



# Mathematical Modeling of Transmission Dynamics with Periodic Contact Rate and Control by Different Vaccination Rates of Hepatitis B Infection in Ghana

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### Authors' contributions

This work was carried out in collaboration among all authors. All of them were highly involved in the concept, modeling, formulation, implementation, analysis and the writing of the paper. All authors read and approved the final manuscript.

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## Abstract

The paper evidenced that Hepatitis B infection is the world's deadliest liver infection and Vaccination is among the principal clinical strategies in fighting it. These have encouraged a lot of researchers to formulate mathematical models to accurately predict the mode of transmission and make deductions for better health decision-making processes. In this paper, an SEIR model is used to model the transmission of the Hepatitis B infection with periodic contact rate and examine the impact of vaccination. The model was validated using estimated data in Ghana and simulated in a MATLAB environment. The results showed that the vaccination rate has a great impact on the transmission mode of the Hepatitis B infection and the periodic contact rate may lead to a chaotic solution which could result in an uncontrolled spreading of the infection. It is concluded that even if the vaccination rate is 70%, the infection rate would reduce to the minimum barest so more newborns must be vaccinated.

*Keywords:* Hepatitis B; susceptible; exposed; differential equation; epidemiology; infectious diseases; vaccination rate.

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## 1 Introduction

Globally, Hepatitis B Infection (HBI) is among the most deadly infectious diseases and it is affecting the world greatly [19],[24]. The HBI is caused by virus or toxins which attacks and inflames the human liver [20],[37]. This is shown in Fig. 1. It is commonly categorized as acute (< 6 months)

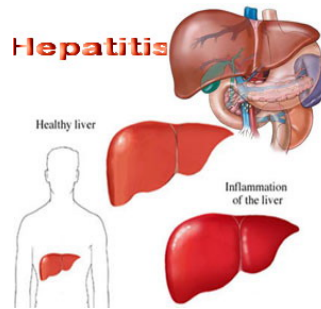


Fig. 1. Symptoms of Hepatitis B: Inflamed liver [19]

and chronic ( $\geq 6$  months). Acute Hepatitis may pass without causing any major damage to liver whereas chronic hepatitis can cause cirrhosis [19]. According to [19], more than two billion people live with Hepatitis B virus and it kills around 820,000 each year.

Among the continents, Africa has one of the highest HBI prevalence [24]. Ghana particularly has been identified as one of the highest prevalent centers of Hepatitis B infections(12.3%) [6]. Approximately 421 out of 117095 cases died between 2009 and 2021 in Ghana [7],[9],[26]. The trend of the Acute Viral Hepatitis Cases and Deaths is shown in Fig. 2.

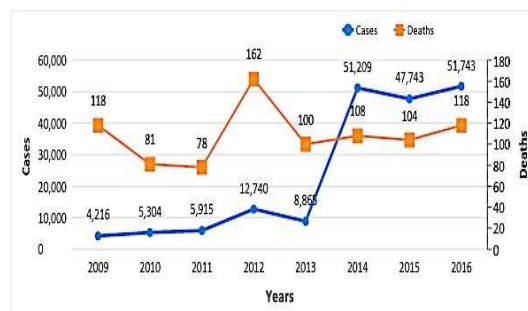


Fig. 2. Trend of Acute Viral Hepatitis Cases and Deaths; Ghana, 2009-2016 [4]

According to studies conducted by [16], [17], one of the most cost-effective, cost-benefit and efficient mechanisms of dealing with the HBI is to identify the precise mode of transmission and accurate level of vaccination of newborns. Though vaccination exercise to check the incidence of HBI has been in existence since 1982, there is still a rapid growth in the transmission rate the HBI [10],[13],[23],[30],[33],[34].

The general transmission mechanism of the HBI is a complex one. It is governed by many variables and parameters such as the large population size (number of susceptible and infectious), migration effect, death, and other medical strategies [25]. The transmission of Hepatitis B in Ghana is mostly studied from medical and statistical directions [12].

Accurate prediction of the HBI prevalence to optimally design and formulate effective general health policies to fight the HBI requires a mathematical modeling strategy as it is an effective theoretical approach to handle the infection dynamic behavior and vaccination impact on the infectious diseases by estimating critical parameters influencing the mode of transmission and suggest suitable solution strategies [1],[14],[17],[18],[21],[27],[29],[32],[38]. Studies such as [9] developed a model to examine the impact of carriers on the HBI transmission. In [28], a transmission model to evaluate the effect of vaccination and other controlling mechanisms to combat the HBI was built and it was concluded that booster doses of HBI vaccine were very crucial. An SIR model to examine the dynamics of HBI in Ghana was built by [38]. Also, the author in the study [1] recently used a transmission model for the investigation of limited medical resources in Ghana. However, many of literature researched on epidemic systems assume that an important parameter such as the disease incubation is negligible which highly unrealistic since according to [3],[19], the incubation period for the Hepatitis B is within 60-90 days. So mathematical models which capture the incubation period (Exposed class) are appropriate in Hepatitis B disease transmission. Some of them are the SEIR (Susceptible- Exposed-Infectious-Recovered) or SEIRS (Susceptible-Exposed-Infectious-Recovered-Susceptible) used in [15],[35],[36]. Also, none of the existing studies critically examined the periodicity in the contact rate of the HBI transmission. Therefore, this paper proposes an SEIR transmission dynamic model suitable to predict and examine effect of vaccination rate on the transmission Hepatitis B with a periodic or nonlinear transmission rate. The findings of this paper would improve upon the well-being of the people, as they would provide preventive mechanisms and other control measures which could make people recover and are able to earn their livelihood.

The paper is outlined in the following manner: section one summarizes the background of the study, the problem statement and the research objectives. In section two, the details of the SEIR mathematical model of the Hepatitis B infection and its stability analysis are presented. The implementation of this modeling strategy with data from Ghana through simulations in the MATLAB environment is carried out in section four. The results from the simulations are also analyzed and discussed in this section. And finally, conclusions and outline recommendations are outlined in the fifth section.

