development of characteristics and inequality pertaining to generalized classes of both convex and nonconvex functions [10–12]. Integral inequalities and interval-valued functions ($\mathscr{IVFS}$) have been linked in recent research, yielding a number of insightful findings. Costa, Roman-Flores, and Chalco-Cano [13] developed the Opial form of disparities through the use of generalized Hukuhara derivatives, Chalco-Cano, Lodwick, and Condori-Equice [14] investigated inequalities of the Ostrowski kind, and Roman-Flores, Chalco-Cano, and Lodwick [15] established the Minkowski inequalities. The authors in [16] defined the term of h-convex $\mathscr{IVFS}$ in 2018 as well as created a generalization of $\mathscr{H}.\mathscr{H}$ and Jensen inequalities. An et al. [17] using the definition of (h_1, h_2)-convex $\mathscr{IVFS}$ developed these inequalities. In [18, 19], the authors introduced the definitions of h and (h_1, h_2)-Godunova–Levin ($\mathscr{GL}$) and extend the results of An et al. in more refined form. Preinvex functions for $\mathscr{IVFS}$ were recently extended to coordinated preinvex mappings by Lai et al. [20]. Saeed et al. [21, 22] utilized the idea of total order relations and developed various new integral bounds of different inequalities. Cortez et al. [23] introduced a new class of generalized convex mappings and developed several new inequalities with applications using the notion of center-to-radius relation. Ahmadini et al. [24] employed the idea of (h_1, h_2)-($\mathscr{GL}$) convex and preinvex function via standard and partial interval order relation to build numerous novel inequalities along with applications to special functions, means, and random variables with a few interesting open problems. For some recent developments related to proposed inequalities using the idea of set-valued maps, see References [25–32].

The terms "order relations" and "convex functions" are crucial to understanding how to modify inequalities inside interval mappings. The order relations like "$\subseteq$" have been employed in recent studies, yet they do not unify results for real value maps. By means of example 3, the authors in reference [33] demonstrated that when interval mappings are warped, the famous Milne inequality is not settled. The authors addressed that problem by introducing a new order relation known as the total order relation, of which all previously proposed order relations in the context of inequalities are special cases of that. The novelty of this work lies in the fact that this order connection has never before been connected to the ($\mathscr{GL}$)-preinvex class of convexity. As a follow-up, we will deal with the new order relation associated with integral inequalities called center order ($\mathfrak{CR}$)-order relation, which was introduced by Bhunia [34]. Following Hanson's [35] studies, Mond and Ben-Israel examined invex and preinvex sets with some interesting properties, see Reference [36]. The concept of distinct classes of preinvex functions with intriguing properties was employed by the authors in [37] to create a number of novel bounds. As opposed to simple convexity, ($\mathscr{GL}$) class of convexity covers a broader range of concepts although we show that this generalize various other classes of convexity as well. For some other properties related to this class of convexity that are not described here, see Reference [38].

Based on the literature discussed above regarding interval analysis in Section 2 and specific articles using ($\mathfrak{CR}$) order for different convexity classes, see References [39–42]. We introduce a novel notion of h-($\mathscr{GL}$) type mappings in preinvex sense and develop some novel version of double inequality with different new forms. The structure of the article is as follows. Following a review of some fundamental interval calculus ideas and some helpful findings required to proceed under Section 2. Some properties and definitions used throughout the article are discussed in Section 3. We develop some new integral bounds of the double inequality with its newly varied variations in Section 4. We present a conclusion addressing the findings and some potential future work in Section 5.

2. Preliminaries

The concept of $\mathfrak{CR}$-order relation for preinvex and ($\mathscr{GL}$)-preinvex $\mathscr{IVFS}$, are extensively studied in this section.

Definition 1 (see Definition 5 [43]). Let $\xi \in \Theta \subset R^n$, then Θ is known as invex at ξ with reference to $\zeta\colon \Theta \times \Theta \longrightarrow R^n$ if

$$\xi + \mathfrak{s}\zeta(\varepsilon, \xi) \in \Theta, \tag{3}$$

for all $\varepsilon \in \Theta$ and $\mathfrak{s} \in [0,1]$.

Definition 2 (see Definition 6 [43]). Consider $\Theta \neq \emptyset \in R$ is an invex with reference to $\zeta\colon \Theta \times \Theta \longrightarrow R$, then $\mathfrak{T}\colon \Theta \longrightarrow R$ is known as preinvex with reference to ζ if

$$\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \leq \mathfrak{s}\mathfrak{T}(\varepsilon) + (1-\mathfrak{s})\mathfrak{T}(\xi), \tag{4}$$

for all $\xi, \varepsilon \in \Theta$ and $\mathfrak{s} \in [0,1]$.

Definition 3 (see Definition 2.3 [10]). Let $\Theta \neq \emptyset \in R$ is invex set with reference to $\zeta\colon \Theta \times \Theta \longrightarrow R$, then $\mathfrak{T}\colon \Theta \longrightarrow R$ is known as h-preinvex with reference to ζ if

$$\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \leq h(\mathfrak{s})\mathfrak{T}(\varepsilon) + h(1-\mathfrak{s})\mathfrak{T}(\xi), \tag{5}$$

for all $\xi, \varepsilon \in \Theta$ and $\mathfrak{s} \in [0,1]$.

Definition 4 (see Definition 6 [38]). Let $\Theta \neq \emptyset \in \mathfrak{R}$ is invex set with reference to $\zeta\colon \Theta \times \Theta \longrightarrow R$, then $\mathfrak{T}\colon \Theta \longrightarrow R$ is known as h-$\mathscr{GL}$ preinvex with reference to ζ if

$$\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \leq \frac{\mathfrak{T}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}(\xi)}{h(1-\mathfrak{s})}, \tag{6}$$

for all $\xi, \varepsilon \in \Theta$ and $\mathfrak{s} \in (0,1)$.

Condition 1 (see [43]) Let $\Theta \subset R^n$ is invex subset with reference to $\zeta\colon \Theta \times \Theta \longrightarrow R$. For all $\xi, \varepsilon \in \Theta$ and $\mathfrak{s} \in [0,1]$,

$$\zeta(\varepsilon, \varepsilon + \mathfrak{s}\zeta(\xi,\varepsilon)) = -\mathfrak{s}\zeta(\xi,\varepsilon), \tag{7}$$

and

$$\zeta(\xi, \varepsilon + \mathfrak{s}\zeta(\xi,\varepsilon)) = (1-\mathfrak{s})\zeta(\xi,\varepsilon). \tag{8}$$

For any $\xi, \varepsilon \in \Theta$ and $\mathfrak{s}_1, \mathfrak{s}_2 \in [0,1]$, we find from Condition 1 that

$$\zeta(\varepsilon + \mathfrak{s}_2\zeta(\xi,\varepsilon), \varepsilon + \mathfrak{s}_1\zeta(\xi,\varepsilon)) = (\mathfrak{s}_2 - \mathfrak{s}_1)\zeta(\xi,\varepsilon). \tag{9}$$

2.1. Basic Properties of $\mathscr{IVFS}$. Here, we will review several fundamental and essential interval analysis principles.

$$\begin{aligned}
&[\Delta] = [\underline{\Delta}, \overline{\Delta}] \quad (x \in \mathrm{R}, \underline{\Delta} \leq x \leq \overline{\Delta}. x \in \mathrm{R}),\\
&[\delta] = [\underline{\delta}, \overline{\delta}] \quad (x \in \mathrm{R}, \underline{\delta} \leq x \leq \overline{\delta}. x \in \mathrm{R}),\\
&[\Delta] + [\delta] = [\underline{\Delta}, \overline{\Delta}] + [\underline{\delta}, \overline{\delta}] = [\underline{\Delta} + \underline{\delta}, \overline{\Delta} + \overline{\delta}],\\
&\eta\Delta = \eta[\underline{\Delta}, \overline{\Delta}] = \begin{cases} [\eta\underline{\Delta}, \eta\overline{\Delta}], & \text{if } \eta > 0;\\ \{0\}, & \text{if } \eta = 0;\\ [\eta\overline{\Delta}, \eta\underline{\Delta}], & \text{if } \eta < 0, \end{cases}
\end{aligned} \tag{10}$$

where $\eta \in \mathrm{R}$.

