# LOCATION OF ADDITIONAL LIBRARY FACILITY

## CASE STUDY: BEREKUM MUNICIPALITY



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Location of Additional Library Facility

Case Study: Berekum Municipality

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## DECLARATION

I, hereby declare that this submission is my own work towards the Master of Science and that to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgement has been in the text.



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## ABSTRACT

The problem of locating a facility on a network to optimize certain objective criteria has been object in the past few years of growing interest for its relevance in the context of minimizing maximum travel distance between demand nodes.

This thesis therefore considers the problem of locating a library facility as a $p$ – centre problem under the condition that there are some existing facilities already located in the Berekum Municipality in the Brong Ahafo Region.

Berman and Drezner (2008) method was used on 18 – nodes network which had two existing library facilities at Berekum and Jininjini. An additional library facility using Berman and Drezner (2008) should be located at Akrofro with an objective function value of 8. The objective function value of 8 means that, the minimum distance travelled the farthest library user to the new library facility at Akrofro is 8 kilometres.

# DEDICATION

To God be the Glory, Great Things He Has Done and Greater Things He will do.

I dedicate this thesis to my family and friends

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# CHAPTER 1

## INTRODUCTION

### 1.1INTRODUCTION

A library is a collection of sources, resources, and services, and the structure in which it is housed; it is organized for use and maintained by a public body, an institution, or a private individual. In a more traditional sense, a library is a collection of books. It is also the collection, the building or room that houses such a collection, or both. The term “Library” has itself acquired a secondary meaning: “a collection of useful material for common use”. This sense is used in fields such as Computer Science, Mathematics, Statistics, Electronics and Biology. It can be used by publishers in naming related books. E.g. The library of Anglo – Catholic Theology.
A library often provides a place of silence for studying. (Wikipedia, 2012).

Public and institutional collections and services may be intended for use by people who choose not to, or cannot afford to purchase an extensive collection themselves and who need material no individual can reasonably be expected to have, or who require professional assistance with their research.

### 1.2 THE EARLY HISTORY OF LIBRARIES

The first libraries were composed, for the most part, of published records, a particular type of library called archives. Archaeological findings from the ancient city – states of summer have revealed temple rooms full of clay tablets in cuneiform script. (Wikipedia, 2012).

These archives were made up of the records of commercial transactions or inventories, with only a few documents touching theological matters, historical records or legends. Things were much the same in the government and temple records on papyrus of Ancient Egypt.

(Wikipedia,2012).

The earliest discovered private archives were kept at Ugarit; besides correspondence and inventories, texts of myths may have been standardized practice – texts for teaching new scribes. (Wikipedia, 2012).

There is also an evidence of libraries at Nippur about 1900 B.C. and those at Nineveh about 700 B.C. (Wikipedia, 2012).

Over 30,000 clay tablets from library of Ashurbanipal have been discovered at Nineveh, providing archaeologists with an amazing wealth of Mesopotamian library, religious and administrative work. (Wikipedia, 2012).

## 1.3 LIBRARY CLASSIFICATION

A library classification is a system of coding and organizing library materials (books, serials, audiovisual materials, computer files, maps, manuscript, realia) according to their subject and allocating of call number to that information resource. Similar to classification systems used inbiology, bibliographic classification systems group entities together that are similar, typically arranged in a hierarchical tree structure. A different kind of classification system, called a faceted classification system, is also widely used which allows the assignment of multiple classification to an object, enabling the classifications to be ordered in multiple ways.

(Wikipedia, 2012).

### 1.4 DESCRIPTION OF LIBRARIES

Library classification form part of the field of library and information Science. It is a form of bibliographic classification (library classification are used in library catalogs, while"bibliographic classification" also covers classification used in other kinds of bibliographic database). It goes hand in hand with library (descriptive) cataloging under the rubric of cataloging and classification, sometimes grouped together as technical services. The library professional who engages in the process of cataloging and classifying library materials is called a cataloguer or catalog librarian. Library classification systems are one of the two tools used to facilitate subject access. The order consists of alphabetical indexing languages such as Thesauri and Subject Headings Systems.

Library classification of a piece of work consists of two steps. Firstly the "aboutness" of the material is asserted. Next, a call number (essentially a book's address), based on the time in use at the particular library will be assigned to the work using the notation of the system.

It is important to note that unlike subject heading where multiple terms can be assigned to the same work, in library classification system, each work can only be placed in one class. This is due to shelving purpose.

### 1.5 TYPES OF LIBRARIES

#### 1.5.1 ACADEMIC LIBRARIES

These libraries are located on the campuses of colleges and universities and serve primarily the students and the faculty of that and other academic institutions. Some academic libraries, especially those at public institutions, are accessible to the members of the general public.

### 1.5.2 PUBLIC LIBRARIES

These libraries provide services to the general public and make at least some of their books available for borrowing, so that readers may use them at home over a period of time. Typically, libraries issue library cards to community members wishing to borrow books.

### 1.5.3RESEARCH LIBRARIES

They are intended for supporting scholarly research, and therefore maintain permanent collections and attempt to provide access to all necessary material. They are often academic libraries or national libraries, but many large special libraries have research libraries within their special filed and a very few of the large public libraries also serve as a research libraries.

### 1.5.4 SCHOOL LIBRARIES

Most public and private primary and secondary schools have libraries designed to support the school curriculum.

### 1.5.5SPECIAL LIBRARIES

All other libraries fall into this category. Many private businesses and public organizations, including hospitals, museums, research laboratories, law firms, and many government departments and agencies, maintain their own libraries for the use of their employees in doing specialized research, related to their work. Special libraries may or may not be accessible to some identified part of the general public.

## 1.6 BEREKUM MUNICIPALITY AND ITS EDUCATIONAL INSTITUTIONS

Berekum Municipal came into existence as a semi-autonomous spatial unit by virtue of the decentralization policy adopted by the Ghana Government in 1988. Berekum is a Municipal located at the western part of the Ghana in the Brong-Ahafo Region. It lies between latitude 7'15' South and 8'00' North and longitudes 2'25' East and 2'50' West. The Municipality shares boundaries with Wenchi Municipal and Jaman District to the Northeast and Northwest respectively, Dormaa Municipal to the South and Sunyani Municipal to the East. Berekum, the Municipal capital is 32km and 437km North West of Sunyani, its regional capital and Accra, the national capital respectively. The human population of the Berekum Municipality in 2006 is 113,650 with population growth rate of 3.3 percent (3.3%). Its total area is nine hundred and fifty-five kilometers square (955 $km^2$) which constitutes about 0.7 percent of the entire two hundred and thirty-three thousand, five hundred and eighty-eight kilometres square (233, 588$km^2$) of Ghana's total area. The municipal's close proximity to Cote d'Ivoire is another remarkable feature which promotes economic and commercial activities between the Municipal and Cote d' Ivoire (Berekum Municipal Assembly, 2006).

The following are the many schools and other important institutions in the municipality; Nurses' Training College, College of Education, Jackson Educational Complex, Presbyterian Senior High School, Berekum Senior High School, Methodist Senior High School, Jinijini Senior High School, Star Business Senior High School and hundreds of Junior High Schools and Primary Schools can be cited in this municipality.

## 1.7 PROBLEM STATEMENT

Due to the nonexistence of libraries in most communities in Berekum Municipality, performance of students is low. Where there are libraries, the locations are poorly sited and that most library users do not get access to the library facility.

The researcher therefore intends to locate an additional library facility to serve the people of Berekum Municipality who had to travel a long distance to any of the only two existing libraries (one at Berekum and the other at Jinijini) and also people who do not have access to library at all.

## 1.8 OBJECTIVES OF THE STUDY

1. To model location of an additional library facility in the Berekum Municipal as a conditional p- center problem

2. To solve the conditional p – center problem using Berman and Drezner's algorithm.

## 1.9 METHODOLOGY

The objective of the study is to locate an additional library facility in Berekum Municipality using Berman and Drezner's algorithm. The data for this study is the road distance between the suburbs of Berekum Municipality. The suburbs of the municipal will be coded and Floyd's algorithm will be used to find the distance matrix, d (i, j) for all pairs shortest path. Berman and Drezner's algorithm will then use the distance matrix to locate an additional library facility.

The researcher will use the following resources for the study; personal laptop, Kwame Nkrumah University of Science and Technology Kumasi main library, Sunyani Polytechnic Library, the Encyclopedia Britannica 2011and the internet.

## 1.10 JUSTIFICATION OF THE STUDY

The study therefore is significant due to the following reasons:

It will help to inculcate the habit of reading in the people of the municipality.

It will improve students' performance.

It will help to reduce students' burden of travelling long distance to library.

It will serve as a reference centre for Schools and Colleges within the municipality.

## 1.11 ORGANIZATION OF THE STUDY

The thesis consists of five chapters: including this chapter, which introduces the study. Chapter 2 is on Literature Review which takes stock of what has already been written on the topic in terms of theories or concepts, scientific research studies and the overall goal of clarifying how the present study intends to address the gap silence or weakness in the existing literature. Chapter 3 explains the methodology that is being used for the study. The findings and discussions will be presented in Chapter 4. Lastly, Chapter 5 will discuss on the conclusions and recommendations.

# CHAPTER 2

## LITERATURE REVIEW

### 2.0 INTRODUCTION

To locate a specific type of facility, one usually looks for the best way to serve a set of communities whose location and demands are known. This implies one needs to consider the following:

i. The number and location of the facilities to serve the demand
ii. Size and capacity of each facility
iii. The allocation of the demand points to open facilities
iv. Optimizing some objective location function.

Most location models deals with desirable facilities, such as warehouse, service and transportation centers, emergency services, etc, which interact with the customers and where distance travel is involved. As a consequence, typical criteria for such decision include minimizing some function of the distance between facilities and/ or clients (i. e., average travel time, average response time, cost function of travel or response time, maximum travel time or cost, etc.). However, during the last two decades, those responsible for the overall development of the area, where the new facility is going to be located (i.e. central government, local authorities) as well as those living in the area (population), are showing an increasing interest in preserving the area's quality of life. Hence, new words have been introduced in the location theory, such as: noxious, obnoxious, semi obnoxious, hazardous, etc. As examples of undesirable facilities we can mention include nuclear and military installations, equipment emitting particular smell or noise, warehouses containing flammable materials, regions containing refuse or waste materials, garbage dumps, sewage plants, correctional centers, etc. The traditional optimality criterion of closeness (to locate the facility as close as possible to the customers) is replaced by the opposite criterion of how far away from the customers can the facility be placed to ensure accessibility to the demand point. This generates the NIMBY syndrome (NOT-IN-MY-BACK-YARD) (Capitivo and Climaco, 2008). Buettcher(2004), described the p-Center problem, as the Min-Max Multicenter problem or the Facility Location problem, to be a famous problem from operations research. He classified the optimization problem into three different types, depending on which of the restrictions applied.

i. The general optimization problem in which the choice of the distance function d is not restricted in any way.

ii. The metric problem in which d satisfies the triangle inequality.

iii. The metric and symmetric problem in which d(x; y) = d(y; x), and d satisfies the triangle inequality.

It was realized that, the metric, asymmetric p-Center problem had remained unstudied even ten years after its symmetric counterpart had been finally solved (by presenting an algorithm with optimal approximation factor) in 1986. In 1998, the O(log*(n)) approximation algorithm found by Panigrahy and Vishwanathan(1998) was published. Thus, it was clear that - in contrast to the general p-Center problem without any restrictions to the distance function - this problem could be approximated. And the approximation was a very good one, although the algorithm could not guarantee a constant-factor approximation. A few years later, in 2003, it turned out that this is the best approximation ratio possible (unless P = NP), (Halperin et al., 2003). So, the p-Center problem is one of the rare problems for which essentially nothing was known, then a first non-trivial algorithm was found that can approximate the problem, and it was already this very first algorithm that achieves the best approximation ratio possible. This alone is already very exciting. What makes the p-Center problem even more fascinating is the approximability of the problem. log*(n) is one of the functions where the discrepancy between theoretical results and practical consequences becomes very clear. Log2* (n) can be assumed ≤ 6 for all practical purposes. Yet, from a theoretical point of view, there is a clear distinction between constant approximation factor and log*(n).



## 2.1 SOME APPROACHES TO FACILITY LOCATION PROBLEMS

Goldengorin et al. (1999), considered the simple plant location problem. This problem often appears as a sub-problem in other combinatorial problems. Several branch and bound techniques have been developed to solve these problems. The thesis considered new approaches called branch and peg algorithms, where pegging refers to assigning values to variables outside the branching process. An exhaustive computational experiment shows that the new algorithms generate less than 60% of the number of sub-problems generated by branch and bound algorithms, and in certain cases requires less than 10% of the execution times required by branch and bound algorithms. Firstly, for each sub-problem generated in the branch and bound tree, a powerful pegging procedure is applied to reduce the size of the sub-problem. Secondly, the branching function is based on predictions made using the Beresnev function of the sub-problem at hand. They saw that branch and peg algorithms comprehensively outperform branch and bound algorithms using the same bound, taking on the average, less than 10% of the execution time of branch and bound algorithms when the transportation cost matrix is dense. The main recommendation from the results of the experiment is that branch and peg algorithms should be used to solve SPLP instances.

Erkut and Neuman (1992), presented a mixed integer linear model for undesirable facility location. The objectives considered are total cost minimization, total opposition minimization and equity minimization.

Caruso et al. (1993), presented a model for planning an urban solid waste management system. Incineration, composition and recycling are considered for the processing phase and sanitary landfills are considered for the disposal phase. Heuristic techniques (embedded in the reference point approximation) are used to solve the model and, as a consequence, “approximate Pareto solutions” are obtained. By varying the reference point, different solutions can be obtained. The results for a case study (Lombardy region in Italy) are presented and discussed.

Wyman and Kuby (1993, 1995), presented a multi-objective mixed integer programming model for the location of hazardous material facilities (including the technologies choice variable) with three objectives functions (cost, risk and equity).

Melachrinoudis et al. (1995), propose a dynamic multi-period capacitated mixed integer programming model for the location of sanitary landfills.

Fonseca and Captivo (1996; 2006; 2007), studied the location of semi obnoxious facilities as a discrete location problem on a network. Several bi-criteria models are presented considering two conflicting objectives, the minimization of obnoxious effect and the maximization of the accessibility of the community to the closest open facility. Each of these objectives is considered in two different ways, trying to optimize its average value over all the communities or trying to optimize its worst value. The Euclidean distance is used to evaluate the obnoxious effect and the shortest path distance is used to evaluate the accessibility. The obnoxious effect is considered inversely proportional to the weighted Euclidean distance between demand points and open facilities, and demand directly proportional to the population in each community. All the models are solved using Chalmet et al. (1986), non- interactive algorithm for Bi-criteria Integer Linear Programming modified to an interactive procedure by Ferreira et al. (1994). Several equity measures are computed for each non-denominated solution presented to the decision-maker, in order to increase the information available to the decision –maker about the set of possible solutions.

Moscibroda and Wattenhofer(2005), initiated the studied of the approximability of the facility location problem in a distributed setting. In particular, they explore a trade between the amount of communication and the resulting approximation ratio. The authors give a distributed algorithm that, for every constant $k$, achieves an $O(pk(m½)1=pk\log\ (m+n))$ approximation in $O(k)$ communication rounds where message size is bounded to $O(\log n)$bits. The number of facilities and clients are $m$ and $n$, respectively, and ½ is a coefficient that depends on the cost values of the instance. Their technique is based on a distributed primal-dual approach for approximating a linear program, that does not form a covering or packing program.

Ferreira et al. (1996), proposed a bi-criteria mixed integer linear model for the facility location where the objectives are the minimization of total cost and the minimization of environmental pollution at facility sites. The interactive approach of Ferreira et al (1994) is used to obtain and analyze non-dominated solutions.

Giannikos (1998), presented a discrete model for the location of disposal or treatment facilities and transporting hazardous waste through a network linking the population centers that produce the waste and the candidate locations for the treatment facilities method to choose the location for a waste treatment facility in a region of Finland.

Krivitski et al. (2005), addressed a well-known facility location problem (FLP) in a sensor network environment. The problem deals with finding the optimal way to provide service to a (possibly) very large number of clients. They show that a variation of the problem can be solved using a *local* algorithm. Local algorithms are extremely useful in a sensor network scenario. This is because they allow the communication range of the sensor to be restricted to the minimum, they can operate in routerless networks, and they allow complex problems to be solved on the basis of very little information, gathered from nearby sensors. The local facility location algorithm presented is entirely asynchronous, seamlessly supports failures and changes in the data during calculation, poses modest memory and computational requirements, and can provide an anytime solution which is guaranteed to converge to the exact same one that would be computed by a centralized algorithm given the entire data.

Costa et al. (2008), developed two bi-criteria models for single allocation hub location problems. In both models the total cost is the first criteria to be minimized. Instead of using capacity constraints to limit the amount of flow that can be received by the hubs, a second objective function is used, trying to minimize the time to process the flow entering the hubs. In the first model, total time is considered as the second criteria and, in the second model, the maximum service time for the hubs are minimized. Non-dominated solutions are generated using an interactive decision-aid approach developed for bi-criteria integer linear programming problems. Both bi-criteria models are tested on a set of instances, analyzing the corresponding non-dominated solutions set and studying the reasonableness of the hubs flow charge for these non-dominated solutions.

Ballou (1998), discusses a selected number of facility location methods for strategic planning. He further classifies the more practical methods into a number of categories in the logistics network, which include single–facility location, multi–facility location, dynamic facility location, retail and service location.

Christopher and Wills (1972), comprehensively present that whether the problem of depot location is static or dynamic, "Infinite Set" approaches and "Feasible Set" approach can be identified. The infinite set approach assumes that a warehouse is flexible to be located anywhere in a certain area. The feasible set approach assumes that only a finite number of known sites are available as warehouse locations. They believe the centre of gravity method is a sort of infinite set model.

Ballou (1998), stated that exact centre of gravity approach is simple and appropriate for locating one depot in a region, since the transportation rate and the point volume are the only location factors. Given a set of points that represent source points and demand points, along with the volumes needed to be moved and the associated transportation rates, an optimal facility location could be found through minimizing total transportation cost. In principle, the total transportation cost is equal to the volume at a point multiplied by the transportation rate to ship to that point multiplied by the distance to that point. Furthermore, Ballou outlines the steps involved in the solution process in order to implement the exact centre of gravity approach properly.

## 2.2 P- CENTRE LOCATION PROBLEM

The conditional location problem is to locate p new facilities to serve a set of demand points given that q facilities are already located. When q is equal to zero (q = 0), the problem is unconditional. In conditional p – center problems, once the new p locations are determined, a demand can be served either by one of the existing or by one of the new facilities whichever is the closest facility to the demand (Berman, 2008). The *p*-center problem seeks the location of *p* facilities. Each demand point receives its service from the closest facility. The objective is to minimize the maximal distance for all demand points. The *p*-center problem consists of choosing *p* facilities among a set of *M* possible locations and assigning *N* clients to them in order to minimize the maximum distance between a client and the facility to which it is allocated.

Elloumi et al. (2004), presented a new integer linear programming formulation for this min-max problem with a polynomial number of variables and constraints, and show that its LP relaxation provides a lower bound tighter than the classical one. Moreover, they showed that an even better lower bound $LB^*$, obtained by keeping the integrality restrictions on a subset of the variables, can be computed in polynomial time by solving at most $O(\log 2(NM))$ linear programs, each having $N$ rows and $M$ columns. They also show that, when the distances satisfy triangle inequalities, $LB^*$ is at least one third of the optimal value. Finally, they used the $LB^*$ in an exact solution method and report extensive computational results on test problems from the literature. For instances where the triangle inequalities are satisfied, their method out performs the running time of other recent exact methods by an order of magnitude. In addition, it is the first one to solve large instances of size up to $N = M = 1, 817$.

Krumke(1995), considered the generalization of the *p*-Center Problem, which is called the α-*Neighbor p-Center Problem* ($p - CENTER^{(\alpha)}$). Given a complete edge-weighted network, the goal is to minimize the maximum distance of a client to it's α nearest neighbor in the set of *p* centers. He shows that in general finding a $O$ ($2^{poly(|V|)}$)-approximation for$p - CENTER^{(\alpha)}$ is NP-hard(Garey and Johnson, 1979), where $|V|$ denotes the number of nodes in the network. If the distances are required to satisfy the triangle inequality, there can be no polynomial time approximation algorithm with a $(2 - \varepsilon)$ performance guarantee for any fixed $\varepsilon > 0$ and any fixed $\alpha \leq p$, unless *P* = *NP*. For this case, He presented a simple yet efficient algorithm that provides a 4-approximation for α≥ 2. Considering the *p*-Center Problem with Connectivity Constraint, let *G* (*V*, *E*, *W*) be a graph with *n*-vertex-set *V* and *m*-edge-set *E* in which each edge *e* is associated with a positive distance *W* (*e*).

Chung-Kung et al. (2006), proposed an additional practical constraint which restricted the *p* vertices, to be connected. The resulting problem is called the connected p-Center problem (the C*p*C problem). They first show that the C*p*C problem is NP-Hard on bipartite graphs and split graphs. Then, an *O* (*n*)-time algorithm for the problem on trees is proposed. Finally, the algorithm was extended to trees with forbidden vertices. That is some vertices in *V* cannot be selected as center vertices, and the time-complexity is also *O* (*n*). Meanwhile, it was identified that other variants of the traditional *p*-Center problem is also a very important issue. For example, just restricting that the *p*-center must be "total", thus, the subgraph induced by the *p*-center has no isolated vertices, is another typical practical variant.