## 2 Materials and Methods

In this section, a mathematical model is presented to predict how the disease evolves over time and the impact on the population. The modeling can help decide appropriate control measures and predicts future growth patterns and transmission of the HBI [5],[11]. Several enhancements have been captured to take into consideration the special cases of the HBI.

### 2.1 The mathematical model description

In this paper, the total population at each time is denoted as  $N_t$  and it is made up of the following four different compartments:

- **Vaccinated** ( $V_t$ ): Fraction of the population that have been immunized.
- **Susceptible** ( $S_t$ ): Fraction of population who are likely to contract the HBI.
- **Exposed** ( $E_t$ ): Fraction of population that have the pathogen and cannot infect others since they are within the incubation period.

- **Infected** ( $I_t$ ): Fraction of individual who transmit the virus.
- **Recovered** ( $R_t$ ): Fraction of population that are no longer have the disease.

**Remarks:**

- The Vaccinated component grows as more recovered get immune and also by the proportion of newborns who have been immunized ( $cP$ ). This proportion also decreases as a result of the vaccine efficacy expiration at the rate given as  $\phi$  and the death as a results of other incidents denoted as  $\mathbf{d}$ .
- Susceptible fraction increases due to the growing of newborns that are not immunized against HB infection at the rate of  $(1-c)P$  and also due to the vaccine efficacy expiration at the rate of  $\phi$ . However, it decays as more individuals are being exposed(latent infection ) at a rate of  $\mathbf{k}$  and also death as results of other incidents at the rate of  $\mathbf{d}$ .
- The exposed fraction of the population increases as the number of infections among the susceptible decreases at the rate of  $\mathbf{k}$  and decreases as a result of the advancement of individuals who are latently infected to into the infection stage at the rate of  $\beta$ . Again, this decreases as more latently infected individuals get relieved without progressing to the infection stage at the rate  $\mathbf{q}$  and the death due to other incidents at the rate  $\mathbf{d}$ .
- The number of infections grows as more latently infected individuals migrate to the infection stage at the rate of  $\beta$ . This proportion decreases; when more HBI infected individuals recover at the rate of  $\mu$ , by death as a result of active HB infection at the rate of  $\eta$  and also by death due to other incidence at the rate of  $\mathbf{d}$ .
- Finally, the number of recoveries increases due to effective and successful cure of latently infected individuals at the rate of  $\mathbf{q}$  and the number of infected at rate of  $\mu$ . It decays as the number of recoveries becomes immune at the rate of  $\alpha$  and as results of death due to other incidents at the rate of  $\mathbf{d}$ .

According to hypo-paper of conservation of population, the entire population is defined by the equation [31]:

$$N_t = V_t + S_t + E_t + I_t + R_t \tag{2.1}$$

## 2.2 Simplifying assumptions on the model

In this paper, we assume the following:

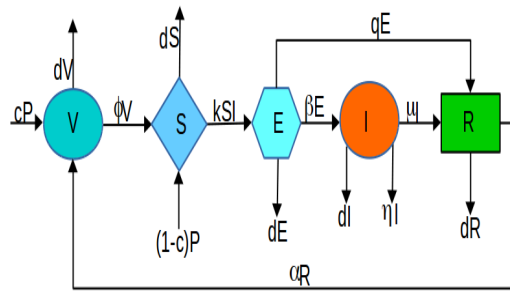
- A portion of newborns get immunization to fight the HBI via vaccination.
- The immunization of newborns via vaccination is not permanent as it expires as time grows.
- The mixing of the individual is in a homogeneous order. This implies that each person in the population is equally likely to be infected.
- Most individuals who recover completely become permanently immune [2] and hence, they rejoin the Susceptible class via the Vaccination class.
- That population in each compartment have equal death rate  $\mathbf{d}$ .
- No newborn is infected initially and thus, they move into the immune class or the susceptible.
- Some demographic data like the migration (immigrants and emigrants) are not captured. The population grows by birth rate only and it decays by either death due to other incidents or death by HBI.
- The birth rate is uniform.

The model parameters are described as follows:

- $\phi$  represents the rate at which the efficacy of the vaccine gets expired,

- $d$  represents the rate of death by other incidents,
- $k$  represents the rate at which each individual in the susceptible class gets latently infected by the HBI,
- $\beta$  is the rate at which the latently infected population progress to the infection stage. In this paper, both periodic and non-periodic contact rates are examined.
- $\mu$  is the recovery rate for the infected population.
- $\alpha$  is the rate at which a proportion of the recovered who do not die by other incident rejoin the vaccinated class since they become permanently immune after recovery.
- $P$ : Total newborns
- the contact rate  $\beta$  may be a function of time.
- $cP$ : The immunized fraction of newborns in the total population.

The general dynamics is depicted by Fig. 3 below.



**Fig. 3. Diagrammatic representation of the SEIR model**

Using the above-mentioned assumptions and the population dynamics in Fig. 3, the following system of ordinary differential equations which is called an SEIR model for Hepatitis B infection is obtained:

$$\begin{aligned} \frac{dV}{dt} &= cP + \alpha R_t - dV_t - \phi V_t \\ \frac{dS}{dt} &= \phi V_t + (1 - c)P - dS_t - kS_t I_t \\ \frac{dE}{dt} &= kS_t I_t - dE_t - qE_t - \beta(t)E_t \\ \frac{dI}{dt} &= \beta E_t - (d + \mu + \eta)I_t \\ \frac{dR}{dt} &= \mu I_t + qE_t - (d + \alpha)R_t \\ V_t + S_t + E_t + I_t + R_t &= N_t \end{aligned}$$

From Equation  $N_t = V_t + S_t + E_t + I_t + R_t$ , it implies that  $\frac{dN_t}{dt} = 0$  and hence, there is a constant population  $\forall t > 0$ .