Assume that the wraps of all positive intervals in R are R_I and R_I^+, respectively. Next, we will discuss certain relational features of the order relation that are used in this note.

Consider $\Delta = [\underline{\Delta}, \overline{\Delta}] \in R_I$, then $\Delta_c = (\overline{\Delta} + \underline{\Delta}/2)$ and $\Delta_r = (\overline{\Delta} + \underline{\Delta}/2)$ are the calculation of $\mathfrak{CR}$ order. The following expression for Δ can be made using $\mathfrak{CR}$:

$$\Delta = \langle \Delta_c, \Delta_r \rangle = \left\langle \frac{\overline{\Delta} + \underline{\Delta}}{2}, \frac{\overline{\Delta} - \underline{\Delta}}{2} \right\rangle. \tag{11}$$

Definition 5 (see Definition 1 [21]). The $(\mathfrak{CR})$-order relation for $\Delta = [\underline{\Delta}, \overline{\Delta}] = \langle \Delta_c, \Delta_r \rangle$ *and* $\delta = [\underline{\delta}, \overline{\delta}] = \langle \delta_c, \delta_r \rangle \in R_I$ is represented as follows (see Figure 1):

$$\Delta \preccurlyeq_{cr} \delta \Longleftrightarrow \begin{cases} \Delta_c < \delta_c, & \text{if } \Delta_c \neq \delta_c,\\ \Delta_r \leq \delta_r, & \text{if } \Delta_c = \delta_c. \end{cases} \tag{12}$$

For these intervals $\Delta, \delta \in \mathrm{R}_I$, this holds $\Delta \preccurlyeq_{cr} \delta$ or $\delta \preccurlyeq_{cr} \Delta$.

Theorem 1 (see Theorem 4 [21]). *Let $\mathfrak{T}, \mathfrak{L}\colon [\xi, \varepsilon]$ be $\mathscr{IVFS}$ given by $\mathfrak{T} = [\underline{\mathfrak{T}}, \overline{\mathfrak{T}}]$ and $\mathfrak{L} = [\underline{\mathfrak{L}}, \overline{\mathfrak{L}}]$. If $\mathfrak{T}(\wp) \preccurlyeq_{cr} \mathfrak{L}(\wp)$ for all $u \in [\xi, \varepsilon]$, then (see Figure 2)*

$$\int_\xi^\varepsilon \mathfrak{T}(\wp)d\wp \preccurlyeq_{\mathfrak{CR}} \int_\xi^\varepsilon \mathfrak{L}(\wp)d\wp. \tag{13}$$

The following example confirms the truth of the preceding theorem.

Example 1. Let $\mathfrak{T} = [\wp, 2\wp]$ and $\mathfrak{L} = [\wp^2, \wp^2 + 2]$. Then, for $\wp \in [0,1]$, $\mathfrak{T}_{\mathscr{C}} = 3\wp/2$, $\mathfrak{T}_{\mathscr{R}} = \wp/2$, $\mathfrak{L}_{\mathscr{C}} = \wp^2 + 1$ and $\mathfrak{L}_{\mathscr{R}} = 1$. As a result, by employing Definition 5, we have $\mathfrak{T}(\wp) \preccurlyeq_{\mathfrak{CR}} \mathfrak{L}(\wp)$ for $\wp \in [0,1]$. Since

$$\begin{aligned}
&\int_0^1 [\wp, 2\wp]d\wp = \left[\frac{1}{2}, 1\right],\\
&\int_0^1 [\wp^2, \wp^2 + 2]d\wp = \left[\frac{1}{3}, \frac{7}{3}\right].
\end{aligned} \tag{14}$$

From Theorem 1, one has

$$\int_0^1 \mathfrak{T}(\wp)d\wp \preccurlyeq_{\mathfrak{CR}} \mathfrak{L}(\wp)d\wp. \tag{15}$$

3. Some Novel Definitions and Special Cases for $\mathfrak{CR}$–h-$(\mathscr{GL})$-Preinvex Functions

In regard to $(\mathfrak{CR})$-order relations, the goal of this section is to define a new notion of $\mathfrak{CR}$–h–$(\mathscr{GL})$-preinvex functions and to show that it is a generalized class that unifies numerous distinct types of convexity.

Definition 6. Let $\mathfrak{T}\colon [\xi, \varepsilon]$ be interval-valued function given by $\mathfrak{T} = [\underline{\mathfrak{T}}, \overline{\mathfrak{T}}]$ and $h\colon (0,1) \longrightarrow R^+$ such that $h \neq 0$. Then, $\mathfrak{T}$ defined on invex set Θ known as $\mathfrak{CR} - h$-$\mathscr{GL}$-preinvex with reference to ζ if

$$\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \preccurlyeq_{\mathfrak{CR}} \frac{\mathfrak{T}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}(\xi)}{h(1-\mathfrak{s})}, \tag{16}$$

for all $\xi, \varepsilon \in \Theta$ and $\mathfrak{s} \in (0,1)$.

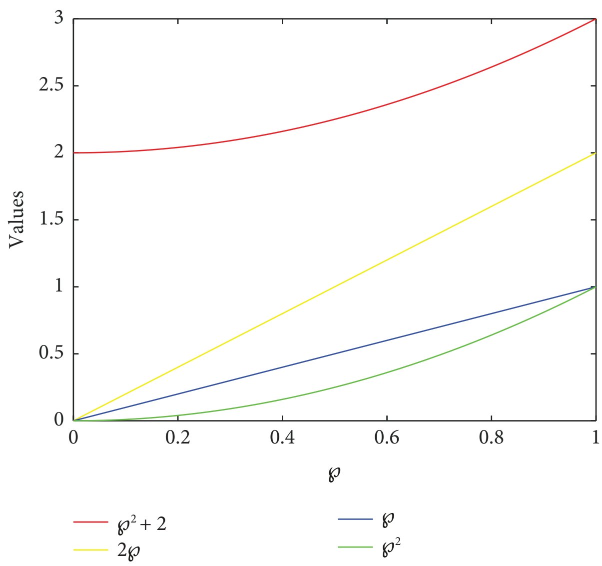

FIGURE 1: The viability of $\mathfrak{CR}$-order relations.