Chen and Chen (2009), presented a new relaxation algorithm for solving the conditional continuous and discrete p-center problems. In the continuous p-center problem, the location of the service facilities can be anywhere in the two-dimensional Euclidean space. In the discrete variant there is a finite set of potential service points to choose from. An analogous representation of the discrete p-center problem is the p-center problem on networks. In the p-center problem on networks, both the demand points and the potential service points are located on a weighted undirected graph, and the distance between any two points is the cost of the shortest path between them. They assumed that, there are a finite number of values for the optimal solution of an unconditional p-center problem. They use the assumption to implement the subroutine Get- Next Bound (Lower-Bound) which returns the smallest value, among the possible values for the optimal solution, which is greater than Lower-Bound. Also the subroutine Find Feasible Solution (Sub, r), which answers the question: ``is there a solution to the sub-problem with value less than r?" (And if so, finds such a solution).

Hassin et al. (2003), introduced a local search strategy that suits combinatorial optimization problems with a min-max (or max-min) objective. According to this approach, solutions are compared lexicographically rather than by their worst coordinate. They apply this approach to the *p*-center problem. Based on a computational study, the lexicographic local search proved to be superior to the ordinary local search. This superiority was demonstrated by a worst-case analysis.

Cheng et al. (2005), worked on the Improved Algorithm for the p-Center Problem on Interval Graphs with Unit Lengths. They presented an O (n) time algorithm for the problem under the assumption that the endpoints of the intervals are sorted, which improves on the existing best algorithm for the problem that has a run time of O (pn).

They modeled the network as a graph G = (V, E), where V is the vertex set with |V | = n and E is the edge set with |E| = m. it was assumed that, the demand points coincide with the vertices, and the location of the facilities was restricted to the vertices. Also they assumed that each edge of E has a unit length. It remains an interesting question whether they could develop an approximation algorithm for the p-center problem on interval graphs with general edge lengths.

## 2.3 CONDITIONAL LOCATION PROBLEM

Minieka (2006), stated that, previous treatments of location problems on a graph have been confined to the optimum location of a single facility or the simultaneous optimum location of multiple facilities. The author addresses the problem of optimally locating a facility on a graph when one or more other facilities have already been located in the graph. The author shows that previous solution techniques can be reused if the distances in the graph are judiciously redefined.

Tamir et al. (2005), deal with the location of extensive facilities on trees, both discrete and continuous, under the condition that existing facilities are already located. They require that the selected new server is a subtree, although we also specialize to the case of paths. They study the problem with the two most widely used criteria in Location Analysis: center and median. Their main results under the center criterion are nestedness properties of the solution and subquadratic algorithms for the location of paths and subtrees. For the case of the median criterion they prove that unlike the case where there is no existing facility, the continuous conditional median sub-tree problem is NP-hard and we develop a corresponding fully polynomial approximation algorithm. They also present subquadratic algorithms for almost all other models.

Wouter et al. (2011), contributed to conditional location by writing; within research on world cities, much attention has been paid to Advanced Producer Services (APS) and their role within both global urban hierarchies and network formation between cities. What is largely ignored is that these APS provide services to firms operating in a range of different sectors. Does sector-specific specialization of advanced producer services influence the economic geography of corporate networks between cities? If so, what factors might explain this geographical pattern? This paper investigates these theoretical questions by empirically focusing on those advanced producer services related to the port and maritime sector. The empirical results show that the location of AMPS is correlated with maritime localisation economies, expressed in the presence of ship owners and port-related industry as well as APS in general, but not by throughput flows of ports. Based upon the findings, policy recommendations are addressed.

Berman and Simchi (2011), described an algorithm to solve conditional location problems (such as the conditional *p*-median problem or the conditional *p*-center problem) on networks, where demand points are served by the closest facility whether existing or new. This algorithm requires the one-time solution of a ($p$ + 1)-unconditional corresponding location problem using an appropriate shortest distance matrix.

Berman and Drezner (2007), discuss the conditional p-median and p-center problems on a network. Demand nodes are served by the closest facility whether existing or new. Rather than creating a new location for an artificial facility and force the algorithm to locate a new facility there by creating an artificial demand point, the distance matrix was just modified. They suggested solving both conditional problems by defining a modified shortest distance matrix.

# CHAPTER 3

# METHODOLOGY

## 3.0 SHORTEST PATH PROBLEM

In graph theory, the shortest path problem is the problem of finding a path between two vertices(nodes) in a graph such that the sum of the weights of its constituent edges is minimized.

An example is finding the quickest way to get from one location to another on a road map; in this case, the vertices represent locations and the edges represent segments of road and are weighted by the time needed to travel that segment.

### 3.1 SHORTEST PATH FROM A SOURCE

#### 3.1.1 DIJKSTRA'S ALGORITHM

Dijkstra's algorithm, conceived by Dutch computer scientist Edsger Dijkstra in 1956 and published in 1959, is a graph search algorithm that solves the single-source shortest path problem for a graph with nonnegative edge path costs, producing a shortest path tree. This algorithm is often used in routing and as a subroutine in other graph algorithms. For a given source vertex (node) in the graph, the algorithm finds the path with lowest cost (i.e. the shortest path) between that vertex and every other vertex. It can also be used for finding costs of shortest paths from a single vertex to a single destination vertex by stopping the algorithm once the shortest path to the destination vertex has been determined. For example, if the vertices of the graph represent cities and edge path costs represent driving distances between pairs of cities connected by a direct road, Dijkstra's algorithm can be used to find the shortest route between one city and all other cities. As a result, the shortest path first is widely used in network routing protocols, most notably IS-IS and OSPF (Open Shortest Path First).

### 3.1.2 Algorithm

Dijkstra's algorithm will assign some initial distance values and will try to improve them step by step.

Step 1. Assign to every node a tentative distance value: set it to zero for our initial node and to infinity for all other nodes.

Step 2. Mark all nodes except the initial node as unvisited. Set the initial node as current. Create a set of the unvisited nodes called the unvisited set consisting of all the nodes except the initial node.

Step 3. For the current node, consider all of its unvisited neighbors and calculate their tentative distances. For example, if the current node A is marked with a distance of 6, and the edge connecting it with a neighbour B has length 2, then the distance to B (through A) will be 6+2=8. If this distance is less than the previously recorded distance, then overwrite that distance. Even though a neighbour has been examined, it is not marked as visited at this time, and it remains in theunvisited set.

Step 4. After considering all of the neighbours of the current node, mark the current node as visited and remove it from the unvisited set. A visited node will never be checked again; its distance recorded now is final and minimal.

Step 5. The next current node will be the node marked with the lowest (tentative) distance in the unvisited set.

Step 6. If the unvisited set is empty, then stop. The algorithm has finished. Otherwise, set the unvisited node marked with the smallest tentative distance as the next "current node" and go back to step 3.

**Solved Example of Dijkstra's algorithm**

In Figure 3.1 below, A, B, C D and E are nodes and the numbers on the edges are the distances between the nodes.

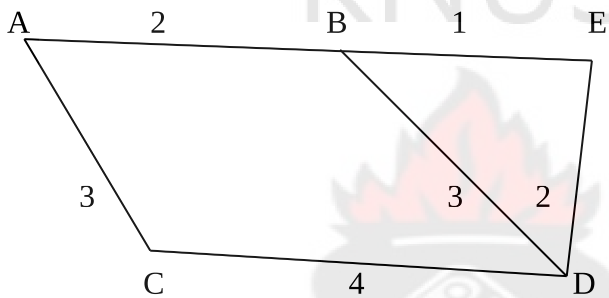
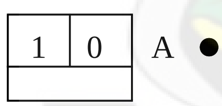


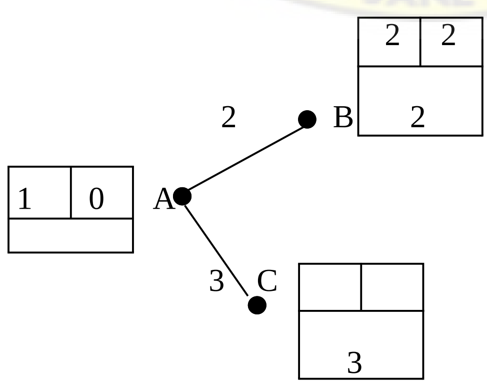

Figure 3.1  Network for Dijkstra's Example

**Solution**

Assign a permanent label 0 to the starting vertex, A

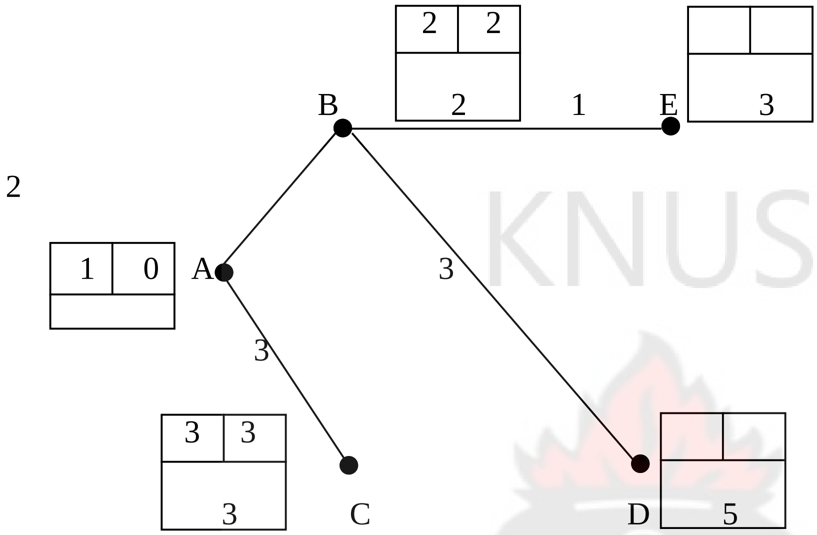



Assign temporary labels to nodes B and C which are directly connected to vertex A. Vertex B is the smallest among all temporary labels so permanently label vertex B as 2.

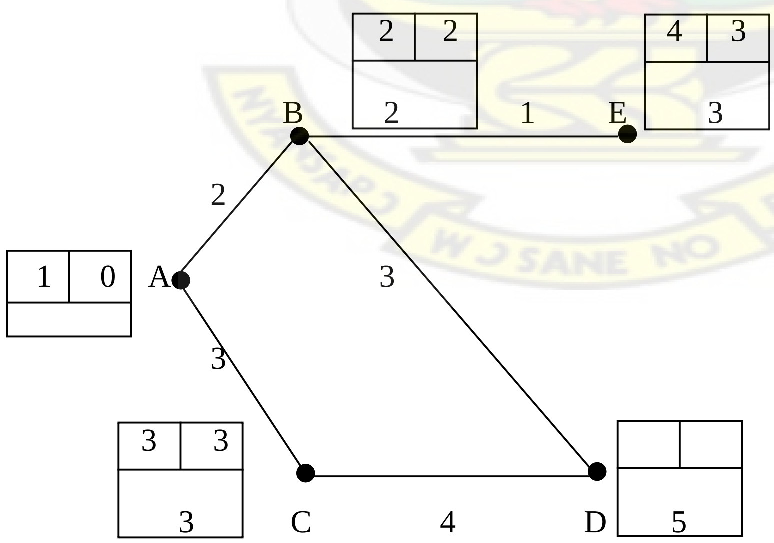


Assign temporary labels to nodes D and E which are directly connected to vertex B.

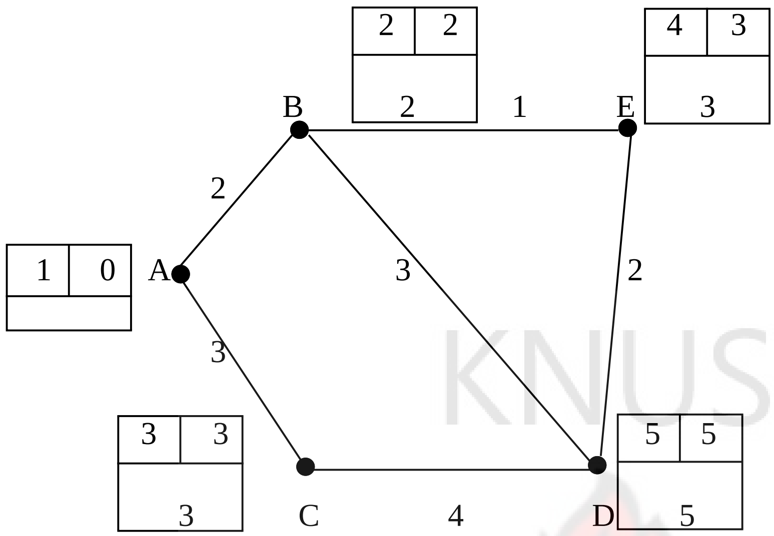
A → B → E = 3, A → C = 3 and A→ B→ D = 5. Since A→ B→ E =3 and A→ C =3, are the same arbitrary permanently assign 3 to vertex C.




Assign temporary labels to nodes which are directly connected to vertex C. A → B → E = 3, A → C → D =9 and A → B → D = 5. Since A → B → E = 3 is smaller than A → B → D = 5 and B → C → D = 9, permanently assign 4 to vertex E.



Finally, assign a permanent label of 5 to node D.




From above, the shortest distance between nodes A and B is 2, that of C and E is 6. The table below summarizes the results.

Table 3.1Distance Matrix, *D* using Dijkstra's algorithm.

| Demand Nodes | Potential Location | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | E |
| A | 0 | 2 | 3 | 5 | 3 |
| B | 2 | 0 | 5 | 3 | 1 |
| C | 3 | 5 | 0 | 4 | 6 |
| D | 5 | 3 | 4 | 0 | 2 |
| E | 3 | 1 | 6 | 2 | 0 |

## 3.2 ALL-PAIR SHORTEST PATH PROBLEM

### 3.2.1 FLOYD-WARSHALLALGORITHM

The Floyd-Warshall algorithm is an efficient matrix method algorithm to find all- pair shortest paths on a graph. That is, it is guaranteed to find the shortest path between every pair of vertices in a graph. The graph may have negative weight edges, but no negative weight cycles (for then the shortest path is undefined).

This algorithm can also be used to detect the presence of negative cycles; the graph has a negative cycles if at the end of the algorithm, the distance from a vertex $v$ to itself is negative.

### 3.2.2 Algorithm

The Floyd-Warshall algorithm is an application of Dynamic Programming.

Let dist$(k, i, j)$ be the length of the shortest path from $i$ and $j$ that uses only the vertices 1,2,3,..........,$k$ as intermediate vertices in N x N graph matrix . The following recurrence:

Step 1.$k = 0$is our base case, thus $dist(0, i, j\ )$ is the length of the edge from vertex $i$ to vertex $j$if it exists and infinite ($\infty$) otherwise.

Step 2, using $dist(0, i, j)$, it then computes $dist(1, i, j)$ for all pairs of nodes $i$ and$j$.

Step 3.using$dist(1, i, j)$, it then computes $dist(2, i, j)$, for all pairs of nodes $i$ and $j$. It then repeats the process until it obtains $dist(k, i, j)$ for all node pairs $i$ and$j$ when it terminates. The algorithm computes; $dist(k, i, j) = \min\big(dist(k-1, i, k) + dist(k-1, k, j), dist(k-1, i, j)\big)$; for any vertex $i$ and vertex $j$, the length of the shortest path from $i$ to$j$ with all intermediate vertices $\leq k$ simply does not involve the vertex $k$ at all (in which case it is the same as dist ($k$ – 1, $k$ ,$j$)),or that the shorter path goes through vertex $k$, so the shortest path between vertex $i$and vertex $j$ is the combination of the path from vertex $i$ to $k$, and from vertex $k$ to $j$.

After $N$ iterations, there is no need any more to go through any more intermediate vertices, so the distance dist ($N, i, j$) represents the shortest distance between $i$and $j$.

**Solved Example of Floyd's Algorithm**

Using Figure 3.1 as an example for Floyd's Algorithm, the matrix below is formed from the network.

Table 3.2 Matrix Representation of Figure 3.1

| Nodes | A | B | C | D | E |
|---|---|---|---|---|---|
| A | 0 | 2 | 3 | ∞ | ∞ |
| B | 2 | 0 | ∞ | 3 | 1 |
| C | 3 | ∞ | 0 | 4 | ∞ |
| D | ∞ | 3 | 4 | 0 | 2 |
| E | ∞ | 1 | ∞ | 2 | 0 |

**<u>From node A</u>**

From node A to B

The direct distance between nodes A and B is 2

$A \xrightarrow{3} C \xrightarrow{\infty} B = 3 + \infty = \infty$

$A \xrightarrow{\infty} D \xrightarrow{3} B = \infty + 3 = \infty$

$A \xrightarrow{\infty} E \xrightarrow{1} B = \infty + 1 = \infty$

The min { ∞} = ∞ which is not less than 2 (the direct distance between node A and B). Therefore 2 in the cells [1 2] and [2 1] is retained.

From node A to C

The direct distance between nodes A and C is 3

$A \xrightarrow{2} B \xrightarrow{\infty} C = 2 + \infty = \infty$

$A \xrightarrow{\infty} D \xrightarrow{4} C = \infty + 4 = \infty$

$A \xrightarrow{\infty} E \xrightarrow{\infty} C = \infty + \infty = \infty$

The min $\{\infty\} = \infty$ which is not less than 3 (the direct distance between nodes A and C). Therefore the value of 3 is retain in the cells [1 3] and [3 1].

From node A to D

The direct distance between nodes A and D is $\infty$

$A \xrightarrow{2} B \xrightarrow{3} D = 2 + 3 = 5$

$A \xrightarrow{3} C \xrightarrow{4} D = 3 + 4 = 7$

$A \xrightarrow{\infty} E \xrightarrow{2} D = \infty + 2 = \infty$

The min $\{5, 7, \infty\} = 5$ which is less than $\infty$ (the direct distance between nodes A and D). Therefore $\infty$ in the cells [1 4] and [4 1] is replaced by 5.

From node A to E

The direct distance between nodes A and E is $\infty$

$A \xrightarrow{2} B \xrightarrow{1} E = 2 + 1 = 3$

$A \xrightarrow{3} C \xrightarrow{\infty} E = 3 + \infty = 4$

$A \xrightarrow{5} D \xrightarrow{2} E = 5 + 2 = 7$

The min $\{3, 4, 7\} = 3$ which is less than $\infty$ (the direct distance between nodes A and E). Therefore $\infty$ in the cells [1 5] and [5 1] is replaced by 3.

**<u>From node B</u>**

From node B to C

The direct distance between nodes B and C is $\infty$

$B \xrightarrow{2} A \xrightarrow{3} C = 2 + 3 = 5$

$B \xrightarrow{3} D \xrightarrow{4} C = 3 + 4 = 7$

$B \xrightarrow{1} E \xrightarrow{\infty} C = 1 + \infty = \infty$

The min {5,7,∞} = 5 which is less than ∞ (the direct distance between nodes B and C). Therefore ∞ in the cells [2 3] and [3 2] is replaced by 5.

From node B to D

The direct distance between nodes B and D is 3

$B \xrightarrow{2} A \xrightarrow{5} D = 2 + 5 = 7$

$B \xrightarrow{\infty} C \xrightarrow{4} D = \infty + 4 = \infty$

$B \xrightarrow{1} E \xrightarrow{2} D = 1 + 2 = 3$

The min {7, ∞,3} = 3 which is not less than 3 (the direct distance between nodes B and D). Therefore 3 in the cells [2 4] and [4 2] is retained.

From node B to E

The direct distance between nodes B and E is 1

$B \xrightarrow{2} A \xrightarrow{3} E = 2 + 3 = 5$

$B \xrightarrow{5} C \xrightarrow{\infty} E = 5 + \infty = \infty$

$B \xrightarrow{3} D \xrightarrow{2} E = 3 + 2 = 5$

The min {5, ∞,} = 5 which is not less than 1 (the direct distance between nodes B and E). Therefore 3 in the cells [2 5] and [5 2] is retained.

**<u>From node C</u>**

From node C to D

The direct distance between nodes C and D is 4

$C \xrightarrow{3} A \xrightarrow{3} D = 3 + 3 = 6$

C $\xrightarrow{5}$ B $\xrightarrow{3}$ D = 5 + 3 = 8

C $\xrightarrow{\infty}$ E $\xrightarrow{2}$ D = ∞+ 2 = ∞

The min {6, 8, ∞} = 6 which is not less than 4 (the direct distance between nodes C and D). Therefore 4 in the cells [3 4] and [4 3] is retained.

From node C to E

The direct distance between nodes C and E is ∞

C $\xrightarrow{3}$ A $\xrightarrow{3}$ E = 3 + 3 = 6

C $\xrightarrow{5}$ B $\xrightarrow{1}$ E = 5 + 1 = 6

C $\xrightarrow{4}$ D $\xrightarrow{2}$ E = 4 + 2 = 6

The min {6} = 6 which is less than ∞ (the direct distance between nodes C and E). Therefore ∞ in the cells [3 5] and [5 3] is replaced by 6.