In this paper, it is assumed every parameter is positive and the models' initial conditions are given by:

$$V_0 \geq 0, \quad S_0 > 0, \quad E_0 \geq 0, \quad I_0 \geq 0, \quad \text{and} \quad R_0 \geq 0 \quad (2.2)$$

## 2.3 Model analysis

We will study the qualitative properties of the solutions of differential equations without solving the equations explicitly. We will do so by first proving the invariant domain in purely mathematical sense and compute the equilibrium point of the system.

### 2.3.1 Positivity of solutions

The model system (2) describes the changes in human population and it is vital to verify that its solutions are non-negative over time using the theorem below:

Given that the initial conditions of system (2) are;  $V_0 \geq 0$ ,  $S_0 > 0$ ,  $E_0 \geq 0$ ,  $I_0 \geq 0$ , and  $R_0 \geq 0$ , then the solutions  $V_t$ ,  $S_t$ ,  $E_t$ ,  $I_t$  and  $R_t$  are positive  $\forall t > 0$ .

*Proof.* Considering the first equation in the system (2), that is,

$$\frac{dV}{dt} = cP + \alpha R_t - dV_t - \phi V_t \quad (2.3)$$

since  $c, \alpha \geq 0$ , it follows that:

$$\begin{aligned} \frac{dV}{dt} &= cP + \alpha R_t - dV_t - \phi V_t \\ &\geq -(d + \phi) V_t \end{aligned}$$

Solving (5) by the method of integrating factors gives a general solution:

$$V_t \geq \mathbf{x} \exp(-(d + \phi)t) \quad (2.4)$$

where  $\mathbf{x}$  is the arbitrary constant. At  $t=0$ , it follows that

$$V_t \geq V_0 \exp(-(d + \phi)t) \quad (2.5)$$

since  $(d + \phi) \geq 0 \implies V_t \geq 0 \forall t$ .

From the second equation in the system (2), since  $1-c, \phi \leq 0$ , it follows that:

$$\begin{aligned} \frac{dS}{dt} &= \phi V_t + (1 - c)P - dS_t - kS_t I_t \\ &\geq -(d + kI_t)S_t \end{aligned}$$

letting  $\Phi = d + kI_t$  and solve by the method integrating factors gives:

$$S_t \geq S_0 \exp(-\Phi)t \quad (2.6)$$

since  $(\Phi) \geq 0$  then  $S_t \geq 0 \forall t$ .

Similarly, it could be deduced that  $I_t \geq 0$ ,  $E_t \geq 0$ ,  $I_t \geq 0$ , hence it could be inferred that the solutions to the equations in system (2) are all non-negative over time.  $\square$

### 2.3.2 Region of feasible solution of the model

Under this section, the domain in which the solution of the model equations lies. However, it is of utmost importance to assume that the parameters used and variables in all classes are positive (ie  $t \geq 0$ ). Let  $\mathfrak{R} = \left\{ (V, S, E, I, R) \in R_+^5 \mid N = V + S + E + I + R \leq \max\left\{N_0, \left(\frac{P}{d}\right)\right\} \right\}$  be the domain of positive invariant set where  $z$  is an arbitrary constant. Then the feasible solution of the model will always lie in  $\mathfrak{R}$ .

*Proof.* Taking derivative with respect to  $t$  on both sides of Equation (2.1) gives:

$$\frac{dN}{dt} = \frac{dV}{dt} + \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} \tag{2.7}$$

This implies that

$$\begin{aligned} \frac{dN}{dt} &= (cP + \alpha R_t - dV_t - \phi V_t) \\ &\quad + (\phi V_t + (1 - c)P - dS_t - kS_t I_t) \\ &\quad + (kS_t I_t - dE_t - qE_t - \beta(t)E_t) \\ &\quad + (\beta E_t - (d + \mu + \eta)I_t) \\ &\quad + (\mu I_t + qE_t - (d + \alpha)R_t) \\ &= P - (V + S + E + I + R) - \eta I \\ &= P - dN - \eta I \end{aligned}$$

In the absence of Hepatitis B infection,  $I=0$  and thus:

$$N_t \leq P - dN \tag{2.8}$$

This is a first order linear ODE with a general solution:

$$N_t \leq \frac{P}{d} + \mathbf{w} \exp(-dt) \tag{2.9}$$

where  $\mathbf{w}$  is an arbitrary constant. At  $t=0$ , it follows that

$$N_t \leq \frac{P}{d} + \left(N_0 - \frac{P}{d}\right) \exp(-dt) \tag{2.10}$$

In equation (2.10), it could be seen that  $t \rightarrow \infty$ ,  $N_t \rightarrow \left(\frac{P}{d}\right)$ . If  $N_0 \leq \left(\frac{P}{d}\right)$ , then

$$\lim_{t \rightarrow \infty} N_t = \frac{P}{d}$$

Otherwise, if  $N_0 > \left(\frac{P}{d}\right)$ , then  $N$  will decrease to  $\left(\frac{P}{d}\right)$  as  $t \rightarrow \infty$ . This means that  $N_t \leq \max\left\{N_0, \left(\frac{P}{d}\right)\right\}$ . This implies that  $N_t$  is bounded above. Subsequently,  $V_t, S_t, E_t, I_t, R_t$  are all bounded above. Thus, in  $\mathfrak{R}$ , system (2) is well posed. Hence, it is sufficient to study the dynamics of the system in  $\mathfrak{R}$  since  $d \neq 0$ . □

We consider the periodic transmission rate as [15]:

$$\beta(t) = \beta_0(1 + \beta_1 \cos 2\pi t) \tag{2.11}$$

for  $\beta_0$  being the base rate transmission, and  $\beta_1 \in [0, 1]$  which quantifies the level of periodicity.