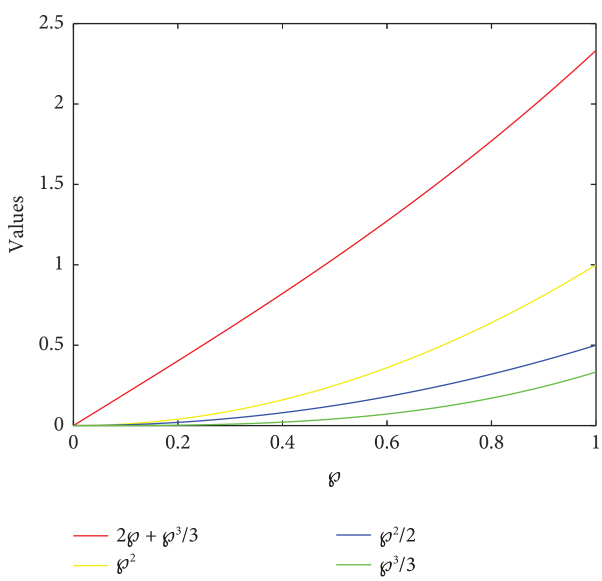

FIGURE 2: The truth of Theorem 1.

Example 2. Let $\mathfrak{T}(\xi) = -|\xi|$ under the assumption $\underline{\mathfrak{T}} = \overline{\mathfrak{T}}$ with

$$\zeta(\varepsilon,\xi) = \begin{cases} \varepsilon - \xi, & \text{if } \varepsilon\xi < 0, \\ \xi - \varepsilon, & \text{if } \varepsilon\xi > 0. \end{cases} \tag{17}$$

If function $h(\mathfrak{s}) = \mathfrak{s}^s, \mathfrak{s} \in (0,1)$, where $s \geq 1$, then by Definition 4, $\mathfrak{T}$ is a h-$\mathcal{GL}$-preinvex function with respect to $\zeta(\cdot,\cdot)$ on $\mathrm{R}\backslash\{0\}$. However, if $h(\mathfrak{s}) = \mathfrak{s}^s, \mathfrak{s} \in (0,1)$, where $s < 1$, by letting $\varepsilon = 1, \xi = 1, \mathfrak{s} = 1/2, s = 1/2$, we have

$$\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) = \mathfrak{T}(1) = -1 > -\sqrt{2} = \frac{\mathfrak{T}(\varepsilon)}{\mathfrak{s}^s} + \frac{\mathfrak{T}(\xi)}{(1-\mathfrak{s})^s}, \tag{18}$$

which shows that $\mathfrak{T}$ is not h-$\mathcal{GL}$-preinvex with respect to the same $\zeta(\cdot,\cdot)$.

Remark 1

i. If we set $\zeta(\varepsilon,\xi) = \varepsilon - \xi$, then Definition 6 becomes a $\mathfrak{CR}$-h-$\mathcal{GL}$ function [44].

$$\mathfrak{T}(\mathfrak{s}\varepsilon + (1-\mathfrak{s})\xi) \preccurlyeq_{\mathfrak{CR}} \frac{\mathfrak{T}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}(\xi)}{h(1-\mathfrak{s})}. \tag{19}$$

ii. If we set $\zeta(\varepsilon,\xi) = \varepsilon - \xi$ and $h(\mathfrak{s}) = 1/\mathfrak{s}$, then Definition 6 becomes a $\mathfrak{CR}$-convex function [45].

$$\mathfrak{T}(\mathfrak{s}\varepsilon + (1-\mathfrak{s})\xi) \preccurlyeq_{\mathfrak{CR}} \mathfrak{s}\mathfrak{T}(\varepsilon) + (1-\mathfrak{s})\mathfrak{T}(\xi). \tag{20}$$

iii. If we set $\underline{\mathfrak{T}} = \overline{\mathfrak{T}}$, then Definition 6 becomes a h-$\mathcal{GL}$-preinvex function [38].

$$\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \leq \frac{\mathfrak{T}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}(\xi)}{h(1-\mathfrak{s})}. \tag{21}$$

iv. If we set $\underline{\mathfrak{T}} = \overline{\mathfrak{T}}$ and $h(\mathfrak{s}) = \mathfrak{s}$, then Definition 6 reduces to $\mathcal{GL}$-preinvex function [10].

$$\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \leq \frac{\mathfrak{T}(\varepsilon)}{\mathfrak{s}} + \frac{\mathfrak{T}(\xi)}{(1-\mathfrak{s})}. \tag{22}$$

Based on the definition and comments above, it is clear that the novel idea $\mathfrak{CR} - h$-$(\mathcal{GL})$-preinvex functions lead to some new ideas for $\mathfrak{CR}$-preinvexities for different classes of convexity.

Proposition 1. *Consider $\mathfrak{T}$: $[\xi,\varepsilon] \longrightarrow R_I$ be an $(\mathcal{IVF})$ given by $\mathfrak{T} = [\underline{\mathfrak{T}}, \overline{\mathfrak{T}}] = \langle \mathfrak{T}_{\mathscr{C}}, \mathfrak{T}_{\mathscr{R}} \rangle$. If $\mathfrak{T}_{\mathscr{C}}$ and $\mathfrak{T}_{\mathscr{R}}$ are h-$\mathcal{GL}$-preinvex, then $\mathfrak{T}$ is $\mathfrak{CR} - h$-$\mathcal{GL}$-preinvex.*

Proof. As $\mathfrak{T}_{\mathscr{C}}$ and $\mathfrak{T}_{\mathscr{R}}$ are h-$\mathcal{GL}$-preinvex mappings, and for all $\mathfrak{s} \in (0,1)$, one has

$$\begin{aligned} \mathfrak{T}_{\mathscr{C}}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) &\leq \frac{\mathfrak{T}_{\mathscr{C}}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}_{\mathscr{C}}(\xi)}{h(1-\mathfrak{s})}, \\ \mathfrak{T}_{\mathscr{R}}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) &\leq \frac{\mathfrak{T}_{\mathscr{R}}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}_{\mathscr{R}}(\xi)}{h(1-\mathfrak{s})}. \end{aligned} \tag{23}$$

If $\mathfrak{T}_{\mathscr{C}}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \neq (\mathfrak{T}_{\mathscr{C}}(\varepsilon)/h(\mathfrak{s})) + (\mathfrak{T}_{\mathscr{C}}(\xi)/h(1-\mathfrak{s}))$, then

$$\mathfrak{T}_{\mathscr{C}}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) < \frac{\mathfrak{T}_{\mathscr{C}}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}_{\mathscr{C}}(\xi)}{h(1-\mathfrak{s})}. \tag{24}$$

This implies

$$\mathfrak{T}_{\mathscr{C}}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \preccurlyeq_{\mathfrak{CR}} \frac{\mathfrak{T}_{\mathscr{C}}(\varepsilon)}{h(\mathfrak{s})} + \frac{\mathfrak{T}_{\mathscr{C}}(\xi)}{h(1-\mathfrak{s})}. \tag{25}$$