**<u>From node D</u>**

From node D to E

The direct distance between nodes D and E is 2

D $\xrightarrow{5}$ A $\xrightarrow{3}$ E = 5 + 3 = 8

D $\xrightarrow{3}$ B $\xrightarrow{1}$ E = 3 + 1 = 4

D $\xrightarrow{4}$ C $\xrightarrow{6}$ E = 4 + 6 = 10

The min {8, 4, 10} = 4 which is not less than 2 (the direct distance between nodes D and E). Therefore 2 in the cells [4 5] and [5 4] is retained.

Table 3.3 Distance Matrix, $D$ using Floyd.

| Demand Nodes | Potential Location | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | E |
| A | 0 | 2 | 3 | 5 | 3 |
| B | 2 | 0 | 5 | 3 | 1 |
| C | 3 | 5 | 0 | 4 | 6 |
| D | 5 | 3 | 4 | 0 | 2 |
| E | 3 | 1 | 6 | 2 | 0 |

## 3.3 NETWORK LOCATION MODELS

Network location problems are concerned with finding the right locations to place one or more facilities in a network of demand points, (customer locations) represented by nodes in the network, that optimize a certain objective function related to the distance between the facilities and the demand points.

## 3.4 BASIC FACILITY LOCATION MODELS

This section presents models classified according to their consideration of distance. The maximum distance models and total (or average) distance.

### 3.4.1 TOTAL OR AVERAGE DISTANCE MODELS

Many facility location planning situations in the public and private sections are concerned with the total travel distance between facilities and demand nodes. An example in the private sector might be the location of production facilities that receive their inputs from established sources by truckload deliveries. In the public sector, one might want to locate a network of service providers such as licensing bureaus in such a way as to minimize the total distance that customers must traverse to reach their closest facility. This approach may be viewed as an "efficiency" objective as opposed to the "equity" objective of minimizing the maximum distance, which is mentioned in other models.

1. **P-median problem**:

The p-median model (Hakimi, 1964; 1965) finds the locations of p facilities to minimize the demand-weighted total distance between demand nodes and the facilities to which they are assigned.

2. **The Maxisum Location Problem:**

The maxisum location problem seeks the locations of p facilities such that the total demand-weighted distance between demand nodes and the facilities to which they are assigned is maximized.

### 3.4.2 MAXIMUM DISTANCE MODELS

In some locations problems, an acceptable distance is set a priori. In the facility location literature, a priori acceptable distances such distances known as "covering" distances. Demand within the covering distance of its closest facility is considered "covered." An underlying assumption of this measure of covering distance is that demand is fully satisfied if the nearest facility is within the coverage distance and is not satisfied if the closest facility is beyond that distance.

1. **Set covering location model:**

The objective of this model is to locate the minimum number of facilities required to "cover" all of the demand nodes (Toregas et al., 1971).

2. **Maximal covering location problem:**

The objective of the Maximal covering location problem (MCLP) is to locate a predetermined number of facilities, p, in such a way as to maximize the demand that is covered. Thus, the MCLP assumes that there may not be enough facilities to cover all of the demand nodes. If all nodes cannot be covered, then the model seeks the sitting scheme that covers the most demand (Church and ReVelle, 1974).

3. **The p-dispersion problem**:

The p-dispersion problem (PDP) is only concerned with the distance between new facilities and the objective is to maximize the minimum distance between any pair of facilities. Potential applications of the PDP include the sitting of military installations where separation makes them more difficult to attack or locating franchise outlets where separation reduces cannibalization among stores (Kuby, 1987).

4. **P-Center Problem**:

The p-center problem (Hakimi, 1964;1965) addresses the problem of minimizing the maximum distance that demand is from its closet facility given that we are sitting a pre-determined number of facilities. There are several possible variations of the basic model. The "vertex" p-center problem restricts the set of candidate facility sites to the nodes of the network while the "absolute" p-center problem permits the facilities to be anywhere along the arcs or the network.

Both versions can be either weighted or unweighted. In the unweighted problem, all demand nodes are treated equally. In the weighted model, the distances between demand nodes and facilities are multiplied by a weight associated with the demand node. For example, this weight might represent a node's importance or, more commonly, the level of its demand.

### 3.5 THE P-CENTER PROBLEM

The *p*-center problem is the problem of locating *p* (facilities) in order to minimizes the maximum response time (the time between a demand site and the nearest facility), using a given number of $p$. With the above definition and the decision variable;

$W$ = The maximum distance between a demand node and the facility to which is assigned.

$$y_{ij} = \begin{cases} 1 \text{ if the demand node is assigned to a facility at node } j \\ 0 \text{ if not} \end{cases}$$

The p – centre problem therefore can be formulated as follows:

Minimize $W$ …………………………………………….. (1)

Subject to:

$$\sum_{j \in J} x_j = p \text{ ………………………..……….. } (2)$$

$$\sum_{j \in J} x_j = 1 \quad \forall i \in I \text{ ……………………….. } (3)$$

$$y_{ij} - x_j \leq 0 \quad \forall i \in I, j \in J \text{ ………………… } (4)$$

$$W - \sum_{j \in J} h_i d_{ij} \geq 0 \quad \forall_i \in I \text{ ………………… } (5)$$

$$x_j \in \{0,1\} \quad \forall_j \in J \text{ ……………………….. } (6)$$

$$y_{ij} \in \{0,1\} \quad \forall i \in I, j \in J \text{ ………………….. } (7)$$

The objective function (1) minimizes the maximum demand – weighted distance between each demand node and its closet open facility. Constraint set (2) stipulates that $p$ facilities are to be located. Constraint set (3) requires that each demand node be assigned to exactly one facility. Constraint set (4) restricts demand node assignments only to open facilities. Constraint set (5) defines the lower bound on the maximum demand – weighted distance, which is being minimized. Constraint set (6) established the sitting decision variable as binary. Constraint set (7) requires the demand at a node to be assigned to one facility only. Constraint set (7) can be replaced by $y_{ij} \geq 0 \ \forall i \in I, j \in J$ because constraint set (4) guarantees that $y_{ij} \leq 1$. If some $y_{ij}$ are fractional, we simply assign node $i$ to its closet open facility (Current et al, 2001).

### 3.6 THE CONDITIONAL P-CENTER PROBLEM

The conditional location problem is to locate $p$ new facilities to serve a set of demand points given that $q$ facilities are already located. When $q$ = 0, the problem is unconditional. In the conditional $p$-center problems, once the new $p$ locations are determined, a demand can be served either by one of the existing or by one of the new facilities whichever is the closest facility to the demand. Consider a network

$G = (N, L)$ Where; $N$ = the set of nodes, $|N| = n$

$L$ = the set of links.

Let $d(x, y)$ be the shortest distance between any $x, y \in G$. Suppose that there is a set $Q$

($|Q| = q$) of existing facilities. Let $Y = (Y_1, Y_2, Y_3, \dots, Y_q)$ and $X = (X_1, X_2, X_3, \dots, X_p)$ be vectors of size $q$and $p$ respectively, where$Y_i$ is the location of existing facility$i$ and $X_i$ is the location of new facility $i$. Without any loss of generality we do not need to assume that $Y_i \in N$. The conditional $p$-center location problem is to;

$Min[G(x) = \max_{i=1,\dots,n} \min\{d(X,i), d(Y,i)\}]$ where $(X,i)$ and $(Y,i)$, is the shortest distance from the closet facility in $X$ and $Y$ respectively to the node $i$, ( Berman and Simchi-Levi, 1990)



### 3.7 BERMAN AND SIMCHI-LEVI ALGORITHM

Berman and Simchi-Levi (1990), suggested to solve the conditional p-center problem on a network by an algorithm that requires one-time solution of an unconditional (p + 1)-center problem.

#### 3.7.1 Algorithm

Step 1: Let$D$ be a distance matrix with rows corresponding to demands and columns corresponding to potential locations. For the $p$-center problem the columns of $D$ correspond to the set of local centers. The idea is to create a new potential location representing all existing facilities. If a demand point is utilizing the services of an existing facility, it will use the services of the closest existing facility. Therefore, the distance between a demand point and the new location is the minimum distance calculated for all existing facilities.

Step 2: To force the creation of a facility at the new location, a new demand point is created with a distance of zero to the new potential location and a large distance to all other potential locations. The new distance matrix, $\hat{\hat{D}}$ is constructed by adding a new location $a_o$ (a new column) to$D$ so that the columns represent the Q existing locations and a new demand point

$v_0$ with an arbitrary positive weight. For each demand point (node) $i, d(i, a_0) = min_{k\in Q}\{d_{ik}\}$ and $d(v_0, a_0) = 0$. For each potential location (node) $j, d(v_0, j) = M(M$ is a large number). Again the nodes in Q and potential locations Q are removed.

Step 3: Find the optimal new location using $\hat{\hat{D}}$ for the network with the objective function

$$Min[G(x) = \max_{i=1,\dots,n} \min\{d(X,i), d(Y,i)\}]$$



**Illustrative example of Berman and Simchi - Levi Algorithm**

Step 1: Table 3.4 below is all pair shortest path (distance matrix), D obtained by using Floyd on figure 3.1. Suppose that nodes B and C are the existing set of facilities and an additional one facility is to be located ( i.e. p = 1).

Column 1 and row 1 in the Table below represent demand nodes and potential location respectively.

Table 3.4 All pair shortest paths distance matrix, *D*

| Demand Nodes | Potential Location | | | | |
|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **E** |
| **A** | 0 | 2 | 3 | 5 | 3 |
| **B** | 2 | 0 | 5 | 3 | 1 |
| **C** | 3 | 5 | 0 | 4 | 6 |
| **D** | 5 | 3 | 4 | 0 | 2 |
| **E** | 3 | 1 | 6 | 2 | 0 |

Step 2: Determine the modified shortest distance matrix, $\hat{\hat{D}}$ by adding a new $a_o$( a new column) to $D$ and adding a new demand point $v_0$( a new row) with an arbitrary positive weight to the rows. For each demand point (node) $i$, $d(i, a_0) = min_{k\in Q}\{d_{ik}\}$ and $d(v_o, a_o) = 0$. For each potential location (node) $j$, $d(v_0, j) = M(M$ is a large number) as shown in Table 3.5 below.

Table 3.5 Modified Distance Matrix, $\hat{\hat{D}}$

| Demand Nodes | Potential Location | | | | | |
|---|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **D** | $\boldsymbol{a_0}$ |
| **A** | 0 | 2 | 3 | 5 | 3 | 2 |
| **B** | 2 | 0 | 5 | 3 | 1 | 0 |
| **C** | 3 | 5 | 0 | 4 | 6 | 0 |
| **D** | 5 | 3 | 4 | 0 | 2 | 3 |
| **E** | 3 | 1 | 6 | 2 | 0 | 1 |
| $\boldsymbol{v_0}$ | M | M | M | M | M | 0 |



Nodes B and C in set $Q$ been the nodes with existing facility are removed and presented in Table 3.6 below.

Table 3.6 Modified Distance Matrix, $\hat{\hat{D}}$ with nodes B and C removed

| Demand Nodes | Potential Location | | | |
|---|---|---|---|---|
| | **A** | **D** | **E** | $\boldsymbol{a_0}$ |
| **A** | 0 | 5 | 3 | 2 |
| **D** | 5 | 0 | 2 | 3 |
| **E** | 3 | 2 | 0 | 1 |
| $\boldsymbol{v_0}$ | M | M | M | 0 |

Step 3: Find the optimal new location using the distance matrix $\hat{\hat{D}}$ and objective function, $Min[G(x) = \max_{i=1,\dots,n} \min\{d(X,i), d(Y,i)\}]$

With $X = \{A, D, E\}$ and $Y = \{B, C\}$



**For $\boldsymbol{X = A}$**

$i = A$, we have

$min\{d(X,i), d(Y_A,i), d(Y_B,i)\}$

$min\{d(A,A), d(B,A), d(C,A)$

$min(0,2,3) = 0$

$i = B$, we have

$min\{d(X,i), d(Y_A,i), d(Y_B,i)\}$

$min\{d(A,B), d(B,B), (C,B)\}$

$\min(2,0,5) = 0$

$i = C$, we have

$min\{d(X, i), d(Y_A, i), d(Y_B, i)\}$

$min\{d(A, C), d(B, C), d(C, C)\}$

$min(3, 5, 0) = 0$

$i = E$, we have

$min\{d(X, i), d(Y_A, i), d(Y_B, i)\}$

$min\{d(A, E), d(B, E), (C, E)\}$

$\min(3, 1, 4) = 1$

$i = D$, we have

$min\{d(X, i), d(Y_A, i), d(Y_B, i)\}$

$min\{d(A, D), d(B, D), d(C, D)\}$

$min(5, 3, 4) = 3$

From the calculations above, maximum of $\min\{d(X,i), d(Y,i)\}]$ at $X = A$ is 3, at$X = D$ , the maximum is 2 and that of $X = E$ is also 2.

The results are then summarized and presented in Table 3.7 with column 5 representing the maximum distance on each row.

Table 3.7 Optimal Location$Min(g(x))$ using $\hat{\hat{D}}$

| Demand Nodes | **A** | **D** | **E** | Maximum |
|---|---|---|---|---|
| **A** | 0 | 3 | 1 | 3 |
| **D** | 2 | 0 | 1 | 2 |
| **E** | 2 | 2 | 0 | 2 |
| Minimum → | | | | **2** |

The optimal new location should be located at either node D or E with an objective function value of 2.

### 3.8 BERMAN AND DREZNER'S ALGORITHM

Berman and Drezner (2008), discussed a very simple algorithm that solves the conditional p-center problem on a network. The algorithm requires one-time solution of an unconditional p-center problem using an appropriate shortest distance matrix. Rather than creating a new location for an artificial facility and force the algorithm to locate a new facility there by creating an artificial demand point, they just modify the distance matrix.

### 3.8.1 Algorithm

Step 1: Let $D$ be a distance matrix with rows corresponding to demands and columns corresponding to potential locations.

Step 2: conditional problem is solved by defining a modified shortest distance matrix, from $D$ to $\hat{D}$,

where $\hat{D} = \min\{d_{ij} \min_{k\in Q}\{d_{ik}\}\} \forall i \in N, j \in C$ (Center). Even though $D$ is symmetric but $\hat{D}$ is not symmetric.

The unconditional p-center problem using the appropriate $\hat{D}$ solves the conditional p-center problem. This is so since if the shortest distance from node $i$ to the new p facilities are larger than $min_{k\in Q}\{d_{ik}\}$, then the shortest distance to the existing facilities is been utilized. Notice that the size of $\hat{D}$ is $n \times |C|$ for the conditional p-center.

Step 3: Find the optimal new location using $\hat{D}$ for the network with the objective function

$$Min[G(x) = \max_{i=1,\dots,n} \min\{d(X,i), d(Y,i)\}]$$

**Illustrative example of Berman and Drezner's Algorithm**.

Step 1: Table 3.8 below is all pair shortest path (distance matrix), D obtained by using Floyd's algorithm on figure 3.1. Suppose that nodes B and C are the existing set of facilities and an additional one facility is to be located ( i.e. p = 1).

Column 1 and row 1 in table 3.8 below represent demand nodes and potential location respectively.

Table 3.8 All pair shortest paths distance matrix, $D$

| Demand Nodes | Potential Location | | | | |
|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **E** |
| **A** | 0 | 2 | 3 | 5 | 3 |
| **B** | 2 | 0 | 5 | 3 | 1 |
| **C** | 3 | 5 | 0 | 4 | 6 |
| **D** | 5 | 3 | 4 | 0 | 2 |
| **E** | 3 | 1 | 6 | 2 | 0 |

Step 2: Modified shortest distance matrix is obtained using $\hat{D} = \min\{d_{ij} min_{k\in Q}\{d_{ik}\}\}$ as shown below:



**<u>For node A</u>**

At $i = A\ and\ j = A,$

$$\hat{D}_{AA} = \min\{d_{AA} min_{\{B,C\}\in Q}\{d_{AB}, d_{AC}\}\}$$

$$\hat{D}_{AA} = \min\left\{0\ min_{\{B,C\}\in Q}\{2,3\}\right\} = 0$$

At $i = A\ and\ j = B,$

$$\hat{D}_{AB} = \min\{d_{BB} min_{\{B,C\}\in Q}\{d_{BB}, d_{BC}\}\}$$

$$\hat{D}_{AB} = \min\{2\ min_{\{B,C\}\in Q}\{2,3\}\} = 2$$

At $i = A\ and\ j = C,$

$$\hat{D}_{AC} = \min\{d_{AC} min_{\{B,C\}\in Q}\{d_{AB}, d_{AC}\}\}$$

$$\hat{D}_{AC} = \min\{3\ min_{\{B,C\}\in Q}\{2,3\}\} = 2$$

At $i = A\ and\ j = D,$

$$\hat{D}_{AD} = \min\{d_{AD} min_{\{B,C\}\in Q}\{d_{AB}, d_{AC}\}\}$$

$$\hat{D}_{AD} = \min\{5\ min_{\{B,C\}\in Q}\{2,3\}\} = 2$$

At $i = A\ and\ j = E,$

$$\hat{D}_{AE} = \min\{d_{AE} min_{\{B,C\}\in Q}\{d_{AB}, d_{AC}\}\}$$

$$\hat{D}_{AE} = \min\{3\ min_{\{B,C\}\in Q}\{2,3\}\} = 2$$

From above, for node $A$: $\hat{D}_{AA} = 0$, $\hat{D}_{AB} = 2$, $\hat{D}_{AC} = 2$, $\hat{D}_{AD} = 2$, $\hat{D}_{AE} = 2$ and the table below then summarizes the results.

Table 3.9 Modified Distance Matrix, $\hat{D}$

| Demand Nodes | Potential location | | | | |
|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **E** |
| **A** | 0 | 2 | 2 | 2 | 2 |
| **B** | 0 | 0 | 0 | 0 | 0 |
| **C** | 0 | 0 | 0 | 0 | 0 |
| **D** | 3 | 3 | 3 | 0 | 2 |
| **E** | 1 | 1 | 1 | 1 | 0 |

Nodes B and C in $Q$ been the nodes with existing facility are removed and presented in Table 3.10 below.

Table 3.10 Modified Distance Matrix, $\hat{D}$ with nodes B and C removed

| Demand Nodes | Potential Location | | |
|---|---|---|---|
| | **A** | **D** | **E** |
| **A** | 0 | 2 | 2 |
| **D** | 3 | 0 | 2 |
| **E** | 1 | 1 | 0 |

Step 3: Find the optimal new location using the modified distance matrix $\hat{D}$ and objective function, $Min[G(x) = \max_{i=1,\ldots,n} \min\{d(X,i), d(Y,i)\}]$

With $X = \{A, D, E\}$ and $Y = \{B, C\}$

**For $X = A$**

$i = A$, we have

$min\{d(X,i), d(Y_A,i), d(Y_B,i)\}$

$min\{d(A,A), d(B,A), d(C,A)$

$min(0,2,2) = 0$

$i = B$, we have

$min\{d(X,i), d(Y_A,i), d(Y_B,i)\}$

$min\{d(A,B), d(B,B), (C,B)\}$

$\min(0,0,0) = 0$

$i = C$, we have

$min\{d(X,i), d(Y_A,i), d(Y_B,i)\}$

$min\{d(A,C), d(B,C), d(C,C)\}$

$min(0,0,0) = 0$

$i = D$, we have

$min\{d(X,i), (Y_A,i), d(Y_B,i)\}$

$min\{d(A,D), d(B,D), d(C,D)\}$

$min(3,3,3) = 3$

$i = E$, we have

$min\{d(X,i), (Y_A,i), d(Y_B,i)\}$

$min\{d(A,E), d(B,E), d(C,E)\}$

$min(1,1,1) = 1$

From the calculations above, maximum of $\min\{d(X,i), d(Y,i)\}]$ at $X = A$ is 3, at $= D$ , the maximum is 2 and that of $X = E$ is also 2.

The results are then summarized and presented in Table 3.11 with column 5 representing the maximum distance on each row.

Table 3.11Optimal Location, $Min(g(x))$ using $\hat{D}$

| Demand Nodes | Potential Location | | | Maximum |
|---|---|---|---|---|
| | **A** | **D** | **E** | |
| **A** | 0 | 3 | 1 | 3 |
| **D** | 2 | 0 | 1 | 2 |
| **E** | 2 | 2 | 0 | 2 |
| Minimum ⟶ | | | | **2** |

The optimal new location should be located at either node D or E with an objective function value of 2.

# CHAPTER 4

# DATACOLLECTION, ANALYSIS AND RESULTS

## 4.1 DATA COLLECTION

Agency that was contacted for distances between towns in this study was the Department of Feeder Roads, Berekum Municipal Assembly. The lengths of the arcs connecting towns in the Berekum Municipality is of interest in this study.

## 4.2 BEREKUM MUNICIPAL ROAD NETWORK

The distance between towns is obtained from the Department of Feeder Roads at the Berekum Municipal Assembly.

The data is then developed into a network. In the network below, numbers in the boxes are the nodes representing the eighteen towns and numbers on the edges are the various road distances between the towns in kilometers.