**Remark:** To able to examine and analyze the impact of vaccination on the transmission, the non-periodic contact rate, i.e,  $\beta(t) = \beta_0$  or  $\beta_1 = 0$  is considered in this paper.

### 2.3.3 Stability of trivial Equilibria and Nontrivial Equilibria of SEIR model for Hepatitis B

From equation (2.11), a non-periodic case is considered by setting  $\beta_1 = 0$  and the system (2) becomes:

$$\begin{aligned}\frac{dV}{dt} &= cP + \alpha R_t - dV_t - \phi V_t \\ \frac{dS}{dt} &= \phi V_t + (1 - c)P - dS_t - kS_t I_t \\ \frac{dE}{dt} &= kS_t I_t - dE_t - qE_t - \beta_0 E_t \\ \frac{dI}{dt} &= \beta_0 E_t - (d + \mu + \eta)I_t \\ \frac{dR}{dt} &= \mu I_t + qE_t - (d + \alpha)R_t \\ V_t + S_t + E_t + I_t + R_t &= N_t\end{aligned}$$

### 2.3.4 The Trivial Equilibrium State

Let the system's trivial equilibrium state be defined as  $T_0(V_0, S_0, E_0, I_0, R_0)$ . Since the terms  $cP \neq 0$  and  $(1 - c)P \neq 0$ , newborns will always be introduced into the population, that is, there couldn't be extinction of the population ( $T_0(V_0, S_0, E_0, I_0, R_0) \neq (0, 0, 0, 0, 0)$ ) and thus there exist no trivial equilibrium state for the model.

## 2.4 Disease-Free Equilibrium State

At the point of equilibrium, it follows from system (15) that;

$$V' = S' = E' = I' = R' = 0$$

and thus:

$$\begin{aligned}cP + \alpha R_t - dV_t - \phi V_t &= 0 \\ \phi V_t + (1 - c)P - dS_t - kS_t I_t &= 0 \\ kS_t I_t - dE_t - qE_t - \beta_0 E_t &= 0 \\ \beta_0 E_t - (d + \mu + \eta)I_t &= 0 \\ \mu I_t + qE_t - (d + \alpha)R_t &= 0 \\ V_t + S_t + E_t + I_t + R_t &= N_t\end{aligned}$$

If  $T^0(V^0, S^0, E^0, I^0, R^0)$  is the disease-free equilibrium state, then

$$I^0 - E^0 = 0 \tag{2.12}$$

since there are no infections and latently infected in the population.

Putting  $I - E = 0$  into equation (16) and solve simultaneously gives the following:  
 From the first equation in the system (16),

$$cP + \alpha R^0 - (\phi + d)V^0 = 0$$

and thus

$$V^0 = \frac{cP + \alpha R^0}{(\phi + d)} \tag{2.13}$$

but from last equation in the system (16), we have:

$$qE + \mu I - (d + \alpha)R^0 = 0 \implies (d + \alpha)R^0 = 0 \implies R^0 = 0 \text{ since } d, \alpha > 0$$

so from equation (2.13),

$$V^0 = \frac{cP}{(\phi + d)} \tag{2.14}$$

From the second equation in the system (16):

$$\begin{aligned} (1 - c)P + \frac{\phi cP}{(\phi + d)} - dS &= 0 \\ S^0 &= \frac{(1 - c)P(\phi + d) + \phi cP}{d(\phi + d)} \\ &= \frac{(\phi + d - cd)P}{d(\phi + d)} \end{aligned}$$

Thus

$$T^0(V^0, S^0, E^0, I^0, R^0) = \left( \frac{cP}{(\phi + d)}, \frac{(\phi + d - cd)P}{d(\phi + d)}, 0, 0, 0 \right) \tag{2.15}$$

### 2.4.1 Hepatitis B Free-Equilibrium Point Stability Analysis

In determining the stability of the Hepatitis B Free- equilibrium point ( $T^0$ ), the behavior of the system (2) is critically examined around the solution of the equilibrium and determine the necessary and sufficient conditions for the stability of Disease-Free Equilibrium point.

To start with, the system (15) is linearized to give the Jacobian matrix  $J$  as:

$$J = \begin{bmatrix} -(\phi + d) & 0 & 0 & 0 & \alpha \\ \phi & -(kI^0 + d) & 0 & -kS^0 & 0 \\ 0 & kI^0 & -(q + \beta_0 + d) & kS^0 & 0 \\ 0 & 0 & \beta_0 & -(\mu + d + \eta) & 0 \\ 0 & 0 & q & \mu & -(d + \alpha) \end{bmatrix} \tag{2.16}$$

At the  $T^0(V^0, S^0, E^0, I^0, R^0)$ , we have  $\mathbf{J}$  as:

$$J_0 = \begin{bmatrix} -(\phi + d) & 0 & 0 & 0 & \alpha \\ \phi & -d & 0 & Q_1 & 0 \\ 0 & 0 & -(q + \beta_0 + d) & Q_2 & 0 \\ 0 & 0 & \beta_0 & -(\mu + d + \eta) & 0 \\ 0 & 0 & q & \mu & -(d + \alpha) \end{bmatrix}$$

where:

$$Q_1 = -k \frac{(\phi + d - cd)P}{d(\phi + d)}$$

$$Q_2 = k \frac{(\phi + d - cd)P}{d(\phi + d)}$$

The characteristic equation  $|J_0 - H\lambda| = 0$  is the determinant of the Jacobian. Where Eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \lambda_4,$  and  $\lambda_5$  are the system's eigenvalues, and H is the identity matrix.