Otherwise, $\mathfrak{T}_{\mathscr{R}}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi)) \leq (\mathfrak{T}_{\mathscr{R}}(\varepsilon)/h(\mathfrak{s})) + (\mathfrak{T}_{\mathscr{R}}(\xi)/h(1-\mathfrak{s}))$ implies

$$\mathfrak{T}_{\mathscr{R}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)) \preccurlyeq_{\mathfrak{CR}} \frac{\mathfrak{T}_{\mathscr{R}}(\varepsilon)}{h(\mathfrak{s})}+\frac{\mathfrak{T}_{\mathscr{R}}(\xi)}{h(1-\mathfrak{s})}. \tag{26}$$

Now, from Definition 6, we have

$$\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)) \preccurlyeq_{\mathfrak{CR}} \frac{\mathfrak{T}(\varepsilon)}{h(\mathfrak{s})}+\frac{\mathfrak{T}(\xi)}{h(1-\mathfrak{s})}. \tag{27}$$

This proves that $\mathfrak{T}$ is a $\mathfrak{CR}-h$-$\mathscr{GL}$-preinvex function if $\mathfrak{T}_{\mathscr{C}}$ and $\mathfrak{T}_{\mathscr{R}}$ are h-$\mathscr{GL}$-preinvex functions. □

4. Some Inequalities as Applications of $\mathfrak{CR}$-h-$(\mathscr{GL})$-Preinvex Functions

In this part, we will talk about how to apply the new idea of $(\mathscr{GL})$-preinvex functions connected with $\mathfrak{CR}$-order relation to show some new variants of Fejér-type and Hermite–Hadamard-type inequalities.

Theorem 2. *Consider* $\mathfrak{T}\colon [\xi,\xi+\zeta(\varepsilon,\xi)] \longrightarrow R$ *be an* $\mathscr{IVF}$ *which is represented as*

$$\mathfrak{T}(\wp)=\left[\underline{\mathfrak{T}}(\wp),\overline{\mathfrak{T}}(\wp)\right], \tag{28}$$

for all $\wp\in[\xi,\varepsilon]$. *If* $\mathfrak{T}\colon [\xi,\xi+\zeta(\varepsilon,\xi)] \longrightarrow R$ *is a* $\mathfrak{CR}-h$-$\mathscr{GL}$*-preinvex function and satisfies Condition 1, then for* $h(1/2)>0$, *as a result, the following inequalities hold:*

$$\frac{h(1/2)}{2}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right) \preccurlyeq_{\mathfrak{CR}} \frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp \preccurlyeq_{\mathfrak{CR}} [\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{d\mathfrak{s}}{h(\mathfrak{s})}. \tag{29}$$

Proof. In accordance with the definition of $\mathfrak{CR}-h$-$\mathscr{GL}$-preinvex function, one has

$$\mathfrak{T}\left(x+\frac{1}{2}\zeta(y,x)\right) \preccurlyeq_{\mathfrak{CR}} \frac{1}{h(1/2)}[\mathfrak{T}(x)+\mathfrak{T}(y)]. \tag{30}$$

Choosing $x=\xi+\mathfrak{s}\zeta(\varepsilon,\xi)$ and $y=\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi)$, it is seen that

$$\mathfrak{T}\left(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)+\frac{1}{2}\zeta(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi),\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\right) \preccurlyeq_{\mathfrak{CR}} \frac{1}{h(1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]. \tag{31}$$

This implies

$$h\left(\frac{1}{2}\right)\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right) \preccurlyeq_{\mathfrak{CR}} [\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]. \tag{32}$$

Integrating inequality (32), we get

$$\begin{aligned}
h\left(\frac{1}{2}\right)\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right) &\preccurlyeq_{\mathfrak{CR}} \left[\int_0^1\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}+\int_0^1\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))d\mathfrak{s}\right]\\
&=\int_0^1\left(\underline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\underline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\right)d\mathfrak{s},\\
&\quad\int_0^1\left(\overline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\overline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\right)d\mathfrak{s}\\
&=\frac{2}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\underline{\mathfrak{T}}(\wp)d\wp,\frac{2}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\overline{\mathfrak{T}}(\wp)d\wp\\
&=\frac{2}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp.
\end{aligned} \tag{33}$$

We can conclude from the preceding developments that

$$\frac{h(1/2)}{2}\mathfrak{T}\left(\frac{2\xi+\zeta(\varepsilon,\xi}{2}\right)\preccurlyeq_{\mathfrak{CR}}\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp. \tag{34}$$

This concludes the first inequality's proof. Next, utilizing the notion of $\mathfrak{CR}-h-\mathscr{GL}$-preinvex functions, we can demonstrate the second inequality.

$$\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\preccurlyeq_{\mathfrak{CR}}\frac{\mathfrak{T}(\varepsilon)}{h(\mathfrak{s})}+\frac{\mathfrak{T}(\xi)}{h(1-\mathfrak{s})}. \tag{35}$$

Integrating the aforementioned inequality, we have

$$\int_0^1\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}\preccurlyeq_{\mathfrak{CR}}\mathfrak{T}(\varepsilon)\int_0^1\frac{d\mathfrak{s}}{h(\mathfrak{s})}+\mathfrak{T}(\xi)\int_0^1\frac{d\mathfrak{s}}{h(1-\mathfrak{s})}. \tag{36}$$

This implies

$$\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{d\mathfrak{s}}{h(\mathfrak{s})}. \tag{37}$$

As a result of (34) and (37), we get the required result, i.e.,

$$\frac{h(1/2)}{2}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\preccurlyeq_{\mathfrak{CR}}\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{d\mathfrak{s}}{h(\mathfrak{s})}. \tag{38}$$

The proof is now completed. □

Remark 2

i. If one has $\underline{\mathfrak{T}}=\overline{\mathfrak{T}}$, then the result presented in Theorem 2 produces an output for h-$\mathscr{GL}$-preinvex functions, i.e.,

$$\frac{h(1/2)}{2}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\leq\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp\leq[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{d\mathfrak{s}}{h(\mathfrak{s})}. \tag{39}$$

ii. If we set $h(t)=(1/t)$, then the result presented in Theorem 2 produces an output for $\mathfrak{CR}$-preinvex, i.e.,

$$\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\preccurlyeq_{\mathfrak{CR}}\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}\frac{\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)}{2}. \tag{40}$$

iii. If we set $h(t)=(1/t)$ and $\zeta(\varepsilon,\xi)=\varepsilon-\xi$, then Theorem 2 produces an output for $\mathfrak{CR}$-convex functions, i.e.,

$$\mathfrak{T}\left(\frac{\xi+\varepsilon}{2}\right)\preccurlyeq_{\mathfrak{CR}}\frac{1}{\varepsilon-\xi}\int_{\xi}^{\varepsilon}\mathfrak{T}(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}\frac{\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)}{2}. \tag{41}$$

Example 3. Let $\mathfrak{T}(\wp)=[1-\sqrt{\wp},3(3-\sqrt{\wp})]$, $\zeta(\varepsilon,\xi)=\varepsilon-\xi$, $\xi=0$, and $\varepsilon=2$. Then, for $h(\mathfrak{s})=(1/\mathfrak{s})$, we have

$$\begin{aligned}&\frac{h(1/2)}{2}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\approx[0,6],\\&\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)d\wp\approx[0.05719,6.17157],\\&[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{d\mathfrak{s}}{h(\mathfrak{s})}\approx[2-\sqrt{2},18-3\sqrt{2}].\end{aligned} \tag{42}$$