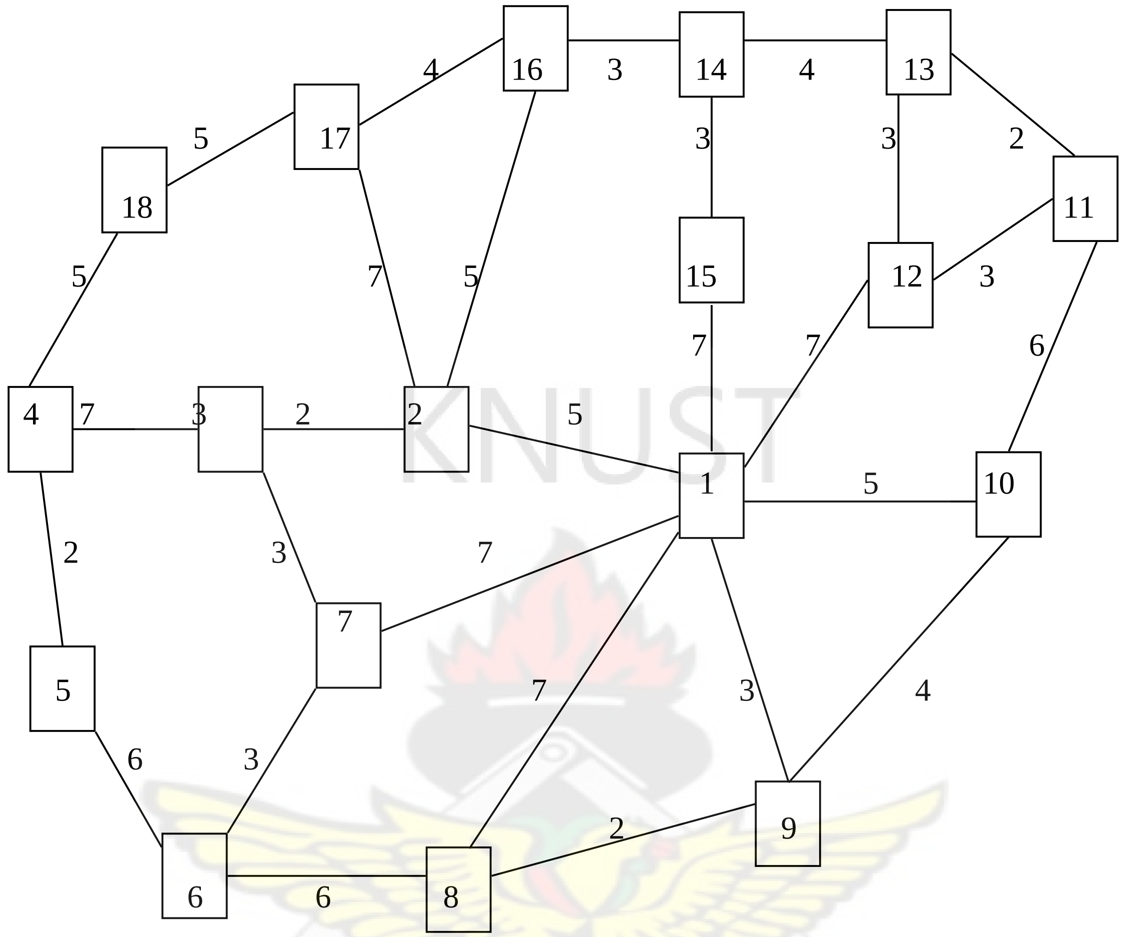



Figure 4.1 Road Network of Berekum Municipality.



## 4.3 DATA PROCESSING

Table 4.1 Towns in Berekum Municipality and their codes

| Number | Town | Number | Town | Number | Town |
|---|---|---|---|---|---|
| **1** | Berekum | **7** | Nsapor | **13** | Kutre No. 1 |
| **2** | Jamdede | **8** | Biadan | **14** | Akrofro |
| **3** | Domfete | **9** | Senase | **15** | Mpatasie |
| **4** | Jinijini | **10** | Kato | **16** | Abisaase |
| **5** | Ayimon | **11** | Kutre No. 2 | **17** | Koraso |
| **6** | Benkasa | **12** | Mpatapo | **18** | Fentantaa |

Currently, there are only two existing libraries; one at Berekum and the other one, community library at Jinijini. These communities form the set of existing facilities, thus node **1** and node **4** respectively.

### 4.3.1 MATRIX FORMATION

The data of inter-towns and suburbs were entered manually into an edge distance matrix of size eighteen by eighteen using Microsoft excel. The edge distances of the nodes (suburbs) which are directly connected were allocated and nodes which were not directly connected have the edge distance entered as 'inf', representing infinite ($\infty$) distance.

A square matrix of size eighteen by eighteen is formed from figure 4.1(the road network) and the matrix is shown in table 4.2 below.

Table 4.2 18 × 18 Matrix input data for the Floyd-Warshall algorithm.

| **Nodes** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 5 | inf | inf | inf | inf | 7 | 7 | 3 | 5 | 7 | inf | inf | inf | 7 | inf | inf | inf |
| **2** | 5 | 0 | 2 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | 5 | 7 | inf |
| **3** | inf | 2 | 0 | 7 | inf | inf | 3 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf |
| **4** | inf | inf | 7 | 0 | 2 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | 5 |
| **5** | inf | inf | inf | 2 | 0 | 6 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf |
| **6** | inf | inf | inf | inf | 6 | 0 | 3 | 6 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf |
| **7** | 7 | inf | 3 | inf | inf | 3 | 0 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf |
| **8** | 7 | inf | inf | inf | inf | 6 | inf | 0 | 2 | inf | inf | inf | inf | inf | inf | inf | inf | inf |
| **9** | 3 | inf | inf | inf | inf | inf | inf | 2 | 0 | 4 | inf | inf | inf | inf | inf | inf | inf | inf |
| **10** | 5 | inf | inf | inf | inf | inf | inf | inf | 4 | 0 | 6 | inf | inf | inf | inf | inf | inf | inf |
| **11** | 7 | inf | inf | inf | inf | inf | inf | inf | inf | 6 | 0 | 3 | 2 | inf | inf | inf | inf | inf |
| **12** | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | 3 | 0 | 3 | inf | 2 | inf | inf | inf |
| **13** | inf | inf | inf | inf | inf | inf | Inf | inf | inf | inf | 2 | 3 | 0 | 4 | inf | inf | inf | inf |
| **14** | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | 4 | 0 | 3 | 3 | inf | inf |
| **15** | 7 | inf | Inf | inf | inf | inf | inf | inf | inf | inf | inf | 2 | inf | 3 | 0 | inf | inf | inf |
| **16** | inf | 5 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | 3 | inf | 0 | 4 | inf |
| **17** | inf | 7 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | 4 | 0 | 5 |
| **18** | inf | inf | inf | 5 | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | inf | 5 | 0 |

## 4.4 COMPUTATIONAL RESULTS

Matlab program software was used for the coding of the Floyd-Warshall algorithm. The codes for Floyd-Warshall algorithm was developed and ran on Dell AMD Athlon (tm) II P360 Dual-Core Processor 2.30GHz of RAM 3.00GB, 64-bit Operating System with Windows Ultimate Laptop Computer.

### 4.4.1 Computation using Floyd-Warshall Algorithm

The 18 x 18 edge distance matrix was used as an input for the Floyd-Warshall algorithm coded in matlab. The algorithm steps for the Floyd-Warshall algorithmis shown below and the matlab code is presented in Appendix A.

Let dist($k, i, j$) be the length of the shortest path from $i$ and $j$ that uses only the vertices 1,2,3,..........,$k$ as intermediate vertices. The following recurrence:

Step 1; $k = 0$ is our base case, thus $dist(0, i, j)$ = DA ($i, j$) is the length of the edge from vertex $i$ to vertex $j$if it exists and infinite (∞) otherwise.

Step 2; using $dist(0, i, j)$, it then computes $dist(1, i, j)$ for all pairs of nodes $i$ and$j$.

Step 3; using $dist(1, i, j)$, it then computes $dist(2, i, j)$, for all pairs of nodes $i$ and $j$. it then repeats the process until it obtains $dist(k, i, j)$ for all node pairs $i$ and$j$ when it terminates. The algorithm computes the shortest paths as;

$$dist(k, i, j) = \min\big(dist(k-1, i, j), dist(k-1, i, k) + dist(k-1, k, j)\big).$$

The calculated all pair shortest paths, Distance Matrix, $D$ (in kilometres) obtained from the run of the Floyd-Warshall algorithm on table 4.2 is summarised in table 4.3.

In table 4.3, column one and row one represent the demand nodes and potential location respectively; the other rows also represent the interconnecting road distances.

Table 4.3 All pair shortest paths Distance Matrix, $D$ using Floyd-Warshall algorithm.

| **Demand Nodes** | **Potential Locations** | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** |
| **1** | 0 | 5 | 7 | 14 | 16 | 10 | 7 | 5 | 3 | 5 | 10 | 7 | 10 | 10 | 7 | 10 | 12 | 17 |
| **2** | 5 | 0 | 2 | 9 | 11 | 8 | 5 | 10 | 8 | 10 | 14 | 12 | 12 | 8 | 11 | 5 | 7 | 12 |
| **3** | 7 | 2 | 0 | 7 | 9 | 6 | 3 | 12 | 10 | 12 | 16 | 14 | 14 | 10 | 13 | 7 | 9 | 12 |
| **4** | 14 | 9 | 7 | 0 | 2 | 8 | 10 | 14 | 16 | 19 | 23 | 21 | 21 | 17 | 20 | 14 | 10 | 5 |
| **5** | 16 | 11 | 9 | 2 | 0 | 6 | 9 | 12 | 14 | 18 | 25 | 23 | 23 | 19 | 22 | 16 | 12 | 7 |
| **6** | 10 | 8 | 6 | 8 | 6 | 0 | 3 | 6 | 8 | 12 | 20 | 17 | 20 | 16 | 17 | 13 | 15 | 13 |
| **7** | 7 | 5 | 3 | 10 | 9 | 3 | 0 | 9 | 10 | 12 | 17 | 14 | 17 | 13 | 14 | 10 | 12 | 15 |
| **8** | 5 | 10 | 12 | 14 | 12 | 6 | 9 | 0 | 2 | 6 | 14 | 12 | 15 | 15 | 12 | 15 | 17 | 19 |
| **9** | 3 | 8 | 10 | 16 | 14 | 8 | 10 | 2 | 0 | 4 | 12 | 10 | 13 | 13 | 10 | 13 | 15 | 20 |
| **10** | 5 | 10 | 12 | 19 | 18 | 12 | 12 | 6 | 4 | 0 | 8 | 11 | 10 | 14 | 12 | 15 | 17 | 22 |
| **11** | 10 | 14 | 16 | 23 | 25 | 20 | 17 | 14 | 12 | 8 | 0 | 3 | 2 | 6 | 6 | 9 | 13 | 18 |
| **12** | 7 | 12 | 14 | 21 | 23 | 17 | 14 | 12 | 10 | 11 | 3 | 0 | 3 | 6 | 3 | 9 | 13 | 18 |
| **13** | 10 | 12 | 14 | 21 | 23 | 20 | 17 | 15 | 13 | 10 | 2 | 3 | 0 | 4 | 6 | 7 | 11 | 16 |
| **14** | 10 | 8 | 10 | 17 | 19 | 16 | 13 | 15 | 13 | 14 | 6 | 6 | 4 | 0 | 3 | 3 | 7 | 12 |
| **15** | 7 | 11 | 13 | 20 | 22 | 17 | 14 | 12 | 10 | 12 | 6 | 3 | 6 | 3 | 0 | 6 | 10 | 15 |
| **16** | 10 | 5 | 7 | 14 | 16 | 13 | 10 | 15 | 13 | 15 | 9 | 9 | 7 | 3 | 6 | 0 | 4 | 9 |
| **17** | 12 | 7 | 9 | 10 | 12 | 15 | 12 | 17 | 13 | 17 | 13 | 13 | 11 | 7 | 10 | 4 | 0 | 5 |
| **18** | 17 | 12 | 12 | 5 | 7 | 13 | 15 | 19 | 15 | 22 | 18 | 18 | 16 | 12 | 15 | 9 | 5 | 0 |

## 4.5 BERMAN AND DREZNER'S ALGORITHM

At this point, we use the Berman and Drezner's algorithm (2008) to solve the problem. We begin by formulating the conditional p- center problem as

$$Min[G(x) = \max_{i=1,..,n} \min\{d(X,i), d(Y,i)\}]$$

Let $d(x, y)$ be the shortest distance between any $x, y \in G$. Suppose that there is a set Q ($|Q| = q$) of existing facilities. Let $Y = (Y_{1,} Y_2, Y_3, \dots, Y_q)$ and $X = (X_1, X_2, X_3, \dots, X_p)$ be vectors of size q and p respectively, where$Y_i$ is the location of existing facility and $X_i$ is the location of new facility. Where $d(X, i)$ and $d(Y, i)$ is the shortest distance from the closest facility in $X$ and $Y$ respectively to the node $i$,(Berman and Simchi-Levi, 1990).

The set of location of new facilities $X = \{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$ and the set of location of existing facilities $Y = \{1, 4\}$, then the conditional p – centre problem is to:

$$\text{Minimize } G(x) = \left[ \max_{i=1,2,3,\ldots,n} min \left\{ \begin{array}{lll} d(2, i), & d(1, i), & d(4, i) \\ d(3, i), & d(1, i), & d(4, i) \\ d(5, i), & d(1, i), & d(4, i) \\ . & . & . \\ . & . & . \\ . & . & . \\ d(16, i), & d(1, i), & d(4, i) \\ d(17, i), & d(1, i), & d(4, i) \\ d(18, i), & d(1, i), & d(4, i) \end{array} \right\} \right]$$

Where $i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$

### 4.5 .1 THE ALGORITHM

Step 1: Let $D$ be a distance matrix with rows corresponding to demands and columns corresponding to potential locations.

Step 2: Conditional problem is solved by defining a modified shortest distance matrix, from $D$ to $\hat{D}$,

where $\hat{D} = \min\{d_{ij} \min_{k\in Q}\{d_{ij}\}\} \forall i \in N, j \in C$ (Center).

Even though $D$ is symmetric but $\hat{D}$ is not symmetric.

Step 3: Find the optimal new location using $\hat{D}$ for the network with the objective function

$$Min[G(x) = \max_{i=1,\dots,n} \min\{d(X,i), d(Y,i)\}]$$

### 4.6 COMPUTATION USING BERMAN AND DREZNER'S ALGORITHM.

A new shortest path distance matrix called modified shortest distance matrix, $\hat{D}$ is formed from table 4.3. Thus from $D$ to $\hat{D}$

#### 4.6.1 MODIFIED SHORTEST DISTANCE MATRIX, $\hat{D}$

By defining a modified shortest distance matrix $\hat{D}$ where

$\hat{D}_{ij} = \min\{d_{ij} \min_{k\in Q}\{d_{ik}\}\} \forall i \in N, j \in C$(Center).

**For node 1**

$i = 1,\ j = 1$ and $Q = \{1,4\}$

$$\hat{D}_{11} = \min\{d_{11}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{0, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 2$ and $Q = \{1,4\}$

$$\hat{D}_{12} = \min\{d_{12}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{5, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 3$ and $Q = \{1,4\}$

$$\hat{D}_{13} = \min\{d_{13}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{7, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 4$ and $Q = \{1,4\}$

$$\hat{D}_{14} = \min\{d_{14}, \min\{d_{11}, d_{14}\}\}$$

$i = 1,\ j = 5$ and $Q = \{1,4\}$

$$\hat{D}_{15} = \min\{d_{15}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{16, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 6$ and $Q = \{1,4\}$

$$\hat{D}_{16} = \min\{d_{16}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{10, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 7$ and $Q = \{1,4\}$

$$\hat{D}_{17} = \min\{d_{17}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{7, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 8$ and $Q = \{1,4\}$

$$\hat{D}_{18} = \min\{d_{18}, \min\{d_{11}, d_{14}\}\}$$

$= \min\{14, \min\{0,14\}\}$
$= 0$

$= \min\{5, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 9$ and $Q = \{1,4\}$

$\hat{D}_{19} \quad = \min\{d_{19}, \min\{d_{11}, d_{14}\}\}$

$= \min\{3, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 10$ and $Q = \{1,4\}$

$\hat{D}_{1\ 10} \quad = \min\{d_{1\ 10}, \min\{d_{11}, d_{14}\}\}$

$= \min\{5, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 11$ and $Q = \{1,4\}$

$\hat{D}_{1\ 11} \quad = \min\{d_{1\ 11}, \min\{d_{11}, d_{14}\}\}$

$= \min\{10, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 12$ and $Q = \{1,4\}$

$\hat{D}_{1\ 12} \quad = \min\{d_{1\ 12}, \min\{d_{11}, d_{14}\}\}$

$= \min\{7, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 13$ and $Q = \{1,4\}$

$\hat{D}_{1\ 13} \quad = \min\{d_{1\ 13}, \min\{d_{11}, d_{14}\}\}$

$= \min\{10, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 14$ and $Q = \{1,4\}$

$\hat{D}_{1\ 14} \quad = \min\{d_{1\ 14}, \min\{d_{11}, d_{14}\}\}$

$= \min\{10, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 15$ and $Q = \{1,4\}$

$\hat{D}_{1\ 15} \quad = \min\{d_{1\ 15}, \min\{d_{11}, d_{14}\}\}$

$= \min\{7, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 16$ and $Q = \{1,4\}$

$\hat{D}_{1\ 16} \quad = \min\{d_{1\ 16}, \min\{d_{11}, d_{14}\}\}$

$= \min\{10, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 17$ and $Q = \{1,4\}$

$\hat{D}_{1\ 17} \quad = \min\{d_{1\ 17}, \min\{d_{11}, d_{14}\}\}$

$= \min\{12, \min\{0,14\}\}$
$= 0$

$i = 1,\ j = 18$ and $Q = \{1,4\}$

$\hat{D}_{1\ 18} \quad = \min\{d_{1\ 18}, \min\{d_{11}, d_{14}\}\}$

$= \min\{17, \min\{0,14\}\}$
$= 0$

Therefore, the modified shortest distance for node 1 at $i = 1, j = 1$ is 0, $i = 1, j = 2$ is 0 and that of $i = 1, j = 3$ is also 0.

Table 4.4 below then summarizes the results of $\hat{D}$ into a modified shortest distance matrix. Column one representing demand nodes and all other columns representing the minimum interconnecting distance when demand nodes are compared with existing facilities nodes. Details of modified shortest distance is elaborated in appendix B



Table 4.4 Modified Shortest Distance Matrix, $\hat{D}$

| Demand Nodes | Potential location | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** |
| **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **2** | 5 | 0 | 2 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **3** | 7 | 2 | 0 | 7 | 7 | 6 | 3 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| **4** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **5** | 2 | 2 | 2 | 2 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| **6** | 8 | 8 | 6 | 8 | 6 | 0 | 3 | 6 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| **7** | 7 | 5 | 3 | 7 | 7 | 3 | 0 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| **8** | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 0 | 2 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **9** | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 0 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| **10** | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 4 | 0 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **11** | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 8 | 0 | 3 | 2 | 6 | 6 | 9 | 10 | 10 |
| **12** | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 3 | 0 | 3 | 6 | 3 | 7 | 7 | 7 |
| **13** | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 2 | 3 | 0 | 4 | 6 | 7 | 10 | 10 |
| **14** | 10 | 8 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 6 | 6 | 4 | 0 | 3 | 3 | 7 | 10 |
| **15** | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 6 | 3 | 6 | 3 | 0 | 6 | 7 | 7 |
| **16** | 10 | 5 | 7 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 9 | 9 | 7 | 3 | 6 | 0 | 4 | 9 |
| **17** | 10 | 7 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 7 | 10 | 4 | 0 | 5 |
| **18** | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 0 |

Comparing various road distances with the existing nodes 1 and 4, the minimum distance is always zero. Hence, in the next table, the set of demand nodes and potential location of the existing facilities are removed from the modified shortest path distance matrix, $\hat{D}$ in table 4.4

Table 4.5 Modified Distance Matrix with Nodes 1 and 4 Removed.

| Demand Nodes | Potential Location | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **2** | **3** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** |
| **2** | 0 | 2 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **3** | 2 | 0 | 7 | 6 | 3 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| **5** | 2 | 2 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| **6** | 8 | 6 | 6 | 0 | 3 | 6 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| **7** | 5 | 3 | 7 | 3 | 0 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| **8** | 5 | 5 | 5 | 5 | 5 | 0 | 2 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **9** | 3 | 3 | 3 | 3 | 3 | 2 | 0 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| **10** | 5 | 5 | 5 | 5 | 5 | 5 | 4 | 0 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| **11** | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 8 | 0 | 3 | 2 | 6 | 6 | 9 | 10 | 10 |
| **12** | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 3 | 0 | 3 | 6 | 3 | 7 | 7 | 7 |
| **13** | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 2 | 3 | 0 | 4 | 6 | 7 | 10 | 10 |
| **14** | 8 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 6 | 6 | 4 | 0 | 3 | 3 | 7 | 10 |
| **15** | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 6 | 3 | 6 | 3 | 0 | 6 | 7 | 7 |
| **16** | 5 | 7 | 10 | 10 | 10 | 10 | 10 | 10 | 9 | 9 | 7 | 3 | 6 | 0 | 4 | 9 |
| **17** | 7 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 7 | 10 | 4 | 0 | 5 |
| **18** | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 0 |

Using table 4.4 above, the optimal new location is found using the objective function;

$Min[G(x) = \max_{i=1,\ldots,n} \min\{d(X,i), d(Y,i)\}]$

With $Y = \{1,4\}$ and $X = \{2,3,5,6,7,8,9,10,11,12,13,14,15,16,17,18\}$

**For $X = 2$**

$i = 1,$
$\min\{d(2,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(2,2), d(1,2), d(4,2)\}$
$= \min\{0,5,5\}$
$= 0$

$i = 3,$
$\min\{d(2,3), d(1,3), d(4,3)\}$
$= \min\{2,7,7\}$
$= 2$

$i = 4,$
$\min\{d(2,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(2,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(2,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(2,7), d(1,7), d(4,7)\}$
$= \min\{5,7,7\}$
$= 5$

$i = 8,$
$\min\{d(2,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(2,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(2,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(2,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(2,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(2,13), d(1,13), d(4,13)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 14,$
$\min\{d(2,14), d(1,14), d(4,14)\}$
$= \min\{8,10,10\}$
$= 8$

$i = 15,$
$\min\{d(2,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(2,16), d(1,16), d(4,16)\}$
$= \min\{5,10,10\}$
$= 5$

$i = 17,$
$\min\{d(2,17), d(1,17), d(4,17)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 18,$
$\min\{d(2,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

Therefore, at $X$ =2, the maximum is 10, the maximum at $X$ =3 is 10 and that of $X$ =5 is 10.The results are then summarized in table 4.6 below with column 18 representing the maximum distance on each row. Details of the calculation are presented in appendix C.