From equation (2.4.1), we have

$$[\lambda^2 + (\phi + 2d)\lambda + (\phi d + d^2) + (\phi + 2d)\alpha](-d - \lambda) \begin{bmatrix} -(q + \beta_0 + d) - \lambda & -k \frac{(\phi + d - cd)P}{d(\phi + d)} \\ \beta_0 & -(\mu + d + \eta) - \lambda \end{bmatrix} = 0 \quad (2.17)$$

From equation (2.17), we can deduced that:

$$[\lambda^2 + (\phi + 2d)\lambda + \phi d + d^2 + (\phi + 2d)\alpha](-d - \lambda) = 0 \quad (2.18)$$

Or

$$\begin{bmatrix} -(q + \beta_0 + d + \lambda) & -k \frac{(\phi + d - cd)P}{d(\phi + d)} \\ \beta_0 & -(\mu + d + \eta + \lambda) \end{bmatrix} = 0 \quad (2.19)$$

From equation (2.18), we have

$$\lambda_1 = -d \quad (2.20)$$

$$\lambda_{2,3} = \frac{-(\phi + 2d) \pm \sqrt{\phi^2 - 4\phi\alpha - 8d\alpha}}{2} \quad (2.21)$$

since  $\phi^2 < 4\alpha(\phi - 2d)$ ,  $\lambda_2$  and  $\lambda_3$  are both imaginary with negative real parts.

Let

$$M = \begin{bmatrix} -(q + \beta_0 + d) - \lambda & -k \frac{(\phi + d - cd)P}{d(\phi + d)} \\ \beta_0 & -(\mu + d + \eta + \lambda) \end{bmatrix} = 0 \quad (2.22)$$

To achieve asymptotically stability for the Hepatitis B Free- equilibrium point, the trace of  $M$  must be less than zero, i.e,  $(tr(M) < 0$  and the determinant must also be positive ( $|M| > 0$ ). But

$$|M| = (q + \beta_0 + d + \lambda)(\mu + d + \eta + \lambda) - k \frac{(\phi + d - cd)P}{d(\phi + d)} \quad (2.23)$$

Also,

$$trace(M) = -(q + \beta_0 + d + \lambda) - (\mu + d + \eta) \quad (2.24)$$

It is clear that  $trace(M) < 0$  since all the parameters  $q, t, d, \mu, \lambda$  and  $\eta$  are all positive. If  $|M| > 0$ , then:

$$(q + \beta_0 + d + \lambda)(\mu + d + \eta) - k\beta_0 \frac{(\phi + d - cd)P}{d(\phi + d)} > 0 \quad (2.25)$$

Thus

$$(q + \beta_0 + d + \lambda)(\mu + d + \eta) > k\beta_0 \frac{(\phi + d - cd)P}{d(\phi + d)} \quad (2.26)$$

From equations (2.20), it can be seen that the real parts of the eigenvalues  $\lambda_{1,2,3}$  are all negative. Now for the real parts of the remaining eigenvalues ( $\lambda_{4,5}$ ) to be negative, conditions that are necessary and sufficient are desired to be established. This is possible if and only if  $|M| > 0$ . Thus

$$(q + \beta_0 + d + \lambda)(\mu + d + \eta) > k\beta_0 \frac{(\phi + d - cd)P}{d(\phi + d)} \quad (2.27)$$

Applying the Routh-Hurwitz theorem which states that the equilibrium state will be asymptotically stable iff all the Eigenvalues of  $|J - H\lambda| = 0$  have their real parts being negative. Based on this theorem, it could be seen Hepatitis B-Free Equilibrium of the system (2) will be asymptotically stable iff

$$(q + \beta_0 + d + \lambda)(\mu + d + \eta) > k\beta_0 \frac{(\phi + d - cd)P}{d(\phi + d)} \quad (2.28)$$

The Necessary and Sufficient condition required to ensure asymptotically stability of the Hepatitis B-Free Equilibrium point for the system (2) is given by inequality (2.28). In other words, the Necessary and Sufficient condition for the asymptotically stability of the Hepatitis B-Free Equilibrium point for the system (2) is that the total rate removal of Exposed and that of infected compartments defined as:

$$(q + \beta_0 + d + \lambda)(\mu + d + \eta) \quad (2.29)$$

must be greater than the product of total contraction and breakdown of the exposed compartment defined as:

$$k\beta_0 \frac{(d + \phi - cd)P}{d(d + \phi)}$$

Alternatively Equation(2.28) can be written as:

$$(q + \beta_0 + d + \lambda) > k\beta_0 \left[ \frac{(\phi + d - cd)P}{d(\phi + d)(\mu + d + \eta)} \right] \quad (2.30)$$

Equation (2.30) provides the Necessary and Sufficient condition for the asymptotically stability of the Hepatitis B-Free Equilibrium point for the system (2) which implies that the addition of the exposed class recovery, the rate the infection, the rate death by other incidents of people which defines the total removal rate form the exposed class should be bounded lowerly by:

$$k\beta_0 \left[ \frac{(\phi + d - cd)P}{d(\phi + d)(\mu + d + \eta)} \right] \quad (2.31)$$

**REMARKS:** In this section, a detailed mathematical methodology has been presented to study the mode of transmission and vaccination rate effect on the Hepatitis B infection. Detailed modeling analysis has been carried out to determine the domain, positivity of solutions of the system developed and the general stability analysis.

In the next section, this modeling strategy is validated using data in Ghana. In the simulations, both periodic nature of the contact rate  $\beta(t) = \beta_0(1 + \beta_1 \cos 2\pi t)$  and non-periodic nature of contact rate  $\beta_1 = 0$  and thus  $\beta(t) = \beta_0$  will be critically examined. This analysis will help identify the solution's behavior at the peak. Again, the impact of vaccination on HB transmission dynamics is analysed.

### 3 Section 3 or Results and Discussion

To observe the dynamics of Hepatitis model over time, numerical simulations are carried out in MATLAB environment on 2.67 GHz and core i5 processor with 4GB of RAM. The implementation is carried out using data in Ghana.