Thus, we have

$$[0,6]\preccurlyeq_{\mathfrak{CR}}[0.05719,6.17157]\preccurlyeq_{\mathfrak{CR}}[2-\sqrt{2},18-3\sqrt{2}]. \tag{43}$$

Theorem 3. *Let* $\mathfrak{T}, \mathfrak{L}$: $[\xi, \xi + \zeta(\varepsilon, \xi)] \longrightarrow R$ *be an* $\mathscr{IVFS}$, *which are given by*

$$\begin{aligned}\mathfrak{L}(\wp) &= \left[\underline{\mathfrak{L}}(\wp), \overline{\mathfrak{L}}(\wp)\right],\\ \mathfrak{T}(\wp) &= \left[\underline{\mathfrak{T}}(\wp), \overline{\mathfrak{T}}(\wp)\right],\end{aligned} \tag{44}$$

for all $\wp \in [\xi, \varepsilon]$ *and* $\mathfrak{T}, \mathfrak{L} \in \mathscr{IR}_{([\xi,\varepsilon])}$. *If* $\mathfrak{T}$: $[\xi, \xi + \zeta(\varepsilon, \xi)] \longrightarrow R$ *is a* $\mathfrak{CR} - h_1$-$\mathscr{GL}$-*preinvex function and* $\mathfrak{L}$: $[\xi, \xi + \zeta(\varepsilon, \xi)] \longrightarrow R$ *is a* $\mathfrak{CR} - h_2$-$\mathscr{GL}$-*preinvex function, then the following disparities exist:*

$$\frac{1}{\zeta(\varepsilon, \xi)} \int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\mathfrak{L}(\wp)d\wp \preccurlyeq_{\mathfrak{CR}} \mathbf{M}(\xi, \varepsilon) \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})} + \mathbf{N}(\xi, \varepsilon) \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}, \tag{45}$$

where

$$\begin{aligned}\mathbf{M}(\xi, \varepsilon) &= \mathfrak{T}(\xi)\mathfrak{L}(\xi) + \mathfrak{T}(\varepsilon)\mathfrak{L}(\varepsilon),\\ \mathbf{N}(\xi, \varepsilon) &= \mathfrak{T}(\xi)\mathfrak{L}(\varepsilon) + \mathfrak{T}(\varepsilon)\mathfrak{L}(\xi).\end{aligned} \tag{46}$$

Proof. Since $\mathfrak{T}$ is a $\mathfrak{CR} - h_1$-$\mathscr{GL}$-preinvex function and $\mathfrak{L}$ is a $\mathfrak{CR} - h_2$-$\mathscr{GL}$-preinvex function, we have

$$\begin{aligned}\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon, \xi)) &\preccurlyeq_{\mathfrak{CR}} \frac{\mathfrak{T}(\varepsilon)}{h_1(\mathfrak{s})} + \frac{\mathfrak{T}(\xi)}{h_2(1-\mathfrak{s})},\\ \mathfrak{L}(\xi + \mathfrak{s}\zeta(\varepsilon, \xi)) &\preccurlyeq_{\mathfrak{CR}} \frac{\mathfrak{L}(\varepsilon)}{h_2(\mathfrak{s})} + \frac{\mathfrak{L}(\xi)}{h_2(1-\mathfrak{s})}.\end{aligned} \tag{47}$$

When the two aforementioned inequalities are multiplied, we have

$$\begin{aligned}&\mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon, \xi))\mathfrak{L}(\xi + \mathfrak{s}\zeta(\varepsilon, \xi))\\ &\preccurlyeq_{\mathfrak{CR}} \left[\frac{\mathfrak{T}(\varepsilon)}{h_1(\mathfrak{s})} + \frac{\mathfrak{T}(\xi)}{h_1(1-\mathfrak{s})}\right]\left[\frac{\mathfrak{L}(\varepsilon)}{h_2(\mathfrak{s})} + \frac{\mathfrak{L}(\xi)}{h_2(1-\mathfrak{s})}\right]\\ &= \frac{[\mathfrak{T}(\varepsilon)\mathfrak{L}(\varepsilon)]}{h_1(\mathfrak{s})h_2(\mathfrak{s})} + \frac{[\mathfrak{T}(\xi)\mathfrak{L}(\xi)]}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})} + \frac{[\mathfrak{T}(\varepsilon)\mathfrak{L}(\xi)]}{h_1(\mathfrak{s})h_2(1-\mathfrak{s})} + \frac{[\mathfrak{T}(\xi)\mathfrak{L}(\varepsilon)]}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}.\end{aligned} \tag{48}$$

Integrating aforementioned inequality (48) over (0, 1), we have

$$\begin{aligned}&\int_0^1 \mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon, \xi))\mathfrak{L}(\xi + \mathfrak{s}\zeta(\varepsilon, \xi))d\mathfrak{s}\\ &\preccurlyeq_{\mathfrak{CR}} [\mathfrak{T}(\varepsilon)\mathfrak{L}(\varepsilon)] \int_0^1 \frac{d\mathfrak{s}}{h_1(\mathfrak{s})h_2(\mathfrak{s})} + [\mathfrak{T}(\xi)\mathfrak{L}(\xi)] \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})}\\ &\quad + [\mathfrak{T}(\varepsilon)\mathfrak{L}(\xi)] \int_0^1 \frac{d\mathfrak{s}}{h_1(\mathfrak{s})h_2(1-\mathfrak{s})} + [\mathfrak{T}(\xi)\mathfrak{L}(\varepsilon)] \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}.\end{aligned} \tag{49}$$

By using Definition 6, we obtain

$$\begin{aligned}\frac{1}{\zeta(\varepsilon, \xi)} \int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\mathfrak{L}(\wp)d\wp &\preccurlyeq_{\mathfrak{CR}} [\mathfrak{T}(\xi)\mathfrak{L}(\xi) + \mathfrak{T}(\varepsilon)\mathfrak{L}(\varepsilon)] \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})}\\ &\quad + [\mathfrak{T}(\xi)\mathfrak{L}(\varepsilon) + \mathfrak{T}(\varepsilon)\mathfrak{L}(\xi)] \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}\\ &= \mathbf{M}(\xi, \varepsilon) \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})} + \mathbf{N}(\xi, \varepsilon) \int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}.\end{aligned} \tag{50}$$

As claimed by Theorem 3, we ultimately arrive at the desired outcome. □

Example 4. Let $\mathfrak{T}(\wp) = [2-\sqrt{\wp}, 3(2-\sqrt{\wp})]$, $\mathfrak{L}(\wp) = [e^{\wp}-\wp, e^{\wp}+\wp]$, $\zeta(\varepsilon,\xi) = \varepsilon-\xi$, $\xi = 0$, and $\varepsilon = 2$. Then, for $h(\mathfrak{s}) = h_2(\mathfrak{s}) = (1/\mathfrak{s})$, we have

$$\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\mathfrak{L}(\wp)d\wp \approx [1.9675, 11.114], \tag{51}$$

and

$$\mathbf{M}(\xi,\varepsilon)\int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})} + \mathbf{N}(\xi,\varepsilon)\int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})} \approx [4.5660, 16.234] \tag{52}$$

Thus, we have

$$[1.9675, 11.114] \preccurlyeq_{\mathfrak{CR}} [4.5660, 16.234]. \tag{53}$$

Thus, the validity of the Theorem 3 is established.