Table 4.6 Optimal Location$Min(g(x))$ using $\hat{D}$

| Demand Nodes | Potential Location | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **2** | **3** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | Maximum |
| **2** | 0 | 2 | 2 | 8 | 5 | 5 | 3 | 5 | 10 | 7 | 10 | 8 | 7 | 5 | 7 | 5 | 10 |
| **3** | 2 | 0 | 2 | 6 | 3 | 5 | 3 | 5 | 10 | 7 | 10 | 10 | 7 | 7 | 9 | 5 | 10 |
| **5** | 5 | 7 | 0 | 6 | 7 | 5 | 3 | 5 | 10 | 7 | 10 | 10 | 7 | 10 | 10 | 5 | 10 |
| **6** | 5 | 6 | 2 | 0 | 3 | 5 | 3 | 5 | 10 | 7 | 10 | 10 | 7 | 10 | 10 | 5 | 10 |
| **7** | 5 | 3 | 2 | 3 | 0 | 5 | 3 | 5 | 10 | 7 | 10 | 10 | 7 | 10 | 10 | 5 | 10 |
| **8** | 5 | 7 | 2 | 6 | 7 | 5 | 2 | 5 | 10 | 7 | 10 | 10 | 7 | 10 | 10 | 5 | 10 |
| **9** | 5 | 7 | 2 | 8 | 7 | 2 | 0 | 4 | 10 | 7 | 10 | 10 | 7 | 10 | 10 | 5 | 10 |
| **10** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 0 | 8 | 7 | 10 | 10 | 7 | 10 | 10 | 5 | 10 |
| **11** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 0 | 3 | 2 | 6 | 6 | 9 | 10 | 5 | 10 |
| **12** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 3 | 0 | 3 | 6 | 3 | 9 | 10 | 5 | 10 |
| **13** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 2 | 3 | 0 | 4 | 6 | 7 | 10 | 5 | 10 |
| **14** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 6 | 6 | 4 | 0 | 3 | 3 | 7 | 5 | 8 |
| **15** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 6 | 3 | 6 | 3 | 0 | 6 | 10 | 5 | 10 |
| **16** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 9 | 7 | 7 | 3 | 6 | 0 | 4 | 5 | 9 |
| **17** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 10 | 7 | 10 | 7 | 7 | 4 | 0 | 5 | 10 |
| **18** | 5 | 7 | 2 | 8 | 7 | 5 | 3 | 5 | 10 | 7 | 10 | 10 | 7 | 9 | 5 | 0 | 10 |
| Minimum | | | | | | | | | | | | | | | | | **8** |



**4.7 DISCUSSION OF RESLUTS**

With the algorithm from Berman and Drezner, considering the eighteen node network depicted in Figure 4.1 and solving the conditional 1- centre problem with the existing facilities at node 1 and node 4 (ie Berekum and Jinijini respectively), Table 4.3,Table 4.4 and Table 4.5 are the all pair shortest paths distance matrix, the modified shortest distance matrix and the modified shortest distance matrix with nodes 1 and 4 removed respectively.

Table 4.6 above clearly indicates that using Berman and Drezner (2008) algorithm on the modified shortest distance matrix, the optimal new location is at node 14 (Akrofro). Therefore the new library can arbitrary be located at Akrofro with an objective function value of 8.

# CHAPTER 5

# CONCLUSION AND RECOMMENDATIONS

## 5.1 CONCLUSION

Floyd-Warshall's algorithm was used to find all pair shortest path between nodes using table 4.1 which resulted in eighteen by eighteen (18×18) matrix in Table 4.2.

The formulation of the model for location of an additional library facility in the Berekum Municipality is indicated on page 49. The model was solved using Berman and Drezner (2008) algorithm as shown in page 50.

Considering the objective function $Min(G(x))$ from Berman and Drezner (2008), the library facility from table 4.6 should be located at Akrofro.

The minimum objective function value obtained was 8 kilometres which means that, the minimum distance travelled by the farthest library user to the new library facility at Akrofro is 8 kilometres.

## 5.2 RECOMMENDATIONS

The following recommendations are made from the study;

This work should serve as basis for further research in the area of conditional p-centre problem.

Finally, since Berekum Municipality was used as case study, the researcher recommends that Berekum Municipal Assembly, Municipal Education Directorate, Non Governmental Organizations (NGOs) and individuals who would like to locate an additional library facility in Berekum Municipality should locate it at Akrofro.

# REFERENCES

1. Ballou, R. H. (1998). "Business Logistics Management – Planning, Organizing, and Controlling the Supply Chain – Fourth Edition", Prentice Hall. Upper Saddle River, New Jersey 07458

2. Berekum Municipal Assembly (2006). Ghana>>Brong Ahafo>>Berekum

http://berekumghanadistricts.gov.gh (Accessed on 22/10/12)

3. Berman, O. and Simchi-Levi, D. (2011). Technical Note – Conditional Location Problems o Networks.

http://transci.journal.informs.org/content/24/1/77.abstract (Accessed on 11/10/12)

4. Berman, O. and Drezner, Z. (2008), A New Formulation of the Conditional p-median and p-centre Problems. Operation Research letters 36: 481-483.

5. Buttcher S. (2004). Approximability Results for the P-centre Problem. University of waterloo, spring

6. Caruso, C.,Colorni, A. and M. Paruccni, ( 1986). "The Regional Urban Solid Waste Management System: A Modeling Approach", European Journal of Operation Research, 70:16-30

7. Chalmet,L. G., L. Lemonidis and D. J. Elzinga, (1986). "An Algorithm for the Bi-criterion Interger ProgrammingProgram". European Journal of Operation Research, 25:292-300

8. Cheng, T. C. E., Kang, L. and Ng, C. T. (2005). An improved Algorithm for the p-centre

Problem on Interval Graphs with Unit Lengths.

9. Christopher, M. and Wills, G. (1972). “Marketing Logistics and Distribution Planning”, George Allen and Unwin Ltd, UK

10. Chung- Kung, Y. W., and Chein-Tsai, C. (2006). The p-centre Problem with Connectivity Constraint, Applied Mathematical Sciences, Volumm 1, No. 27 pp 1311 – 1324

http://www.m-hikari.com/ams/ams-password-2007 (Accessed on 11/10/12)

11. Costa, M. G., Captivo, M. E., and Climaco, J. (2008). “Capacitated Single Allocation Hub Location Problem – A Bicriteria Approach”. Computers and Operations Research, 35(11):3671-3695.

12. Elloumi, S., Martine, L. and Yves, P. (2004). A New Formulation and Resolution Method for the p-Centre Problem, Informs Journal of Computing, Volume 16, No. 1, Winter, pp.84 – 94.

http://joc.journal.informs.org/content/16/1/84.abstract (Accessed on 09/10/12)

13. Erkut, E. and Neuman, S. (1992) “A Multiobjective Model for Location Undesirable Facilities”. Annals of operations Research. 40:209-227.

14. Ferreire, C., Santos,B. S., Captivo, M. E., Climaco, J., Sliva, C.C. and Paixão, J. (1997). “Bicriteria Covering and Related Problems – aArototype of a Decisions Support System”. In proceedings of the 5th Conference of CEMAPRE –Aplicações da Matemàtica à Economia e Gestão, Lisboa, 415-428

15. Ferreira, C., Santos, B. S., Captivo, M. E., Climaco, J. and Sliva, C. C. (1996).

"Multiobjective Location of Unwelcome or Central Facilities Involving Environmental Aspects – a Prototype of a Decision Support System". Beligium Journal of Operations Researche, Stastistics and Computer Science, 36:159-72

16. Ferreira, C., Climaco, J. and Paixão, J. (1994). "The Location-covering Problem: a Bicriterion Interactive Approach", InvestigaciónOperativa, 4:119-139

17. Fonseca, M.C. and Captivo,M. E. (2007), "Analysis of Bicriteria Semi Obnoxious location models". Paper presented at TercerEncuentro de la Red Iberoamericana de Evaluación y DecisiónMulticeriterio, R. E.D.-M. Culiacàn, México,2007

18. Fonseca, M. C. And Captivo, M. E. (1996). "Location of Semi-obnoxious Facility with Capacity Constraints", Studies in Locational Analysis, 9:51-52

19. Fonseca, M.C. and Captivo, M. E. (2006). "Models for Semi-obnoxious Facility Location ", Proceeding of the XIIII CLAIO, Montevideo, Uruguay

20. Giannikos, I. (1998), "A multiobjective Programming model for locating treatment sitrs and routing hazardous waste", European Journal of Operation Research, 104:333-342.

21. Goldengorin, B., Diptesh G. and Sierksma, G. (1999). Branch and Peg Algorithms for the Simple Plant Location Problem. Faculty of Economic Sciences, University of Groningen, The Netherlands.

22. Hakimi, S. (1964). "Optimum Location of Switching Centres and the Absolute Centres and medians of a graph," Operations Research, 12: 450 – 459.

23. Hakimi, S. (1965). "Optimum Location of Switching Centres in a Communications

Network and some Related Graph Theoretic Problems," Operation Research, 13, 462 – 475.

24. Hassin, R. and Tamir, A. (1991), Improved Complexity Bounds for Location Problems on the Real Line. Operations Research Letters, 10, 395-402.

25. Halperin, E., Kortsarz, G. and Krauthgamer, R. (2003). Tight Lower Bounds for the Asymmetric k-center problem. Electronic Colloquium on Computational Complexity. Report No. 35

26. Krumke, S. O. (1995). On a Generalization of the p-centreproblem, Information Processing Letters, Volume 56, Issue 2, 7, pages 67-71(5)

http://www.ingentaconnect.com/content/els/00200190 (Accessed 0n 09/10/12)

27. Krivitski, D., Schuster, A. and Woilf, R. (2005) A Local Facility Location Algorithm for Sensor Networks.

https://docs.google.com/viewer?a=v&q=cache: (Accessed on 09/10/12).

28. Kuby, M. (1987) "The p dispersion and Maximum dispersion Problems," Geographical Analysis. 16: 315 – 329.

29. Melachrinoudis, E., Min, H. and Wu, X. (1995). "A Multiobjective Model for the Dynamic Location of Landfills", Location Science, 3:143-166.

30. Minieka, E. (2006). Conditional Centres and Medians of a Graphhttp://onlinelibrary.wiley.com/doi/10.1002/net.3230100307/abstract

(Accessed on 11/10/12)

31. Moscibroda, T. and Wttenhofer, R. (2005). Facility Location: Distribution Approximation, Networks Laboratory ETH zurich. Switzerland.

32. Tamir, A., Puerto, J., Mesa, J. A. and Rodriguez-Chia, A. M. (2005). Conditional Location of Path and Tree Shaped Facilities on Trees, Elsevier Inc.

http://www.google.com/#sclient=psy-ab&hl (Accessed on 11/10/12)

33. Toregas, C., Swain, R., ReVelle, C. and Bergman, L. (1971). "The Location of Emergency Services Facilities," Operations Research, 19: 1363-1373.

34. Wikipedia (2012). What is Library?

http://www.librarylaws.com (Accessed on 04/09/2012)

35. Wikipedia (2012). Description of Libraries.

http://en.wikipedia.org/wiki/Library_and_information_Science

(Accessed on 04/09/2012)

36. Wikipedia (2012). Library Classification

http://en.wikipedia.org/wiki/library_classification#mw-head (Accessed on 04/09/2012)

37. Wikipedia (2012). Early History of Library

http://en.wikipedia.org/wiki/Document (Accessed on 04/09/2012)

38. Wouter, J., Koster, H. and Hall, P. (2011). The Location and Global Network Structure of Maritime Advanced Producer Services.

http://usj.sagepub.com/content/early/2011/03/08/0042098010391294.abstract

(Accessed on 03/10/12)

39. Wymam, M. and Kuby, M. (1993), "A multiobjective location- Allocation Model for Assesing Toxic Waste Processing Technologies", Studies in location Analysis, 4:193-196.

40. Wyman, M. and Kuby, M. (1995), "Proactive Optimization of Toxic Waste Transportation, Location and Techologies", Location Science, 3(3):167-185.

# APPENDICES

## APPENDIX A: Matlab code for all pair shortest path. (Floyd-Warshall Algorithm)

```
functionfloy()
% solving the whole network
C = load( 'matrixx.m' );
 n = size (C, 1);
D = repmat ( inf, n, n ); % Direct path are infinite at first
P = repmat ( -1, n, n ); % (-1) pointer matrix to shortest intermediate
node
% Create initial matrix with direct path
for k = 1 : n;
v = find (C ( k, : ) > 0); % column vector with non zero entries on row k
D (k, v) = C ( k, v ); % Copy real direct path to this matrix
D (k, k) = 0; % zero path on diagonal nodes
P ( k, v ) = k; % Direct shortest linked paths, point to the destination
node
end
% Calculate shortest distances and path matrix
for k = 1 : n;
for i = 1 : n;
for j = 1 : n;
if (D ( i, k ) + D (k, j)) < D ( i, j );
            D ( i, j ) = D ( i, k ) + D ( k, j )
P( i, j ) = k; % Last intermediate node on the shortest path
end
end
end
end
end
```

## APPENDIX B: MODIFIED SHORTEST DISTANCE MATRIX, $\hat{D}$

By defining a modified shortest distance matrix $\hat{D}$ where

$\hat{D}_{ij} = \min\{d_{ij} \min_{k\in Q}\{d_{ik}\}\} \forall i \in N, j \in C$ (Center).

**For node 1**

$i = 1,\ j = 1$ and $Q = \{1,4\}$

$$\hat{D}_{11} = \min\{d_{11}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{0, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 2$ and $Q = \{1,4\}$

$$\hat{D}_{12} = \min\{d_{12}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{5, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 3$ and $Q = \{1,4\}$

$$\hat{D}_{13} = \min\{d_{13}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{7, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 4$ and $Q = \{1,4\}$

$$\hat{D}_{14} = \min\{d_{14}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{14, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 5$ and $Q = \{1,4\}$

$$\hat{D}_{15} = \min\{d_{15}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{16, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 6$ and $Q = \{1,4\}$

$$\hat{D}_{16} = \min\{d_{16}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{10, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 7$ and $Q = \{1,4\}$

$$\hat{D}_{17} = \min\{d_{17}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{7, \min\{0,14\}\}$$
$$= 0$$

$i = 1,\ j = 8$ and $Q = \{1,4\}$

$$\hat{D}_{18} = \min\{d_{18}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{5, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=9$ and $Q=\{1,4\}$

$$\hat{D}_{19} = \min\{d_{19}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{3, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=10$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 10} = \min\{d_{1\ 10}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{5, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 11} = \min\{d_{1\ 11}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{10, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 12} = \min\{d_{1\ 12}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{7, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=13$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 13} = \min\{d_{1\ 13}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{10, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=14$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 14} = \min\{d_{1\ 14}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{10, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=15$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 15} = \min\{d_{1\ 15}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{7, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=16$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 16} = \min\{d_{1\ 16}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{10, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=17$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 17} = \min\{d_{1\ 17}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{12, \min\{0,14\}\}$$
$$= 0$$

$i=1,\ j=18$ and $Q=\{1,4\}$

$$\hat{D}_{1\ 18} = \min\{d_{1\ 18}, \min\{d_{11}, d_{14}\}\}$$
$$= \min\{17, \min\{0,14\}\}$$
$$= 0$$

**For Node 2**

$i=2,\ j=1$ and $Q=\{1,4\}$

$\hat{D}_{21} \quad = \min\{d_{21}, \min\{d_{21}, d_{24}\}\}$

$= \min\{5, \min\{5,9\}\}$
$= 5$

$i=2,\ j=2$ and $Q=\{1,4\}$

$\hat{D}_{22} \quad = \min\{d_{22}, \min\{d_{21}, d_{24}\}\}$

$= \min\{0, \min\{5,9\}\}$
$= 0$

$i=2,\ j=3$ and $Q=\{1,4\}$

$\hat{D}_{23} \quad = \min\{d_{23}, \min\{d_{21}, d_{24}\}\}$

$= \min\{2, \min\{5,9\}\}$
$= 2$

$i=2,\ j=4$ and $Q=\{1,4\}$

$\hat{D}_{24} \quad = \min\{d_{22}, \min\{d_{21}, d_{24}\}\}$

$= \min\{9, \min\{5,9\}\}$
$= 5$

$i=2,\ j=5$ and $Q=\{1,4\}$

$\hat{D}_{25} \quad = \min\{d_{25}, \min\{d_{21}, d_{24}\}$

$= \min\{11, \min\{5,9\}\}$
$= 5$

$i=2,\ j=6$ and $Q=\{1,4\}$

$\hat{D}_{26} \quad = \min\{d_{26}, \min\{d_{21}, d_{24}\}\}$

$= \min\{8, \min\{5,9\}\}$
$= 5$

$i=2,\ j=7$ and $Q=\{1,4\}$

$\hat{D}_{27} \quad = \min\{d_{27}, \min\{d_{21}, d_{24}\}\}$

$= \min\{5, \min\{5,9\}\}$
$= 5$

$i=2,\ j=8$ and $Q=\{1,4\}$

$\hat{D}_{28} \quad = \min\{d_{28}, \min\{d_{21}, d_{24}\}\}$

$= \min\{10, \min\{5,9\}\}$
$= 5$

$i=2,\ j=9$ and $Q=\{1,4\}$

$\hat{D}_{29} \quad = \min\{d_{29}, \min\{d_{21}, d_{24}\}\}$

$= \min\{8, \min\{5,9\}\}$
$= 5$

$i=2,\ j=10$ and $Q=\{1,4\}$

$\hat{D}_{1\ 10} \quad = \min\{d_{1\ 10}, \min\{d_{21}, d_{24}\}\}$

$= \min\{10, \min\{5,9\}\}$
$= 5$

$i=2,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 11} = \min\{d_{2\ 11}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{14, \min\{5,9\}\}$$
$$= 5$$

$i=2,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 12} = \min\{d_{2\ 12}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{12, \min\{5,9\}\}$$
$$= 5$$

$i=2,\ j=13$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 13} = \min\{d_{1\ 13}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{12, \min\{5,9\}\}$$
$$= 5$$

$i=2,\ j=14$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 14} = \min\{d_{1\ 14}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{8, \min\{5,9\}\}$$
$$= 5$$

$i=2,\ j=15$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 15} = \min\{d_{1\ 15}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{11, \min\{5,9\}\}$$
$$= 5$$

$i=2,\ j=16$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 16} = \min\{d_{2\ 16}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{5, \min\{5,9\}\}$$
$$= 5$$

$i=3,\ j=2$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 17} = \min\{d_{2\ 17}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{7, \min\{5,9\}\}$$
$$= 5$$

$i=2,\ j=18$ and $Q=\{1,4\}$

$$\hat{D}_{2\ 18} = \min\{d_{2\ 18}, \min\{d_{21}, d_{24}\}\}$$

$$= \min\{12, \min\{5,9\}\}$$
$$= 5$$

**For Node 3**

$i=3,\ j=1$ and $Q=\{1,4\}$

$$\hat{D}_{31} = \min\{d_{35}, \min\{d_{31}, d_{34}\}\}$$

$$= \min\{7, \min\{7,7\}\}$$
$$= 7$$

$i=3,\ j=2$ and $Q=\{1,4\}$

$$\hat{D}_{32} = \min\{d_{32}, \min\{d_{31}, d_{34}\}\}$$

$$= \min\{2, \min\{7,7\}\}$$
$$= 2$$

$i = 3,\ j = 3$ and $Q = \{1,4\}$

$$\hat{D}_{33} = \min\{d_{33}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{0, \min\{7,7\}\}$$
$$= 0$$