#### 3.1 An application of the model to the data of Hepatitis B in Ghana

In this paper, the data recorded in 2016 was used for the validation and the details are given in Table 1. There is limited data available for the viral Hepatitis B in Ghana [9] so some of the parameters were from the existing studies [4], [8],[38].

**Table 1. Reported Acute Viral Hepatitis Cases and Deaths by Region, 2016 [4]**

Region	Cases	Deaths
Ashanti	13556	40
Brong Ahafo	7157	13
Central	3213	9
Eastern	0	0
Greater Accra	6693	37
Northern	86	0
Upper East	1356	0
Upper West	7,601	6
Volta	2036	0
Western	10045	13
Total	51743	118

The value of the parameters used for the analysis of SEIR model is presented in Table 2.

**Table 2. Parameters for the Model SEIR**

Parameter	Value
Recovery rate ( $\mu$ )	0.984155
Hepatitis B death rate ( $\eta$ )	0.0004142584
rate vaccination ( $c$ )	$c \in [0, 1]$
Death rate ( $d$ )	7.627
rate exposure ( $k$ )	0.7
rate vaccination expiry ( $\phi$ )	0.0045
Number of new borns ( $P$ )	3984000
Entire population ( $N$ )	28480000
Periodic contact rate( $\beta(t)$ )	$\beta(t) = \beta_0(1 + \beta_1 \cos 2\pi t)$

**Remark:** The simulation process is in two forms. We first examine the nature of solution for the system (2) considering the periodic contact rate by perturbing the initial conditions and different values of  $\beta_1$  because a vital characteristic of chaotic systems is that their solutions are very sensitive to changes in the initial conditions. Also, to be able to examine the impact of vaccination on the mode of transmission of the Hepatitis B virus, we assume  $\beta(t) = \beta_0$  or  $\beta_1 = 0$  because of the chaotic nature of the solution. These graphs represent the solution after integrating for a long time and

thus, they provide the asymptotically stable behavior. The chosen values of  $\beta_1$  are for convenient sake and the minus logarithm times is for better visualization.

From the Fig. 4, it is observed that when  $\beta_1 = 0.05$ , the period of  $I$  is equal to the period of  $\beta$ , that is 1. If  $\beta_1 = 0.2$ , the period of  $I$  doubles as  $\beta$ 's period. This is called the period of doubling bifurcation. For  $\beta_1 = 0.26$ , again, a period of doubling bifurcation is observed, the period of  $I$  is four times the period of  $\beta$ . Between  $\beta_1 = 0.26$  and  $\beta_1 = 0.27$ , an infinite number of such period of doubling bifurcations occurs which is called a period doubling cascade. At  $\beta_1 = 0.27$ , the solution becomes chaotic and very difficult to handle the Hepatitis B infection. This could result in periodic behavior for the Hepatitis B model discussed in the section (3) and could lead to a quick spreading of Hepatitis B infection. Thus, greatest proportion of the population in the country will attract the Hepatitis B infection which might result in massive death and hence, it is desired to put up strategies to control the periodic contact rate. At  $\beta_1 = 0.00$ , the solution becomes quite stable and therefore we use this value to examine the effect of vaccination on the Hepatitis B transmission. This is shown in Fig. 14.

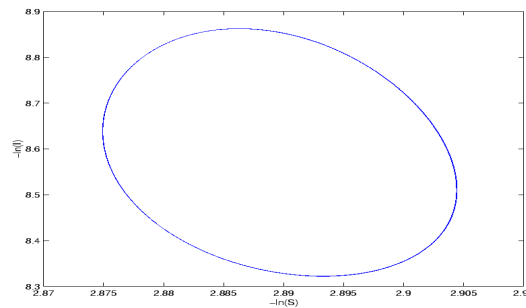


Fig. 4. Susceptible Vs Infected

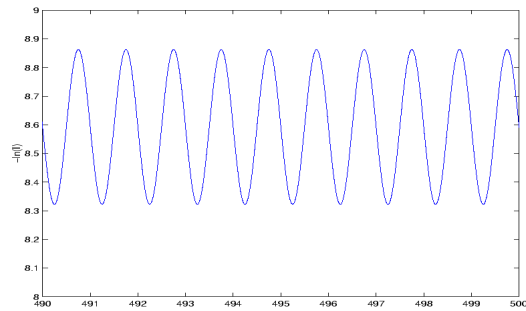
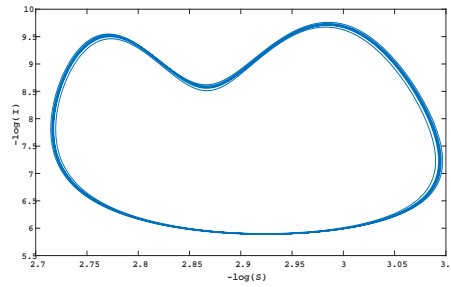


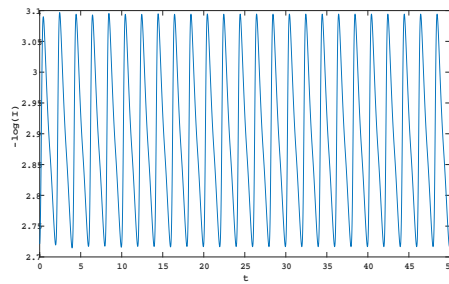
Fig. 5. Solution in phase space for  $\beta_1 = 0.05$

### 3.2 Vaccination Impact on the Hepatitis B Transmission

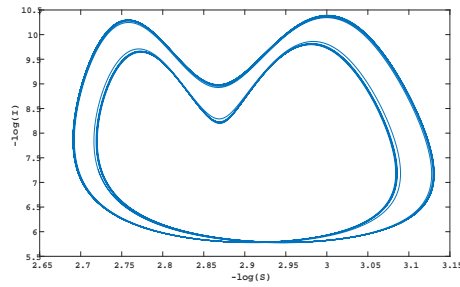
We consider two cases: Presence and absence of vaccination on the spread. Now considering the Fig. 16, after 10 weeks, about 0.5% of the susceptible individuals started getting exposed and after 17 weeks their signs started manifesting. The infection grew until it reached 25th week. That is, more individuals got infected between the weeks of 15-25. As at 25th week, about 20% of the entire population has been infected and that seems to be the peak.