Theorem 4. *Based on the identical hypothesis as provided in Theorem 3, we have the following relationship:*

$$\begin{aligned}&\frac{H(1/2,1/2)}{2}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\mathfrak{L}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\\ &\preccurlyeq_{\mathfrak{CR}}\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\mathfrak{L}(\wp)d\wp + \mathbf{M}(\xi,\varepsilon)\int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})} + \mathbf{N}(\xi,\varepsilon)\int_0^1 \frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})}.\end{aligned} \tag{54}$$

Proof. Since $\mathfrak{T}$ is $\mathfrak{CR}-h$-$\mathscr{GL}$-preinvex, then one has

$$h\left(\frac{1}{2}\right)\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right) \preccurlyeq_{\mathfrak{CR}} [\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)) + \mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]. \tag{55}$$

Since $\mathfrak{T}$ and $\mathfrak{L}$ are interval-valued $\mathfrak{CR}-h_1$-$(\mathscr{GL})$-preinvex function and $\mathfrak{CR}-h_2$-$(\mathscr{GL})$-preinvex function, respectively, and using Condition 1, we have

$$\begin{aligned}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right) &= \mathfrak{T}\left(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)+\frac{1}{2}\zeta(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi), \xi+\mathfrak{s}\zeta(\varepsilon,\xi))\right)\\ &\preccurlyeq_{\mathfrak{CR}}\frac{1}{h_1(1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)) + \mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))].\end{aligned} \tag{56}$$

Similarly,

$$\begin{aligned}\mathfrak{L}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right) &= \mathfrak{L}\left(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)+\frac{1}{2}\zeta(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi), \xi+\mathfrak{s}\zeta(\varepsilon,\xi))\right)\\ &\preccurlyeq_{\mathfrak{CR}}\frac{1}{h_2(1/2)}[\mathfrak{L}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)) + \mathfrak{L}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))].\end{aligned} \tag{57}$$

Next, multiplying (56) and (57), we have

$$
\begin{aligned}
&\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\mathfrak{L}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\\
&\preccurlyeq_{\mathfrak{CR}}\frac{1}{h_1(1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]\frac{1}{h_2(1/2)}[\mathfrak{L}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\mathfrak{L}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]\\
&=\frac{1}{H(1/2,1/2)}[\underline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\underline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\underline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\underline{\mathfrak{L}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\\
&\quad+\underline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\underline{\mathfrak{L}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))+\underline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\underline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\\
&\quad\overline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\overline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\overline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\overline{\mathfrak{L}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\\
&\quad+\overline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\overline{\mathfrak{L}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))+\overline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\overline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))]\\
&=\frac{1}{H(1/2,1/2)}\left[\underline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\underline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)),\overline{\mathfrak{T}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\overline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\right]\\
&\quad+\frac{1}{H(1/2,1/2)}\left[\underline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\underline{\mathfrak{L}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi)),\overline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\overline{\mathfrak{L}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\right]\\
&\quad+\frac{1}{H(1/2,1/2)}\left[\underline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\underline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)),\overline{\mathfrak{T}}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\overline{\mathfrak{L}}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\right]\\
&=\frac{1}{H(1/2,1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]\\
&\quad+\frac{1}{H(1/2,1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))]\\
&\preccurlyeq_{\mathfrak{CR}}\frac{1}{H(1/2,1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]\\
&\quad+\frac{1}{H(1/2,1/2)}\left[\left(\frac{\mathfrak{T}(\varepsilon)}{h_1(\mathfrak{s})}+\frac{\mathfrak{T}(\xi)}{h_1(1-\mathfrak{s})}\right)\left(\frac{\mathfrak{L}(\xi)}{h_2(\mathfrak{s})}+\frac{\mathfrak{L}(\varepsilon)}{h_2(1-\mathfrak{s})}\right)\right.\\
&\quad\left.+\left(\frac{\mathfrak{T}(\xi)}{h_1(\mathfrak{s})}+\frac{\mathfrak{T}(\varepsilon)}{h_1(1-\mathfrak{s})}\right)\left(\frac{\mathfrak{L}(\varepsilon)}{h_2(\mathfrak{s})}+\frac{\mathfrak{L}(\xi)}{h_2(1-\mathfrak{s})}\right)\right]\\
&=\frac{1}{H(1/2,1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\mathfrak{L}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))]\\
&\quad+\frac{1}{H(1/2,1/2)}\left[\mathbf{M}(\xi,\varepsilon)\left[\frac{1}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}+\frac{1}{h_1(\mathfrak{s})h_2(1-\mathfrak{s})}\right]+\mathbf{N}(\xi,\varepsilon)\left[\frac{1}{h_1(\mathfrak{s})h_2(\mathfrak{s})}+\frac{1}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})}\right]\right].
\end{aligned}
\tag{58}
$$

As a result of integrating and changing variables, we get

$$
\begin{aligned}
\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\mathfrak{L}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\preccurlyeq_{\mathfrak{CR}}\frac{2}{h_1(1/2)h_2(1/2)}&\left\{\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\mathfrak{L}(\wp)d\wp\right.\\
&\left.+\mathbf{M}(\xi,\varepsilon)\int_0^1\frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}+\mathbf{N}(\xi,\varepsilon)\int_0^1\frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})}\right\}.
\end{aligned}
\tag{59}
$$

This readily gives

$$\frac{H(1/2,1/2)}{2}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\mathfrak{L}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)$$
$$\preccurlyeq_{\mathfrak{CR}}\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\mathfrak{L}(\wp)\,d\wp+\mathbf{M}(\xi,\varepsilon)\int_0^1\frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}+\mathbf{N}(\xi,\varepsilon)\int_0^1\frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})}. \tag{60}$$

The proof of the desired outcome is now completed. □

Example 5. Let $\mathfrak{T}(\wp)=[2-\sqrt{\wp},3(2-\sqrt{\wp})]$, $\mathfrak{L}(\wp)=[e^{\wp}-\wp,e^{\wp}+\wp]$, $\zeta(\varepsilon,\xi)=\varepsilon-\xi$, $\xi=0$, and $\varepsilon=2$. Then, for $h_1(\mathfrak{s})=h_2(\mathfrak{s})=1/\mathfrak{s}$, we have

$$\frac{h_1(1/2)h_2(1/2)}{2}\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\mathfrak{L}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\approx\left[\frac{86}{25},\frac{2231}{100}\right],$$
$$\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\mathfrak{L}(\wp)d\wp+\mathbf{M}(\xi,\varepsilon)\int_0^1\frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(1-\mathfrak{s})}+\mathbf{N}(\xi,\varepsilon)\int_0^1\frac{d\mathfrak{s}}{h_1(1-\mathfrak{s})h_2(\mathfrak{s})}\approx\left[\frac{14171}{2000},\frac{337531}{10000}\right]. \tag{61}$$

Thus, we have

$$\left[\frac{86}{25},\frac{2231}{100}\right]\preccurlyeq_{\mathfrak{CR}}\left[\frac{14171}{2000},\frac{337531}{10000}\right]. \tag{62}$$

Thus, the validity of the Theorem 4 is established.