$i = 3,\ j = 4$ and $Q = \{1,4\}$

$$\hat{D}_{34} = \min\{d_{34}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{7, \min\{7,7\}\}$$
$$= 7$$

$i = 3,\ j = 5$ and $Q = \{1,4\}$

$$\hat{D}_{35} = \min\{d_{35}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{9, \min\{7,7\}\}$$
$$= 7$$

$i = 3,\ j = 6$ and $Q = \{1,4\}$

$$\hat{D}_{36} = \min\{d_{36}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{6, \min\{7,7\}\}$$
$$= 6$$

$i = 3,\ j = 7$ and $Q = \{1,4\}$

$$\hat{D}_{37} = \min\{d_{37}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{3, \min\{7,7\}\}$$
$$= 3$$

$i = 3,\ j = 8$ and $Q = \{1,4\}$

$$\hat{D}_{38} = \min\{d_{38}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{12, \min\{7,7\}\}$$
$$= 7$$

$i = 3,\ j = 9$ and $Q = \{1,4\}$

$$\hat{D}_{39} = \min\{d_{39}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{10, \min\{7,7\}\}$$
$$= 7$$

$i = 3,\ j = 10$ and $Q = \{1,4\}$

$$\hat{D}_{3\ 10} = \min\{d_{3\ 10}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{12, \min\{7,7\}\}$$
$$= 7$$

$i = 3,\ j = 11$ and $Q = \{1,4\}$

$$\hat{D}_{3\ 11} = \min\{d_{3\ 11}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{16, \min\{7,7\}\}$$
$$= 7$$

$i = 3,\ j = 12$ and $Q = \{1,4\}$

$$\hat{D}_{3\ 12} = \min\{d_{3\ 12}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{14, \min\{7,7\}\}$$
$$= 7$$

$i = 3,\ j = 13$ and $Q = \{1, 4\}$

$$\hat{D}_{3\ 13} = \min\{d_{3\ 13}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{14, \min\{7, 7\}\}$$
$$= 7$$

$i = 3,\ j = 14$ and $Q = \{1, 4\}$

$$\hat{D}_{3\ 14} = \min\{d_{3\ 14}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{10, \min\{7, 7\}\}$$
$$= 7$$

$i = 3,\ j = 15$ and $Q = \{1, 4\}$

$$\hat{D}_{3\ 15} = \min\{d_{3\ 15}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{15, \min\{7, 7\}\}$$
$$= 7$$

$i = 3,\ j = 16$ and $Q = \{1, 4\}$

$$\hat{D}_{3\ 16} = \min\{d_{3\ 16}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{7, \min\{7, 7\}\}$$
$$= 7$$

$i = 3,\ j = 17$ and $Q = \{1, 4\}$

$$\hat{D}_{3\ 17} = \min\{d_{3\ 17}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{9, \min\{7, 7\}\}$$
$$= 7$$

$i = 3,\ j = 18$ and $Q = \{1, 4\}$

$$\hat{D}_{3\ 18} = \min\{d_{3\ 18}, \min\{d_{31}, d_{34}\}\}$$
$$= \min\{12, \min\{7, 7\}\}$$
$$= 7$$

**For Node 4**

$i = 4,\ j = 1$ and $Q = \{1, 4\}$

$$\hat{D}_{41} = \min\{d_{41}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{14, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 2$ and $Q = \{1, 4\}$

$$\hat{D}_{42} = \min\{d_{42}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{9, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{43} = \min\{d_{43}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{7, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 4$ and $Q = \{1, 4\}$

$$\hat{D}_{44} = \min\{d_{44}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{0, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{45} = \min\{d_{45}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{2, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{46} = \min\{d_{46}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{8, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{47} = \min\{d_{47}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{10, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{48} = \min\{d_{48}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{14, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{49} = \min\{d_{49}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{16, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 10$ and $Q = \{1, 4\}$

$$\hat{D}_{4\ 10} = \min\{d_{4\ 10}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{19, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 11$ and $Q = \{1, 4\}$

$$\hat{D}_{4\ 11} = \min\{d_{4\ 11}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{23, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 12$ and $Q = \{1, 4\}$

$$\hat{D}_{4\ 12} = \min\{d_{4\ 12}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{21, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 13$ and $Q = \{1, 4\}$

$$\hat{D}_{4\ 13} = \min\{d_{4\ 13}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{21, \min\{14, 0\}\}$$
$$= 0$$

$i = 4,\ j = 14$ and $Q = \{1, 4\}$

$$\hat{D}_{4\ 14} = \min\{d_{4\ 14}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{17, \min\{14, 0\}\}$$
$$= 0$$

$i=4,\ j=15$ and $Q=\{1,4\}$

$$\hat{D}_{4\ 15} = \min\{d_{4\ 15}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{20, \min\{14, 0\}\}$$
$$= 0$$

$i=4,\ j=16$ and $Q=\{1,4\}$

$$\hat{D}_{4\ 16} = \min\{d_{4\ 16}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{14, \min\{14, 0\}\}$$
$$= 0$$

$i=4,\ j=17$ and $Q=\{1,4\}$

$$\hat{D}_{4\ 17} = \min\{d_{4\ 17}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{10, \min\{14, 0\}\}$$
$$= 0$$

$i=4,\ j=18$ and $Q=\{1,4\}$

$$\hat{D}_{4\ 13} = \min\{d_{4\ 18}, \min\{d_{41}, d_{44}\}\}$$
$$= \min\{5, \min\{14, 0\}\}$$
$$= 0$$

$i=5,\ j=1$ and $Q=\{1,4\}$

$$\hat{D}_{51} = \min\{d_{51}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{16, \min\{16, 2\}\}$$
$$= 2$$

$i=5,\ j=2$ and $Q=\{1,4\}$

$$\hat{D}_{52} = \min\{d_{52}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{11, \min\{16, 2\}\}$$
$$= 2$$

$i=5,\ j=3$ and $Q=\{1,4\}$

$$\hat{D}_{53} = \min\{d_{53}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{9, \min\{16, 2\}\}$$
$$= 2$$

$i=5,\ j=4$ and $Q=\{1,4\}$

$$\hat{D}_{54} = \min\{d_{54}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{2, \min\{16, 2\}\}$$
$$= 2$$

$i=5,\ j=5$ and $Q=\{1,4\}$

$$\hat{D}_{55} = \min\{d_{55}, \min\{d_{51}, d_{54}\}$$
$$= \min\{0, \min\{16, 2\}\}$$
$$= 0$$

$i=5,\ j=6$ and $Q=\{1,4\}$

$$\hat{D}_{56} = \min\{d_{56}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{6, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{57} = \min\{d_{57}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{9, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{58} = \min\{d_{58}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{12, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{59} = \min\{d_{59}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{14, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 10$ and $Q = \{1, 4\}$

$$\hat{D}_{5\ 10} = \min\{d_{5\ 10}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{18, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 11$ and $Q = \{1, 4\}$

$$\hat{D}_{5\ 11} = \min\{d_{5\ 11}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{25, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 12$ and $Q = \{1, 4\}$

$$\hat{D}_{5\ 12} = \min\{d_{5\ 12}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{23, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 13$ and $Q = \{1, 4\}$

$$\hat{D}_{5\ 13} = \min\{d_{5\ 13}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{23, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 14$ and $Q = \{1, 4\}$

$$\hat{D}_{5\ 14} = \min\{d_{5\ 14}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{19, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 15$ and $Q = \{1, 4\}$

$$\hat{D}_{5\ 15} = \min\{d_{5\ 15}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{22, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 16$ and $Q = \{1, 4\}$

$$\hat{D}_{5\ 16} = \min\{d_{5\ 16}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{16, \min\{16, 2\}\}$$
$$= 2$$

$i = 5,\ j = 17$ and $Q = \{1,4\}$

$$\hat{D}_{5\ 17} = \min\{d_{5\ 17}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{12, \min\{16,2\}\}$$
$$= 2$$

$i = 5,\ j = 18$ and $Q = \{1,4\}$

$$\hat{D}_{5\ 18} = \min\{d_{5\ 18}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{7, \min\{16,2\}\}$$
$$= 2$$

**For Node 6**

$i = 6,\ j = 1$ and $Q = \{1,4\}$

$$\hat{D}_{61} = \min\{d_{61}, \min\{d_{51}, d_{54}\}\}$$
$$= \min\{10, \min\{10,8\}\}$$
$$= 8$$

$i = 6,\ j = 2$ and $Q = \{1,4\}$

$$\hat{D}_{62} = \min\{d_{62}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{8, \min\{10,8\}\}$$
$$= 8$$

$i = 6,\ j = 3$ and $Q = \{1,4\}$

$$\hat{D}_{63} = \min\{d_{63}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{6, \min\{10,8\}\}$$
$$= 6$$

$i = 6,\ j = 4$ and $Q = \{1,4\}$

$$\hat{D}_{64} = \min\{d_{64}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{8, \min\{10,8\}\}$$
$$= 8$$

$i = 6,\ j = 5$ and $Q = \{1,4\}$

$$\hat{D}_{65} = \min\{d_{65}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{6, \min\{10,8\}\}$$
$$= 6$$

$i = 6,\ j = 6$ and $Q = \{1,4\}$

$$\hat{D}_{66} = \min\{d_{66}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{0, \min\{10,8\}\}$$
$$= 0$$

$i = 6,\ j = 7$ and $Q = \{1,4\}$

$$\hat{D}_{67} = \min\{d_{67}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{3, \min\{10,8\}\}$$
$$= 3$$

$i = 6,\ j = 8$ and $Q = \{1,4\}$

$$\hat{D}_{68} = \min\{d_{68}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{6, \min\{10,8\}\}$$
$$= 6$$

$i=6,\ j=9$ and $Q=\{1,4\}$

$$\hat{D}_{69} = \min\{d_{69}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{8, \min\{10,8\}\}$$
$$=8$$

$i=6,\ j=10$ and $Q=\{1,4\}$

$$\hat{D}_{6\ 10} = \min\{d_{6\ 10}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{12, \min\{10,8\}\}$$
$$=8$$

$i=6,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{6\ 11} = \min\{d_{6\ 11}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{20, \min\{10,8\}\}$$
$$=8$$

$i=6,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{6\ 12} = \min\{d_{6\ 12}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{17, \min\{10,8\}\}$$
$$=8$$

$i=6,\ j=13$ and $Q=\{1,4\}$

$$\hat{D}_{6\ 13} = \min\{d_{6\ 13}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{20, \min\{10,8\}\}$$
$$=8$$

$i=6,\ j=14$ and $Q=\{1,4\}$

$$\hat{D}_{6\ 14} = \min\{d_{6\ 14}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{16, \min\{10,8\}\}$$
$$=8$$

$i=6,\ j=15$ and $Q=\{1,4\}$

$$\hat{D}_{6\ 15} = \min\{d_{6\ 15}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{17, \min\{10,8\}\}$$
$$=8$$

$i=6,\ j=16$ and $Q=\{1,4\}$

$$\hat{D}_{6\ 16} = \min\{d_{6\ 16}, \min\{d_{61}, d_{64}\}\}$$
$$= \min\{13, \min\{10,8\}\}$$
$$=8$$

$i = 6,\ j = 17$ and $Q = \{1, 4\}$

$$\hat{D}_{6\ 17} = \min\{d_{6\ 17}, \min\{d_{61}, d_{64}\}\}$$

$$= \min\{15, \min\{10, 8\}\}$$
$$= 8$$

$i = 6,\ j = 18$ and $Q = \{1, 4\}$

$$\hat{D}_{6\ 18} = \min\{d_{6\ 18}, \min\{d_{61}, d_{64}\}\}$$

$$= \min\{13, \min\{10, 8\}\}$$
$$= 8$$

$i = 7,\ j = 1$ and $Q = \{1, 4\}$

$$\hat{D}_{71} = \min\{d_{71}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{7, \min\{7, 10\}\}$$
$$= 7$$

$i = 7,\ j = 2$ and $Q = \{1, 4\}$

$$\hat{D}_{72} = \min\{d_{72}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{5, \min\{7, 10\}\}$$
$$= 5$$

$i = 7,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{73} = \min\{d_{73}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{3, \min\{7, 10\}\}$$
$$= 3$$

$i = 7,\ j = 4$ and $Q = \{1, 4\}$

$$\hat{D}_{74} = \min\{d_{74}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{10, \min\{7, 10\}\}$$
$$= 7$$

$i = 7,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{75} = \min\{d_{75}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{9, \min\{7, 10\}\}$$
$$= 7$$

$i = 7,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{76} = \min\{d_{76}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{3, \min\{7, 10\}\}$$
$$= 3$$

$i = 7,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{77} = \min\{d_{77}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{0, \min\{7, 10\}\}$$
$$= 0$$

$i = 7,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{78} = \min\{d_{78}, \min\{d_{71}, d_{74}\}\}$$

$$= \min\{9, \min\{7, 10\}\}$$
$$= 7$$

$i=7,\ j=9$ and $Q=\{1,4\}$

$$\hat{D}_{79} = \min\{d_{71}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{10, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=10$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 10} = \min\{d_{7\ 10}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{12, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 11} = \min\{d_{7\ 11}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{17, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 12} = \min\{d_{7\ 12}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{14, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=13$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 13} = \min\{d_{7\ 13}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{17, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=14$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 14} = \min\{d_{7\ 14}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{13, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=15$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 15} = \min\{d_{7\ 15}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{14, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=16$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 16} = \min\{d_{7\ 16}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{10, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=17$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 17} = \min\{d_{7\ 17}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{12, \min\{7,10\}\}$$
$$= 7$$

$i=7,\ j=18$ and $Q=\{1,4\}$

$$\hat{D}_{7\ 18} = \min\{d_{7\ 18}, \min\{d_{71}, d_{74}\}\}$$
$$= \min\{15, \min\{7,10\}\}$$
$$= 7$$

**For Node 8**

$i=8,\ j=1$ and $Q=\{1,4\}$

$$\hat{D}_{81} = \min\{d_{81}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{5, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=2$ and $Q=\{1,4\}$

$$\hat{D}_{82} = \min\{d_{82}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{10, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=3$ and $Q=\{1,4\}$

$$\hat{D}_{83} = \min\{d_{83}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{12, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=4$ and $Q=\{1,4\}$

$$\hat{D}_{84} = \min\{d_{84}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{14, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=5$ and $Q=\{1,4\}$

$$\hat{D}_{85} = \min\{d_{85}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{12, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=6$ and $Q=\{1,4\}$

$$\hat{D}_{86} = \min\{d_{86}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{6, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=7$ and $Q=\{1,4\}$

$$\hat{D}_{87} = \min\{d_{87}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{9, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=8$ and $Q=\{1,4\}$

$$\hat{D}_{88} = \min\{d_{88}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{0, \min\{5,14\}\}$$
$$= 0$$

$i=8,\ j=9$ and $Q=\{1,4\}$

$$\hat{D}_{89} = \min\{d_{89}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{2, \min\{5,14\}\}$$
$$= 2$$

$i=8,\ j=10$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 10} = \min\{d_{8\ 10}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{6, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 11} = \min\{d_{8\ 11}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{14, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 12} = \min\{d_{8\ 12}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{12, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=13$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 13} = \min\{d_{8\ 13}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{15, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=14$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 14} = \min\{d_{8\ 14}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{15, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=15$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 15} = \min\{d_{8\ 15}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{12, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=16$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 16} = \min\{d_{8\ 16}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{15, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=17$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 17} = \min\{d_{8\ 17}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{17, \min\{5,14\}\}$$
$$= 5$$

$i=8,\ j=18$ and $Q=\{1,4\}$

$$\hat{D}_{8\ 18} = \min\{d_{8\ 18}, \min\{d_{81}, d_{84}\}\}$$
$$= \min\{19, \min\{5,14\}\}$$
$$= 5$$

**For Node 9**

$i=9,\ j=1$ and $Q=\{1,4\}$

$$\hat{D}_{91} = \min\{d_{91}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{3, \min\{3,16\}\}$$
$$= 3$$

$i=9,\ j=2$ and $Q=\{1,4\}$

$$\hat{D}_{92} = \min\{d_{92}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{8, \min\{3,16\}\}$$
$$= 3$$

$i=9,\ j=3$ and $Q=\{1,4\}$

$$\hat{D}_{93} \quad =\min\{d_{93},\min\{d_{91},d_{94}\}\}$$

$$=\min\{3,\min\{3,16\}\}$$
$$=3$$

$i=9,\ j=4$ and $Q=\{1,4\}$

$$\hat{D}_{94} \quad =\min\{d_{94},\min\{d_{91},d_{94}\}\}$$

$$=\min\{16,\min\{3,16\}\}$$
$$=3$$

$i=9,\ j=5$ and $Q=\{1,4\}$

$$\hat{D}_{95} \quad =\min\{d_{95},\min\{d_{91},d_{94}\}\}$$

$$=\min\{14,\min\{3,16\}\}$$
$$=3$$

$i=9,\ j=6$ and $Q=\{1,4\}$

$$\hat{D}_{96} \quad =\min\{d_{96},\min\{d_{91},d_{94}\}\}$$

$$=\min\{8,\min\{3,16\}\}$$
$$=3$$

$i=9,\ j=7$ and $Q=\{1,4\}$

$$\hat{D}_{97} \quad =\min\{d_{97},\min\{d_{91},d_{94}\}\}$$

$$=\min\{10,\min\{3,16\}\}$$
$$=3$$

$i=9,\ j=8$ and $Q=\{1,4\}$

$$\hat{D}_{98} \quad =\min\{d_{93},\min\{d_{91},d_{94}\}\}$$

$$=\min\{2,\min\{3,16\}\}$$
$$=2$$

$i=9,\ j=9$ and $Q=\{1,4\}$

$$\hat{D}_{99} \quad =\min\{d_{99},\min\{d_{91},d_{94}\}\}$$

$$=\min\{0,\min\{3,16\}\}$$
$$=0$$

$i=9,\ j=10$ and $Q=\{1,4\}$

$$\hat{D}_{9\ 10} \quad =\min\{d_{9\ 10},\min\{d_{91},d_{94}\}\}$$

$$=\min\{4,\min\{3,16\}\}$$
$$=3$$

$i=9,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{9\ 11} \quad =\min\{d_{9\ 11},\min\{d_{91},d_{94}\}\}$$

$$=\min\{12,\min\{3,16\}\}$$
$$=3$$

$i=9,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{9\ 12} \quad =\min\{d_{9\ 12},\min\{d_{91},d_{94}\}\}$$

$$=\min\{10,\min\{3,16\}\}$$
$$=3$$

$i = 9,\ j = 13$ and $Q = \{1,4\}$

$$\hat{D}_{9\ 13} = \min\{d_{9\ 13}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{13, \min\{3,16\}\}$$
$$= 3$$

$i = 9,\ j = 14$ and $Q = \{1,4\}$

$$\hat{D}_{9\ 14} = \min\{d_{9\ 14}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{13, \min\{3,16\}\}$$
$$= 3$$

$i = 9,\ j = 15$ and $Q = \{1,4\}$

$$\hat{D}_{9\ 15} = \min\{d_{9\ 15}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{10, \min\{3,16\}\}$$
$$= 3$$

$i = 9,\ j = 16$ and $Q = \{1,4\}$

$$\hat{D}_{9\ 16} = \min\{d_{9\ 16}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{13, \min\{3,16\}\}$$
$$= 3$$

$i = 9,\ j = 17$ and $Q = \{1,4\}$

$$\hat{D}_{9\ 17} = \min\{d_{9\ 17}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{15, \min\{3,16\}\}$$
$$= 3$$

$i = 9,\ j = 18$ and $Q = \{1,4\}$

$$\hat{D}_{9\ 18} = \min\{d_{9\ 18}, \min\{d_{91}, d_{94}\}\}$$
$$= \min\{20, \min\{3,16\}\}$$
$$= 3$$

**For Node 10**

$i = 10,\ j = 1$ and $Q = \{1,4\}$

$$\hat{D}_{10\ 1} = \min\{d_{10\ 1}, \min\{d_{10\ 1}, d_{10\ 4}\}$$
$$= \min\{5, \min\{5,19\}\}$$
$$= 5$$

$i = 10,\ j = 2$ and $Q = \{1,4\}$

$$\hat{D}_{10\ 2} = \min\{d_{10\ 2}, \min\{d_{10\ 1}, d_{10\ 4}\}$$
$$= \min\{10, \min\{5,19\}\}$$
$$= 5$$

$i = 10,\ j = 3$ and $Q = \{1,4\}$

$$\hat{D}_{10\ 3} = \min\{d_{10\ 3}, \min\{d_{10\ 1}, d_{10\ 4}\}$$
$$= \min\{12, \min\{5,19\}\}$$
$$= 5$$

$i = 10,\ j = 4$ and $Q = \{1,4\}$

$$\hat{D}_{10\ 4} = \min\{d_{10\ 4}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{19, \min\{5,19\}\}$$
$$= 5$$

$i = 10,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 5} = \min\{d_{10\ 5}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{18, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 6} = \min\{d_{10\ 6}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{12, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 7} = \min\{d_{10\ 7}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{12, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 8} = \min\{d_{10\ 8}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{6, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 9} = \min\{d_{10\ 9}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{4, \min\{5, 19\}\}$$
$$= 4$$