**Fig. 6. Suceptible Vs Infected**



**Fig. 7. Solution in phase space for  $\beta_1 = 0.2$**



**Fig. 8. Suceptible Vs Infected**

Though there were no vaccines, many individuals (about 10%) started recovering after the 20th week. This may be due to other factors like strong immune system. The number of exposed and infected individuals started declining after the 25th week. After the 40th week, the proportion of recoveries getting closer to 1 means almost everyone is getting immune but unfortunately, almost everyone is also being infected. Everyone gets sick but they recover.

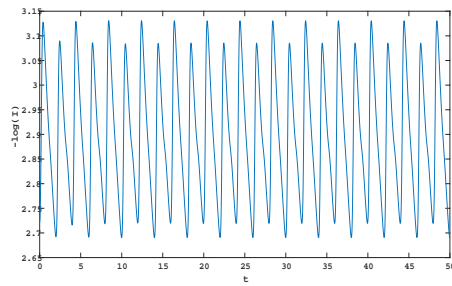


Fig. 9. Solution in phase space for  $\beta_1 = 0.25$

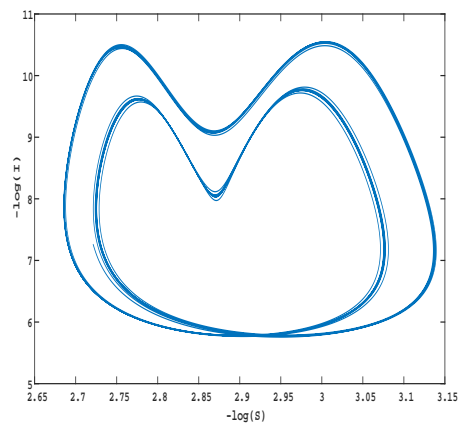


Fig. 10. Dynamics of Infection in time

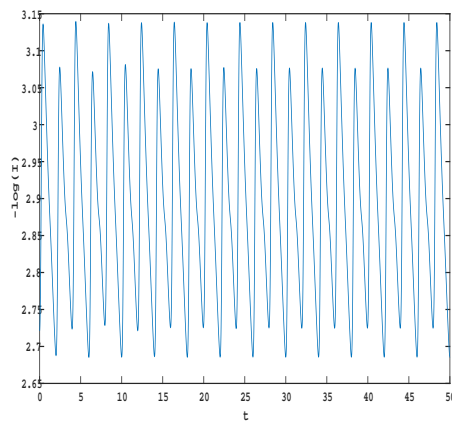
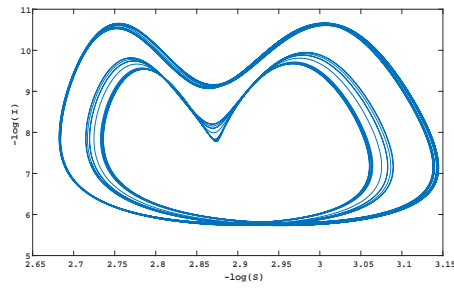
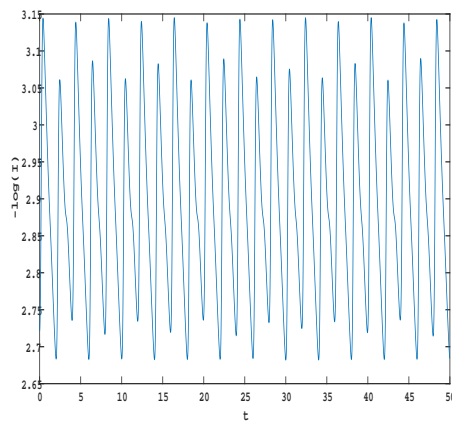


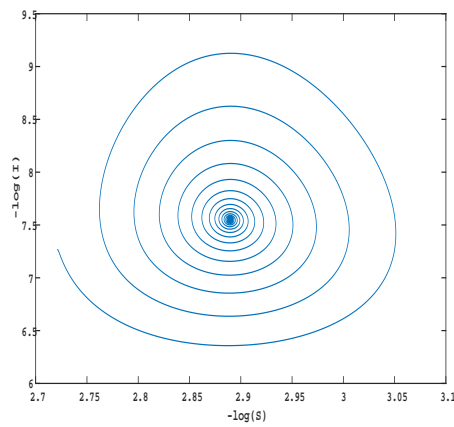
Fig. 11. Solution in phase space for  $\beta_1 = 0.26$



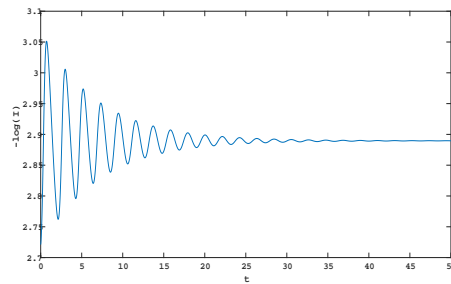
**Fig. 12.** Dynamics of Infection in time



**Fig. 13.** Solution in phase space for  $\beta_1 = 0.27$

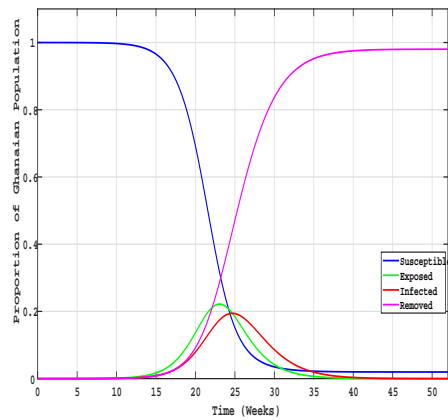


**Fig. 14.** Dynamics of Infection in time



**Fig. 15. Solution in phase space for  $\beta_1 = 0.00$**

More people becoming susceptible and infected but as time goes on they get immune or recover.