Theorem 5 (2^{nd} $\mathscr{H}.\mathscr{H}$-Fejér-type inequality for $\mathfrak{CR}-h$-$\mathscr{GL}$-preinvex function). *Let* $\mathfrak{T}:[\xi,\xi+\zeta(\varepsilon,\xi)]\longrightarrow R$ *be an* $\mathscr{IVF}$*, which is defined as*

$$\mathfrak{T}(\wp)=\left[\underline{\mathfrak{T}}(\wp),\overline{\mathfrak{T}}(\wp)\right], \tag{63}$$

for all $\wp\in[\xi,\varepsilon]$. *If* $\mathfrak{T}:[\xi,\xi+\zeta(\varepsilon,\xi)]\longrightarrow R$ *is a* $\mathfrak{CR}-h$-$\mathscr{GL}$*-preinvex function and* $\chi:[\xi,\xi+\zeta(\varepsilon,\xi)]\longrightarrow R,\chi>0$ *is proportioned with reference to* $\xi+(1/2)\zeta(\varepsilon,\xi)$, *then the following disparities exist:*

$$\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\chi(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}}{h(\mathfrak{s})}. \tag{64}$$

Proof. As $\mathfrak{T}$ is a $\mathfrak{CR}-h$-$\mathscr{GL}$-preinvex function and χ is proportioned with reference to $\xi+(1/2)\zeta(\varepsilon,\xi)$, we have

$$\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\preccurlyeq_{\mathfrak{CR}}\left[\frac{\mathfrak{T}(\varepsilon)}{h(\mathfrak{s})}+\frac{\mathfrak{T}(\xi)}{h(1-\mathfrak{s})}\right]\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)),$$
$$\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\preccurlyeq_{\mathfrak{CR}}\left[\frac{\mathfrak{T}(\xi)}{h(\mathfrak{s})}+\frac{\mathfrak{T}(\varepsilon)}{h(1-\mathfrak{s})}\right]\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi)). \tag{65}$$

After adding the two previously given inequalities and integrating, we obtain

$$\begin{aligned}
&\int_0^1 \mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}+\int_0^1 \mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))d\mathfrak{s}\\
&\preccurlyeq_{\mathfrak{CR}} \int_0^1\left[\frac{\mathfrak{T}(\xi)(\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)))}{h(1-\mathfrak{s})}+\frac{\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi)))}{h(\mathfrak{s})}\right.\\
&\quad\left.+\frac{\mathfrak{T}(\varepsilon)(\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi)))}{h(\mathfrak{s})}+\frac{\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi)))}{h(1-\mathfrak{s})}\right]d\mathfrak{s}\\
&=2\mathfrak{T}(\xi)\int_0^1\frac{\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))}{h(\mathfrak{s})}d\mathfrak{s}+2\mathfrak{T}(\varepsilon)\int_0^1\frac{\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))}{h(\mathfrak{s})}d\mathfrak{s}\\
&=2[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))}{h(\mathfrak{s})}d\mathfrak{s}.
\end{aligned} \tag{66}$$

Since

$$\begin{aligned}
&\int_0^1 \mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}+\int_0^1 \mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))d\mathfrak{s}\\
&=\frac{2}{\zeta(\varepsilon,\xi)}\int_\xi^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\chi(\wp)d\wp.
\end{aligned} \tag{67}$$

Taking into account (66) and (67) allows us to obtain the desired outcome. □

Remark 3

i. If $h(\mathfrak{s})=1/\mathfrak{s}$, then Theorem 5 produces an output for $\mathfrak{CR}$-preinvex, i.e.,

$$\frac{1}{\zeta(\varepsilon,\xi)}\int_\xi^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\chi(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\mathfrak{s}\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}. \tag{68}$$

ii. If $\zeta(\varepsilon,\xi)=\varepsilon-\xi$, then Theorem 5 produces an output for $\mathfrak{CR}$ − h-$\mathscr{GL}$, i.e.,

$$\frac{1}{\varepsilon-\xi}\int_\xi^\varepsilon\mathfrak{T}(\wp)\chi(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\frac{\chi((1-\mathfrak{s})\xi+\mathfrak{s}\varepsilon)}{\mathrm{h}(\mathfrak{s})}d\mathfrak{s}. \tag{69}$$

iii. If $h(\mathfrak{s})=1/\mathfrak{s}$ and $\zeta(\varepsilon,\xi)=\varepsilon-\xi$, then Theorem 5 produces an output for $\mathfrak{CR}$-convex, i.e.,

$$\frac{1}{\varepsilon-\xi}\int_\xi^\varepsilon\mathfrak{T}(\wp)\chi(\wp)d\wp\preccurlyeq_{\mathfrak{CR}}[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_0^1\mathfrak{s}\chi((1-\mathfrak{s})\xi+\mathfrak{s}\varepsilon)d\mathfrak{s}. \tag{70}$$

Example 6. Let $\mathfrak{T}(\wp)=[2-\sqrt{\wp},3(2-\sqrt{\wp})]$, $\zeta(\varepsilon,\xi)=\varepsilon-\xi$, $\xi=0$, and $\varepsilon=2$. Then, for $h(\mathfrak{s})=1/\mathfrak{s}$ and proportioned function $\chi(\wp)=\wp$ for $\wp\in[0,1]$ and $\chi(\wp)=-\wp+2$ for $\wp\in[1,2]$, we have

$$\frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\chi(\wp)d\wp=\frac{1}{2}\int_{0}^{2}\mathfrak{T}(\wp)\chi(\wp)d\wp$$
$$=\frac{1}{2}\int_{0}^{1}\left[\left(2-\wp^{1/2}\right)\wp,3\wp\left(2-\wp^{1/2}\right)\right]d\wp$$
$$+\frac{1}{2}\int_{1}^{2}\left[\left(2-\wp^{1/2}\right)(-\wp+2),3(-\wp+2)\left(2-\wp^{1/2}\right)\right]d\wp \tag{71}$$
$$\approx[0.512,1.537],$$

and

$$[\mathfrak{T}(\xi)+\mathfrak{T}(\varepsilon)]\int_{0}^{1}\frac{\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))}{h(\mathfrak{s})}d\mathfrak{s}=\left([2,6]+\left[2-2^{1/2},3\left(2-2^{1/2}\right)\right]\right)\int_{0}^{1}\mathfrak{s}\chi(2\mathfrak{s})d\mathfrak{s}$$
$$=\left[4-2^{1/2},6+3\left(2-2^{1/2}\right)\right]\left(\int_{0}^{1/2}2\mathfrak{s}^{2}dt+\int_{1/2}^{1}\mathfrak{s}(-2\mathfrak{s}+2)d\mathfrak{s}\right) \tag{72}$$
$$\approx[0.646,1.939].$$

Thus, we have

$$[0.512,1.537]\preccurlyeq_{\mathfrak{CR}}[0.646,1.939]. \tag{73}$$

Theorem 5 validity is, therefore, confirmed.