$i = 10,\ j = 10$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 10} = \min\{d_{10\ 10}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{0, \min\{5, 19\}\}$$
$$= 0$$



$i = 10,\ j = 11$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 11} = \min\{d_{10\ 11}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{8, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 12$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 12} = \min\{d_{10\ 12}, \min\{d_{10\ 1}, d_{10}$$
$$= \min\{11, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 13$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 13} = \min\{d_{10\ 13}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{10, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 14$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 14} = \min\{d_{10\ 14}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$
$$= \min\{14, \min\{5, 19\}\}$$
$$= 5$$

$i = 10,\ j = 15$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 15} = \min\{d_{10\ 15}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$

$$= \min\{12, \min\{5,19\}\}$$
$$= 5$$

$i = 10,\ j = 16$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 16} = \min\{d_{10\ 16}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$

$$= \min\{15, \min\{5,19\}\}$$
$$= 5$$

$i = 10,\ j = 17$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 17} = \min\{d_{10\ 17}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$

$$= \min\{17, \min\{5,19\}\}$$
$$= 5$$

$i = 10,\ j = 18$ and $Q = \{1, 4\}$

$$\hat{D}_{10\ 18} = \min\{d_{10\ 18}, \min\{d_{10\ 1}, d_{10\ 4}\}\}$$

$$= \min\{22, \min\{5,19\}\}$$
$$= 5$$

**For Node 11**

$i = 11,\ j = 1$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 1} = \min\{d_{11\ 1}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{22, \min\{5,19\}\}$$
$$= 5$$

$i = 11,\ j = 2$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 2} = \min\{d_{11\ 2}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{14, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 3} = \min\{d_{11\ 3}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{16, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 4$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 4} = \min\{d_{11\ 4}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{23, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 5} = \min\{d_{11\ 5}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{25, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 6} = \min\{d_{11\ 6}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{20, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 7} = \min\{d_{11\ 7}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{17, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 8} = \min\{d_{11\ 8}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{14, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 9} = \min\{d_{11\ 9}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{12, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 10$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 10} = \min\{d_{11\ 10}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{8, \min\{10, 23\}\}$$
$$= 8$$

$i = 11,\ j = 11$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 11} = \min\{d_{11\ 11}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$
$$= \min\{0, \min\{10, 23\}\}$$
$$= 0$$

$i = 11,\ j = 12$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 12} = \min\{d_{11\ 12}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{3, \min\{10, 23\}\}$$
$$= 3$$

$i = 11,\ j = 13$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 13} = \min\{d_{11\ 13}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{2, \min\{10, 23\}\}$$
$$= 2$$

$i = 11,\ j = 14$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 14} = \min\{d_{11\ 14}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{6, \min\{10, 23\}\}$$
$$= 6$$

$i = 11,\ j = 15$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 15} = \min\{d_{11\ 15}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{6, \min\{10, 23\}\}$$
$$= 6$$

$i = 11,\ j = 16$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 16} = \min\{d_{11\ 16}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{9, \min\{10, 23\}\}$$
$$= 9$$

$i = 11,\ j = 17$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 17} = \min\{d_{11\ 17}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{13, \min\{10, 23\}\}$$
$$= 10$$

$i = 11,\ j = 18$ and $Q = \{1, 4\}$

$$\hat{D}_{11\ 18} = \min\{d_{11\ 18}, \min\{d_{11\ 1}, d_{11\ 4}\}\}$$

$$= \min\{18, \min\{10, 23\}\}$$
$$= 10$$

**For Node 12**

$i = 12,\ j = 1$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 1} = \min\{d_{12\ 1}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$

$$= \min\{7, \min\{7, 21\}\}$$
$$= 7$$

$i = 12,\ j = 2$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 2} = \min\{d_{12\ 2}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$

$$= \min\{12, \min\{7, 21\}\}$$
$$= 7$$

$i = 12,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 3} = \min\{d_{12\ 3}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$

$$= \min\{14, \min\{7, 21\}\}$$
$$= 7$$

$i=12,\ j=4$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 4}=\min\{d_{12\ 4},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{21,\min\{7,21\}\}$$
$$=7$$

$i=12,\ j=5$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 5}=\min\{d_{12\ 5},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{23,\min\{7,21\}\}$$
$$=7$$

$i=12,\ j=6$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 6}=\min\{d_{12\ 6},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$
$$=\min\{17,\min\{7,21\}\}$$
$$=7$$

$i=12,\ j=7$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 7}=\min\{d_{12\ 7},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{14,\min\{7,21\}\}$$
$$=7$$

$i=12,\ j=8$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 8}=\min\{d_{12\ 8},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{12,\min\{7,21\}\}$$
$$=7$$

$i=12,\ j=9$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 9}=\min\{d_{12\ 9},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{10,\min\{7,21\}\}$$
$$=7$$

$i=12,\ j=10$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 10}=\min\{d_{12\ 10},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{11,\min\{7,21\}\}$$
$$=7$$

$i=12,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 11}=\min\{d_{12\ 11},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$
$$=\min\{3,\min\{7,21\}\}$$
$$=3$$

$i=12,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 12}=\min\{d_{12\ 12},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{0,\min\{7,21\}\}$$
$$=0$$

$i=12,\ j=13$ and $Q=\{1,4\}$

$$\hat{D}_{12\ 13}=\min\{d_{12\ 13},\ \min\{d_{12\ 1},\ d_{12\ 4}\}\}$$

$$=\min\{3,\min\{7,21\}\}$$
$$=3$$

$i = 12,\ j = 14$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 14} = \min\{d_{12\ 14}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$
$$= \min\{6, \min\{7, 21\}\}$$
$$= 6$$

$i = 12,\ j = 15$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 15} = \min\{d_{12\ 15}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$
$$= \min\{3, \min\{7, 21\}\}$$
$$= 3$$

$i = 12,\ j = 16$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 16} = \min\{d_{12\ 16}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$
$$= \min\{9, \min\{7, 21\}\}$$
$$= 7$$

$i = 12,\ j = 17$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 17} = \min\{d_{12\ 17}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$
$$= \min\{13, \min\{7, 21\}\}$$
$$= 7$$

$i = 12,\ j = 18$ and $Q = \{1, 4\}$

$$\hat{D}_{12\ 18} = \min\{d_{12\ 18}, \min\{d_{12\ 1}, d_{12\ 4}\}\}$$
$$= \min\{18, \min\{7, 21\}\}$$
$$= 7$$

**For Node 13**

$i = 13,\ j = 1$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 1} = \min\{d_{13\ 1}, \min\{d_{13\ 1}, d_{13\ 4}\}\}$$
$$= \min\{10, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 2$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 2} = \min\{d_{13\ 2}, \min\{d_{13\ 1}, d_{13\ 4}\}\}$$
$$= \min\{12, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 3} = \min\{d_{13\ 3}, \min\{d_{13\ 1}, d_{13\ 4}\}\}$$
$$= \min\{14, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 4$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 4} = \min\{d_{13\ 4}, \min\{d_{13\ 1}, d_{13\ 4}\}\}$$
$$= \min\{21, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 5} = \min\{d_{13\ 5}, \min\{d_{13\ 1}, d_{13\ 4}\}\}$$
$$= \min\{23, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 6} = \min\{d_{13\ 6},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{20, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 7} = \min\{d_{13\ 7},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{17, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 8} = \min\{d_{13\ 8},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{15, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 9} = \min\{d_{13\ 9},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{13, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 10$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 10} = \min\{d_{13\ 10},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{10, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 11$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 11} = \min\{d_{13\ 11},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{2, \min\{10, 21\}\}$$
$$= 2$$

$i = 13,\ j = 12$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 12} = \min\{d_{13\ 12},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{3, \min\{10, 21\}\}$$
$$= 3$$

$i = 13,\ j = 13$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 13} = \min\{d_{13\ 13},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{0, \min\{10, 21\}\}$$
$$= 0$$

$i = 13,\ j = 14$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 14} = \min\{d_{13\ 14},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{4, \min\{10, 21\}\}$$
$$= 4$$

$i = 13,\ j = 15$ and $Q = \{1, 4\}$

$$\hat{D}_{13\ 15} = \min\{d_{13\ 15},\ \min\{d_{13\ 1},\ d_{13\ 4}\}\}$$
$$= \min\{6, \min\{10, 21\}\}$$
$$= 6$$

$i = 13,\ j = 16$ and $Q = \{1,4\}$

$$\hat{D}_{13\ \ 16} = \min\{d_{13\ \ 16}, \min\{d_{13\ \ 1}, d_{13\ \ 4}\}\}$$

$$= \min\{7, \min\{10, 21\}\}$$
$$= 7$$

$i = 13,\ j = 17$ and $Q = \{1,4\}$

$$\hat{D}_{13\ \ 17} = \min\{d_{13\ \ 17}, \min\{d_{13\ \ 1}, d_{13\ \ 4}\}\}$$

$$= \min\{11, \min\{10, 21\}\}$$
$$= 10$$

$i = 13,\ j = 18$ and $Q = \{1,4\}$

$$\hat{D}_{13\ \ 18} = \min\{d_{13\ \ 18}, \min\{d_{13\ \ 1}, d_{13\ \ 4}\}\}$$

$$= \min\{16, \min\{10, 21\}\}$$
$$= 10$$

**For Node 14**

$i = 14,\ j = 1$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 1} = \min\{d_{14\ \ 1}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$

$$= \min\{10, \min\{10, 17\}\}$$
$$= 10$$

$i = 14,\ j = 2$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 2} = \min\{d_{14\ \ 2}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$

$$= \min\{8, \min\{10, 17\}\}$$
$$= 8$$

$i = 14,\ j = 3$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 3} = \min\{d_{14\ \ 3}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$

$$= \min\{10, \min\{10, 17\}\}$$
$$= 10$$

$i = 14,\ j = 4$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 4} = \min\{d_{14\ \ 4}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$

$$= \min\{17, \min\{10, 17\}\}$$
$$= 10$$

$i = 14,\ j = 5$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 5} = \min\{d_{14\ \ 5}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$

$$= \min\{19, \min\{10, 17\}\}$$
$$= 10$$

$i = 14,\ j = 6$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 6} = \min\{d_{14\ \ 6}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$

$$= \min\{16, \min\{10, 17\}\}$$
$$= 10$$

$i = 14,\ j = 7$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 7} = \min\{d_{14\ \ 7}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$

$$= \min\{13, \min\{10, 17\}\}$$
$$= 10$$

$i = 14,\ j = 8$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 8} = \min\{d_{14\ \ 8},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{15, \min\{10,17\}\}$$
$$= 10$$

$i = 14,\ j = 9$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 9} = \min\{d_{14\ \ 9},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{13, \min\{10,17\}\}$$
$$= 10$$

$i = 14,\ j = 10$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 10} = \min\{d_{14\ \ 10},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{14, \min\{10,17\}\}$$
$$= 10$$

$i = 14,\ j = 11$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 11} = \min\{d_{14\ \ 11},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{6, \min\{10,17\}\}$$
$$= 6$$

$i = 14,\ j = 12$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 12} = \min\{d_{14\ \ 12},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{6, \min\{10,17\}\}$$
$$= 6$$

$i = 14,\ j = 13$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 13} = \min\{d_{14\ \ 13},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{4, \min\{10,17\}\}$$
$$= 4$$

$i = 14,\ j = 14$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 14} = \min\{d_{14\ \ 14},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{0, \min\{10,17\}\}$$
$$= 0$$

$i = 14,\ j = 15$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 15} = \min\{d_{14\ \ 15},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{3, \min\{10,17\}\}$$
$$= 3$$

$i = 14,\ j = 16$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 16} = \min\{d_{14\ \ 16},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{3, \min\{10,17\}\}$$
$$= 3$$

$i = 14,\ j = 17$ and $Q = \{1,4\}$

$$\hat{D}_{14\ \ 17} = \min\{d_{14\ \ 17},\ \min\{d_{14\ \ 1},\ d_{14\ \ 4}\}\}$$
$$= \min\{7, \min\{10,17\}\}$$
$$= 7$$

$i = 14,\ j = 18$ and $Q = \{1, 4\}$

$$\hat{D}_{14\ \ 18} = \min\{d_{14\ \ 18}, \min\{d_{14\ \ 1}, d_{14\ \ 4}\}\}$$
$$= \min\{12, \min\{10, 17\}\}$$
$$= 10$$

$i = 15,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 5} = \min\{d_{15\ \ 5}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{22, \min\{7, 20\}\}$$
$$= 7$$

**For Node 15**

$i = 15,\ j = 1$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 1} = \min\{d_{15\ \ 1}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{7, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 6} = \min\{d_{15\ \ 6}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{17, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 2$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 2} = \min\{d_{15\ \ 2}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{11, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 7} = \min\{d_{15\ \ 7}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{14, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 3} = \min\{d_{15\ \ 3}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{13, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 8} = \min\{d_{15\ \ 8}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{12, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 4$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 4} = \min\{d_{15\ \ 4}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{20, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{15\ \ 9} = \min\{d_{15\ \ 9}, \min\{d_{15\ \ 1}, d_{15\ \ 4}\}\}$$
$$= \min\{10, \min\{7, 20\}\}$$
$$= 7$$

$i = 15,\ j = 10$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 10} = \min\{d_{15\ 10}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{12, \min\{7,20\}\}$$
$$= 7$$

$i = 15,\ j = 11$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 11} = \min\{d_{15\ 11}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{6, \min\{7,20\}\}$$
$$= 6$$

$i = 15,\ j = 12$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 12} = \min\{d_{15\ 12}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{3, \min\{7,20\}\}$$
$$= 3$$

$i = 15,\ j = 13$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 13} = \min\{d_{15\ 13}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{6, \min\{7,20\}\}$$
$$= 6$$

$i = 15,\ j = 14$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 14} = \min\{d_{15\ 14}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{3, \min\{7,20\}\}$$
$$= 3$$

$i = 15,\ j = 15$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 15} = \min\{d_{15\ 15}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{0, \min\{7,20\}\}$$
$$= 0$$

$i = 15,\ j = 16$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 16} = \min\{d_{15\ 16}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{6, \min\{7,20\}\}$$
$$= 6$$

$i = 15,\ j = 17$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 17} = \min\{d_{15\ 16}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{10, \min\{7,20\}\}$$
$$= 7$$

$i = 15,\ j = 18$ and $Q = \{1,4\}$

$$\hat{D}_{15\ 18} = \min\{d_{15\ 18}, \min\{d_{15\ 1}, d_{15\ 4}\}\}$$
$$= \min\{15, \min\{7,20\}\}$$
$$= 7$$

**For Node 16**

$i = 16,\ j = 1$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 1} = \min\{d_{16\ 1}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{10, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 2$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 2} = \min\{d_{16\ 2}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{5, \min\{10, 14\}\}$$
$$= 5$$

$i = 16,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 3} = \min\{d_{16\ 3}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{7, \min\{10, 14\}\}$$
$$= 7$$

$i = 16,\ j = 4$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 4} = \min\{d_{16\ 4}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{14, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 5} = \min\{d_{16\ 5}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{16, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 6} = \min\{d_{16\ 6}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{13, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 7} = \min\{d_{16\ 7}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{10, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 8} = \min\{d_{16\ 8}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{15, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 9} = \min\{d_{16\ 9}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{13, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 10$ and $Q = \{1, 4\}$

$$\hat{D}_{16\ 10} = \min\{d_{16\ 10}, \min\{d_{16\ 1}, d_{16\ 4}\}\}$$
$$= \min\{15, \min\{10, 14\}\}$$
$$= 10$$

$i = 16,\ j = 11$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 11} = \min\{d_{16\ \ 11}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{9, \min\{10, 14\}\}$
$= 9$

$i = 16,\ j = 12$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 12} = \min\{d_{16\ \ 12}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{9, \min\{10, 14\}\}$
$= 9$

$i = 16,\ j = 13$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 13} = \min\{d_{16\ \ 13}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{7, \min\{10, 14\}\}$
$= 7$

$i = 16,\ j = 14$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 14} = \min\{d_{16\ \ 14}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{3, \min\{10, 14\}\}$
$= 3$

$i = 16,\ j = 15$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 15} = \min\{d_{16\ \ 15}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{6, \min\{10, 14\}\}$
$= 6$

$i = 16,\ j = 16$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 16} = \min\{d_{16\ \ 16}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{0, \min\{10, 14\}\}$
$= 0$

$i = 16,\ j = 17$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 17} = \min\{d_{16\ \ 17}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{4, \min\{10, 14\}\}$
$= 4$

$i = 16,\ j = 18$ and $Q = \{1, 4\}$

$\hat{D}_{16\ \ 18} = \min\{d_{16\ \ 18}, \min\{d_{16\ \ 1}, d_{16\ \ 4}\}\}$

$= \min\{9, \min\{10, 14\}\}$
$= 9$

**For Node 17**

$i = 17,\ j = 1$ and $Q = \{1, 4\}$

$\hat{D}_{17\ \ 1} = \min\{d_{17\ \ 1}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$

$= \min\{12, \min\{12, 10\}\}$
$= 10$

$i = 17,\ j = 2$ and $Q = \{1, 4\}$

$\hat{D}_{17\ \ 2} = \min\{d_{17\ \ 2}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$

$= \min\{7, \min\{12, 10\}\}$
$= 7$

$i = 17,\ j = 3$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 3} = \min\{d_{17\ 3}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{9, \min\{12,10\}\}$$
$$= 9$$

$i = 17,\ j = 4$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 4} = \min\{d_{17\ 4}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{10, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 5$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 5} = \min\{d_{17\ 5}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{12, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 6$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 6} = \min\{d_{17\ 6}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{15, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 7$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 7} = \min\{d_{17\ 7}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{12, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 8$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 8} = \min\{d_{17\ 8}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{17, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 9$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 9} = \min\{d_{17\ 9}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{15, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 10$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 10} = \min\{d_{17\ 10}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{17, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 11$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 11} = \min\{d_{17\ 11}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{13, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 12$ and $Q = \{1, 4\}$

$$\hat{D}_{17\ 12} = \min\{d_{17\ 12}, \min\{d_{17\ 1}, d_{17\ 4}\}\}$$

$$= \min\{13, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 13$ and $Q = \{1,4\}$

$$\hat{D}_{17\ \ 13} = \min\{d_{17\ \ 13}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$$

$$= \min\{11, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 14$ and $Q = \{1,4\}$

$$\hat{D}_{17\ \ 14} = \min\{d_{17\ \ 14}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$$

$$= \min\{7, \min\{12,10\}\}$$
$$= 7$$

$i = 17,\ j = 15$ and $Q = \{1,4\}$

$$\hat{D}_{17\ \ 15} = \min\{d_{17\ \ 15}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$$
$$= \min\{10, \min\{12,10\}\}$$
$$= 10$$

$i = 17,\ j = 16$ and $Q = \{1,4\}$

$$\hat{D}_{17\ \ 16} = \min\{d_{17\ \ 16}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$$

$$= \min\{4, \min\{12,10\}\}$$
$$= 4$$

$i = 17,\ j = 17$ and $Q = \{1,4\}$

$$\hat{D}_{17\ \ 17} = \min\{d_{17\ \ 17}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$$

$$= \min\{0, \min\{12,10\}\}$$
$$= 0$$

$i = 17,\ j = 18$ and $Q = \{1,4\}$

$$\hat{D}_{17\ \ 18} = \min\{d_{17\ \ 18}, \min\{d_{17\ \ 1}, d_{17\ \ 4}\}\}$$

$$= \min\{5, \min\{12,10\}\}$$
$$= 5$$

**For Node 18**

$i = 18,\ j = 1$ and $Q = \{1,4\}$

$$\hat{D}_{18\ \ 1} = \min\{d_{18\ \ 1}, \min\{d_{18\ \ 1}, d_{18\ \ 4}\}\}$$

$$= \min\{17, \min\{17,5\}\}$$
$$= 5$$

$i = 18,\ j = 2$ and $Q = \{1,4\}$

$$\hat{D}_{18\ \ 2} = \min\{d_{18\ \ 2}, \min\{d_{18\ \ 1}, d_{18\ \ 4}\}\}$$
$$= \min\{12, \min\{17,5\}\}$$
$$= 5$$

$i = 18,\ j = 3$ and $Q = \{1,4\}$

$$\hat{D}_{18\ \ 3} = \min\{d_{18\ \ 3}, \min\{d_{18\ \ 1}, d_{18\ \ 4}\}\}$$

$$= \min\{12, \min\{17,5\}\}$$
$$= 5$$

$i = 18,\ j = 4$ and $Q = \{1,4\}$

$$\hat{D}_{18\ \ 4} = \min\{d_{18\ \ 4}, \min\{d_{18\ \ 1}, d_{18\ \ 4}\}\}$$

$$= \min\{5, \min\{17,5\}\}$$
$$= 5$$

$i=18,\ j=5$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 5}=\min\{d_{18\ \ 5},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{7,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=6$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 6}=\min\{d_{18\ \ 6},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{13,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=7$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 7}=\min\{d_{18\ \ 7},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{15,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=8$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 8}=\min\{d_{18\ \ 8},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{19,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=9$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 9}=\min\{d_{18\ \ 9},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{20,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=10$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 10}=\min\{d_{18\ \ 10},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{22,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=11$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 11}=\min\{d_{18\ \ 11},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{18,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=12$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 12}=\min\{d_{18\ \ 12},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{18,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=13$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 13}=\min\{d_{18\ \ 13},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{16,\min\{17,5\}\}$$
$$=5$$