**Fig. 16. The Dynamics of the Hepatitis B Virus without a vaccine**

To examine the effect of vaccination, we simulated our model under four conditions: a vaccination rate 10%, 20%, 40%, 50%, 70% and 100%

With vaccination rate 10%, it could be seen that just after 24 weeks, the number of susceptible individuals reduces by 15%. With the increment in the vaccination rate, few people get exposed and infected and the recovery rate increases. These are demonstrated in the Figs. 17-21.

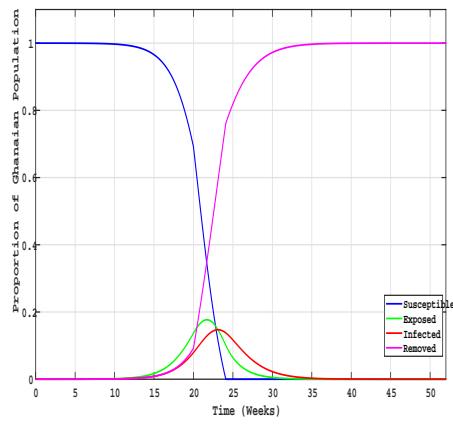


Fig. 17. With vaccine rate 0.1

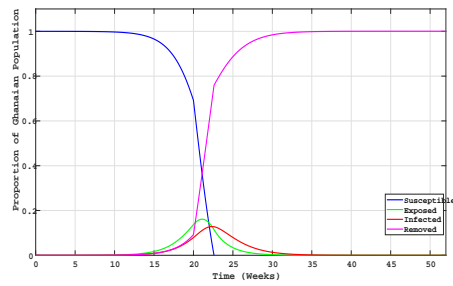


Fig. 18. With vaccine rate 0.2

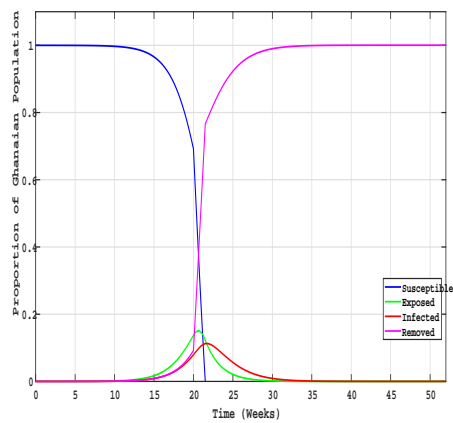


Fig. 19. With vaccine rate 0.4

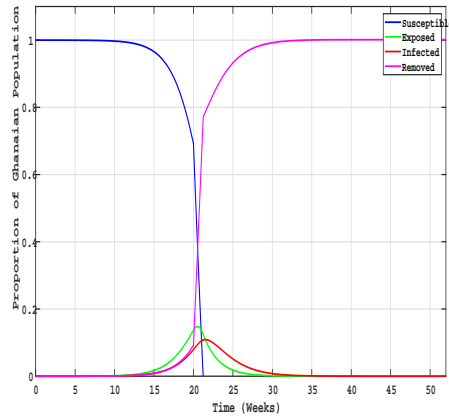


Fig. 20. With vaccine rate 0.5

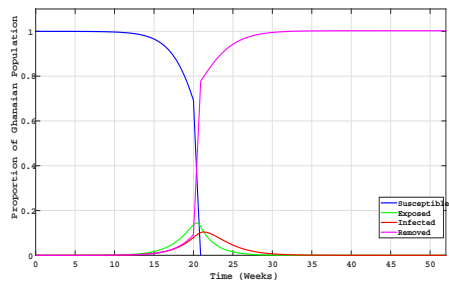


Fig. 21. With vaccine rate 0.7

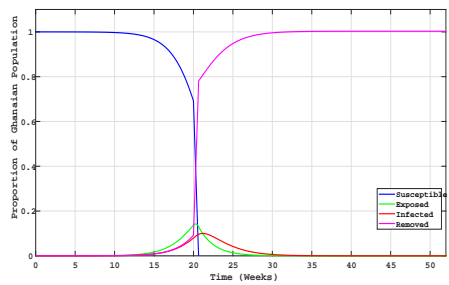


Fig. 22. With vaccine rate 1

## 4 Conclusions

In this paper, a standard SEIR epidemiology model with periodic contact rate has been developed to examine the mode of transmission (at the peak) and the rate of Vaccination impact on HBI. It is classified into Vaccination, Susceptible, Exposed, Infection, and Recovery. The Routh-Hurwitz theorem was employed in the stability analysis to determine the state of disease-free of the Hepatitis

B infection. In order to test the methodology, simulations were carried out and the results were analyzed and discussed. The results showed that a periodic contact rate could lead to chaotic behavior for the SEIR model, thereby providing an explanation for seasonal variations of the number of infected individuals for Hepatitis B infection. In fact, when the  $\beta_1 \neq 0$  the system becomes highly hyper-chaotic. Again, the different vaccination rates impact on the HBI in Ghana was examined. The results from the simulations suggested that the vaccination strategies in newborns played the most important role in reducing Hepatitis B prevalence. It was discovered that even if the vaccination rate is 70%, the infection rate would be reduced to the minimum barest. The periodic contact rate could create dynamical phenomena which are highly inimical for biological systems as it mostly leads to a spike in the disease out-break and therefore, it is recommended that further research would include a control strategy to handle chaotic dynamical behavior.

## Data Availability Statement

All data used can be found in the manuscript.

## Competing Interests

The authors declare that they have no competing interests and there was no funding for this study.

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