Theorem 6. *(1^{st} $\mathscr{H}.\mathscr{H}$-Fejér inequality for $\mathfrak{CR}-h$-$\mathscr{G}\mathscr{L}$-preinvex function). Let $\mathfrak{T}$: $[\xi,\xi+\zeta(\varepsilon,\xi)]\longrightarrow R$ be an $\mathscr{IVF}$ which is defined as*

$$\mathfrak{T}(\wp)=\left[\underline{\mathfrak{T}}(\wp),\overline{\mathfrak{T}}(\wp)\right], \tag{74}$$

for all $\wp\in[\xi,\varepsilon]$. If $\mathfrak{T}$: $[\xi,\xi+\zeta(\varepsilon,\xi)]\longrightarrow R$ is $\mathfrak{CR}-h$-$\mathscr{G}\mathscr{L}$-preinvex function and χ: $[\xi,\xi+\zeta(\varepsilon,\xi)]\longrightarrow$ R, $\chi>0$ be proportioned with reference to $\xi+(1/2)\zeta(\varepsilon,\xi)$, assuming $\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\chi(\wp)d\wp>0$, then the following disparities exist:

$$\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\preccurlyeq_{\mathfrak{CR}}\frac{2}{h(1/2)\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\chi(\wp)d\wp}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\chi(\wp)d\wp. \tag{75}$$

Proof. Since $\mathfrak{T}$ is $\mathfrak{CR}-h$-$\mathscr{G}\mathscr{L}$-preinvex function, we have

$$\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\preccurlyeq_{\mathfrak{CR}}\frac{1}{h(1/2)}[\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}+\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))d\mathfrak{s}]. \tag{76}$$

Multiplying aforementioned disparities by $\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))=\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))$ and integrating, we have

$$\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right)\int_{0}^{1}\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}\preccurlyeq_{\mathfrak{CR}}\frac{1}{h(1/2)}\left[\int_{0}^{1}\mathfrak{T}(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))\chi(\xi+\mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s}\right.$$
$$\left.+\int_{0}^{1}\mathfrak{T}(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))\chi(\xi+(1-\mathfrak{s})\zeta(\varepsilon,\xi))d\mathfrak{s}\right]. \tag{77}$$

Since

$$\int_0^1 \mathfrak{T}(\xi + \mathfrak{s}\zeta(\varepsilon,\xi))\chi(\xi + \mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s} = \int_0^1 \mathfrak{T}(\xi + (1-\mathfrak{s})\zeta(\varepsilon,\xi))\chi(\xi + (1-\mathfrak{s})\zeta(\varepsilon,\xi))d\mathfrak{s} = \frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\chi(\wp)d\wp, \tag{78}$$

and

$$\int_0^1 \chi(\xi + \mathfrak{s}\zeta(\varepsilon,\xi))d\mathfrak{s} = \frac{1}{\zeta(\varepsilon,\xi)}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \chi(\wp)d\wp. \tag{79}$$

Using (78) and (79) in (77), we have

$$\mathfrak{T}\left(\xi + \frac{1}{2}\zeta(\varepsilon,\xi)\right) \preccurlyeq_{\mathfrak{CR}} \frac{2}{h(1/2)\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\chi(\wp)d\wp}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\chi(\wp)d\wp. \tag{80}$$

The proof is now complete. □

Remark 4

i. If $\underline{\mathfrak{T}} = \overline{\mathfrak{T}}$, then Theorem 6 produces an output for h-$\mathscr{GL}$-preinvex functions, i.e.,

$$\mathfrak{T}\left(\xi + \frac{1}{2}\zeta(\varepsilon,\xi)\right) \leq \frac{2}{h(1/2)\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\chi(\wp)d\wp}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\chi(\wp)d\wp. \tag{81}$$

ii. If $h(\mathfrak{s}) = 1/\mathfrak{s}$, then Theorem 6 produces an output for $\mathfrak{CR}$-preinvex functions, i.e.,

$$\mathfrak{T}\left(\xi + \frac{1}{2}\zeta(\varepsilon,\xi)\right) \preccurlyeq_{\mathfrak{CR}} \frac{1}{\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\chi(\wp)d\wp}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)} \mathfrak{T}(\wp)\chi(\wp)d\wp. \tag{82}$$

iii. If $\zeta(\varepsilon,\xi) = \varepsilon - \xi$, then Theorem 6 produces an output for $\mathfrak{CR} - h$-$\mathscr{GL}$ functions, i.e.,

$$\mathfrak{T}\left(\frac{\xi+\varepsilon}{2}\right) \preccurlyeq_{\mathfrak{C}\mathfrak{R}} \frac{2}{h(1/2)\int_{\xi}^{\varepsilon}\chi(\wp)d\wp}\int_{\xi}^{\varepsilon}\mathfrak{T}(\wp)\chi(\wp)d\wp. \quad (83)$$

Example 7. Let $\mathfrak{T}(\wp) = [2-\sqrt{\wp}, 3(2-\sqrt{\wp})]$, $\zeta(\varepsilon,\xi) = \varepsilon - \xi$, $\xi = 0$, and $\varepsilon = 2$. Then, for $h(\mathfrak{s}) = 1/\mathfrak{s}$ and proportioned function $\chi(\wp) = \wp$ for $\wp \in [0,1]$ and $\chi(\wp) = -\wp + 2$ for $\wp \in [1,2]$, we have

$$\begin{aligned}
&\mathfrak{T}\left(\xi+\frac{1}{2}\zeta(\varepsilon,\xi)\right) = \mathfrak{T}(1) = [1,3] \\
&\frac{2}{h(1/2)\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\chi(\wp)d\wp}\int_{\xi}^{\xi+\zeta(\varepsilon,\xi)}\mathfrak{T}(\wp)\chi(\wp)d\wp \\
&= \frac{1}{\int_0^2\chi(\wp)d\wp}\int_0^2[2-\sqrt{\wp}, 3(2-\sqrt{\wp})]\chi(\wp)d\wp \\
&= \int_0^1[\wp(2-\sqrt{\wp}), 3\wp(2-\sqrt{\wp})]d\wp + \int_1^2\left[(-\wp+2)\left(2-\wp^{1/2}\right), 3(-\wp+2)\left(2-\wp^{1/2}\right)\right]\chi(\wp)d\wp \\
&\approx [1.024, 3.074].
\end{aligned} \quad (84)$$

Thus, we have

$$[1,3] \preccurlyeq_{\mathfrak{C}\mathfrak{R}} [1.02484, 3.07452]. \quad (85)$$

Theorem 6 validity is, therefore, confirmed.

5. Conclusion

In the development of the theory of inequalities, interval-valued concepts have made a significant contribution. This study explores new refinements of well-known integral inequalities by utilizing interval-valued concepts and $(\mathscr{G}\mathscr{L})$-preinvex mappings. A novel aspect of this class and study is that the results obtained from under-consideration are more accurate and unified. Based on these notions, we developed Hermite–Hadamard's weighted and product forms, and together with remarks, we showed how our newly developed results generalize various previous results with different setups. We hope that these findings will play a critical role in the optimization theory and error analysis in the future. This concept will be extended to fuzzy numbers and time-scale calculus in the future.

Data Availability Statement

All the data will be provided on request.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

W.A., W.N., and Z.A.K conceptualized the study; J.K.K.A. was responsible for validation; W.N., W.A., and Z.A.K performed the investigation; W.A wrote the original draft; J.K.K.A. and W.N reviewed the manuscript; W.N was responsible for supervision; W.A. and Z.A.K were responsible for project administration. All the authors have read and agreed to the published version of the manuscript.

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Use of AI Tools Declaration. The authors declare that they have not used artificial intelligence tools in the creation of this article.

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