$i=18,\ j=14$ and $Q=\{1,4\}$

$$\hat{D}_{18\ \ 14}=\min\{d_{18\ \ 14},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$
$$=\min\{12,\min\{17,5\}\}$$
$$=5$$

$i = 18,\ j = 15$ and $Q = \{1, 4\}$

$$\hat{D}_{18\ \ 15} = \min\{d_{18\ \ 15},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$

$$= \min\{15, \min\{17, 5\}\}$$
$$= 5$$

$i = 18,\ j = 16$ and $Q = \{1, 4\}$

$$\hat{D}_{18\ \ 16} = \min\{d_{18\ \ 16},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$

$$= \min\{9, \min\{17, 5\}\}$$
$$= 5$$

$i = 18,\ j = 17$ and $Q = \{1, 4\}$

$$\hat{D}_{18\ \ 17} = \min\{d_{18\ \ 17},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$

$$= \min\{5, \min\{17, 5\}\}$$
$$= 5$$

$i = 18,\ j = 18$ and $Q = \{1, 4\}$

$$\hat{D}_{18\ \ 18} = \min\{d_{18\ \ 18},\ \min\{d_{18\ \ 1},\ d_{18\ \ 4}\}\}$$

$$= \min\{0, \min\{17, 5\}\}$$
$$= 0$$

**APPENDIX C: OPTIMAL NEW LOCATION**

For optimal new location, use the objective function, $Min[G(x) = \max_{i=1,...,n} \min\{d(X,i), d(Y,i)\}]$

With $Y = \{1,4\}$ and $X = \{2,3,5,6,7,8,9,10,11,12,13,14,15,16,17,18\}$,

where $i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$

**For $X = 2$**

$i = 1,$
$\min\{d(2,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(2,2), d(1,2), d(4,2)\}$
$= \min\{0,5,5\}$
$= 0$

$i = 3,$
$\min\{d(2,3), d(1,3), d(4,3)\}$
$= \min\{2,7,7\}$
$= 2$

$i = 4,$
$\min\{d(2,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(2,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(2,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(2,7), d(1,7), d(4,7)\}$
$= \min\{5,7,7\}$
$= 5$

$i = 8,$
$\min\{d(2,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(2,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(2,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(2,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(2,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(2,13), d(1,13), d(4,13)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 14,$
$\min\{d(2,14), d(1,14), d(4,14)\}$
$= \min\{8,10,10\}$
$= 8$

$i = 15,$
$\min\{d(2,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(2,16), d(1,16), d(4,16)\}$
$= \min\{5,10,10\}$
$= 5$

$i = 17,$
$\min\{d(2,17), d(1,17), d(4,17)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 18,$
$\min\{d(2,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 3$**

$i = 1,$
$\min\{d(3,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(3,2), d(1,2), d(4,2)\}$
$= \min\{2,5,5\}$
$= 5$

$i = 3,$
$\min\{d(3,3), d(1,3), d(4,3)\}$
$= \min\{0,7,7\}$
$= 0$

$i = 4,$
$\min\{d(3,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(3,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(3,6), d(1,6), d(4,6)\}$
$= \min\{6,8,8\}$
$= 6$

$i = 7,$
$\min\{d(3,7), d(1,7), d(4,7)\}$
$= \min\{3,7,7\}$
$= 3$

$i = 8,$
$\min\{d(3,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(3,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(3,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(3,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(3,12), d(1,12), d(4,12)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 13,$
$\min\{d(3,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 14,$
$\min\{d(3,14), d(1,14), d(4,14)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 15,$
$\min\{d(3,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(3,16), d(1,16), d(4,16)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 17,$
$\min\{d(3,17), d(1,17), d(4,17)\}$
$= \min\{9,10,10\}$
$= 9$

$i = 18,$
$\min\{d(3,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 5$**

$i = 1,$
$\min\{d(5,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(5,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(5,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(5,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(5,5), d(1,5), d(4,5)\}$
$= \min\{0, 2, 2\}$
$= 2$

$i = 6,$
$\min\{d(5,6), d(1,6), d(4,6)\}$
$= \min\{6, 8, 8\}$
$= 6$

$i = 7,$
$\min\{d(5,7), d(1,7), d(4,7)\}$
$= \min\{7, 7, 7\}$
$= 7$

$i = 8,$
$\min\{d(5,8), d(1,8), d(4,8)\}$
$= \min\{5, 5, 5\}$
$= 5$

$i = 9,$
$\min\{d(5,9), d(1,9), d(4,9)\}$
$= \min\{3, 3, 3\}$
$= 3$

$i = 10,$
$\min\{d(5,10), d(1,10), d(4,10)\}$
$= \min\{5, 5, 5\}$
$= 5$

$i = 11,$
$\min\{d(5,11), d(1,11), d(4,11)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 12,$
$\min\{d(5,12), d(1,12), d(4,12)\}$
$= \min\{7, 7, 7\}$
$= 7$

$i = 13,$
$\min\{d(5,13), d(1,13), d(4,13)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 14,$
$\min\{d(5,14), d(1,14), d(4,14)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 15,$
$\min\{d(5,15), d(1,15), d(4,15)\}$
$= \min\{7, 7, 7\}$
$= 7$

$i = 16,$
$\min\{d(5,16), d(1,16), d(4,16)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 17,$
$\min\{d(5,17), d(1,17), d(4,17)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 18,$
$\min\{d(5,18), d(1,18), d(4,18)\}$
$= \min\{5, 5, 5\}$
$= 5$

**For $X = 6$**

$i = 1,$
$\min\{d(6,1), d(1,1), d(4,1)\}$
$= \min\{0, 0, 0\}$
$= 0$

$i = 2,$
$\min\{d(6,2), d(1,2), d(4,2)\}$
$= \min\{5, 5, 5\}$
$= 5$

$i = 3,$
$\min\{d(6,3), d(1,3), d(4,3)\}$
$= \min\{6, 7, 7\}$
$= 6$

$i = 4,$
$\min\{d(6,4), d(1,4), d(4,4)\}$
$= \min\{0, 0, 0\}$
$= 0$

$i = 5,$
$\min\{d(6,5), d(1,5), d(4,5)\}$
$= \min\{2, 2, 2\}$
$= 2$

$i = 6,$
$\min\{d(6,6), d(1,6), d(4,6)\}$
$= \min\{0, 8, 8\}$
$= 0$

$i = 7,$
$\min\{d(6,7), d(1,7), d(4,7)\}$
$= \min\{3, 7, 7\}$
$= 3$

$i = 8,$
$\min\{d(6,8), d(1,8), d(4,8)\}$
$= \min\{5, 5, 5\}$
$= 5$

$i = 9,$
$\min\{d(6,9), d(1,9), d(4,9)\}$
$= \min\{3, 3, 3\}$
$= 3$

$i = 10,$
$\min\{d(6,10), d(1,10), d(4,10)\}$
$= \min\{5, 5, 5\}$
$= 5$

$i = 11,$
$\min\{d(6,11), d(1,11), d(4,11)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 12,$
$\min\{d(6,12), d(1,12), d(4,12)\}$
$= \min\{7, 7, 7\}$
$= 7$

$i = 13,$
$\min\{d(6,13), d(1,13), d(4,13)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 14,$
$\min\{d(6,14), d(1,14), d(4,14)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 15,$
$\min\{d(6,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(6,16), d(1,16), d(4,16)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 17,$
$\min\{d(6,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(6,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 7$**

$i = 1,$
$\min\{d(7,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(7,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(7,3), d(1,3), d(4,3)\}$
$= \min\{3,7,7\}$
$= 3$

$i = 4,$
$\min\{d(7,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(7,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(7,6), d(1,6), d(4,6)\}$
$= \min\{3,8,8\}$
$= 3$

$i = 7,$
$\min\{d(7,7), d(1,7), d(4,7)\}$
$= \min\{0,7,7\}$
$= 0$

$i = 8,$
$\min\{d(7,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(7,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(7,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(7,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(7,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(7,13), d(1,13), d(4,13)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 14,$
$\min\{d(7,14), d(1,14), d(4,14)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 15,$
$\min\{d(7,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(7,16), d(1,16), d(4,16)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 17,$
$\min\{d(7,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(7,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 8$**

$i = 1,$
$\min\{d(8,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(8,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(8,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(8,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(8,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(8,6), d(1,6), d(4,6)\}$
$= \min\{6,8,8\}$
$= 6$

$i = 7,$
$\min\{d(8,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 8,$
$\min\{d(8,8), d(1,8), d(4,8)\}$
$= \min\{0,5,5\}$
$= 0$

$i = 9,$
$\min\{d(8,9), d(1,9), d(4,9)\}$
$= \min\{2,3,3\}$
$= 2$

$i = 10,$
$\min\{d(8,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(8,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(8,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(8,13), d(1,13), d(4,13)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 14,$
$\min\{d(8,14), d(1,14), d(4,14)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 15,$
$\min\{d(8,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(8,16), d(1,16), d(4,16)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 17,$
$\min\{d(8,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(8,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 9$**

$i = 1,$
$\min\{d(9,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(9,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(9,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(9,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(9,5), d(1,5), d(4,5)\}$
$= \min\{2, 2, 2\}$
$= 2$

$i = 6,$
$\min\{d(9,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(9,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 8,$
$\min\{d(9,8), d(1,8), d(4,8)\}$
$= \min\{2,5,5\}$
$= 2$

$i = 9,$
$\min\{d(9,9), d(1,9), d(4,9)\}$
$= \min\{0,3,3\}$
$= 0$

$i = 10,$
$\min\{d(9,10), d(1,10), d(4,10)\}$
$= \min\{4,5,5\}$
$= 4$

$i = 11,$
$\min\{d(9,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(9,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(9,13), d(1,13), d(4,13)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 14,$
$\min\{d(9,14), d(1,14), d(4,14)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 15,$
$\min\{d(9,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(9,16), d(1,16), d(4,16)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 17,$
$\min\{d(9,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(9,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 10$**

$i = 1,$
$\min\{d(10,1), d(1,1), d(4,1)\}$
$= \min\{0, 0, 0\}$
$= 0$

$i = 2,$
$\min\{d(10,2), d(1,2), d(4,2)\}$
$= \min\{5, 5, 5\}$
$= 5$

$i = 3,$
$\min\{d(10,3), d(1,3), d(4,3)\}$
$= \min\{7, 7, 7\}$
$= 7$

$i = 4,$
$\min\{d(10,4), d(1,4), d(4,4)\}$
$= \min\{0, 0, 0\}$
$= 0$

$i = 5,$
$\min\{d(10,5), d(1,5), d(4,5)\}$
$= \min\{2, 2, 2\}$
$= 2$

$i = 6,$
$\min\{d(10,6), d(1,6), d(4,6)\}$
$= \min\{8, 8, 8\}$
$= 8$

$i = 7,$
$\min\{d(10,7), d(1,7), d(4,7)\}$
$= \min\{7, 7, 7\}$
$= 7$

$i = 8,$
$\min\{d(10,8), d(1,8), d(4,8)\}$
$= \min\{5, 5, 5\}$
$= 5$

$i = 9,$
$\min\{d(10,9), d(1,9), d(4,9)\}$
$= \min\{3, 3, 3\}$
$= 3$

$i = 10,$
$\min\{d(10,10), d(1,10), d(4,10)\}$
$= \min\{0, 5, 5\}$
$= 0$

$i = 11,$
$\min\{d(10,11), d(1,11), d(4,11)\}$
$= \min\{8, 10, 10\}$
$= 8$

$i = 12,$
$\min\{d(10,12), d(1,12), d(4,12)\}$
$= \min\{7, 7, 7\}$
$= 7$

$i = 13,$
$\min\{d(10,13), d(1,13), d(4,13)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 14,$
$\min\{d(10,14), d(1,14), d(4,14)\}$
$= \min\{10, 10, 10\}$
$= 10$

$i = 15,$
$\min\{d(10,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(10,16), d(1,16), d(4,16)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 17,$
$\min\{d(10,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(10,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 11$**

$i = 1,$
$\min\{d(11,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(11,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(11,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(11,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(11,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(11,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(11,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 8,$
$\min\{d(11,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(11,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(11,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(11,11), d(1,11), d(4,11)\}$
$= \min\{0,10,10\}$
$= 0$

$i = 12,$
$\min\{d(11,12), d(1,12), d(4,12)\}$
$= \min\{3,7,7\}$
$= 3$

$i = 13,$
$\min\{d(11,13), d(1,13), d(4,13)\}$
$= \min\{2,10,10\}$
$= 2$

$i = 14,$
$\min\{d(11,14), d(1,14), d(4,14)\}$
$= \min\{6,10,10\}$
$= 6$

$i = 15,$
$\min\{d(11,15), d(1,15), d(4,15)\}$
$= \min\{6,7,7\}$
$= 6$

$i = 16,$
$\min\{d(11,16), d(1,16), d(4,16)\}$
$= \min\{9,10,10\}$
$= 9$

$i = 17,$
$\min\{d(11,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(11,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 12$**

$i = 1,$
$\min\{d(12,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(12,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(12,3), d(1,3), d(4,3)\}$
$= \min\{3,10,10\}$
$= 3$

$i = 4,$
$\min\{d(12,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(12,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(12,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(12,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 8,$
$\min\{d(12,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(12,9), d(1,9), d(4,)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(12,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(12,11), d(1,11), d(4,11)\}$
$= \min\{3,10,10\}$
$= 3$

$i = 12,$
$\min\{d(12,12), d(1,12), d(4,12)\}$
$= \min\{0,7,7\}$
$= 0$

$i = 13,$
$\min\{d(12,13), d(1,13), d(4,13)\}$
$= \min\{3,10,10\}$
$= 3$

$i = 14,$
$\min\{d(12,14), d(1,14), d(4,14)\}$
$= \min\{6,10,10\}$
$= 6$

$i = 15,$
$\min\{d(12,15), d(1,15), d(4,15)\}$
$= \min\{3,7,7\}$
$= 3$

$i = 16,$
$\min\{d(12,16), d(1,16), d(4,16)\}$
$= \min\{9,10,10\}$
$= 9$

$i = 17,$
$\min\{d(12,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(12,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 13$**

$i = 1,$
$\min\{d(13,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(13,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(13,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(13,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(13,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(13,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(13,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 8,$
$\min\{d(13,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(13,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(13,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 10,$
$\min\{d(13,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(13,11), d(1,11), d(4,11)\}$
$= \min\{2,10,10\}$
$= 2$

$i = 12,$
$\min\{d(13,12), d(1,12), d(4,12)\}$
$= \min\{3,7,7\}$
$= 3$

$i = 13,$
$\min\{d(13,13), d(1,13), d(4,13)\}$
$= \min\{0,10,10\}$
$= 0$

$i = 14,$
$\min\{d(13,14), d(1,14), d(4,14)\}$
$= \min\{4,10,10\}$
$= 4$

$i = 15,$
$\min\{d(13,15), d(1,15), d(4,15)\}$
$= \min\{6,7,7\}$
$= 6$

$i = 16,$
$\min\{d(13,16), d(1,16), d(4,16)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 17,$
$\min\{d(13,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(13,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 6,$
$\min\{d(14,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(14,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

**For $X = 14$**

$i = 1,$
$\min\{d(14,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(14,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(14,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(14,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(14,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 8,$
$\min\{d(14,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(14,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(14,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(14,11), d(1,11), d(4,11)\}$
$= \min\{6,10,10\}$
$= 6$

$i = 12,$
$\min\{d(14,12), d(1,12), d(4,12)\}$
$= \min\{6,7,7\}$
$= 6$

$i = 13,$
$\min\{d(14,13), d(1,13), d(4,13)\}$
$= \min\{4,10,10\}$
$= 4$

$i = 14,$
$\min\{d(14,14), d(1,14), d(4,14)\}$
$= \min\{0,10,10\}$
$= 0$

$i = 15,$
$\min\{d(14,15), d(1,15), d(4,15)\}$
$= \min\{3,7,7\}$
$= 3$

$i = 16,$
$\min\{d(14,16), d(1,16), d(4,16)\}$
$= \min\{3,10,10\}$
$= 3$

$i = 17,$
$\min\{d(14,17), d(1,17), d(4,17)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 18,$
$\min\{d(14,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 2,$
$\min\{d(15,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(15,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(15,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(15,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(15,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(15,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

**For $X = 15$**

$i = 1,$
$\min\{d(15,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 8,$
$\min\{d(15,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(15,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(15,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(15,11), d(1,11), d(4,11)\}$
$= \min\{6,10,10\}$
$= 6$

$i = 12,$
$\min\{d(15,12), d(1,12), d(4,12)\}$
$= \min\{3,7,7\}$
$= 3$

$i = 13,$
$\min\{d(15,13), d(1,13), d(4,13)\}$
$= \min\{6,10,10\}$
$= 6$

$i = 14,$
$\min\{d(15,14), d(1,14), d(4,14)\}$
$= \min\{3,10,10\}$
$= 3$

$i = 15,$
$\min\{d(15,15), d(1,15), d(4,15)\}$
$= \min\{0,7,7\}$
$= 0$

$i = 16,$
$\min\{d(15,16), d(1,16), d(4,16)\}$
$= \min\{6,10,10\}$
$= 6$

$i = 17,$
$\min\{d(15,17), d(1,17), d(4,17)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 18,$
$\min\{d(15,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 16$**

$i = 1,$
$\min\{d(16,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(16,12), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(16,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(16,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(16,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(16,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(16,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 8,$
$\min\{d(16,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(16,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(16,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(16,11), d(1,11), d(4,11)\}$
$= \min\{9,10,10\}$
$= 9$

$i = 12,$
$\min\{d(16,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(16,13), d(1,13), d(4,13)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 14,$
$\min\{d(16,14), d(1,14), d(4,14)\}$
$= \min\{3,10,10\}$
$= 3$

$i = 15,$
$\min\{d(16,15), d(1,15), d(4,15)\}$
$= \min\{6,7,7\}$
$= 6$

$i = 16,$
$\min\{d(16,16), d(1,16), d(4,16)\}$
$= \min\{0,10,10\}$
$= 0$

$i = 17,$
$\min\{d(16,17), d(1,17), d(4,17)\}$
$= \min\{4,10,10\}$
$= 4$

$i = 18,$
$\min\{d(16,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 7,$
$\min\{d(17,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

**For $X = 17$**

$i = 1,$
$\min\{d(17,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(17,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(17,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(17,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(17,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(17,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 8,$
$\min\{d(17,8), d(1,8), d(4,8)\}$
$= \min\{5,5,\}$
$= 5$

$i = 9,$
$\min\{d(17,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(17,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(17,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(17,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(17,13), d(1,13), d(4,13)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 14,$
$\min\{d(17,14), d(1,14), d(4,14)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 14,$
$\min\{d(17,14), d(1,14), d(4,14)\}$
$= \min\{7,10,10\}$
$= 7$

$i = 15,$
$\min\{d(17,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(17,16), d(1,16), d(4,16)\}$
$= \min\{4,10,10\}$
$= 4$

$i = 17,$
$\min\{d(17,17), d(1,17), d(4,17)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 18,$
$\min\{d(17,18), d(1,18), d(4,18)\}$
$= \min\{5,5,5\}$
$= 5$

**For $X = 18$**

$i = 1,$
$\min\{d(18,1), d(1,1), d(4,1)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 2,$
$\min\{d(18,2), d(1,2), d(4,2)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 3,$
$\min\{d(18,3), d(1,3), d(4,3)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 4,$
$\min\{d(18,4), d(1,4), d(4,4)\}$
$= \min\{0,0,0\}$
$= 0$

$i = 5,$
$\min\{d(18,5), d(1,5), d(4,5)\}$
$= \min\{2,2,2\}$
$= 2$

$i = 6,$
$\min\{d(18,6), d(1,6), d(4,6)\}$
$= \min\{8,8,8\}$
$= 8$

$i = 7,$
$\min\{d(18,7), d(1,7), d(4,7)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 8,$
$\min\{d(18,8), d(1,8), d(4,8)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 9,$
$\min\{d(18,9), d(1,9), d(4,9)\}$
$= \min\{3,3,3\}$
$= 3$

$i = 10,$
$\min\{d(18,10), d(1,10), d(4,10)\}$
$= \min\{5,5,5\}$
$= 5$

$i = 11,$
$\min\{d(18,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 11,$
$\min\{d(18,11), d(1,11), d(4,11)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 12,$
$\min\{d(18,12), d(1,12), d(4,12)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 13,$
$\min\{d(18,13), d(1,13), d(4,13)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 14,$
$\min\{d(18,14), d(1,14), d(4,14)\}$
$= \min\{10,10,10\}$
$= 10$

$i = 15,$
$\min\{d(18,15), d(1,15), d(4,15)\}$
$= \min\{7,7,7\}$
$= 7$

$i = 16,$
$\min\{d(18,16), d(1,16), d(4,16)\}$
$= \min\{9,10,10\}$
$= 9$

$i = 17,$
$\min\{d(18,17), d(1,17), d(4,17)\}$
$= \min\{5,10,10\}$
$= 5$

$i = 18,$
$\min\{d(18,18), d(1,18), d(4,18)\}$
$= \min\{0,5,5\}$
$= 0$